

A space–time in toroidal coordinates

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We present an exact solution of Einstein’s field equations in toroidal coordinates. The solution has three regions: an interior with a string equation of state; an Israel boundary layer; and an exterior with constant isotropic pressure and constant density, locally isometric to anti–de Sitter space–time. The exterior can be a cosmological vacuum with negative cosmological constant. The size and mass of the toroidal loop depend on the size of Λ . © 2003 American Institute of Physics.
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I. INTRODUCTION

There has been increasing interest in space–times with nonspherical topology and negative cosmological constant. Much of the discussion has focused on structures with horizons in anti–de Sitter (AdS) space–times.^{1–5} Vanzo⁶ pointed out that, in AdS space, there are black hole solutions with genus g horizons. Aminneborg *et al.* (ABHP)¹ discussed space–times locally isometric to AdS with horizons of arbitrary genus. While many current models of the universe seem to indicate that Λ is positive, there are some models with $\Lambda < 0$.⁷ Aside from their physical relevance to the actual structure of the Universe, solutions in AdS are very interesting as a comparison case to asymptotically flat solutions. For example, Hawking and Page⁸ have discussed the relevance of a negative Λ to black hole thermal stability. The 2 + 1 Bañados–Teitelboim–Zanelli⁹ black hole solution and its 3 + 1 black string¹⁰ lift have generated a large literature.¹¹

In this work we discuss a toroidal fluid solution embedded in a locally AdS exterior. There is an overall metric scale factor which depends on the size of Λ . The solution has three regions:

- (i) an interior solution with an equation of state, $\rho + p_\varphi = 0$;
- (ii) an Israel¹² boundary layer with surface stress energy S_{ij} and stringlike content $S_{00} + S_{\varphi\varphi} = 0$; and
- (iii) an exterior with constant isotropic pressure, constant density, and a negative cosmological constant. Just as in the ABHP study, the exterior metric is locally isometric to AdS. The solution models an extended loop with interior structure. The size of the loop and its mass depend on the cosmological constant. The solution can be used to model both micro loops or very large loop structures, depending on the size of Λ .

There have been other discussions of circular string structures. Frolov, Israel and Unruh¹³ started with an axially symmetric space–time and discussed the relation between internal string structure and angular deficit, then transformed the metric to toroidal coordinates to discuss the mass structure of circular cosmic strings. Using toroidal coordinates, Hughes *et al.*¹⁴ studied weak field loops. Sen and Banerjee¹⁵ have discussed a solution for a circular cosmic string loop in cylindrical coordinates. Because often a particular choice of surfaces can simplify the solution of the field equations, we begin with toroidal coordinates.

Cartesian toroids are discussed in the next section. In Sec. III we write the field equations for the space–time and develop the interior and exterior solutions. Matching conditions are presented in the fourth section. The Israel boundary layer is described in the fifth section. In Sec. VI we

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discuss the mass, and the final section concludes with a general discussion.

II. CARTESIAN TOROIDS

The relation between Cartesian coordinates (x, y, z) and toroidal coordinates $(\eta, \vartheta, \varphi)$ on \mathcal{R}^3 is¹⁶

$$x = a \frac{\sinh(\eta) \cos(\varphi)}{\cosh(\eta) - \cos(\vartheta)}, \tag{1a}$$

$$y = a \frac{\sinh(\eta) \sin(\varphi)}{\cosh(\eta) - \cos(\vartheta)}, \tag{1b}$$

$$z = a \frac{\sin(\vartheta)}{\cosh(\eta) - \cos(\vartheta)}, \tag{1c}$$

with $0 \leq \eta < \infty$, $0 \leq \vartheta \leq 2\pi$, $0 \leq \varphi \leq 2\pi$. Here “ a ” is a constant scale factor.

In toroidal coordinates, the Euclidean metric $dx^2 + dy^2 + dz^2$ becomes

$$dL^2 = \frac{a^2}{[\cosh(\eta) - \cos(\vartheta)]^2} [d\eta^2 + d\vartheta^2 + \sinh^2(\eta) d\varphi^2]. \tag{2}$$

The torus $\eta = \eta_0$ described by dL^2 has a circular cross section with circumference $2\pi a \operatorname{csch}(\eta_0)$, and the center of the toroid circular cross section is a distance $a \operatorname{coth}(\eta_0)$ from the origin. The equation of the $\varphi = 0, y = 0$ circles, Eq. (1b), is¹⁷

$$[x - a \operatorname{coth}(\eta_0)]^2 + z^2 = a^2 \operatorname{csch}^2(\eta_0).$$

As η_0 increases, the radius of the loop decreases and the torus approaches the flat torus, a distance “ a ” from the origin. Looking down the z -axis (about which φ has range $0 \leq \varphi \leq 2\pi$) at the torus, one sees two concentric circles. The $\vartheta = \text{constant}$ surfaces, $0 \leq \vartheta \leq 2\pi$, are spheres centered on the z -axis. From Eq. (1) these spheres have equation

$$(x^2 + y^2 + z^2 - a^2)/2az = \cot(\vartheta),$$

which defines the relation of ϑ to the torus.

