# Shear Stabilization of Drift Cyclotron Instabilities in Plasma

JAMES N. DAVIDSON AND TERRY KAMMASH
*Department of Nuclear Engineering, University of Michigan, Ann Arbor, Michigan*
(Received 23 June 1969)

Shear stabilization of low-frequency drift modes in plasma has been studied intensively in recent years, but little if any work has been done regarding the effect of shear on the drift cyclotron instabilities. The linearized Vlasov equations are used to examine the stability of electrostatic oscillations with frequency near integral multiples of the ion cyclotron frequency, propagating in an infinite inhomogeneous collisionless plasma situated in a sheared magnetic field. A normal mode analysis is employed to investigate the stability of waves propagating nearly normal to the magnetic field, and it is found that shear has a destabilizing effect although it does not alter the over-all condition for marginal stability obtained in the uniform field case. A wave packet approach is then utilized to investigate the stability of these high-frequency oscillations when they propagate at an arbitrary angle to the field lines. For low-density plasma, it is shown that the drift cyclotron modes are stabilized by approximately the same shear required to stabilize the low-frequency modes, while at high densities, a more stringent but physically realizable shear is required for stabilization.

## I. INTRODUCTION

It has been known for some time that low-frequency drift instabilities, driven by density and temperature gradients in plasma, can be stabilized by the introduction of magnetic shear. Normal mode drift waves are found to be stable when $\Theta \gtrsim (\rho_i/R)^{2/3}$, where $\Theta$ is the angle by which the field lines are sheared, $\rho_i$ is the ion Larmor radius, and $R$ is the distance characterizing the gradients.[1,2] For highly localized wave packet disturbances the stability condition has been shown[3,4] to be $\Theta \gtrsim (m_e/m_i)^{1/3}$ where $m_e$ and $m_i$ are the electron and ion masses, respectively. It is also well known[5] that temperature and density gradients give rise to drift instabilities at or near integral multiples of the ion cyclotron frequency, and it is the purpose of this paper to examine the effect of sheared fields on these oscillations.

In most present day fusion devices drift instabilities at the higher frequencies are viewed as unimportant or perhaps innocuous since, at the particle densities which characterize these systems, the drift cyclotron modes have much smaller growth rates than their low-frequency counterparts. However, as densities increase, the high-frequency drift modes become of comparable or greater importance, and some method of stabilization must be found. Owing to the already demonstrated effectiveness of magnetic shear stabilization of low-frequency drift modes, it is natural to consider shear as a possible method of stabilizing cyclotron drifts. The inclusion of shear into the analysis of cyclotron drifts, however, is not as arbitrary as it is for the low-frequency waves. Since the low-frequency drift instabilities are nearly independent of density, a certain minimum shear will be necessary to stabilize these waves regardless of the requirements of the density-dependent cyclotron drifts. Therefore, in addition to looking for shear stabilization it is necessary to consider the possibility that the shear which stabilizes low-frequency waves has a destabilizing effect on the cyclotron drifts.

In the following sections we examine the effect of shear on electrostatic waves with frequencies near multiples of the ion cyclotron frequency in collisionless plasmas characterized by single temperature, isotropic Maxwellian velocity distributions, and small linear spatial gradients in temperature and density. The general equation for cyclotron drift waves is derived in Sec. II by the usual linearization techniques. In Secs. III and IV, this equation is treated under two approximation schemes which are consonant with the two treatments of cyclotron drifts in the paper of Mikhailovsky and Timofeev.[5] The effects of shear are examined for each case, and the results and conclusions are presented in Sec. V.

## II. BASIC EQUATION

The basic equation is derived by the standard techniques for dealing with electrostatic oscillations.[6] The Vlasov equations describing the particle distribution functions in terms of an electrostatic perturbation potential are linearized, and the resulting equations are integrated over the unperturbed particle orbits to determine the perturbed distributions for substitution into Poisson's equation. The result is an integrodifferential equation for the potential of the perturbation,

$$-\nabla^2\Phi(\mathbf{r}, t) = \sum_i \frac{4\pi e_i^2}{m_i} \int d^3v \int_{-\infty}^{t} \nabla\Phi[\mathbf{r}_i'(t'), t'] \cdot \nabla_{\mathbf{v}_i'} f_{0i}[\mathbf{r}_i'(t'), \mathbf{v}_i'(t')]\, dt', \qquad (1)$$

where $e_j$ and $m_j$ are the charge and mass, respectively, of the $j$th particle species, $f_{0j}(\mathbf{r}, \mathbf{v})$ is the equilibrium distribution function for the $j$th species, and it is a function only of the constants of the unperturbed particle motion. The quantities $\mathbf{r}_j'(t')$ and $\mathbf{v}_j'(t')$ are the position and velocity solutions of the orbit equations for a particle of type $j$ in the equilibrium fields, with the conditions that at $t' = t$, $\mathbf{r}_j' = \mathbf{r}$, and $\mathbf{v}_j' = \mathbf{v}$.

The particular equilibrium geometry considered in this paper is indicated in Fig. 1. Three distances are of importance in describing this equilibrium:

(1) the Larmor radius of the ions,

$$\rho_i = [(2T_i m_i c^2)/(e_i^2 B^2)]^{1/2},$$

where $T_i$ is the ion temperature, (2) the distance characterizing density gradients in the plasma,

$$R = \left[\left(\frac{\partial f_{0i}}{\partial x}\right) \Big/ f_{0i}\right]^{-1},$$

and (3) the distance characterizing the magnetic shear,

$$L_s = \left[\left(\frac{dB_z}{dx}\right) \Big/ B_z\right]^{-1}.$$

These lengths are ordered as follows:

$$\rho_i \ll R \ll L_s$$

and in subsequent manipulations of the equations quantities of order $(\rho_i/R)$ and $(R/L_s)$ are retained, while terms of order $(\rho_i/L_s)$ are ignored. The last assumption means that the orbit equations reduce to simple spirals along the local field lines; i.e., the shear does not affect the individual equilibrium particle motion.

We consider a perturbation potential of the form

$$\Phi(\mathbf{r}, t) = \varphi(x) \exp(iky - i\omega t),$$

where $\varphi(x)$ is to be determined, and an equilibrium

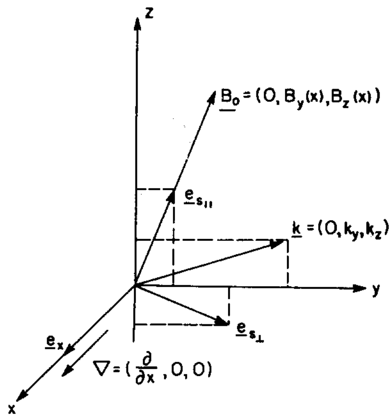


FIG. 1. Coordinate system.

characterized by single temperature Maxwellian velocity distributions, with linear spatial variations in density and temperature, i.e.,

$$f_{0j}(\mathbf{r}, \mathbf{v}) = \left\{1 + \left[\frac{n'}{n} + \frac{T'}{T}\left(\frac{v_\perp^2 + v_\parallel^2}{\alpha_j^2} - \frac{3}{2}\right)\right]\right.$$
$$\left.\cdot\left(x + \frac{v_\perp}{\Omega_j}\sin(\theta)\right)\right\} \frac{n}{\pi^{3/2}\alpha_j^3} \exp\left(-\frac{(v_\perp^2 + v_\parallel^2)}{\alpha_j^2}\right).$$

In this expression $\alpha_j = (2T/m_j)^{1/2}$ is the thermal velocity and $\Omega_j = (e_j B/m_j c)$ is the cyclotron frequency of the $j$th species, and the velocity variables are expressed in cylindrical coordinates $(v_\perp, v_\parallel, \theta)$ defined with respect to the local magnetic field.

