# Effect of the Temperature Gradient on the Drift Instabilities in a High-$\beta$ Collisionless Plasma

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The effect of temperature gradient on the drift instabilities in an inhomogeneous high-$\beta$ plasma is discussed. The analysis shows that in the presence of a temperature-gradient, ion-acoustic, and Alfvénic instabilities can be developed.

The purpose of the present note is to report a recent finding on drift instabilities in a high-$\beta$ plasma which might clarify some confusion pertaining to the temperature-gradient effect. The drift effect on the collisionless plasma due to the magnetic field and number density gradient has been extensively discussed in the literature,[1] most of which are restricted to the low $\beta$ case ($\beta \ll 1$), where $\beta$ is the ratio of plasma pressure to the magnetic pressure. However, in some controlled-fusion experiment and outer-space solar wind plasma, $\beta$ is known to be of the order of one. Mikhailovskaya and Mikhailovskii[2] show that the drift effect is highly suppressed by the enhanced Landau damping for high $\beta$ plasma if the temperature gradient is not taken into account.

In this note we plan to calculate the drift effect in an inhomogeneous high $\beta$ plasma with a temperature gradient. In a high $\beta$ plasma, the induced magnetic field effect must be included unlike that in the low $\beta$ approximation. Hence,

$$E_\alpha = -[(\partial\phi/\partial r_\alpha) + c^{-1}(\partial A_\alpha/\partial t)], \qquad (1)$$

where $\phi$ is a scalar potential, $A_\alpha$ is a vector potential which is related to the magnetic field, $B_\alpha = \mathcal{E}_{\alpha\beta\gamma}\partial A_\gamma/\partial r_\beta$, and $\mathcal{E}_{\alpha\beta\gamma}$ is a unit cyclic tensor. By using the definition of the dielectric tensor $\mathcal{E}_{\alpha\beta}$ and the polarizability vector $\chi_\alpha$ of the medium

$$\mathcal{E}_{\alpha\beta} = \delta_{\alpha\beta} + (4\pi i/\omega)\sigma_{\alpha\beta}; \qquad e(n_i - n_e) = \chi_\alpha E_\alpha;$$
$$j_\alpha = \sigma_{\alpha\beta}E_\beta, \qquad (2)$$

where $\sigma_{\alpha\beta}$ is the conductivity tensor, Maxwell's field equations become[3]

$$k^2\phi = -4\pi i\chi_\alpha k_\alpha\phi + (4\pi i\omega/c)\chi_\alpha A_\alpha, \qquad (3)$$

$$k^2 A_\alpha = (\omega^2/c^2)\mathcal{E}_{\alpha\beta}A_\beta - (\omega/c)\mathcal{E}_{\alpha\beta}k_\beta\phi \qquad (\alpha = x, y, z). \qquad (4)$$

Let us choose the unperturbed inhomogeneous magnetic field along the $Z$ axis. The spatial dependence of the field, density, and temperature vary along the $X$ axis. It is noted that under the low frequency, $\omega < \Omega_i$, and high $\beta$ assumption, all the plasma waves, except the drift waves with $k_z \ll k_\perp$ are highly damped by the Cerenkov absorption in an inhomogeneous plasma,[3,4] where $\Omega_i$ is ion Larmor frequency. For simplicity, let $k_\perp$ align with the $Y$ axis: Eqs. (3) and (4) may be reduced by means of the condition of existence of a nontrivial solution which requires the vanishing of the determinant of the coefficients of the equations,

$$\begin{vmatrix} -k^2(1+4\pi i\chi_\alpha k_\alpha/k^2) & 4\pi i\omega\chi_\alpha/c & 4\pi i\omega\chi_z/c \\ \mathcal{E}_{x\beta}k_\beta\omega/c & k^2-\omega^2\mathcal{E}_{xx}/c^2 & -\omega^2\mathcal{E}_{xz}/c^2 \\ \mathcal{E}_{z\beta}k_\beta\omega/c & -\omega^2\mathcal{E}_{zx}/c^2 & k^2-\omega^2\mathcal{E}_{zz}/c^2 \end{vmatrix} = 0. \qquad (5)$$