III. SPACE-TIME

For the curved space torus, one must construct two different metrics, an exterior for $0 \leq \eta \leq \eta_0$ and an interior for $\eta_0 \leq \eta \leq \infty$. The metric that we use to describe the space-time is a simple generalization of the flat space metric:

$$ds^2 = \frac{a^2}{[\cosh(\eta) - \cos(\vartheta)]^2} [-h^2(\eta) dt^2 + e^{2\mu(\eta)} d\eta^2 + d\vartheta^2 + h^2(\eta) d\varphi^2]. \tag{3}$$

Note that metric (3) cannot reduce to the Minkowski metric.

A. Field equations

We write Einstein’s field equations using the conventions of Misner, Thorne, and Wheeler¹⁸ and Wald.¹⁹ The field equations are ($G = c = 1$)

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}. \tag{4}$$

Using flow vector $\hat{u}^\alpha \hat{u}_\alpha = -1$, the energy-momentum tensor for a fluid is given in terms of principal pressures as

$$T_{\alpha\beta} = \rho \hat{u}_\alpha \hat{u}_\beta + p_1 \hat{x}_\alpha \hat{x}_\beta + p_2 \hat{y}_\alpha \hat{y}_\beta + p_3 \hat{z}_\alpha \hat{z}_\beta. \quad (5)$$

In the following development, we write the field equations allowing for fluid anisotropy. We do not include Λ explicitly in the stress-energy tensor but will interpret the stress-energy associated with a metric solution in terms of Λ if appropriate. Using metric (3) above with $\Psi = \cosh(\eta) - \cos(\vartheta)$ and $u^\alpha \partial_\alpha = (\Psi/ah) \partial_t$, the field equations are

$$\begin{aligned} 8\pi\rho a^2 e^{2\mu} &= -8\pi p_\varphi a^2 e^{2\mu} \\ &= -\cosh^2(\eta) - 2\cosh(\eta)\cos(\vartheta) + 3 + 2\sinh(\eta)\Psi(h'/h) - \Psi^2(h''/h) + \Psi^2\mu'(h'/h) \\ &\quad - 2\Psi\mu' \sinh(\eta) + e^{2\mu}[-3 + \cos^2(\vartheta) + 2\cosh(\eta)\cos(\vartheta)], \end{aligned} \quad (6a)$$

$$\begin{aligned} 8\pi p_\eta a^2 e^{2\mu} &= 3\cosh^2(\eta) - 3 - 4\Psi \sinh(\eta)(h'/h) + \Psi^2(h'/h)^2 \\ &\quad + e^{2\mu}[3 - 2\cosh(\eta)\cos(\vartheta) - \cos^2(\vartheta)], \end{aligned} \quad (6b)$$

$$\begin{aligned} 8\pi p_\vartheta a^2 e^{2\mu} &= \cosh^2(\eta) - 3 + 2\cosh(\eta)\cos(\vartheta) - 2\Psi \sinh(\eta)[2(h'/h) - \mu'] \\ &\quad + \Psi^2[2(h''/h) - 2\mu'(h'/h) + (h'/h)^2] + 3e^{2\mu} \sin^2(\vartheta), \end{aligned} \quad (6c)$$

where $\partial h/\partial\eta$ and $\partial\mu/\partial\eta$ are abbreviated by h' and μ' .

B. Interior solution

Let $h^2 = [d_0 \sinh(\eta) - b_0]^2$, $e^{2\mu} = 1$. The interior metric is

$$g_{\alpha\beta}^{\text{in}} dx^\alpha dx^\beta = (a/\Psi)^2 [-h^2 dt^2 + d\eta^2 + d\vartheta^2 + h^2 d\varphi^2]. \quad (7)$$

The energy-momentum components for g^{in} are

$$8\pi a^2 \rho = -8\pi a^2 p_\varphi = (b_0/h) [\cosh^2(\eta) - \cos^2(\vartheta)], \quad (8a)$$

$$8\pi a^2 p_\eta = (\Psi/h^2) \{ (d_0^2 + b_0^2) \Psi - 2b_0 h [\cosh(\eta) + \cos(\vartheta)] \}, \quad (8b)$$

$$8\pi a^2 p_\vartheta = (\Psi/h^2) [(d_0^2 + b_0^2) \Psi - 4b_0 h \cos(\vartheta)]. \quad (8c)$$

