Matrix Elements of the Linearized Collision Operator

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The matrix elements of the linearized Boltzmann collision operator with respect to the Burnett functions are constructed for arbitrary power law potentials. The result of Mott-Smith for elastic spheres appears as a special case.

1. INTRODUCTION

The matrix elements of the linearized Boltzmann collision operator formed with respect to the so-called Burnett functions are useful for calculation of kinetic and transport properties for nearly ideal gases. They were first introduced by Burnett, and were used extensively by Chapman and Cowling, who give general expressions for the lower-order matrix elements.[1,2] Some years ago Wang Chang and Uhlenbeck and, independently, Mott-Smith, proved that the matrix is diagonal for the case of Maxwell molecules, i.e., the Burnett functions are eigenfunctions of the linearized collision operator.[3-5] These authors also found the general expression for the eigenvalues. In the same work Mott-Smith obtained a general expression for the matrix elements for the case of elastic spheres.

This result of Mott-Smith can be generalized to the case of arbitrary power law forces, and it is the purpose of this paper to present a succinct derivation of this general result, which at the same time encompasses the cases of hard spheres, Maxwell molecules, and all intermediate power laws. In the next section the linearized collision operator and also the related Hilbert operator are defined; the point here is to fix and clarify the notation. The following section is devoted to the derivation of the formula for the matrix elements. In the final section the relations of this result to the results and definitions of other authors is made explicit.

2. LINEARIZED COLLISION OPERATOR

The linearized Boltzmann collision operator J may be written in the form[6]

$$Jh = \pi^{-\frac{3}{2}} \int d\mathbf{c}_1\, e^{-c_1^2} \cdot \int d\Omega\, gI(g, \theta)[h(\mathbf{c}') + h(\mathbf{c}_1') - h(\mathbf{c}) - h(\mathbf{c}_1)], \quad (2.1)$$

where dimensionless velocity variables are used. Thus,

$$\mathbf{c} = (m/2kT)^{\frac{1}{2}}\mathbf{v}, \quad (2.2)$$

with m the mass of a gas particle, T the local temperature, k Boltzmann's constant, and $\mathbf{v}$ the velocity of a gas particle. The four velocities in (2.1) are the velocities of a binary collision $(\mathbf{c}, \mathbf{c}_1) \leftrightarrows (\mathbf{c}', \mathbf{c}_1')$. These are related by the conservation laws; from conservation of linear momentum we have that the velocity of the center of mass is constant,

$$\mathbf{G} = \tfrac{1}{2}(\mathbf{c} + \mathbf{c}_1) = \tfrac{1}{2}(\mathbf{c}' + \mathbf{c}_1'), \quad (2.3)$$

while from conservation of energy we have that the relative velocity is unchanged in magnitude,

$$\mathbf{g} = \mathbf{c} - \mathbf{c}_1, \quad \mathbf{g}' = \mathbf{c}' - \mathbf{c}_1', \quad |\mathbf{g}| = |\mathbf{g}'| = g. \quad (2.4)$$

Thus a binary collision is characterized by the rotation of the relative velocity. In (2.1) the quantity $I(g, \theta)$ is the collision cross section for a collision in which the relative velocity is rotated through angle θ into solid angle $d\Omega$; the integral is over all directions of $\mathbf{g}'$. Note that

$$\mathbf{g}\cdot\mathbf{g}' = g^2 \cos\theta. \quad (2.5)$$

The collision cross section is determined from the dynamics of a binary collision. If the interparticle potential energy for two particles separated by a distance r is $\Phi(r)$, then

$$I(g, \theta) = \frac{p}{\sin\theta}\left|\frac{dp}{d\theta}\right|, \quad (2.6)$$

where p is the impact parameter, determined as a function of θ through the relation[7]

$$\theta = \pi - 2\int_0^{\eta_0} d\eta\left[1 - \eta^2 - \frac{2}{kTg^2}\,\Phi\left(\frac{p}{\eta}\right)\right]^{-\frac{1}{2}}. \quad (2.7)$$

[1] D. Burnett, Proc. London Math. Soc. **39**, 385 (1935).
[2] S. Chapman and T. E. Cowling, *The Mathematical Theory of Nonuniform Gases* (Cambridge University Press, Cambridge, 1958), 2nd ed.
[3] C. S. Wang Chang and G. E. Uhlenbeck, University of Michigan Report, Project M999 (1952).
[4] H. M. Mott-Smith, Lincoln Laboratory Group Report V-2 (1954).
[5] See also L. Waldman in *Handbuch der Physik*, S. Flugge, Ed. (Springer-Verlag, Berlin, 1960), Vol. 12, Sec. 38.
[6] The notation used here is that of G. E. Uhlenbeck and G. W. Ford, *Lectures in Statistical Mechanics* (American Mathematical Society, Providence, Rhode Island, 1963).
[7] See, e.g., L. Landau and S. Lifshitz, *Mechanic* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1960), p. 18. Note that for binary scattering the mass must be replaced by the reduced mass $m/2$ in their formulas. Equation (3) on p. 78 of Ref. 6 is correct only with this replacement.

