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TIME DEPENDENT NEUTRON ENERGY SPECTRA IN PULSED MEDIA

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CHAPTER I. INTRODUCTION

The study of the behavior of neutrons from a pulsed source is of great physical interest. The wide use of pulsed neutrons to measure reactor physics parameters has increased the importance of theoretical studies. The parameters obtained from these experiments are averaged over the neutron distribution existing inside the medium. Complete knowledge of the neutron distribution as a function of space, time and energy variables is required to analyse and to interpret the results of these experiments. The measurement of the neutron distribution as a function of space, time and energy variables in all ranges is a formidable task. The rigorous determination of neutron distribution can be obtained analytically as well as numerically under certain physical approximations.

A burst of high energy neutrons, when incident on a medium, undergoes three physical processes in succession: moderation, thermalization and diffusion. In the first stage, neutrons lose energy by elastic and inelastic collisions. This stage is called moderation of neutrons. In the second stage, neutrons gain as well as lose energy during the collisions and undergo thermalization. When the neutrons are completely thermalized, the exchange of energy stops, and neutrons acquire an equilibrium energy distribution. In the last stage, neutrons diffuse in the medium. The diffusion process is governed by the absorption and the leakage properties of the medium. During diffusion, the shape of the equilibrium energy distribution is maintained. The process of diffusion is terminated by absorption or leakage of neutrons. Neutrons spend less time during slowing down than

during thermalization and diffusion. On the other hand, the distance travelled is larger while slowing down. It becomes evident, therefore, that the study of the spatial distribution of neutrons is perferred for understanding the slowing down process. In order to understand thermalization and diffusion, however, time dependent studies are desirable.

Time-dependent studies of slowing down of neutrons above one electron-volt have been made by Ornstein and Uhlenbeck[36], Marshak[31], Waller[50], Kazarnovsky[23] and others. Reference to their work shall be made in the third chapter. No references for such studies below one electron-volt exist in the literature.

This thesis deals with the study of time dependent energy spectrum below 1/4 electron-volt. Analytical and numerical methods have been employed to study these problems. The characteristics of thermalization and diffusion processes, namely, the thermalization time constant, the thermalization time, the asymptotic energy spectrum, the decay constant, the average speed and the diffusion cooling coefficient, have been determined.

We briefly describe pulsed neutron studies undertaken by other authors, both experimentally and theoretically.

I. Experimental Studies

In 1953, von Dardel[48] introduced pulsed neutron techniques to measure the diffusion parameters: the absorption cross section and the diffusion constant. A beam of pulsed neutrons is allowed to fall upon a finite medium. The integrated neutron density is measured as a function of time, and the decay constant is determined as a function of geometrical

buckling, B^2. The decay constant λ is fitted as a function of B^2 as follows.

$$\lambda = \overline{\Sigma_a v} + \overline{Dv} B^2 (1 - c B^2) \qquad (1.1)$$

where: Σ_a = absorption cross section

D = diffusion coefficient

v = speed of neutrons

c = diffusion cooling coefficient.

$\overline{\Sigma_a v}$ and $\overline{Dv}$ are averaged over the Maxwellian distribution. It was suggested by von Dardel that the energy distribution is not Maxwellian in the case of a finite medium, and that the deviation of the spectrum is determined by the diffusion cooling coefficient c. In the finite medium, there is a preferential leakage of neutrons of high energy, and, therefore, the energy spectrum shifts toward the low energy side. This decreases the mean energy associated with the spectrum, causing a phenomenon termed "diffusion cooling."

Using pulsed neutron sources, the diffusion parameters of beryllium, beryllium oxide, graphite and water have been studied. Review articles by Amaldi[1] and also by von Dardel and Sjöstrand[49], summarize these studies.

Formula (1.1) is based upon the existence of a fundamental spatial mode in a medium characterized by geometrical buckling. The experiments of Campbell and Stelson[11] suggest the existence of a single decaying spatial mode at long times. However, the need for an experiment which measures the time dependence as well as the spatial distributions as a function of energy has yet to be fulfilled.

In addition to the non-multiplying medium experiments, measurements have been made in multiplying medium by Campbell and Stelson[11], Kloverstrom and Komoto[26] and others.

II. Theoretical Studies

Theoretical studies reported in the literature have been limited to the determination of the decay constant λ and the diffusion cooling coefficient c. In these studies all authors have made the assumption that at sufficiently long times the decay of the asymptotic energy spectrum in the pulsed medium can be represented by a single decay constant.

Mathematically, this means:

$$\phi(E,t) = \phi(E)\, e^{-\lambda t} \tag{1.2}$$

Hurwitz and Nelkin[21], using perturbation theory, determined the diffusion cooling coefficient c for a heavy gas model. The value of c for graphite and for beryllium was calculated and found to be lower than the experimental value.

Nelkin[32] repeated the calculations for c using a variational technique. He employed an energy distribution function $\varphi(E)$ given by the following modified Maxwellian distribution:

$$\phi(E) = M(E)\, e^{-\beta E} \tag{1.3}$$

In Equation (1.3), M(E) is the Maxwellian energy distribution. The energy is measured in units of KT_m. The moderator temperature is given by T_m and the neutron temperature by T_n. The variational parameter β in the

trial function is defined as follows:

$$\beta = \frac{T_m - T_n}{T_n} \tag{1.4}$$

Nelkin obtained the following result for c:

$$C = (\alpha + 1/2)^2 \sqrt{\pi}\, D_o\, (v_o M_2)^{-1} \tag{1.5}$$

where: $D_o = \dfrac{\int_0^\infty D(E)\, M(E)\, dE}{\int_0^\infty \frac{M(E)}{v}\, dE}$

$M(E)$ is the Maxwellian distribution

v_o is the speed corresponding to the most probable energy

M_2 is the thermalization parameter:

$$M_2 = \int_0^\infty \int_0^\infty \Sigma_s (E \to E') M(E) (E'-E)^2\, dE dE'$$

In deriving Equation (1.5), the transport mean free path was taken as proportional to E^α.

Singwi and Kothari[44] also derived a relation similar to 1.5, using an arbitrary energy dependence for the transport mean free path. According to them, c is given as follows:

$$C = \sqrt{\pi}\, D_o \left(\frac{A_1}{A_0} - \frac{3}{2}\right)^2 (v_o M_2)^{-1} \tag{1.6}$$

where: $A_i(E) = \int_0^\infty \lambda_{tr}(E)\, E^{i+1} \exp(-E)\, dE$

$\lambda_{tr}(E)$ = transport mean free path.

The above integral has to be determined from a knowledge of $\lambda_{tr}(E)$.

For the case of a constant diffusion coefficient, α is equal to zero, and A_1/A_o is equal to two. Under these conditions, Equations (1.5) and (1.6) become identical.

The concept of neutron temperature was used to obtain both expressions for c given above. This concept of assigning temperature to neutrons has been criticized on physical grounds because "temperature" is defined thermodynamically only for equilibrium distributions.

Considerable advance has been made in neutron thermalization and diffusion theory by the application of Laguerre polynomials. This is due to the fact that the eigen functions of the heavy gas model differential operator are associated Laguerre polynomials of the first order. A large class of neutron thermalization and diffusion problems in heavy media were solved by Kazarñovsky et al.[24]

Häfele and Dresner[19] determined the diffusion cooling coefficient in all orders, for the heavy gas model. Their value of c is 33 per cent higher than the value obtained using Equation (1.5), for the case of a constant diffusion coefficient. Singwi[43] using the Laguerre polynomial expansion derived the same expression for the diffusion cooling coefficient as given by the variational method. Singwi's work represents a considerable advance in the theory of diffusion cooling, as it is not based upon the concept of neutron temperature.

All the above results were obtained using the diffusion theory approximation. A transport correction for the diffusion cooling coefficient has been considered by Sjöstrand[45], Singwi[43] and Nelkin[35]. This correction was determined using single velocity neutrons; and it has been found to have the opposite sign of the multi-velocity correction mentioned above. The correction is given as follows:

$$C_{tr} = \frac{1}{15\,\Sigma_{tr}^2} \qquad (1.7)$$

where: Σ_{tr} = transport mean free path.

The diffusion cooling coefficient has been related to the thermalization time constant (t_{th}). We define the thermalization time constant as the time constant with which neutrons reach thermal equilibrium with the atoms of the moderator.

On the basis of elementary thermodynamic considerations, Beckurts[4] gave the following expression relating the diffusion-cooling coefficient and the thermalization time constant:

$$t_{th} = 3C \left[2 T_m^2 \left(\frac{\partial D}{\partial T_n} \right)^2_{T_n = T_m} \right]^{-1} \qquad (1.8)$$

$\partial D / \partial T_n$ represents the variation of the diffusion coefficient with the neutron temperature.

von Dardel[48] gave the following result for t_{th} for the case of a monoatomic gas of mass M.

$$t_{th} = \frac{3}{16} \sqrt{\pi}\, \ell_b \left(\frac{2 K T_m}{m} \right)^{-1/2} \frac{M}{m} \left(1 + \frac{m}{M} \right)^{7/2} \qquad (1.9)$$

where: m = the mass of the neutron

M = the mass of the moderator

ℓ_b = the mean free path of the neutrons for rigidly bound nuclei.

Nelkin[32], using the variational method mentioned above, derived the following result for the thermalization time constant.

$$t_{th} = \frac{3}{2} C (\alpha + \tfrac{1}{2})^2 (D_0 U_0^2)^{-1} \qquad (1.10)$$

In Chapter IV, we have derived a relation for the thermalization time constant which is identical with Nelkin's result, without using the concept of neutron temperature.

Antonov et al.[2] gave a two-energy group theory for the analysis of their pulsed neutron experiments in non-multiplying media. A theory of pulsed neutron experiments in multiplying media has been given by Krieger and Zweifel.[29] In their treatment, thermal neutrons have been assumed to be of a single group. The problem of slowing down neutrons was treated by using the kernel method. A two-group analysis of the transient behavior of thermal flux in a subcritical assembly has been made by Fultz[18] and applied to the experiments of Bengston et al.[6], Kirschbaum[25] and Reynolds.[40]

III. Treatment of the Problem

The objective of this thesis is the determination of the time-dependent low-energy spectrum of a pulse of neutrons, and the study of the characteristics of neutron thermalization and diffusion. Analytical and numerical methods have been employed to carry out this study. The analytical method involves an eigen-value technique. In the numerical method, the time behavior of the neutron energy spectrum is generated for all times greater than the slowing down time. We describe here, in brief, both methods and the problems to which they are applied.

A time-dependent problem can be reduced to a stationary problem by taking the Laplace transform. The transformed problem can then be solved by the eigen-value method, with the transform variable as a parameter. The complete determination of the neutron distribution involves the knowledge of all the eigen-functions and eigen-values. In addition, the initial distribution at a specified time is needed to determine the amplitudes of the

eigen functions. Each eigen function is expanded as a sum of an infinite number of associated Laguerre polynomials of first order. (The choice of the Laguerre polynomial, however, is arbitrary.) By using this method, we shall determine the first two eigen values for any physical model. Determination of the higher eigen values involves a knowledge of energy transfer moments greater than two. These are not readily available due to a lack of information about energy-exchange scattering cross sections. However, for the heavy gas model it is possible to determine all eigen values, since all of the energy transfer moments are known.

It is not always possible to construct the complete neutron distribution by using two or three of the lower eigen values. These lower eigen values, however, help us in the study of the characteristics of the last stage of neutron thermalization and diffusion. The first eigen value determines the rate of thermalization and the zeroth eigen value determines the final decay of a pulse of neutrons in the finite medium. It is of great physical interest to study analytically the first eigen value and its associated eigen function.

Practical considerations limit the use of the eigen value method, although it is rigorous. As far as the analytical method is concerned, we shall restrict ourselves to the determination of lower eigen values.

A multigroup method has been developed to obtain the time dependent low energy neutron spectrum in a heavy gas medium. A burst of neutrons of high energy is followed in time during the thermalization and diffusion periods. The initial source of neutrons at the slowing down time is determined from Fermi age slowing down theory. Using this source,

we generate the neutron energy distribution at different times. From these energy spectra, it is possible to study the characteristics of thermalization and diffusion. To obtain these spectra, the energy interval between 10 KT and zero has been divided into groups. The problem is solved for the heavy gas model, since this requires only the solution of the differential equation which is easy to handle. The electric analog computer was employed to generate these distributions, with time as the continuous variable.

Using the numerical method, we shall solve the infinite medium problem. The thermalization time constant and the thermalization time are determined from the energy spectrum curves. The results obtained analytically are compared with those obtained by the numerical method. The same numerical method has been applied to the study of the finite medium problem. Two cases have been dealt with: the constant diffusion coefficient case of graphite, and the energy dependent diffusion coefficient case of beryllium. The time behavior of the energy spectrum for a number of geometrical bucklings has been obtained. From this study, we attempt to answer the following questions.

1. Is there any equilibrium spectrum at the time of measurement in the finite medium?
2. At what time does this equilibrium set in?

In all the previous studies it has been assumed that at sufficiently long times the equilibrium spectrum does exist. We test the validity of this statement at the time of measurement.

The following characteristics of the equilibrium spectrum for a given buckling shall be studied.

1. The shape of the equilibrium spectrum.
2. The average speed.
3. The decay constant.
4. The diffusion cooling coefficient.

The numerical method developed here is used for the heavy gas model. However, by similar use of the multigroup technique, the integral equation for the crystalline medium can also be solved. A medium other than the heavy gas requires the scattering frequency of thermal neutrons before the integral equation can be solved. The determination of the scattering frequency, for media other than the heavy gas, is in itself a laborious task. Thus, it is not undertaken here.

A general formulation for treating the multi-velocity problem in a pulsed, multiplying medium is given. Campbell and Stelson's[11] experimental data for water and uranium mixtures are analyzed.

In the following chapters, we shall attempt to study the problems, and answer the questions, that have been outlined briefly here.

CHAPTER II. THE MATHEMATICAL FORMULATION OF THE PROBLEM

In this chapter we present the mathematical formulation which is used in the subsequent chapters. The behavior of neutrons in a medium can be completely described by the classical Boltzmann Equation. This equation is based upon the neutron conservation principle. We shall write down the basic equation for the neutron distribution as a function of energy, space, angle and time variables. We restrict our consideration here to the non-multiplying medium.

I. The Boltzmann Equation

This equation has been discussed in such great detail by Weinberg and Wigner[52] and also by Davison[14], that only a brief discussion need be given here. The physical behavior of neutrons can be represented by the following equation:

$$\frac{1}{v}\frac{\partial \phi(E,\vec{r},\vec{\Omega},t)}{\partial t} = -[\Sigma_a(E) + \Sigma_s(E) + \vec{\Omega}\cdot\vec{\nabla}]\,\phi(E,\vec{r},\vec{\Omega},t) + S(E,\vec{r},\vec{\Omega},t) + \int_{E'}\int_{\Omega'} \Sigma_s'(E')\,\phi'(E',\vec{r},\vec{\Omega}',t)\,F(E'\to E;\vec{\Omega}'\to\vec{\Omega})\,dE'd\Omega' \quad (2.1)$$

We define the following quantities:

$\varphi(E, \vec{r}, \vec{\Omega}, t)\, dE d\vec{r} d\vec{\Omega}$ is the angular flux, i.e., the number of neutrons at time t of energy in the interval E to E + dE, having their direction of motion along $\vec{\Omega}$ in the solid angle interval $\vec{\Omega}$ to $\vec{\Omega} + d\vec{\Omega}$, in the volume element $d\vec{r}$ around $\vec{r}$, multiplied by their speed v.

$F(E' \rightarrow E; \vec{\Omega}' \rightarrow \vec{\Omega})\, dE d\vec{\Omega}$ is the probability that the neutrons of energy E' along the direction $\vec{\Omega}'$ shall have the final energy E in the interval E to E + dE along the direction of motion $\vec{\Omega}$ in the solid angle interval $\vec{\Omega}$ to $\vec{\Omega} + d\vec{\Omega}$, after the collision with the atoms of the moderator. We shall assume the collisions to be scattering collisions only. The above quantity is called the scattering frequency.

$S(E, \vec{r}, \vec{\Omega}, t)\, d\vec{r} d\vec{\Omega} dE dt$ is the number of neutrons of energy E between E and E + dE in the volume element $d\vec{r}$ around $\vec{r}$, along the direction $\vec{\Omega}$ in the solid angle interval $\vec{\Omega}$ to $\vec{\Omega} + d\vec{\Omega}$ produced in the time interval between t and t + dt, due to external sources.

$\Sigma_a(E)$ and $\Sigma_s(E)$ are the energy dependent macroscopic absorption and scattering cross sections, respectively. The cross sections are assumed to be independent of space and angle variables.

The rigorous solution of Equation (2.1) is in general not possible. It has only been given for a few special cases. We shall examine the equation under different approximations.

The derivation of the Boltzmann Equation (2.1) is based upon the following assumptions.

a) Neutron- neutron interaction is neglected. Since the volume occupied by the neutrons is very small compared to the total volume, we can neglect the mutual interaction between neutrons. Even for the high intensity sources, the neutron density is of the order of 10^7 or 10^8 neutrons per cubic centimeter.

b) The neutron moves with a constant velocity between two collisions, along a straight line. A change in velocity occurs only at the instant of collision.

c) The collision of neutrons are instantaneous, and not delayed. Only in the case of fissionable material do we consider the delay effects.

d) The medium is isotropic and homogeneous.

e) The neutron density in the medium is very large, so that the effect of fluctuations can be neglected.

II. Angular Distribution

The problems connected with the angular distribution of neutrons lie in the domain of the transport theory. If the size of the scattering medium is large compared to the scattering mean free path of the neutrons, then diffusion theory can be used. We shall derive the diffusion theory equation from the integral Equation (2.1). We expand the angular flux, the scattering frequency and the external source in series of spherical harmonics.

$$\phi(E,\vec{r},\vec{\Omega},t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \phi_{\ell,m}(\vec{r},E,t)\, P_{\ell,m}(\vec{\Omega}) \tag{2.2}$$

$$S(E,\vec{r},\vec{\Omega},t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} S_{\ell,m}(\vec{r},E,t)\, P_{\ell,m}(\vec{\Omega}) \tag{2.3}$$

$$\begin{aligned} F(E'\to E;\vec{\Omega}'\to\vec{\Omega}) &= F(E'\to E;\vec{\Omega}'\cdot\vec{\Omega}) \\ &= \sum_{\ell=0}^{\infty} F_{\ell}(E'\to E)\, P_{\ell}(\vec{\Omega}\cdot\vec{\Omega}') \end{aligned} \tag{2.4}$$

We make the following assumptions. The external source is isotropic.

$$S(E,\vec{r},\vec{\Omega},t) = S_{0}(\vec{r},E,t) \tag{2.5}$$

The scattering frequency is a function of the angle between the initial and final directions, $\mu_o = \vec{\Omega} \cdot \vec{\Omega}'$. This assumption holds good in an isotropic, homogeneous medium.

$$\mu_o = \text{Cos}\,\theta \cdot \text{Cos}\,\theta' + \text{Sin}\,\theta\ \text{Sin}\theta' \text{Cos}(\phi - \phi') \quad (2.6)$$

For the sperical harmonics, we make use of the following relations.

$$P_{\ell,-m}(\vec{\Omega}) = (-)^m P_{\ell,m}(\vec{\Omega})^* \quad \{ P_{\ell,m}(\vec{\Omega})^* = \text{Complex Conjugate of } P_{\ell,m}(\vec{\Omega}) \quad (2.7)$$

$$P_\ell(\mu_o) = \sum_{m=-\ell}^{m=+\ell} P_{\ell,m}(\vec{\Omega})\ P_{\ell,m}(\vec{\Omega}')^* \quad \{\text{Addition Theorem} \quad (2.8)$$

$$\int_{\vec{\Omega}} P_{\ell,m'}(\vec{\Omega})\ P_{\ell,m}(\vec{\Omega})^*\, d\vec{\Omega} = \frac{4\pi}{2\ell+1}\, \delta_{\ell,\ell'}\, \delta_{m,m'} \quad (2.9)$$

$\{$Orthogonality Relation.

$$F_\ell(E' \to E) = \int F(E' \to E; \mu_o)\, d\mu_o \quad (2.10)$$

We also write down the scalar flux and current.

$$\int_{\vec{\Omega}} \phi(E, \vec{r}, \vec{\Omega}, t)\, d\vec{\Omega} = \phi(E, \vec{r}, t) \quad = \text{Flux} \quad (2.11)$$

$$\int_{\vec{\Omega}} \phi(E, \vec{r}, \vec{\Omega}, t)\, \vec{\Omega}\, d\vec{\Omega} = \vec{J}(E, \vec{r}, t) = \text{Current} \quad (2.12)$$

We integrate Equation (2.1) over all angles. Making use of the above relations, we get

$$\frac{1}{v}\frac{\partial \phi(E, \vec{r}, t)}{\partial t} = -\vec{\nabla} \cdot \vec{J}(E, \vec{r}, t) - \{\Sigma_a(E) + \Sigma_s(E)\}\phi(E, \vec{r}, t) \quad (2.13)$$
$$+ \int_{E'} F_o(E' \to E)\ \Sigma_s'(E')\ \phi'(E', \vec{r}, t)\, dE' + S(E, \vec{r}, t)$$

In order to determine the neutron current $\vec{J}(E,\vec{r},t)$, we multiply Equation (2.1) by $\vec{\Omega}$ and integrate over all angles. Then we get

$$\frac{1}{v}\frac{\partial \vec{J}(E,\vec{r},t)}{\partial t} = -\left[\{\Sigma_a(E)+\Sigma_s(E)\}\vec{J}(E,\vec{r},t)+\vec{\nabla}\cdot\int\vec{\Omega}_i\vec{\Omega}_j\phi\, d\vec{\Omega}\right] + \int_{E'}\Sigma_s'(E')\,F_1(E'\to E)\,\vec{J}(E',\vec{r},t)\,dE' \qquad (2.14)$$

The external source term is absent, since the source is isotropic. We further assume

$$\frac{\partial \vec{J}(E,\vec{r},t)}{\partial t} = 0 \qquad (2.15)$$

$$\nabla\cdot\int_{\vec{\Omega}}\vec{\Omega}_i\vec{\Omega}_j\,\phi(E,\vec{r},\vec{\Omega},t)\,d\vec{\Omega} = \frac{1}{3}\,\text{grad}\,\phi(E,\vec{r},t) \qquad (2.16)$$

Equation (2.16) is justified because we propose to use only the diffusion theory approximation, i.e., we will consider only the P_0 and P_1 components of the angular flux. Then Equation (2.16) follows from the fact that

$$\int\vec{\Omega}_i\vec{\Omega}_j\,d\vec{\Omega} = \frac{4\pi}{3}\,\delta_{ij} \qquad (2.17)$$

and

$$\int\Omega_i\Omega_j P_1(\Omega)\,d\vec{\Omega} = 0 \qquad (2.18)$$

When Equations (2.15), (2.16) and (2.17-18) are substituted into Equation (2.14) we obtain:

$$\int_{E'}F_1(E'\to E)\,\Sigma_s'(E')\,\vec{J}'(E',\vec{r},t)\,dE' - \frac{1}{3}\,\overrightarrow{\text{grad}}\,\phi(E,\vec{r},t) - \{\Sigma_a(E)+\Sigma_s(E)\}\,\vec{J}(E,\vec{r},t) = 0 \qquad (2.19)$$

If we further assume that $\Sigma_s'(E')J'(E',\vec{r},t) \simeq \Sigma_s J(E,\vec{r},t)$, a basic assumption in the Selengut-Gortzel[22] and Fermi Age slowing down models, we obtain:

$$\vec{J}(E,\vec{r},t) = -\frac{1}{3}\left[\Sigma_s(E) + \Sigma_a(E) - \bar{\mu}_o \Sigma_s(E)\right]^{-1} \vec{\nabla}\phi(E,\vec{r},t) \quad (2.20)$$

$$\vec{J}(E,\vec{r},t) = -D(E)\,\vec{\nabla}\phi(E,\vec{r},t) \quad (2.21)$$

$\bar{\mu}_o$, the average cosine of the scattering angle, is given by $\int F_1(E' \to E)dE'$.[31]

Equations (2.20) and (2.21) are sometimes referred to as Fick's law. D(E) is the diffusion coefficient. It depends upon the energy (E) and the scattering kernel of the medium $F(E' \to E;\mu_o)$ through the mean cosine of the scattering angle μ_o.

We shall restrict our considerations to only the diffusion theory approximation. Equation (2.14) reduces to the following expression when the expression (2.21) for the neutron current is substituted into it.

$$\frac{1}{v}\frac{\partial\phi(E,\vec{r},t)}{\partial t} = -\left[\Sigma_a(E) + \Sigma_s(E) - D(E)\nabla^2\right]\phi(E,\vec{r},t) + \int_{E'} \Sigma_s'(E')\,F(E' \to E)\,\phi'(E',r,t)\,dE' + S'(E,\vec{r},t) \quad (2.22)$$

It should be remarked that the above expression is exact, except for the expression of the neutron current, which contains the assumption of diffusion theory.

III. Space Distribution

It is possible to expand $\varphi(E,\vec{r},t)$ into a set of orthonormal functions of space variable. We have to assume that the neutron distribution

goes to zero at an extrapolated distance from the boundary. The extrapolated distance is assumed to be constant with energy and given by an average value. We expand $\varphi(E,\vec{r},t)$ and $S(E,\vec{r},t)$ as follows:

$$\phi(E,\vec{r},t) = \sum_{n=0}^{\infty} \phi_n(\vec{r})\,\phi_n(E,t)$$
$$S(E,\vec{r},t) = \sum_{n=0}^{\infty} \phi_n(\vec{r})\,S_n(E,t) \qquad (2.23)$$

$$\nabla^2\phi_n(\vec{r}) = -B_n^2\phi_n(\vec{r}) \quad \{B_n^2 = \text{Geometrical Buckling}$$
$$\phi_n(\tilde{r}) = 0 \qquad \text{at the extrapolated boundary } (\tilde{r}). \qquad (2.24)$$

where n represents the number of the spatial mode.

Substituting expansion (2.23) into Equation (2.22) we obtain

$$\sum_{n=0}^{\infty} \phi_n(\vec{r})\left[\frac{\partial\phi_n(E,t)}{\partial t} = -\{\Sigma_a(E)+\Sigma_s(E)+D(E)B_n^2\}\phi_n(E,t)\right. \qquad (2.25)$$
$$\left. + \int_{E'} \Sigma_s'(E')\,F(E'\to E)\,\phi_n(E',t)\,dE' + S_n(E,t)\right]$$

For the fundamental spatial mode n equals zero. If we consider only the fundamental spatial mode, we have the following equation.

$$\frac{1}{v}\frac{\partial\phi(E,t)}{\partial t} = -\left[\Sigma_a(E)+\Sigma_s(E)+D\,B^2\right]\phi(E,t) \qquad (2.26)$$
$$+ \int_{E'} \Sigma_s'(E')\,F(E'\to E)\,\phi(E',t)\,dE' + S(E,t)$$

Equation (2.26) is also true for any n-th spatial mode, and can be used to obtain the neutron distribution for that spatial mode, provided we use the corresponding B_n^2.

We shall later use the above equation for studying the time behavior of the neutron energy spectra in finite media characterized by a fixed geometrical buckling.

IV. Time Dependent Problem

The time dependent problem can be treated in principle by taking the Laplace transform with respect to the time variable. The Laplace transform of Equation (2.1) is as follows:

$$\left[\frac{S}{v} + \Sigma_a(E) + \Sigma_s(E) + \vec{\Omega}\cdot\vec{\nabla}\right]\phi(E,\vec{r},\vec{\Omega},S) + \bar{S}(E,\vec{r},\vec{\Omega}) = \int_{E'}\int_{\Omega'} \Sigma_s'(E')\,\phi'(E',\Omega',r',S)\,F(E'\to E;\vec{\Omega}'\to\vec{\Omega})\,dE'd\Omega' + \phi(E,\vec{r},\vec{\Omega},0) \quad (2.27)$$

where: $\varphi(E,\vec{r},\vec{\Omega},s)$ = Laplace transform of angular flux.

$$= \int_0^\infty \varphi(E,\vec{r},\vec{\Omega},t)\, e^{-st}\, dt$$

$S(E,\vec{r},\vec{\Omega})$ = Laplace transform of source.

$$= \int_0^\infty e^{-st}\, S(E,r,\Omega,t)\delta(t)\, dt$$

$\varphi(E,\vec{r},\vec{\Omega},t = 0) = \varphi(E,\vec{r},\vec{\Omega},0)$ = Angular flux at $t = 0$

$= 0$

The solution of Equation (2.27) can be reduced to the equivalent stationary problem by replacing the 1/v absorption term by an effective absorption term which includes the transform variable 's'. The effective absorption becomes a sum of two terms.

$$\frac{\Sigma_a^*}{v} = \frac{\Sigma_a}{v} + \frac{S}{v}$$

Though in principle the solution can be obtained, yet the problem of inversion of the transform is not a trivial one. This problem shall be discussed in the next chapter, for the case of hydrogen.

If the source term is represented by a Dirac delta function in energy by equating $s(\vec{E},\vec{r},\vec{\Omega})$ to $s(\vec{r},\vec{\Omega})\ \delta(E-E_0)$, then for an energy smaller than E_0, Equation (2.27) becomes a homogeneous equation. The problem reduces to an eigen value problem with s as a parameter. This gives the solution for the Laplace transform of the angular flux and the angular flux itself, as follows:[14]

$$\phi(E,\vec{r},\vec{\Omega},s) = \sum_{m=0} A_m \phi_m(\vec{r},E,\vec{\Omega})\,\delta(s-\lambda_m) \qquad (2.28)$$

$$\phi(E,\vec{r},\vec{\Omega},t) = \sum_{m=0} A_m \phi_m(E,\vec{r},\vec{\Omega})\, e^{\lambda_m t} \qquad (2.29)$$

λ_m is the eigen value associated with the eigen function $\varphi_m(\vec{r},E,\vec{\Omega})$. We shall apply the eigen value method to Equation (2.26) in the fourth and fifth chapters.

We shall assume the λ_m corresponds to an operator half bounded from above, i.e., set of eigen values for m = 0,1..., the zeroth eigen value is algebraically larger than any other eigen value, so that at long times it predominates. In the case of non-multiplying or subcritical systems, the neutron distribution decays in time. λ_m is negative for these cases. Equation (2.29) can be rewritten as follows:

$$\phi(E,\vec{r},\vec{\Omega},t) = \sum_{m=0} A_m \phi_m(\vec{r},E,\vec{\Omega})\, e^{-\lambda_m t} \qquad (2.29)$$

We shall make use of the above results in the fourth chapter.

V. Energy Independent Solution

Upon integrating Equation (2.26) over all energies, it reduces to:

$$\int_0^\infty \frac{1}{v}\frac{\partial \phi(E,t)}{\partial t}dE = -\int_0^\infty dE\left\{\Sigma_a(E) + D(E)B^2\right\}\phi(E,t) - \int_0^\infty \Sigma_s(E)\,\phi(E,t)dE + \int_0^\infty\int_0^\infty \Sigma_s'(E')\,\phi'(E',t)F(E'\to E)dE'dE + \int_0^\infty S(E,t)\,dE \tag{2.30}$$

For a pulsed neutron source, $S(E,t) = \delta(t)\,S_o$

For time $t > 0$ the source term is absent.

$$\int_0^\infty \Sigma_s(E)\,\phi(E,t)\,dE = \int_0^\infty \Sigma_s(E)\int_0^\infty \phi(E,t)\,F(E\to E')\,dE'dE = \int_0^\infty\int_0^\infty \Sigma_s'(E)\,\phi'(E',t)F(E'\to E)\,dE\,dE' \tag{2.31}$$

We further relate the flux $\varphi(E,t)$ with the neutron density $n(E,t)$.

$$\phi(E,t) = v\,n(E,t) \tag{2.32}$$

Substituting Equations (2.31) and (2.32) into (2.30)

$$\frac{\partial n(t)}{\partial t} = -\left[\overline{\Sigma_a(E)\,v} + \overline{D(E)\,v}\,B^2\right]n(t) \tag{2.33}$$

The solution of the Equation (2.33) is given by

$$n(t) = n_o\, e^{-\left\{\overline{\Sigma_a(E)v} + \overline{D(E)v}\,B^2\right\}t} \tag{2.34}$$

The decay constant is given by:

$$\lambda = \overline{\Sigma_a v} + \overline{D(E)\,v}\,B^2 \tag{2.35}$$

$$\overline{\Sigma_a(E)\,v} = \frac{\int_0^\infty n(E,t)\,\Sigma_a(E)\,v\,dE}{\int_0^\infty n(E,t)\,dE} \tag{2.36}$$

$$n(t) = \int_0^\infty n(E,t)\, dE \tag{2.37}$$

$$\overline{v(t)} = \frac{\int_0^\infty v\, n(E,t)\, dE}{\int_0^\infty n(E,t)\, dE} \tag{2.38}$$

$$\overline{D v B^2} = \frac{\int_0^\infty v\, D(E) B^2 n(E,t)\, dE}{\int_0^\infty n(E,t)\, dE} \tag{2.39}$$

For constant diffusion coefficient and 1/v absorption, the decay constant λ reduces to

$$\lambda = \Sigma_{a_0} v_0 + D \overline{v} B^2 \tag{2.40}$$

The determination of the diffusion coefficient D and the absorption cross section Σ_{a_0} from the measurements of the decay constant λ in the pulsed assembly requires the knowledge of the average speed $\overline{v}$. We shall generate $\varphi(E,t)$ for a few geometrical bucklings, B^2, and obtain $\overline{v}$. We note that since $\overline{v}$ is a function of time, there will not be pure exponential decay until n(E,t) [and thus $\overline{v}(t)$] attains its asymptotic value.

VI. The Scattering Integral

We shall discuss the scattering integral in the next chapter. It must be pointed out that the complexity of the integral Equation (2.1) arises from the complexity of the scattering frequency $F(E' \to E; \vec{\Omega}' \to \vec{\Omega})$. The physics of the problem is determined by this kernel. A detailed theoretical or experimental knowledge of this kernel is essential for

the rigorous solution of the Equation (2.1).