The equation of state is

$$\rho + p_\varphi = 0. \quad (9)$$

The interior metric has quadratic Weyl invariant

$$C_{\alpha\beta\mu\nu} C^{\alpha\beta\mu\nu} = \frac{4}{3} d_0^2 \frac{\Psi^4}{a^4 h^4} [b_0 \sinh(\eta) + d_0]^2, \quad (10)$$

and Ricci scalar

$$R_{\alpha\beta} g_{\text{in}}^{\alpha\beta} = -\frac{2\Psi}{a^2 h^2} \{ (d_0^2 + b_0^2) \Psi - 2b_0 h [\cosh(\eta) + 2\cos(\vartheta)] \}. \quad (11)$$

C. Exterior solution

The solution to be used in the toroid exterior is

$$g_{\alpha\beta}^{\text{ex}} dx^\alpha dx^\beta = \frac{a^2}{[\cosh(\eta) - \cos(\vartheta)]^2} [-h^2(\eta) dt^2 + e^{2\mu(\eta)} d\eta^2 + d\vartheta^2 + h^2(\eta) d\varphi^2]. \quad (12)$$

In order to describe a cosmological vacuum, p_η will have to be constant. The cosine terms should vanish. From the general field equations we write p_η , grouping the terms:

$$8\pi p_\eta a^2 e^{2\mu} = 3 \cosh^2(\eta) - 3 - 4 \sinh(\eta) \cosh(\eta) (h'/h) + 3e^{2\mu} + \cosh^2(\eta) (h'/h)^2 + \cos(\vartheta) [-2 \cosh(\eta) e^{2\mu} + 4 \sinh(\eta) (h'/h) - 2 \cosh(\eta) (h'/h)^2] + \cos^2(\vartheta) [-e^{2\mu} + (h'/h)^2].$$

To eliminate the \cos^2 term, take $(h'/h)^2 = e^{2\mu}$. The cosine term then becomes

$$4 \cos(\vartheta) e^\mu [-\cosh(\eta) e^\mu + \sinh(\eta)].$$

Requiring this term to vanish provides one nontrivial solution

$$e^\mu = \sinh(\eta) / \cosh(\eta), \quad h = \cosh(\eta). \tag{13}$$

Substituting (13), the energy-momentum components of g^{ex} are

$$8\pi\rho = -3/a^2, \tag{14a}$$

$$8\pi p_\eta = 8\pi p_\vartheta = 8\pi p_\varphi = 3/a^2. \tag{14b}$$

This can be a space-time with negative cosmological constant $\Lambda = -3/a^2$. The metric is conformally flat and has constant negative Ricci scalar $R = -12/a^2$. $g^{\alpha\beta}_{\text{ex}}$ is locally isometric to the AdS metric.

IV. MATCHING INTERIOR TO EXTERIOR

The two metrics to be joined are

$$g^{\text{in}}_{\alpha\beta} dx^\alpha dx^\beta = \frac{a^2}{\Psi^2} \{ -[d_0 \sinh(\eta) - b_0]^2 dt^2 + d\eta^2 + d\vartheta^2 + [d_0 \sinh(\eta) - b_0]^2 d\varphi^2 \}, \tag{15}$$

$$g^{\text{ex}}_{\alpha\beta} dx^\alpha dx^\beta = \frac{a^2}{\Psi^2} \left[-\cosh^2(\eta) dt^2 + \frac{\sinh^2(\eta)}{\cosh^2(\eta)} d\eta^2 + d\vartheta^2 + \cosh^2(\eta) d\varphi^2 \right].$$

Matching the metrics one obtains

$$\cosh(\eta_0) = d_0 \sinh(\eta_0) - b_0.$$

Matching the extrinsic curvature yields

$$d_0 \cosh(\eta_0) = \sinh(\eta_0).$$

The bounding surface is thus defined by

$$\cosh(\eta_0) = \frac{b_0}{d_0^2 - 1}, \tag{16a}$$

$$\sinh(\eta_0) = \frac{d_0 b_0}{d_0^2 - 1}, \tag{16b}$$

with

$$b_0^2 + d_0^2 = 1. \quad (17)$$

This implies that both b_0 and d_0 are less than 1. On the boundary the stresses are

$$8\pi a^2 \rho = -8\pi a^2 p_\varphi = [b_0^2 \cos^2(\vartheta) - 1], \quad (18a)$$

$$8\pi a^2 p_\eta = \Psi b_0 [-3 + b_0 \cos(\vartheta)], \quad (18b)$$

$$8\pi a^2 p_\vartheta = \Psi b_0 [-1 + 3b_0 \cos(\vartheta)]. \quad (18c)$$

A problem with the matching is that p_η does not smoothly join to the exterior stress. This mismatch would lead to a dynamic boundary. Therefore, an Israel boundary layer will be developed.