Here η_0 is the smallest zero of the quantity in square brackets. For the special case of a repulsive power law potential, where $\Phi(r)$ is of the form

$$\Phi(r) = \Phi_0\left(\frac{a}{r}\right)^s, \quad (2.8)$$

with Φ_0 and a both positive constants, one can readily see from (2.6) and (2.7) that

$$I(g, \theta) = a^2\left(\frac{2\Phi_0}{g^2kT}\right)^{2/s} f_s(\theta), \quad (2.9)$$

where $f_s(\theta)$ is a purely numerical function of θ alone.

Note that in Eq. (2.1) the interchange of $\mathbf{c}$ and $\mathbf{c}_1$ corresponds to changing the sign of $\mathbf{g}$, i.e., to replacing θ by $\pi - \theta$. Since the quantity in square brackets is symmetric under this interchange, only the part of the cross section which is symmetric under this interchange contributes to the integral. Hence, if we replace $gI(g, \theta)$ by the quantity

$$F(g, \theta) = \tfrac{1}{2}g[I(g, \theta) + I(g, \pi - \theta)], \quad (2.10)$$

we can write (2.1) in the form

$$\mathbf{J}h = \pi^{-\frac{3}{2}} \int d\mathbf{c}_1\, e^{-c_1^2} \cdot \int d\Omega\, F(g, \theta)[2h(\mathbf{c}') - h(\mathbf{c}) - h(\mathbf{c}_1)]. \quad (2.11)$$

This expression will save some writing in the following discussion.

The linearized collision operator (2.1) is a self-adjoint linear operator, providing we define the scalar product of two functions $\psi(\mathbf{c})$ and $\Phi(\mathbf{c})$ to be

$$(\psi, \Phi) = \pi^{-\frac{3}{2}} \int d\mathbf{c}\, e^{-c^2}\psi(\mathbf{c})\Phi(\mathbf{c}). \quad (2.12)$$

The self-adjoint property of $\mathbf{J}$ means that

$$(\psi, \mathbf{J}\Phi) = (\mathbf{J}\psi, \Phi). \quad (2.13)$$

The so-called Burnett functions form a complete set in velocity space. They are given by

$$\chi_{rlm}(\mathbf{c}) = c^l L_r^{(l+\frac{1}{2})}(c^2) Y_{lm}(\hat{c}), \quad (2.14)$$

where $Y_{lm}(\hat{c})$ is the spherical harmonic and

$$L_r^{(\alpha)}(z) = \frac{1}{r!} z^{-\alpha} e^z \frac{d^r}{dz^r} e^{-z} z^{r+\alpha} \quad (2.15)$$

is the Laguerre polynomial.[8,9] The functions χ_{rlm} are orthogonal but not normalized; we have that

$$(\chi^*_{rlm}, \chi_{r'l'm'}) = \frac{\Gamma(r + l + \frac{3}{2})}{4\pi r!\Gamma(\frac{3}{2})} \delta_{rr'}\, \delta_{ll'}\, \delta_{mm'}. \quad (2.16)$$

The normalized Burnette functions are denoted by ψ_{rlm};

$$\psi_{rlm}(\mathbf{c}) = \left[\frac{4\pi r!\Gamma(\frac{3}{2})}{\Gamma(r + l + \frac{3}{2})}\right]^{\frac{1}{2}} \chi_{rlm}(\mathbf{c}). \quad (2.17)$$

The linearized collision operator is a scalar operator. That is, when operating on a function of the form

$$h(\mathbf{c}) = f(c)Y_{lm}(\hat{c}), \quad (2.18)$$

where $f(c)$ is a function of the magnitude of $\mathbf{c}$ alone, then

$$\mathbf{J}h = g(c)Y_{lm}(\hat{c}), \quad (2.19)$$

where $g(c)$ is also a function of c alone. This has the consequence that the matrix elements of $\mathbf{J}$ with respect to the Burnett functions are diagonal in l and m, and independent of m. Thus

$$(\chi^*_{rlm}, \mathbf{J}\chi_{r'l'm'}) = M^l_{rr'}\, \delta_{ll'}\, \delta_{mm'}, \quad (2.20)$$

and

$$(\psi^*_{rlm}, \mathbf{J}\psi_{r'l'm'}) = J^l_{rr'}\, \delta_{ll'}\, \delta_{mm'}, \quad (2.21)$$

where

$$J^l_{rr'} = 4\pi\Gamma(\tfrac{3}{2}) \cdot \left[\frac{r!r'!}{\Gamma(r + l + \frac{3}{2})\Gamma(r' + l + \frac{3}{2})}\right]^{\frac{1}{2}} M^l_{rr'}. \quad (2.22)$$