Equation (5) gives the dispersion relation for the wave modes of an inhomogeneous plasma. To evaluate the dispersion relation the explicit forms of dielectric tensor $\mathcal{E}_{\alpha\beta}$ and polarizability vector $\chi_\alpha$ must be obtained beforehand. The following relations are postulated that connect between the distribution function and dielectric tensor or polarizability vector:

$$\mathcal{E}_{\alpha\beta} = \delta_{\alpha\beta} + (4\pi i/\omega E_\beta)\sum_j e_j\int v_\alpha f_j\,d\mathbf{v};$$
$$\chi_\alpha = E_\alpha^{-1}\sum_j e_j\int f_j\,d\mathbf{v}, \qquad (6)$$

where $\sum_j$ is the summation of the species of electrons and ions.

The inhomogeneities of $\mathbf{B}_0$, $n$, and $T$ are assumed to be much smaller than their average values such that Taylor series expansion with respect to the inhomogeneity is applicable

$$\mathbf{B}_0 = \mathbf{B}_{00}(1+G_Bx+\cdots); \qquad n = n_0(1+G_nx+\cdots);$$
$$T = T_0(1+G_Tx+\cdots), \qquad (7)$$

where

$$G_B = B_{00}^{-1}\frac{dB}{dx}, \qquad G_T = T_0^{-1}\frac{dT}{dx}, \qquad G_n = n_0^{-1}\frac{dn}{dx},$$

and $B_{00}$, $T_0$, $n_0$ are the average values at the reference point. These gradients are the causes of drift currents[5,6] across the magnetic field, e.g.,

$$(v_{dj})_B = G_Ba_{j\perp}^2(2\Omega_j)^{-1}; \qquad (v_{dj})_n = G_na_{j\perp}^2(2\Omega_j)^{-1};$$
$$(v_{dj})_T = G_Ta_{j\perp}^2(2\Omega_j)^{-1}, \qquad (8)$$

where $a_{j\perp}^2 = 2T_{j\perp}/m_j$ is the thermal velocity of species $j$. The propagating frequencies of drift waves across a magnetic field are defined as

$$\omega_{Bj}^* = (v_{dj})_B/k_\perp; \qquad \omega_{nj}^* = (v_{dj})_n/k_\perp; \qquad \omega_{Tj}^* = (v_{dj})_T/k_\perp. \qquad (9)$$

The important wave dissipation in a collisionless plasma is the thermal phase-mixing resulting from wave-particle resonant interactions,[7] which are treated with the linearized Vlasov equation suitable for the inhomogeneous plasma oscillations. The zeroth-order solution to the Vlasov equation is a function of the constants of motion of the unperturbed orbits[6]

$$f_{j0}(\mathbf{v}, x) = f_{j0}(\mathcal{E}_{j\perp}, v_z, x+v_y/\Omega_j), \qquad (10)$$

where $\mathcal{E}_{j\perp} = \frac{1}{2}m_jv_\perp^2$. The perturbed distribution function may be obtained from the first-order Vlasov equation by integrating along a path of an unperturbed orbit. After lengthy but routine calculations,[1,3,6] the perturbed distribution function is reduced to