After some relatively straightforward manipulations, illustrated for the low-frequency case in a previous paper,[7] Eq. (1) reduces to a second-order differential equation for the $x$-dependent part of the perturbation potential, $\varphi(x)$, or

$$A(x)\frac{d^2\varphi(x)}{dx^2} + k_\perp B(x)\frac{d\varphi(x)}{dx} + k_\perp^2 C(x)\varphi(x) = 0,$$

where, for a general frequency, the coefficients are given by

$$A(\omega, \mathbf{k}, x) = k_\perp^2 d^2 + \sum_j \sum_{l=-\infty}^{\infty} \exp(-\lambda_j) I_l(\lambda_j) \frac{1}{k_\parallel \alpha_j}\left\{-\omega\lambda_j\left[1 - \frac{I_l'(\lambda_j)}{I_l(\lambda_j)}\right] Z(\xi_j)\right.$$
$$+ \left(\frac{1}{k_\perp}\frac{n'}{n}\right)\left[-l\omega - [(l + k_\perp x)\omega - \lambda_j\Omega_j]\lambda_j\left(1 - \frac{I_l'(\lambda_j)}{I_l(\lambda_j)}\right)\right] Z(\xi_j) + \left(\frac{1}{k_\perp}\frac{T'}{T}\right)$$
$$\cdot\left[l\omega + (l^2 - \lambda_j)[(l + k_\perp x)\omega - \lambda_j\Omega_j] + 2l\omega\lambda_j\left(1 - \frac{I_l'(\lambda_j)}{I_l(\lambda_j)}\right) + k_\perp x\omega\lambda_j\left(1 - \frac{I_l'(\lambda_j)}{I_l(\lambda_j)}\right)\right.$$
$$\left. + 2\lambda_j[(l + k_\perp x)\omega - \lambda_j\Omega_j]\lambda_j\left(1 - \frac{I_l'(\lambda_j)}{I_l(\lambda_j)}\right)\right] Z(\xi_j)$$
$$\left. + \left(\frac{1}{k_\perp}\frac{T'}{T}\right)\left[l\omega + [(l + k_\perp x)\omega - \lambda_j\Omega_j]\lambda_j\left(1 - \frac{I_l'(\lambda_j)}{I_l(\lambda_j)}\right)\right][\tfrac{1}{2}Z(\xi_j) - \xi_j - \xi_j^2 Z(\xi_j)]\right\}, \quad (2)$$

$$B(\omega, \mathbf{k}, x) = \sum_i \sum_{l=-\infty}^{\infty} \exp(-\lambda_i) I_l(\lambda_i) \frac{\omega}{k_\| \alpha_i} \left\{ -\left(\frac{1}{k_\perp} \frac{n'}{n}\right) \lambda_i \left(1 - \frac{I_l'(\lambda_i)}{I_l(\lambda_i)}\right) Z(\xi_i) + \left(\frac{1}{k_\perp} \frac{T'}{T}\right) \left[ (l^2 - \lambda_i) + (1 + 2\lambda_i)\lambda_i \left(1 - \frac{I_l'(\lambda_i)}{I_l(\lambda_i)}\right)\right] Z(\xi_i) + \left(\frac{1}{k_\perp} \frac{T'}{T}\right) \lambda_i \left(1 - \frac{I_l'(\lambda_i)}{I_l(\lambda_i)}\right) [\tfrac{1}{2} Z(\xi_i) - \xi_i - \xi_i^2 Z(\xi_i)] \right\}, \tag{3}$$

$$C(\omega, \mathbf{k}, x) = -\left[2 + k^2 d^2 + 2\left(\frac{n'}{n} - \frac{T'}{T}\right)x\right] - \sum_i \sum_{l=-\infty}^{\infty} \exp(-\lambda_i) I_l(\lambda_i) \frac{1}{k_\| \alpha_i} \cdot \left\{ \omega Z(\xi_i) + \left(\frac{1}{k_\perp} \frac{n'}{n}\right) [(l + k_\perp x)\omega - \lambda_i \Omega_i] Z(\xi_i) + \left(\frac{1}{k_\perp} \frac{T'}{T}\right) \left[ -(l + k_\perp x)\omega - [(l + k_\perp x)\omega - \lambda_i \Omega_i] \lambda_i \left(1 - \frac{I_l'(\lambda_i)}{I_l(\lambda_i)}\right)\right] Z(\xi_i) + \left(\frac{1}{k_\perp} \frac{T'}{T}\right) [-(l + k_\perp x)\omega + \lambda_i \Omega_i][\tfrac{1}{2} Z(\xi_i) - \xi_i - \xi_i^2 Z(\xi_i)] \right\}. \tag{4}$$

In these expressions, $d = (T/4\pi ne^2)^{1/2}$ is the Debye length, $\lambda_i = \frac{1}{2} k_\perp^2 \rho_i^2$, $I_l$ is the Bessel function with imaginary argument of order $l$, and $Z(\xi_i)$ is the usual plasma dispersion function[8]

$$Z(\xi_i) = \pi^{-1/2} \int_{-\infty}^{\infty} \frac{\exp(-x^2)\, dx}{x - \xi_i}$$

with

$$\xi_i = \frac{\omega - l\Omega_i}{k_\| \alpha_i}.$$

The quantities $k_\perp$ and $k_\|$, as illustrated in Fig. 1, are the components of the wave vector $k$ perpendicular and parallel to the local magnetic field, respectively, and hence are functions of $x$. In particular, we will restrict the field variation to be of the form

$$B_y = B_0 \sin\left(\frac{x}{L_s}\right)$$

and

$$B_z = B_0 \cos\left(\frac{x}{L_s}\right)$$

so that

$$k_\| = \frac{1}{B_0} (\mathbf{k} \cdot \mathbf{B}) = k \sin\left(\frac{x}{L_s}\right)$$

and

$$k_\perp = k \cos\left(\frac{x}{L_s}\right).$$

For the small shear we are considering these quantities can be further reduced to

$$k_\| \approx k\left(\frac{x}{L_s}\right), \quad k_\perp \approx k,$$

where $k$ is a constant.