For the numerical solution of the problem, we decompose the scattering integral into two parts.

$$\int_{E=0}^{\infty} \Sigma_s'(E')\,\phi'(E',t)\,F(E'\to E)\,dE' = \int_{E_T}^{\infty} \Sigma_s'(E')\,\phi'(E',t)\,F_{fast}(E'\to E)\,dE' + \int_{0}^{E_T} \Sigma_s'\,\phi'\,F_{thermal}(E'\to E)\,dE' \qquad (2.41)$$

where: E_T = thermal cut off energy

$F_{thermal}(E'\to E)\,dE$ = thermal neutron scattering frequency

$F_{fast}(E'\to E)\,dE$ = fast neutron scattering frequency.

We shall evaluate the fast neutron scattering integral at the slowing down time (t_s) in the third chapter. We also discuss the thermal neutron scattering integral in the next chapter.

CHAPTER III. FAST AND THERMAL NEUTRON SCATTERING

We shall discuss fast and thermal neutron scattering, in detail, in this chapter. For the study of the slowing down of fast neutrons, we can neglect the low energy effects and assume the atoms of the moderator to be free and at rest. The lower limit for the validity of the slowing down model is taken to be about 10 KT, which corresponds to 0.25 ev. Below this energy, one must take into account the thermal motion of the atoms of the moderator, the crystalline effects and the chemical binding of the moderator. In the low energy region below 10 KT, the neutrons lose their energy due to elastic as well as inelastic scattering with the atoms of the moderator and with the medium as a whole. We shall speak of fast neutron scattering as the slowing down process and of thermal neutron scattering as thermalization. These are two distinct processes valid in two different energy intervals. In the slowing down process, the neutron can lose its energy only by elastic scattering with the atoms of the moderator. Inelastic nuclear scattering is only important for high energy ($\geq$1Mev) neutrons, therefore it will be neglected. In the thermalization process, the neutron can lose as well as gain energy due to elastic or inelastic scattering with the atoms of the medium. The physical state of the medium plays an important role in neutron thermalization.

I. The Slowing Down Process

The slowing down of neutrons in heavy element and hydrogenous moderators constitute two separate problems. A neutron can lose all of

its energy in a single collision with an atom of hydrogen, but requires a large number of collisions with an atom of a heavy moderator, before it can lose all its energy. No attempt is made to review slowing down theory here. We discuss two specific problems.

1. Determination of the heavy moderator fast neutron scattering integral.

2. Determination of the space-and time-dependent slowing down solution for hydrogen.

The fast scattering integral will be used in the following way. A burst of fast neutrons is inserted into the assembly at t = 0, then at some later time of the order of the slowing down time t_s, these neutrons will begin to enter the thermal group. Those last collision slowing down neutrons arriving at energy $E < E_T$ will serve as a source for the thermalization problem. Clearly those neutrons will arrive from the collisions at some energy between E_T and E_T/α, so we will consider the fast integral between those limits. Furthermore, this "source" for the thermalization problem must vanish below $E = \alpha E_T$.

The scattering frequency for the elastic scattering of neutrons is:

$$F(E' \rightarrow E)\, dE = \frac{dE}{E'(1-\alpha)} \tag{3.1}$$

This scattering frequency is obtained under the following assumptions:

1. Neutrons lose energy by elastic scattering which is isotropic in the center of mass system.

2. Atoms of the moderator are free and at rest. Low energy effects are ignored.

A. The Heavy Element Scattering Integral ($A \gg 1$)

We determine the following integral encountered in Equation (2.41).

$$S(E,t) = \int \Sigma_s(E')\,\phi'(E',t)\,F_{fast}(E' \to E)\,dE' \qquad (3.2)$$

The ranges of energies for E' and E as indicated in the discussion above, are as follows:

$$\frac{E}{\alpha} \gtrsim E' \gtrsim E_T \qquad E_T \gtrsim E \gtrsim \alpha E_T$$

Substituting the scattering frequency (3.1) into Equation (3.2), we obtain

$$S(E,t) = \int_{E_T}^{E/\alpha} \Sigma_s'(E')\,\frac{\phi'(E',t)}{E'(1-\alpha)}\,dE' \qquad (3.3)$$

The determination of S(E,t) involves the knowledge of $\varphi(E',t)$ as a function of energy E' at time t. The time dependent slowing down problem has been studied by Marshak,[31] von Dardel[48], Svartholm[46], Kazarnovsky[23], Waller[50], Eriksson[16], and others. All these authors have attempted to obtain the velocity distribution of neutrons as a function of the dimensionless quantity $vt\Sigma_s$ in a non-absorbing, homogeneous, infinite medium of heavy mass. An excellent review of most of these studies has been made by Amaldi[1]. The basic physics of these studies is the same, but they differ in the methods employed to obtain final solutions.

We give in Figure 3.1 the velocity distribution curve for a moderator of mass equal to 12, as given by Eriksson[16] based upon Waller's[50] exact solution. For a very large mass ($A \gg 1$) the velocity distribution given by Kazarnovsky[23] reduces to the following distribution.

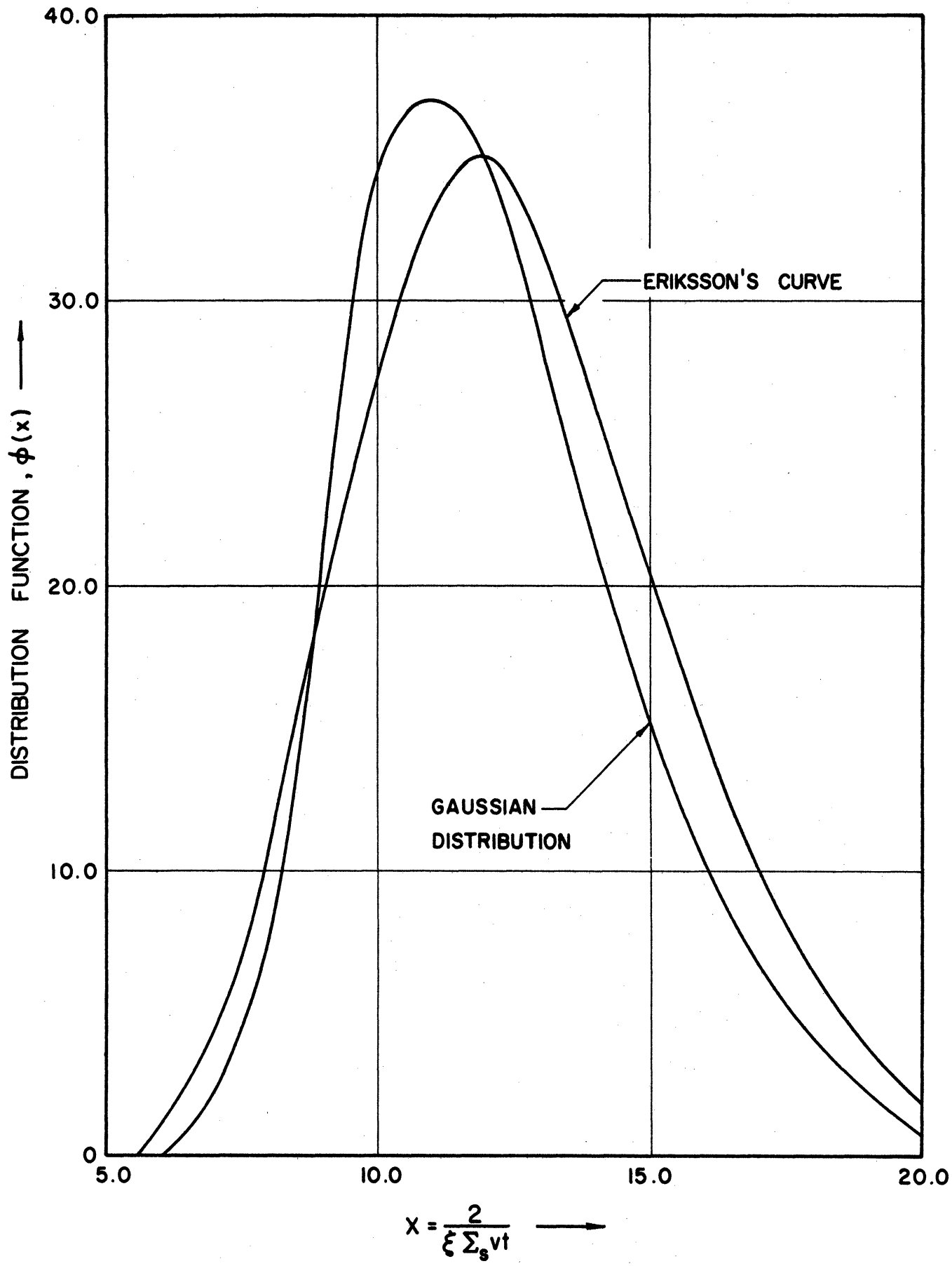

Figure 3.1 Comparison of Time Dependent Distribution Function $\varphi(x)$ for Mass A = 12.

$$\Psi(Z) = \sqrt{\frac{3A}{4\pi}} \exp\left\{-\frac{3A}{4}(Z-1)^2\right\} \qquad (3.4)$$

where: $\psi(Z) = \eta\, t\, \phi(u,t)\, \Sigma_s(u)$

$\eta = \frac{2}{A+1}$

$Z = \frac{t_s(E)}{t}$

u = lethargy corresponding to energy E.

$t_s(E)$ = slowing down time corresponding to energy E.

When converted into energy units, the above equation reduces to the following expression.

$$\phi(E',t) = \frac{1}{\Sigma_s E' t \eta}\left[\sqrt{\frac{3A}{4\pi}} \exp\left\{-\frac{3A}{4}\left(\frac{t_s(E')}{t}-1\right)^2\right\}\right] \qquad (3.5)$$

We compare the Gaussian distribution with Eriksson's exact curve in Figure 3.1.

In contrast with the Gaussian distribution, the Fermi age slowing down model gives a line distribution in time. According to this model $\varphi(E',t)$ can be given as follows:

$$\phi(E',t) = \phi(E')\, \delta\left(t - t_s'(E')\right) \qquad (3.6)$$

We determine $\varphi(E')$ by the time independent Fermi age model. The elementary calculations yield $\varphi(E')$ as follows.

$$\phi(E') = S(E_0)\, p'\, e^{-B^2\tau'}\left[E'\, \xi\, \Sigma_s'\right]^{-1} \qquad (3.7)$$

In Equation (3.7) the following terms need to be defined:

$S(E_o)$ = the source strength of neutrons of energy E_o.

p' = the resonance escape probability for the energy interval E_o to E'.

$e^{-B^2\tau'}$ = the non-escape probability factor

ξ = logarithmic decrement

τ' = the Fermi age for the energy interval E_o to E.

The Dirac delta function $\delta(t - t_s'(E'))$ in Equation (3.6) is characteristic of the Fermi age model.

We evaluate S(E,t) based upon Equations (3.5) and (3.6) for t equal to $\bar{t}_s$, which is defined as the mean slowing down time for neutrons having a final energy between E_T and E_T/α.

1. Determination of $S(E,\bar{t}_s)$ by the Fermi Age Model

We obtain $S(E,\bar{t}_s)$ using the approximations given by the Fermi age model as mentioned above.

$$S(E,\bar{t}_s) = \delta(t-\bar{t}_s) \int_{E_T}^{E/\alpha} \frac{\Sigma_s' e^{-B^2\tau'} p' S(E_o)\, dE'}{\xi\, \Sigma_s'\, E'^2 (1-\alpha)} \qquad (3.8)$$

In writing Equation (3.8) we have made one more assumption. The slowing down time of neutrons having final energy in the interval between E_T and E_T/α does not change with energy. It can be represented by $\bar{t}_s$. This is valid, provided E_T/α does not differ appreciably from E_t. It is true for heavy elements, for which α is close to unity.

We also assume that the resonance escape and non-escape probabilities, p' and $e^{-B^2\tau'}$ do not change in the above energy interval but this is an excellent assumption. We integrate the right hand side of Equation (3.8).

$$S(E,\bar{t}_s) = \delta(t-\bar{t}_s) \frac{S(E_o)\, p\, e^{-B^2\tau\left(1-\alpha\frac{E_T}{E}\right)}}{\xi(1-\alpha)E_T} \qquad (3.9)$$

2. Determination of $S(E,\bar{t}_s)$ using Gaussian Distribution

Substituting the Gaussian distribution, as given by Equation (3.5), into Equation (3.3) we obtain

$$S(E,\bar{t}_s) = \int_{E_T}^{E/\alpha} \sqrt{\frac{3A}{4\pi}} \left\{ \exp - \frac{3A}{4} \left(\frac{t_s'(E')}{\bar{t}_s} - 1 \right)^2 \right\} \left\{ \bar{t}_s (1-\alpha) E'^{-1/2} \right\} dE' \quad (3.10)$$

In order to integrate Equation (3.10) we make the following substitutions.

$$\frac{t_s'(E')}{\bar{t}_s} = \frac{a}{(E'^{1/2})} \quad (3.11)$$

$$\left(\frac{a}{E'^{1/2}} - 1 \right) = x \quad (3.12)$$

$$S(E,\bar{t}_s) = - \int_{x(E_T)}^{x(E/\alpha)} (x+1) \exp\left(-\frac{3A}{4}x^2\right) dx \quad (3.13)$$

We use the following formulae to integrate Equation (3.13).

$$\int x e^{-ax^2} dx = - \frac{e^{-ax^2}}{2a} \quad (3.14)$$

$$\int e^{-ax^2} dx = \int (1 - ax^2)\, dx \quad (3.15)$$

for $ax^2 < 1$

$$\begin{aligned} S(E,\bar{t}_s) = c \Bigg[& \frac{2}{3A} \left\{ \exp - \left\{ \frac{3A}{4} \left(\frac{t_s(E/\alpha)}{\bar{t}_s} - 1 \right)^2 \right\} \right. \\ & \left. - \exp - \left\{ \frac{3A}{4} \left(\frac{t_s(E_T)}{\bar{t}_s} - 1 \right)^2 \right\} \right\} \\ & - \left\{ \frac{t_s(E/\alpha)}{\bar{t}_s} - \frac{t_s(E_T)}{\bar{t}_s} \right\} \\ & + \frac{A}{4} \left\{ \left(\frac{t_s(E/\alpha)}{\bar{t}_s} - 1 \right)^3 - \left(\frac{t_s(E_T)}{\bar{t}_s} - 1 \right)^3 \right\} \Bigg] \end{aligned} \quad (3.16)$$

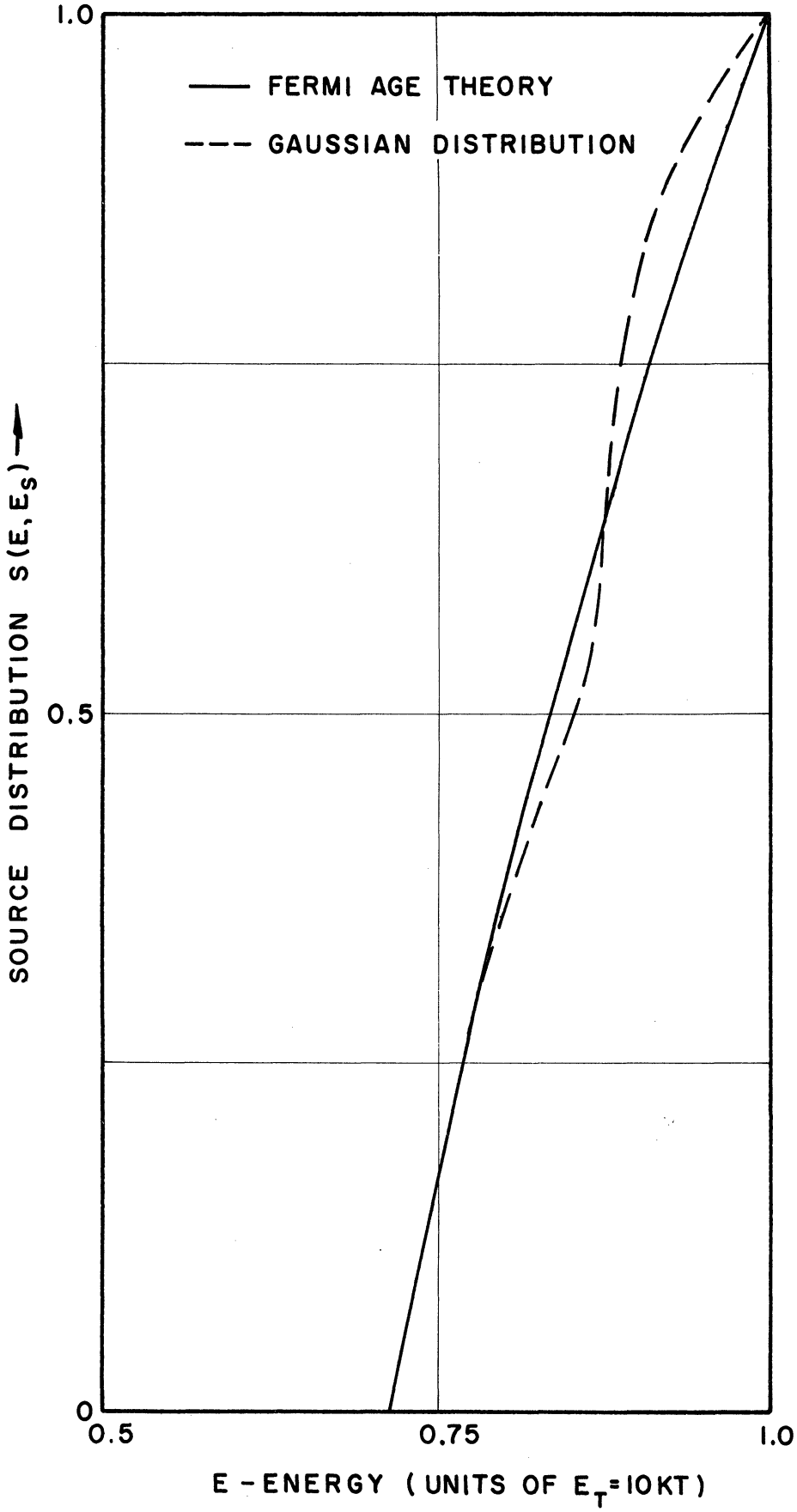

Figure 3.2 Source Distribution $S(E,\bar{t}_s)$ for Graphite (A = 12).

Values of $S(E,\bar{t}_s)$ are listed in Table 3.1, for a few values of energy, obtained by using Equations (3.9) and (3.16). In these calculations, A was taken equal to 12 and $\bar{t}_s$ equal to the mean slowing down time between E_T and E_T/α.

TABLE 3.1

THE SOURCE DISTRIBUTION $S(E,\bar{t}_s)$

E	$S(E,\bar{t}_s)_{Gaussian}$	$S(E,\bar{t}_s)_{Fermi\ age}$
E_t	1.000	1.000
$0.9E_t$	0.818	0.718
$0.86E_t$	0.510	0.588
$0.8E_t$	0.362	0.370
$\alpha\ E_t$	0	0

In Figure 3.2 $S(E,\bar{t}_s)_{Gaussian}$ and $S(E,\bar{t}_s)_{Fermi\ age}$ are plotted for the sake of comparison. The basic features of the shape of the energy distribution are almost the same in both curves so that the simpler source, determined by the Fermi age model, and given by Equation (3.9), is used for the generation of the time dependent spectrum.

It must be pointed out here, that the behavior of neutrons for times longer than $\bar{t}_s$ is governed by the physics of the problem and is independent of the details of the initial source. The neutrons lose memory of details of the source after a few collisions.

B. Discussion of the Space and Time Dependent Slowing Down Solution for Hydrogen

The time dependent slowing down solution in a non-absorbing, infinite homogeneous medium of hydrogen was given exactly by Ornstein and

Uhlenbeck[36]. We modify their solution for the case of a finite medium having 1/v absorption, using the diffusion theory approximation. The neutron transport equation, as given in the second chapter, is as follows.

$$\frac{1}{v}\frac{\partial \phi(E,\vec{r},t)}{\partial t} = -(\Sigma_a + \Sigma_s)\,\phi(E,\vec{r},t) + D\nabla^2\phi(E,\vec{r},t) + \int_E^{E_0} dE'\,\Sigma_s(E')\,F(E'\to E)\,\phi(E',\vec{r},t) + S(E,\vec{r},t) \tag{2.22}$$

Assuming the source to be a delta function in space and time variables, we get

$$S(E,\vec{r},t) = \delta(\vec{r})\,\delta(t)\,S(E) \tag{3.17}$$

We take the Fourier transform of Equation (2.22) with respect to the space variable, obtaining

$$\frac{1}{v}\frac{\partial \bar{\phi}(E,K,t)}{\partial t} = -(\Sigma_a + \Sigma_s)\,\bar{\phi}(E,K,t) - DK^2\bar{\phi}(E,K,t) + \int_E^{E_0}\Sigma_s'(E')\,\phi'(E',K,t)\,F(E'\to E)\,dE' + \delta(t)\,S(E) \tag{3.18}$$

$$\bar{\phi}(E,K,t) = \int_0^{\infty}\phi(E,\vec{r},t)\,e^{iK\cdot\vec{r}}\,d\vec{r} \tag{3.19}$$

= Fourier transform

K = Transform variable

On taking the Laplace transform of Equation (3.18) with respect to the time variable, we get

$$\left[\frac{S}{v} + \Sigma_a(E) + \Sigma_s(E) + DK^2\right]\tilde{\bar{\phi}}(E,K,S) = \int_E^{E_0}\Sigma_s'(E')\,\tilde{\bar{\phi}}(E',K,S)\,F(E'\to E)\,dE' + S(E) \tag{3.20}$$

Rearranging Equation (3.20)

$$\left[\frac{S}{v} + \Sigma_a(E) + \Sigma_s(E) + DK^2\right]\tilde{\phi}(E,K,S) = \int_E^{E_0} \Sigma_s' \tilde{\phi}(E',K,S)\,\frac{dE'}{E'} + \tilde{S}(E) \qquad (3.21)$$

On differentiating Equation (3.21) we obtain a differential equation.

$$\frac{dX(E)}{dE} = \frac{d\tilde{S}(E)}{dE} - \frac{\Sigma_s X(E)}{\left[\frac{S}{v} + \Sigma_a + \Sigma_s + DK^2\right] E} \qquad (3.22)$$

where $X = \left(\frac{S}{v} + \Sigma_a + \Sigma_s + DK^2\right)\tilde{\phi}(E,K,S)$

The solution of Equation (3.22) can be given in a simple manner.

$$X(E) = \tilde{S}(E) + \frac{1}{E}\int_E^{E_0} \frac{\tilde{S}(E')\,dE'\,\Sigma_s'\ \exp\left[-\int_E^{E'} \frac{(S/v'' + \Sigma_a'' + D''K^2)\,dE''}{(S/v'' + \Sigma_a'' + \Sigma_s'' + D''K^2)E''}\right]}{\left(\frac{S}{v'} + \Sigma_a' + \Sigma_s' + D'K^2\right)} \qquad (3.23)$$

We assume the source S(E) equal to $\delta(E - E_0)$. Transforming X to $\tilde{\Phi}(E,k,s)$ we get

$$\tilde{\phi}(E,K,S) = \frac{\delta(E-E_0)}{\left(\frac{S}{v} + \Sigma_a + \Sigma_s + DK^2\right)} + \frac{\Sigma_s(E_0)}{E\left[\frac{S}{v_0} + \Sigma_a(E_0) + \Sigma_s(E_0) + D_0K^2\right]} \times \exp - \int_E^{E_0} \left(\frac{S/v' + \Sigma_a' + D'K^2}{S/v' + \Sigma_a' + \Sigma_s' + D'K^2}\right)\frac{dE'}{E'} \times \frac{}{\left[\frac{S}{v} + \Sigma_a(E) + \Sigma_s(E) + D(E)K^2\right]} \qquad (3.24)$$

For energies E less than E_0, Equation (3.24) reduces to the following equation.

$$\tilde{\phi}(E,K,S) = \frac{\Sigma_s(E_0)\exp - \left\{\int_E^{E_0} \left[\frac{S/v' + \Sigma_a' + D'K^2}{S/v' + \Sigma_a' + \Sigma_s' + D'K^2}\right]\frac{dE'}{E'}\right\}}{E\left[\frac{S}{v_0} + \Sigma_a(E_0) + \Sigma_s(E_0) + D_0K^2\right]\left[\frac{S}{v} + \Sigma_a(E) + \Sigma_s(E) + DK^2\right]} \qquad (3.25)$$

The following comments about Equation (3.25) be noted:

1. For $\Sigma_a = 0$ and $K^2 = 0$ (infinite, homogeneous medium containing a uniform distribution of source): Equation (3.25) reduces to the expression given by Ornstein and Uhlenbeck.(36)
2. For the time independent case (S = 0); this equation corresponds to the solution given by the Selengut-Gortzel model as discussed by Hurwitz and Zweifel(22) and also by Simon.(42)
3. If the spatial dependent part can be separated from $\varphi(E,r,t)$ then the Fourier transform variable can be shown to be equal to the geometrical buckling of the medium.

The Fourier Laplace transform of the slowing down kernel is related with the neutron flux as follows:

$$\tilde{q}(E,K^2;S) = \frac{\Sigma_s(E_0)\exp\left[-\int_E^{E_0}\frac{dE'}{E'}\left[\frac{\Sigma_a(E')+S/v'+D'K^2}{\frac{S}{v'}+\Sigma_a(E')+\Sigma_s(E')+D'K^2}\right]\right]}{\left[\Sigma_a(E_0)+D(E_0)K^2+\Sigma_s(E_0)+\frac{S}{v_0}\right]} \quad (3.26)$$

The above equation can be written in words as follows:

$$\tilde{q}(E,K^2,s) = (\text{First Flight Factor}) \times (\text{Resonance Captive Escape Factor}) \times (\text{Non-leakage Factor}) \times (\text{Time Dependent Factor}) \quad (3.27)$$

II. Thermal Neutron Scattering

Neutron scattering in the thermal energy region depends upon the low energy effects mentioned earlier, namely, the thermal motion of the atoms of the moderator, the chemical binding effects, and the crystalline nature of the medium. In the thermal energy region, neutrons can gain as well as lose energy. This process is characteristic of the thermalization phenomenon. If we consider only thermal motion of the atom and neglect chemical and crystalline effects, then the thermalization is carried out by means of elastic scattering only. When the chemical bond and crystalline effects are taken into account, then inelastic scattering also becomes important. The neutrons lose or gain energy through exchange of quantas in molecules and phonons in crystals. The scattering of thermal neutrons is a big subject in itself and covers a wide range of problems. A number of excellent articles by Amaldi[1], Kothari and Singwi[27], Zemach and Glauber[55] and Nelkin and Cohen[34] and several other authors exist.

We shall consider the effects of low energy neutron scattering upon the neutron transport problem. For 1/v absorption, the total scattering cross section and scattering frequency are the only parameters which are dependent upon the low energy effects. The thermal scattering term of the transport equation can be given as follows.

$$\int_{E'} \Sigma_s'(E')\,\phi(E',t)\,F(E'\to E)\,dE' - \bar{\Sigma}_s(E)\,\phi(E,t) \qquad (3.28)$$

We express $\varphi(E,t)$ as a product of two terms

$$\phi(E,t) = M(E)\,\Psi(E,t) \qquad (3.29)$$

where M(E) is the Maxwellian distribution.

The total scattering cross section $\Sigma_s(E)$ can be given by the following integral

$$\Sigma_S(E) = \int_{E'} \Sigma_S(E) \; F(E \to E') \, dE' \tag{3.30}$$

We make use of the Detailed Balance Theorem of statistical thermodynamics. This theorem gives the balance between two small energy intervals for the equilibrium distribution. Accordingly, the following equality holds:

$$M(E') \; \Sigma_S(E') \; F(E' \to E) = M(E) \; \Sigma_S(E) \; F(E \to E') \tag{3.31}$$

We substitute Equations (3.29), (3.30) and (3.31) into Equation (3.28), then the following result is obtained.

$$\int_{E'} \Sigma_S'(E') \, F(E' \to E) \, \phi(E',t) \, dE' - \Sigma_S(E) \, \phi(E,t) = \int_{E'} M(E) \; \Sigma_S(E) \; F(E \to E') \left[\psi(E',t) - \psi(E,t) \right] dE' \tag{3.32}$$

Expanding $\psi(E',t)$ in a Taylor series about E, we get

$$\psi(E',t) = \psi(E,t) + (E'-E) \frac{\partial \psi(E,t)}{\partial E} + \cdots (E'-E)^K \frac{\partial^K \psi(E,t)}{\partial E^K} \tag{3.33}$$

On substituting the above expansion into Equation (3.32), we obtain the following result for the scattering term.

$$\int_{E'} \Sigma_S'(E') \, \phi(E',t) \, F(E' \to E) \, dE' - \Sigma_S(E) \, \phi(E,t) = \sum_{K=1}^{\infty} \frac{1}{K!} \left\{ \frac{\partial^K \psi(E,t)}{\partial E^K} \right\} \int_{E'} \Sigma_S(E) \, M(E) (E'-E)^K \, F(E \to E') \, dE \tag{3.34}$$

The K-th energy transfer moment is defined as follows

$$A_K(E) = \frac{\int_{E'} (E'-E)^K \; \Sigma_S(E) \; F(E \to E') \, dE'}{\Sigma_S(E)} \tag{3.35}$$

In order to determine A_k integrals, we need the energy exchange (partial) scattering cross section. The physical approximations involved

in the determination of this quantity defines the basic physics of the problem. We shall discuss briefly the energy transfer moments.

The main interest of the reactor physicist lies in the determination of the energy transfer moments represented by the integrals $A_K(E)$. It is these integrals, which appear in the neutron transport problem. Determination of these integrals involve the knowledge of the partial scattering cross sections. Zemach and Glauber[55] have given a general formalism for the determination of these cross sections for a large class of scattering nuclei. Kothari and Singwi[27] have extensively reviewed the interaction of thermal neutrons with solids. All of these formalisms are based upon the concept of Fermi pseudo-potential, introduced by Fermi[17] to explain the chemical binding effect of the hydrogen atom bound as a harmonic oscillator. Pseudo-potential is characterized by the Dirac Delta function. In addition to Fermi, Vaughn and Cohen[47] have also determined the scattering cross sections for the atom bound as an harmonic oscillator. This model has been found to be adequate for solid zirconium hydride. The quantum of vibration is of 0.13 ev. No attempt is made here to discuss the various models employed to determine the partial scattering cross sections for liquids, solids and polyatomic gases.

In the case of a monoatomic gas, Wigner and Wilkins[53] derived the partial scattering cross section, assuming that the velocity distribution of the atoms of the moderator has a Maxwellian distribution. They also assumed the scattering cross section to be constant and independent of relative speed between the neutron and the atom.

Brown and St. John[9] modified the Wigner-Wilins kernel by taking into account the effect of the relative speed upon the scattering cross section. A general method for obtaining the thermal neutron scattering frequency has been given by Osborn.[37] It must be pointed out that, under the assumptions involved, the Wigner-Wilkins scattering kernel is exact. The energy transfer moments can be easily obtained, provided the heavy mass approximation is made and terms of order $(1/A)^2$ are neglected. Under this approximation, the integral equation is reduced to the Wilkins[54] second order differential equation. In this thesis, we shall deal with the heavy gas differential equation extensively. An excellent review article summarizing the neutron thermalization studies dealing with the monoatomic gas has been given by Cohen.[13]

For moderators of greatest interest one integral quantity, defined by Nelkin[32,33] as the thermalization parameter, has been determined. This parameter is given by the following integral

$$M_2 = \int_0^\infty \int_0^\infty \Sigma_s(E)\, M(E)\, (E'-E)\, F(E \to E')\, dE'\, dE \qquad (3.36)$$

It will be shown in the fourth chapter that neutron thermalization is governed by M_2. In the expression for the time constant of neutron thermalization, M_2 appears as the basic physical parameter, which determines the rate of thermalization.

CHAPTER IV. EIGEN VALUE SOLUTION OF THE TIME DEPENDENT PROBLEM

The use of the eigen value method to solve the time dependent problem was pointed out in the second chapter. A general formalism for determining eigen values governing the time behavior of neutrons, in any physical medium, is presented in this chapter. We shall determine the first two eigen values; the zeroth eigen value governs the decay of a neutron pulse in the diffusion period, while the first eigen value is associated with the last stage of neutron thermalization.

I. General Formulation

The neutron distribution given by Equation (2.26) for a single spatial mode shall be employed to study the time dependent problem. Rewriting Equation (2.26), we get

$$\frac{1}{v}\frac{\partial \phi(E,t)}{\partial t} = -\left[\Sigma_a(E) + D(E)B^2\right]\phi(E,t) + \int_{E'} \Sigma_s'(E')\,\phi'(E',t)\,F(E'\to E)\,dE' - \Sigma_s(E)\,\phi(E,t) + S'(E,t) \qquad (2.26)$$

We expand $\varphi(E,t)$ as a product of two functions as follows.

$$\phi(E,t) = \psi(E,t)\,M(E) \qquad (4.1)$$

$$M(E)\,dE = E\,e^{-E}\,dE \quad \text{Maxwellian distribution} \qquad (4.2)$$

The neutron source S(E,t) is pulsed in time and is of the following form.

$$S'(E,t) = S'(E_o)\,\delta(E-E_o)\,\delta(t) \qquad (4.3)$$

For energies less than the source neutron energy E_o, the source term disappears. On substituting Equations (4.1) and (4.2) into Equation

(2.26) we get:

$$\frac{1}{v}\frac{\partial \psi(E,t)\, M(E)}{\partial t} = -\left[\Sigma_a(E) + D(E)\, B^2\right] \psi(E,t)\, M(E) + \int_{E'} \Sigma_s'(E')\, \psi(E',t)\, M(E')\, F(E' \to E)\, dE' - \Sigma_s(E)\, \psi(E,t)\, M(E) \quad (4.4)$$

Scattering terms can be represented in terms of the energy transfer moment integrals $A_k(E)$, which have been defined in Equation (3.35). Using the results of Equations (3.34) and (3.35), we represent the scattering terms as follows.

$$\int_{E'} \Sigma_s'(E')\, \psi(E',t)\, M(E')\, F(E' \to E)\, dE' - \Sigma_s(E)\, \psi(E,t)\, M(E) = \sum_{k=1}^{\infty} \frac{1}{k!} \left\{ \frac{\partial^k \psi(E,t)}{\partial E^k} \right\} \left\{ M(E)\, \Sigma_s(E)\, A_k(E) \right\} \quad (4.5)$$

Recapitulating, the definition of $A_k(E)$ is given by Equation (3.35) as follows.

$$A_k(E) = \frac{\int_{E'} (E'-E)^k\, \Sigma_s(E)\, F(E \to E')\, dE'}{\Sigma_s(E)} \quad (3.35)$$

We substitute the result given by Equation (4.5) into Equation (4.1), getting

$$\frac{1}{v}\frac{\partial \psi(E,t)}{\partial t}\, M(E) = -\left[\Sigma_a(E) + D(E)\, B^2\right] \psi(E,t)\, M(E) + \sum_{k=1}^{\infty} \frac{1}{k!} \left\{ \frac{\partial^k \psi(E,t)}{\partial E^k} \right\} \left\{ M(E)\, \Sigma_s(E)\, A_k(E) \right\} \quad (4.6)$$

We make the following ansatz:

$$\phi(E,t) = \sum_i e^{-\lambda_i t} \left\{ \sum_{n=0}^{\infty} a_{ni}\, L_n^{(1)}(E)\, M(E) \right\} \quad (4.7)$$

$\varphi(0,t) = 0$ Neutron distribution for energy zero is equal to zero at all times.

