LETTERS

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A new class of fluid instabilities: Vorticity-induced waveforms on falling parabolic jets

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(Received 22 March 1982; accepted 5 April 1982)

Steady streams issuing from horizontal circular ducts form stationary surface waves whose angular symmetry depends on the vorticity distribution within the duct. For Poiseuille flow, the observed sequences of wavelengths and frequencies are shown to be consistent with Crocco's theorem: With fall the stream narrows, inducing secondary flow patterns dependent on the rate of narrowing. These findings suggest a method for obtaining scaling estimates of prevalent vorticity patterns in ducts.

I. INTRODUCTION

An established method for determining liquid surface-tension coefficients exploits the formation of stationary waves on jets issuing from horizontal ducts with elliptical apertures.[1] The jet appears in cross section as a sequence of ellipses, with major axes oriented alternately vertically, then horizontally along the parabolic trajectory. Rayleigh,[2] and later Bohr,[3] demonstrated that such oscillating jets result from an instability mechanism driven by inertial and surface tension forces; the particular angular symmetry of the waveform is found experimentally and theoretically to depend on the shape of the aperture.[4] Most pertinent to this report, no waveform is generated by a circular duct if the exit velocity profile is uniform. Instead, the narrowing stream retains a circular cross section throughout its parabolic trajectory.[5]

If a nonuniform velocity profile is present at a conduit orifice, we have found waveforms reminiscient of the oscillating jet instability are produced even if the duct cross section is circular.[6] Two examples of such waveforms are shown in Fig. 1: Poiseuille flow produces a bilaterally symmetric waveform (a), a pattern also seen in low-velocity jets emerging from coils. However, as velocity is increased in these latter sections, stream patterns undergo a transition to rotational waves, as shown in (b). Reversing the direction of flow through a coil reverses the rotation sense of the wave.

These results constitute experimental evidence that duct vorticity, both axial and azimuthal, produces a class of inertial wave instabilities in horizontal jets issuing from circular orifices. In this communication a few quantitative features of the waveforms are discussed.

II. METHODS AND RESULTS

Steady streams were generated by passing various aqueous solutions through conduits positioned on a horizontal bench. Conduit exits extended 2 cm beyond the bench edge and some 30 cm above a collecting tank, permitting nodal patterns on the falling jets to be observed conveniently. The required pressures were supplied by a "constant-head" reservoir. Exit flow was measured by timed volume collection. A straight pipe and a coil (each 1.11 cm i.d.) were used to compare the effect of different vorticity distributions at equal flow rates.

Wavelengths were defined by the positions of nodes along a jet's trajectory. Vertical distances from the duct centerline to successive nodes were measured on a scale placed alongside the fluid column, then corrected for arc curvature and nodal length[7] to obtain a sequence of wavelengths for each stream. Corresponding wave frequencies were computed by converting the measured vertical distance into an equivalent time-of-fall. Error bars assigned to wavelengths and frequencies reflect the estimated uncertainties stemming from parallax, finite duct size, and stream unsteadiness.

Wavelengths generated by a straight pipe and by a coil are compared in Figs. 1(a) and 1(b) for a sizeable range of laminar flow conditions. Wavelengths associated with Poiseuille flow are approximately independent of time-of-fall, and therefore invariant on a given stream. By contrast, rotational waves generated by coils lengthen about 10% over five nodes. Therefore, both cases are represented in Fig. 1 by wavelengths averaged over the first four nodes. Two points are clear: wavelength increases markedly with exit velocity, but viscosity has a considerable effect only on the

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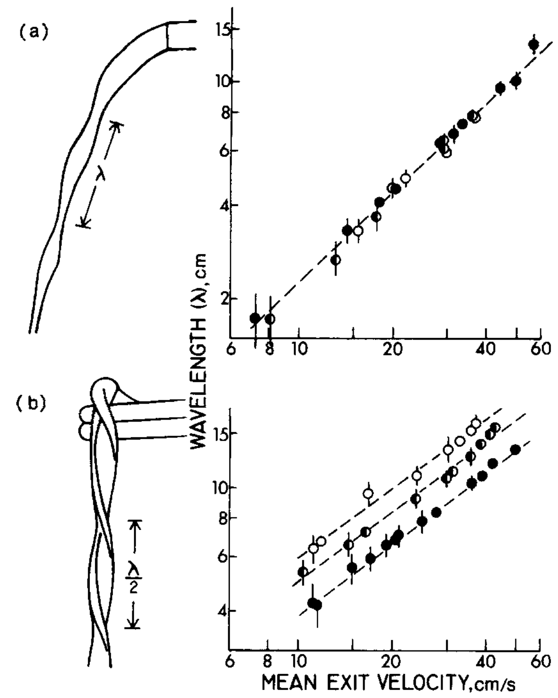