It is often convenient to write the linearized collision operator in the form

$$\mathbf{J}h = -m(c)h + \mathbf{K}h, \quad (2.23)$$

where

$$\mathbf{K}h = \pi^{-\frac{3}{2}} \int d\mathbf{c}_1\, e^{-c_1^2} \int d\Omega\, gI(g, \theta) \cdot [h(\mathbf{c}') + h(\mathbf{c}_1') - h(\mathbf{c}_1)], \quad (2.24)$$

and

$$m(c) = \pi^{-\frac{3}{2}} \int d\mathbf{c}_1\, e^{-c_1^2} \int d\Omega\, gI(g, \theta). \quad (2.25)$$

The operator $\mathbf{K}$, called the Hilbert operator, has many simple properties: it can be written as an integral operator, for the case of hard spheres it is compact and positive definite, etc. The matrix elements of this operator are of the form

$$(\psi^*_{rlm}, \mathbf{K}\psi_{r'l'm'}) = K^l_{rr'}\, \delta_{ll'}\, \delta_{mm'}. \quad (2.26)$$

[8] For spherical harmonics the notation is that of A.R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957), especially pp. 19–24.

[9] For Laguerre polynomials the notation is that of W. Magnus and F. Oberhettinger, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Chelsea Publishing Company, New York, 1949), especially pp. 84–85.

In the next section an expression for the matrix elements $M^l_{rr'}$, will be obtained for case of repulsive power law forces, for which $I(g, \theta)$ is the form (2.11).

3. THE MATRIX ELEMENTS $M^l_{rr'}$

From (2.20) we can write

$$M^l_{rr'} = \frac{1}{2l+1} \sum_{m=-l}^{l} (\chi^*_{rlm}, J\chi_{r'lm}). \tag{3.1}$$

If we recall the addition theorem for spherical harmonics[10];

$$\sum_{m=-l}^{l} Y^*_{lm}(\hat{c}) Y_{lm}(\hat{c}_1) = \frac{2l+1}{4\pi} P_l(\hat{c}\cdot\hat{c}_1), \tag{3.2}$$

where $P_l(x)$ is the Legendre polynomial, then using (2.1), (2.13), and (2.14), we can write

$$M^l_{rr'} = \frac{1}{4\pi^4} \int d\mathbf{c} \int d\mathbf{c}_1\, e^{-c^2-c_1^2} c^l L_r^{(l+\frac{1}{2})}(c^2) \cdot \int d\Omega\, F(g, \theta)[2c'^l L_{r'}^{(l+\frac{1}{2})}(c'^2) P_l(\hat{c}\cdot\hat{c}') - c^l L_{r'}^{(l+\frac{1}{2})}(c^2) - c_1^l L_{r'}^{(l+\frac{1}{2})}(c_1^2) P_l(\hat{c}\cdot\hat{c}_1)]. \tag{3.3}$$

We note the generating function for the Laguerre polynomials[9]

$$(1-x)^{-\alpha-1} e^{-xz/1-x} = \sum_{r=0}^{\infty} L_r^{(\alpha)}(z) x^r. \tag{3.4}$$

Hence, the matrix elements (3.3) may be obtained from the generating function

$$M^l(x, y) = \sum_{r=0}^{\infty} \sum_{r'=0}^{\infty} M^l_{rr'} x^r y^{r'} = \frac{1}{4\pi^4[(1-x)(1-y)]^{l+\frac{3}{2}}} \cdot \int d\mathbf{c} \int d\mathbf{c}_1\, e^{-c^2-c_1^2} \int d\Omega\, F(g, \theta) \cdot \Big[2c^l c'^l \exp\Big(-\frac{x}{1-x}c^2 - \frac{y}{1-y}c'^2\Big) P_l(\hat{c}\cdot\hat{c}') - c^{2l} \exp\Big[-\Big(\frac{x}{1-x} + \frac{y}{1-y}\Big)c^2\Big] - c^l c_1^l \exp\Big(-\frac{x}{1-x}c^2 - \frac{y}{1-y}c_1^2\Big) P_l(\hat{c}\cdot\hat{c}_1)\Big]. \tag{3.5}$$

Next we note the integral representation of Laplace and Mehler for the Legendre polynomials[11]

$$P_l(\cos\beta) = \frac{1}{2\pi} \int_0^{2\pi} d\alpha\, (\cos\beta + i \sin\beta \cos\alpha)^l. \tag{3.6}$$