V. THE BOUNDARY LAYER

A. Position of the layer

If the interior and exterior solutions do not match derivatives but join over an Israel surface layer,¹² then the position of the boundary will be set by matching only h at $\eta = \eta_0$. For the exterior we have

$$h = \cosh(\eta), \quad e^\mu = \sinh(\eta)/\cosh(\eta).$$

For the interior

$$h = d_0 \sinh(\eta) - b_0, \quad e^{2\mu} = 1.$$

Matching the interior and exterior at $\eta = \eta_0$ provides

$$\cosh(\eta_0) = d_0 \sinh(\eta_0) - b_0.$$

Note that the $e^{2\mu}$ term need not match, since it is the coefficient of $d\eta^2$ and the match is for η constant surfaces. Rearranging, we have the bounding surface

$$\cosh(\eta_0) = \frac{b_0 + k d_0 (b_0^2 + d_0^2 - 1)^{1/2}}{d_0^2 - 1}, \quad k = (\pm 1), \quad (19a)$$

$$\sinh(\eta_0) = \frac{d_0 b_0 + k (b_0^2 + d_0^2 - 1)^{1/2}}{d_0^2 - 1}. \quad (19b)$$

B. Parameter constraints

Constraints can be set on d_0 and b_0 by requiring

$$\sinh(\eta_0) > 0, \quad \cosh(\eta_0) > 0, \quad \rho_{\text{interior}} > 0.$$

Both of the hyperbolic functions in Eq. (19) have a sign choice which is the same for both functions. There are eight possible parameter (k, d_0, b_0) combinations for both $d_0^2 > 1$ and $d_0^2 < 1$ for a total of 16 cases. The hyperbolic conditions eliminate eight and the density constraint five more. The three remaining allowed parameter combinations with their constraints are

$$(1) \quad d_0^2 > 1: [k = +1, d_0 > 0, b_0 > 0], \quad \text{no constraints,}$$

$$(2) \quad d_0^2 > 1: [k = -1, d_0 > 0, b_0 > 0], \quad \sqrt{b_0^2 + d_0^2 - 1} < \left| \frac{b_0}{d_0} \right|, \quad \sqrt{b_0^2 + d_0^2 - 1} < |d_0 b_0|, \quad (20)$$

$$(3) \quad d_0^2 < 1: [k = -1, d_0 > 0, b_0 > 0], \quad \sqrt{b_0^2 + d_0^2 - 1} > \left| \frac{b_0}{d_0} \right|, \quad \sqrt{b_0^2 + d_0^2 - 1} > |d_0 b_0|.$$

The algebraic details are in the Appendix.

C. Extrinsic curvature

We are interested in a space-time that could describe a loop of matter with an energy density equal to the loop tension over a bounding Israel surface layer at $\eta = \eta_0$. The stress-energy content of the surface layer S_{ij} (Ref. 12) is given by

$$8 \pi S_{ij} = \gamma_{ij} - \gamma h_{ij}^{(b)} \quad (21)$$

with $h_{ij}^{(b)}$ the metric of the bounding torus. γ_{ij} is the difference between the extrinsic curvatures of the exterior and interior metrics on the boundary

$$\gamma_{ij} = K_{ij}^{\text{ex}} - K_{ij}^{\text{in}} = \langle K_{ij} \rangle.$$

Calculating the general extrinsic curvature on the bounding torus $\eta = \eta_0$ with unit normal n_α we have

$$K_{ij} = -n_{\alpha;\beta} h_i^\alpha h_j^\beta,$$

$$K_{ij} = n_\alpha \Gamma_{ij}^\alpha = -(n_\alpha/2) g^{\alpha\beta} g_{ij,\beta}.$$

With $\Psi = \cosh(\eta) - \cos(\vartheta)$ and $\eta^\alpha \partial_\alpha = \partial/\partial\eta$, we have for the extrinsic curvatures on the boundary

$$K_{00} = (n_\alpha \eta^\alpha) \frac{\Psi^2}{2e^{2\mu}} \frac{\partial}{\partial\eta} (h^2/\Psi^2), \quad (22a)$$

$$K_{\varphi\varphi} = -K_{00}, \quad (22b)$$

$$K_{\vartheta\vartheta} = (n_\alpha \eta^\alpha) \frac{\Psi^2}{2e^{2\mu}} \frac{\partial}{\partial\eta} (1/\Psi^2). \quad (22c)$$

