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A STUDY OF A MODEL FOR PARALLEL COMPUTATION

by

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Abstract

This paper deals with a generalization of a model for parallel computation as formulated by Karp and Miller. A parallel computation is viewed as a directed graph in which a node represents a sequence of operations to be performed upon data on the node input branches with the results of an operation being placed upon the output branches. An operation associated with a node n may initiate if and only if, for each input branch d_p to n, the data queue length on d_p is greater than, or equal to, some threshold T_p. It is shown that a unique computation is defined independently of the timing of the node operations. Methods are derived for determining whether a computation terminates and for finding the number of performances of each node computation step involved. In particular, an algorithm is given which yields the number of performances of each node computation step for the set of terminating nodes of a computation graph. Finally, necessary and sufficient conditions for data queue lengths to remain bounded are derived.

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1. Introduction

This paper generalizes a model for parallel computation formulated by Karp and Miller [1]. Briefly, this model represents a computation as a directed graph in which a node n_i represents an operation to be performed upon data which lie upon the input branches to n_i. The results of the operation by node n_i are placed upon the output branches of n_i. A branch, therefore, represents a data queue. With each branch d_p directed from node n_i to node n_j is associated a quadruple (A_p,U_p,W_p,T_p), the elements of which are interpreted as follows:

A_p - the initial number of data items on branch d_p

U_p - the number of data items placed on d_p upon the termination of the operation associated with n_i.

W_p - the number of data items removed from d_p, under a first-in first-out queue discipline, upon the initiation of the operation associated with node n_j.

T_p - a threshold. In order to initiate the operation associated with node n_j, the queue length on d_p must be greater than, or equal to T_p.

It is shown in [1] that such a computation graph represents a unique computation, determined independently of operation times. Methods of determining termination properties and the number of performances of each computation step are developed. Finally, necessary and sufficient conditions for data queue lengths to remain bounded are derived.

In the present paper, a model for parallel computation based upon that in [1] is given. This model is a generalization in that the operation associated with a given node need not be fixed, but may vary throughout the computation, provided that it is uniquely specified as a function of the number of previous performances of the given node during the course of the computation. For this general model, determinacy is established in the sense that a unique computation is determined independently of operation times. The model is then specialized to be data dependent upon the lengths of data queues as follows: with each branch d_p, directed from node n_i to node n_j, is associated a quadruple $(A_p, \{U_p(n)\}, \{W_p(n)\}, T_p)$ where A_p and T_p have the previous interpretation. The sequence $\{U_p(n)\}$ specifies the number of data words, placed on d_p upon the termination of the n-th operation of node n_i. $\{W_p(n)\}$ specifies the number of data words, under a first-in first-out queue discipline, removed from d_p upon the initiation of the n-th operation of node n_j.

With this model we address ourselves to finding methods of determining whether such a computation terminates and to finding the number of performances of each node computation step involved. In particular, an algorithm is given which yields the number of performances of each node computation step for the set of terminating nodes of a computation graph. It is found, as was the case in [1], that the maximal strongly connected subgraphs of a computation graph, and their loops, play a central role in the analysis. Finally, it is assumed that the sequences $\{U_p(n)\}$ and $\{W_p(n)\}$ are periodic. Under this assumption, methods of determining termination properties and the number of performances of each node computation step are given, and necessary and sufficient conditions for data queue lengths to remain bounded are derived.

In the development it will be assumed that the reader is familiar with the definitions and principal results of [1]. The proofs of all results which follow in an obvious fashion from the proofs of analogous results in [1] will be omitted.

2. The Model

Definition 1

A computation graph is a directed graph consisting of:

(i) nodes $n_1, n_2, \ldots, n_\ell$.

(ii) branches $d_1, d_2, \ldots, d_t$, where any given branch $d_p = (n_i, n_j)$ is directed from a specified node n_i to a specified node n_j.

(iii) a sextuple $(\mathcal{A}_p, H_p, a_p, h_p, t_p, \alpha_p)$ associated with each branch $d_p = (n_i, n_j)$ where:

(a) $\mathcal{A}_p$ is a given set, the alphabet.

(b) $H_p \subseteq \pi_p$ where π_p is the free semigroup on $\mathcal{A}_p$, with identity $\emptyset$. If $\sigma \epsilon H_p$, $\tau \epsilon \pi_p$, then $\sigma\tau \epsilon H_p$.

(c) $a_p \ \epsilon \ \pi_p$.

(d) h_p is a mapping: $\pi_p \to \pi_p$ such that for $\sigma \epsilon H_p$, $\tau \epsilon \pi_p$, $h_p(\sigma\tau) = h_p(\sigma)$ and for $\sigma \epsilon \pi_p - H_p$, $h_p(\sigma) = \emptyset$.

(e) t_p is a mapping: $\pi_p \to \pi_p$ such that for $\sigma \epsilon H_p$, $\tau \epsilon \pi_p$, $t_p(\sigma\tau) = t_p(\sigma)\tau$ and for $\sigma \epsilon \pi_p - H_p$, $t_p(\sigma) = \sigma$.

(f) α_p is a mapping: $\pi_{p_1} \times \pi_{p_2} \times \ldots \times \pi_{p_r} \to \pi_p$ where n_i has input branches $d_{p_1}, d_{p_2}, \ldots, d_{p_r}$, and for $\sigma(p_i) \epsilon \pi_{p_i}$, $i=1,2,\ldots,r$; α_p: $(h_{p_1}(\sigma(p_1)), \ldots, h_{p_r}(\sigma(p_r))) \to \sigma(p) \epsilon \pi_p$.

The sextuple associated with each branch may be interpreted as follows: $\mathcal{A}_p$ is the set of all data words upon which O_j, the operation associated with node j, can operate. π_p represents the set of all sequences of data words which can appear as a queue on the branch d_p. H_p is interpreted as the class of all sequences acceptable to O_j; i.e. O_j cannot initiate unless the data queue on branch d_p is an element of H_p, so that H_p defines a threshold for the initiation of O_j. The data string a_p represents the initial data queue on the branch d_p. The function h_p defines, for a given data queue, the minimal acceptable head which permits the initiation of O_j, and the function t_p defines the data queue remaining after O_j removes data from the head of the queue upon its initiation. Finally the function α_p defines the data sequence which is placed on the tail of the branch $d_p = (n_i, n_j)$ upon termination of the operation O_i.

We shall require that the operation O_j associated with a given node n_j be eligible for initiation if and only if for each branch d_p directed into n_j, the data sequence on d_p is acceptable, i.e. is an element of H_p, and that no previously initiated performance of O_j is still in progress. After O_j becomes eligible for initiation, there may be some delay before initiation actually occurs. For each branch d_p directed into n_j, O_j, upon initiation, removes an initial sequence of data words on the queue associated with d_p. There may then be a further delay before O_j terminates, at which time for each branch d_q directed out of n_j, a sequence of data words determined by the function α_q is placed on d_q. The delays mentioned above are left unspecified by the model and, in fact, may differ for different initiations and performances of the same operation.