$$f_j=\left(\frac{e_j}{m_j}\right)|E_x,\ E_y,\ E_z|\sum_l \exp-i\left[\left(\frac{k_\perp v_\perp}{\Omega_j}\right)\sin(\theta-\Omega_j t-\psi)-l\Omega_j t\right]\left\{(\omega-k_y v_{Mj}-k_z v_z-l\Omega_j)^{-1}\left(\frac{M_j}{T_{j\perp}}\right)\right.$$
$$\times\left[-\left(1+k_y T_{j\perp}m_j^{-1}\omega^{-1}\Omega_j^{-1}\frac{\partial}{\partial x}\right)f_{j0}+\left(\frac{v_y T_{j\perp}}{\Omega_j}\right)\frac{\partial}{\partial x}\left(T_{j\perp}^{-1}+k_y m_j^{-1}\omega^{-1}\Omega_j^{-1}\frac{\partial}{\partial x}\right)f_{j0}\right]$$
$$\left.\times\begin{vmatrix} v_\perp[(l\Omega_j/k_\perp v_\perp)J_l(k_\perp v_\perp/\Omega_j)\cos\psi-iJ_l'(k_\perp v_\perp/\Omega_j)\sin\psi \\ -v_\perp[iJ_l'(k_\perp v_\perp/\Omega_j)\cos\psi+(l\Omega_j/k_\perp v_\perp)J_l(k_\perp v_\perp/\Omega_j)\sin\psi] \\ v_z J_l(k_\perp v_\perp/\Omega_j)\end{vmatrix}-\begin{vmatrix}0\\1\\0\end{vmatrix}\frac{i}{\omega\Omega_j}\left(\frac{\partial}{\partial x}-\frac{v_y}{\Omega_j}\frac{\partial^2}{\partial x^2}\right)f_{j0}\right\},\quad(11)$$

where $\psi=\tan^{-1}(k_y/k_x)$ and $v_{Mj}=G_{Bj}(v_\perp^2/2\Omega_j)$. Under the low frequency, $\omega<\Omega_j$, and long wavelength, $\rho_j k_\perp<1$, assumptions (where $\rho_j$ is the Larmor radius of species $j$), the explicit forms of the dielectric tensor $\mathcal{E}_{\alpha\beta}$ and polarizability vector $\chi_\alpha$ can be obtained by substituting (11) in (6) and using the relation

$$\int d\mathbf{v}=\int_0^{2\pi}d\theta\int_0^\infty v_\perp\,dv_\perp\int_{-\infty}^{\infty}dv_z.$$

The dispersion relation for the wave modes of an inhomogeneous high $\beta$ plasma is obtained by substituting the explicit forms of dielectric tensor $\mathcal{E}_{\alpha\beta}$ and polarizability vector $\chi_\alpha$ into Eq. (5). The calculations are rather cumbersome but straightforward.[3,6] After simplification, the dispersion relation becomes

$$|N_{\alpha\beta}|=0\qquad(\alpha,\beta=1,2,3),\tag{12}$$

where

$$N_{11}=-k^2+\frac{\omega_{pe}^2}{a_{e||}^2}\left[-\left(1+\frac{i\pi^{1/2}\omega}{k_z a_{e||}}\right)+\frac{i\pi^{1/2}\omega_{Be}^*a_{e\perp}^4}{k_z a_{e||}^5}+\frac{i\pi^{1/2}\omega_{ne}^*a_{e\perp}^2}{k_z a_{e||}^3}\left(-1+\frac{\omega_{Be}^*}{\omega}\right)\right.$$
$$\left.+i\frac{\pi^{1/2}\omega_{Te}^*a_{e\perp}^2}{2k_z a_{e||}^3}\left(1+\frac{\omega_{Be}^*}{\omega}\right)-\frac{\omega_{Ti}^*C_s^2}{\omega a_{i||}^2}(1-\tfrac{1}{2}\rho_i^2k_\perp^2)\right]$$

$$N_{12}=-\frac{2\omega_{pi}^2k_\perp a_{i\perp}^2}{c\Omega_i a_{i||}^2}\left(1+\frac{\omega_{ni}^*}{\omega}-\frac{\omega_{Ti}^*}{\omega}\right),$$

$$N_{13}=\frac{\omega_{pe}^2\omega}{a_{e||}^2c}\left[\frac{a_{e\perp}^2}{k_z a_{e||}^2}\left(1+\frac{\omega_{ne}^*}{\omega}\right)\left(1+\frac{i\pi^{1/2}\omega}{k_z a_{e||}}-\frac{i\pi^{1/2}\omega_{Be}^*}{k_z a_{e||}}\right)-\frac{i\pi^{1/2}\omega_{Te}^*a_{e\perp}^2}{2k_z^2a_{e||}^3}\left(1+\frac{\omega_{Be}^*}{\omega}\right)+\frac{\omega_{Ti}^*C_S^2}{k_z\omega a_{i||}^2}(1-\tfrac{1}{2}\rho_i^2k_\perp^2)\right],$$