$K_{\vartheta\vartheta}$ will match across the boundary with the metrics we have found. Using Eq. (22a) and forming K_{00} we have

$$K_{00} = \frac{h}{\Psi} [\Psi h' - h \sinh(\eta)]. \quad (23)$$

Establishing the difference between inner and outer spaces and matching h on the boundary, the discontinuity in the extrinsic curvature is

$$\langle K_{00} \rangle = h [\sinh(\eta_0) - d_0 \cosh(\eta_0)].$$

Therefore the boundary layer has a stress energy content

$$8 \pi S_{00} = \cosh(\eta_0) [d_0 \cosh(\eta_0) - \sinh(\eta_0)] = -8 \pi S_{\varphi\varphi}. \quad (24)$$

Requiring $S_{00} > 0$ and substituting for $\cosh(\eta_0)$ and $\sinh(\eta_0)$ from Eq. (19) implies $k = 1$. Thus one parameter set remains:

$$d_0^2 > 1 : [k = +1, d_0 > 0, b_0 > 0]. \tag{25}$$

The stress energy content of the boundary layer represents a toroidal loop with a stringlike equation of state.

VI. MASS

When the generator of time translations is Killing vector ξ^ν , then the Einstein four-momentum $p^\mu = \sqrt{-g} T^\mu_\nu \xi^\nu$ is conserved and a mass can be associated with three-volume dV_μ ,

$$M = \int_{3vol} \sqrt{-g} T^\mu_\nu \xi^\nu dV_\mu,$$

where $dV_\mu = t_{,\mu} d\eta d\vartheta d\varphi$. Substituting we have the mass inside the torus:

$$\begin{aligned} M &= \frac{2\pi b_0 a^2}{8\pi} \int_{\eta_0}^\infty \int_0^{2\pi} \frac{h}{\Psi^3} [\cosh(\eta) + \cos(\vartheta)] d\vartheta d\eta \\ &= \frac{\pi b_0 a^2}{8 \sinh^4(\eta_0)} \{4d_0 \sinh(\eta_0) \cosh^2(\eta_0) - b_0 [2 \sinh^2(\eta_0) + 3]\}. \end{aligned} \tag{26}$$

A similar calculation can be repeated for the mass associated with the surface layer. In the Israel formalism the surface stress energy is defined in geodesic coordinates as the thickness ε of the layer approaches zero:

$$S_{\mu\nu} = \lim_{\varepsilon \rightarrow 0} \int_0^\varepsilon T_{\mu\nu} dx. \tag{27}$$

Start with the definition of the mass in a three-volume and take the limit as the distance between tori goes to zero:

$$M' = \int_{3vol} \sqrt{-g} T^\mu_\nu \xi^\nu dV_\mu = \int_{3vol} \sqrt{-g} T^0_\nu \xi^\nu d\eta d\vartheta d\varphi.$$

In the limit of zero layer thickness

$$M' = \lim_{\varepsilon \rightarrow 0} \int \int \int_{\eta_0-\varepsilon}^{\eta_0} \sqrt{-g_{tt} g_{\vartheta\vartheta} g_{\varphi\varphi} g_{\eta\eta}} T^0_\nu \xi^\nu d\eta d\vartheta d\varphi. \tag{28}$$

Assume that the limit can be taken inside the integral and that over the range of the η - integral that $\sqrt{-g_{tt} g_{\vartheta\vartheta} g_{\varphi\varphi}}$ is approximately constant and takes its value on η_0 :

$$M' = \int \int \sqrt{-g_{tt}(\eta_0, \vartheta) g_{\vartheta\vartheta}(\eta_0, \vartheta) g_{\varphi\varphi}(\eta_0, \vartheta)} d\vartheta d\varphi \lim_{\varepsilon \rightarrow 0} \int_{\eta_0-\varepsilon}^{\eta_0} (T^0_\nu \xi^\nu \sqrt{g_{\eta\eta}} d\eta),$$

$$M' = \int \int \sqrt{-g_{tt} g_{\vartheta\vartheta} g_{\varphi\varphi}} S^0_\nu \xi^\nu d\vartheta d\varphi.$$

Integration results in

$$M' = \frac{ah^2}{4} [d_0 - \tanh(\eta_0)] \frac{2\pi}{\sinh(\eta_0)}. \tag{29}$$