The above constraints on initiation lead to the following definitions of the possible sequences of initiations associated with a given computation graph G.

Let $\mathcal{E}$ be a sequence of nonempty sets $S_1, S_2, \ldots, S_n, \ldots$, such that each $S_n \subseteq \{1, 2, \ldots, \ell, \bar{1}, \bar{2}, \ldots, \bar{\ell}\}$ where ℓ is the number of nodes in G. Let

$x(j,0) = 0$ for $j \in \{1,2,\dots,\ell\}$ and for $N > 0$ let $x(j,N)$ denote the number of sets S_m, $1 \leq m \leq N$ of which j is an element. A similar definition holds for $x(\bar{j},N)$.

Let $d_p = (n_i,n_j)$ and let n_i have input branches $d_{p_1},d_{p_2},\dots,d_{p_r}$. We define a sequence $\sigma(p,N) \epsilon \pi_p$ for $N=0,1,\dots,$ as follows: $\sigma(p,0) = a_p$. For $N \geq 1$, assume that for all p, $\sigma(p,0)$, $\sigma(p,1)$, $\dots$, $\sigma(p,N-1)$ have been defined and let $N' = \max\{T \mid T \leq N \text{ and } i \epsilon S_T\}$. Then if $\bar{i} \epsilon S_N$ and N' exists define $\alpha(p,N-1) = \alpha_p[h_{p_1}(\sigma(p_1,N'-1)),\dots,h_{p_r}(\sigma(p_r,N'-1))]$. Otherwise, $\alpha(p,N-1) = \emptyset$. Then, if

$$j \epsilon S_N,\ \sigma(p,N) = t_p[\sigma(p,N-1)\alpha(p,N-1)]$$
$$j \not\epsilon S_N,\ \sigma(p,N) = \sigma(p,N-1)\alpha(p,N-1).$$

Definition 2

The sequence $\mathcal{E} = S_1,S_2,\dots,$ is an execution of G if and only if, for all $N \geq 0$, the following conditions hold.

(i) $x(j,N) - x(\bar{j},N) \in \{0,1\}$.

(ii) If $j \epsilon S_N$, there exists $R \geq N$ such that $\bar{j} \epsilon S_R$.

(iii) If $j \epsilon S_{N+1}$ and G has a branch d_p from n_i to n_j, then $\sigma(p,N) \epsilon H_p$.

(iv) If $\mathcal{E}$ is finite and of length P, then for each j, there exists a node n_i and a branch d_p from n_i to n_j such that $\sigma(p,P) \not\epsilon H_p$.

Definition 3

An execution $\mathcal{E}$ of G is called a proper execution if the following implication holds.

(v) If $\sigma(p,N) \epsilon H_p$ for every branch $d_p = (n_i,n_j)$ directed into n_j, then $j \epsilon S_P$ for some $P > N$.

The sequence $\mathcal{E}$ is interpreted as giving a temporal sequence of initiations of operations throughout a performance of the parallel computation specified by G. The occurrence of S_N denotes the simultaneous initiation of O_j for all $j \epsilon S_N$ and the simultaneous termination of O_j for all $\bar{j} \epsilon S_N$. The recursive definition of $\sigma(p,N)$ is given with this interpretation in mind so that $\sigma(p,N)$ is viewed as the sequence of data words occurring on branch d_p immediately after all initiations and terminations associated with S_N have been effected. Thus, for example, if $\bar{i} \epsilon S_N$ and $j \epsilon S_N$, the sequence $\alpha(p,N-1)$ which is the result of the previous initiation of O_i, is placed upon the queue $\sigma(p,N-1)$, yielding the data sequence $\sigma(p,N-1)\alpha(p,N-1)$. The result of initiating O_j then leaves on d_p the sequence $t_p(\sigma(p,N-1)\alpha(p,N-1))$ which is $\sigma(p,N)$. Conditions (i) and (ii) of Definition 2 insure that to each initiation of O_j there corresponds a termination and that no termination may occur which is not preceded by an initiation. Condition (iii) states that a node of G may initiate only if it is eligible for initiation. Condition (iv) gives the only situation under which an execution may terminate. Condition (v) of Definition 3 states that any node of G which is eligible for initiation must eventually initiate. Finally, it should be noted that the simultaneous initiation and termination of any operation O_j is permissible.

3. Determinacy of Computation Graphs

In this section we prove that, for proper executions of a computation graph with given initial data, the sequence of data words occurring on any branch is unique. Thus a computation graph represents a unique computation regardless of which proper execution occurs. We refer to this property as the determinacy of computation graphs. This result is embodied in Theorem 1, for the proof of which we require the following:

Lemma

Let $\mathcal{E} = S_1, S_2, \ldots, S_N, \ldots,$ be an execution of a computation graph and let $x(j,N) = n$. Then $\sigma(p,N) = t_p^{(n)}[a_p\ \alpha(p,0)\alpha(p,1),\ldots \alpha(p,N-1)]$.

PROOF: We shall use induction on N. For N=1 we must have $x(j,1) \epsilon \{0,1\}$. If $x(j,1) = 0$, $j \notin S_1$, so that $\sigma(p,1) = a_p\ \alpha(p,0) = t_p^{(0)}\ [a_p\ \alpha(p,0)]$. If $x(j,1) = 1$, $j \epsilon S_1$ so that $\sigma(p,1) = t_p[a_p\ \alpha(p,0)]$. In either case, the lemma holds.

Assume the truth of the lemma for N=P. Then if $x(j,P) = n$, $\sigma(p,P) = t_p^{(n)}[a_p\ \alpha(p,0)\ldots\alpha(p,P-1)]$. Suppose $j \notin S_{P+1}$. Then $\sigma(p,P+1) = \sigma(p,P)\alpha(p,P)$, i.e. $\sigma(p,P+1) = t_p^{(n)}[a_p\ \alpha(p,0)\ldots\alpha(p,P-1)]\alpha(p,P)$. If n=0, the lemma follows immediately. If $n \geq 1$, $t_p^{(n)}[a_p\ \alpha(p,0) \ldots \alpha(p,P-1)] = t_p[t_p^{(n-1)}(a_p\ \alpha(p,0) \ldots \alpha(p,P-1))]$ and by (iii) of Definition 2 we must have $t_p^{(n-1)}\ (a_p\ \alpha(p,0) \ldots \alpha(p,P-1)) \epsilon H_p$. Hence $t_p[t_p^{(n-1)}(a_p\ \alpha(p,0) \ldots \alpha(p,P-1)]\alpha(p,P) = t_p[t_p^{(n-1)}(a_p\alpha(p,0) \ldots \alpha(p,P-1))\alpha(p,P)]$. By iterating this argument we finally obtain $\sigma(p,P+1) = t_p^{(n)}[a_p\ \alpha(p,0) \ldots \alpha(p,P)]$ so that for $j \notin S_{P+1}$ the lemma holds.