Since our interest is in frequencies near multiples of the ion cyclotron frequency and in instabilities driven by the equilibrium gradients, expressions (2), (3), and (4) may be considerably simplified. First, if $|\omega - l_0\Omega_i| \ll \Omega_i$, only the $l = l_0$ term need be retained in the ion contribution to the coefficient, and only the $l = 0$ term need be kept for the electron contribution. Second, since the ratio of gradient terms to homogeneous terms is on the order of

$$\left|\frac{1}{k_\perp R} \frac{l_0\omega - \lambda_i \Omega_i}{\omega}\right| \approx \left|\frac{\lambda_i}{k_\perp R}\right|$$

and the effects of the drifts will be most prominent when this ratio is approximately unity, we require that $\lambda_i$ be large; i.e., terms of order $\lambda_i^{-1}$ will subsequently be ignored. In effect, this condition is the requirement that the ion drift velocity

$$v_{Di} \equiv \frac{1}{2} \frac{\rho_i}{R} \alpha_i = -v_{De} \tag{5}$$

be nearly equal to the perpendicular phase velocity of the wave, $(\omega/k_\perp)$. We still assume, however, that $\lambda_e$ is small, and ignore terms of order $\lambda_e$ in the electron contribution to the coefficients.

With the above assumptions, the terms which explicitly contain $x$ are eliminated except for the term $\{[(n'/n) - (T'/T)]x\}$ in expression (4). This term will be of order $(x/R)$ and, therefore, will be significant over some portion of the region in which the equation is applicable. However, for the sake of simplicity, this term is ignored. This is commonly done, even if not expressly stated, and hopefully does not invalidate the final results.

The coefficients (2), (3), and (4) reduce to

$$A(\omega, \mathbf{k}, x) = k^2d^2 - \frac{1}{2}\left(\frac{1}{2\pi\lambda_i}\right)^{1/2} \cdot\left\{\left[\omega - kv_{D_i}\left(1 - \frac{\eta}{2}\right)\right]\frac{1}{k_\parallel\alpha_i} Z(\xi_i) + \frac{kv_{D_i}}{k_\parallel\alpha_i}\eta[(\tfrac{1}{2} - \xi_i^2)Z(\xi_i) - \xi_i]\right\}, \quad (6)$$

$$B(\omega, \mathbf{k}, x) = 0,$$

$$C(\omega, \mathbf{k}, x) = -(2 + k^2d^2) - \left(\frac{1}{2\pi\lambda_i}\right)^{1/2}\left\{\left[\omega - kv_{D_i}\left(1 - \frac{\eta}{2}\right)\right] \cdot\frac{1}{k_\parallel\alpha_i} Z(\xi_i) + \frac{kv_{D_i}}{k_\parallel\alpha_i}\eta[(\tfrac{1}{2} - \xi_i^2)Z(\xi_i) - \xi_i]\right\} - \left((\omega + kv_{D_i})\frac{Z(\xi_e)}{k_\parallel\alpha_e} - \frac{kv_{D_i}}{k_\parallel\alpha_e}\eta[(\tfrac{1}{2} - \xi_e^2)Z(\xi_e) - \xi_e]\right), \quad (7)$$

where $\eta = (d \ln T/d \ln n)$, $\xi_e = (\omega/k_\parallel\alpha_e)$, $\xi_i = (\omega - l\Omega_i)/(k_\parallel\alpha_i)$, $v_{D_i}$ is defined by Eq. (5), and the $x$ dependence is totally contained in $k_\parallel = k(x/L_s)$. Therefore, the basic equation for the study of magnetic shear effects on ion cyclotron drift instabilities is

$$\frac{d^2\varphi(x)}{dx^2} + k_\perp^2 \frac{C(\omega, \mathbf{k}, x)}{A(\omega, \mathbf{k}, x)}\varphi(x) = 0, \quad (8)$$

where $A$ and $C$ are given by (6) and (7), respectively. The boundary condition for Eq. (8) is the requirement that $\varphi(x) \to 0$ as $|x| \to \infty$.

Of the many significant assumptions that restrict the applicability of Eq. (8), the major ones are:

(1) The equilibrium plasma is characterized by (i) a single temperature Maxwellian velocity distribution for both species, (ii) linear spatial gradients in temperature and density perpendicular to the magnetic field, and (iii) the absence of collisions.

(2) The applied magnetic field is sheared so slightly that individual particles behave locally as in a constant field. There is no zero-order electric field.

(3) Only small electrostatic, normal mode perturbations with wave vectors perpendicular to the gradients are considered.

## III. SHEAR EFFECTS ON WAVES PROPAGATING NEARBY PERPENDICULAR TO THE MAGNETIC FIELD

In the absence of shear, $k_\parallel$ is a constant and the stability of the plasma for cyclotron drift waves may be determined by considering the roots of $C(\mathbf{k}, \omega) = 0$ where $C$ is given by expression (7). If propagation perpendicular to the magnetic field is considered, i.e., $k_\parallel = 0$, this equation reduces to

$$1 + k^2d^2 - \frac{kv_{D_i}}{\omega} = \left(\frac{1}{2\pi\lambda_i}\right)^{1/2}\frac{\omega - [1 - (\eta/2)]kv_{D_i}}{\omega - l\Omega_i}. \quad (9)$$

Mikhailovsky and Timofeev[5] have shown that if

$$\left|\frac{\rho_i}{R}\right| > 4l\left|\frac{d}{\rho_i}\right| \quad (10)$$

the following unstable solutions are possible:

$$\frac{\mathrm{Re}\,(\omega - l\Omega_i)}{l\Omega_i} = -\frac{1}{2}\left(\frac{k_l^2d^2 - 1}{k_l^2d^2 + 1} - \frac{\eta}{2}\right)\left(\frac{1}{2\pi\lambda_i}\right)^{1/2} + \cdots, \quad (11)$$

$$\frac{\mathrm{Im}\,(\omega - l\Omega_i)}{l\Omega_i} = \left(\frac{k_l^2d^2}{k_l^2d^2 + 1} - \frac{\eta}{2}\right)^{1/2}\left(\frac{1}{2\pi\lambda_i}\right)^{1/4} + \cdots, \quad (12)$$

where $k_l$ is a solution of the equation

$$k_lv_{D_i} = (1 + k_l^2\, d^2)l\Omega_i.$$

In this section, we examine the effect of shear on the particular instability described by Eqs. (11) and (12). Of course, shear will introduce a small $k_\parallel$ which is missing from Eq. (9), but if the shear is small, the arguments of the $Z$ functions $\xi_e$ and $\xi_i$ will be large and, therefore, the $Z$ functions may be expanded asymptotically for both the ions and electrons, i.e.,[8]

$$\xi_iZ(\xi_i) \simeq -1 - \frac{1}{2\xi_i^2} - \cdots.$$

If terms to order $(\xi_i)^{-2}$ and $(\xi_e)^{-2}$ are retained, Eq. (8) becomes