$\varphi(\infty,t) = 0$ Neutron distribution for energy E_o is zero at all times.

In the above ansatz, the eigen function corresponding to the i-th eigen value is given by the sum of the associated Laguerre polynomials of first order. $L_n^{(1)}(E)$ is the n-th Laguerre polynomial of first order. λ_i is the corresponding eigen value. Substituting the above ansatz into Equation (4.6), we get

$$\sum_i -\frac{\lambda_i}{v} e^{-\lambda_i t}\left\{\sum_n a_{ni} L_n^{(1)}(E)\, M(E)\right\} = -\sum_i e^{-\lambda_i t}\left[\sum_n a_{ni}\left\{\left(\Sigma_a(E) + D B^2\right) L_n^{(1)}(E) M(E) + \left(\sum_{k=1}^{\infty} \frac{1}{k!} \frac{\partial^k L_n^{(1)}(E)}{\partial E^k} A_k(E) M(E) \Sigma_s(E)\right)\right\}\right] \tag{4.8}$$

Multiplying by $L_m^{(1)}(E)$ and integrating over all energies, we get

$$\sum_i e^{-\lambda_i t}\left\{\sum_n a_{ni} \int_0^{\infty} -\frac{\lambda_i}{v} L_n^{(1)}(E)\, M(E)\, dE\right\} = \sum_i e^{-\lambda_i t}\left[\sum_n a_{ni}\left\{\left\{-\int_0^{\infty}\left(\Sigma_a(E) + D(E) B^2\right) L_n^{(1)} M(E) L_m^{(1)} dE\right\} + \left\{\sum_{k=1} \frac{1}{K!} \int_0^{\infty} \Sigma_s(E) A_k(E)\left(\frac{\partial^k L_n^{(1)}(E)}{\partial E^k}\right) L_m^{(1)}(E)\, M(E)\, dE\right.\right.\right. \tag{4.9}$$

We define the following integrals.

$$\int_0^{\infty} \frac{L_n^{(1)}(E)\, M(E)\, L_m^{(1)}(E)}{E^{1/2}}\, dE = W_{mn} \tag{4.10}$$

$$\int_0^{\infty} L_n^{(1)}(E)\, M(E)\, L_m^{(1)}(E)\, D B^2 dE = (D B^2)_{mn} \tag{4.11}$$

$$\sum_{K=1}^{\infty} \frac{1}{K!}\left\{\int_0^{\infty} \frac{\partial^K L_n^{(1)}(E)}{\partial E^K} L_m^{(1)}(E) M(E) \Sigma_s A_k(E) dE = F_{mn}\right. \tag{4.12}$$

$$\Sigma_a(E) = \frac{\Sigma_{a0}}{E^{1/2}} \tag{4.13}$$

Equation (4.13) is given by the 1/v absorption law.

$$U = E^{1/2} U_0 \tag{4.14}$$

Substituting these integrals in to (4.9) we get

$$\sum_i e^{-\lambda_i t} \left[\sum_n a_{ni} \left\{ -\frac{\lambda_i}{U_0} W_{mn} + \Sigma a_0 W_{mn} + (\underline{D}B^2)_{mn} - F_{mn} \right\} \right] = 0 \tag{4.15}$$

The above set of equations holds good for every coefficient a_{ni}. It will have a non-vanishing solution provided the secular determinant of its coefficients vanishes. The secular determinant is given as:

$$\left| \left(-\frac{\lambda}{U_0} + \Sigma a_0 \right) W_{mn} + (\underline{D}B^2)_{mn} - F_{mn} \right| = 0 \tag{4.16}$$

In principle, the above determinant can be solved, provided we know all the matrix elements involved. W_{mn} for all m and n are known. However, $(DB^2)mn$ and F_{mn} require a detailed knowledge of the scattering kernels. For a heavy gas model, all the energy moments of the scattering kernel can be obtained by using the Wigner-Wilkins kernel, therefore, F_{mn} can be obtained for the heavy gas model. In the case of crystals and liquids, extensive numerical calculations using digital computers are required. Even then, in these cases, the problem is a formidable one, because of the lack of experimental data, and inadequacy of the theoretical models used.

It is possible to solve the above determinant for the 2 x 2 case, since the F_{mn}'s involved are known. However, the above solution is also possible for a definite energy dependence of the diffusion coefficient. When D is proportional to v^{α}, then $(DB^2)_{mn}$ can be determined. For the case of a constant diffusion coefficient, we shall solve the 2 x 2

determinant and obtain the first two eigen values λ_o and λ_1.

Hurwitz and Nelkin[21] determined λ_o for the infinite heavy gas medium using perturbation techniques. With the help of a variational principle, using a Maxwellian distribution as a trial function, Nelkin[32] also determined λ_o. This method, however, employed the concept of 'neutron temperature.' The use of this concept has been found objectionable. Singwi[43] obtained λ_o without using the idea of 'neutron temperature.' Singwi's expression is exactly the same as that of Nelkin's, and therefore, represents a considerable advance. Häfele and Dresner[19] calculated λ_o for a heavy gas model with a better approximation.

As pointed out above, the determination of λ_o and λ_1 involves the calculation of W_{mn} and F_{mn}. The first few values of m and n have been calculated and are given below. A few associated Laguerre polynomials are also listed.

We define the n-th associated Laguerre polynomial of the first order as follows:[30]

$$L_n^{(1)}(E) = \sqrt{n+1}\left[\sum_{p=0}^{n}(-1)^p \frac{n!\,E^p}{p!\,(p+1)!\,(n-p)!}\right]$$

We list the first four Laguerre polynomial $L_n^1(E)$

$$\begin{aligned} L_0^{(1)}(E) &= 1 \\ L_1^{(1)}(E) &= \sqrt{2}\left(1-\frac{E}{2}\right) \\ L_2^{(1)}(E) &= \sqrt{3}\left(1-E+\frac{E^2}{3!}\right) \\ L_3^{(1)}(E) &= \sqrt{4}\left(1-\frac{3}{2}E+\frac{E^2}{2}-\frac{E^3}{4!}\right) \end{aligned} \qquad (4.17)$$

The following matrix elements have been calculated.

$$W_{oo} = 0.500\sqrt{\pi}$$
$$W_{o1} = 0.1768\sqrt{\pi}$$
$$W_{11} = 0.4375\sqrt{\pi}$$
$$W_{1o} = 0.1768\sqrt{\pi}$$
$$W_{o2} = 0.1083\sqrt{\pi}$$
$$W_{12} = 0.1914\sqrt{\pi}$$
$$W_{22} = 0.3984\sqrt{\pi}$$
$$W_{21} = 0.1914\sqrt{\pi}$$
$$W_{30} = W_{o3} = 0.0781\sqrt{\pi}$$
$$W_{31} = W_{13} = 0.1271\sqrt{\pi}$$
$$W_{32} = W_{23} = 0.1928\sqrt{\pi}$$
$$W_{33} = 0.3706\sqrt{\pi}$$

also for $(n \geq m)$

$$W_{mn} = \frac{1}{\sqrt{m+1}\sqrt{n+1}\,\pi}\left[\sum_{k=0}^{m}\left\{\frac{\Gamma(m-k+1/2)\,\Gamma(n-k+1/2)}{(m-k)!\,(n-k)!}\right\}\times\left\{\frac{\Gamma(k+3/2)}{k!}\right\}\right]$$

We determine the F_{mn} integrals.

$$F_{oo} = 0$$
$$F_{o1} = 0$$
$$F_{1o} = -\frac{\sqrt{2}}{2}\int_0^\infty\left\{\int_0^\infty M(E)\,\Sigma_s(E)\,F(E\to E')(E'-E)\,dE'\,dE\right\}$$
$$= 0 \qquad \{\text{By symmetry and detailed balance theorem.}$$

$$F_{11} = -\int_0^\infty \left\{1 - \frac{E}{2}\right\} \left\{ \int_0^\infty M(E)\ \Sigma_s(E)\ F(E \to E')(E'-E)\, dE' \right\} dE$$

$$= \int_0^\infty \int_0^\infty \Sigma_s(E)\ M(E)\ F(E \to E')(E-E')\, dE'\, dE$$

$$+ \frac{1}{2} \int_0^\infty E \int_0^\infty M(E)\ \Sigma_s(E)\ F(E \to E')(E'-E)\, dE'\, dE$$

$$= -\frac{1}{4} \int_0^\infty \int_0^\infty M(E)\ \Sigma_s(E)\ (E-E')^2\ F(E \to E')\, dE'\, dE$$

$$= -\frac{M_2}{4}$$

II. Determination of Eigen Values λ_0 and λ_1.

We shall determine λ_0 and λ_1 from Equation (4.16) by considering m equal to zero and one, and n equal to zero and one. The secular determinant (4.16) reduces to the 2 x 2 determinant.

$$\begin{vmatrix} \left\{\left(\Sigma a_0 - \frac{\lambda}{v_0}\right) W_{00} + (DB^2)_{00} - F_{00}\right\} & \left\{\left(\Sigma a_0 - \frac{\lambda}{v_0}\right) W_{01} + (DB^2)_{01} - F_{01}\right\} \\ \left\{\left(\Sigma a_0 - \frac{\lambda}{v_0}\right) W_{10} + (DB^2)_{10} - F_{10}\right\} & \left\{\left(\Sigma a_0 - \frac{\lambda}{v_0}\right) W_{11} + (DB^2)_{11} - F_{11}\right\} \end{vmatrix} = 0 \qquad (4.18)$$

The solution of Equation (4.18) gives two values for λ. We shall call them λ_0 and λ_1. The zeroth eigen value λ_0 is smaller than λ_1. We give the solutions for λ_0 and λ_1 assuming the diffusion coefficient to be constant with energy.

$$\lambda_0 = \Sigma a_0 v_0 + \frac{v_0 (DB^2)}{W_{00}} \left[1 - 4 \left(\frac{W_{01}}{W_{00}}\right)^2 \frac{DB^2}{M_2}\right] \qquad (4.19)$$

$$\lambda_1 = \Sigma a_0 v_0 + \frac{M_2 W_{00} v_0}{4(W_{11} W_{00} - W_{01}^2)} + \frac{DB^2 v_0}{W_{00}} \left\{\frac{W_{00}^2 - W_{01}^2}{W_{11} W_{00} - W_{01}^2}\right\} + 4 D^2 B^4 \left(\frac{W_{01}^2}{W_{00}^2}\right) \frac{v_0}{W_{00} M_2} \qquad (4.20)$$

These expressions give the upper limits for the two eigen values.

We make the following observations:

1. For the case of zero absorption and infinite medium.

$$\lambda_o = 0$$

$$\lambda_1 = \frac{M_2 W_{oo} \bar{v}_o}{4(W_{11}W_{oo} - W_{o1}^2)} \qquad (4.21)$$

This means that the neutron energy spectrum emitted from a pulsed neutron source reaches the final equilibrium spectrum with the decay constant λ_1, which is characteristic of the physical model of the medium. For this particular case, the final energy spectrum is Maxwellian and the approach to the Maxwellian distribution is governed by the first eigen value λ_1.

2. For the case of infinite medium and 1/v absorber.

In this case the two eigen values are as follows.

$$\lambda_o = \Sigma_{a_o} \bar{v}_o$$

$$\lambda_1 = \Sigma_{a_o} \bar{v}_o + \frac{M_2 W_{oo} \bar{v}_o}{4(W_{11}W_{oo} - W_{o1}^2)}$$

For small absorptions, λ_1 is much larger than λ_o, therefore the final distribution is given by the smaller of these two eigen values, i.e. λ_o. The final spectrum is Maxwellian, but decays exponentially with a decay constant λ_o, which is proportional to the amount of absorption inside the medium.

For large absorptions, λ_1 is not large compared to λ_o, therefore, both decay constants play important roles in the decay of the spectrum. The decaying spectrum is not Maxwellian in this case.

3. For the case of a finite, absorbing medium.

λ_o and λ_1 depend upon B^2 and B^4. From Equations (4.19) and

(4.20).

$$\lambda_o = a_o + b_o B^2 - C_o B^4$$

$$\lambda_1 = a_1 + b_1 B^2 + C_1 B^4$$

If λ_o and λ_1 are plotted against B^2, we find that the deviation of λ_o and λ_1 from straight lines are exactly of the same magnitude but of opposite signs. Coefficient C_o is the well known 'diffusion cooling coefficient.' If the departure from the straight line for λ_o is due to the shift of the zeroth eigen function associated with λ_o towards the low energy side, then for λ_1, it is due to the shift of the first eigen function associated with λ_1 to the high energy side.

The slopes in both cases are of positive signs but of different magnitudes.

III. Relation Between the Diffusion Cooling Coefficient and λ_1

The expression for the diffusion cooling coefficient, for constant diffusion coefficient is as follows.

$$C_c = \frac{4 D^2 U_o}{M_2 W_{oo}} \left(\frac{W_{o1}}{W_{oo}} \right)^2 \qquad (4.22)$$

It is possible to obtain a relation between the diffusion cooling coefficient C_o and the first eigen value λ_1 in the infinite non-absorbing medium case.

$$\lambda_1 = \frac{M_2 W_{oo} U_o}{4(W_{11} W_{co} - W_{o1}^2)} \qquad (4.23)$$

Eliminating M_2, we get

$$C_o = \frac{D^2 U_o^2}{\lambda_1} \left[\left(\frac{W_{o1}}{W_{oo}} \right)^2 \frac{1}{(W_{11} W_{oo} - W_{o1}^2)} \right] \qquad (4.24)$$

If we substitute the values for W_{oo}, W_{ol} and W_{ll} we get

$$C_o = \frac{2 D^2 U_o^2}{3 \lambda_1 \pi} \qquad (4.25)$$

If λ_1 is defined as the reciprocal of the thermalization time constant (t_{th}) in the infinite, non-absorbing medium, then

$$\lambda_1 = \frac{1}{t_{th}} \quad ; \quad t_{th} = \frac{3 \pi C_o}{2 D^2 U_o^2} \qquad (4.26)$$

Nelkin[32] derived the same relation between the diffusion cooling coefficient and the time constant, with which the 'neutron temperature' approaches the moderator temperature in the infinite, non-absorbing medium. He assumed that the neutron temperature can be given by the following relation.

$$T_n(t) = T_{Moderator} \{1 + \beta\}^{-1} \qquad (4.27)$$

Using the variational principle, with β' as the variational parameter and a modified Maxwellian trial function, he obtained the following result for the neutron temperature

$$T_n(t) = T_{Moderator} \{1 + e^{-\gamma t}\} \qquad (4.28)$$

where $\gamma = \frac{2}{3} \frac{U_c M^2}{\sqrt{\pi}}$

If γ^{-1} is the time constant, then the relation between γ^{-1} and the diffusion cooling coefficient is as follows.

$$\gamma^{-1} = \frac{\frac{3}{2} \pi C_o}{D^2 U_o^2} \qquad \text{Nelkin}^{(32)} \qquad (4.29)$$

This is exactly the same relation as given for t_{th} by Equation (4.26). The difference in the two results is due to the methods employed to obtain them. In obtaining t_{th}, we have not assumed the concept of neutron temperature, which is the weak point in Nelkin's derivation. t_{th} gives the rate with which the eigen function associated with λ_1 is decaying in time. The information about λ_1 will give the diffusion cooling coefficient C_o and vice versa.

Equation (4.21) gives λ_1 for the case of an infinite, non-absorbing medium.

$$\lambda_1 = 8.278 \times 10^4 M_2 \quad (\text{sec}^{-1}) \qquad (4.30)$$

Thermalization time constant can be given as follows:

$$t_{th} = 12.08/M_2 \quad (\mu \text{ sec}) \qquad (4.31)$$

Determination of λ_1 or t_{th} by Equation (4.30) and (4.31) involves the knowledge of M_2, which has to be obtained from the energy exchange scattering moments. For a few interesting moderators, M_2 has been determined, using extensive numerical calculations. Table 4.1 gives the list of values of M_2 for a few moderators.

TABLE 4.1

THERMALIZATION PARAMETER

No.	Moderator	M_2 (cm^{-1})	Reference	Remark
1	beryllium	0.30 0.36	Singwi & Kothari[44] Nelkin[32]	Θ_D = 1000°K Θ_D = 930°K
2	beryllium oxide	0.20	Singwi & Kothari	
3	graphite	0.068	Nelkin[32]	Θ_D = 900°K Vibrations $\perp$ to latice planes. Θ_D = 2500°K Vibrations $\parallel$ to latice planes.
4	water	1.05×10^{-1}	Nelkin[33]	Effective mass = 18

Θ_D = Debye temperature

For the heavy gas model,

$$M_2 = 4 \xi \Sigma_{so} \qquad \text{where} \quad \xi = 2/A$$

Σ_{so} = free atom scattering cross section.

We shall list in Table 4.2 λ_1 (in kilocycles/second) and t_{th} in μ seconds, along with earlier determinations by various other authors using different methods.

TABLE 4.2

THE THERMALIZATION TIME CONSTANT (t_{th})

No.	Moderator	λ_1 (kilocycles/sec)	t_{th}(μ sec)	Remarks
1	beryllium	24.8	40.266	Singwi & Kothari[27] obtained 28 μ secs.
		29.8	33.56	based upon neutron temperature idea
2	beryllium oxide	16.56	60.40	R.C. Bhandari et al.[8] obtained 67 μ secs
3	graphite	5.629	177.65	K.H. Beckurts[4] obtained experimentally 185 $\pm$ 45 μ secs. Antonov et al.[2] obtained thermalization time $\simeq$ 200 μ sec.
4	water	87.00	11.94	von Dardel's[49] result = 7 μ sec.

As indicated above, the thermalization time constant determined by the eigen value method agrees with the independent results of other authors.

The case of the heavy gas moderator will not be discussed here. It is discussed in detail in the following chapter.

IV. Determination of λ_o/λ_1

Since the experimentalists are interested in the ratio of λ_o/λ_1, in order to estimate the contamination of λ_o by λ_1 in their measurements, we shall calculate this ratio for a given example. We shall take the experiments of de Saussure and Silver[15] for beryllium assemblies. We assume the following constants for beryllium.

$$\Sigma_{ao} v_o = 0.288 \times 10^3/\text{sec}$$

$$M_2 = 0.36 \text{ cm}^{-1}$$

$$2\frac{Dv_o}{\sqrt{\pi}} = 1.25 \times 10^5 \text{ cm}^2/\text{sec}.$$

$$\lambda_o = 0.288 \times 10^3 + 1.25 \times 10^5 B^2 - 0.8653 \times 10^5 B^4 \qquad (4.32)$$

$$\lambda_1 = 30.088 \times 10^3 + 1.458 \times 10^5 B^2 + 0.8653 \times 10^5 B^4 \qquad (4.33)$$

We have the following results for various size assemblies listed in Table 4.3.

TABLE 4.3

COMPARISON OF λ_o AND λ_1 IN BERYLLIUM

No.	B^2 (10^{-2}cm^{-2})	λ_o(10^3/sec)	λ_1(10^3/sec)	λ_o/λ_1
1	0	0.288	30.088	9.57×10^{-3}
2	1.05	1.589	31.627	5.024×10^{-2}
3	2.02	2.778	33.068	8.40×10^{-2}
4	3.31	4.331	35.009	1.237×10^{-1}
5	5.36	6.739	38.152	1.776×10^{-1}
6	7.18	8.817	41.002	2.150×10^{-1}

In Figure 4.1 we have plotted λ_o/λ_1 against B^2 obtained from (4.32) and (4.33) equations for beryllium assemblies used by de Saussure and Silver[15] in their pulsed neutron experiments. For the smallest

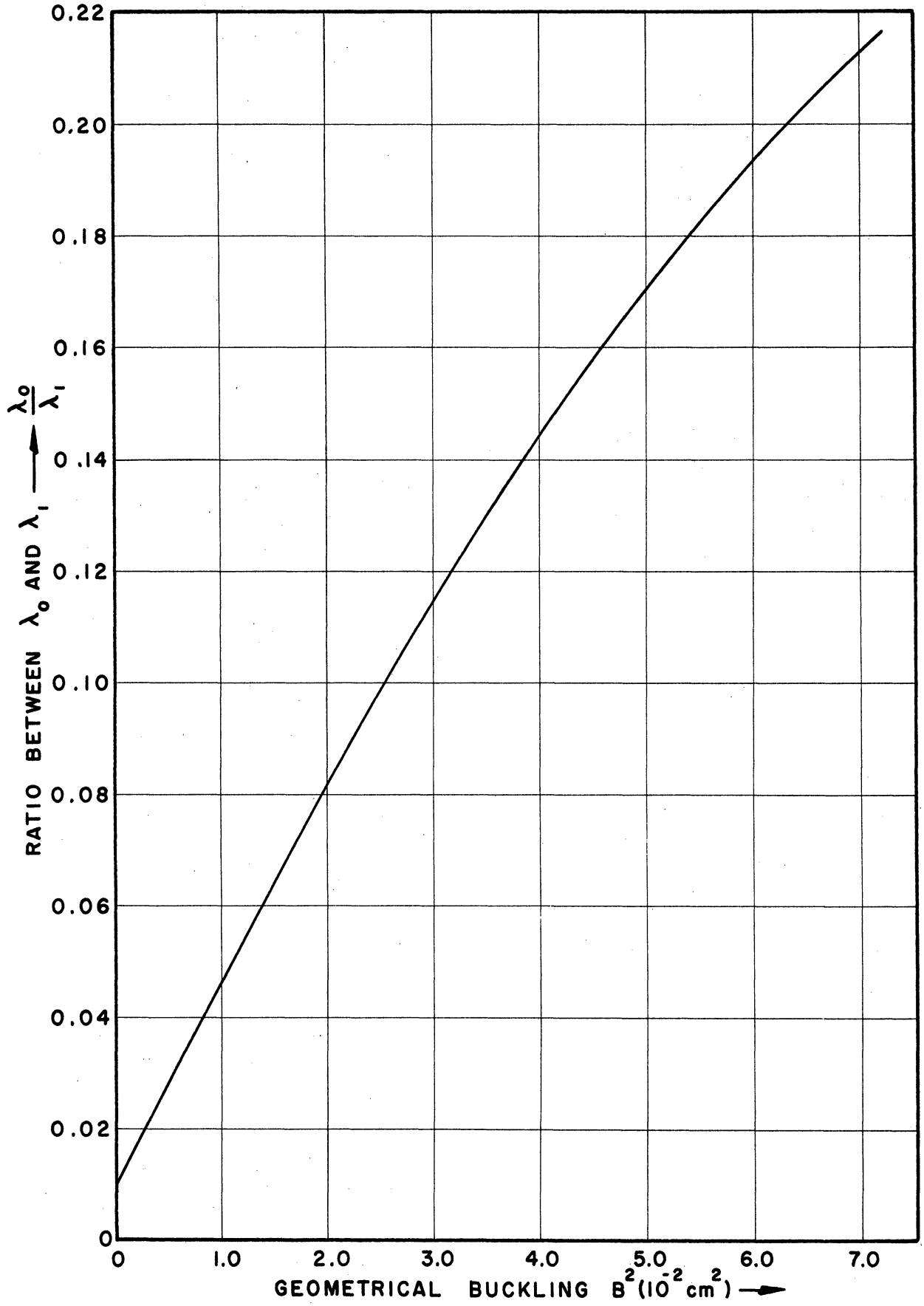

Figure 4.1 Variation of λ_o/λ_1 with B^2 for Beryllium.

assembly of geometrical buckling, B^2 equal to 7.18×10^2 cm^{-2}, λ_o/λ_1 is equal to 1/4.65. This clearly demonstrates that the contribution by the first and other higher eigen values are small compared to the zeroth eigen value and, thus, may be neglected.

V. Determination of M_2

The neutron thermalization parameter is characterized by the integral M_2.

$$M_2 = \int_0^\infty \int_0^\infty M(E)\, \Sigma_s(E)\, (E - E')^2\, F(E \rightarrow E')\, dE'\, dE$$

Nelkin[32] proposed the determination of its value from the measurement of the diffusion cooling coefficient. We propose an absorption method for determining its value. We shall discuss the merits of the two methods.

A. Diffusion Cooling Method

Equation (4.22) relates M_2 with C_o, the diffusion cooling coefficient.

$$C_c = 4 D^2 U_o W_{o1}^2 (M_2 W_{oo}^3)^{-1}$$

The determination of the diffusion cooling coefficient depends on the accuracy of the determination of the decay constant λ_o, given by Equation (4.19). A small error in λ_o can cause a large error in the value of C_o.

The diffusion coefficient is assumed to be independent of energy in Equation (4.22). This is not rigorously true at low energies.

The geometrical buckling, B^2, depends upon the extrapolated distance. The lack of complete knowledge of the extrapolated distance introduces a small uncertainty in the determination of B^2.

Equation (4.22) has been obtained considering only the fundamental spatial mode. The effects of higher modes have been entirely neglected.

A rigorous determination of M_2 should be based upon the corrections for the finite medium effects introduced by the leakage and boundary. The above method is handicapped due to lack of information for obtaining these corrections.

B. Infinite Medium 1/v Absorption Method

For a very large assembly, which may be considered as infinite, two eigen values λ_0 and λ_1 were given earlier.

$$\lambda_0 = \Sigma a_0 v_0 \quad ; \quad \lambda_1 = \Sigma a_0 v_0 + M_2 W_{00} v_0 \left[4(W_{11}W_{00}-W_{01}^2)\right]^{-1}$$

In the presence of a small 1/v absorber, $\lambda_1 \gg \lambda_0$. Then the final decay of the spectrum is governed by the zeroth eigen value λ_0.

If we increase the 1/v absorption by using boric acid in water, for example, then λ_1 becomes comparable to λ_0. The absorption term in the expression of λ_1 becomes larger than the thermalization term. In that case the final decay of the spectrum is governed not only by λ_0 but also by λ_1. It is possible to determine λ_0 and λ_1 by analyzing the decay curve. Since the absorption term is known, M_2 can be determined rigorously.

This method avoids the finite medium corrections but requires intense neutron sources, as a large amount of absorption will be needed before the decay can be governed by λ_0 and λ_1.

In the next chapter we shall give an estimate of the absorption required for determining M_2 for the heavy gas model.

CHAPTER V. NEUTRON THERMALIZATION IN THE HEAVY GAS MEDIUM

In the fourth chapter, a general method to obtain eigen values for any physical model was discussed. The explicit expressions for the zeroth and the first eigen values were obtained, considering only the first three energy transfer moments. In this chapter, the problem of the heavy gas model will be discussed in detail. Wigner and Wilkins[53] derived the scattering kernel based upon the Maxwellian distribution of velocities for the atoms of the moderator. Since they ignored the chemical and crystalline effects, their kernel can be used only for the monoatomic gases. As will be shown in this chapter, energy transfer moments numbering more than three involve terms of the order of $(1/A^2)$, where A is the mass of the scattering nucleus. For the case of a heavy gas, therefore, the energy transfer moments higher than three can be neglected. This leads to the reduction of the integral equation into a differential equation. The differential operator of the heavy gas model has the associated Laguerre polynomials of order one as its eigen function. In the case of a medium having 1/v absorption and of known energy dependence of the diffusion coefficient, the results can be obtained to any degree of accuracy, and all the eigen values can be determined. We shall determine the first three eigen values λ_0, λ_1, and λ_2. The zeroth eigen value was determined by Häfele and Dresner.[19] The first eigen value, λ_1, and the associated eigen function, $\varphi_1(E)$, are obtained here rigorously.

I. Transformation of the Integral Equation into the Differential Equation

We have already outlined the method for the transformation of the integral equation into the differential equation in Chapter IV.

Rewriting Equation (4.6)

$$\frac{1}{v}\frac{\partial \psi(E,t)}{\partial t} M(E) = -\left\{\Sigma_a(E) + D(E) B^2\right\} \psi(E,t) M(E) + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{\partial^k \psi(E,t)}{\partial E^k} \Sigma_s(E) M(E) A_k(E)$$

Where $A_k(E)$ has been defined previously as the k-th energy transfer moment.

For the heavy gas model, we have the following values for the various energy transfer moments, based upon the Wigner-Wilkins kernel.

$$\begin{aligned} \Sigma_s(E) A_0 &= \Sigma_s(E) \\ \Sigma_s(E) A_1 &= 2 \Sigma_{s_0} \mu (2-E) \\ \Sigma_s(E) A_2 &= 4 \mu E \Sigma_{s_0} \\ \Sigma_s(E) A_k &= (\mu^2) \qquad \text{for } k \geq 3 \\ \mu^2 &= \left(\frac{1}{A}\right)^2 \end{aligned} \tag{5.1}$$

Substituting these moments into Equation (4.6), we get Wilkin's[54] differential equation.

$$\frac{1}{v \xi \Sigma_{s_0}} \frac{\partial \psi(E,t) M(E)}{\partial t} = \left[E \frac{\partial^2 \psi(E,t)}{\partial E^2} + (2-E) \frac{\partial \psi(E,t)}{\partial E} - \frac{1}{\xi \Sigma_{s_0}} \left\{\Sigma_a(E) + D(E) B^2\right\} \psi(E,t) \right] M(E) \tag{5.2}$$

The heavy gas differential operator has as its eigen values the associated Laguerre polynomial of order one. This result has been used by Häfele and Dresner[19] in their derivation of the diffusion cooling coefficient. Kazarnovsky et al.[24] also made use of this result in their studies. The energy portion of the differential operator in Equation (5.2) satisfies the following equation, according to Murphy and

Margenau[30]

$$\left[E\frac{\partial^2}{\partial E^2} + (2-E)\frac{\partial}{\partial E}\right] L_n^{(1)}(E) = -n\, L_n^{(1)}(E) \qquad (5.3)$$

where $L_n^{(1)}(E)$ is the associated Laguerre polynomial of first order. 'n' is the eigen value associated with the n-th eigen function. The eigen value takes the following values: zero, one two ....

We make the following ansatz:

$$\phi(E,t) = \sum_i e^{-\lambda_i t} \sum_n a_{ni} L_n^{(1)}(E)\, M(E)$$
$$\phi(0,t) = 0 \quad ; \quad \phi(\infty,t) = 0 \qquad \text{for } t > 0 \qquad (5.4)$$

Substituting the above ansatz into Equation (5.2), multiplying by $L_m^{(1)}(E)$ and integrating over all energies, we get

$$\left[\sum_i e^{-\lambda_i t} \sum_n a_{ni} \left\{ n\delta_{mn} + \left(\Sigma a_0 - \frac{\lambda_i}{\xi \Sigma_{s_0} v_0}\right) W_{mn} + (DB^2)_{mn} \right\}\right] = 0 \qquad (5.5)$$

We have used the notations introduced in Chapter IV. A new symbol, δ_{mn} is introduced in Equation (5.5). It is the Kronecker delta function.

$$\int_0^\infty L_n^{(1)}(E)\, E e^{-E}\, L_m^{(1)}(E)\, dE = \delta_{mn} \qquad (5.6)$$

The secular determinant given by Equation (5.5) is as follows

$$\left| n\delta_{mn} + \left(\Sigma a_0 - \frac{\lambda}{\xi \Sigma_{s_0} v_0}\right) W_{mn} + (DB^2)_{mn} \right| = 0 \qquad (5.7)$$

II. Determination of Eigen Values

For the case of a known diffusion coefficient, all the eigen values can be obtained, since all the matrix elements are known. Häfele and Dresner[19] obtained the zeroth eigen value and the diffusion cooling coefficient 'c' by expanding the secular determinant in the minors of the first row. This is possible because the off-diagonal terms contain B^2 or its higher powers. It is not possible, for the case of higher eigen values, to do higher order calculations except by tedious numerical calculations.