If $j \epsilon S_{P+1}$, $\sigma(p,P+1) = t_p[\sigma(p,P)\alpha(p,P)]$

$$= t_p[t_p^{(n)}\{a_p\ \alpha(p,0)\ \ldots\ \alpha(p,P-1)\}\alpha(p,P)]$$

$$= t_p[t_p^{(n)}\{\ a_p\ \alpha(p,0)\ \ldots\ \alpha(p,P)\}]$$

by the above argument, so that $\sigma(p,P+1) = t_p^{(n+1)}[a_p\ \alpha(p,0)\ \ldots\ \alpha(p,P)]$ and the induction is completed.

For the proof of Theorem 1 we shall require some notation. If σ has the form $\sigma = \theta\tau$ with $\theta,\tau\epsilon\pi_p$, we write $\theta \subseteq \sigma$. Let $\mathcal{E}$ be an execution. We write $N(j,s) = \text{Min}\ \{N \mid x(j,N) = s\}$.

In what follows, primed symbols will indicate that these are symbols defined with respect to the proper execution $\mathcal{E}'$ (e.g. $\sigma'(p,N')$, $x'(j,N')$) while unprimed symbols either relate to the proper execution $\mathcal{E}$ (e.g. $\sigma(p,N)$, $x(j,N)$) or are independent of the executions considered, (e.g. h_p, t_p, a_p).

Theorem 1

Let $\mathcal{E} = S_1,S_2,\ldots,S_N,\ldots$, and $\mathcal{E}' = S_1',S_2',\ldots,S_N',\ldots$, be two proper executions of a computation graph G. If $x(j,N) = s$ then there exists N' such that $x'(j,N') = s$. Moreover, if $j\epsilon S_N$, then N' can be chosen such that $x'(j,N'-1) = s-1$ and $h_p(\sigma(p,N-1)) = h_p(\sigma'(p,N'-1))$.

PROOF: We apply induction to $N(j,s)$ with respect to the proper execution $\mathcal{E}$. Suppose $N(j,s) = 1$. Then $s\epsilon\{0,1\}$. If $s=0$, i.e. $x(j,1) = 0$, then, since $x'(j,0) = 0$, we take $N' = 0$ and the theorem holds in this case. If $s=1$, then $j\epsilon S_1$ which implies, by (iii) of Definition 2, $a_p \epsilon H_p$. This implies, by (v) of Definition 3, that there exists N' such that $x'(j,N') = 1$ with

$j \epsilon S'_{N'}$. By the above lemma we have $\sigma'(j,N'-1) = a_p\alpha'(p,0) \ldots \alpha'(p,N'-2)$ and since $a_p \epsilon H_p$, we have $h_p(\sigma(p,0)) = h_p(a_p) = h_p(\sigma'(j,N'-1))$ and again the theorem holds.

Now assume the theorem holds for $N(j,s) \leq P$ and suppose $N(j,s) = P+1$. Clearly, we must have $s \geq 1$. Then by the lemma, $\sigma(p,P) = t_p^{(s-1)}[a_p\ \alpha(p,0) \ldots \alpha(p,P-1)]$ and since $j \epsilon S_{P+1}$, by (iii) of Definition 2, we must have $\sigma(p,P) \epsilon H_p$. Now $N(j,s-1) \leq P$ so that, by the induction hypothesis, there exists M' such that $x'(j,M') = s-1$ with $j \epsilon S'_{M'}$. We shall prove that there exists N' such that $x'(j,N') = s$. Suppose, on the contrary, that N' does not exist. Then, for all $R' \geq M'$, we must have $\sigma'(p,R') = t_p^{(s-1)}[a_p\alpha'(p,0) \ldots \alpha'(p,R'-1)]$. Let $x(i,P) = t$. Then $N(i,t) \leq P$, so that by the induction hypothesis and the definition of $\alpha(p,\cdot)$, to each nonnull $\alpha(p,T)$, $T \leq P-1$, there corresponds an $\alpha'(p,T)$, such that $\alpha(p,T) = \alpha'(p,T')$. Finally, for $M_1, M_2 \leq P-1$, suppose that $\alpha(p,M_1)$ and $\alpha(p,M_2)$ are the results of two consecutive occurrences of $i \epsilon S_{T_1}$ and $i \epsilon S_{T_2}$, i.e. $T_1 < T_2$. Then by (i) of Definition 2, we must have $M_1 < M_2$. By the induction hypothesis there exist T'_1 and T'_2 with $T'_1 < T'_2$ with $i \epsilon S_{T'_1}$ and $i \epsilon S_{T'_2}$. By (i) and (ii) of Definition 2 there exist M'_1 and M'_2 such that $i \epsilon S'_{M'_1+1}$ and $\bar{i} \epsilon S'_{M'_2+1}$ and $M'_1 < M'_2$. Hence, to each nonnull sequence $\alpha(p,M)$ of $\sigma(p,P)$ there corresponds a sequence $\alpha'(p,M') = \alpha(p,M)$ of $\sigma'(p,R')$ and their ordering is preserved. It follows that, for sufficiently large R', $\sigma(p,P) \subseteq \sigma'(p,R')$. But $\sigma(p,P) \epsilon H_p$ so that, by (v) of Definition 3, there exists N' such that $x'(j,N') = s$ and $j \epsilon S'_{N'}$. Clearly, $x'(j,N'-1) = s-1$. Finally, if $x(\bar{i},P) \leq x'(\bar{i},N'-1)$, then $\sigma(p,P) \subseteq \sigma'(p,N'-1)$ while if $x(\bar{i},P) > x'(\bar{i},N'-1)$, then $\sigma'(p,N'-1) \subseteq \sigma(p,P)$. In either case, since both $\sigma(p,P)$ and $\sigma'(p,N'-1)$ are in H_p, it follows that $h_p(\sigma(p,P)) = h_p(\sigma'(p,N'-1))$. This completes the induction.

Theorem 1 establishes that the number of performances of an operation O_j is the same for all proper executions of a given computation graph G. Moreover, the n-th performance of O_j under two proper executions operates upon the same data in both. It follows that the results of a proper execution of G are determined solely by the initial data regardless of the timing of the operations, i.e. regardless of the proper execution considered. One more point should be noted. The proof of Theorem 1 and its preceding lemma requires only that the set H_p be the same at the n-th initiation of O_j for all proper executions, and that each of the functions t_p, α_p, and h_p be single valued and perform the same mapping for all proper executions at the n-th initiation of O_j. Therefore consider modifying the definition of a computation graph by replacing the single "threshold set" H_p by a collection $\{H_p(n)\}$ such that the threshold associated with $d_p = (n_i, n_j)$ is $H_p(n)$ upon the n-th initiation of O_j. Similarly, replace the functions $t_p(\cdot)$ and $h_p(\cdot)$ by functions $t_p(\cdot,n)$ and $h_p(\cdot,n)$ respectively, and modify the recursive definition of $\sigma(p,N)$ in the obvious way. Then by the above remark, Theorem 1 holds for this more general definition of a computation graph and its associated proper executions.

