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Technical Note

COMPUTER NUMBER SYSTEMS: LINEAR AND NON-LINEAR CATEGORIES

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SUMMARY

This report is intended to answer some fundamental questions such as: what is the structure of a number system in general? and how is this structure modified with linearity and redundancy? Their properties and usefulness with respect to the arithmetic operations have been derived. The discussion includes some non-linear types of number systems to enable the reader to understand the advantages of linear systems over others.

INTRODUCTION

Here we may pose the question: what is a number system? To answer the question, it will be fitting to obtain the necessary and sufficient conditions a system must satisfy. A number system could well be considered as a method of representing and assigning names to the integers. Precisely, it is the structure of the system of integers. When our discussion is limited to computer application, we mean it as the system representation of a finite subset of the integers, namely Z_M (integers modulo M, or $Z_M = \{0, 1, \ldots M-1\}$).

A NUMBER SYSTEM IN GENERAL

A system of representation N will be a number system if and only if

(1) N is a cartesian product of n tuples and is denoted as

$N = D_1 \times D_2 \times \ldots \times D_n$

where $D_i = \{0, 1, \ldots m_i - 1\}$

D_i is called the ith digit set, and m_i the ith modulus.

(2) N is closed under addition.

$x, y \in N \Longrightarrow x + y \in N$ (closure law)

$\exists 0 \in N$ such that $x + 0 = 0 + x = x$ (identity)

It will be proved later that, in addition to the above two, the other axioms of an abelian group are obeyed by non-redundant linear number systems.

(3) $\exists$ a mapping $w : N \to Z_M$

w is a function of the n coordinate variables such that

$w(x + y) \equiv w(x) + w(y) \pmod{M}$ for all $x, y \in N$.

This function, which we will call hereafter the weight function, is the most significant property of the number system and is responsible largely in determining the arithmetic and carry properties of N. We will observe further that the division of number systems into different categories is based on this function. The following definitions which are quite familiar are included as a basis for further discussion on number systems.

Definition 1. A number system N (obeying the three axioms stated above) is complete $\Longleftrightarrow$ w is onto Z_M. This is to say that for all $a \in Z_M$ $\exists x \in N$ such that $w(x) = a$.

Also $$M > \prod_{i=1}^{n} m_i \Longrightarrow N \text{ is incomplete.}$$

Definition 2. N is a redundant system $\Longleftrightarrow$ $\exists x, y \in N$ such that $x \neq y$ and $w(x) = w(y)$.

The above two definitions can be combined to obtain the lemma:

Lemma. A number system is complete and non-redundant $\Longleftrightarrow$ w is an isomorphism.

Definition 3. N is said to be a linear system if and only if w is a linear function of n coordinate variables, the coefficients coming from Z_M.

Using the lemma, we can divide systems into redundant and non-redundant types. Definition 3 will enable us to classify the systems into linear and non-linear types.

Definition 4. A number system N is weighted $\Longleftrightarrow$ $\exists \rho_i \in Z_M$ for i = 1, 2...n such that for any $X = (x, \ldots x_n) \in N$ $w(x) \equiv |\sum \rho_i x_i|_M$ the weight function for weighted systems is a linear homogeneous function.

Thus all the weighted systems are linear. However, all linear systems are not weighted. Some examples typifying the above statement are given in the next section.

NON-WEIGHTED CODES

Before we go into the advantages of weighted and non-weighted systems, we shall examine the weight functions w of some non-weighted codes. Given below are tables of representation of the code known as excess three representing $Z_{10}(N_1 \to Z_{10})$, and the four-bit reflected binary code for $Z_{16}(N_2 \to Z_{16})$. It is well known that the excess three code has the advantage over the conventional binary (which is linear homogeneous) system, in that the 9's complement is obtained by interchanging 0's and 1's. The advantage of the reflected binary code is that it is an unit distance code and any single error in transfers would cause a change of one in magnitude. However, we will soon find that the weight function of the excess three code is linear and non-homogeneous, and that of the reflected binary code is non-linear.