For the infinite medium having zero absorption, λ_0 equals zero, therefore, the next higher eigen value, λ_1, becomes important. It governs the rate of thermalization in this case. If we use the assumption of zero absorption and infinite medium, then Equation (5.7) reduces to the following simple determinant.

$$\left| n\,\delta_{mn} - \frac{\lambda}{\xi \Sigma_{s_0} v_0} W_{mn} \right| = 0 \qquad (5.8)$$

The two by two determinant is obtained from determinant (5.8), by putting m equal to zero and one, and n equal to zero and one.

$$\begin{vmatrix} -\frac{\lambda W_{00}}{\xi \Sigma_{s_0} v_0} & -\frac{\lambda W_{01}}{\xi \Sigma_{s_0} v_0} \\ -\frac{\lambda W_{10}}{\xi \Sigma_{s_0} v_0} & \left(1 - \frac{\lambda W_{11}}{\xi \Sigma_{s_0} v_0}\right) \end{vmatrix} = 0 \qquad (5.9)$$

This gives two values for λ

$$\lambda_0 = 0 \quad ; \quad \lambda_1 = \frac{W_{00}\, \xi \Sigma_{s_0} v_0}{W_{00} W_{11} - W_{01}^2} \qquad (5.10)$$

If we substitute the value of M_2 equal to $4\xi\Sigma_{s_0}$ into Equation (4.21), it then becomes identical to Equation (5.10) for the heavy gas model. By expressing the secular determinant in the minors of the first row, we get matrices of the k-th order, for n equal to k.

$$\left[\lambda W_{oo} \prod_{n=1}^{k} \alpha_{nn} - \frac{\alpha_{o1}\alpha_{1o}}{\alpha_{11}} \prod_{n=1}^{k} \alpha_{nn} \cdots\cdots - \frac{\alpha_{ok}\alpha_{ko}}{\alpha_{kk}} \prod_{n=1}^{k} \alpha_{nn}\right] = 0 \quad (5.11)$$

where $$\alpha_{mn} = \lambda W_{mn}$$

$$\alpha_{nn} = -n + \lambda W_{nn}$$

$$\prod_{n=1}^{k} \alpha_{nn} = \alpha_{11} \cdot \alpha_{22} \cdots\cdots \alpha_{kk}$$

This leads to the following equation for λ.

$$\lambda W_{oo} = \sum_{n=1}^{k} \frac{\lambda^2 W_{no}^2}{-n + \lambda W_{nn}} \quad (5.12)$$

The above equation can be reduced to the following two equations.

$$\lambda = 0 \quad (5.13)$$

$$W_{oo} = \sum_{n=1}^{k} \frac{\lambda W_{no}^2}{-n + \lambda W_{nn}} \quad (5.14)$$

λ equals zero gives the zeroth eigen value, therefore, we have the final equilibrium distribution constant in time. Equation (5.14) gives the first eigen value, which determines the rate of thermalization. We have calculated the following eigen values in various orders. They are listed in Table 5.1.

TABLE 5.1

EIGEN VALUES IN HEAVY GAS MODEL

Determinant	λ_o	λ_1 $(\xi\Sigma_{s_o}v_o)$	λ_2 $(\xi\Sigma_{s_o}v_o)$
2 x 2	0	1.506	-
3 x 3	0	1.329	4.088
4 x 4	0	1.274	-

We have calculated the first eigen value, λ_1, in various approximations. It is guessed that the next value of λ_1 will be within two percent of the value for the four by four case. λ_2 has also been calculated to estimate its relative importance compared to λ_1.(38)

III. Determination of Eigen Functions

The eigen functions associated with the eigen values λ_o and λ_1 will be determined. With the help of Equation (5.5) all the coefficients a_{ni} for the i-th eigen value can be determined. Rewriting Equation (5.5)

$$\sum_{i=0}^{\infty} \epsilon^{-\lambda_i t} \left[\sum_{n=0}^{\infty} a_{ni} \left\{ n\,\delta_{mn} - \frac{\lambda_i}{\xi\,\Sigma_{s_o} v_o} W_{mn} \right\} \right] = 0 \qquad (5.8)$$

Zeroth eigen function $\varphi_o(E)$.

The zeroth eigen value, λ_o, is equal to zero.

$$\sum_{n=0}^{\infty} a_{no}\, n\, \delta_{mn} = 0$$

This leads to all a_{no} equal to zero, except for n equal to zero.

$$a_{oo} \neq 0 \qquad a_{no} = 0 \qquad \text{for } n \geq 1$$

The zeroth eigen function is $\phi_0(E) = a_{00} E e^{-E}$

This eigen function is the Maxwellian distribution.

First eigen function $\varphi_1(E)$

The first eigen function associated with the first eigen value λ_1 is obtained by determining the various coefficients, a_{n1}. Using the 4 x 4 case, the ratios of various a_{n1}'s to a_{01}'s have been determined. The coefficients a_{11}/a_{01}, a_{21}/a_{01} and a_{31}/a_{01} were evaluated. With the help of these coefficients we construct the first eigen function in the following manner.

$$\phi_1(E) = E e^{-E} a_{01} \left[L_0^{(1)}(E) + \frac{a_{11}}{a_{01}} L_1^{(1)}(E) + \frac{a_{21}}{a_{01}} L_2^{(1)}(E) + \cdots \right] \qquad (5.15)$$

The following set of equations, obtained from Equation (5.5), was used to obtain the above coefficients.

$$\begin{aligned}
&-\{a_{01} W_{00} + a_{11} W_{01} + a_{21} W_{02} + a_{31} W_{03}\} \frac{\lambda_1}{\xi \Sigma_{s_0} v_0} = 0 \\
&-a_{01} \frac{\lambda_1}{\xi \Sigma_{s_0} v_0} W_{10} + a_{11}\left(1 - \frac{\lambda_1}{\xi \Sigma_{s_0} v_0} W_{11}\right) - a_{21} \frac{\lambda_1}{\xi \Sigma_{s_0} v_0} W_{12} - a_{31} \frac{\lambda_1 W_{13}}{\xi \Sigma_{s_0} v_0} = 0 \\
&-a_{01} \frac{\lambda_1}{\xi \Sigma_{s_0} v_0} W_{20} - a_{11} \frac{\lambda_1}{\xi \Sigma_{s_0} v_0} W_{21} + a_{21}\left(2 - \frac{\lambda_1}{\xi \Sigma_{s_0} v_0} W_{22}\right) - a_{31} \frac{\lambda_1}{\xi \Sigma_{s_0} v_0} W_{23} = 0 \qquad (5.16) \\
&-a_{01} \frac{\lambda_1}{\xi \Sigma_{s_0} v_0} W_{30} - a_{11} \frac{\lambda_1 W_{31}}{\xi \Sigma_{s_0} v_0} - a_{21} \frac{\lambda_1}{\xi \Sigma_{s_0} v_0} W_{32} + a_{31}\left(3 - \frac{\lambda_1}{\xi \Sigma_{s_0} v_0} W_{33}\right) = 0
\end{aligned}$$

The last three equations were used to obtain a_{11}/a_{01}, a_{21}/a_{01} and a_{31}/a_{01}. The values of these coefficients are listed in Table 5.2.

TABLE 5.2

COEFFICIENTS OF THE FIRST EIGENFUNCTION

Coefficients	Values
a_{11}/a_{01}	-2.127
a_{21}/a_{01}	-0.7516
a_{31}/a_{01}	-0.3515

The first equation may be used to determine the contribution of these coefficients to the eigen function. The first equation of set (5.16) is as follows:

$$\frac{a_{11}}{a_{01}} W_{01} + \frac{a_{21}}{a_{01}} W_{02} + \frac{a_{31}}{a_{01}} W_{03} = - W_{00} \qquad (5.17)$$

Substituting the values for the coefficients from Table 5.2 and the values of the matrix elements W_{mn} from Chapter IV, we get

$$\frac{a_{11}}{a_{01}} W_{01} + \frac{a_{21}}{a_{01}} W_{02} + \frac{a_{31}}{a_{01}} W_{03} = - 0.485\sqrt{\pi} \qquad (5.18)$$

and $$- W_{00} = - 0.5\sqrt{\pi}$$

The difference is about three percent.

Substituting the values of the coefficients into Equation (5.15) leads to

$$\phi_1(E) = E e^{-E} a_{01} [- 4.013 + 3.8605E - 0.5685E^2 + 0.0293E^3 \ldots] \qquad (5.19)$$

The adjoint of the above eigen function $\varphi_1(E)$ is

$$\phi_1^{\dagger}(E) = a_{01}\left[-4.013 + 3.8605E - 0.5685E^2 + 0.0293E^3\right] \qquad (5.20)$$

The eigen functions $\varphi_0(E)$ and $\varphi_1(E)$ have been numerically determined for different values of energy. In Table 5.3, these values are listed.

TABLE 5.3

EIGENFUNCTIONS IN HEAVY GAS MODEL

Energy (KT)	$\varphi_0(E) = Ee^{-E}a_{00}$	$\varphi_1(E) = Ee^{-E}f(E)a_{01}$ x a_{01}
0.1	0.09	-0.327
0.2	0.164	-0.535
0.3	0.222	-0.646
0.4	0.268	-0.686
0.5	0.303	-0.674
0.6	0.329	-0.624
0.7	0.348	-0.549
0.8	0.357	-0.455
0.9	0.366	-0.358
1.0	0.368	-0.255
1.5	0.335	0.1998
2.0	0.271	0.452
2.5	0.205	0.522
3.0	0.149	0.486
4.0	0.073	0.308
5.0	0.034	0.158

The zeroth eigen function is a pure Maxwellian distribution with a maximum at 1.0 KT. On the other hand, when one examines $\varphi_1(E)$, one discovers that it has a maximum at about 2.5 KT and a minimum at about 0.4 KT. The zero of the curve occurs at 1.3 KT.

The eigen value method discussed in the fourth and fifth chapters does not give the ratios between the amplitudes of the various eigen functions. It does give however, the eigen values and also the shape of the eigen functions, accurately. In order to obtain the ratios between the amplitudes of the eigen functions, one must have the complete knowledge of the function $\varphi(E,t)$ at some time. In the case of pulsed neutron source assemblies, this information is not available unless extensive numerical calculations are made to generate $\varphi(E,t)$ at different times. In the next few chapters the results of extensive numerical calculations, carried out to generate $\varphi(E,t)$, will be given.

CHAPTER VI. THE TIME DEPENDENT ENERGY SPECTRUM IN THE INFINITE MEDIUM

The rigorous solution of the time dependent neutron energy spectrum can only be given by numerical methods. The analytical methods have not proved to be sufficient to answer all questions, due to convergence difficulties. The fast neutrons exhibit 1/E energy behavior, and the slow neutrons have Maxwellian distribution represented by Ee^{-E}. It has not been found possible to represent the complete behavior of neutrons in all the energy intervals by a single function.

In the fourth and fifth chapters we used the eigen value method to study neutron thermalization. Though elegant to use, the eigen value method has its own limitations. It can not give the complete time behavior of the neutron energy spectrum, unless one has the following information,:

a) A complete set of eigen values $\lambda_0, \lambda_1, \ldots \lambda_n \ldots$

b) A complete set of the corresponding eigen functions

$$\varphi_0, \varphi_1 \ldots \varphi_n$$

c) A boundary condition to determine the coefficients of the eigen functions.

We used the associated Laguerre polynomials of first order to determine the eigen values. In the case of the general model, we limited ourselves to the first two Laguerre polynomials, due to the lack of information about the higher energy transfer moments. In the case of a heavy gas, however, we used the first four associated Laguerre polynomials of the first order. It is not easy to get higher eigen

values than λ_1, for crystals and liquids, unless the problem of the scattering kernel is rigorously solved. For the heavy gas model, it is possible to determine all the eigen values and eigen functions analytically. This, however, would require the expansion of the eigen functions by a large number of Laguerre polynomials. The eigen value method is useful in determining the lower eigen values and eigen functions but is inadequate to determine the coefficients of the eigen functions due to lack of accurate knowledge of the boundary conditions.

We generate the neutron distribution function $\varphi(E,t)$ numerically in the following pages, for a few interesting cases. With the help of this distribution, we shall determine the thermalization time, the rate of thermalization, the equilibrium spectra and their decay rates. The diffusion cooling phenomenon in finite media will also be studied. We propose, in addition, to test the validity of the statement, used by earlier workers in the analysis of pulsed neutron experiments, that "after sufficiently long time the asymptotic neutron energy spectrum can be given as follows:"

$$\phi(E,t) = \phi_0(E)\, e^{-\lambda_0 t}$$

I. Details of the Numerical Method.

We shall discuss, in this section, the details of the numerical method used to generate the neutron distribution function $\varphi(E,t)$. The electrical analog computer was used to obtain these distributions. We shall use the heavy gas model for our problem. In the fundemental spatial

mode, the time behavior of the neutrons in the above model is given by Equation (5.3). Rewriting it, we obtain:

$$\left[E\frac{\partial^2}{\partial E^2} + E\frac{\partial}{\partial E} + 1 - \frac{(\Sigma_a + D\beta^2)}{2\mu\Sigma_{s_0}}\right]\phi(E,t) = \frac{1}{2\mu\Sigma_{s_0}v_0}\frac{\partial\phi(E,t)}{\partial t} \quad (6.1)$$

We solve Equation (6.1) using the multigroup method. For one group of problems, the energy interval between E = 10KT to E = 0 is divided into fourteen groups. For the other group of problems we used twenty groups, in the above energy interval. In the latter case, the energy interval was divided in the following manner. Between E = 0 and E = KT, ten groups of 0.1 KT interval, and between E = KT and E = 10 KT, ten groups of 10 KT interval, were made.

The functions, $\frac{\partial\phi(E,t)}{\partial E}$ and $\frac{\partial^2\phi(E,t)}{\partial E^2}$, for each group are expressed by the finite difference method in the following way. Let ΔE_m be the energy interval between the (n + m) th and n th groups; let $\Delta E_{m'}$ be the energy interval between the n th and (n - m) th groups. If $\varphi_{n-m'}$; φ_m and φ_{n+m} represent the distributions for n-m, n and n+m groups respectively, then we have for $\left(\frac{\partial\phi}{\partial E}\right)_n$

$$\left(\frac{\partial\phi}{\partial E}\right)_n = \frac{1}{2}\left[\frac{\phi_{n+m} - \phi_n}{\Delta E_m} + \frac{\phi_n - \phi_{n-m'}}{\Delta E_{m'}}\right] \quad (6.2)$$

The above expression has been employed due to different group intervals between the (n+m) th and n th groups and the (n-m) th and n th groups.

For $\left(\frac{\partial^2\phi}{\partial E^2}\right)_n$, we have the following expression:

$$\left(\frac{\partial^2\phi}{\partial E^2}\right)_n = \frac{1}{2(\Delta E_m + \Delta E_{m'})}\left[\frac{\phi_{n+m} - \phi_n}{\Delta E_m} - \frac{\phi_n - \phi_{n-m'}}{\Delta E_{m'}}\right] \qquad (6.3)$$

The above expressions are substituted into Equation (6.1). This gives a set of coupled differential equations.

$$\phi_{n+m}\left[\frac{2E_n^{3/2}}{\Delta E_m(\Delta E_m+\Delta E_{m'})} + \frac{1}{2}\frac{E_m^{3/2}}{\Delta E_m}\right]$$

$$+ \phi_n\left[E_n^{3/2}\left\{-\frac{2(\Delta E_m+\Delta E_{m'})}{\Delta E_{m'}\,\Delta E_m(\Delta E_m+\Delta E_{m'})} - \frac{1}{2}\left(\frac{1}{\Delta E_m}-\frac{1}{\Delta E_{m'}}\right)\right\}\right.$$

$$\left. + E_n^{\frac{1}{2}}\left\{1-\gamma(E_n)\right\}\right] \qquad (6.4)$$

$$+ \phi_{n-m'}\left[\frac{2E_n^{\frac{3}{2}}}{\Delta E_{m'}(\Delta E_m+\Delta E_{m'})} - \frac{E_n^{3/2}}{2\Delta E_{m'}}\right] = \frac{\alpha}{2}\frac{\partial\phi_n}{\partial t}$$

The above equation is the representative of the n th group, which is coupled to the (n+m) th higher group and the (n-m) th lower group. We have expressed the energy in KT units. The symbols introduced stand for the following quantities.

$$\alpha = \frac{1}{\mu\Sigma_{s_0}v_0} \qquad (6.5)$$

$$\gamma(E_n) = \left[\frac{\Sigma_a(E_n) + D(E_n)B^2}{2\mu\Sigma_{s_0}}\right] \qquad (6.6)$$

Equation (6.4) is the basic equation for generating $\varphi(E,t)$. We solve the set with the help of the analog computer, using time as the continuous variable.

We use the following boundary conditions for generating the solution.

1. Neutrons of zero energy have zero distribution at all times.

$$\varphi(0,t) = 0 \qquad \text{for} \qquad t \geq 0$$

2. The neutron distribution function at time $t = \bar{t}_s$ is given. ($\bar{t}_s$ is the mean slowing down time for the source neutrons to have final energy in the interval E_T to E_T/α). In the third chapter, we calculated this distribution. Let this given distribution be represented by $S(E,t)$

$$S(E,t) = \frac{\delta(t-\bar{t}_s)\, A \left[1-\alpha\frac{E_T}{E}\right]}{\xi(1-\alpha)\, E_T} \qquad (3.9)$$

The neutron distribution function using the above source condition $S(E,t)$ will give the complete time behavior of the neutron energy spectrum, for times greater than the mean slowing down time $\bar{t}_s$.

We have generated the distribution function $\varphi(E,t)$ for beryllium and graphite in the following cases.

A. Infinite medium, zero absorption

B. Infinite medium, 1/v absorption

C. Finite medium, zero absorpiton

a) Graphite - constant diffusion coefficient.

b) Beryllium - energy dependent diffusion coefficient.

The asymptotic behavior can be used to obtain general results for any other heavy medium in the above cases, except for the special case of beryllium. This is possible since the initial source condition does not influence the asymptotic behavior, which is given by the physics of the medium. In order to transfer the distribution function from the known case to the unknown one, we have to carry out the following transformation for the time and geometrical buckling variables.

Let t_o, B_o^2, ξ_o, $(\Sigma_{so})_o$, and D_o be the time, geometrical buckling, average logarithmic energy decrement, free atom scattering cross section, and diffusion coefficient respectively, for the known medium. For the unknown medium, the corresponding quantities are t_x, $(B_o^2)_x$, ξ_x, $(\Sigma_{so})_x$ and D_x. The relation between the time scales for the two cases is:

$$t_x = t_o \left[\frac{\xi_o (\Sigma_{so})_o}{\xi_x (\Sigma_{so})_x} \right] \tag{6.7}$$

When the diffusion coefficient is constant, we get a relationship for the geometrical bucklings by equating the leakage of neutrons in the two cases.

$$(B_o^2)_x = \frac{D_o (B_o^2)_o}{D_x} \tag{6.8}$$

II. The Infinite Medium, Zero Absorption Case.

In Figure 6.1 the neutron energy spectrum at different times in the interval of $t = \bar{t}_s$ to $t = 350 + \bar{t}_s$ micro seconds has been plotted for beryllium. In determination of these spectra, twenty group calculations were used. For graphite we obtained the spectra in the time interval of $t = \bar{t}_s$ to $t = 510 + \bar{t}_s$ micro seconds, using fourteen groups.[39] This is plotted in Figure 6.2.

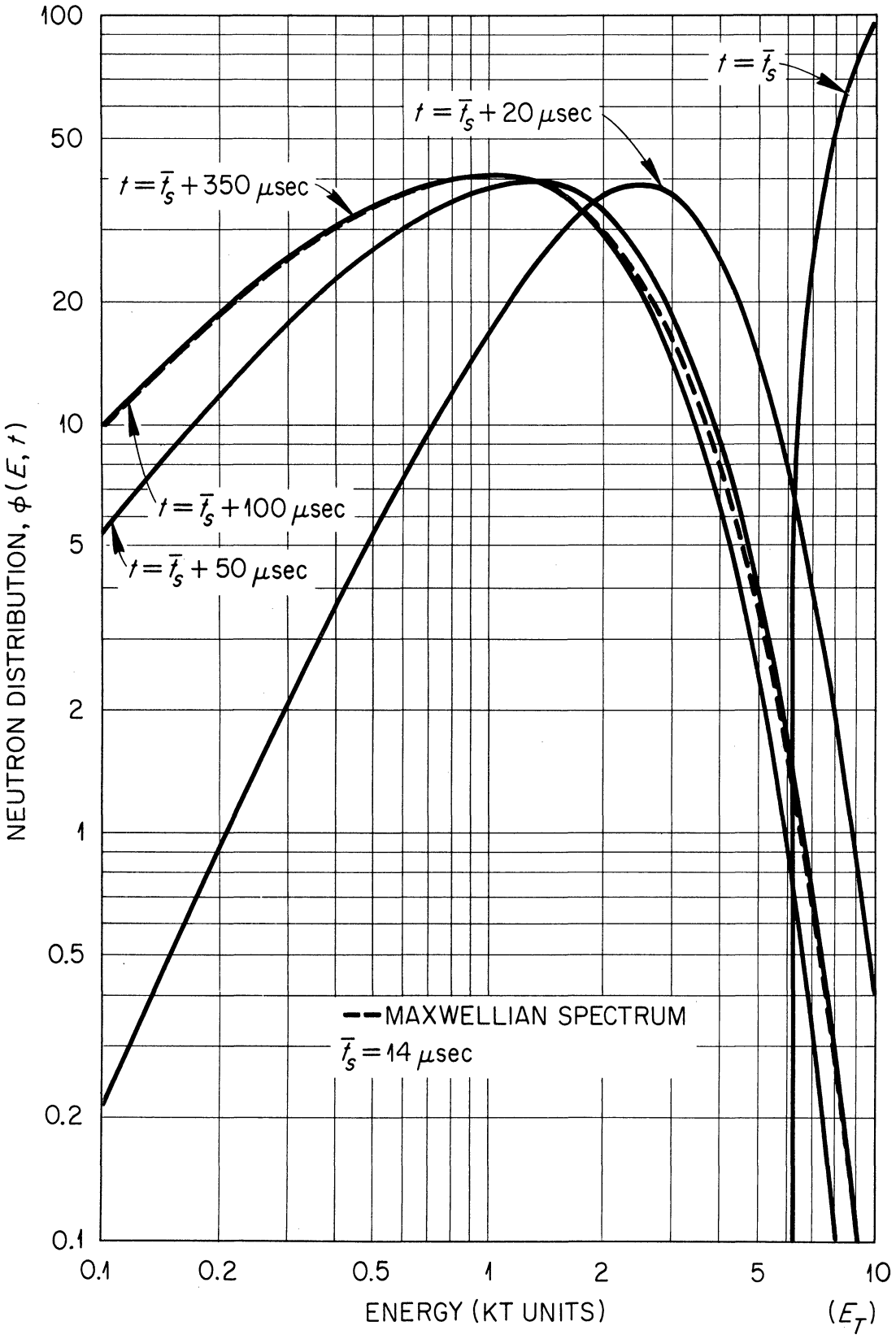

Figure 6.1 Time Behavior of Neutron Energy Spectrum in Beryllium for Infinite Medium and Zero Absorption.

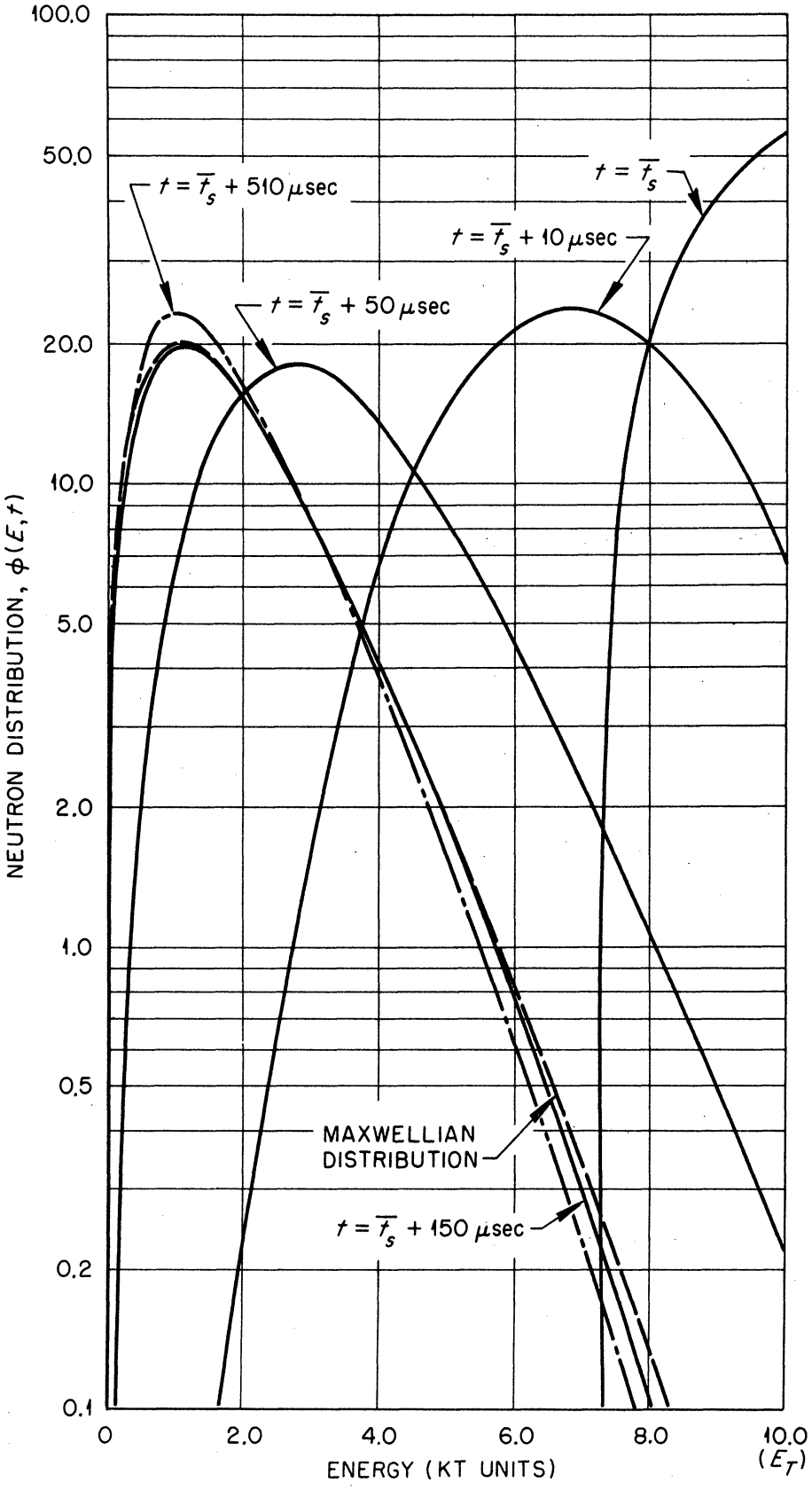

Figure 6.2 Time Behavior of Neutron Energy Spectrum in Graphite for Infinite Medium and Zero Absorption.

The energy spectrum at earlier times approximates a Gaussian distribution, but converges finally into the Maxwellian distribution. The degree of convergence is very good. After attaining the Maxwellian distribution, the neutrons retain this spectrum at all further times. In the non-absorbing infinite medium, the equilibrium spectrum is the Maxwellian spectrum. This is a basic result of the H Theorem of Statistical Mechanics and holds good for any scattering model. It is also consistent with the detailed balance theorem. A verification of this important result is furnished by the distributions plotted in Figures 6.1 and 6.2.

A. Asymptotic Energy Spectrum

The time behavior of a few energy groups is given in Figures 6.3 and 6.4. We have plotted the behavior of the individual groups in order to demonstrate the equilibrium state. Parallel curves indicate the existance of the equilibrium. The slopes of these curves indicate that the spectrum is constant in time after a certain time known as the thermalization time. This thermalization time will be defined and discussed subsequently. The energy groups having energy equal to, or lower than one KT, always increase in time till they reach the equilibrium state. The groups of energy higher than one KT increase first, reaching a maximum value before decaying to the asymptotic value.

It is interesting to know the rate with which the energy groups are reaching their asymptotic distribution. The curves in Figures 6.3 and 6.4 provide excellent data for determining the thermalization time constant. In the fifth chapter we have determined λ_1 for the non-absorbing, infinite heavy gas model. If the contribution of the eigen values higher

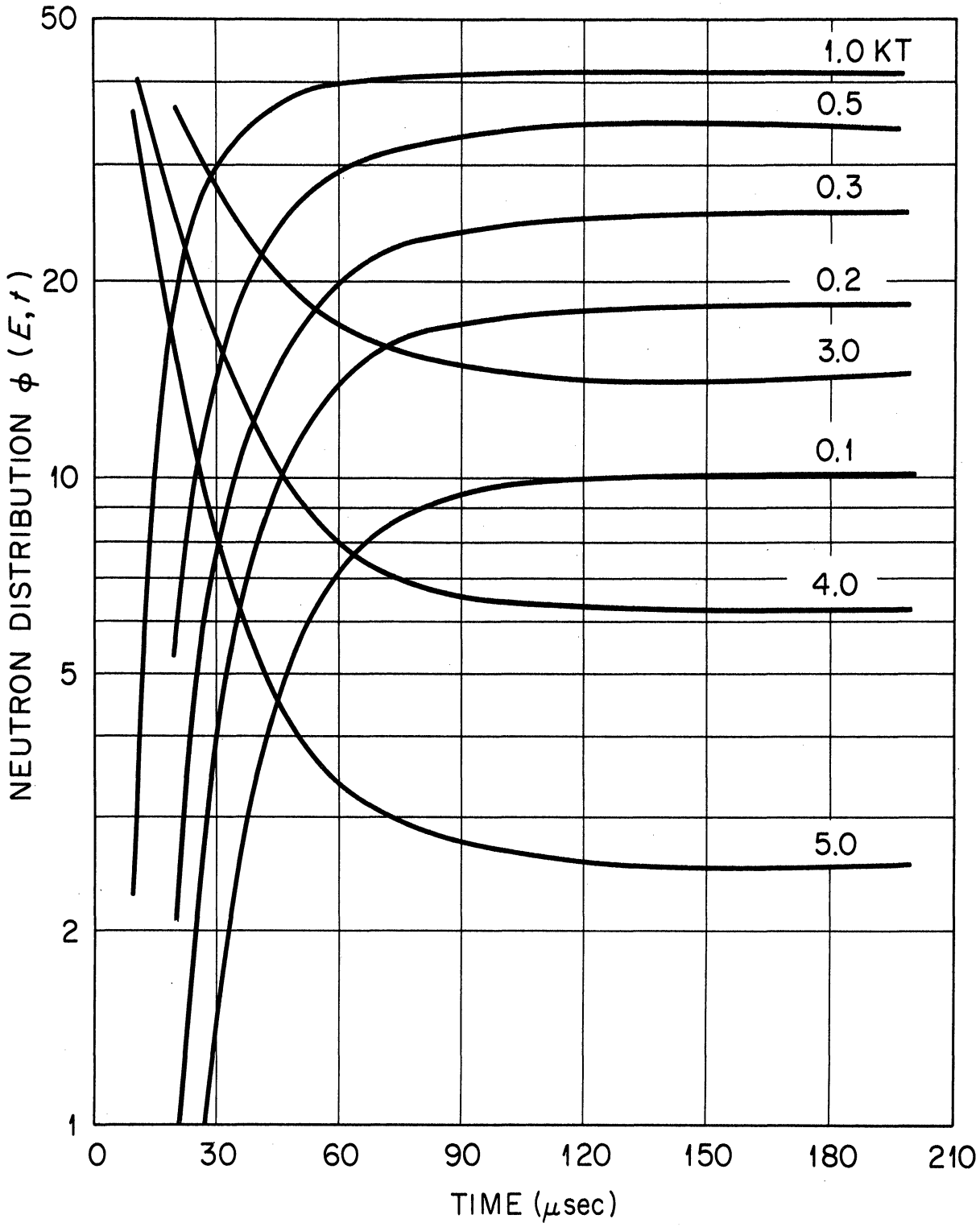

Figure 6.3 Time Behavior of Energy Groups in Beryllium for Infinite Medium and Zero Absorption.

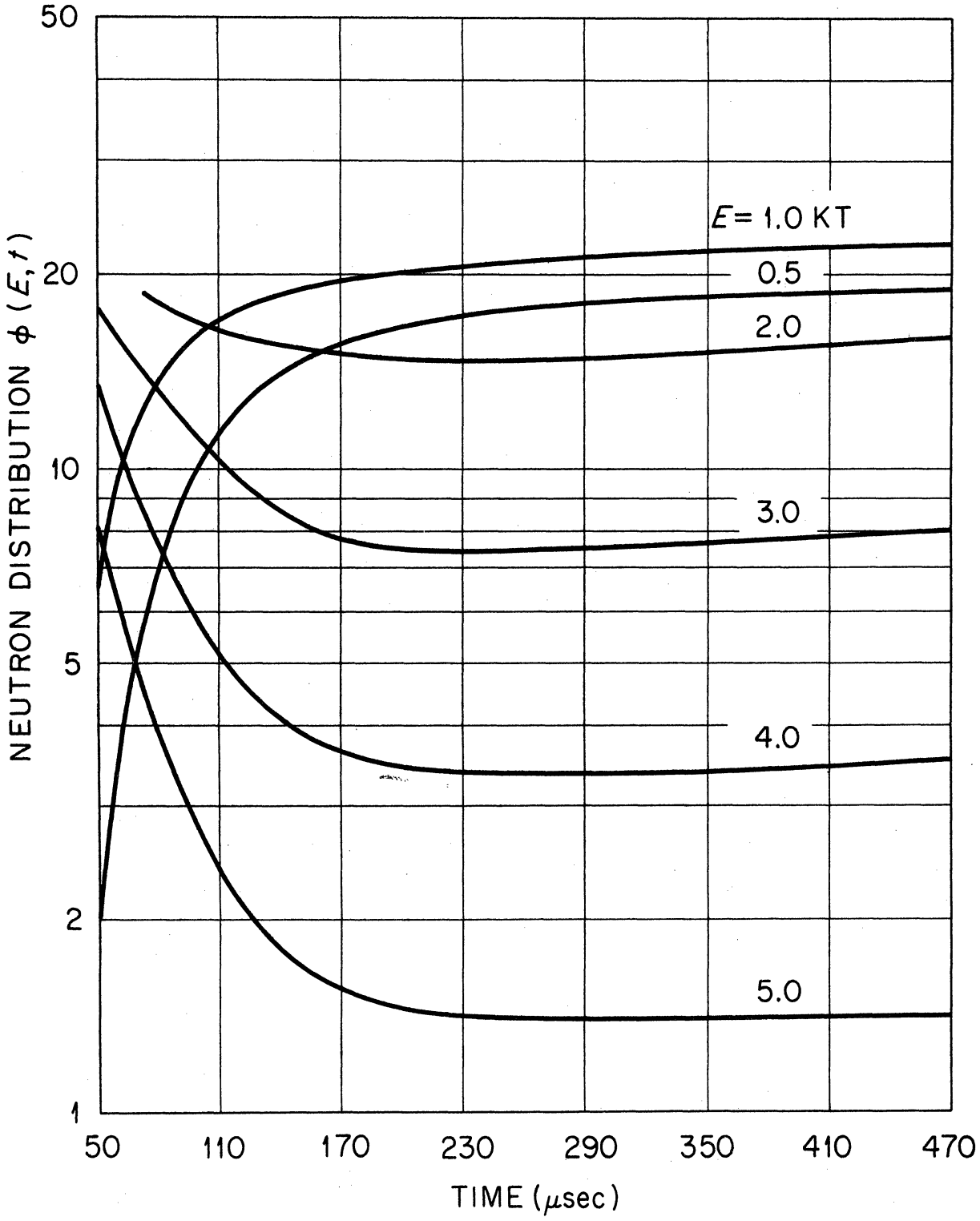

Figure 6.4 Time Behavior of Energy Groups in Graphite for Infinite Medium and Zero Absorption.

than λ_1 is neglected then the distribution function will be represented as follows.

$$\phi(E,t) = \phi_0(E) + \phi_1(E)\, e^{-\lambda_1 t} \qquad (6.9)$$

The first eigen value, λ_1, gives the thermalization time constant. We shall fit our data to the type of distribution given by Equation (6.9). The exponential fit of the distribution function is discussed in detail in a following section of this chapter.