In the following development we shall investigate a particular realization of this more general form of a computation graph. More precisely, we shall deal with a model which is specified by the following choice of $H_p(n)$ and $t_p(\cdot,n)$: $H_p(n) = H_p = \{\sigma_p \mid \sigma_p \epsilon \pi_p \text{ and } \ell(\sigma_p) \geq T_p\}$. Here $\ell(\sigma_p)$ is the length of the sequence σ_p with the convention $\ell(\emptyset) = 0$, and T_p is a nonnegative integer. Let $\{W_p(n) \mid n=1,2,\ldots\}$ be a sequence of nonnegative integers with $W_p(n) \leq T_p$. If $\sigma_p \epsilon H_p$, we have $\sigma = a_1 a_2 \ldots a_{W_p(n)} \ldots a_{T_p} \ldots a_r$

where $a_i \in \mathcal{a}_p$, $i=1,2,\ldots,r$. Define $t_p(\sigma_p,n) = a_{W_p(n)+1} \cdots a_r$. For $\sigma \notin H_p$, $t_p(\sigma_p,n) = \sigma_p$. Let $\mathcal{E}$ be an execution for the above form of a computation graph. A simple induction yields

$$\ell[\sigma(p,N)] = \ell(a_p) + \sum_{R=0}^{N-1} \ell[\alpha(p,R)] - \sum_{n=1}^{x(j,N)} W_p(n).$$

But $\alpha(p,R) = \emptyset$ except possibly when $\bar{i} \epsilon S_{R+1}$. For $\bar{i} \epsilon S_{R+1}$ define $U_p[x(\bar{i},R+1)] = \ell[\alpha(p,R)]$. Then

$$\ell[\sigma(p,N)] = A_p + \sum_{n=1}^{x(\bar{i},N)} U_p(n) - \sum_{n=1}^{x(j,N)} W_p(n) \text{ where } A_p = \ell(a_p).$$

It follows that $\sigma(p,N) \epsilon H_p$ if and only if $\ell[\sigma(p,N)] \geq T_p$. Thus for $d_p = (n_i,n_j)$, T_p, A_p, $U_p(n)$ and $W_p(n)$ may be interpreted, respectively, as a threshold which prevents O_j from initiating whenever the threshold exceeds the queue length on d_p, the initial data length on branch d_p, the length of data placed on d_p upon the n-th termination of O_i, and the length of data removed from d_p upon the n-th initiation of O_j.

Finally, we note, by (i) of Definition 2, that $x(i,N) - x(\bar{i},N) \epsilon \{0,1\}$, so that $\sigma(p,N) \epsilon H_p$ implies that

$$A_p + \sum_{n=1}^{x(i,N)} U_p(n) - \sum_{n=1}^{x(j,N)} W_p(n) \geq T_p$$

while $\mathcal{E}$ finite implies $x(i,N) = x(\bar{i},N)$ by (i) and (ii) of Definition 2. We may then reformulate Definition 2 and 3 of an execution and proper execution to conform to the present model as follows:

Definition 2'

The sequence $\mathcal{E} = S_1, S_2, \ldots,$ is an execution of G if and only if, for all $N \geq 0$ the following conditions hold.

(i) $x(j,N) - x(\bar{j},N) \epsilon \{0,1\}$.

(ii) If $j \epsilon S_N$, there exists $R \geq N$ such that $\bar{j} \epsilon S_R$.

(iii) If $j \epsilon S_{N+1}$ and G has a branch d_p from n_i to n_j, then

$$A_p + \sum_{1}^{x(i,N)} U_p(n) - \sum_{1}^{x(j,N)} W_p(n) \geq T_p.$$

(iv) If $\mathcal{E}$ is finite and of length P, then for each j there exists a node n_i and a branch d_p from n_i to n_j, such that

$$A_p + \sum_{1}^{x(i,N)} U_p(n) - \sum_{1}^{x(j,N)} W_p(n) < T_p.$$

Definition 3'

An execution $\mathcal{E}$ of G is called a proper execution if the following implication holds.

(v) If

$$A_p + \sum_{1}^{x(i,N)} U_p(n) - \sum_{1}^{x(j,N)} W_p(n) \geq T_p$$

for every branch $d_p = (n_i, n_j)$ directed into n_j, then $j \epsilon S_P$ for some $P > N$.

An Example

Consider matrix multiplication C = AB where

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

$i=1,\ldots,m;\ j=1,\ldots,p.$

Figure 1 shows a computation graph for this calculation where each branch d_p is labeled with its associated quadruple $(A_p, \{U_p(n)\}, \{W_p(n)\}, T_p)$.

Each of the indices i,j, and k is assumed to have zero as its initial value. Index modification is performed by the operations $O_1,O_2,\ldots,O_7$ as follows:

O_1 replaces k by (k+1) mod n

O_2 replaces j by (j+1) mod p

O_3 replaces i by i+1.

The initial data on the branches $(n_1,n_1),(n_2,n_2)$ and (n_3,n_3) consist of markers sequencing the indices through their appropriate values. Termination of the queue associated with branch (n_3,n_3) is caused by the depletion of the queue associated with branch (n_3,n_3). The values U = np and W=1 on the branch (n_3,n_4) specify that any given value of the index i is used in np consecutive performances of O_4. Similarly, for branch (n_2,n_5).