$$\frac{d^2\varphi(x)}{dx^2} - k^2\left(\frac{C_0 - C_2x^2}{A_0 + A_2x^2}\right)\varphi(x) = 0, \quad (13)$$

where

$$C_0 = \left[1 + k^2d^2 - \frac{kv_{D_i}}{\omega} - \left(\frac{1}{2\pi\lambda_i}\right)^{1/2}\frac{\{\omega - kv_{D_i}[1 - (\eta/2)]\}}{\omega - l\Omega_i}\right],$$
$$C_2 = \left[\frac{[\omega + kv_{D_i}(1 + \eta)]}{\omega^3}\left(\frac{m_i}{m_e}\right) + \left(\frac{1}{2\pi\lambda_i}\right)^{1/2}\frac{\{\omega - kv_{D_i}[1 + (\eta/2)]\}}{(\omega - l\Omega_i)^3}\right]\frac{\lambda_i\Omega_i^2}{L_s^2}, \quad (14)$$

$$A_0 = \left[k^2d^2 + \frac{1}{2}\left(\frac{1}{2\pi\lambda_i}\right)^{1/2} \cdot \frac{\{\omega - kv_{Di}[1 - (\eta/2)]\}}{(\omega - l\Omega_i)}\right],$$
$$A_2 = \left[\frac{1}{2}\left(\frac{1}{2\pi\lambda_i}\right)^{1/2} \cdot \frac{\{\omega - kv_{Di}[1 - (\eta/2)]\}}{(\omega - l\Omega_i)^3}\right]\frac{\lambda_i\Omega_i^2}{L_s^2}. \qquad (15)$$

It is a simple task to show that for $(\omega - l\Omega_i)$ near the solutions given by (11) and (12), the ion contribution to $C_2$ is of order $\lambda_e$ less than the electron contribution, so that

$$C_2 \simeq \frac{[\omega + kv_{Di}(1 + \eta)]}{\omega^3}\left(\frac{m_i}{m_e}\right)\frac{\lambda_i\Omega_i^2}{L_s^2} \qquad (16)$$

and that near the origin of the $x$ plane, $|A_0| \gg |A_2x^2|$, so that Eq. (13) reduces to

$$\frac{d^2\varphi(x)}{dx^2} - k^2\left(\frac{C_0 - C_2x^2}{A_0}\right)\varphi(x) = 0.$$

The simple change of independent variable to $\zeta = k\beta x$ allows this equation to be written as Weber's equation[9]

$$\frac{d^2\varphi}{d\zeta^2} + \left(n + \frac{1}{2} - \frac{\zeta^2}{4}\right)\varphi = 0 \qquad (17)$$

if

$$\left(-\frac{C_0}{\beta^2A_0}\right) = n + \tfrac{1}{2} \qquad (18)$$

and

$$\left(-\frac{C_2}{k^2\beta^4A_0}\right) = \frac{1}{4}. \qquad (19)$$

The solutions of Eq. (17) are the parabolic cylinder functions, $D_n(\zeta)$, which are given asymptotically by

$$D_n(\zeta) \simeq \exp\left(-\frac{\zeta^2}{4}\right)\zeta^n - \frac{(2\pi)^{1/2}}{\Gamma(-n)}\exp(in\pi)\exp\left(+\frac{\zeta^2}{4}\right)\zeta^{-n-1}.$$

In order to satisfy the boundary condition $\varphi \to 0$ as $x \to \pm\infty$, it is necessary that: (1) $n = 0, 1, 2, \cdots$, a nonnegative integer, and (2) Re $(\zeta^2) > 0$, when $x$ is real. This latter condition requires that Re $(\beta^2) > 0$, or from Eq. (18),

$$\text{Re}\left(\frac{C_0}{A_0}\right) < 0. \qquad (20)$$

Elimination of $\beta^2$ from (18) and (19) then gives an eigenvalue relation for $\omega$, or

$$C_0^2 + \frac{(2n+1)^2}{k^2}A_0C_2 = 0. \qquad (21)$$

The above analysis assumes that Eq. (13) is valid in a region of the $x$ plane that extends from the origin at least as far as $|x| \leq |C_0/C_2|^{1/2}$. Even though $C_2x^2$ is a small term resulting from the asymptotic expansion of the $Z$ function, this assumption is valid as long as $(\omega - l\Omega_i)$ is assumed to be near the roots of $C_0$, i.e., near the values of expressions (11) and (12).

With coefficients $C_0$ and $A_0$ given by (14) and (15), condition (20) becomes

$$\left[\text{Im}\left(\frac{(\omega - l\Omega_i)}{l\Omega_i}\right)\right]^2\left(k^2d^2 + 1 - \frac{kv_{Di}}{l\Omega_i}\right) + O\left(\frac{1}{2\pi\lambda_i}\right) < 0, \qquad (22)$$

where, from Eqs. (11) and (12), it is assumed that

$$\text{Re}\left(\frac{(\omega - l\Omega_i)}{l\Omega_i}\right) \simeq O\left[\left(\frac{1}{2\pi\lambda_i}\right)^{1/2}\right]$$

and

$$\text{Im}\left(\frac{(\omega - l\Omega_i)}{l\Omega_i}\right) \approx O\left[\left(\frac{1}{2\pi\lambda_i}\right)^{1/4}\right].$$

Since $\lambda_i \gg 1$, condition (22) further becomes

$$k^2d^2 - \frac{kv_{Di}}{l\Omega_i} + 1 \leq 0 \qquad (23)$$

which can only be satisfied for real $k$ if

$$\left(\frac{v_{Di}}{2l\Omega_id}\right)^2 > 1.$$

This condition is exactly that given for the existence of instabilities in a uniform field, Eq. (10). Therefore, the introduction of a small shear will not disrupt the stability of a plasma which is stable against these disturbances in a uniform field.

If condition (10) is satisfied, the stability of the plasma is determined by examining the roots of the eigenvalue equation (21), for all wavenumbers satisfying condition (23). With $C_0$, $A_0$, and $C_2$ given by expressions (14), (15), and (16), respectively, Eq. (21) becomes, upon multiplication by $\omega^3(\omega - l\Omega_i)^2$,

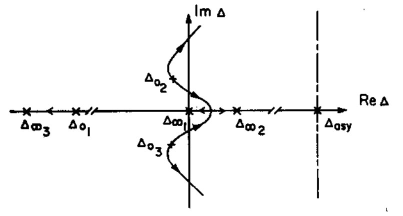


FIG. 2. The effect of shear on the unstable roots of the uniform field dispersion relation for perpendicular propagation.