TABLES OF REPRESENTATION

Excess Three Code N_1

Z_{10}	x_4	x_3	x_2	x_1
0	0	0	1	1
1	0	1	0	0
2	0	1	0	1
3	0	1	1	0
4	0	1	1	1
5	1	0	0	0
6	1	0	0	1
7	1	0	1	0
8	1	0	1	1
9	1	1	0	0

Four-Bit Reflected Binary Code N_2

Z_{16}	x_4	x_3	x_2	x_1
0	0	0	0	0
1	0	0	0	1
2	0	0	1	1
3	0	0	1	0
4	0	1	1	0
5	0	1	1	1
6	0	1	0	1
7	0	1	0	0
8	1	1	0	0
9	1	1	0	1
10	1	1	1	1
11	1	1	1	0
12	1	0	1	0
13	1	0	1	1
14	1	0	0	1
15	1	0	0	0

$N_i = D_4 \times D_3 \times D_2 \times D_1$ for $t_i = 1,2.$

$D_j = \{0, 1\}$ for $j = 1, 2, 3, 4.$

The mappings w of $N_1 \to Z_{10}$

$N_2 \to Z_{16}$

are defined as shown in the table.

Both of these mappings are (1-1) and onto, and hence the addition in N is defined by the following relation:

for $X, Y \epsilon N$

$X + Y = w^{-1} [w(X) + w(Y)].$

This shows that it is closed under addition and the existence of an identity is trivially simple. Therefore, these two codes come under the class of general number systems.

We can easily find that the weight function for the excess three code is such that

for $X = (x_1, x_2, x_3, x_4)$

$w(X) = 2^3x_1 + 2^2x_2 + 2x_3 + x_4 - 3.$

the coefficients $2^3, 2^2, 2, 1, M - 3 \epsilon Z_M$

and w is a non-homogeneous linear function on the variables $x_4,\ldots,x_1$.

However, in the case of the reflected binary code it is not so straightforward to obtain. The suggested procedure is as follows:

$$w(0, 0, 0, 0) = 0$$
$$w(0, 0, 0, 1) = 1 = \rho_1$$
$$w(0, 0, 1, 0) = 3 = \rho_2$$
$$w(0, 1, 0, 0) = 7 = \rho_3$$
$$w(1, 0, 0, 0) = 15 = \rho_4$$

for an n-bit code

$$w(0, 0, \quad , 0, \underset{\substack{\uparrow \\ \text{ith place.}}}{1}, 0, \ldots. 0) = 2^i - 1 = \rho_i$$

Next we denote the weight function for the n-bit reflected binary code as f_n. Taking note of the alternate negation in the weights of the digits in the positions where 1's are present, we can write as below:

In the case of a 1-bit code

$$f_1 = w(X_1) = \rho_1 X_1 = X_1$$

And for a 2-bit code

$$f_2 = w(X_2, X_1) = X_2\rho_2 + X_1\rho_1 - 2X_2X_1\rho_1$$

$$= X_2\rho_2 + (1 - 2X_2) f_1$$

For a 3-bit code

$$f_3 = w(X_3, X_2, X_1)$$

$$= X_3\rho_3 + X_2\rho_2 + X_1\rho_1 - 2X_3X_2\rho_2 - 2X_2X_1\rho_1 - 2X_3X_1\rho_1$$

$$+ 4X_3X_2X_1\rho_1$$

$$= X_3\rho_3 + (1 - 2X_3)[X_2\rho_2 + (1 - 2X_2)X_1\rho_1]$$

$$= X_3\rho_3 + (1 - 2X_3) f_2$$

By induction or otherwise, we can obtain

for $f_4 = w(X_4 \ldots X_1)$

$$= X_4\rho_4 + (1 - 2X_4) f_3$$

$$
\begin{aligned}
\text{and} \quad f_n &= w(X_n \dots X_1) \\
&= X_n\rho_n + (1 - 2X_n) f_{n-1} \\
&= X_n\rho_n + (1 - 2X_n) X_{n-1}\rho_{n-1} + (1 - 2X_n) X_{n-2}\rho_{n-2} \\
&\quad + \dots\dots\dots \\
&\quad + (1 - 2X_n)(1 - 2X_{n-1}) \dots (1 - 2X_2) X_1\rho_1
\end{aligned}
$$

which is clearly a non-linear function of order n.