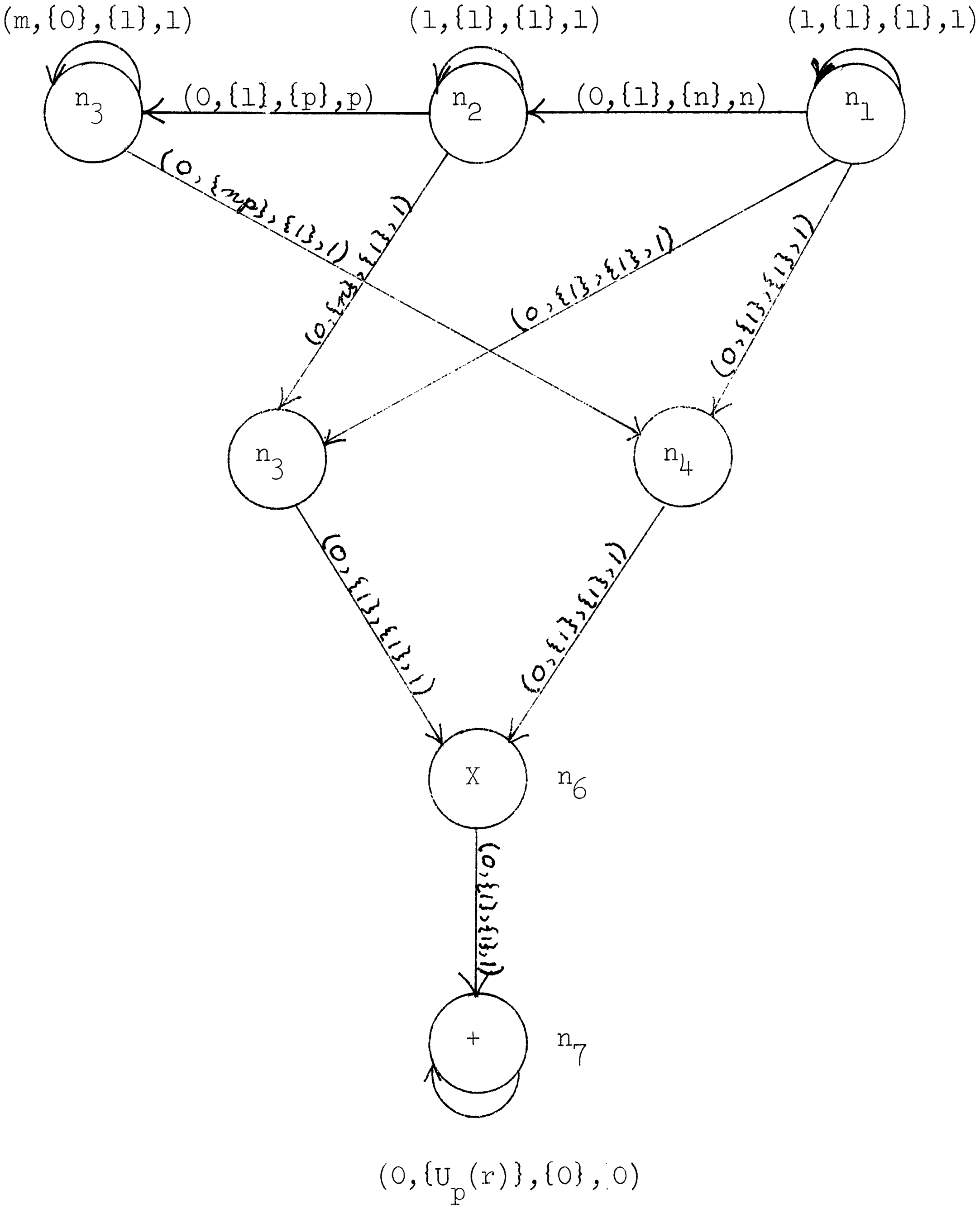
O_4 fetches a_{ik} from storage

O_5 fetches b_{kj} from storage

O_6 forms the product $a_{ik}b_{kj}$

O_7 adds $a_{ik}b_{kj}$ to the partial sum in a sum bucket and puts out c_{ij} on (n_7,n_7) only when it has performed n summations.

At the end of the calculation (n_7,n_7) contains the sequence $c_{11},c_{12},\ldots,c_{1p},c_{21},\ldots,c_{2p},\ldots,c_{m1},\ldots,c_{mp}$.



$U_p(r)$ = 1 if r | n
= 0 otherwise

Figure 1. A computation graph for matrix multiplication.

4. Termination Properties of Executions

We shall assume throughout that $U_p(n) \neq 0$ and $W_p(n) \neq 0$ for all n. The case $U_p(n) = W_p(n) = 0$ for all n is treated in [1]. We write $\sigma_p = A_p - T_p + 1$, and $\Sigma_j = \{(i,P) \mid d_p$ is a branch from node n_i to node $n_j\}$.

Theorem 3

Let G be a computation graph. Consider the system of inequalities

$$\min_{(i,P)\epsilon\Sigma_j} [\sigma_p + \sum_1^{x(i)} U_p(n) - \sum_1^{x(j)} W_p(n)] \leq 0 \qquad \text{for } j=1,2,\ldots,\ell \qquad (1)$$

If Σ_j is empty for any j or if (1) has no solution in nonnegative integers then every execution $\mathcal{E}$ of G is of infinite length.

If (1) has a nonnegative integer solution, then every execution $\mathcal{E}$ of G is of finite length, and, for any j, the number of performances of O_j is $x^*(j)$, the minimum integer solution of (1).

From this point on we assume that $\Sigma W_p(n) = \infty$.

Lemma 1

If, for some $(i,p)\epsilon\Sigma_j$, $\Sigma U_p(n) < \infty$, n_j terminates. (Note in this case n_i need not terminate.) If, for some $(i,p)\epsilon\Sigma_j$, n_i terminates, then n_j terminates.

Lemma 2

Let G' be a strongly connected subgraph of G. Then in any execution $\mathcal{E}$ of G either every node of G' terminates or none does.

5. Termination Properties of Proper Executions

We consider, in what follows, the case $\Sigma\, U_p(n) = \infty$, as well as $\Sigma\, W_p(n) = \infty$.

Lemma 3

The node $n_j \epsilon G$ terminates if and only if, for some $(i,p) \epsilon \Sigma_j$ n_i terminates. Also, if n_j terminates, there exists $(i,p) \epsilon \Sigma_j$ such that n_i terminates and

$$\sum_{1}^{\underline{x}(j)} W_p(n) \geq \sigma_p + \sum_{1}^{\underline{x}(i)} U_p(n) \tag{2}$$

where $\underline{x}(k) = \max x(k,N)$ in any proper execution of G.

Theorem 4

Let S be the set of indices of nodes terminating in G. For each $j \epsilon S$, $\underline{x}(j)$ is given by the minimum value of $x(j)$ in any nonnegative integer solution of

$$\min_{\substack{(i,p)\epsilon\Sigma_j \\ i \epsilon S}} [\sigma_p + \sum_{1}^{x(i)} U_p(n) - \sum_{1}^{x(j)} W_p(n)] \leq 0. \tag{3}$$

Note: The proof of Theorem 4 establishes that the set $\{\underline{x}(j), j \epsilon S\}$ is a solution of (3). Since $\underline{x}(j) = x^*(j)$, the minimum value of $x(j)$ in any nonnegative integer solution of (3), the set $\{\underline{x}(j),\ j \epsilon S\}$ simultaneously minimizes all of the quantities $x(j)$ over the set of all nonnegative integer solutions of (3). We call this the minimal solution of (3).