Hence,

$$c^l c_1^l P_l(\hat{c}\cdot\hat{c}_1) = \frac{1}{2\pi} \int_0^{2\pi} d\alpha\, (\mathbf{c}\cdot\mathbf{c}_1 + i\, |\mathbf{c}\times\mathbf{c}_1| \cos\alpha)^l. \tag{3.7}$$

But we can write

$$|\mathbf{c}\times\mathbf{c}_1| \cos\alpha = \hat{l}\cdot\mathbf{c}\times\mathbf{c}_1, \tag{3.8}$$

where $\hat{l}$ is a unit vector which is perpendicular to some fixed direction in the plane of $\mathbf{c}$ and $\mathbf{c}_1$, say, to be definite, the direction of $\mathbf{c} - \mathbf{c}_1$ (see Fig. 1). Therefore, using this expression in Eq. (3.6) and then multiplying both sides by $t^l/l!$ and summing, we obtain the result

$$\sum_{l=0}^{\infty} c^l c_1^l P_l(\hat{c}\cdot\hat{c}_1) = \frac{1}{2\pi} \int_0^{2\pi} d\alpha\, \exp[t(\mathbf{c}\cdot\mathbf{c}_1 + i\hat{l}\cdot\mathbf{c}\times\mathbf{c}_1)], \tag{3.9}$$

where, to repeat, the integral is over all directions of $\hat{l}$ perpendicular to $\mathbf{c} - \mathbf{c}_1$;

$$\hat{l}\cdot(\mathbf{c} - \mathbf{c}_1) = 0. \tag{3.10}$$

Since the relation (3.9) is valid for arbitrary $\mathbf{c}$ and $\mathbf{c}_1$, if we form the generating function

$$M(x, y; t) = \sum_{l=0}^{\infty} M^l(x, y) \frac{t^l}{l!}, \tag{3.11}$$

then, using Eqs. (3.5) and (3.9) we can write

$$M(x, y; t) = \frac{1}{8\pi^5[(1-x)(1-y)]^{\frac{3}{2}}} \int d\mathbf{c} \int d\mathbf{c}_1 \int d\Omega\, F(g, \theta) \int_0^{2\pi} d\alpha \cdot\Big\{2 \exp\Big[-\frac{1}{1-x}c^2 - c_1^2 - \frac{y}{1-y}c'^2 + \frac{t(\mathbf{c}\cdot\mathbf{c}' + i\hat{l}\cdot\mathbf{c}\times\mathbf{c}')}{(1-x)(1-y)}\Big] - \exp\Big(-\Big[\frac{1}{1-x} + \frac{y}{1-y} - \frac{t}{(1-x)(1-y)}\Big]c^2 - c_1^2\Big) - \exp\Big[-\frac{1}{1-x}c^2 - \frac{1}{1-y}c_1^2 + \frac{t(\mathbf{c}\cdot\mathbf{c}_1 + i\hat{l}\cdot\mathbf{c}\times\mathbf{c}_1)}{(1-x)(1-y)}\Big]\Big\}. \tag{3.12}$$

[10] Reference 8, p. 63.

[11] Reference 9, p. 52.

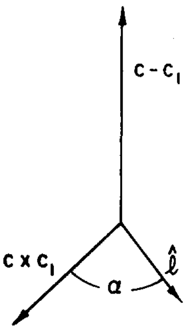

FIG. 1. The angle α lies in the plane perpendicular to $\mathbf{c} - \mathbf{c}_1$.

Here in the first term the α integration is over all directions of $\hat{l}$ perpendicular to $\mathbf{c} - \mathbf{c}'$ and in the last over all directions of $\hat{l}$ perpendicular to $\mathbf{c} - \mathbf{c}_1$; the second term is independent of α so the α integration just gives a factor of 2π.

In this last expression we change variables to $\mathbf{G}$ and $\mathbf{g}$ given by (2.3) and (2.4). That is, we put

$$\mathbf{c} = \mathbf{G} + \tfrac{1}{2}\mathbf{g},$$
$$\mathbf{c}_1 = \mathbf{G} - \tfrac{1}{2}\mathbf{g}, \tag{3.13}$$