$$N_{21}=i(\omega_{pi}^2\omega_{Ti}^*a_{i\perp}^2/2c\omega\Omega_i a_{i||}^2)k_\perp,$$

$$N_{22}=k^2-\frac{\omega^2}{c^2}-\left(1+\frac{\omega_{pi}^*}{\omega}\right)\frac{a_{i\perp}^4k_\perp^2}{2c_A^2a_{i||}^2}-\frac{3\omega_{Ti}^*a_{i\perp}^4k_\perp^2}{4\omega c_A^2a_{i||}^2},$$

$$N_{23}=-i(\omega_{Ti}^*\Omega_i a_{i\perp}^2k_\perp/2C_A^2k_z a_{i||}^2),$$

$$N_{31}=\frac{\omega k_z}{c}-\frac{\omega_{pe}^2\omega}{ca_{e||}^2k_z}\left\{\frac{a_{e\perp}^2}{a_{e||}^2}\left[\left(1-\frac{\omega_{Be}^*}{\omega}\right)\left(1+\frac{\omega_{ne}^*}{\omega}\right)+\frac{\omega_{Te}^*\omega_{Be}^*}{\omega^2}\right]+\frac{C_S^2\omega_{Ti}^*}{a_{i||}^2\omega}(1-\tfrac{1}{2}\rho_i^2k_\perp^2)\right\},$$

$$N_{32}=i(\omega_{Ti}^*\Omega_i a_{i\perp}^2/2c_A^2k_z a_{i||}^2)k_\perp,$$

$$N_{33}=k^2-\frac{\omega^2}{c^2}+\frac{\omega_{pe}^2\omega^2}{c^2a_{e||}^2k^2}\left\{\frac{a_{e\perp}^2}{a_{e||}^2}\left[\left(1+\frac{\omega_{ne}^*}{\omega}\right)\left(1-\frac{\omega_{Be}^*}{\omega}\right)-\frac{\omega_{Te}^*\omega_{Be}^*}{\omega^2}\right]+\frac{C_S^2\omega_{Ti}^*}{a_{i||}^2\omega}\right\}-\frac{\omega_{Ti}^*\omega k_\perp^2a_{i\perp}^2}{4C_A^2k_z^2a_{i||}^2}.$$

Here, $C_A$ and $C_S$ are Alfvén and ion-acoustic velocity, respectively.

It is noted that wave drift effect is important under the following condition[2]:

$$a_e\gg\omega/k_z;\qquad C_A\gg\omega/k_\perp.\tag{13}$$

With the inclusion of low frequency $\omega<\Omega_i$, long wavelength $\rho_i k_\perp<1$, nonrelativistic effect $\omega/k_z\ll c$, and drift-effect condition mentioned in (13), Eq. (12) may be further simplified. Let $\omega=\omega_k+i\gamma_k$, $\gamma_k$ be the imaginary part of the frequency or the growth rate is reduced under the above simplifications and we obtain

$$\gamma_k=2k(T_{e\perp}/T_{e||})(a_{i||}/c_A)^2\{[(v_{di})_T-(v_{di})_n]-\omega/k\}.\tag{14}$$

Here, $a_{i||}/c_A\leq1$ for the high $\beta$ plasma. For waves of small growth rate we can make $\gamma_k$ satisfy the condition $\gamma_k/\omega_k\ll1$. It is postulated that electron temperature ratio $T_{e\perp}/T_{e||}$ shall be either equal to or smaller than unity. When the electron temperature ratio $T_{e\perp}/T_{e||}$ is of the order of one, the real part of the frequency is reduced to