Theorem 5

The following iteration scheme converges in a finite number of steps to the minimal solution of (3), for $j \epsilon S$.

$$x^{(0)}(j) = 0.$$

Choose $x_p^{(n+1)}(j)$ so that:

(i) Whenever

$$\sum_{k=1}^{x^{(n)}(j)} W_p(k) \geq \sigma_p + \sum_{k=1}^{x^{(n)}(i)} U_p(k)$$

$$x_p^{(n+1)}(j) = x^{(n)}(j).$$

(ii) Otherwise

$$\sum_{k=1}^{x_p^{(n+1)}(j)} W_p(k) \geq \sigma_p + \sum_{k=1}^{x^{(n)}(i)} U_p(k) \qquad \text{and}$$

$$\sum_{k=1}^{x_p^{(n+1)}(j)-1} W_p(k) < \sigma_p + \sum_{k=1}^{x^{(n)}(i)} U_p(k) \qquad \text{where}$$

$$x^{(n)}(r) = \operatorname*{Min}_{(i,p)\epsilon\Sigma_r} x_p^{(n)}(r) \qquad r\epsilon S.$$

PROOF: We use induction on n to show that for all n and for $j \epsilon S$, $\underline{x}(j) \geq x^{(n)}(j)$. For n=0, $x^{(0)}(j) = 0 \leq \underline{x}(j)$.

Assume the result for n=r. We show it true for n=r+1. Clearly $x^{(r+1)}(j) \geq x^{(r)}(j)$. If $x^{(r-1)}(j) = x^{(r)}(j)$, the result is immediate. Otherwise $x^{(r+1)}(j) > x^{(r)}(j)$ so that for all $(i,p)\epsilon\Sigma_j$, (ii) of the iteration scheme applies.

Suppose, now, that the induction hypothesis fails for n=r+1. Then $x^{(r+1)}(j) > \underline{x}(j)$. Hence, for any $(i,p)\epsilon\Sigma_j$, $x_p^{(r+1)}(j) > \underline{x}(j)$. Then by (ii) of the iteration scheme,

$$\sum_{k=1}^{\underline{x}(j)} W_p(k) \leq \sum_{k=1}^{x_p^{(r+1)}(j)-1} W_p(k) < \sigma_p + \sum_{k=1}^{x^{(r)}(i)} U_p(k) \quad \text{so that}$$

$$\sum_{k=1}^{\underline{x}(j)} W_p(k) < \sigma_p + \sum_{k=1}^{x^{(r)}(i)} U_p(k) \leq \sigma_p + \sum_{k=1}^{\underline{x}(i)} U_p(k),$$

the last inequality following from the induction hypothesis for n=r. But then, for all $(i,p)\epsilon\Sigma_j$,

$$\sigma_p + \sum_{k=1}^{x(i)} U_p(k) - \sum_{k=1}^{x(j)} W_p(k) > 0$$

contradicting (3).

Hence $x^{(n)}(j) \leq \underline{x}(j)$ for all $j\epsilon S$. Since, for each j, $x^{(n)}(j)$ is bounded above by $\underline{x}(j)$, and since $x^{(n+1)}(j) \geq x^{(n)}(j)$, the iteration must terminate with some n such that $x^{(n+1)}(j) = x^{(n)}(j)$ for all $j\epsilon S$. For this value of n, (i) of the iteration scheme must hold for all $j\epsilon S$ and some $(i,p)\epsilon\Sigma_j$, otherwise (ii) would yield $x^{(n+1)}(j) > x^{(n)}(j)$. For each such j, and some $(i,p)\epsilon\Sigma_j$, (i) yields

$$\sum_{k=1}^{x^{(n)}(j)} W_p(k) \geq \sigma_p + \sum_{k=1}^{x^{(n)}(i)} U_p(k) \qquad \text{i.e.}$$

$$\sigma_p + \sum_{k=1}^{x^{(n)}(i)} U_p(k) - \sum_{k=1}^{x^{(n)}(j)} W_p(k) \leq 0$$

i.e. $\{x^{(n)}(j), j \epsilon S\}$ is a solution of (3).

Combining this with the result $x^{(n)}(j) \leq x(j)$ yields $x^{(n)}(j) = \underline{x}(j)$.

Theorem 6

Let G' be a strongly connected subgraph of a computation graph G. Then G' is terminating if and only if:

(i) For some $n_j \epsilon G'$ there exists $(i,p) \epsilon \Sigma_j$ such that n_i is terminating and $n_i \notin G'$, or

(ii) The system of inequalities

$$\min_{\substack{(i,p)\epsilon\Sigma_j \\ d_p \epsilon G'}} [\sigma_p + \sum_{1}^{x(i)} U_p(n) - \sum_{1}^{x(j)} W_p(n)] \leq 0 \qquad (5)$$

for all j such that $n_j \epsilon G'$ has a solution in nonnegative integers.

Theorem 7

A maximal strongly connected subgraph G^s terminates in G if and only if there exists G^r such that G^r is self-terminating and $G^r \geq G^s$.

6. Conditions for Self-Termination

The condition for self-termination for a loop with ℓ nodes is the existence of a nonnegative integer solution of the following inequalities, obtained from Equation (5) of Theorem 6.

$$\left.\begin{array}{l} \displaystyle\sum_{1}^{x(1)} W_\ell(n) \geq \sigma_\ell + \sum_{1}^{x(\ell)} U_\ell(n) \\ \text{and for } k=1,2,\ldots,\ell-1 \\ \displaystyle\sum_{1}^{x(k+1)} W_k(n) \geq \sigma_k + \sum_{1}^{x(k)} U_k(n) \end{array}\right\} \qquad (6)$$

Theorem 8

A maximal strongly connected subgraph G' of a computation graph is self-terminating if and only if G' contains a self-terminating loop.

Corollary 2

Let $\tilde{G}$ be a computation graph contained in a computation graph G. If $\tilde{G}$ is terminating in $\tilde{G}$, then $\tilde{G}$ is terminating in G, and every node of G accessible from a node of $\tilde{G}$ is terminating in G. Also, if $n_j \epsilon \tilde{G}$, $\underline{x}(j,\tilde{G}) \geq \underline{x}(j,G)$.

Lemma 4

Let $\mathcal{E} = S_1,S_2,\ldots,S_N,\ldots$ be an execution of a loop L. Then, if $x(k+1,N) > 0$

$$A_k + \sum_{1}^{x(k,N)} U_k(n) - \sum_{1}^{x(k+1,N)} W_k(n) \geq T_k - W_k$$

where $W_k = \max_n W_k(n)$.

PROOF: W_k exists since we have, for all n, $T_k \geq W_k(n)$.

We use induction on N. For N=0, x(k+1,0) = 0 and there is nothing to be proved. Assuming the result for N=P, we prove it for N=P+1. If x(k+1,P+1) = x(k+1,P) the result follows from the induction hypothesis. Otherwise $k+1 \in S_{P+1}$. Thus x(k+1,P+1) = x(k+1,P)+1 and from Definition 2',

$$A_k + \sum_1^{x(k,P)} U_k(n) - \sum_1^{x(k+1,P)} W_k(n) \geq T_k$$

i.e.

$$A_k + \sum_1^{x(k,P)} U_k(n) - \sum_1^{x(k+1,P+1)} W_k(n) \geq T_p - W_k(x(k+1,P+1)) \geq T_k - W_k.$$

The desired result now follows since $x(k,P+1) \geq x(k,P)$.