while

$$\mathbf{c}' = \mathbf{G} + \tfrac{1}{2}\mathbf{g}', \tag{3.14}$$

with $\mathbf{g}'$ given by (2.5). It is easy to show that

$$d\mathbf{c}\, d\mathbf{c}_1 = d\mathbf{G}\, d\mathbf{g}, \tag{3.15}$$

and, therefore, that (3.12) becomes

$$\begin{aligned} M(x, y; t) = {} & \frac{1}{8\pi^5[(1-x)(1-y)]^{\frac{3}{2}}} \int d\mathbf{G} \int d\mathbf{g} \int d\Omega\, F(g, \theta) \exp\left[-\frac{2-x-y-t}{(1-x)(1-y)} G^2\right] \\ & \int_0^{2\pi} d\alpha \left\{ 2 \exp\left[-\frac{x(1-y)\mathbf{g} + y(1-x)\mathbf{g}' - \frac{1}{2}t(\mathbf{g}+\mathbf{g}') - \frac{1}{2}it\hat{l}\times(\mathbf{g}-\mathbf{g}')}{(1-x)(1-y)} \cdot \mathbf{G} \right.\right. \\ & \left. - \frac{(2-x-y)g^2 - t(\mathbf{g}\cdot\mathbf{g}' + i\hat{l}\cdot\mathbf{g}\times\mathbf{g}')}{4(1-x)(1-y)} \right] \\ & - \exp\left[-\frac{x(1-y) + y(1-x) - t}{(1-x)(1-y)} \mathbf{g}\cdot\mathbf{G} - \frac{2-x-y+t}{4(1-x)(1-y)} g^2 \right] \\ & \left. - \exp\left[-\frac{(x-y)\mathbf{g} - it\hat{l}\cdot\mathbf{g}}{(1-x)(1-y)} \cdot \mathbf{G} - \frac{2-x-y+t}{4(1-x)(1-y)} g^2 \right] \right\}. \end{aligned} \tag{3.16}$$

Here the α integration is over-all directions of $\hat{l}$ such that $\hat{l}\cdot(\mathbf{g} - \mathbf{g}') = 0$ in the first term and $\hat{l}\cdot\mathbf{g} = 0$ in the last term. We now perform the $\mathbf{G}$ integration, using the integral formula

$$\int d\mathbf{G}\, e^{-aG^2 + \mathbf{b}\cdot\mathbf{G}} = \left(\frac{\pi}{a}\right)^{\frac{3}{2}} e^{b^2/4a}. \tag{3.17}$$

Then (3.16) becomes, after working out the algebra and using (2.5),

$$\begin{aligned} M(x, y; t) = {} & \frac{1}{4\pi^{5/2}(2-x-y-t)^{\frac{3}{2}}} \int d\mathbf{g} \int d\Omega\, F, (g, \theta) \\ & \cdot\left\{ \frac{1}{\pi}\int_0^{2\pi} d\alpha \exp\left[-\frac{[1-(xy+t)\cos^2\frac{1}{2}\theta]g^2 - \frac{1}{2}it\hat{l}\cdot\mathbf{g}\times\mathbf{g}'}{2-x-y-t} \right] \right. \\ & \left. - \exp\left(-\frac{1-xy-t}{2-x-y-t} g^2 \right) - \exp\left(-\frac{1}{2-x-y-t} g^2 \right) \right\}. \end{aligned} \tag{3.18}$$

Remember that the α integration is over all directions of $\hat{l}$ perpendicular to $\mathbf{g} - \mathbf{g}'$. Since $\mathbf{g}\times\mathbf{g}'$ is itself perpendicular to $\vec{\mathbf{g}} - \vec{\mathbf{g}}'$, we have

$$\begin{aligned} \hat{l}\cdot\mathbf{g}\times\mathbf{g}' &= |\mathbf{g}\times\mathbf{g}'| \cos\alpha \\ &= g^2 \sin\theta \cos\alpha = 2g^2 \sin\frac{\theta}{2} \cos\frac{\theta}{2} \cos\alpha. \end{aligned} \tag{3.19}$$

If we put this in Eq. (3.18), and then replace

$$\mathbf{g} \rightarrow (2-x-y-t)^{\frac{1}{2}}\mathbf{g}, \tag{3.20}$$

we get

$$\begin{aligned} M(x,y;t) = {} & \frac{1}{4\pi^{5/2}} \int d\mathbf{g} \int d\Omega\, F([2-x-y-t]^{\frac{1}{2}}g, \theta) \\ & \cdot\left[\frac{1}{\pi}\int_0^{2\pi} d\alpha \exp\left\{ -\left[1 - xy\cos^2\frac{\theta}{2} + t\cos\frac{\theta}{2} \right.\right.\right. \\ & \left.\left. \cdot\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\cos\alpha \right) \right] g^2 \right\} \\ & \left. - \exp\left[-(1-xy-t)g^2\right] - \exp(-g^2) \right]. \end{aligned} \tag{3.21}$$