$$\omega_k \sim -c_S k_\perp (\rho_i G_{Ti})(T_i/T_e)^{1/2}. \quad (15)$$

This means that when ion drift velocity $[(v_{di})_T-(v_{di})_n]$ overcomes the phase velocity of ion-acoustic wave, the drift effect will excite the instability ion-acoustic wave across the field. Physically speaking, the gradient effects of ion temperature and ion density generate the ion current $j=en_i[(v_{di})_T-(v_{di})_n]$ flowing across the magnetic field. These crossing-field currents make the ion-acoustic wave unstable when the ion-drift velocity $[(v_{di})_T-(v_{di})_n]$ is greater than the phase velocity of ion-acoustic wave. Notice that from Eq. (14) when the temperature gradient vanishes, the growth rate of the system $\gamma_k$ takes negative values. This implies that the strong Landau damping suppresses the drift instability of high $\beta$ plasma oscillation when the temperature gradient is negligible. This strong damping mechanism of high $\beta$ plasma oscillation with vanishing temperature gradient complies with the result obtained by Mikhailovskaya and Mikhailovskii.[2]

When the electron temperature ratio $T_{e\perp}/T_{e\parallel}$ is much smaller than unity, the real part of the frequency is reduced to

$$\omega_k \sim -2c_A k_z(\rho_i^2 k_\perp^2). \quad (16)$$

Here, the coefficient inside the bracket denotes the finite Larmor radius correction which is very important in the high $\beta$ case. This expression shows that when the electron temperature directed along the field is much greater than the electron temperature across the field, the firehose instability drives the Alfvén wave unstable.

From the analysis we have shown that the existence of temperature gradient generates the currents across the magnetic field. It turns out that the drift effect induces the instability of ion-acoustic wave when the electron temperature ratio $T_{e\perp}/T_{e\parallel}$ is the order of one and the drift velocity is higher than the phase velocity of ion acoustic waves. Similarly, the drift effect also makes the electron temperature along the field much greater than that across the field,[8] this, in turn, causes the instability of the Alfvén wave.

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# Stabilization of Instabilities by Mode Coupling

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The nonlinear interaction between unstable and damped ion acoustic waves is investigated numerically. Energy couples to damped waves, stabilizing the system. The manner of stabilization depends strongly on the functional dependence of the growth (or damping) rate on wavenumber.

This note introduces a numerical study of the role played by resonant mode coupling in the stabilization of instabilities.

The equation for the $k$th mode can be written as[1]

$$\begin{aligned}\frac{dC(k)}{dt} &= \gamma(k)C(k) \\ &+ \sum_{k'} iV(k,k',k+k')C^*(k')C(k+k') \\ &\times(\exp\{-i[\omega(k)+\omega(k')-\omega(k+k')]\}t) \\ &+ \sum_{k'} iV(k',k-k',k)C(k')C(k-k') \\ &\times(\exp\{i[\omega(k')+\omega(k-k')-\omega(k)]\}t). \quad (1)\end{aligned}$$

Above $C(k)=\mathcal{E}(k,t)(4\pi)^{-1/2}(\partial\epsilon/\partial\omega)^{1/2}$, $\epsilon(k,\omega)$ is the plasma dielectric constant, and the total electric field is given by

$$\mathcal{E}(x,t)=\sum_k \mathcal{E}(k,t)\exp[i(kx-\omega_k t)]+\text{c.c.} \quad (2)$$

The matrix elements, e.g., $V(k,k',k-k')$, are symmetric with respect to argument interchange and the $\gamma_k$'s are the linear growth or damping rates. In the small amplitude limit, electric fields at each wavenumber grow or damp according to linear theory. As the amplitude of the unstable fields grows, the nonlinear terms will couple energy to damped modes and the system will presumably stabilize.

Equation (1) can be solved or greatly simplified in two cases. First, if there are only three interacting waves and no linear growth or damping, a solution in terms of elliptic functions is possible. When growth and damping are included, one could surmise that the equation for an unstable wave looks basically like

$$\frac{d}{dt}n(k)\sim 2\gamma n(k)-Vn^{3/2}(k), \quad (3)$$

where $n(k)=|C(k)|^2$. At nonlinear saturation, then,

$$n(k)\sim(\gamma/V)^2. \quad (4)$$