VII. DISCUSSION

In summary, we have obtained a fluid solution to the field equations that describes a positive density torus with a boundary layer, embedded in a locally AdS exterior. The solution has two parameters, d_0 and b_0 with a restricted range. The fluid and boundary layers both have a stringlike equation of state. The solution can describe a variety of structures, depending on the parameter value chosen. First consider the size of the loop, $R_\vartheta = a \operatorname{csch}(\eta_0)$. For the allowed parameter set we have, in the limit $b_0^2 \gg |d_0^2 - 1|$,

$$\frac{R_\vartheta[k=+1, d_0^2 > 1]}{a} \sim \frac{d_0 - 1}{b_0}.$$

R_ϑ/a can become very small and the torus will approach the flat torus a distance “ a ” from the center of the torus loop. The size of the loop depends on the scale parameter, “ a .” The size of the scale factor is determined by the cosmological constant. From the field equations we have

$$\frac{8\pi G}{c^2} \rho_{\text{exterior}} = -\frac{3}{a^2}, \quad |\Lambda| = \frac{3}{a^2}. \tag{30}$$

For example, if this density is roughly the same order as the critical density, we would have $|\rho| \sim 10^{-27} \text{ kg/m}^3$ and one finds that $a \sim 10^{28} \text{ m}$. If the solution is used to describe a primordial universe with a large negative Λ , the scale factor could be much smaller and micro loops could be possible.

The mass description is also dependent on the size of the scale factor. We have from Eq. (26) for the fluid interior

$$M = \frac{\pi b_0 a^2}{8} \left[\frac{4d_0}{\sinh(\eta_0)} + \frac{4d_0}{\sinh^3(\eta_0)} - \frac{2b_0}{\sinh^2(\eta_0)} - \frac{3b_0}{\sinh^4(\eta_0)} \right].$$

For the surface layer we have Eq. (29),

$$M' = \frac{\pi}{2} a h^2 [d_0 - \tanh(\eta_0)] \frac{1}{\sinh(\eta_0)}.$$

One thing that is immediately obvious is the different dependence on the scale parameter. In the large b_0 limit taken above we have

$$M' \sim \frac{\pi}{2} a b_0,$$

$$M \sim \frac{\pi}{4} a^2 (d_0^2 - 1).$$

The fluid inside the torus does not depend on b_0 in this limit. In the current universe, if $a \gg 1$ and if $b_0 \ll a$, the fluid inside the torus can dominate the mass because of the scale factor. If $b_0 \sim a$ and $d_0 \rightarrow 1$, the mass in the surface layer could dominate the loop structure. While the size of the thin-loop torus depends on “ a ,” the “fat” torus can extend much closer in to the origin. As above, if, in the primordial universe, the cosmological constant was negative and much larger, the scale factor, “ a ,” could be quite small. The solution could then describe micro loops with the surface layer the dominant mass contribution.

Several extensions of this solution might be possible. Adding time dependence to generate an oscillating loop for a Casimir calculation would be quite interesting. Time dependence could also

be used to check the evolution and stability over time of a primordial loop. This solution could also be regarded as a step toward generating multi segment Brevik–Nielson²⁰ loops with metric dependent tensions.

APPENDIX: MATCHING CONSTRAINTS

The hyperbolic functions are, with $S(b_0, d_0) := (b_0^2 + d_0^2 - 1)^{1/2}$,

$$\cosh(\eta_0) = \frac{b_0 + k d_0 S}{d_0^2 - 1}, \quad k = (\pm 1), \quad (\text{A1a})$$

$$\sinh(\eta_0) = \frac{d_0 b_0 + k S}{d_0^2 - 1}. \quad (\text{A1b})$$

The conditions to be satisfied are

$$\sinh(\eta_0) > 0,$$

$$\cosh(\eta_0) > 0.$$

The cosh function is always positive and $\sinh(\eta_0)$ is positive because the range for the interior metric is $\eta_0 < \eta < \infty$. The parameters must always satisfy the condition

$$b_0^2 + d_0^2 > 1.$$

The equal sign with $S=0$ is not a possibility since that would imply an exact match of interior and exterior.

1. $\sinh(\eta_0) > 0$

$$\frac{d_0 b_0 + k S}{d_0^2 - 1} > 0$$

A: $d_0^2 > 1$, $k = +1$, $0 < d_0 b_0 + S$

- (1) ($d_0 > 0$, $b_0 > 0$) condition satisfied
- (2) ($d_0 < 0$, $b_0 < 0$) condition satisfied
- (3) ($d_0 > 0$, $b_0 < 0$) condition satisfied if $|d_0 b_0| < S$
- (4) ($d_0 < 0$, $b_0 > 0$) condition satisfied if $|d_0 b_0| < S$

B: $d_0^2 > 1$, $k = -1$, $0 < d_0 b_0 - S$

- (5) ($d_0 > 0$, $b_0 > 0$) condition satisfied if $S < |d_0 b_0|$
- (6) ($d_0 < 0$, $b_0 < 0$) condition satisfied if $S < |d_0 b_0|$
- (7) ($d_0 > 0$, $b_0 < 0$) condition excluded
- (8) ($d_0 < 0$, $b_0 > 0$) condition excluded