B. The Thermalization Time.

The thermalization time may be defined as the total time required by the source neutrons to have slowed down and to have been completely thermalized, so as to attain the equilibrium distribution and to retain it for all times thereafter. It can be determined with the help of Figures 6.3 and 6.4 for beryllium and graphite, respectively. It may be noted that the precise determination of the thermalization time is not possible. A good estimate, however, based upon the slopes of the curves mentioned can be given. The uncertainty involved is between five to ten percent.

For beryllium, the thermalization time can be taken as equal to 100 micro seconds plus 14 micro seconds, the mean slowing down time defined previously, making the total equal to 114 micro seconds. Kothari and Singwi[27] report the thermalization time in beryllium to be about 145 micro seconds. The time required by the neutrons to acquire the moderator temperature 310K from 2000°K, was calculated by these authors to be about 130 micro seconds.

We estimate the thermalization time to be equal to about 240 micro seconds in the case of graphite. It takes about 200 micro seconds for the thermalization process and 38.6 micro seconds for the slowing down process. Antonov et al.[2] determined the thermalization time to be equal to 200 micro seconds. Beckurts[4] gave the estimate of the thermalization time constant in graphite equal to 180 ± 40 micro seconds. If one were to take the thermalization time constant to be of the same order as the thermalization time, then the results of Beckurts agree with Antonov's results. Both values were obtained experimentally.

The beryllium curves were obtained using twenty energy groups, with the help of the Oak Ridge National Laboratory analog computer. In the case of graphite only fourteen groups were employed. This problem was carried out on the analog computer of the Nuclear Engineering Department of the University of Michigan. It is interesting to note that the estimate for the thermalization times for beryllium and graphite given by the neutron distributions $\varphi(E,t)$ are of the same order as given by the other authors. It may be suspected that the thermalization time is independent of the nature of the scattering kernel and depends upon the mean velocities of the initial and final distributions. This statement is of a speculative nature. The agreement between the results based upon the heavy gas model and the results obtained for the crystalline medium, forces one to conclude the above statement.

C. The Exponential Fit of the Distribution Function $\varphi(E,t)$

We fit the distribution function $\varphi(E,t)$ as a sum of two exponentials. We use the beryllium data in the interval of 60 micro seconds.

$$\phi(E,t) = \phi_0(E)\, e^{-\lambda_0 t} + \phi_1(E)\, e^{-\lambda_1 t} \qquad (6.10)$$

Since λ_0 is very small (around 0.1×10^{-2} sec^{-1}), the above expression reduces to Equation (6.9). The fit was made using Cornell's method.[12] It involved the determination of $\varphi_0(E)$, $\varphi_1(E)$ and λ_1, which correspond to the zeroth eigen function, first eigen function and first eigen value of the analytical method. We record these quantities for each energy group in Table 6.1.

The arithmetic mean of the decay constant λ_1 is equal to 4.75×10^4 sec^{-1} in beryllium. When expressed in terms of $\xi \Sigma_{s_0} v_0$ units, λ_1 is found to be $1.176\ \xi \Sigma_{s_0} v_0$. The reciprocal λ_1, defined as the thermalization time constant, is 21.05 micro seconds in beryllium. For graphite t_{th} is 56.8 μ secs. On the heavy gas model, t_{th} for beryllium given by Kothari and Singwi[27] is 21 μ secs.

The examination of $\varphi_1(E)$ indicates that it has maximum between 2 and 3 KT, at around 2.5 KT, and a minimum at 0.4KT. The zero is at about 1.3 KT. We plot $\varphi_0(E)$ and $\varphi_1(E)$ in Figure 6.5.

From the values of $\varphi_0(E)$; $\varphi_1(E)$ and λ_1 listed in Table 6.1 we can construct neutron distribution function $\varphi(E,t)$ for times greater than 60 μ secs by the following equation.

$$\phi(E,t) = \phi_0(E) - \phi_1(E)\, e^{-\lambda_1(t-60)} \qquad (6.11)$$

TABLE 6.1

NUMERICAL VALUES FOR φ_0 and φ_1 IN A HEAVY GAS MEDIUM OF (BERYLLIUM)

E(KT)	$\varphi_0(E)$	$\varphi_1(E)$	$\lambda_1 \times 10^{-5}$ Sec^{-1}
0.1	10.13	-3.09	0.448
0.2	18.42	-4.79	0.446
0.3	25.14	-5.22	0.456
0.4	30.59	-5.39	0.472
0.5	34.29	-5.03	0.479
0.6	37.10	-4.495	0.476
0.7	38.74	-3.86	0.490
0.8	40.18	-3.12	0.496
0.9	40.74	-2.37	0.518
1.0	41.04	-1.62	0.503
1.5	37.89	1.635	0.439
2.0	29.07	3.122	0.461
3.0	13.44	2.953	0.476
4.0	6.19	1.78	0.483
5.0	2.53	0.903	0.477
6.0	0.957	0.438	0.477

Arithmetic mean of λ_1 equals 0.475×10^5 sec.$^{-1}$

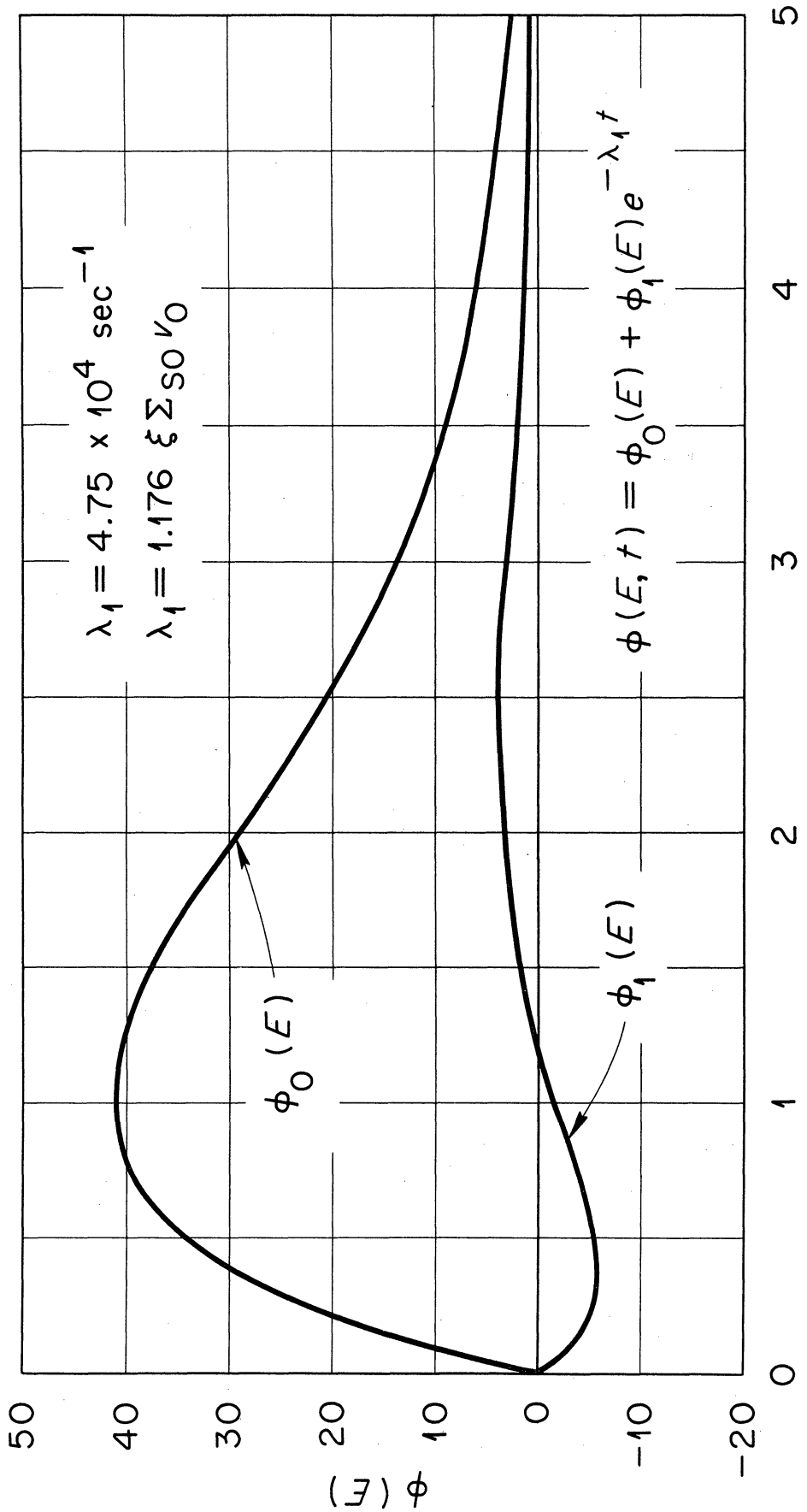

Figure 6.5 Exponential Fit of the Infinite Medium Data for Berryllium.

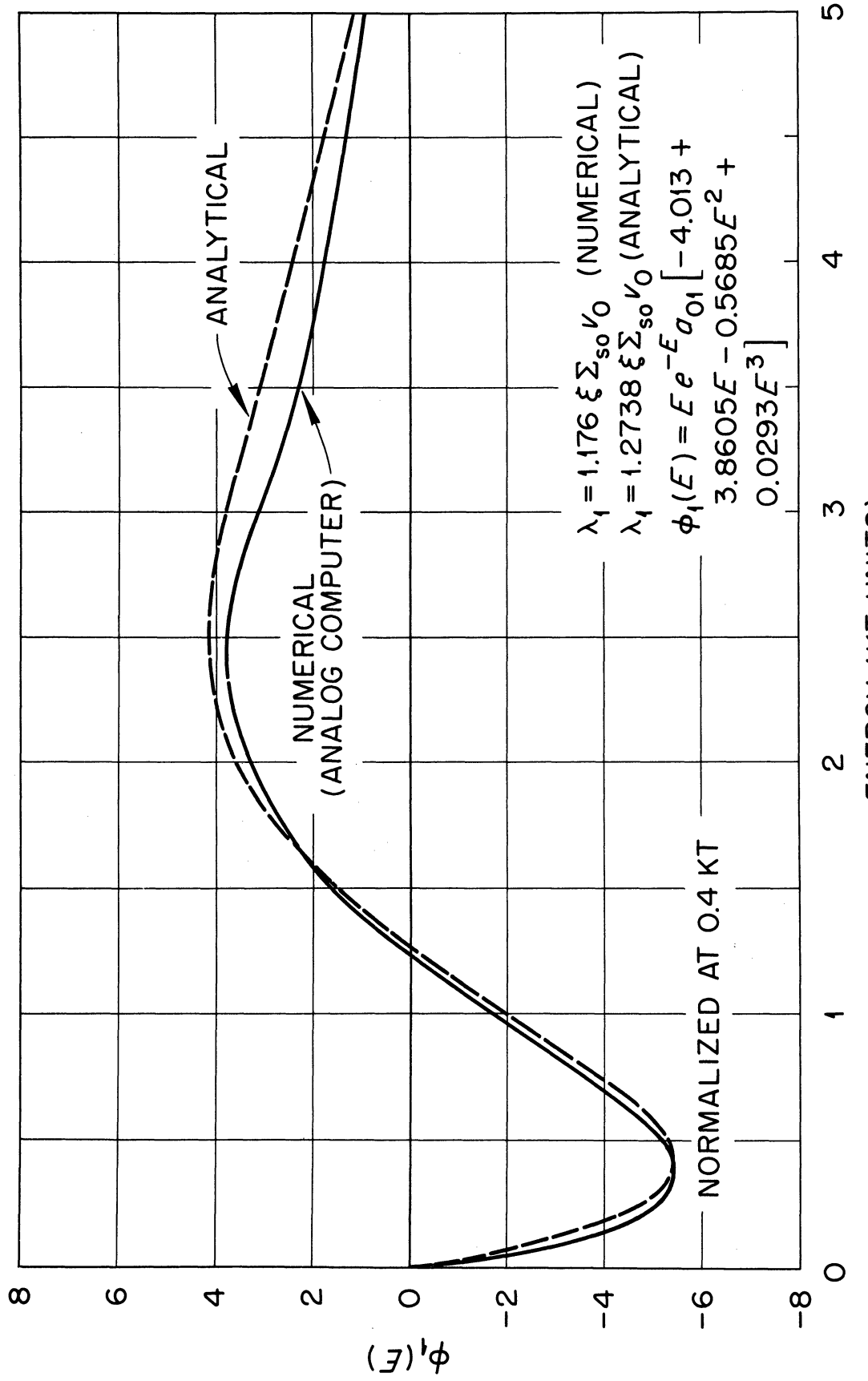

Figure 6.6 Comparison of Numerical and Analytical Eigenfunction $\Phi_1(E)$.

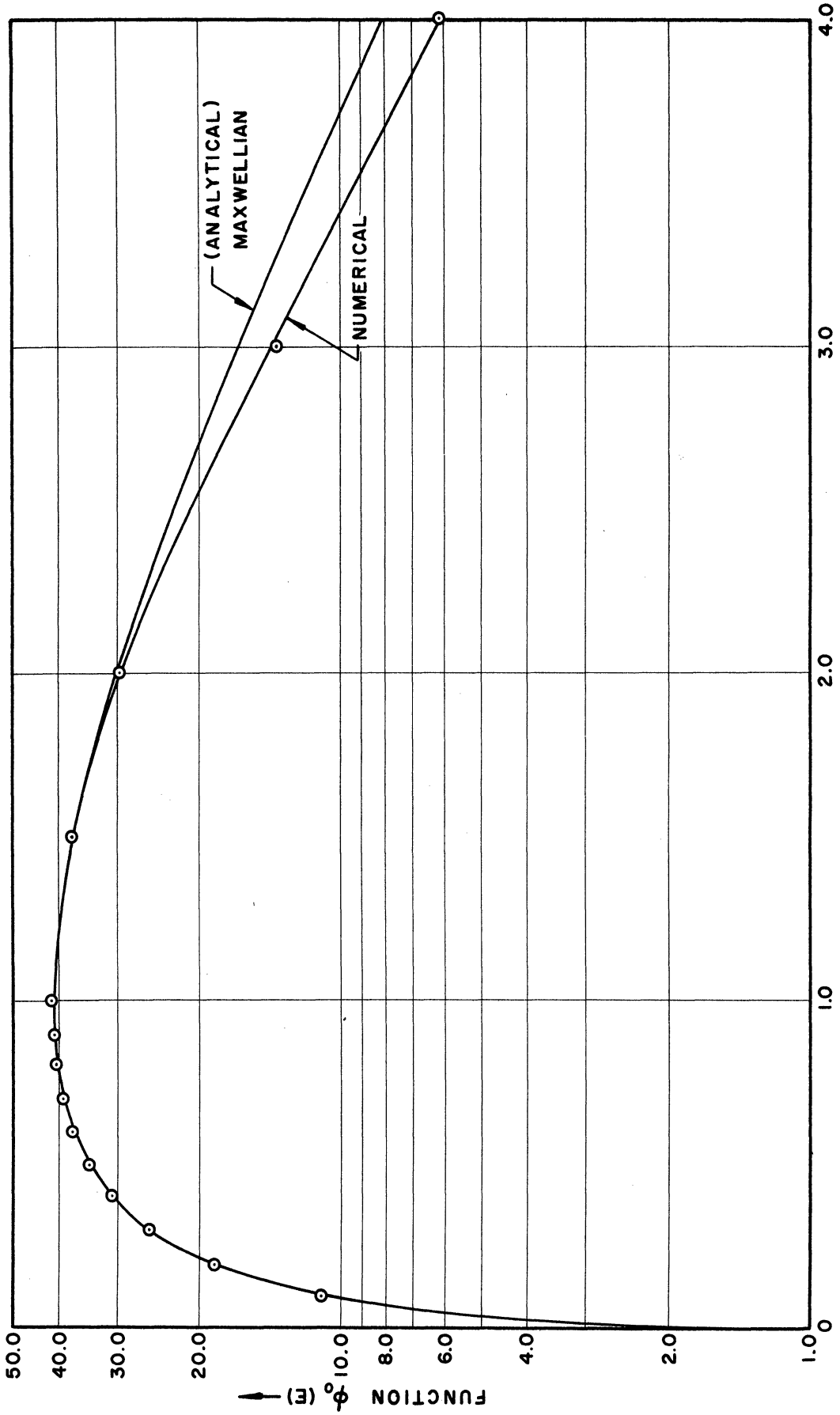

Figure 6.7 Comparison of Numerical and Analytical $\Phi_o(E)$.

D. Comparison Between the Analytical and Numerical Results.

The analytical results obtained in the fifth chapter for the heavy gas model, by the eigen value method, are compared with the numerical results determined in the previous section.

$1.176\ \xi\Sigma_{s_0}\sigma_0$ Numerical result for λ_1 by the analog computer (6.12)

$1.2738\ \xi\Sigma_{s_0}\sigma_0$ Analytical result for λ_1 by the eigen value method

The difference between these two results is about eight percent. The analytical value will decrease when more than four Laguerre polynomials are used, thus bringing the two values closer together. In Figure 6.6, we compare $\varphi_1(E)$ obtained by the two different methods. The agreement is very good on the low energy side, but is only fair on the high energy side. The analytical curve will converge to the numerical curve when a large number of Laguerre polynomials are used to describe the first eigen function. We normalize the analytical and numerical distributions, $\phi_1(E)$, at 0.4 KT. The numerical curve has a minimum at 0.4 KT, a maximum at about 2.5 KT and zero at 1.23 KT, whereas the analytical curve has a minimum at 0.4 KT, a maximum at about 2.5 KT and zero at 1.27 KT.

In addition, we also compare $\varphi_0(E)$ given by the analytical and numerical methods, in Figure 6.7. The analytical method gives $\varphi_0(E)$ equal to a pure Maxwellian distribution. We use the values of $\varphi_0(E)$ as listed in Tables 5.3 and 6.1 for purposes of comparison.

III. The Infinite Medium 1/v Absorber

It was pointed out in the fourth chapter that the asymptotic energy spectrum is Maxwellian only in the presence of a small 1/v absorber, and not for a large absorption. We give the ratio of λ_0 and λ_1 for the various values of absorption, for the heavy gas model, in Table 6.2. λ_0 and λ_1 are given according to the following equations.

$$\lambda_0 = \Sigma_{a_0} v_0 = \Delta \xi \Sigma_{s_0} v_0 \tag{6.13}$$

$$\lambda_1 = (\Delta + 1.2738) \xi \Sigma_{s_0} v_0 \tag{6.14}$$

TABLE 6.2

THE VALUES OF $\frac{\lambda_1}{\lambda_0}$ FOR DIFFERENT ABSORPTIONS

Δ	$\frac{\lambda_0}{\lambda_1}$
0.0	∞
0.1	13.74
0.4	4.185
1.0	2.27
5.0	1.2548
10.0	1.12738

We conclude from Table 6.2, that λ_1 is of the same order as λ_0 for Δ larger than 1.0. The importance of this result has been discussed in Chapter IV.

The neutron distribution function $\varphi(E,t)$ has been generated for various values of the absorption. It was not possible to obtain the distribution for Δ larger than 0.4. This was due to the rapid decay of the spectrum. In Figure 6.8 we plot the energy spectrum at 300 micro seconds, for the different values of absorption. The spectra were normalized for E = KT. All the cases have the same spectrum except for the case $\Delta = 0.4$. The agreement is very good on the low energy side, but not on the high energy side. The neutron distribution for high energy is so small that the uncertainty of the computer data becomes of the order of 15 to 25 percent of the actual value. The deviation on the high energy side may, therefore, be neglected for Δ equals 0.4.

The decay constants calculated from the distribution function for the various values of Δ are given in Table 6.3.

TABLE 6.3

THE DECAY CONSTANTS FOR DIFFERENT ABSORPTIONS IN THE INFINITE MEDIUM

Δ	λ_1 (sec^{-1})	λ_0 (sec^{-1})
0.08	2.850×10^3	3.56×10^4
0.10	3.572×10^3	3.57×10^4
0.20	7.185×10^3	3.59×10^4
0.40	14.275×10^3	3.57×10^4

The constancy of λ_0 indicates the decay rate to be proportional to the amount of absorption.

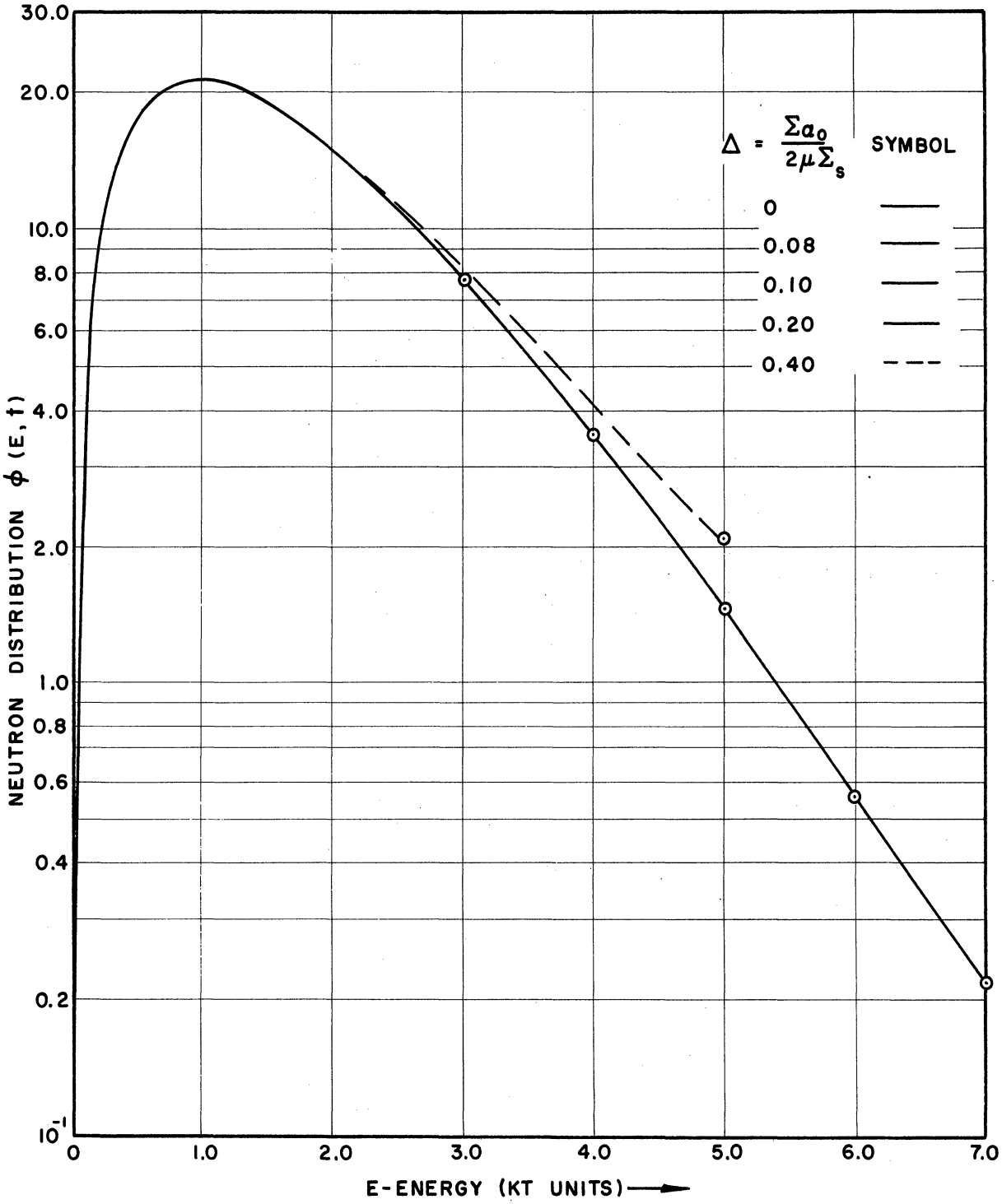

Figure 6.8 Neutron Energy Spectrum at 300 μ secs in Infinite Medium of Graphite for Different Amounts of Absorptions.

CHAPTER VII. THE TIME DEPENDENT ENERGY SPECTRUM IN FINITE MEDIUM

The study of the time dependent energy spectra in the finite media assumes special significance in the light of pulsed neutron experiments. In order to understand and interpret the results obtained from these experiments, it is essential to study the characteristics of the energy spectra existing in the medium at different times. The experimentalist measures the decaying spectrum (which is integrated over all energies) as a function of time. We attempt to simulate the physics of the experiment by determining the energy spectrum as a function of time. With the help of the multigroup formalism introduced in the last chapter, we generate the distribution function $\varphi(E,t)$. We deal with two cases. One is the constant diffusion coefficient, graphite case and the second is the energy dependent diffusion coefficient beryllium case. We use the heavy gas model for the treatment of these cases.

I. Constant Diffusion Coefficient, Graphite Case

The assumption of the constant diffusion coefficient (D) for graphite is very good. It follows from the constancy of the scattering cross section (4.8 ± 0.2 barns) in the energy interval between 0.005 ev. and 1 Mev.[56] The Bragg cut off for graphite occurs at 0.0015 ev. We may further assume that the absorption cross section is very small, since it is about 0.5 ± 0.2 millibarns. It may therefore be neglected.

The energy spectra, as pointed out earlier, are based upon the determination of the energy transfer in the heavy gas model. The spatial distribution of neutrons is assumed to be given by the fundamental spatial

mode. The higher spatial modes have been assumed to have small contributions, compared to the fundamental one. The time behavior of the neutron energy spectra for a few geometrical bucklings have been obtained.

In Figure 7.1, we give the spectra for the case where the geometrical buckling, B^2, is equal to 18.5×10^{-3} cm^{-2}. The time behavior of the individual energy groups has been plotted in Figure 7.2. We find that the energy spectrum decays with a single decay constant after 250 to 300 micro seconds after the introduction of the source pulse. The existence of the same decay constant, within two to three percent, for each energy group, indicates the presence of an equilibrium spectrum. The following characteristics of the equilibrium spectra have been studied in detail: the equilibrium spectrum; the average speed, $\overline{E^{1/2}}$; and the decay constant λ.

A. The Equilibrium Spectrum

The normalized equilibrium spectra for the following geometrical bucklings, B^2 equal to zero, 2.96×10^{-3}, 4.62×10^{-3}, 8.21×10^{-3} and 18.5×10^{-3} cm^{-2}, are recorded in Table 7.1. For $B^2 = 0$ and $B^2 = 18.5 \times 10^{-3}$ cm^{-2} equilibrium spectra are plotted in Figure 7.3. We note from these curves that the energy spectrum shifts to the low energy side, with increase in the geometrical buckling, B^2. The maximum of the spectrum for B^2 equal to zero occurs at 1.0 KT, while for the case of B^2 equal to 18.5×10^{-3} cm^{-2}, the maximum appears at 0.9 KT. We have determined the average speed, $\overline{E^{1/2}}$, for different geometrical bucklings.

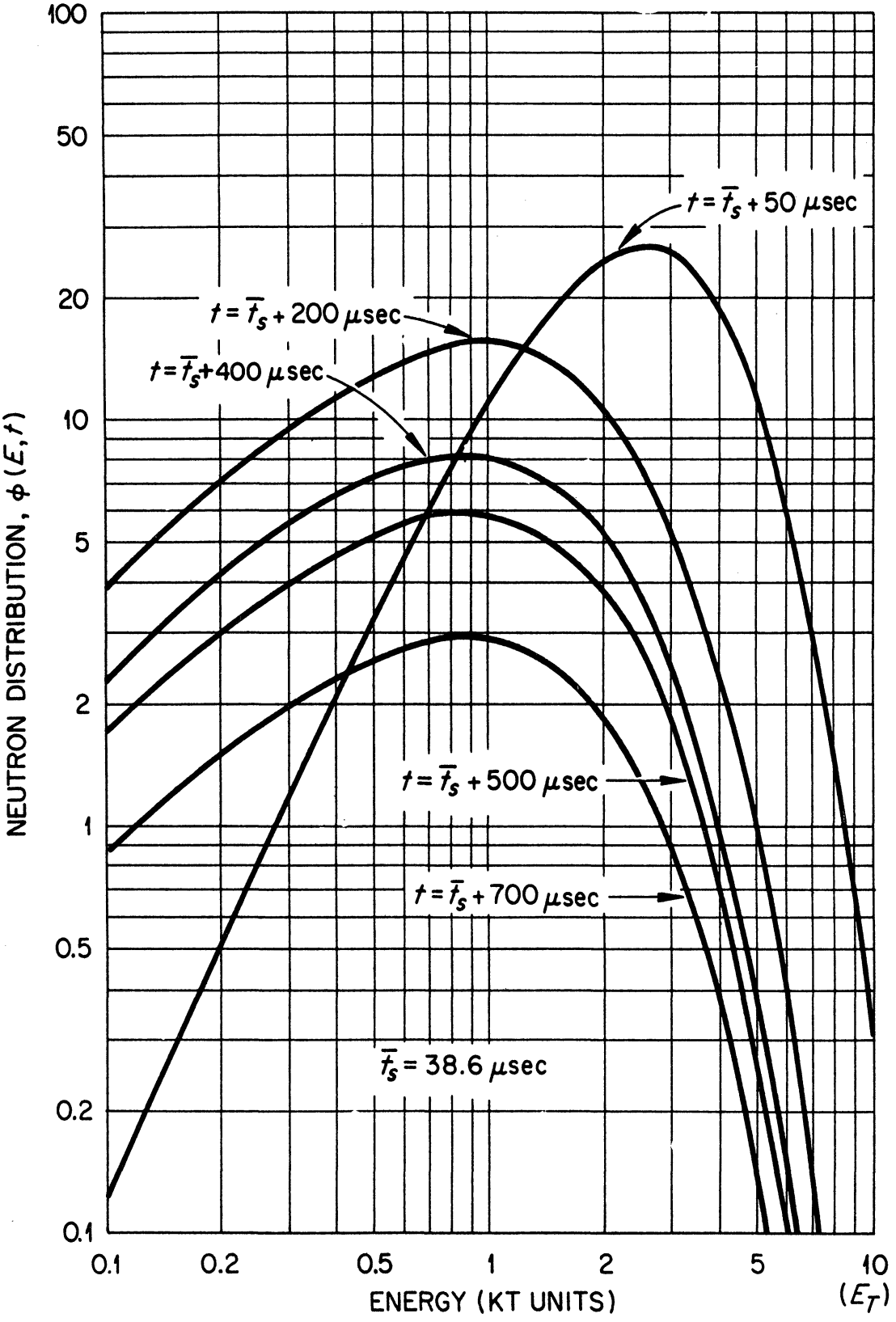

Figure 7.1 Time Behavior of Neutron Energy Spectrum in Finite Medium of Geometrical Buckling $B^2 = 18.5 \times 10^{-3} cm^{-2}$ in Graphite.

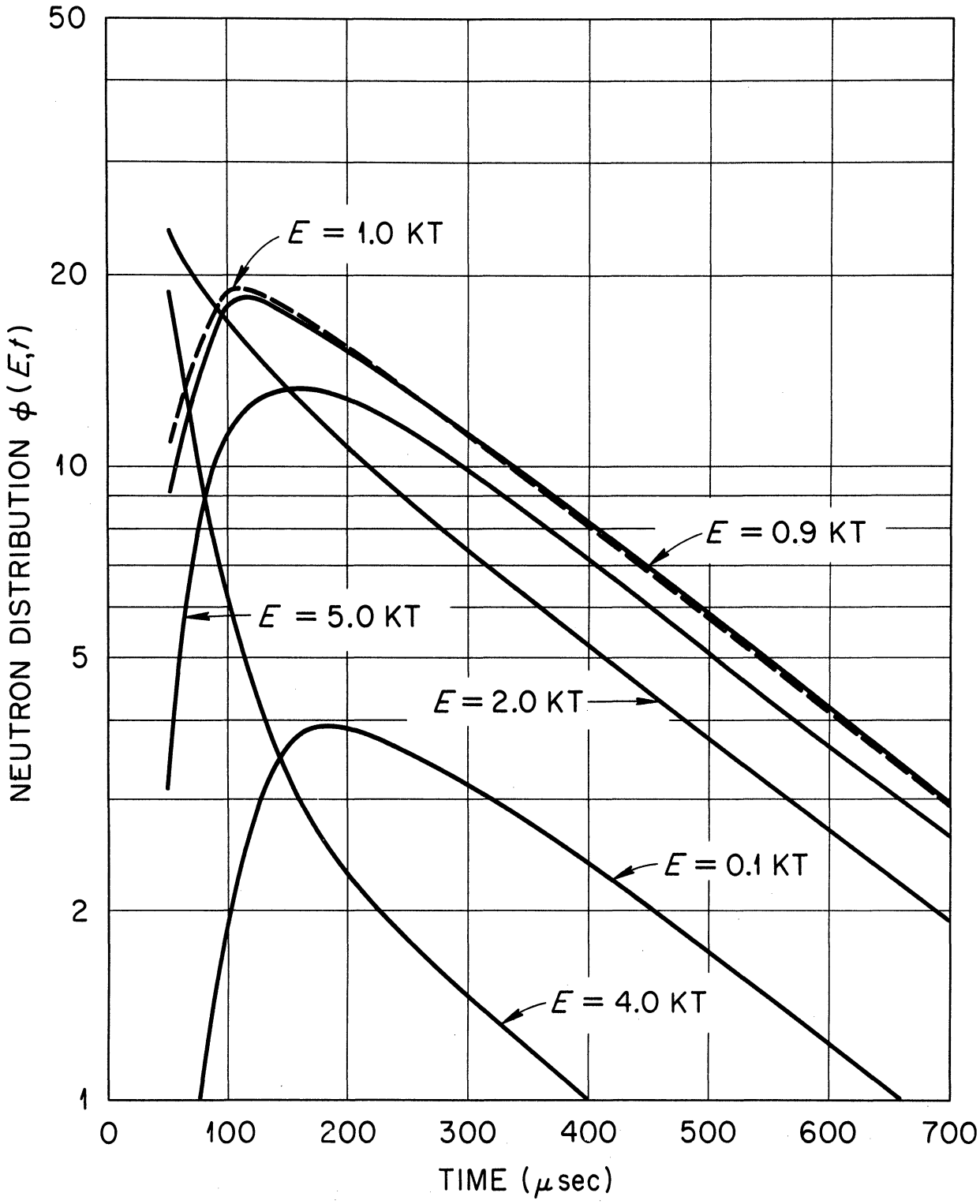

Figure 7.2 Time Behavior of Each Energy Group in the Finite Medium of Graphite of Geometrical Buckling $B^2 = 18.5 \times 10^{-3} cm^{-2}$.