FIG. 1. Examples of waveforms with nonuniform velocity profile present at a conduit orifice. (a) Bilaterally symmetric waves on a stream emergent from a straight pipe, and the resulting dependence of wavelength on mean conduit velocity (log-log scales). (b) Rotational stream from a coil, curvature ratio 16.1, and corresponding wavelengths. Waveforms begin to appear rotational at an exit velocity of 15 cm/sec. Symbols: ●, 3.9 cS; ◐, 7.2–7.6 cS; ○, 16–18.4 cS. Aqueous solutions of either polymerized sodium acetate or ethylene glycol.

rotational waves produced by coils. In neither case can wavelength be correlated with Reynolds number.

These results lead to a conceptual difficulty. Stationary waves on falling streams must be propagated with a continually increasing wave speed. It follows that wave frequency is not constant on a given stream, but increases monotonically with time-of-fall[8] (Table I).

III. DISCUSSION

Rather than accept the notion that a wave would be propagated with no definite frequency, I postulated that these stream patterns derive from a precessional motion of the jet, induced by the interaction of gravitational acceleration with the primary duct vorticity. Assuming that the frequencies of these precessional waves depend on stream vorticity, this hypothesis could be tested by comparing how wave frequency and vorticity evolve with fall.

For axisymmetric flow, of which Poiseuille flow is a particular example, it is a consequence of Helmholtz's vortex laws that the azimuthal vorticity component $\eta(r)$ of each fluid element must obey the relation η/r = constant, where r is the radial distance from the stream centerline.[9] It follows that the azimuthal vorticity at the jet surface R must decrease with fall as R/R_0, the ratio of jet-to-duct radii. This change in vorticity might not in itself perturb the jet cross section; however, the narrowing cross section also indicates the onset of a radial velocity component $u(R)$. Using Crocco's theorem,[9] it is easily shown that these two quantities interact to produce a *streamwise* gradient in the jet Bernoulli number B:

$$\begin{aligned}(\operatorname{grad} B)_{\text{axial}} &= \eta(R)u(R)\\ &= (4V_0R/R_0^2)(-R^5g/2R_0^4V_0)(R_0^4/R^4-1)^{1/2}\\ &= 2g[\text{function (jet-duct area ratio)}], \qquad (1)\end{aligned}$$

where g is the gravitational acceleration and V_0 the mean exit duct velocity. Since the Bernoulli number along the jet stream must be conserved,[10] Eq. (1) implies that secondary flows, and associated surface deformations, must be set up within the jet cross section to compensate a gradient that depends only on area ratio, and not velocity. Similar considerations apply for the more complex vortex pair present in a coil.

By hypothesis, therefore, the observed wave frequency evolves with fall as a function of jet cross-sectional area. The functional dependence is still unresolved, but a simple assumption, consistent (when applicable) with Kelvin's circulation theorem, is that

$$\text{Measured wave frequency} = f_I\,[(\text{duct/jet) area ratio}], \qquad (2)$$

where f_I, an "intrinsic frequency", is an empirical constant of proportionality. This quantity can be evaluated using the experimentally-determined wave frequency associated with each nodal position downstream of the first (Table I), averaged over the displacement from the first node.

TABLE I. Example of a nodal sequence, straight-pipe section (1.11 cm ID; exit velocity 30.1 ± 1.5 cm/s; viscosity 3.9 cS).

Vertical distance of node from duct, cm	Nodal-averaged wave frequency, Hz	"Intrinsic frequency" f_I, Hz
2.3 ± .5	. . .	. . .
7.5	13.9 ± 2.4	4.13
13.5	18.1	4.35
19.5	21.1	4.38
25.5 ± 1.	23.5 ± 1.8	4.36 ± .33

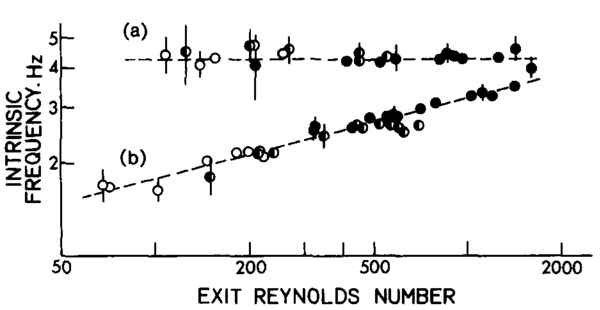

FIG. 2. Log-log plots of intrinsic frequency (f_I) versus conduit Reynolds number for (a) straight pipe; (b) coil; using data of Fig. 1. Dashed lines are least-squares-fitted.