C: $d_0^2 < 1$, $k = +1$, $0 < -d_0 b_0 - S$

- (9) ($d_0 > 0$, $b_0 > 0$) condition excluded
- (10) ($d_0 < 0$, $b_0 < 0$) condition excluded
- (11) ($d_0 > 0$, $b_0 < 0$) condition satisfied if $S < |d_0 b_0|$
- (12) ($d_0 < 0$, $b_0 > 0$) condition satisfied if $S < |d_0 b_0|$

D: $d_0^2 < 1$, $k = -1$, $0 < -d_0 b_0 + S$

- (13) ($d_0 > 0$, $b_0 > 0$) condition satisfied if $|d_0 b_0| < S$
- (14) ($d_0 < 0$, $b_0 < 0$) condition satisfied if $|d_0 b_0| < S$
- (15) ($d_0 > 0$, $b_0 < 0$) condition satisfied
- (16) ($d_0 < 0$, $b_0 > 0$) condition satisfied

Summary of Condition 1

$d_0^2 > 1, k = -1, (d_0 > 0, b_0 < 0)$ and $(d_0 < 0, b_0 > 0)$ are excluded
 $d_0^2 < 1, k = +1, (d_0 > 0, b_0 > 0)$ and $(d_0 < 0, b_0 < 0)$ are excluded

2. cosh(η_0) > 0

$$\frac{b_0 + d_0 k S}{d_0^2 - 1} > 0$$

- A: $d_0^2 > 1, k = +1, 0 < b_0 + d_0 S$
 - (1) $(d_0 > 0, b_0 > 0)$ condition satisfied
 - (2) $(d_0 < 0, b_0 < 0)$ condition excluded
 - (3) $(d_0 > 0, b_0 < 0)$ condition satisfied if $|b_0| < d_0 S$
 - (4) $(d_0 < 0, b_0 > 0)$ condition satisfied if $|b_0| > |d_0| S$
- B: $d_0^2 > 1, k = -1, 0 < b_0 - d_0 S$
 - (5) $(d_0 > 0, b_0 > 0)$ condition satisfied if $d_0 S < b_0$
 - (6) $(d_0 < 0, b_0 < 0)$ condition satisfied if $|d_0| S > |b_0|$
 - (7) $(d_0 < 0, b_0 > 0)$ condition satisfied
 - (8) $(d_0 > 0, b_0 < 0)$ condition excluded
- C: $d_0^2 < 1, k = +1, 0 < -b_0 - d_0 S$
 - (9) $(d_0 > 0, b_0 > 0)$ condition excluded
 - (10) $(d_0 < 0, b_0 < 0)$ condition satisfied
 - (11) $(d_0 > 0, b_0 < 0)$ condition satisfied if $|b_0| > d_0 S$
 - (12) $(d_0 < 0, b_0 > 0)$ condition satisfied if $b_0 < |d_0| S$
- D: $d_0^2 < 1, k = -1, 0 < -b_0 + d_0 S$
 - (13) $(d_0 > 0, b_0 > 0)$ condition satisfied if $|b_0| < d_0 S$
 - (14) $(d_0 < 0, b_0 < 0)$ condition satisfied if $|b_0| > |d_0| S$
 - (15) $(d_0 < 0, b_0 > 0)$ condition excluded
 - (16) $(d_0 > 0, b_0 < 0)$ condition satisfied

Summary of Condition 2

- $d_0^2 > 1, k = +1, (d_0 < 0, b_0 < 0)$ is excluded
- $d_0^2 > 1, k = -1, (d_0 > 0, b_0 < 0)$ is excluded
- $d_0^2 < 1, k = +1, (d_0 > 0, b_0 > 0)$ is excluded
- $d_0^2 < 1, k = -1, (d_0 < 0, b_0 > 0)$ is excluded

When the constraints for the two conditions are put together, the cases

$k = +1, d_0 < 0, b_0 > 0$, are eliminated for both $d_0^2 > 1$ and $d_0^2 < 1$.