$$(\Delta + 1)\Big\{(\mu + 1 - \epsilon)\Delta^2 + \left[\mu + 1 - \omega_D - 2\epsilon + \epsilon\omega_D\left(1 - \frac{\eta}{2}\right)\right]\Delta + \epsilon\left[\omega_D\left(1 - \frac{\eta}{2}\right) - 1\right]\Big\}^2 + S\Delta\left\{\left(\mu + \frac{\epsilon}{2}\right)\Delta - \frac{\epsilon}{2}\left[\omega_D\left(1 - \frac{\eta}{2}\right) - 1\right]\right\} \cdot\{\Delta + [\omega_D(1 + \eta) + 1]\} = 0, \qquad (24)$$

where

$$\Delta = \frac{(\omega - l\Omega_i)}{l\Omega_i}, \qquad \omega_D = \frac{kv_{Di}}{l\Omega_i},$$

$$\epsilon = \left(\frac{1}{2\pi\lambda_i}\right)^{1/2}, \qquad \mu = k^2d^2,$$

and

$$S = \frac{(2n + 1)^2\lambda_i}{k^2L_s^2}\left(\frac{m_i}{m_e}\right) \geq 0.$$

Equation (24) is a fifth-degree polynomial in Δ, and, therefore, no method exists for analytically determining the roots in terms of the coefficients. However, this equation has real coefficients and is of the form

$$(\Delta + 1)[p(\Delta)]^2 + S\,\Delta q(\Delta)r(\Delta) = 0,$$

where $p(\Delta)$ is a second-order polynomial, and $q(\Delta)$ and $r(\Delta)$ are first-order polynomials. We are interested in the small roots of $C_0(\omega)$ or equivalently, the roots of $p(\Delta)$ and their change as $S$ increases from zero. Since the roots of $p(\Delta)$, $q(\Delta)$, and $r(\Delta)$ are readily determined, the root locus method suggests itself as a way to handle this problem. In particular, we note that: (1) the number of finite roots of Eq. (24) decreases from the five to three as $S$ goes from 0 to $+\infty$, and, therefore, the root locus has two branches going to infinity, and (2) the roots of interest, i.e., those of $p^2(\Delta)$ are complex roots in the unstable uniform field case and are double, indicating that two branches of the root locus begin at each root. From the properties of the root locus[10] it can readily be seen that the two branches going to infinity are asymptotically perpendicular to the real axis, and that one of them has a monotonically increasing imaginary part. Furthermore, these branches must originate at the complex double roots of $p^2(\Delta) = 0$. Therefore, as $S$ increases from zero, i.e., shear is introduced, one of the unstable roots of the eigenvalue Eq. (24) has a monotonically increasing imaginary part. A typical example of such a root locus is shown in Fig. 2. Finally, we may say that not only does the small shear not stabilize these instabilities, but it actually increases their growth rates. We recall, however, that the introduction of shear does not change the condition for the onset of instability given by Eq. (10).

## IV. SHEAR EFFECTS ON WAVES PROPAGATING OBLIQUELY TO THE FIELD

In examining cyclotron drift waves in uniform fields, it is more realistic to consider waves for which $k_\parallel \neq 0$, since perpendicular propagation is unlikely for a machine of finite length. If $(1/k_\parallel)$ is restricted to be less than some machine length $L_m$, an upper limit is placed on the magnitude of the argument of the electron $Z$ function, $\xi_e$, i.e.,

$$|\xi_e| \lesssim \left|\frac{l\Omega_i}{\alpha_e}L_m\right| \lesssim \left|l\left(\frac{L_m}{\rho_i}\right)\left(\frac{m_e}{m_i}\right)^{1/2}\right|.$$

The mass ratio factor assures us that this is a small number, and in fact, we assume that $|\xi_e| \ll 1$, allowing the electron $Z$ function in expression (7) to be expanded in a power series, and replaced by a constant, i.e.,

$$Z(\xi_e) = i\pi^{1/2} + O(\xi_e).$$

If the ion $Z$ functions are expanded asymptotically as before, Mikhailovsky and Timofeev[5] have shown that a solution of $C = 0$, consistent with these expansions, is given by

$$\mathrm{Re}\left(\frac{(\omega - l\Omega_i)}{l\Omega_i}\right) \approx \left(\frac{1}{2\pi\lambda_i}\right)^{1/2}\frac{\{1 - [1 - (\eta/2)](k_\perp v_{Di}/l\Omega_i)\}}{(2 + k^2d^2)} \qquad (25)$$

and

$$\mathrm{Im}\left(\frac{(\omega - l\Omega_i)}{l\Omega_i}\right) \approx -\left(\frac{1}{2\pi\lambda_i}\right)^{1/2}\left|\frac{l\Omega_i}{k_\parallel\alpha_e}\right| \cdot\frac{\{1 - [1 - (\eta/2)]^2(k_\perp v_{Di}/l\Omega_i)^2\}}{(2 + k^2d^2)^2}. \qquad (26)$$

Since the stability condition

$$\left|\left(1-\frac{\eta}{2}\right)\frac{k_\perp v_{D_i}}{l\Omega_i}\right| < 1$$

can always be violated for some value of $k_\perp$, these solutions represent a universal instability. We now wish to examine the effect of shear on this particular disturbance.

We return to Eq. (8), where $C$ and $A$ are given by Eqs. (6) and (7), respectively. Our task is considerably simplified if we anticipate that the result for $(\omega - l\Omega_i)$ in the presence of shear does not significantly differ from the uniform field result, expressions (25) and (26), and thus approximate $A(x)$ by a constant

$$A(x) \approx 1 + \tfrac{3}{2}(k^2 d^2)$$

obtained by substitution of Eq. (25) into expression (6). Using the expansions of the $Z$ functions mentioned above, we can put Eq. (8) in the form

$$\frac{d^2\varphi(x)}{dx^2} - \frac{k^2}{1+\frac{3}{2}k^2d^2}\Bigg[(2+k^2d^2) - \left(\frac{1}{2\pi\lambda_i}\right)^{1/2}\frac{\{\omega - kv_{D_i}[1-(\eta/2)]\}}{\omega - l\Omega_i} - \left(\frac{1}{2\pi\lambda_i}\right)^{1/2}\frac{\{\omega - kv_{D_i}[1+(\eta/2)]\}}{(\omega - l\Omega_i)^3}\frac{\lambda_i\Omega_i^2}{L_s^2}x^2 + i\pi^{1/2}\frac{\{\omega + [1-(\eta/2)]kv_{D_i}\}}{|k\alpha_e|}\frac{L_s}{|x|}\Bigg]\varphi = 0. \quad (27)$$

We now proceed to study the effect of sheared fields on highly localized wave packets composed of a superposition of solutions of Eq. (27), by using a method suggested by Rutherford and Frieman[4] and explained in detail in Ref. 7. First, we note that if an equation such as

$$\frac{d^2\varphi(x)}{dx^2} - Q(x, k, \omega - l\Omega_i)\varphi(x) = 0$$

is derived from a consideration of potentials of the form

$$\Phi(\mathbf{r}, t) = \varphi(x)\exp(iky - il\Omega_i t)\exp[-i(\omega - l\Omega_i)t],$$

then wave packets of the form

$$\Phi(\mathbf{r}, t) = \varphi(x, t)\exp(iky - il\Omega_i t),$$

where

$$\varphi(x, t) = \int_{-\infty}^{\infty} F(x, \omega - l\Omega_i)\cdot\exp[-i(\omega - l\Omega_i)t]\,d(\omega - l\Omega_i)$$

will lead to the partial differential equation:

$$\frac{\partial^2\varphi(x,t)}{\partial x^2} - Q\left(x, k, i\frac{\partial}{\partial t}\right)\varphi(x, t) = 0.$$