In the case of a loop, the iteration scheme of Theorem 5 becomes:

$$x^{(0)}(k+1) = 0.$$

Choose $x^{(n+1)}(k+1)$ so that:

(i) Whenever

$$\sum_{r=1}^{x^{(n)}(k+1)} W_k(r) \geq \sigma_k + \sum_{r=1}^{x^{(n)}(k)} U_k(r), \quad x^{(n+1)}(k+1) = x^{(n)}(k+1) \qquad (10)$$

(ii) Otherwise

$$\sum_{r=1}^{x^{(n+1)}(k+1)} W_k(r) \geq \sigma_k + \sum_{r=1}^{x^{(n)}(k)} U_k(r) \quad \text{and}$$

$$\sum_{r=1}^{x^{(n+1)}(k+1)-1} W_k(r) < \sigma_k + \sum_{r=1}^{x^{(n)}(k)} U_k(r).$$

In order to prove Lemma 5 we require the following.

Lemma

$$x^{(n+1)}(k+1) > x^{(n)}(k+1), \implies x^{(n)}(k) > x^{(n-1)}(k).$$

PROOF: Since $x^{(n+1)}(k+1) > x^{(n)}(k+1)$, condition (ii) of the iteration scheme applies so that

$$\sum_{r=1}^{x^{(n+1)}(k+1)} W_k(r) \geq \sigma_k + \sum_{r=1}^{x^{(n)}(k)} U_k(r) > \sum_{r=1}^{x^{(n)}(k+1)} W_k(r).$$

But, in general,

$$\sum_{r=1}^{x^{(n)}(k+1)} W_k(r) \geq \sigma_k + \sum_{r=1}^{x^{(n-1)}(k)} U_k(r).$$

Combining these two results yields

$$\sigma_k + \sum_{r=1}^{x^{(n)}(k)} U_k(r) > \sigma_k + \sum_{r=1}^{x^{(n-1)}(k)} U_k(r)$$

i.e. $$\sum_{r=1}^{x^{(n)}(k)} U_k(r) > \sum_{r=1}^{x^{(n-1)}(k)} U_k(r)$$

i.e. $$x^{(n)}(k) > x^{(n-1)}(k).$$

Lemma 5

If ℓ is the number of nodes in the loop L, then, for k=1,2,...,ℓ $\underline{x}(k,L) \geq 1$ if and only if $x^{(\ell)}(k) \geq 1$. Also, if $\underline{x}(k,L) = 0$, then L is self-terminating. Since $x^{(\ell)}(k) = 0$ implies $\underline{x}(k,L) = 0$, the condition $x^{(\ell)}(k) = 0$ is sufficient for self-termination.

PROOF: Clearly, $x^{(\ell)}(k) \geq 1 \implies \underline{x}(k,L) \geq 1$. Conversely, suppose $\underline{x}(k+1,L) \geq 1$. Then there exists n such that $x^{(i)}(k+1) = 0$ for $i=0,1,2,\ldots,n-1$ and $x^{(n)}(k+1) > 0$. Assume $n > \ell$.

Now $x^{(n)}(k+1) > x^{(n-1)}(k+1)$ so that by the above lemma, $x^{(n-1)}(k) > x^{(n-2)}(k)$. Applying the Lemma ℓ times yields $x^{(n-\ell)}(k+1) = 0 > x^{(n-\ell-1)}(k+1) = 0$, a contradiction.

Hence $\underline{x}(k+1,L) \geq 1 \implies x^{(\ell)}(k+1) \geq 1$. The rest of the theorem follows.

We assume, in what follows, that $\{U_k(n)\}$ is periodic with period p_k and $\{W_k(n)\}$ is periodic with period q_k. Write

$$\bar{U}_k = \frac{\sum_{n=1}^{p_k} U_k(n)}{p_k}$$

$$\bar{W}_k = \frac{\sum_{n=1}^{q_k} W_k(n)}{q_k}$$

$$\alpha_k = \frac{\bar{U}_k}{\bar{W}_k} .$$

The following inequalities are self evident.

$$\left.\begin{array}{l}
\sum\limits_{n=1}^{x(k+1)} W_k(n) \le \bar{W}_k \; \{x(k+1) + (q_k - 1)\} \\[2ex]
\sum\limits_{n=1}^{x(k)} U_k(n) \ge \bar{U}_k \; \{x(k) - (p_k - 1)\} \\[2ex]
\sum\limits_{n=1}^{x(k+1,N)} W_k(n) \ge \bar{W}_k \; \{x(k+1,N) - (q_k - 1)\} \\[2ex]
\sum\limits_{n=1}^{x(k,N)} U_k(n) \le \bar{U}_k \; \{x(k,N) + (p_k - 1)\}
\end{array}\right\} \qquad (8)$$

By applying the first two of these inequalities to the fundamental loop inequalities (6), we obtain the following system:

$$x(k+1) \ge \alpha_k \; x(k) + \frac{\sigma_k}{\bar{W}_k} - \alpha_k(p_k-1) - (q_k-1).$$

Put

$$\beta_{k+1} = \frac{\sigma_k}{\bar{W}_k} - \alpha_k(p_k - 1) - (q_k - 1).$$

Then this system can be written in a matrix form as

$$\begin{bmatrix}
1 & 0 & . & . & . & 0 & . & . & -\alpha_\ell \\
-\alpha_1 & 1 & 0 & . & . & . & . & & 0 \\
0 & -\alpha_2 & 1 & 0 & . & . & . & & 0 \\
 & & & & & & & & \\
 & & . & . & . & & & & \\
0 & . & . & . & . & & -\alpha_{\ell-1} & & 1
\end{bmatrix}
\begin{bmatrix}
x(1) \\ x(2) \\ . \\ . \\ . \\ x(\ell)
\end{bmatrix}
\ge
\begin{bmatrix}
\beta_1 \\ \beta_2 \\ . \\ . \\ . \\ \beta_\ell
\end{bmatrix}$$

i.e.

$$(I - A) \; X \ge \beta \qquad (7)$$

Now apply the last two inequalities of (8) to the inequality of Lemma 4. We obtain

$$x(k+1,N) \leq \alpha_k \; x(k,N) + \frac{A_k - T_k + W_k}{\bar{W}_k} + \alpha_k(p_k-1) + (q_k-1)$$

Put

$$\gamma_{k+1} = \frac{A_k - T_k + W_k}{\bar{W}_k} + \alpha_k(p_k-1) + (q_k-1).$$

Then this system can be written

$$(I - A)X_N \leq \gamma \tag{9}$$

where $x(k,N)$ is the k-th component of X_N and γ_k is the k-th component of γ.