Intrinsic frequencies associated with Poiseuille flow turn out to be a single constant of motion, independent of time-of-fall (Table I) or jet Reynolds number [Fig. 2, curve (a)], and therefore are consistent with Eqs. (1) and (2). Possibly because their axial velocity profiles are not axisymmetric, coil intrinsic frequencies diminish approximately 3% through successive nodes, still in satisfactory agreement with Eq. (2). Furthermore, for a coil note that f_I exhibits a consistent 1/4-power dependence on Reynolds number in Fig. 2 [curve (b)], despite the fact that neither wavelength nor frequency do so. By contrast, intrinsic frequencies generated in aqueous solutions by a 60° wye (not shown) are found to be almost independent of Reynolds number, even though the large-scale swirl pattern within this conduit qualitatively resembles that of a coil.[11]

In Poiseuille flow the strength of vorticity increases linearly with axial velocity, so that the ratio η/V_0 is independent of fluid velocity or viscosity. Since such flows generate waveforms whose intrinsic frequencies are also invariant to velocity or viscosity, f_I probably is an empirical measure of the strength of a prevailing vorticity component relative to flow rate. If this new aspect of oscillating jets can be systematized, large-scale patterns of conduit vorticity could be scanned to infer appropriate scaling parameters. For example, results seen in Fig. 2 would suggest vorticity varies nonlinearly with flow rate in a coil. In wyes, the relative invariance of f_I to viscosity or flow requires a linear relation between vorticity and flow, in agreement with direct observations of swirl patterns.[12]

ACKNOWLEDGMENTS

Professor W. R. Debler, Department of Mechanical Engineering and Applied Mechanics, has contributed much insight to this topic. A portion of this work was completed while the author was a Parker B. Francis Research Fellow in the Department of Physiology, The University of Florida.

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[1] For a recent critique of this method, see: W. D. E. Thomas and L. Potter, J. Colloid. Interface Sci. **50**, 397 (1975).

[2] J. W. Strutt (Lord Rayleigh), *The Theory of Sound* (Dover, New York, 1945), 2nd ed., Vol. 2, pp. 343–375.

[3] N. Bohr, Phil. Trans. Roy. Soc., London Ser. A **209**, 281 (1909).

[4] See Refs. 2 and 3. Also: J. F. Geer and J. C. Strikwerda, J. Fluid Mech. **101**, 53 (1980) for a numerical study of vertical-jet waves produced by a variety of apertures in the absence of surface tension.

[5] Bidone's oft-cited experiments are summarized in Refs. 1 and 2. However, we find waveforms do appear on low-velocity, uniform, horizontal jets (Ref. 6).

[6] B. Snyder and W. R. Debler (unpublished).

[7] Nodal midpoints must be determined by a suitable averaging, as is clear in Fig. 1 (a).

[8] Both Rayleigh and Bohr assume in their analyses that horizontal jet speed is high enough that wave speed is constant. Horizontal jets are used "to avoid the complications due to gravity" (Ref. 2, p. 357).

[9] See, for example: A. Shapiro, in *Illustrated Experiments in Fluid Mechanics*, edited by the National Commitee for Fluid Mechanics Films (MIT Press, Cambridge, Massachusetts, 1972), p. 63.

[10] See: E. O. Tuck, J. Fluid Mech. **76**, 625 (1976) on the application of Bernoulli number to inviscid jets.

[11] R. C. Schroter and M. F. Sudlow, Respir. Physiol. **7**, 341 (1969).

[12] See Ref. 11. These authors report that a 70° wye generates exit "secondary flows ... strong enough to complete at least one helical cycle within 3 diameters downstream (of the flow divider)" in air, for Reynolds numbers between 50 and 4500. By inference the pitch of these helices must be nearly independent of Reynolds number, over the 90-fold range of flow rates they observed.