TABLE 7.1

EQUILIBRIUM ENERGY DISTRIBUTION (GRAPHITE)

E (KT)	$B^2 = 0$	$B^2 = 2.96 \times 10^{-3}$ (cm^{-2})	$B^2 = 4.62 \times 10^{-3}$ (cm^{-2})	$B^2 = 8.21 \times 10^{-3}$ (cm^{-2})	$B^2 = 18.5 \times 10^{-3}$ (cm^{-2})
0.1	9.80	10.38	11.00	11.65	11.86
0.2	17.94	18.50	18.60	20.67	21.03
0.3	24.63	29.39	26.00	27.26	27.63
0.4	30.00	30.93	32.00	32.31	32.75
0.5	33.83	34.88	34.95	35.86	36.30
0.6	36.70	37.31	37.50	38.42	38.64
0.7	38.65	39.20	39.40	39.97	40.20
0.8	39.85	40.19	40.40	40.80	40.85
0.9	40.60	40.69	40.90	41.04	41.11
1.0	40.91	40.91	40.91	40.91	40.91
1.5	36.68	36.13	35.97	34.65	34.83
2.0	29.38	28.70	28.52	27.62	26.66
3.0	14.49	14.16	13.99	13.58	12.57
4.0	6.35	6.26	6.17	5.96	5.16
5.0	2.62	2.57	2.56	2.49	2.08
6.0	1.00	1.00	1.00	0.98	0.76
7.0	0.35	0.35	0.35	0.30	0.26
8.0	0.10	0.10	0.10	0.10	0.10
9.0	0.03	0.03	0.03	0.03	0.03
10.0	-	-	-	-	-

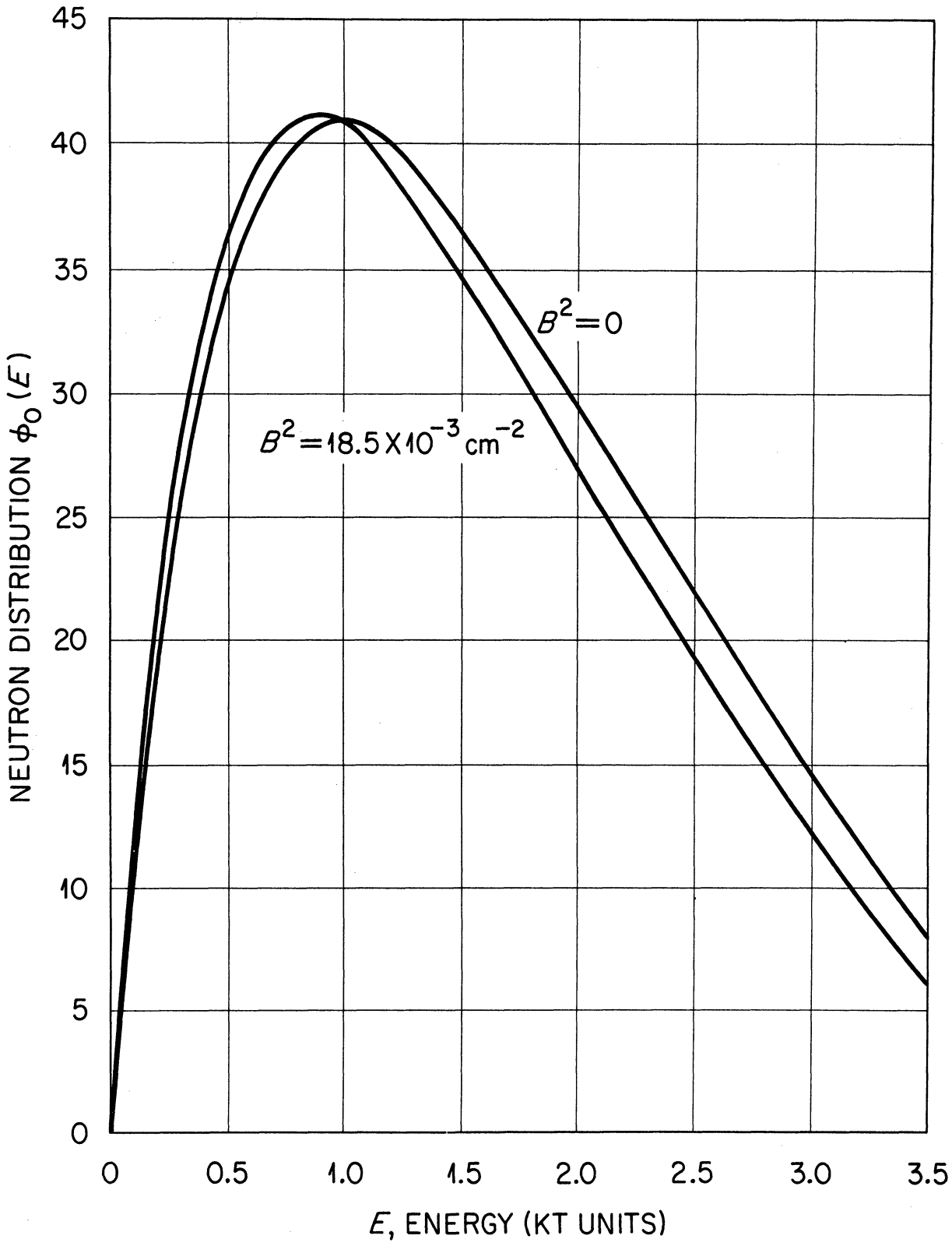

Figure 7.3 Equilibrium Neutron Energy Spectra in Graphite for Geometrical Bucklings. $B^2 = 0$ and $B^2 = 18.5 \times 10^{-3} cm^{-2}$.

B. Average Speed, $\overline{E^{1/2}}$

The average speed in a medium having the equilibrium energy spectrum $\varphi_o(E)$ can be given as follows.

$$\overline{E^{1/2}} = \frac{\int_0^{E_0} E^{1/2} n_o(E) dE}{\int_0^{E_0} n_o(E)\, dE} \tag{7.1}$$

$$E^{1/2} n_o(E)\, dE = \phi_o(E)\, dE \tag{7.2}$$

In Table 7.2 the average speed is given for a few geometrical bucklings.

TABLE 7.2

AVERAGE SPEED FOR VARIOUS VALUES OF B^2 (GRAPHITE)

B^2 (cm^{-2})	Average Speed*
0	1.0914
2.96×10^{-3}	1.0820
4.62×10^{-3}	1.0789
8.21×10^{-3}	1.0655
18.50×10^{-3}	1.0513

* in units of v_o = speed for most probable energy (KT).

For the Maxwellian distribution, $\overline{E^{1/2}}$ can be given exactly:

$$\overline{E^{1/2}} = 1.129 \tag{7.3}$$

When compared to the case where B^2 equals zero, in Table 7.2, we discover that the difference between the two values is about 3.30 percent, which is due to numerical errors in the solution.

The average speed given in Table 7.2 characterizes the equilibrium spectrum. With an increase in geometrical buckling, the average speed decreases.

The data given for $\overline{E^{1/2}}$ in Table 7.2 has been fitted by the least square method in the following manner.

$$\overline{E^{1/2}} = 1.088 \left[1 - 1.852 B^2\right] \tag{7.4}$$

Beckurts[5] also obtained the expression for $\frac{\bar{v}}{v_0}$ on the heavy gas model by the iteritive process assuming that the equilibrium energy spectrum exists. His results are as follows:

$$\overline{E^{1/2}} \simeq 1.129 \left\{1 - \frac{\Gamma}{25} B^2\right\} \qquad \text{Beckurts' equation}^{(5)} \tag{7.5}$$

$$\Gamma = \frac{D}{\xi \Sigma_{s0}} \tag{7.6}$$

Transforming Equation (7.4) in an analogous manner, we obtain:

$$\overline{E^{1/2}} = 1.088 \left\{1 - \frac{\Gamma}{27.3} B^2\right\} \tag{7.7}$$

$$\frac{\Gamma}{27.3} = 1.852 \tag{7.8}$$

The expression for $\bar{v}/v_0$ given in this study has been obtained after establishing the existence of an equilibrium energy spectrum during the last stage of the decay of the neutron pulse.

C. Average Energy

The average energy of the neutrons can be defined as follows

$$\bar{E} = \frac{\int_0^{E_0} E\, n_0(E)\, dE}{\int_0^{E_0} n_0(E)\, dE} \tag{7.9}$$

The values for the average energy are given in Table 7.3.

TABLE 7.3

AVERAGE ENERGY

B^2 (cm^{-2})	Average Energy* (E)
0	1.4176
2.96×10^{-3}	1.3727
4.62×10^{-3}	1.3659
8.21×10^{-3}	1.3410
18.50×10^{-3}	1.2960

* in units of KT.

The least square fit for $\bar{E}$ versus B^2 gives

$$\bar{E} = 1.425 \left[1 - 5.615 B^2 \right] \qquad (7.10)$$

For a Maxwellian, $\bar{E} = 1.5$ KT, so again there is an error, this time $\sim 5\%$ in the $B^2 = 0$ value. However, it is felt that the relative values may be accurately expressed by Table 7.3.

D. 'Neutron Temperature'

The concept of neutron temperature, though useful in characterizing the spectrum, is not rigorous, except in one case. This case is the one of infinite medium and zero absorption, where the final energy distribution of the neutrons is Maxwellian. For this case, the moderator temperature T_o may be taken equal to the neutron temperature T_n for the neutron gas. For the sake of convenience, we extend this concept to the cases of non-Maxwellian distributions. For the Maxwellian distribution, $\bar{E}$ is equal to 3/2 KT_o and is

also equal to 3/2 KT_n for ($B^2 = 0$). Extending this to the non-Maxwellian distribution, we get

$$\frac{(\bar{E})_{B^2 \neq 0}}{(\bar{E})_{B^2 = 0}} = \frac{T_n (B^2 \neq 0)}{T_n (B^2 = 0)} \qquad (7.11)$$

With the help of Table 7.3 we calculate the percentage change in the neutron temperature $\delta T_n / T_o$ for different values of B^2. These values are listed in Table 7.4.

TABLE 7.4

NEUTRON TEMPERATURE

B^2 (cm^{-2})	(K°)
0	0
2.96×10^{-3}	-3.17 %
4.62×10^{-3}	-3.65 %
8.21×10^{-3}	-5.40 %
18.50×10^{-3}	-8.58 %

E. Decay Constant

We have calculated the decay constants from the time dependent energy spectrum curves. After about 250 to 300 micro seconds, there exists an equilibrium spectrum for all cases of the geometrical bucklings. The neutron energy spectrum decays thereafter. In Table 7.5 we list the decay constants.

TABLE 7.5

DECAY CONSTANTS

B^2 (cm^{-2})	(sec^{-1})
0	0
2.05×10^{-3}	4.84×10^{2}
2.96×10^{-3}	7.08×10^{2}
3.65×10^{-3}	9.00×10^{2}
4.62×10^{-3}	10.16×10^{2}
6.04×10^{-3}	13.83×10^{2}
8.21×10^{-3}	17.94×10^{2}
18.5×10^{-3}	34.40×10^{2}

The uncertainty involved in the determination of the decay constants, for all cases except the last one, was within 3 percent. In the last case, the decay was very fast due to the small size of the assembly (about 40 cm^3). The uncertainty in the determination of λ was more than 5 percent. This was due to the time lag between starting the computer and the stop watch. The time scale was 10 micro seconds equals about 2.6 seconds.

It must be pointed out that the error involved in the determination of λ has to be minimized by obtaining the time measurements accurately. In the case of small assemblies, the decay is so fast that the uncertainty in the value of λ becomes very large. In Beckurts' experiments on graphite, the smallest size assembly was of B^2 equal to 8.21×10^{-3} cm^{-2}.

We have made a least square fit using the first six values from Table 7.5 in the range of bucklings $\leq 8.21 \times 10^{-3}$ cm^{-3}.

$$\lambda = 2.29 \times 10^{5} B^{2} [1 - 2.068 B^{2}] \qquad (7.12)$$

II. Energy Dependent Diffusion Coefficient (Beryllium) Case.

In the case of beryllium, Bhandari[7] calculated the transport mean free path as a function of energy. The diffusion coefficient for beryllium is taken to be energy dependent as given by Bhandari. The time dependent neutron energy spectra are generated for a few geometrical bucklings. In Figure 7.4, the time behavior of the neutron energy spectrum is given for B^2 equal to 7.18×10^{-2} cm^{-2}. The behavior of individual energy groups is plotted in Figure 7.5. The parallel lines indicate the same decay constant for all energy groups. After about 100 micro seconds plus the slowing down time, there does exist an equilibrium spectrum. The neutron energy spectrum maintains its shape, but decays with a single decay constant λ_o equal to 8.39×10^{3} sec.$^{-1}$. We have studied the characteristics of the equilibrium spectrum for a few geometrical bucklings.

A. Equilibrium Spectrum

For the sake of comparison, we plot the equilibrium spectra for B^2 equal to zero and B^2 equal to 7.18×10^{-2} cm^{-2} in Figure 7.6. The energy spectrum shifts toward the low energy side. The shift is small compared to the case of graphite. We calculate the average speed associated with the above spectra.

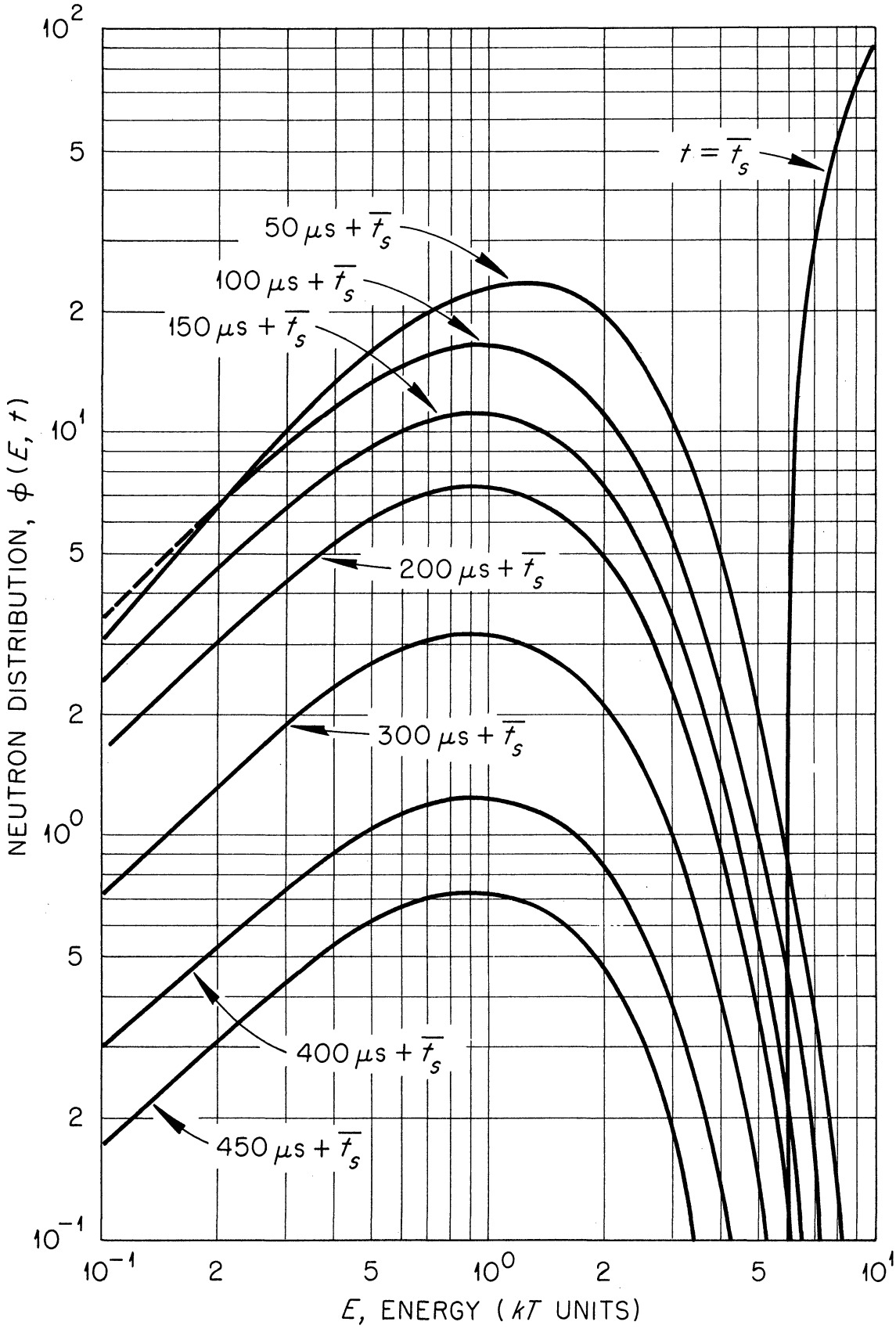

Figure 7.4 Time Behavior of Neutron Energy Spectrum in Finite Medium of Beryllium of Geometrical Buckling $B^2 = 7.18 \times 10^{-2}$ cm^{-2}.

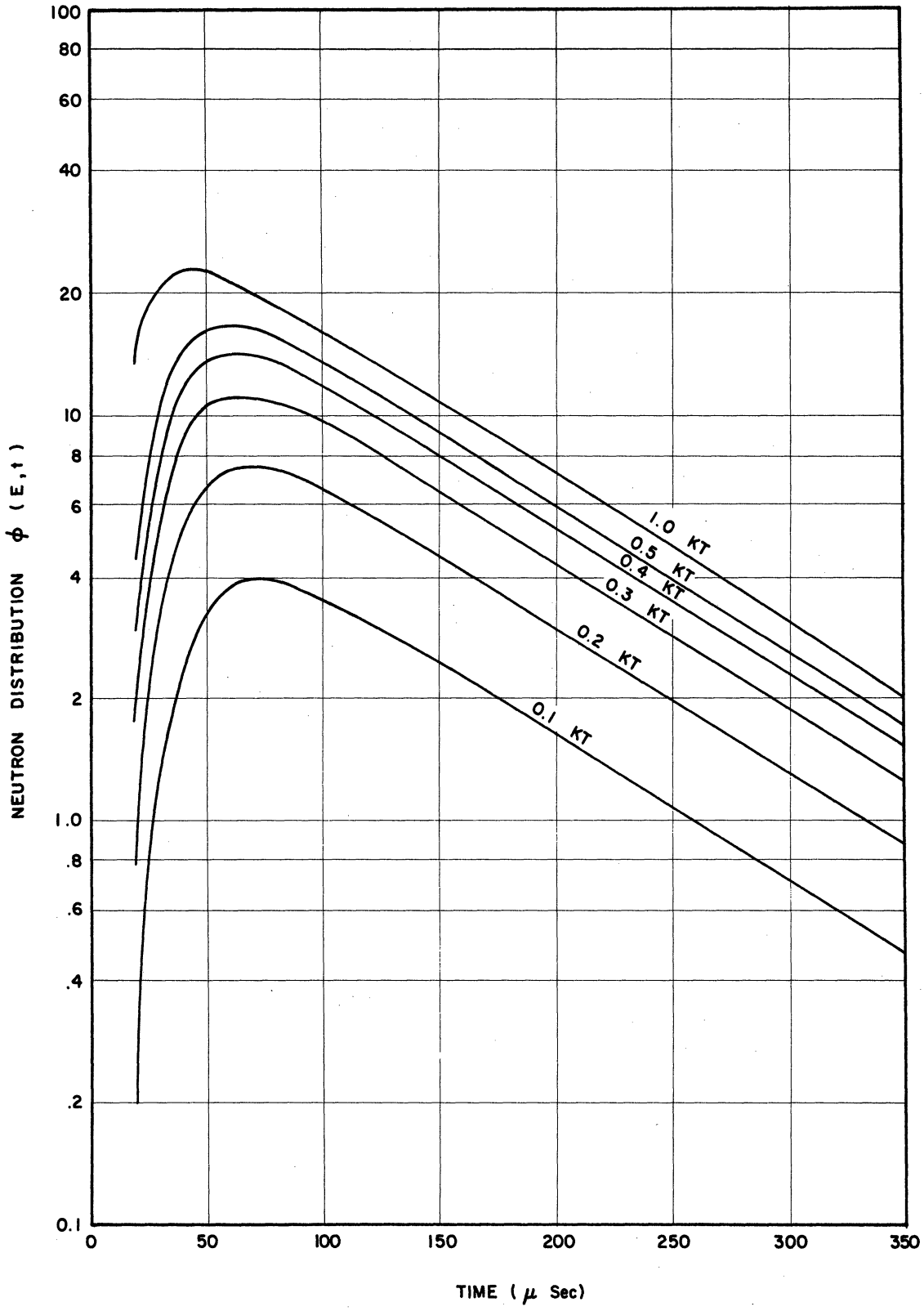

Figure 7.5 Time Behavior of Each Energy Group in the Finite Medium of Beryllium of Geometrical Buckling $B^2 = 7.18 \times 10^{-2} cm^{-2}$.

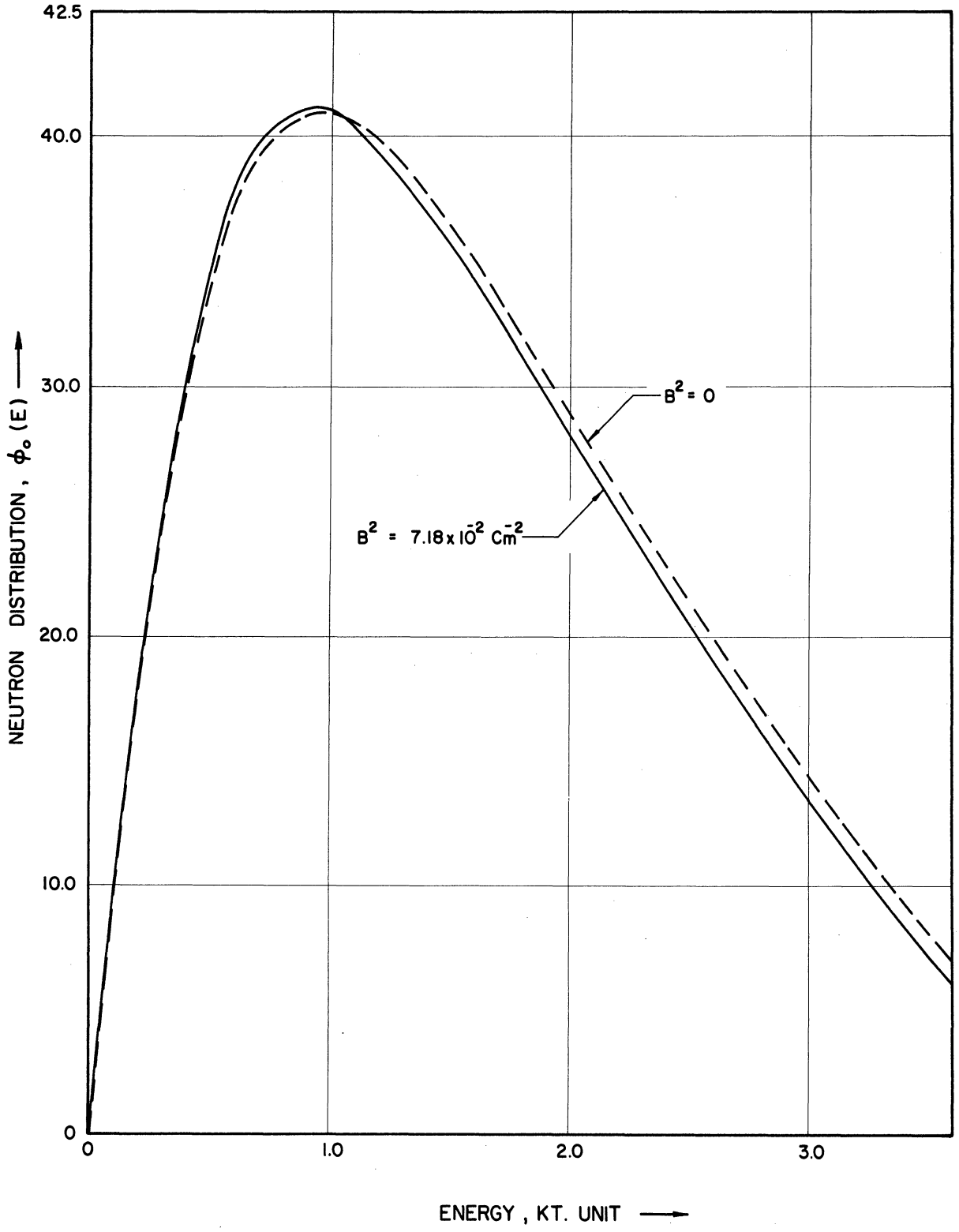

Figure 7.6 Equilibrium Neutron Energy Spectra in Beryllium for Geometrical Buckling $B^2 = 0$ and $B^2 = 7.18 \times 10^{-2} cm^{-2}$.

B. Average Speed $\bar{v}/v_o$ or $\overline{E^{1/2}}$

We have already defined the average speed by Equation (7.1). In Table 7.6 we record the values for the average speed.

TABLE 7.6

B^2 cm^{-2}	Average Speed
0	1.106
2.94×10^{-2}	1.0941
5.36×10^{-2}	1.0919
7.18×10^{-2}	1.0586

The least square fit gives

$$\overline{E^{1/2}} = 1.1099 \left[1 - 0.5179\, B^2\right] \quad (7.13)$$

The difference between the average speed for the Maxwellian distribution and for the case of B^2 equal to zero, from the Equation (7.13) is about 1.7%.

The coefficient of B^2 for graphite from the Equation (7.4) is -1.852 compared to -0.5179 for beryllium from Equation (7.11). We can, therefore, conclude that the shift in the energy spectrum toward the low energy side is larger for graphite than for beryllium.

C. Neutron Temperature

The neutron temperature can be estimated from Equation (7.13). The average speed $\overline{E^{1/2}}$ is proportional to the square root of the neutron temperature, if one assumes the Maxwellian distribution. With this

assumption, we have

$$\left(\frac{T_n}{T_o}\right)^{\frac{1}{2}} = 1.1099\ [1 - 0.5179\ B^2] \qquad (7.14)$$

where T_o is the moderator temperature.

For the smallest assembly of B^2 equal to $7.18 \times 10^{-2} cm^{-2}$ used by de Saussure, the change in neutron temperature is about -7.27%.

D. Decay Constant λ

The value of the decay constant for a few geometrical bucklings has been obtained. These values are given in Table 7.7.

TABLE 7.7

DECAY CONSTANT (BERYLLIUM)

B^2 (cm^{-2})	λ (sec^{-1})
2.94×10^{-2}	3.64×10^3
3.96×10^{-2}	4.53×10^3
5.36×10^{-2}	6.86×10^3
7.18×10^{-2}	8.39×10^3

The least square fit of λ versus B^2 gives the following expression for λ.

$$\lambda = 1.267 \times 10^5\ [1 - 0.890\ B^2\ ]\ B^2 \qquad (7.15)$$

III. Diffusion Cooling Coefficient

According to Equation (2.35) the decay of a pulse of neutrons in a non-absorbing finite medium, can be given by the following decay constant.

$$\lambda = \overline{Dv}\ B^2 \qquad (7.16)$$

(Dv) in the above equation is averaged over the energy distribution of the decaying pulse in the medium. We expand $\overline{Dv}$ as follows:

$$\overline{Dv} = (\overline{Dv})_o [1 - CB^2] \tag{7.17}$$

where $(\overline{Dv})_o$ is averaged over the Maxwellian distribution,

C determines the deviation from the Maxwellian spectrum

If the diffusion coefficient is constant, then $(\overline{Dv})_o$ is equal to $(\overline{D})_o(\overline{v})_o$. We shall determine the diffusion cooling coefficient C using the following two equations for $\bar{v}$ and λ.

$$\lambda = (Dv)_o [1 - CB^2] B^2 \tag{7.18}$$

$$\bar{v} = (\overline{v})_o [1 - CB^2] \tag{7.19}$$

In the case of a constant diffusion coefficient, the two values of C must be equal. We shall discuss in brief the results for graphite and beryllium.

A. Graphite (Constant Diffusion Coefficient)

From two methods, we get the following values of C for graphite, from Equations (7.4) and (7.12).

$$\begin{aligned} C &= 1.852 \quad \text{from } \bar{v} \\ C &= 2.068 \quad \text{from } \lambda \end{aligned} \tag{7.20}$$

These two values differ by almost 10 percent. We observe that for $(\bar{v})_o$ and $(\overline{Dv})_o$ we get 1.088 and 2.29×10^5 $cm^2/sec.$ instead of 1.129 and 2.13×10^5 $cm^2/sec.$, respectively. If we make corrections of these orders to the diffusion cooling coefficients given in Equation (7.20), we obtain C as equal to 1.924 (from λ), and C equal to 1.922 (from $\bar{v}$). We shall consider C equal to 1.922 for graphite.

B. Beryllium (Energy Dependent Diffusion Coeffient).

For beryllium we get the following values of C from Equations (7.13) and (7.15).

$$\begin{aligned} C &= 0.518 \quad \text{from } \bar{v} \\ C &= 0.890 \quad \text{from } \lambda \end{aligned} \tag{7.21}$$

We shall compare the values of C obtained by Hurwitz and Nelkin[21] and by Häfele and Dresner[19] to our value. We use the following values for the constants.

$$\Sigma_{s_0}(Be) = 8.28 \times 10^{-1} \text{ cm}^{-1}$$
$$\Sigma_{s_0}(C) = 4.1 \times 10^{-1} \text{ cm}^{-1}$$
$$\xi(Be) = 0.222$$
$$\xi(C) = 0.16667$$

Based upon heavy gas model

$$(\overline{DU})_0(Be) = 1.26 \times 10^5 \text{ cm}^2/\text{sec}$$
$$(\overline{DU})_0(C) = 2.13 \times 10^5 \text{ cm}^2/\text{sec}$$

We use the following formulae, obtained under the constant diffusion coefficient assumption.

$$C_{H.N} = \frac{(\overline{DU})_0 (0.148)}{\xi \Sigma_{s_0} U_0} \tag{7.22}$$

$$C_{H.D} = \frac{(\overline{DU})_0 (0.147)}{\xi \Sigma_{s_0} U_0} \tag{7.23}$$

Values obtained for graphite and beryllium are as follows:

$$\begin{aligned} C_{H.N}(Be) &= 0.46 \\ C_{H.D}(Be) &= 0.459 \end{aligned} \tag{7.24}$$

$$C_{H.N}(\text{graphite}) = 2.102$$
$$C_{H.D}(\text{graphite}) = 2.092 \qquad (7.25)$$

We conclude from the above study, that the diffusion cooling coefficient, as calculated in this study, differs by 9 percent from that of other authors. This observation is true for the case of constant diffusion coefficient. The value of the diffusion cooling coefficient obtained from λ for beryllium is considerably higher than from $\bar{v}$. This can be explained by the energy dependence of the transport mean free path.

The experimental values of C for beryllium, according to de Saussure and Silver[15] and for graphite according to Beckurts[4] are as follows:

$$C\ (\text{Be}) = 1.1 \pm 0.9$$
$$C\ (\text{graphite}) = 7.0 \pm 1 \qquad (7.26)$$

The experimental values for graphite are considerably higher than the heavy gas model results. But for beryllium the calculated value for the energy dependent D(E) is closer to the experimental value than the constant diffusion coefficient value.

CHAPTER VIII. THE TIME DEPENDENT DISTRIBUTION IN THE PULSED MULTIPLYING MEDIUM

In this chapter, we discuss the time behavior of the neutron energy spectrum, in the multiplying medium. In addition to the decrease of neutron density, there is a regeneration of neutrons due to the presence of fissionable material. The process of regeneration is characteristic only of the multiplying medium.

I. General Formulation

The distribution of the neutrons according to the diffusion theory approximation can be given as follows, where $\phi(E,r,t)$ is the thermal flux. The energy E lies in thermal region.

$$\begin{aligned}\frac{1}{v}\frac{\partial\phi(E,r,t)}{\partial t} &= -\left\{\Sigma_a(E)+\Sigma_s(E)-D\nabla^2\right\}\phi(E,r,t)\\ &\quad+\nu\Sigma_f\int_{t'}\int_{r'}\phi(E,r',t')\,q(r'\to r;\,t-t';E_0\to E)\,dr'dt'\\ &\quad+\int_{E'}\Sigma_s'(E')\,\phi'(E',t,r)\,F(E'\to E)\,dE'\\ &\quad+S_{ext}(E,r,t)\end{aligned}\tag{8.1}$$

The external source term $S_{ext}(E,r,t)$ is represented by the delta function in time.

$$S_{ext}(E,r,t) = S_{ext}(E,r)\;\delta(t)$$

Equation (8.1) contains the extra source of neutrons due to the presence of the fissionable material. We shall assume that the fission of the neutrons takes place at energy E. The slowing down of the neutrons

born with fission energy E_o is given by the slowing down kernel $q(r' \to r;\ t' \to t;\ E_o \to E)$. We have neglected the effect of the delayed neutrons, which is about 0.75%. In order to include this effect, the extra source term should be multiplied by a factor of $(1-\beta)$, where β equals 0.0075 delayed neutron fraction.