Up to this point we have made no assumptions about the form of the cross section; the expression (3.21) is quite general. If we specialize to the case of repulsive power law potentials, for which the cross section is of the form (2.9), we see from (2.10) that

$$F(g,\theta) = a^2\left(\frac{2\Phi_0}{kT}\right)^{2/s} g^{1-(4/s)}F_s(\theta), \qquad (3.22)$$

where

$$F_s(\theta) = \tfrac{1}{2}[f_s(\theta) + f_s(\pi-\theta)] \qquad (3.23)$$

is a purely numerical function of θ alone. Inserting (3.22) in (3.21) and using the integral formula

$$\int d\mathbf{g}\, g^{1-(4/s)}e^{-ag^2} = \pi^{\frac{3}{2}}a^{(2/s)-2}\,\frac{\Gamma[2-(2/s)]}{\Gamma(\frac{3}{2})}, \qquad (3.24)$$

we obtain

$$M(x, y; t)$$
$$= a^2\left(\frac{2\Phi_0}{kT}\right)^{2/s}\frac{\Gamma[2-(2/s)]}{4\pi\Gamma(\frac{3}{2})}(2-x-y-t)^{\frac{1}{2}-(2/s)}$$
$$\cdot\int d\Omega\, F_s(\theta)\left\{\frac{1}{\pi}\int_0^{2\pi} d\alpha\left[1 - xy\cos^2\frac{\theta}{2} + t\cos\frac{\theta}{2}\right.\right.$$
$$\left.\cdot\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\cos\alpha\right)\right]^{(2/s)-2}$$
$$\left.-(1-xy-t)^{(2/s)-2} - 1\right\}. \qquad (3.25)$$

Next we note the formula

$$(1-a-b)^{-\alpha}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\Gamma(n+m+\alpha)}{n!m!\Gamma(\alpha)}a^nb^m. \qquad (3.26)$$

So we can write

$$\left[1 - xy\cos^2\frac{\theta}{2}\right.$$
$$\left.-t\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\cos\alpha\right)\right]^{(2/s)-2}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\Gamma[2-(2/s)+n+m]}{\Gamma[2-(2/s)]n!m!}\cos^{2n+m}\frac{\theta}{2}$$
$$\cdot\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\cos\alpha\right)^m(xy)^nt^m. \qquad (3.27)$$

Hence, using (3.6) we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} d\alpha\left[1 - xy\cos^2\frac{\theta}{2}\right.$$
$$\left.-t\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\cos\alpha\right)\right]^{(2/s)-2}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\Gamma[n+m+2-(2/s)]}{\Gamma[2-(2/s)]n!m!}$$
$$\cdot\cos^{2n+m}\frac{\theta}{2}P_m\left(\cos\frac{\theta}{2}\right)(xy)^nt^m. \qquad (3.28)$$

Similarly, we obtain from (3.26) the expansion

$$(1-xy-t)^{(2/s)-2}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\Gamma[n+m+2-(2/s)]}{\Gamma[2-(2/s)]n!m!}(xy)^nt^m. \qquad (3.29)$$

Using these last two expansions in (3.25), we have

$$M(x, y; t)$$
$$= \frac{a^2}{2}\left(\frac{2\Phi_0}{kT}\right)^{2/s}\frac{\Gamma[2-(2/s)]}{\Gamma(\frac{3}{2})}(2-x-y-t)^{\frac{1}{2}-(2/s)}$$
$$\cdot\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}B_m^n(s)(xy)^n(\tfrac{1}{2}t)^m, \qquad (3.30)$$

where

$$B_m^n(s) = \frac{2^{m+1}\Gamma[n+m+2-(2/s)]}{\Gamma[2-(2/s)]n!m!}\frac{1}{4\pi}\int d\Omega\, F_s(\theta)$$
$$\cdot\left[2\cos^{2n+m}\frac{\theta}{2}P_m\left(\cos\frac{\theta}{2}\right) - 1 - \delta_{n,0}\delta_{m,0}\right]. \qquad (3.31)$$

But, by an obvious extension of Eq. (3.26), we can write

$$(2-x-y-t)^{\frac{1}{2}-(2/s)} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{2^{\frac{1}{2}-(2/s)-i-j-k}\Gamma[i+j+k+(2/s)-\frac{1}{2}]}{\Gamma[(2/s)-\frac{1}{2}]i!j!k!}x^iy^jt^k. \qquad (3.32)$$

Using this in Eq. (3.30) we get

$$M(x, y; t) = a^2\left(\frac{2\Phi_0}{kT}\right)^{2/s}\frac{\Gamma[2-(2/s)]}{\Gamma(\frac{3}{2})}\sum_{i,j,k,n,m=0}^{\infty}$$
$$\cdot\frac{2^{-\frac{1}{2}-(2/s)-i-j-k-m}\Gamma[i+j+k+(2/s)-\frac{1}{2}]}{\Gamma[(2/s)-\frac{1}{2}]i!j!k!}B_m^n(s)x^{i+n}y^{j+n}t^{k+m}. \qquad (3.33)$$