This slight variation of the argument used in the low-frequency case allows us to treat Eq. (27) exactly as Rutherford and Frieman treat their comparable low-frequency equation. In fact, Eq. (27) can be reduced to the same equation treated by these authors, i.e.,

$$\delta^2\frac{\partial^2\varphi(y,\tau)}{\partial y^2} + K\left(y, i\delta\frac{\partial}{\partial\tau}\right)\varphi(y, \tau) = 0 \quad (28)$$

by introducing the following nondimensional parameters:

$$\kappa = \frac{\{l\Omega_i - kv_{D_i}[1-(\eta/2)]\}}{(2\pi\lambda_i)^{1/2}(2+k^2d^2)l\Omega_i},$$

$$\Omega = \frac{(\omega - l\Omega_i)}{\kappa l\Omega_i},$$

$$\delta = \left|\left(\frac{R}{L_s}\right)\frac{(1+\frac{3}{2}k^2d^2)^{1/2}kv_{D_i}}{\lambda_i^{1/2}\kappa l\Omega_i}\right|,$$

$$y = \frac{\sqrt{2}\,R}{\rho_i L_s}\frac{kv_{D_i}}{\kappa l\Omega_i}x,$$

$$\nu = \frac{\{l\Omega_i - kv_{D_i}[1+(\eta/2)]\}}{\{l\Omega_i - kv_{D_i}[1-(\eta/2)]\}},$$

$$\theta = \frac{\{l\Omega_i + kv_{D_i}[1-(\eta/2)]\}}{|l\Omega_i - kv_{D_i}[1-(\eta/2)]|}\pi\left(\frac{\lambda_i}{2}\frac{m_e}{m_i}\right)^{1/2},$$

$$\tau = \delta\kappa l\Omega_i t.$$

Of course, the functional form of $K$ is different for the ion cyclotron case,

$$K(y, \Omega) = \left[\left(\frac{1}{\Omega} - 1\right) + \nu\frac{y^2}{\Omega^3}\right] - i\frac{\theta}{|y|}, \quad (29)$$

but Eq. (28) is the same, and the Rutherford and Frieman results pertinent to equations of this form may be carried over without change. They are

1. The criterion for stabilization of highly localized disturbances characterized by the parameter $\Omega$ is $\delta > \delta_{\text{crit}}$, where

$$\delta_{\text{crit}} = \int^{y}\frac{\operatorname{Im}K(y,\Omega)}{[\operatorname{Re}K(y,\Omega)]^{1/2}}\,dy \quad (30)$$

for $|\operatorname{Im}K/\operatorname{Re}K| \ll 1$, and

$$\delta_{\text{crit}} = \int^{y}\operatorname{Im}[K^{1/2}(y,\Omega)]\,dy \quad (31)$$

if $|\operatorname{Im}K/\operatorname{Re}K| \approx O(1)$. In both cases, the integration is performed for all $y$ for which the integrand is

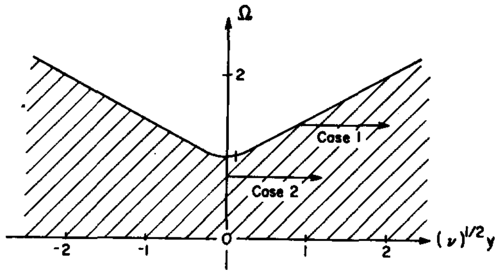


FIG. 3. The region of the $(y, \Omega)$ plane for which the integrand of Eq. (33) is real; $\nu > 0$.

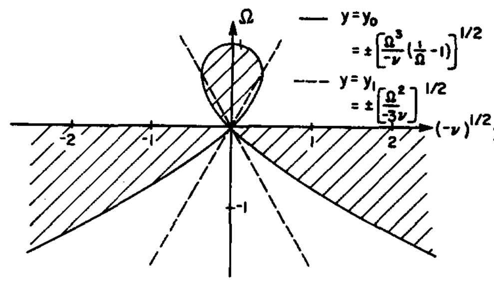


FIG. 4. The region of the $(y, \Omega)$ plane for which the integrand of Eq. (33) is real; $\nu < 0$.

real and positive. As is generally the case when dealing with convective instabilities, the term *stabilization* means that the energy increase is restricted to one exponentiation during the time in which the perturbation is in a region of positive growth. The parameter $\delta_{crit}$ must of course, be maximized over $\Omega$.

2. The motion of the wave packet is governed by the characteristic equation,

$$\frac{dy}{d\tau} = -2\frac{[\mathrm{Re}\,K(y,\,\Omega)]^{1/2}}{(\partial/\partial\Omega)[\mathrm{Re}\,K(y,\,\Omega)]} \tag{32}$$

for a given $\Omega$.

Using the cyclotron expression for $K$, from Eq. (29), we write Eq. (30) as

$$\delta_{crit} = \int^{y} \frac{(-\theta)}{\{[(1/\Omega) - 1] + \nu(y^2/\Omega^3)\}^{1/2}}\frac{dy}{|y|}, \tag{33}$$

and the integrand will be real only when

$$\left[\left(\frac{1}{\Omega} - 1\right) + \nu\frac{y^2}{\Omega^3}\right] > 0. \tag{34}$$

The boundary of the region of the $(y, \Omega)$ plane in which this condition is satisfied, $|y| = y_0$ where

$$y_0 = +\left[\frac{\Omega^3}{\nu}\left(1 - \frac{1}{\Omega}\right)\right]^{1/2}$$

is shown in Figs. 3 and 4 for positive and negative values of $\nu$, respectively. The sign of the integrand of Eq. (33) will be positive when the square root in the denominator and $\theta$ have opposite signs. The sign of the square root is determined by the requirement that the integration in expression (34) must correspond to a forward movement of the wave packet in real time, i.e., $dt > 0$. Integration along the positive $y$ axis means that

$$dy = \left(\frac{dy}{d\tau}\right)\left(\frac{d\tau}{dt}\right) dt > 0$$

and, therefore, $(dy/d\tau)$, given by expression (32), and $(d\tau/dt) = \delta\kappa l\Omega_i$, must have the same sign. In other words, the product

$$\theta\kappa\left(\frac{\partial}{\partial\Omega}\,\mathrm{Re}\,K(y,\,\Omega)\right)$$

must be positive for the integrand to be positive. From Eq. (29), we have

$$\frac{\partial}{\partial\Omega}\,\mathrm{Re}\,K = -\frac{1}{\Omega^2}\left(1 + 3\nu\frac{y^2}{\Omega^2}\right)$$

and, therefore, when $\nu > 0$, $(\partial/\partial\Omega)(\mathrm{Re}\,K)$ is negative for all $\Omega$ and $y$, while if $\nu < 0$, $(\partial/\partial\Omega)(\mathrm{Re}\,K)$ is positive unless $|y| < y_1$, with

$$y_1 = +\left(\frac{1}{3(-\nu)}\right)^{1/2}|\Omega|.$$

This curve is shown in Fig. 4. We note that the intersection of this curve with the region defined by Eq. (34) occurs at $\Omega = 0$ and $\Omega = (\frac{2}{3})$, independent of $\nu$. If these conditions are collected and combined with the reality condition (34), one finds six unique cases of integration, divided on the basis of integration limits. These cases are given in Table I,