Summary of existing cases after hyperbolic conditions are imposed

$$d_0^2 > 1 : k = +1 \tag{A2}$$

$$(d_0 > 0, b_0 > 0)$$

$$(d_0 > 0, b_0 < 0) : |d_0 b_0| < S, \left| \frac{b_0}{d_0} \right| < S$$

$$d_0^2 > 1 : k = -1 \tag{A3}$$

$$(d_0 > 0, b_0 > 0) : S < |d_0 b_0|, S < \left| \frac{b_0}{d_0} \right|$$

$$(d_0 < 0, b_0 < 0) : S < |d_0 b_0|, S > \left| \frac{b_0}{d_0} \right|$$

$$d_0^2 < 1 : k = +1 \tag{A4}$$

$$(d_0 > 0, b_0 < 0): S < |d_0 b_0|, S < \left| \frac{b_0}{d_0} \right|$$

$$d_0^2 < 1: k = -1 \tag{A5}$$

$$(d_0 > 0, b_0 > 0): S > |d_0 b_0|, S > \left| \frac{b_0}{d_0} \right|$$

$$(d_0 < 0, b_0 < 0): S > |d_0 b_0|, S < \left| \frac{b_0}{d_0} \right|$$

$$(d_0 > 0, b_0 < 0).$$

Now we require the fluid density inside the torus to be positive:

$$8 \pi a^2 \rho = (b_0/h)[\cosh^2(\eta_0) - \cos^2(\vartheta)] > 0.$$

cosh(η_0) will always be greater than 1 since it equals 1 at $\eta=0$, which is outside of the torus interior. In the interior $\eta_0 \leq \eta \leq \infty$. We have

$$\frac{b_0}{d_0 \sinh(\eta_0) - b_0} > 0,$$

$$\frac{1}{\frac{d_0}{b_0} \sinh(\eta_0) - 1} > 0,$$

$$\frac{d_0 b_0 d_0 + k S}{b_0 d_0^2 - 1} > 1.$$

3. $d_0^2 > 1$

$$d_0^2 + k \frac{d_0}{b_0} S > d_0^2 - 1$$

$$-k \frac{d_0}{b_0} S < 1$$

$k = +1, (b_0 > 0, d_0 > 0)$ and $(b_0 < 0, d_0 < 0)$. No constraints

$k = -1, (b_0 > 0, d_0 > 0)$ and $(b_0 < 0, d_0 < 0)$ with constraint $\left| \frac{d_0}{b_0} \right| S < 1$

4. $d_0^2 < 1$

$$-d_0^2 - k \frac{d_0}{b_0} S > 1 - d_0^2$$

$$-k \frac{d_0}{b_0} S > 1$$

$$k = +1, (b_0 < 0, d_0 > 0) \text{ with constraint } S > \left| \frac{b_0}{d_0} \right| \quad (\text{A6})$$

$$k = -1, (b_0 > 0, d_0 > 0) \text{ and } (b_0 < 0, d_0 < 0) \text{ with constraint } S > \left| \frac{b_0}{d_0} \right|$$

Summarizing all constraints provides

$$d_0^2 > 1 : k = +1 \quad (\text{A7})$$

$$(d_0 > 0, b_0 > 0)$$

$$(d_0 > 0, b_0 < 0) : |d_0 b_0| < S, \left| \frac{b_0}{d_0} \right| < S, S < \left| \frac{b_0}{d_0} \right| \text{ is excluded}$$

$$d_0^2 > 1 : k = -1 \quad (\text{A8})$$

$$(d_0 > 0, b_0 > 0) : S < |d_0 b_0|, S < \left| \frac{b_0}{d_0} \right|$$

$$(d_0 < 0, b_0 < 0) : S < |d_0 b_0|, S > \left| \frac{b_0}{d_0} \right|, S < \left| \frac{b_0}{d_0} \right| \text{ is excluded}$$

$$d_0^2 < 1 : k = +1 \quad (\text{A9})$$

$$(d_0 > 0, b_0 < 0) : S < |d_0 b_0|, S < \left| \frac{b_0}{d_0} \right|, S > \left| \frac{b_0}{d_0} \right| \text{ is excluded}$$

$$d_0^2 < 1 : k = -1 \quad (\text{A10})$$

$$(d_0 > 0, b_0 > 0) : S > |d_0 b_0|, S > \left| \frac{b_0}{d_0} \right|$$

$$(d_0 < 0, b_0 < 0) : S > |d_0 b_0|, S < \left| \frac{b_0}{d_0} \right|, S > \left| \frac{b_0}{d_0} \right| \text{ is excluded}$$

$$(d_0 > 0, b_0 < 0) \text{ is excluded}$$

The three allowed parameter combinations are

$$d_0^2 > 1 : k = +1 \quad (d_0 > 0, b_0 > 0)$$

$$d_0^2 > 1 : k = -1 \quad (d_0 > 0, b_0 > 0) : S < |d_0 b_0|, S < \left| \frac{b_0}{d_0} \right|$$

$$d_0^2 < 1 : k = -1 \quad (d_0 > 0, b_0 > 0) : S > |d_0 b_0|, S > \left| \frac{b_0}{d_0} \right|$$

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