If in these summations we replace

$$i = r-n, \quad j = r'-n, \quad k = l-m, \qquad (3.34)$$

we can write (3.33) in the form

$$M(x, y; t) = \sum_{r=0}^{\infty} \sum_{r'=0}^{\infty} \sum_{l=0}^{\infty} M_{rr'}^{l} x^r y^{r'} \frac{t^l}{l!}, \tag{3.35}$$

where

$$M_{rr'}^{l} = a^2 \left(\frac{\Phi_0}{kT}\right)^{2/s} \frac{\Gamma[2-(2/s)] l!}{\Gamma(\frac{3}{2}) 2^{r+r'+l+\frac{1}{2}}} \sum_{n=0}^{r,r'} \sum_{m=0}^{l} \frac{4^n \Gamma[r+r'-2n+l-m+(2/s)-\frac{1}{2}]}{\Gamma[(2/s)-\frac{1}{2}](r-n)!(r'-n)!(l-m)!} B_m^n(s). \tag{3.36}$$

Here the upper limit of the n summation is r or r', whichever is smaller. This is the desired expression for $M_{rr'}^{l}$.

4. EXPRESSIONS FOR THE MATRIX ELEMENTS

Using the result (3.36) of the previous section, we obtain the following expression for the matrix elements of **J** with respect to the normalized Burnett functions, defined in (2.21):

$$J_{rr'}^{l} = \pi a^2 \left(\frac{\Phi_0}{kT}\right)^{2/s} 2^{\frac{1}{2}-r-r'-l} \Gamma\left(2-\frac{2}{s}\right) l! \left[\frac{r! r'!}{\Gamma(r+l+\frac{3}{2})\Gamma(r'+l+\frac{3}{2})}\right]^{\frac{1}{2}} \cdot \sum_{n=0}^{r,r'} \sum_{m=0}^{l} \frac{4^n \Gamma[r+r'-2n+l-m+(2/s)-\frac{1}{2}]}{\Gamma[(2/s)-\frac{1}{2}](r-n)!(r'-n)!(l-m)!} B_m^n(s), \tag{4.1}$$

where the coefficients $B_m^n(s)$ are given by (3.31). Alternatively, we can use the expression (3.23) to write

$$B_m^n(s) = \frac{2^{m+1}\Gamma[n+m+2-(2/s)]}{\Gamma[2-(2/s)]n!m!} \frac{1}{4\pi} \int d\Omega\, f_s(\theta) \cdot \left[\cos^{2n+m}\frac{\theta}{2} P_m\left(\cos\frac{\theta}{2}\right) + \sin^{2n+m}\frac{\theta}{2} P_m\left(\sin\frac{\theta}{2}\right) - 1 - \delta_{n,0}\,\delta_{m,0}\right]. \tag{4.2}$$

Here $f_s(\theta)$ is the purely numerical function of θ defined by (2.9).

The matrix elements of the Hilbert operator, defined in (2.26), are easily obtained; we need merely drop the "1" in square brackets in Eq. (4.2). Thus we have

$$K_{rr'}^{l} = \pi a^2 \left(\frac{\Phi_0}{kT}\right)^{2/s} 2^{\frac{1}{2}-r-r'-l} \Gamma\left(2-\frac{2}{s}\right) l! \left[\frac{r! r'!}{\Gamma(r+l+\frac{3}{2})\Gamma(r'+l+\frac{3}{2})}\right]^{\frac{1}{2}} \cdot \sum_{n=0}^{r,r'} \sum_{m=0}^{l} \frac{4^n \Gamma[r+r'-2n+l-m+(2/s)-\frac{1}{2}]}{\Gamma[(2/s)-\frac{1}{2}](r-n)!(r'-n)!(l-m)!} A_m^n(s), \tag{4.3}$$

where

$$A_m^n(s) = \frac{2^{m+1}\Gamma[n+m+2-(2/s)]}{\Gamma[2-(2/s)]n!m!} \frac{1}{4\pi} \int d\Omega\, f_s(\theta) \cdot \left[\cos^{2n+m}\frac{\theta}{2} P_m\left(\cos\frac{\theta}{2}\right) + \sin^{2n+m}\frac{\theta}{2} P_m\left(\sin\frac{\theta}{2}\right) - \delta_{n,0}\,\delta_{m,0}\right]. \tag{4.4}$$

We should note that this last expression diverges for the repulsive power law potential, since it can be shown that for small angles,[12]

$$f_s(\theta) \cong \frac{1}{s}\left[\frac{\Gamma(\frac{1}{2})\Gamma[(s+1)/2]}{\Gamma(s/2)}\right]^{2/s} \theta^{-2-(2/s)}. \tag{4.5}$$

It is still, however, a useful expression for model calculations where $f_s(\theta)$ is chosen to be an integrable function of θ.