TABLE I. The six distinct cases of integration for which the critical shear parameter $\delta_{crit}$ is defined.

| Case number | Range of $\Omega$ | Limits of integration | Range of $\nu$ | Figure number |
|---|---|---|---|---|
| 1 | $1 < \Omega$ | $-\infty$ to $-y_0$; $+y_0$ to $+\infty$ | $\nu > 0$ | 3 |
| 2 | $0 < \Omega < 1$ | $-\infty$ to $+\infty$ | $\nu > 0$ | 3 |
| 3 | $\frac{2}{3} < \Omega < 1$ | $-y_0$ to $+y_0$ | $\nu < 0$ | 5 |
| 4 | $0 < \Omega < \frac{2}{3}$ | $-y_1$ to $+y_1$ | $\nu < 0$ | 5 |
| 5 | $0 < \Omega < \frac{2}{3}$ | $-y_0$ to $-y_1$; $+y_1$ to $+y_0$ | $\nu < 0$ | 6 |
| 6 | $\Omega < 0$ | $-\infty$ to $-y_0$; $+y_0$ to $+\infty$ | $\nu < 0$ | 6 |

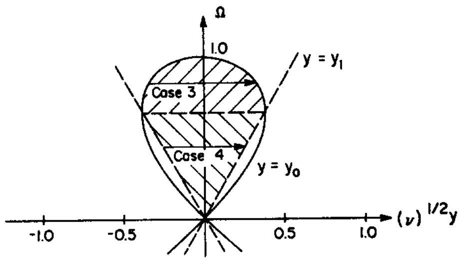


FIG. 5. The limits of integration for Eq. (30) for cases 3 and 4 of Table I.

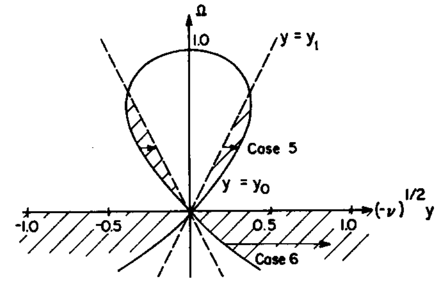


FIG. 6. The limits of integration for Eq. (30) for cases 5 and 6 of Table I.

and illustrated in Figs. 3, 5, and 6. Since in each case, the integrand is assuredly real and positive, the sign of the square root always being opposite to that of $\theta$, we may replace $\theta$ by its absolute value and consider only the positive square root of the denominator. The integrations can then be performed case by case.

### A. Case 1: $\nu > 0, \Omega > 1$

The value of $\delta_{crit}$ is given by

$$\delta_{crit} = |\theta| \int_{y_0}^{\infty} \frac{dy}{y} \left(\frac{1}{\Omega} - 1 + \nu \frac{y^2}{\Omega^3}\right)^{-1/2}$$

which is immediately integrated to give

$$\delta_{crit} = \frac{\pi}{2} |\theta| \left(\frac{\Omega}{\Omega - 1}\right)^{1/2}.$$

Therefore, except at $\Omega = 1$, $\delta_{crit}$ will have the same order of magnitude as $\theta$, i.e.,

$$\delta_{crit} \sim \left(\frac{m_e}{m_i}\right)^{1/2} \lambda_i^{1/2}$$

and the singularity at $\Omega = 1$ will be examined later.

### B. Case 2: $\nu > 0, 0 < \Omega < 1$

In this case, the integral is over all $y$ and appears to have a logarithmic singularity at the origin. However, the assumption that $|\mathrm{Im}\, K/\mathrm{Re}\, K| \ll 1$, breaks down at the origin and Eq. (31) must be used instead of (30) in this area. The dominent contribution comes from the range $y_{min} < y < y_{max}$, where $y_{min}$ is the point at which Im $K$ is of the order of Re $K$, i.e.,

$$y_{min} \sim \frac{|\theta|\ \Omega^{1/2}}{(1 - \Omega)^{1/2}}$$

and $y_{max}$ is the point at which the $y^2$ term in the denominator begins to be appreciable, or

$$y_{max} \sim \frac{\Omega}{\nu^{1/2}} (1 - \Omega)^{1/2}.$$

Integration of (30) between these limits gives

$$\delta_{crit} = |\theta| \left(\frac{\Omega}{1 - \Omega}\right)^{1/2} \ln\left(\frac{\Omega^{1/2}(1 - \Omega)^{1/2}}{\nu^{1/2}\, |\theta|}\right).$$

Once more, except for $\Omega$ near its limiting values of 0 and 1, $\delta_{crit}$ is given by

$$\delta_{crit} \sim \left(\frac{m_e}{m_i}\right)^{1/2} \lambda_i^{1/2} \ln\left(\frac{m_i}{m_e}\frac{1}{\lambda_i}\right)^{1/2}.$$

Near $\Omega = 1$, the same singularity found in case 1 arises and will also be treated later. The apparent singularity at $\Omega = 0$ is removed by noting that $\Omega$ is restricted to values greater than $(\nu\theta^2)$, so that $y_{max}$ will be greater than $y_{min}$. For smaller values of $\Omega$, we may approximately calculate $\delta_{crit}$ to be

$$\delta_{crit} \sim |\theta| \int_{y_{max}}^{\infty} \frac{dy}{y} \left(\nu \frac{y^2}{\Omega^3}\right)^{-1/2}$$
$$\sim |\theta|\ \Omega^{1/2}$$

which goes to zero as $\Omega$ vanishes.

### C. Case 3: $\nu < 0, \frac{2}{3} < \Omega < 1$

The integration limits are $(\pm y_0)$, and the apparent singularity of the origin is treated as in case 2, the limits being reduced to $[y_{min}, y_0]$. The resulting value of $\delta_{crit}$ is again of the order of $\theta$, and does not have a singularity as $\Omega$ goes to one.

### D. Case 4: $\nu < 0, 0 < \Omega < \frac{2}{3}$

Again the limits of integration are reduced from $[-y_1, +y_1]$ to $[+y_{min}, +y_1]$ with results comparable to case 3.

### E. Case 5: $\nu < 0, 0 < \Omega < \frac{2}{3}$

For the integration limits $[y_1, y_0]$ no approximate formulas are needed, and Eq. (30) directly gives a value of $\delta_{crit}$ of the order of $\theta$.