We define the kernel as follows. The slowing down kernel gives the probability that the fission neutrons of energy E_o, born at r' and at t', will slow down to energy dE about E at dr about $\vec{r}$ and at a time dt at t.

Let us make the following ansatz.

$$\phi(E,r,t) = \sum_p \phi_p(r)\ \phi_p(E,t)$$

$$\nabla^2 \phi_p(r) = -B_p^2\ \phi_p(r) \quad ;\ \phi_p(\tilde{r}) = 0 \qquad (8.2)$$

$\varphi_p(r)$ vanishes at a fixed boundary $\tilde{r}$. This boundary is given by the actual geometrical dimension, plus the extrapolated distance equal to $0.71\ \lambda_{tr}$. The symbol λ_{tr} stands for the transport mean free path. B_p^2 in Equation (8.2) stands for the geometrical buckling for the p-th spatial mode. We have assumed in Equation (8.2) that the neutron distribution goes to zero at the fixed boundary $\tilde{r}$ for all energies. That means that the extrapolated distance is independent of energy.

Substituting the set (8.2) into Equation (8.1) we get

$$\sum_p \left[\frac{1}{v} \frac{\partial \phi_p(E,t)}{\partial t} \phi_p(r) = -\phi_p(r) \left\{ \Sigma_a(E) + \Sigma_s(E) + D(E) B_p^2 \right\} \phi_p(E,t) \right.$$
$$\left. + \int_{E'} \Sigma_s'(E')\, \phi_p(E',t)\, F(E' \to E)\, dE' \right\} \qquad (8.3)$$
$$+ v \Sigma_f \int_{t'} \int_{r'} \phi_p(r')\, \phi_p(E,t')\, q(r' \to r;\ t-t';\ E_o \to E)\, dr'dt'$$

We shall use the results of the fundamental theorems of the asymptotic reactor theory to simplify the slowing down kernel. These theorems first given by Weinberg[51] are discussed in detail by Weinberg and Wigner.[52] We give these theorems in brief here.

First theorem:

This theorem has two postulates.

1. The finite medium slowing down kernel can be replaced by a displacement kernel. The slowing down kernel is a displacement kernel in the infinite medium. In the finite medium, away from the boundary, the kernel can be taken to be the displacement kernel.
2. The effect of the finite medium is taken into account by making the distribution zero at a fixed boundary $\tilde{r}$. Mathematically this is expressed as

$$\int_R \phi(r')\ P(r' \to r)\, dr' = \int_{-\infty}^{+\infty} \phi(r')\, P_\infty(|r'-r|)\, dr' \quad (8.4)$$

Second theorem:

This theorem states that the non escape probability during moderation in a uniform bare assembly is given by the Fourier transform of the slowing down kernel.

In essence the second theorem replaces the infinite medium integral by the Fourier transform.

$$\sum_p \left\{ \int_{-\infty}^{+\infty} \phi_p(r')\ P_\infty(|r'-r|)\, dr' \right\} = \sum_p \bar{P}_\infty(B_p^2)\, \phi_p(r) \quad (8.5)$$

Using the results of both theorems, the slowing down integral can be represented as follows.

$$\sum_p \int_{t'} \int_{r'} \phi_p(r')\, \phi_p(E,t')\, q(r' \to r;\ t'-t;\ E_0 \to E)\, dr'dt' = \sum_p \phi_p(r) \int_{t'} \bar{q}_\infty(B_p^2;\ t-t';\ E_0 \to E)\, \phi_p(E,t')\, dt' \tag{8.6}$$

where $\bar{q}_\infty(B_p^2;\ t'-t;\ E_0 \to E)$ is equal to the Fourier transform of the slowing down kernel in the infinite medium with B_p as the Fourier transform variable.

Substituting Equation (8.6) into Equation (8.3) for the p-th spatial mode, we get

$$\begin{aligned} \frac{1}{v}\frac{\partial \phi_p(E,t)}{\partial t} = &- \left[\Sigma_a(E) + \Sigma_s(E) + D(E)B^2\right]\phi_p(E,t) \\ &+ v\Sigma_f(E) \int_{t'} \phi_p(E,t')\, \bar{q}_\infty(B_p^2;\ t-t';\ E_0 \to E)\, dt' \\ &+ \int_{E'} \Sigma_s'(E')\, \phi_p'(E',t)\, F(E' \to E)\, dE' \end{aligned} \tag{8.7}$$

We make the following expansion for the time dependent function.

$$\phi_p(E,t) = \sum_i \phi_{pi}(E)\, e^{-\lambda_i t}$$
$$\phi_p(0,t) = 0 \quad ; \quad \phi_p(\infty,t) = 0 \tag{8.8}$$

λ_i is the eigen value for the i-th energy eigen function, for the p-th spatial mode. Our assembly is subcritical, therefore, the time behavior is given by the decaying exponential.

Substituting the expansion for $\varphi_p(E,t)$ into (8.7)

$$\sum_i \left[-\frac{\lambda_{pi}}{v} \phi_{pi}(E)\, e^{-\lambda_{pi} t} \right] = -\left[\Sigma_a + \Sigma_s + D B_{pi}^2\right] \phi_{pi}(E)\, e^{-\lambda_{pi} t} + \int_{E'} \Sigma_s'(E') \phi_{pi}'(E')\, e^{-\lambda_{pi} t} F(E' \to E)\, dE' + v \Sigma_f \int_{t'} \phi_{pi}(E)\, \bar{q}_\infty(B_{pi}^2;\, t-t';\, E_0 \to E)\, e^{-\lambda_{pi} t'}\, dt' \tag{8.9}$$

We simplify the slowing down integral.

$$I = \int_{t'=-\infty}^{t} \bar{q}_\infty(B_{pi}^2;\, t-t';\, E_0 \to E)\, e^{-\lambda_{pi}(t'-t)}\, dt' \tag{8.10}$$

Let t-t' equal t_0. We substitute this into the integral (8.10).

$$I = \int_0^\infty \bar{q}_\infty(B_{pi}^2;\, t_0;\, E_0 \to E)\, e^{t_0 \lambda_{pi}}\, dt_0$$

We can identify the above integral as the Laplace-Fourier transform of the slowing down kernel in the infinite medium with $s = -\lambda_{pi}$ being the Laplace transform variable.

$$I = \int_0^\infty \bar{q}_\infty(B_{pi}^2;\, t_0;\, E_0 \to E)\, e^{-s t_0}\, dt_0$$

$$I = \tilde{\bar{q}}_\infty(B_{pi}^2;\, S;\, E_0 \to E) \tag{8.11}$$

We substitute (8.11) into Equation (8.9). Then we get for the p-th spatial mode and the i-th energy mode:

$$-\frac{\lambda}{v}\phi(E) = -\left[\Sigma_a + \Sigma_s + D B^2\right]\phi(E) + v\Sigma_f \tilde{\bar{q}}_\infty(B_p^2;\, -\lambda;\, E_0 \to E)\phi(E) + \int_{E'} \Sigma_s'\, F(E' \to E)\, \phi(E')\, dE' \tag{8.12}$$

We have removed the subscripts p and i for the sake of brevity.

II. The Monoenergetic Case

We specialize Equation (8.12) for the monoenergetic thermal case. We assume that the experimentalist is measuring the density of monoenergetic neutrons. Then the decay constant λ is given by the following expression.

$$\lambda = (\Sigma_a + DB^2)v - v\nu\Sigma_f \tilde{\bar{q}}_\infty(B^2; -\lambda; E_0 \to E) \quad (8.13)$$

In order to use the above equation, we must determine the Laplace-Fourier transform of the slowing down kernel. The determination of this transform requires a detailed knowledge of the slowing down process. We shall treat two cases, namely, the heavy element moderated and hydrogen moderated systems. In the case of heavy element moderation, we can use the continuous slowing down model, i.e., the Fermi age model. For hydrogen moderated systems, this model is not applicable, because the neutron can lose all its energy in a single collision with the hydrogen nucleus. We require a different model for this case.

A. Heavy Element Moderation. (A >> 1)

On the basis of the Fermi age slowing down model, the Laplace-Fourier transform is here calculated. We rewrite Equation (8.10).

$$\tilde{\bar{q}}_\infty(B^2; -\lambda; E_0 \to E) = \int_0^\infty \bar{q}_\infty(B^2; t_0; E_0 \to E) e^{\lambda t_0} dt_0 \quad (8.10)$$

The Fourier transform of the slowing down kernel, in the infinite medium, can be given in a simple manner as follows:

$$\bar{q}_\infty(B^2; E_0 \to E) = 4\pi \int_0^\infty q_\infty(r; E_0 \to E) \frac{(\sin Br) r^2}{Br} dr \quad (8.14)$$

The slowing down kernel according to the Fermi age model is as follows.

$$q_\infty(r;\ E_o \to E) = \frac{e^{-\frac{r^2}{4\tau}} p}{(4\pi\tau)^{3/2}} \tag{8.15}$$

where τ equals the Fermi age corresponding to the energy interval from E_o to E, and p equals the resonance escape probability.

Substituting Equation (8.15) into Equation (8.14) we get

$$\bar{q}_\infty(B^2;\ E_o \to E) = p\, e^{-B^2\tau} \tag{8.16}$$

When the neutrons are slowing down, according to the Fermi age model, the time taken by neutrons of speed v_o to have slowed down to speed v is as follows.

$$t_s(U) = \frac{1}{2\xi\Sigma_s}\left[\frac{1}{U} - \frac{1}{U_o}\right]$$

It is characteristic of the Fermi age model that all neutrons having a speed v_o at time zero, have the same speed v at the time t_s. Mathematically, it can be expressed as follows.

$$\bar{q}_\infty(B^2;\ t_o;\ E_o \to E) = e^{-B^2\tau}\, p\, \delta(t_o - t_s) \tag{8.17}$$

where $\delta(t_o - t_s)$ is the Dirac delta function.

Using Equation (8.15) in Equation (8.10), we get

$$\tilde{\bar{q}}_\infty(B^2;\ -\lambda;\ E_o \to E) = \int_0^\infty p\, e^{-B^2\tau}\, \delta(t_o - t_s)\, e^{\lambda t_o}\, dt_o$$
$$= p\, e^{-B^2\tau}\, e^{\lambda t_s} \tag{8.18}$$

The Laplace-Fourier transform of the slowing down kernel in the infinite medium, is given by Equation (8.19). We substitute this into the expression for the decay constant λ in Equation (8.13).

$$\lambda = v(\Sigma_a + DB^2) - v\nu\Sigma_f p e^{-B^2\tau} e^{\lambda t_s} \qquad (8.19)$$

If we take into account the delayed neutrons and express $\nu\Sigma_f$ in the conventional form, Equation (8.19) reduces to

$$\lambda = v(\Sigma_a + DB^2) - \eta p \Sigma_a^U (1-\beta) v e^{-B^2\tau} e^{\lambda t_s} \qquad (8.19)$$

where $$\eta = \nu \Sigma_f^U / \Sigma_a^U$$

Equation (8.19) should be used to analyse pulse neutron data in a multiplying medium containing the heavy element moderator, i.e., graphite or beryllium. It must be pointed out that it is not possible to omit the exponential factor $e^{\lambda t_s}$, unless $\lambda t_s \ll 1$. This is not true in the case of small assemblies, which have large decay constants.

B. Hydrogen Moderation (A = 1)

As pointed out earlier, the hydrogen moderated systems can not be treated on the basis of the Fermi age model. The study of slowing down neutrons in these systems is still actively pursued. A great deal of effort is being spent to study this problem theoretically and experimentally. It is well known that the age of the fission neutrons to the indium resonance energy (1.44 e.v.) in water is 26 cm^2, as determined by theoretical methods.[22] The earlier experimental values are higher, as obtained by Hill et al.[20] (30 cm^2) and by Barkov and Mukhin[3]

(29.4 ± 1.5 cm^2). Recent experiments at Argonne National Laboratory give a lower value of the age, about 27 cm^2. It must be pointed out, however, that the Selengut-Gortzel model gives a theoretical value of 30 cm^2, in agreement with the earlier experimental values. A discussion of the various models has been given by Hurwitz and Zweifel.[22]

Due to lack of agreement about the use of a particular model, we represent the Fourier transform of the slowing down kernel in an arbitrary manner for hydrogeneous systems.

$$\bar{q}_\infty(B^2; E_0 \to E) = p\left[\sum_{k=0}^{\infty} \frac{(-1)^k B^{2k}}{(2k+1)!} \overline{r^{2k}}\right] \quad (8.20)$$

We define the 2K-th spatial moment as follows.

$$\overline{r^{2k}} = \frac{\int_0^\infty q_\infty(r; E_0 \to E)\, r^{2k+2}\, dr}{\int_0^\infty q_\infty(r; E_0 \to E)\, r^2 dr} \quad (8.21)$$

The spatial moments are given by the type of model used to describe the slowing down kernel. In the case of water and other hydrogeneous systems Equation (8.20) should be employed. A choice must be made between an experimental or a theoretical model.

For analyzing pulse neutron experiments in a multiplying medium moderated by hydrogeneous material, we need the Laplace-Fourier transform of the slowing down kernel. We have given the solution for this case in Chapter III. The results obtained were given in Equation (3.26) and are as follows.

$$\tilde{\bar{q}}_\infty(E; B^2; S=-\lambda) = \frac{\Sigma_s(E_0)\, \exp\left\{-\int_E^{E_0} \frac{(\Sigma_{a_0} + S + v'D'B^2)\, dE'}{E'\{S + \Sigma_{a_0} + v'(D'B^2 + \Sigma_s')\}}\right\}}{\left[\Sigma_a(E_0) + \frac{S}{v_0} + D(E_0)B^2 + \Sigma_s(E_0)\right]}$$

$$\Sigma_a(E) = \frac{\Sigma_{a_0}}{v} \quad (8.22)$$

We write the above equation as the product of three factors.

$$\tilde{q}_\infty(E;B^2;-\lambda) = \frac{\Sigma_s(E_0)\exp\left\{-\int_E^{E_0}\frac{\Sigma_{a_0}\,dE'}{\{\Sigma_{a_0}-\lambda+v'(D'B^2+\Sigma_s')\}E'}\right\}}{\left[\Sigma_a(E)-\frac{\lambda}{v}+D(E)B^2+\Sigma_s(E)\right]_{E=E_0}} \times$$

$$\times \exp -B^2\int_E^{E_0}\frac{D'v'\,dE'}{\{\Sigma_{a_0}-\lambda+v'(D'B^2+\Sigma_s')\}E'} \qquad (8.23)$$

$$\times \exp \lambda\int_E^{E_0}\frac{dE'}{E'[\Sigma_{a_0}-\lambda+v'(D'B^2+\Sigma_s')]}$$

We simplify the above equation.

$$\tilde{q}_\infty(E;B^2;-\lambda) = p\,e^{-B^2\tau}\,e^{\lambda t_s} \qquad (8.24)$$

Equation (8.24) is true only if it is possible to separate the time and space dependent solutions. We assume that in the exponentials involving resonance escape probability p and non-escape probability $e^{-B^2\tau}$, the decay constant λ is small compared to $\Sigma_{a_0} + v(DB^2 + \Sigma_s)$. This leads to a separation of Equation (8.23) into two parts.

$$\tilde{q}_\infty(E;B^2;-\lambda) = \left[\frac{\exp\lambda\int_E^{E_0}\frac{dE'}{E'[-\lambda+\Sigma_{a_0}+v'(D'B^2+\Sigma_s)]}}{\left\{1-\frac{\lambda}{v_0\{\Sigma_a(E_0)+D(E)B^2+\Sigma_s(E_0)\}}\right\}}\right]$$

$$\times\left[\frac{\Sigma_s(E_0)\exp-\int_E^{E_0}\left[\frac{\Sigma_{a_0}+v'D'B^2}{\Sigma_{a_0}+v'(D'B^2+\Sigma_s')}\right]\frac{dE'}{E'}}{\{\Sigma_a(E_0)+D(E_0)B^2+\Sigma_s(E_0)\}}\right] \qquad (8.25)$$

Equation (8.25) can be expressed as the product of two functions, as follows.

$$\tilde{\bar{q}}_{\infty}(E; B^2; -\lambda) = \bar{P}(B^2)\ \tilde{P}(B^2; -\lambda) \qquad (8.26)$$

The function $P(B^2; -\lambda)$ may be regarded as a function of λ alone, since the leakage term vDB^2 may be neglected in comparison to $\Sigma_{a_o} + v\Sigma_s - \lambda$. This leads to a simplification of Equation (8.26).

$$\tilde{\bar{q}}_{\infty}(E; B^2; -\lambda) = \bar{P}(B^2)\ \tilde{P}(-\lambda) \qquad (8.27)$$

$$\bar{P}(B^2) = \frac{\Sigma_s(E) \exp -\left\{ \int_E^{E_o} \left[\frac{\Sigma a_o + v' D' B^2}{\Sigma a_o + v'(D' B^2 + \Sigma_s')} \right] \frac{dE'}{E'} \right\}}{\left[\Sigma a + (DB^2 + \Sigma_s) \right]_{E=E_o}} \qquad (8.28)$$

assuming $\Sigma a_o + v(DB^2 + \Sigma_s) \gg \lambda$

$$\tilde{P}(-\lambda) = \frac{\exp \lambda \left\{ \int_E^{E_o} \frac{dE'}{(\Sigma_{a_o} + v\Sigma_s - \lambda) E'} \right\}}{\left[1 - \left(\frac{1}{v_o}\right) \left\{ \frac{\lambda}{\Sigma_a(E_o) + \Sigma_s(E_o)} \right\} \right]} \qquad (8.29)$$

assuming $\Sigma a_o + v\Sigma_s - \lambda \gg v D B^2$; $\Sigma_a(E_o) + \Sigma_s(E_o) \gg D_o B^2$

The decay constant in pulse neutron assemblies consisting of hydrogenous and fissionable material can be given by the following equation.

$$\lambda = v\Sigma a + vDB^2 - v\eta \Sigma a^{v} \bar{P}(B^2) \tilde{P}(-\lambda)(1-\beta) \qquad (8.30)$$

It must be pointed out that Equation (8.28) corresponds to the expression for the Selengut-Gortzel slowing down model in hydrogen. It is essentially the same expression as derived by Simon[42] for the case of constant cross section using the Selengut-Gortzel model for the time-independent infinite medium problem.

In the analysis of pulse neutron experiments on subcritical systems, Campbell and Stelson[11] used the following equation for the decay constant λ.

$$\lambda = \upsilon \Sigma_a + \upsilon D B^2 - \eta \Sigma_a^U \upsilon \overline{P}(B^2) \qquad (8.31)$$

When the expressions for λ given by Equations (8.30) and (8.31) are compared, we find that the factor $P(-\lambda)$ is missing in Equation (8.31). The effect of this factor is to decrease $\overline{P}(B^2)$. Campbell and Stelson[11] expressed $\overline{P}(B^2)$ as equal to $1/1 + B^2\tau$ and obtained τ as equal to 21.2 cm^2. If, however, we take into account the correction factor $P(-\lambda)$, then the value of τ increases. We shall determine this correction subsequently.

III. The Multi-Energy Case

We shall use Equations (8.9) and (8.11) to study this case. Combining these two equations, we get

$$\sum_i e^{-\lambda_i t}\left[-\frac{\lambda_i}{\upsilon}\phi_i(E) = -\left\{\Sigma_a(E)+\Sigma_s(E)+D(E)B^2\right\}\phi_i(E) + \upsilon\Sigma_f \tilde{\overline{q}}_\infty(B^2; -\lambda; E_0 \to E)\phi_i(E) + \int_{E'}\Sigma_s'(E')\phi_i(E')F(E'\to E)dE'\right. \qquad (8.32)$$

A detailed knowledge of the variation of the kernel with the final energy E is required, in the multi-energy case. This variation depends upon the physical model used to describe the slowing down phenomenon. Even if the Fermi age slowing down model is used, the problem does not become a simple one unless a few drastic assumptions are made. In the final stage of the slowing down process, we assume that the slowing down time can be an average, $\bar{t}_s$. In the energy interval between 10 KT and 1 KT, the slowing down time $t_s(E)$ varies by almost a factor of five. The slowing down length, however, does not change more than a few percent in the above interval, thus it can justifiably be given by an average value.

We shall use the above assumptions for simplifying the Laplace-Fourier transform of the slowing down kernel.

Let us make the following ansatz:

$$\phi(E) = \sum_n a_n L_n^{(1)}(E)\, M(E) \qquad \phi_0(0) = 0 \qquad \phi(\infty) = 0 \tag{8.33}$$

For the scattering integral, we shall use the expression given previously,

$$-\Sigma_s \phi(E) + \int_{E'} \Sigma_s'(E')\, \phi(E')\, F(E' \to E)\, dE' = \sum_{K=1}^{\infty} \frac{1}{K!} \frac{\partial^K L_n^{(1)}(E)}{\partial E^K} \Sigma_s M(E) A_K(E)$$

We substitute ansatz (8.33) into Equation (8.32), then multiply by $L_m^{(1)}(E)$ and finally integrate over all energies. Using the notation of the fourth chapter, we have

$$\sum_i e^{-\lambda_i t} \sum_n a_{ni} \left[-\frac{\lambda_i W_{mn}}{v_0} = -\Sigma a_0 W_{mn} - (DB^2)_{mn} + F_{mn} + G_{mn} \right] \tag{8.34}$$

$$G_{mn} = \int_0^\infty \nu \Sigma_f \tilde{q}_\infty (B^2; E_0 \to E; -\lambda) L_m^{(1)}(E) M(E) L_n^{(1)}(E) dE \quad (8.35)$$

We shall simplify the G_{mn} integral for a special case.

The secular determinant for (8.34) is as follows:

$$\left| \left(\Sigma a_c - \frac{\lambda}{v_0} \right) W_{mn} + (D B^2)_{mn} - F_{mn} - G_{mn} \right| = 0 \quad (8.36)$$

The solution of the secular determinant (8.36) will give the values for λ. It will, however, require a detailed knowledge of F_{mn} and G_{mn} integrals.

For the Fermi age slowing down model, we have:

$$G_{mn} = \int_0^\infty e^{\lambda t_s(E)} \nu \Sigma_f p e^{-B^2 \tau} L_m^{(1)}(E) M(E) L_n^{(1)}(E) dE \quad (8.37)$$

As discussed in detail above, we assume:

1. The neutron age $\tau_{E_0 \to E}$ does not change appreciably in the energy region ($E_T \geq E \geq 0$). E_T equals thermal cut off energy.
2. The slowing down time $\bar{t}_s$ can be given by an average value $\bar{t}_s$ weighted over the Maxwellian distribution.

$$\bar{t}_s = \frac{\int_0^\infty t_s(E) E^{1/2} e^{-\frac{E}{E}} dE}{\int_0^\infty E^{1/2} e^{-E} dE} \quad (8.38)$$

On substituting these values into Equation (8.37), we get

$$G_{mn} = \left[p e^{-B^2 \tau} e^{\lambda \bar{t}_s} \int_0^\infty \upsilon \Sigma_f L_m^{(1)} M(E) L_n^{(1)} dE \right] \quad (8.39)$$

For 1/v cross section law, G_{mn} integrals can be evaluated.

IV. Analysis of Experiments

We shall analyze the data given by the experiments of Campbell and Stelson.[11] The value of the decay constant λ for different geometrical bucklings B^2 at a concentration of 26.5 grams U^{235} per liter are given in Table 8.1.

TABLE 8.1

THE VALUE OF THE DECAY CONSTANT FOR DIFFERENT B^2 VALUES[10]

B^2 cm^{-2}	10^{-4}/sec^{-1}
0.0436	0.550
0.0483	0.630
0.0562	0.704
0.072	0.883
0.135	1.386
0.192	1.768
0.296	2.358
0.506	3.194

We calculate the experimental non-leakage probability $\bar{P}_{exp}(B^2)$ using the following equation.

$$\lambda = \upsilon \Sigma_a + \upsilon D B^2 + \upsilon \Sigma_a^U (1 - \eta \bar{P}_{exp}(B^2) \quad (8.40)$$

We use the following values for the constants.

$V\Sigma_a$ = 0.48×10^5 sec^{-1} for water

$V\Sigma_a^u$ = 0.965×10^5 sec^{-1} for U^{235} (26.5 gms U^{235}/liter)

η = 2.07

vD = 3.49×10^4 $cm^2 sec^{-1}$

$v\Sigma_s$ = 29.48×10^4 sec^{-1}

v = 2.2×10^5 cms/sec Thermal neutron speed

We express $\overline{P}_{exp}(B^2)$ as the product of two functions, $\overline{P}(B^2)$. $P(-\lambda)$ as given by Equation (8.27). $P(-\lambda)$ is calculated according to Equation (8.29). In Table 8.2 we give $\overline{P}_{exp}(B^2)$; $\overline{P}(B^2)$ and $P(-\lambda)$ for different values of the geometrical buckling, B^2.

TABLE 8.2

NON-ESCAPE PROBABILITIES

B^2	$\overline{P}_{exp}(B^2)$	$P(-\lambda)$	$\overline{P}(B^2)$
0.0436	0.524	1.03697	0.5053
0.0483	0.490	1.04285	0.4699
0.0562	0.469	1.0480	0.4475
0.0720	0.407	1.0646	0.3823
0.135	0.265	1.0986	0.2412
0.192	0.174	1.1156	0.1560
0.296	0.060	1.1676	0.0514
0.506	0.009	1.2318	0.0073

In the region where B^2 is less than 0.14 cm^{-2}, Campbell and Stelson fitted $\overline{P}_{exp}(B^2)$ by $1/1 + B^2\tau$ and obtained τ as equal to 21.2 cm^2.[11]

If we take $\overline{P}(B^2)$ instead of $\overline{P}_{exp}(B^2)$ we get τ equal to 22.82 cm^2. It is important to note that the $P(-\lambda)$ correction increases τ in the right direction. In small finite systems, the experimental value of $\overline{P}(B^2)$, rather than the neutron age, should be compared with the theoretical calculations. For small systems it is no longer possible to express $\overline{P}(B^2)$ as equal to $1/1 + B^2\tau$; Equation (8.20) should be used.

In the case of heavy element moderators, $P(-\lambda)$ is equal to $e^{\lambda t_s}$. The experimental value of $\overline{P}_{exp}(B^2)$ can be used to obtain the slowing down time t_s. Kloverstrom and Komoto[26] carried out the experiments on graphite enriched uranium systems and analyzed their data taking into account the $e^{\lambda t_s}$ factor.

CHAPTER IX. DISCUSSION OF RESULTS.

We have studied the time behavior of the low energy spectrum, for a pulse of fast neutrons, in a homogeneous medium. Analytical and numerical methods have been employed to undertake this study. Characteristics of neutron thermalization and diffusion of a pulse of neutrons in a medium have been determined. The analytical method is based upon the expansion of the neutron distribution function $\varphi(E,t)$ for one spatial mode, in the following manner.

$$\phi(E,t) = \sum_i e^{-\lambda_i t} a_i \phi_i(E) \qquad (9.1)$$

In the above expansion, $\varphi_i(E)$ is the eigen function associated with the i th eigen value λ_i. Each eigen function is expanded by a sum of an infinite number of associated Laguerre polynomials of first order, weighted by the Maxwellian energy distribution. The choice of the Laguerre polynomial is arbitrary.

I. First Eigen Value

We have given a general formulation of the method for determining eigen values which characterize the decay of a pulse of neutrons in any medium. The zeroth eigen value was determined by Singwi.[43] The expression for the first eigen value, according to Equation (4.20), is as follows:

$$\lambda_1 = \Sigma_{a_0} v_0 + \frac{M_2 W_{00} v_0}{4(W_{11}W_{00} - W_{01}^2)} + \frac{D B^2 v_0}{W_{00}} \left\{ \frac{W_{00}^2 - W_{01}^2}{W_{11}W_{01} - W_{01}^2} \right\} + \frac{4 D^2 B^4 v_0}{W_{00} M_2} \left(\frac{W_{01}}{W_{00}} \right)^2 \qquad (9.2)$$

In a non-absorbing, infinite medium, the zeroth eigen value is zero, but the first eigen value is not zero and is equal to the following expression.

$$\lambda_1 = \frac{M_2 W_{00} U_0}{4(W_{11}W_{00} - W_{01})^2} = 8.278 \times 10^4 M_2 \quad (\text{Sec}^{-1}) \tag{9.3}$$

λ_1 determines the rate of thermalization. The reciprocal of λ_1 is the thermalization time constant, with which the neutron energy distribution approaches the Maxwellian distribution. We have derived the following relation between the thermalization time constant and the diffusion cooling coefficient, as given by Equation (4.25).

$$t_{th} = \frac{3}{2} \frac{C_0 \pi}{D^2 U_0^2} \tag{9.4}$$

Nelkin[32] derived this equation based upon the concept of neutron temperature, using the variational method. We have rigorously derived this expression without using the concept of neutron temperature, which has been criticized on physical grounds. The thermalization time constants for a number of moderators have been determined and are found to be in good agreement with the results of other authors.

II. Determination of M_2

It was pointed out by Nelkin[32] that the experimental value of the diffusion cooling coefficient can be used to determine the thermalization parameter M_2. This parameter is related to the second

energy transfer moment, as shown by the following equation.

$$M_2 = \int_0^\infty \int_0^\infty \Sigma_s(E)\, M(E)\, F(E \to E')\, (E'-E)^2 dE' dE \qquad (9.5)$$

Finite medium corrections, coupled with the dependence of the diffusion coefficient upon energy, limit the usefulness of this method. We have pointed out the importance of the determination of M_2 by the absorbtion method. The eigen values for an infinite medium, containing a 1/v absorber, are given as follows.

$$\begin{aligned} \lambda_0 &= \bar{\Sigma}_{a_0} v_0 \\ \lambda_1 &= \Sigma_{a_0} v_0 + \frac{M_2 W_{00} v_2}{4(W_{11} W_{00} - W_{01}^2)} \end{aligned} \qquad (9.6)$$

For large absorptions, λ_1 is comparable to λ_0 and the final decay of the energy spectrum is governed by both, not by λ_0 alone, which is the case for small absorptions. The advantage of the absorption method lies in the determination of M_2 by direct measurement of decay constants, which can be measured accurately.

III. Thermalization in a Heavy Gas Medium

For the heavy gas, we have determined λ_1 and λ_2 for a non-absorbing and infinite medium case. The fact that the eigen functions of the heavy gas differential operator are the associated Laguerre polynomials of first order, simplifies the problem. The two eigen values for this case are as follows:

$$\lambda_1 \simeq 1.2738\, \xi \Sigma_{s_0} v_0 \quad \text{Four Laguerre polynomials} \qquad (9.7)$$

$$\lambda_2 \simeq 4.088\, \xi \Sigma_{s_0} v_0 \quad \text{Three Laguerre polynomials}$$

The eigen function corresponding to the first eigen value was also obtained. The expression for $\varphi_1(E)$ is as follows.

$$\phi_1(E) = E e^{-E} a_{01} [-4.013 + 3.861 E - 0.569 E^2 + 0.029 E^3 ..] \quad (9.8)$$

IV. The Time Dependent Energy Spectrum in the Infinite Medium

A unique method of studying the time behavior of the energy spectrum of a pulse of neutrons has been developed. With its help, we have generated the neutron energy spectra for times greater than the slowing down time. The source at the slowing down time was calculated on the basis of the Fermi age slowing down model. It must, however, be pointed out that the results obtained here are independent of the details of the initial source at the slowing down time. In the infinite medium, we have studied two cases: zero absorption and 1/v absorption.

For the case of zero absorption, we determined the total thermalization time required by the neutrons to reach the Maxwellian distribution. The thermalization time for beryllium and graphite was found to be 114 micro seconds and 240 micro seconds, respectively. Kothari and Singwi's calculations for beryllium gave 143 micro seconds, and Antonov et al's. experiments gave for graphite 200 micro seconds. It is speculated that the total thermalization time depends upon the mean speeds of the initial and final distributions, and is insensitive to the scattering kernel. The time constant with which the neutron energy

distribution approaches the Maxwellian distribution is found to be equal to 1.176 $\xi \Sigma_{s_0} v_0$ compared to 1.2738 $\xi \Sigma_{s_0} v_0$, as given by the eigen value method. By means of an exponential fit, we have determined $\varphi_1(E)$ and have compared it with the first eigen function $\varphi_1(E)$. The agreement is fairly good.

V. The Time Dependent Energy Spectrum in a Finite Medium

The finiteness of the medium introduces the effects of leakage of neutrons. In all the previous theoretical studies, it was assumed that the asymptotic energy spectrum can be described by a single decay constant. In this thesis, however, it is established that the asymptotic energy spectrum decays with a single decay constant. The energy spectrum attains an equilibrium spectrum at about the thermalization time. The shape of the spectrum remains constant in time, but the amplitude decreases. A shift of the equilibrium spectrum toward low energy was observed with the increase in geometrical buckling. This is an evidence of the 'cooling of the spectrum'. The following characteristics of the equilibrium specta were studied.

1. Average Speed
2. Average Energy E and ' Neutron Temperature'
3. Decay Constant
4. Diffusion Cooling Coefficient

Two special cases were studied: first, the constant diffusion coefficient case of graphite, and secondly, the energy dependent diffusion coefficient case of beryllium. The average speed $\bar{v}$ and the decay constant λ, when fitted by the least square fit give the 'diffusion cooling coefficient'.

The value for the diffusion cooling coefficient for graphite obtained here was 1.922. This differs by about 9 percent from the results of other authors, namely a value of 2.10 for the diffusion cooling coefficient.

For beryllium, we obtained the following values of the diffusion cooling coefficient.