The case of Maxwell molecules, for which $s = 4$, is especially simple. Here, from (4.1) we see that $J_{r,r'}^{l}$ vanishes unless $r = r'$, in which case the only nonvanishing term in the sums is that for which $n = r$ and $m = l$. Hence, using (4.2), we have

$$J_{rr}^{l} = a^2 \left(\frac{2\Phi_0}{kT}\right)^{\frac{1}{2}} \int d\Omega\, f_4(\theta) \left[\cos^{2r+l}\frac{\theta}{2} P_l\left(\cos\frac{\theta}{2}\right) + \sin^{2r+l}\frac{\theta}{2} P_l\left(\sin\frac{\theta}{2}\right) - 1 - \delta_{r,0}\,\delta_{l,0}\right], \tag{4.6}$$

which is the well-known expression for the eigen-

[12] Reference 7, pp. 56–57.

values of the Maxwell collision operator.[13] Here

$$\sin\theta f_4(\theta) = -\frac{d\cot 2\phi}{d\theta}, \tag{4.7}$$

where ϕ is given as a function of θ by

$$\theta = \pi - 2(\cos 2\phi)^{\frac{1}{2}}K(\sin\phi). \tag{4.8}$$

Here

$$K(k) = \int_0^{\pi/2} d\alpha\,[1 - k^2\sin^2\alpha]^{-\frac{1}{2}} \tag{4.9}$$

is the complete elliptic integral of the first kind.

The case of elastic spheres of diameter a is obtained by taking the limit $s \rightarrow \infty$. Here

$$f_\infty(\theta) = \tfrac{1}{4} \tag{4.10}$$

and, using the integral formula[14]

$$\int_0^1 dx\, x^{2n+m+1}P_m(x) = 2^{-m-1}\frac{(2n+m+1)!n!}{(2n+1)!(n+m+1)!}, \tag{4.11}$$

one obtains from Eq. (4.2),

$$B_m^n(\infty) = \begin{cases} 0, & n = m = 0, \\ \dfrac{(2n+m+1)!}{(2n+1)!m!} - 2^{m-1}\dfrac{(n+m+1)!}{n!m!} & \text{otherwise.}\end{cases} \tag{4.12}$$

Finally, we should compare our notation for the matrix elements for the hard-sphere case with that used by other authors. The original result of Mott-Smith is quoted in a paper by Pekeris *et al.*, where the matrix elements are expressed in terms of a symbol $[rlr'l]$.[15] The relation with our notation is

$$[rlr'l] = -\frac{4\pi}{2l+1}M_{rr'}^l. \tag{4.13}$$

The well-known book of Chapman and Cowling introduces a notation for the first few matrix elements.[16] Thus, in their notation,

$$[S_{\frac{3}{2}}^{(r)}(c_1^2)c_1,\ S_{\frac{3}{2}}^{(r')}(c_1^2)c_1]_1 = -4\pi\left(\frac{2kT}{m}\right)^{\frac{1}{2}}M_{rr'}^1, \tag{4.14}$$

and

$$[S_{5/2}^{(r)}(c_1^2)c_1^0c_1,\ S_{5/2}^{(r')}(c_1^2)c_1^0c_1]_1 = -\frac{8\pi}{3}\left(\frac{2kT}{m}\right)^{\frac{1}{2}}M_{rr'}^2. \tag{4.15}$$

Finally, Sirovich and Thurber have tabulated numerical values of the first few matrix elements, still for the case of hard spheres of diameter a.[17] They introduce the symbol $B_{rl;r'l'}$ which is related to the $J_{rr'}^l$ of this paper by

$$B_{rl;r'l} = \frac{1}{4\pi}J_{rr'}^l. \tag{4.16}$$

[13] Z. Alterman, K. Frankowski, and C. L. Pekeris, Astrophys. J. Suppl. **7**, 291 (1962). In their expressions one must put $F(\theta) = 2^{-1/2}f_4(\theta)$ and $\lambda_{rl} = a^{-2}(2\Phi_0/kT)^{-1/2}J^l{}_{rr}$. The same $F(\theta)$ is introduced in Ref. 3, p. 43 and Ref. 6, p. 88, but each of these expressions is in error by a factor of $2^{1/2}$.

[14] Reference 9, p. 52.

[15] C. L. Pekeris, Z. Alterman, L. Finkelstein, and K. Frankowski, Phys. Fluids **5**, 1608 (1962).

[16] Reference 2, Chap. 9. See especially pp. 161 and 162.

[17] L. Sirovich and J. K. Thurber, in *Rarefied Gas Dynamics*, J. H. de Leeuw, Ed. (Academic Press Inc., New York, 1966), Vol. I, p. 21.