In a non-redundant system (w is a (1-1) mapping), the addition in N is defined by w. This is because that for $X, Y \in N$

$$
\begin{aligned}
w(X + Y) &= [w(X) + w(Y)]_M \\
(X + Y) &= w^{-1}\{[w(X) + w(Y)]_M\}
\end{aligned}
$$

and w^{-1} is (1-1) and, as Z_M is an additive abelian group, so is N. More important is the property of all linear homogeneous systems that for all $X, Y \in N$

$$
w(X + Y) = w(x_1 + y_1, x_2 + y_2 \dots x_n + y_n) \qquad (*)
$$

This is trivial if $x_i + y_i < m_i$ for all $i = 1, \dots n$. In which case $X + Y = (x_1 + y_1, x_2 + y_2, \dots x_n + y_n) \in N$. But in cases where for any i $x_i + y_i \geq m_i$, then $(x_1 + y_1, \dots, x_n + y_n) \notin N$. However, (*) still holds.

$$
\begin{aligned}
w(X) &= [x_1\rho_1 + x_2\rho_2 + \dots + x_n\rho_n]_M \\
w(Y) &= [y_1\rho_2 + y_2\rho_2 + \dots + y_n\rho_n]_M \\
w(X + Y) = w(X) + w(Y) &= [(x_1 + y_1)\rho_1 + \dots + (x_n + y_n)\rho_n]_n \\
&= w(x_1 + y_1, \dots, x_i + y_i, \dots, x_n + y_n).
\end{aligned}
$$

In an excess three code which is non-homogeneous

$$X = (x_4, x_3, x_2, x_1), \quad Y = (y_4, y_3, y_2, y_1), \quad X,Y\epsilon N_1$$

$$w(X) = 8x_4 + 4x_3 + 2x_2 + x_1 - 3$$

$$w(Y) = 8y_4 + 4y_3 + 2y_2 + y_1 - 3$$

$$w(X + Y) = w(X) + w(Y) = 8(x_4 + y_4) + 4(x_3 + y_3) + 2(x_2 + y_2) + (x_1 + y_1) - 3 - 3$$

$$= w(x_4 + y_4, x_3 + y_3, x_2 + y_2, x_1 + y_1) - 3$$

Thus

$$w(X + Y) \neq w(x_4 + y_4, x_3 + y_3, x_2 + y_2, x_1 + y_1).$$

In the reflected two-bit binary code $(N_2 \to Z_4)$

$$X = (x_2, x_1) \quad Y = (y_2, y_1) \quad X,Y\epsilon N_2$$

$$w(X) = 3x_2 + x_1 - 2x_2x_1$$

$$w(Y) = 3y_2 + y_1 - 2y_2y_1$$

$$w(X + Y) = w(X), w(Y) = 3(x_2 + y_2) + (x_1 + y_1) - 2(x_2x_1 + y_2y_1)$$

$$\neq w(x_2 + y_2, x_1 + y_1).$$

This only proves that digit-wise addition is possible only in linear homogeneous systems.

CONCLUSION

In linear homogeneous systems, we know that digit-wise addition can be carried out. Then if there is any overflow of the digit, we know the result is not in our number system. Therefore, it is necessary to transform the result back into the system. In conventional arithmetic this is done by carry generation and then carry assimilation. To understand the complete algebraic structure of this process of addition, it is necessary to introduce the theory of modules and other notions of abstract algebra. Using the theory of modules it is proved that any linear number system can be given the structure of a difference module of the type M/S; where M is a Z_m - module and S is a submodule of M. A set of generators of the submodule S placed in a matrix form will be called the carry matrix, and readily gives the carry propagation structure of the number system. For a full discussion on this topic, the interested reader may refer to the technical report on "Linear Number Systems" by R. F. Arnold, soon to be published by the Information Systems Laboratory, The University of Michigan.