### F. Case 6: $\nu < 0, \Omega < 0$

The integration is identical to that of case 1, with identical results, except that there is no singularity for any negative value of $\Omega$.

The general conclusion is that $\delta_{\text{crit}}$ is of the same order of magnitude as $\theta$, except when $\Omega$ approaches one, for $\nu > 0$. As $\Omega \rightarrow 1$, approximation $(\text{Im } K/\text{Re } K) \ll 1$ is no longer valid over a large range of $y$, and (31) must be used to calculate $\delta_{\text{crit}}$. Roughly, the range of $y$ is $y < y^*$ where

$$|\text{Im } K(y^*, 1)| \approx |\text{Re } K(y^*, 1)|$$

or

$$y^* \sim \left(\frac{|\theta|}{\nu}\right)^{1/3}.$$

Therefore, the integral for $\delta_{\text{crit}}$ can be divided into two parts

$$\delta_{\text{crit}} = \text{Im} \int_0^{y^*} \left(\nu y^2 + i\frac{|\theta|}{y}\right)^{1/2} dy + |\theta| \int_{y^*}^{\infty} \frac{dy}{y(\nu y^2)^{1/2}}.$$

The second term is easily evaluated to give a term of the order of $|\theta|^{2/3}$. The first term can be estimated to be

$$\text{Im } (\nu y^* + i\,|\theta|)^{1/2} \int_0^{y^*} \frac{dy}{y^{1/2}}$$

which is also of the order of $|\theta|^{2/3}$. Finally, one finds that as $\Omega \rightarrow 1$, with $\nu > 0$, the largest value of $\delta_{\text{crit}}$ is calculated to be

$$\delta_{\text{crit}} \approx \frac{|\theta|^{2/3}}{\nu^{1/6}}. \tag{35}$$

We note that a plasma with constant temperature is no more stable than the above general case, since in that instance, $\nu$ is still positive, in fact $\nu = 1$. From the definitions of the nondimensional parameters, we see that the worst case, $\Omega = 1$, corresponds to the wave whose real frequency $(\omega - l\Omega_i)$ is equal to the frequency obtained for drift waves in the absence of shear. This result is similar to the low-frequency drift results calculated by Rutherford and Frieman.

If Eq. (35) is converted back into dimensional quantities, we obtain the following condition for shear stabilization of ion cyclotron drift waves:

$$\left(\frac{R}{L_s}\right) \gtrsim \left(\frac{m_e}{m_i}\right)^{1/3} \frac{\lambda_i^{1/3}}{(2 + k^2d^2)^{1/2}(1 + \frac{3}{2}k^2d^2)^{1/2}}. \tag{36}$$

## V. CONCLUSIONS

To properly interpret the results of Secs. III and IV, we must be cognizant of two important facts concerning ion cyclotron drift waves. First, they are characterized by very short wavelengths. In fact, we have assumed that $\lambda_i \gg 1$, so that the wavelengths are short even in comparison with the average ion Larmor radius. This is in contrast to the low-frequency results in which it is assumed that $\lambda_i \lesssim 1$. Secondly, on purely physical grounds, we expect that a plasma will have trouble supporting any electrostatic waves with wavelengths shorter than the Debye length characterizing the plasma. That is, we expect waves to be troublesome only when $k^2d^2 \lesssim 1$. The combination of these two facts leads us to expect that cyclotron drift waves will only be of importance when the ratio $(\rho_i/d)$ is relatively large, at least greater than unity. The importance of this requirement in satisfying condition (10) for the onset of one type of drift instability is obvious. We also see that the size of this ratio is going to have a large effect on the interpretation of Eq. (36). If $(\rho_i/d)$ is very small, $(kd)^2$ will be much greater than one, and condition (36) becomes

$$\left(\frac{R}{L_s}\right) \gtrsim \left(\frac{m_e}{m_i}\right)^{1/3} \left(\frac{1}{\lambda_i}\right)^{2/3} \left(\frac{\rho_i}{d}\right)^2.$$

The right-hand side of this relation is the product of three small quantities, and the shear required for stability will be minimal. In fact, it is likely that the shear required for the stabilization of low-frequency drift waves will be much greater, although no direct comparison is possible due to the different assumptions about $\lambda_i$ in the two cases. However, if $(\rho_i/d)$ is large enough for $(kd)^2$ to be of the order of unity or less, condition (36) becomes

$$\left(\frac{R}{L_s}\right) \gtrsim \left(\frac{m_e}{m_i}\right)^{1/3} \lambda_i^{1/3}.$$

This is still a small quantity, since $\lambda_i(m_e/m_i) = \lambda_e \ll 1$, but this condition is significantly more stringent than the low-frequency condition, since $\lambda_i$ is large. Therefore, in this case, the necessity of stabilizing ion cyclotron drifts becomes the dominant consideration in determining the minimum required shear for plasma stability.

The quantity $(\rho_i/d)$ can be expressed in terms of particle density and magnetic field strength as

$$\left(\frac{\rho_i}{d}\right) = A\,\frac{n^{1/2}}{B},$$

where $n$ is in particles per cubic centimeter, $B$ is in gauss, and $A$ is a constant of order one. With an operating field strength of 30 kG we note that densities of the order $10^{11}$ are needed to make this ratio just unity. Indeed, significantly higher densities must be reached in order for the cyclotron drifts to be the dominant factor in determining stability.

It is reasonable to conclude, therefore, that although the drift cyclotron instabilities are not a particularly important consideration at the present time, they will become so as densities increase, and sheared magnetic fields will be available as possible means of stabilization.

## ACKNOWLEDGMENT

This work is supported by the United States Atomic Energy Commission under Contract AT(11-1)-1478 with the University of Michigan.

[1] L. V. Mikhailovskaya and A. B. Mikhailovsky, Nucl. Fusion **3**, 113 (1963).
[2] A. A. Galeev, Zh. Eksp. Teor. Fiz. **44**, 1920 (1963) [Sov. Phys.—JETP **17**, 1292 (1963)].
[3] B. Coppi, G. Laval, R. Pellat, and M. N. Rosenbluth, Nucl. Fusion **6**, 261 (1966).
[4] P. H. Rutherford and E. A. Frieman, Phys. Fluids **10**, 1007 (1967).
[5] A. B. Mikhailovsky and A. V. Timofeev, Zh. Eksp. Teor. Fiz. **44**, 919 (1963) [Sov. Phys.—JETP **17**, 626 (1963)].
[6] M. N. Rosenbluth, N. A. Krall, and N. Rostoker, Nucl. Fusion Suppl. Pt. 1, 143 (1962).
[7] J. N. Davidson and T. Kammash, Nucl. Fusion **8**, 203 (1968).
[8] B. D. Fried and S. D. Conte, *The Plasma Dispersion Function* (Academic Press Inc., New York, 1961).
[9] E. T. Whittaker and G. M. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, England, 1962), 4th ed., p. 347.
[10] W. Kaplan, *Operational Methods for Linear Systems* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1962), p. 422.