0.52	from	$\bar{v}$
0.89	from	λ

Since we used the energy dependent diffusion coefficient, the two values are not expected to be identical. The results of other authors, based upon a constant diffusion coefficient, are equal to 0.46.

The Multiplying Medium

We have analyzed the data of Campbell and Stelson's experiments on U^{235} and water mixtures. The Laplace- Fourier transform should be used to analyze the pulse neutron experiments instead of the Fourier transform of the slowing down kernel. Taking into account this correction, the age is found to be 22.82 sq cm^2 instead of 21.2 cm^2, as calculated by Campbell and Stelson.(11)

The pulse neutron experiments on multiplying media can be used to determine a large number of reactor physics parameteres: the slowing down time, the slowing down length, and η. However, the determination of these quantities requires a careful analysis of the experiments, as pointed out above.

VI. Conclusion

The eigen value method is useful in determining the upper limits for lower eigen values, which characterize the last stage of neutron thermalization and diffusion processes. This method fails to give the neutron distribution function at all times. In order to obtain results of higher order, a large number of Laguerre polynomials and the higher energy transfer moments are required. However, the usefulness of this method lies in its exactness and in its general applicability. For the heavy gas model, this method can give exact results of all orders, though the calculations involved are tedious and require large machine calculations.

In addition to the eigen value method, a numerical method to generate the behavior of the neutron energy spectrum at times greater than the slowing down time, has been developed. This method has been applied to the heavy gas model, and the time dependent energy spectra were obtained for the various cases mentioned earlier. This method is unique, since it simulates the experiment and follows a pulse of neutrons during thermalization and diffusion periods. The advantage of this method is that it describes the neutron distribution function $\varphi(E,t)$ completely, provided the initial source distribution is given exactly. It also gives the correct asymptotic behavior, which describes the last stage of neutron thermalization and diffusion, and is independent of the details of the initial source. Results obtained by this method differ by 5 to 10 percent when compared to the analytical results. The limitations of this method are determined by the machine used to generate these distributions. It might be possible to generate these distributions for the crystalline model by a similar multigroup technique.

It should be pointed out that the two methods described are complimentary to one another.

We have established in this study the existence of an asymptotic energy spectrum, characterized by a single decay constant λ_0. Evidence of the 'diffusion cooling', based upon the time dependent energy spectra for the finite media, has been given. The energy spectrum deviates from the Maxwellian distribution and shifts toward the low energy side in the finite media. In the case of an infinite medium, for small absorbtions, the energy spectrum is Maxwellian and is characterized by a single decay constant λ_0. However, for large 1/v absorption, the energy spectrum is not Maxwellian, and its decay is governed by two decay constants, one of which depends upon the energy transfer properties of the medium.

It should also be pointed out here, that for small absorptions, the pulse neutron experiments give the absorption cross section and diffusion coefficient averaged over the Maxwellian distribution. The deviation from the Maxwellian distribution is contained in the 'diffusion cooling' term.

It is hoped that this study would further the understanding of neutron thermalization and diffusion, and would stimulate and aid in the study of thermalization and diffusion by pulse neutron techniques.

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APPENDIX I. PROPERTIES OF ASSOCIATED LAGUERRE POLYNOMIALS

We list the properties of the associated Laguerre polynomial of α-th order, as given by Copson[1] and Sneddon.[2]

$$L_n(Z) = \frac{\Gamma(\alpha+1+n)}{n!\,\Gamma(\alpha+1)} \; {}_1F_1\,(-n;\ \alpha+1;\ Z) \tag{I.1}$$

$${}_1F_1\,(-n,\ \alpha+1,\ Z) = \sum_{r=0}^{\infty} \frac{(-n)_r\, Z^r}{r!\,(\alpha+1)_r} \qquad \text{Hypergeometric Function} \tag{I.2}$$

$$(\alpha)_r = \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \tag{I.3}$$

$$\sum_{0}^{\infty} t^n\, L_n^{\alpha}\,(Z) = (1-t)^{-\alpha-1}\, e^{-\frac{Zt}{1-t}} \tag{I.4}$$

$$L_n^{\alpha}\,(Z) = \frac{e^{-Z}\, Z^{-\alpha}}{n!} \frac{d^n}{dZ^n}\,(e^{-Z}\, Z^{n+\alpha}) \tag{I.5}$$

If $(\alpha+n)$ is positive integer or zero, then

$$L_n^{\alpha}\,(Z) = \frac{(-1)^{\alpha}\, e^{Z}}{n!} \frac{d^{n+\alpha}}{dZ^{n+\alpha}}\; e^{-Z}\, Z^n \tag{I.6}$$

$$\int_{0}^{\infty} Z e^{-Z}\, L_m^{\alpha}\,(Z)\; L_n^{\alpha}(Z)\; dZ = \frac{\Gamma(\alpha+n+1)}{n!}\; \delta_{nm} \tag{I.7}$$

If $f(Z) = \sum_{0}^{\infty} C_n L_n^{\alpha}(Z)$ and term by term integration is valid, then

$$C_n = \frac{n!}{\Gamma(n+\alpha+1)} \int_0^{\infty} e^{Z} Z^{\alpha} f(Z) L_n^{\alpha}(Z)\, dZ \tag{I.8}$$

$$L_n^{(1)}(Z) = \frac{\Gamma(n+2)}{n!} \sum_{r=0}^{\infty} \frac{(-n)_r}{r!} \frac{Z^r}{(2)_r} \tag{I.9}$$

$$L_n^{(1)}(Z) = (n+1) \sum_{r=0}^{n} \frac{(-1)^n\, n!\, Z^r}{(n-r)!\,(r!)(r+1)!} \tag{I.10}$$

$$\int_0^{\infty} L_n^{(1)}(Z)\, L_m^{(1)}(Z)\, Ze^{-Z}\, dZ = n+1\ \delta_{mn} \tag{I.11}$$

The normalizing factor is $\frac{1}{\sqrt{n+1}}$. The normalized associated Laguerre polynomial of order one, is given as follows.

$$L_n^{(1)}(Z) = \sqrt{n+1} \sum_{r=0}^{n} \frac{(-1)^n\, Z^r\, n!}{(n-r)!\,(r!)(r+1)!} \tag{I.12}$$

$L_n^{(1)}(Z)$ is the solution of the following differential equation.(30)

$$Z \frac{d^2y}{dZ^2} + (2-Z) \frac{dy}{dz} + ny = 0 \tag{I.13}$$

The differential operator of the heavy gas model has the associated Laguerre polynomial of first order as its eigen function.

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APPENDIX II. TIME DEPENDENT ENERGY SPECTRA FOR DIFFERENT B^2

We give the additional time dependent energy spectra for beryllium assemblies in Figures II.1, 2 and 3. All these spectra were obtained using twenty energy groups with the ORNL analog computer.

In Figure II.4 we give the neutron energy spectrum at 300 micro seconds in graphite for various geometrical bucklings, B^2. All these curves were obtained using fourteen energy groups with the help of the analog computer of the Nuclear Engineering Department of the University of Michigan. In Chapter VII, a complete set of curves were given for the case of $B^2 = 18.5 \times 10^{-2}$ cm^{-2}. These curves were obtained using twenty energy groups by the ORNL analog computer

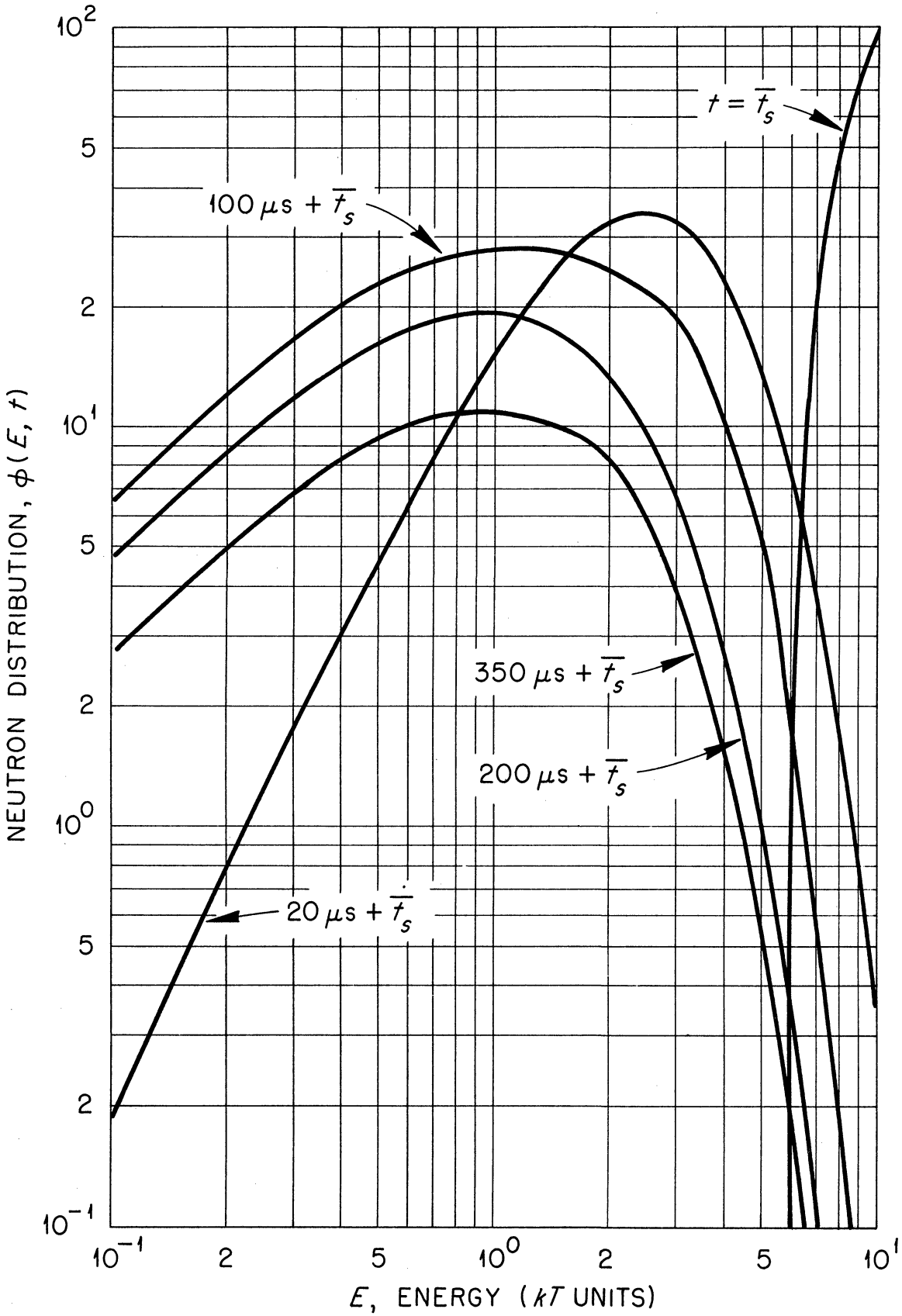

Figure II.1 Time Behavior of Neutron Energy Spectrum in Finite Medium of Beryllium of Geometrical Buckling $B^2 = 2.96 \times 10^{-2}$ cm^{-2}.

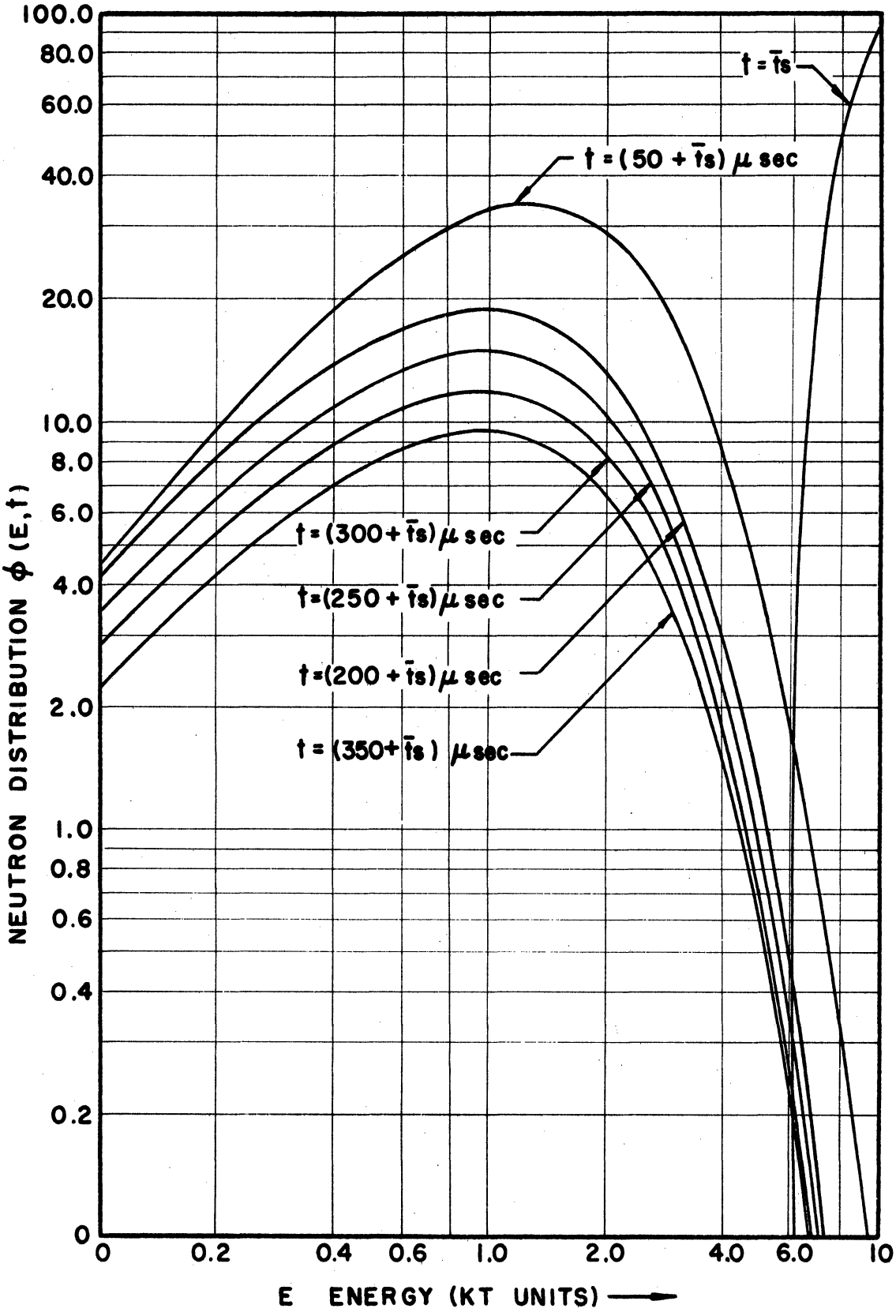

Figure II.2 Time Behavior of Neutron Energy Spectrum in Finite Medium of Geometrical Buckling $B^2 = 3.96 \times 10^{-2} cm^{-2}$.

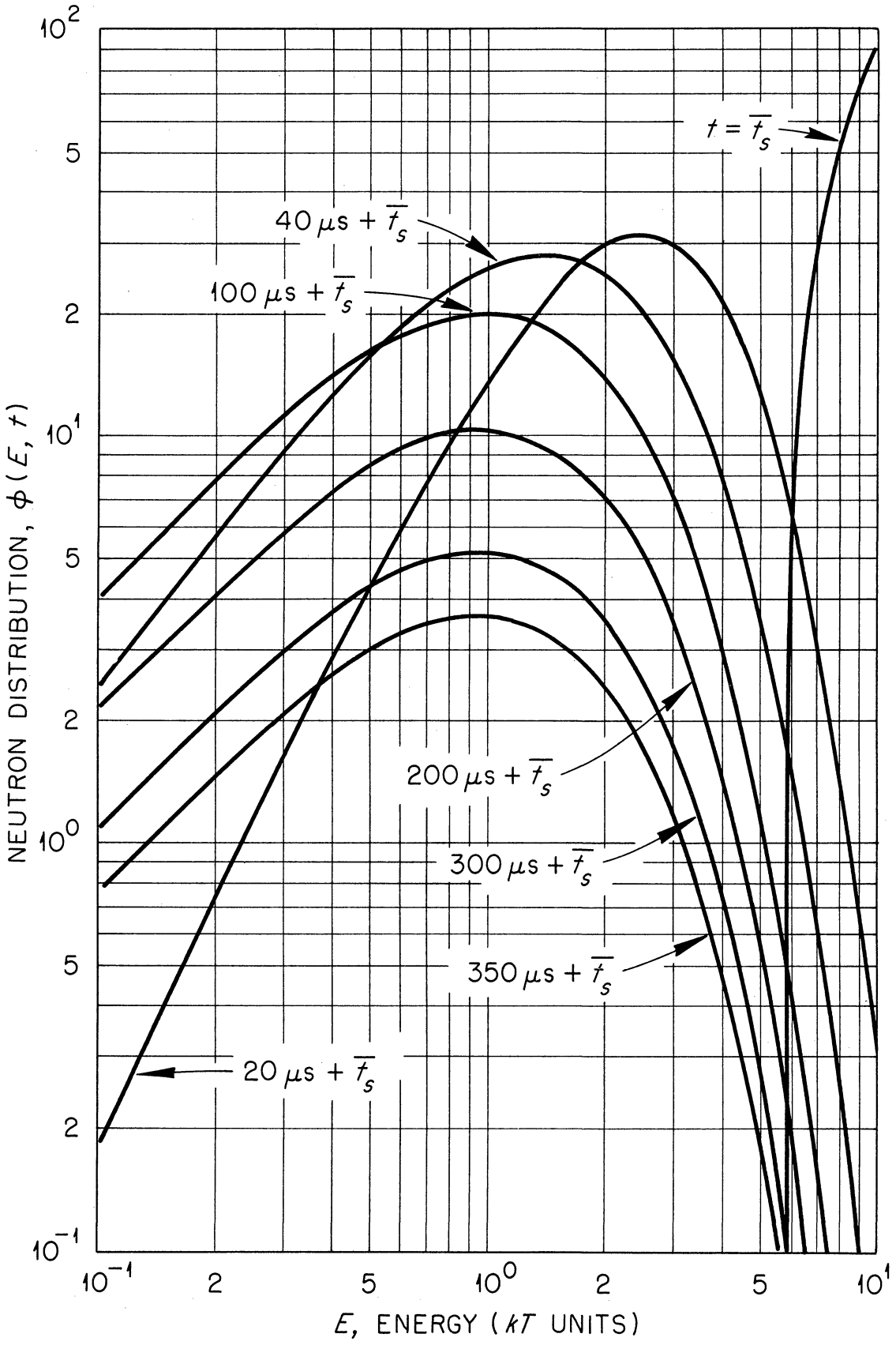

Figure II.3 Time Behavior of Neutron Energy Spectrum in Finite Medium of Beryllium of Geometrical Buckling $B^2 = 5.36 \times 10^{-2}$ cm^{-2}.

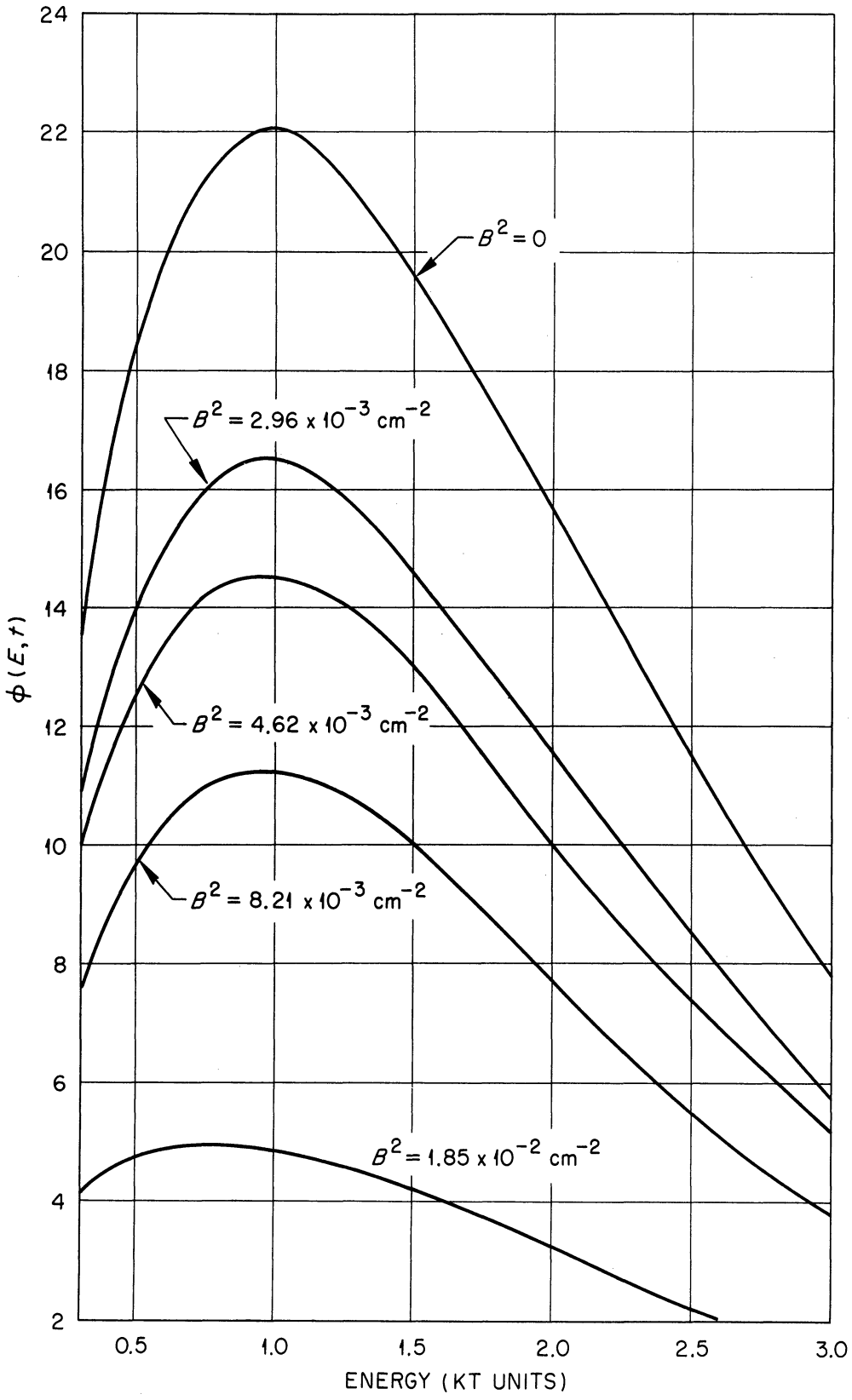

Figure 11.4 Neutron Energy Spectrum at 300 μ sec in Graphite for Various Geometrical Bucklings, B^2.

APPENDIX III. HIGHER SPATIAL MODE CONTRIBUTION

The neutron distribution as a function of energy, space and time variables can be represented as follows:

$$\varphi(E,r,t) = \sum_m A_m \varphi_m(r) \varphi_m(E,t) \qquad \text{(III.1)}$$

where:

m represents the m-th mode,

Am is the amplitude of m-th spatial mode,

$\varphi_m(r)$ spatial distribution of m-th mode,

$\varphi_m(E,t)$ energy-time distribution of m-th mode.

In order to determine $\varphi(E,r,t)$ completely, we need $\varphi_m(E,t)$ and Am for each m-th mode. We have shown in Chapter VII that the neutron distribution $\varphi(E,t)$ at times greater than the thermalization time can be represented by a single exponential decay. We also note that for the Fermi Age slowing down model, the slowing down time is constant and does not vary with the spatial mode or the geometrical buckling. This does not hold good for the other slowing down model.

For times greater than the thermalization time III.1 can be expressed in the following manner for a rectangular block.

$$\varphi(E,r,t) = \sum_{\ell,m,n} e^{-B^2_{\ell,m,n}\tau} \varphi_{\ell,m,n}(r)\, e^{-\lambda_{\ell,m,n}t} \qquad \text{(III.2)}$$

where:

ℓ,m,n represent the integers which characterize the spatial mode in a rectangular assembly.

$e^{-B^2_{\ell,m,n}\tau}$ = amplitude factor for ℓ,m,n spatial mode,

$\lambda_{\ell,m,n}$ = decay constant for ℓ,m,n spatial mode.

We take the specific example of $B^2_{111} = 2.96 \times 10^{-2} cm^{-2}$. This is one of the assemblies used by de Saussure and Silver (ORNL-2641) in their experiment on beryllium. We take $\tau = 80\ cm^2$ and calculate $\lambda_{\ell,m,n}$ for different spatial modes using the decay constant formulas given in Chapter VII. We rewrite it:

$$\lambda = 1.267 \times 10^5 [1-0.89B^2] B^2$$

For t = 114 μ secs, we have:

$$\varphi(E,r,t) = [\varphi_{111}(r) + 6.37 \times 10^{-2} \varphi_{112}(r) + 4.17 \times 10^{-3} \varphi_{122}(r) + 6.97 \times 10^{-4} \varphi_{113}(r) + 2.77 \times 10^{-4} \varphi_{222}(r) + ..] \quad \text{(III.3)}$$

For t = 200 μ secs, we have:

$$\varphi(E,r,t) = [\varphi_{111}(r) + 4.77 \times 10^{-2} \varphi_{112}(r) + 2.37 \times 10^{-3} \varphi_{122}(r) + 3.30 \times 10^{-4} \varphi_{113}(r) + 1.21 \times 10^{-4} \varphi_{222}(r)] \quad \text{(III.4)}$$

From the above analysis, it is apparent that all the higher spatial modes except (112) contribute less than 1/2%. The contribution by (112) diminishes with time, leaving the fundamental as the dominant mode. Some of the next higher spatial modes are suppressed by the judicious placement of the source and the detector. In de Saussure and Silver's experiment next higher mode to the fundamental is (113). The contribution to this mode is less than 0.1% of the fundamental.

It may, therefore, be concluded that at times greater than the thermalization time, only the fundamental spatial mode is important.

During the interval between the slowing down time and the thermalization time, one has to use the exact $\varphi_{\ell,m,n}(E,t)$ for each (ℓ,m,n) mode. We have limited this study to the generation of $\varphi(E,t)$ for a few geometrical bucklings used in the experiments. In order to undertake a complete modal analysis, $\varphi_{\ell,m,n}(E,t)$ will have to be generated by the additional machine calculations or by the interpolation method using the curves given in this study. It should, however, be pointed out that for times slightly greater than the slowing down time, the distribution function will be strongly dependent upon the initial source distribution. The initial source distribution given by the Fermi Age model will also have to be replaced by Eriksson's distribution described in Chapter III. No attempt is made to undertake this study, here; it is a complete problem in itself.

APPENDIX IV COMPARISON OF THE STEADY AND TIME-DEPENDENT ENERGY SPECTRUM IN THE FINITE MEDIUM

The energy spectrum of a decaying pulse of neutrons was found to be softened in the finite medium compared to the Maxwellian distribution. However, the energy spectrum due to a steady source is hardened spectrum. In the heavy gas model, under the diffusion theory approximation, the neutron energy distribution for the time independent case can be given as follows:

$$E \frac{d^2\varphi(E)}{dE^2} + E \frac{d\varphi(E)}{dE} + \varphi(E) = \frac{\Sigma_a + DB^2}{2\mu\Sigma_{s_o}} \varphi(E) - S(E) \qquad (IV.1)$$

where

$$S(E) = \text{External Source}$$

$$= \delta(E - E_o)$$

This equation has been studied extensively by Hurwitz et al.[1] and also by Cohen[2] in some special cases.

We shall deal with two cases.

Case A: Absorption cross section Σ_a = constant

Diffusion coefficient $D(E)$ = constant

Case B: Absorption cross section follows 1/v law

Diffusion coefficient is constant.

For Case A, the differential Equation (IV.1) can be transformed into the confluent hypergeometric equation form by using $\varphi(E) = \psi(E)Ee^{-E}$. The solution of the transformed equation is given by

Jahnke and Emde.[3] It is as follows:

$$\varphi(E) = A\, E e^{-E}\, {}_1F_1\,[\alpha,2,E] \qquad \text{(IV.2)}$$

where:

$$\alpha(E) = \frac{D(E)B^2 + \Sigma_a}{2\mu\Sigma_{s_o}} = \text{constant}$$

$${}_1F_1\,(\alpha,2,E) = 1 + \frac{\alpha}{2}E + \frac{\alpha(\alpha+1)\,E^2}{2!\;2(2+1)} + \ldots .$$

For the Case B, we make the following additional transformation $E^{1/2} = x$. Transformed equation is as follows:

$$x\frac{d^2\psi(E)}{dx^2} + (3 - 2x^2)\frac{d\psi(E)}{dx} - 4(x\,\alpha_{B^2} - \alpha_{\Sigma_a})\psi(x) = 0 \qquad \text{(IV.3)}$$

where:

$$\alpha_{B^2} = \frac{D(E)\,B^2}{2\mu\Sigma_{s_o}}$$

$$\alpha_{\Sigma_a} = \frac{\Sigma_{a_o}}{2\mu\Sigma_{s_o}} \quad \text{and} \quad \Sigma_{a_o} = \Sigma_a\, x$$

We seek the series solution for the differential Equation (IV.3).

$$\psi(x) = \sum_{n=0} a_n\, x^n \qquad \text{(IV.4)}$$

On substituting the above series into the differential Equation (IV.3), and equating the coefficients of x^{n+1}, we get the following recurrence relation between the coefficients.

$$a_n = \frac{\{2(n-2) + 4\alpha_{B^2}\}\, a_{n-2} + 4\alpha\, \Sigma_a a_{n-1}}{n(n+2)} \qquad \text{(IV.5)}$$

On evaluating the coefficients, we get the following solution for $\varphi(E)$

$$\varphi(E) = a_o E_e^{-E} \left[1 + \frac{4\alpha\Sigma_a}{3} E^{1/2} + \frac{1}{8}\left\{4\alpha_{B^2} + \frac{(4\alpha\Sigma_a)^2}{3}\right\} E \right.$$

$$+ \frac{E^{3/2}}{15}\left\{\frac{4\alpha\Sigma_a}{3}(4\alpha_{B^2}+2) + \frac{4\alpha\Sigma_a}{8}\left(4\alpha_{B^2}+\frac{(4\alpha\Sigma_a)^3}{3}\right)\right\} \qquad \text{(IV.6)}$$

$$\left. + \frac{E^2}{24}\left\{\frac{(1+\alpha_B{}^2)}{2}(4\alpha_{B^2} + \frac{(4\alpha\Sigma_a)^2}{3}) + \frac{4\alpha\Sigma_a}{15}\left(\frac{4\alpha\Sigma_a}{3}(4\alpha_{B^2}+2) + \frac{4\alpha\Sigma_a}{8}\left(4\alpha_{B^2}+\frac{(4\alpha\Sigma_a)^2}{}\right)\right)\right\} + \ldots\right]$$

For the case of zero absorption, the solutions given by the Equations (IV.6) and (IV.2) become identical.

We note the following features of the solution for $\varphi(E)$ as given by the Equation (IV.6).

1. For zero absorption and infinite medium, the neutron energy distribution is Maxwellian.
2. For small 1/v absorption and infinite medium, the spectrum is Maxwellian.
3. For large 1/v absorption and for $B^2 \neq 0$, the spectrum is hardened.

For a special case of $B^2 = 2.96 \times 10^{-3}$ cm^{-2} for graphite, we calculated $(\overline{\frac{1}{v}})$ according to the pulsed and steady state sources according to the following definition.

$$\left(\overline{\frac{1}{v}}\right) = \frac{\int_o^{E = 10\ KT} \varphi(E)\ \frac{1}{v}\ dE}{\int_o^{E = 10\ KT} \varphi(E)\ dE} \qquad \text{(IV.7)}$$

We have the following results for the pulsed, steady state and Maxwellian cases.

$$\left(\frac{\bar{1}}{v}\right)_{\text{Pulse}} = 9.36 \times 10^{-1}$$

$$\left(\frac{\bar{1}}{v}\right)_{\text{Steady state}} = 8.06 \times 10^{-1}$$

$$\left(\frac{\bar{1}}{v}\right)_{\text{Maxwellian}} = 8.84 \times 10^{-1}$$

The degree of hardness of the spectrum is not equal to the degree of softness. Shift of the energy spectrum occurs in two different directions for these two cases. The deviation is not symmetrical about the Maxwellian spectrum.

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1. Hurwitz, H. H. Jr., Nelkin, H. S., and Habetler, G. J. (1956) Nuclear Sci. Engng. 1, 280.

2. Cohen, E. R., (1957) Nuclear Sci. Engng. 2, 227.

3. Jahnke, E., and Emde, F., "Tables of Functions." (1945) Dover Publications, New York.

APPENDIX V. DIFFUSION PARAMETERS OF BERYLLIUM AND GRAPHITE

We list the diffusion parameters of beryllium, as compiled by de Saussure and Silver[1] and of graphite as given by Dardel and Sjöstrand.[2]

TABLE V.1

BERYLLIUM (DENSITY 1.85 gms/cc)[1]

Authors	Diffusion Coefficient (D_o) x 10^{-5} (cm^2/sec)	Diffusion Cooling (C) Coefficient
Antonov et al.	1.17 ± 0.05	2.5 ± 0.9
Campbell et al.	1.25	0
Kloverstrom et al.	1.24 ± 0.04	3.15 ± 0.65
de Saussure and Silver	1.25 ± 0.06	1.1 ± 0.9

TABLE V.2

GRAPHITE (DENSITY 1.60 gms/cc)[2]

Authors	D_o(cm^2/sec) x 10^{-5}	C	D_o(cm^4/sec) x 10^{-5}
Beckurts	2.13 ± 0.017		16.3 ± 2.5
Antonov et al.	2.07 ± 0.03		12.5 ± 2.0

Decay constant $\lambda = \Sigma_{a_o} v_o + D_o B^2 (1 - CB^2)$

REFERENCES

1 de Saussure, G. and Silver, E.G. ORNL-2641 (1959).

2 von Dardel, G. and Sjöstrand, N.G. Progress in Nuclear Energy Series I, Vol. II, (1958) Pergamon Press, New York.