The fact that A is a positive matrix yields the following:

Lemma 6

If the matrix inequality (7) holds, then for any positive integer n

$(I - A^n) X \geq (A^{n-1} + A^{n-2} + \dots + A+1)\beta.$

If (9) holds, then for any positive integer n

$(I - A^n)X_N \leq (A^{n-1} + A^{n-2} + \dots \div A+I)\gamma.$

Define $g = \alpha_1\alpha_2\dots\alpha_\ell$. The matrix $I-A^\ell$ is diagonal with each diagonal element equal to $1-g$.

A. Loops with $g < 1$

Theorem 9

$g < 1 \Longrightarrow$ L is self-terminating.

PROOF: Suppose L is not self-terminating. Then Lemma 5 implies that there exists P such that if $N > P$, $x(k,N) \geq 1$ for all k. For all such N, $(I-A)X_N \leq \gamma$ by Lemma 4 and inequality (9).

By Lemma 6

$$(I-A^\ell)X_N = (1-g)X_N \leq (A^{\ell-1} + \dots + A+I)\gamma.$$

Thus, each component of X_N has an upper bound independent of N so that L is self-terminating.

B. Loops with g = 1

When g=1, det (I-A) = 0 so that the homogeneous system

$$(I - A)X = 0$$

has a nontrivial solution, as follows.

$$\begin{aligned} x_1 &= a \\ x_2 &= \alpha_1 a \\ &\cdots\cdots\cdots\cdot \\ x_\ell &= \alpha_1\alpha_2\ldots\alpha_{\ell-1}a \end{aligned}$$

where a is an arbitrary parameter. Thus, by judiciously choosing a, we can arrange that each x_i, $i=1,2,\ldots,\ell$ be a multiple of p_jq_j, $j=1,2,\ldots,\ell$. i.e. $p_jq_j \mid x_i$, $i, j=1,2,\ldots,\ell$. This is possible since each α_k is rational. Let $\tilde{X}$ be the minimal positive integer solution of $(I-A)X = 0$ subject to these conditions. $\tilde{X}$ exists, for the minimal a which yields a solution satisfying the above conditions will yield a component-wise minimal positive integer solution $\tilde{X}$.

Theorem 10

If L is self-terminating with g=1, then at least one component of $\underline{X}$ is less than the corresponding component of $\tilde{X}$.

PROOF: L is self-terminating, which implies that $\underline{X}$ is the smallest non-negative integer solution of (6), i.e.

$$\sum_{n=1}^{\underline{x}(k+1)} W_k(n) \geq \sigma_k + \sum_{n=1}^{\underline{x}(k)} U_k(n).$$

Therefore

$$\sum_{n=1}^{\underline{x}(k+1)} W_k(n) - \sum_{n=1}^{\tilde{x}(k+1)} W_k(n) \geq \sigma_k + \sum_{n=1}^{\underline{x}(k)} U_k(n) - \sum_{n=1}^{\tilde{x}(k+1)} W_k(n).$$

Assume, for all k, that $\underline{x}(k) \geq \tilde{x}(k)$. Then

$$\sum_{\tilde{x}(k+1)+1}^{\underline{x}(k+1)} W_k(n) \geq \sigma_k + \sum_{1}^{\underline{x}(k)} U_k(n) - \sum_{1}^{\tilde{x}(k+1)} W_k(n).$$

But $\tilde{x}(k+1)$ is a multiple of q_K. Hence

$$\sum_{1}^{\underline{x}(k+1)-\tilde{x}(k+1)} W_k(n) \geq \sigma_k + \sum_{1}^{\underline{x}(k)} U_k(n) - \bar{W}_k \, \tilde{x}(k+1) .$$

Finally, $\tilde{x}(k+1) = \alpha_k \, \tilde{x}(k)$

$$= \frac{\bar{U}_k}{\bar{W}_k} \tilde{x}(k)$$

so that $\bar{W}_k \, \tilde{x}(k+1) = \bar{U}_k \, \tilde{x}(k)$

$$= \sum_{1}^{\tilde{x}(k)} U_k(n)$$

since $\tilde{x}(k)$ is a multiple of p_k. Then we have

$$\sum_{1}^{\underline{x}(k+1)-\tilde{x}(k+1)} W_k(n) \geq \sigma_k + \sum_{1}^{\underline{x}(k)} U_k(n) - \sum_{1}^{\tilde{x}(k)} U_k(n)$$

$$= \sigma_k \sum_{\tilde{x}(k)+1}^{\underline{x}(k)} U_k(n)$$

$$= \sigma_k + \sum_{1}^{\underline{x}(k)-\tilde{x}(k)} U_k(n)$$

again since $\tilde{x}(k)$ is a multiple of p_k. But then $\underline{x}(k+1)-\tilde{x}(k+1)$ is a smaller nonnegative integer solution of (6), a contradiction.

This theorem leads to a finite procedure for determining self-termination. The iteration procedure (10), is followed until either:

(i) it terminates, yielding the number of performances of each operation, or else

(ii) for all k, $x^{(n)}(k) \geq \tilde{x}(k)$, establishing nontermination.

C. Loops with $g > 1$

Put $P = A^{\ell-1} + A^{\ell-2} + \ldots + A + I$. Then Lemma 6 yields $(I-A^{\ell})X \geq P\beta$, or, $(1-g)X \geq P\beta$, or, since $g > 1$

$$X \leq \frac{1}{1-g} P\beta .$$

From this inequality it follows that if any component of $P\beta$ is positive, the loop is not self-terminating. For if the loop is self-terminating, let $\underline{X}$ be the minimum nonnegative integer solution of (6). Then $\underline{X}$ satisfies

$$\underline{X} \leq \frac{1}{1-g} \; P\beta \tag{15}$$

so that some component of $\underline{X}$ is negative, a contradiction.

The upper bound on the values of $\underline{x}(k,L)$ given by (15) leads to the following finite procedure to determine whether the loop L is self-terminating or not: Apply the iteration scheme (10) until either

(i) it terminates, yielding the number of performances of each operation, or else

(ii) some component of $x^{(n)}(k)$ exceeds the upper bound obtained from (15).

7. Properties of Queue Lengths

Define

$$q(p,N) = A_p + \sum_{1}^{x(i,N)} U_p(n) - \sum_{1}^{x(j,N)} W_p(n).$$

The queue length on d_p immediately after the $x(j,N)$-th initiation of O_j is $q(p,N)$ if the $x(i,N)$-th performance of O_i has been completed; otherwise, the queue length is $q(p,N) - U_p(x(i,N))$.

Theorem 12

Let d_p be a branch of G from n_i to n_j. If $n_i \epsilon G^r$ and $n_j \epsilon G^s$, where G^r and G^s are different maximal strongly connected subgraphs of G, then the queue length of d_p is uniformly bounded if and only if n_i terminates.

Theorem 13

Let G' be a nonterminating maximal strongly connected subgraph of G. Every loop of G' for which $g > 1$ contains at least one branch for which the queue length is not uniformly bounded. Also, the queue length for every branch which lies in a loop with $g=1$ is uniformly bounded.

PROOF: Let L be a loop in G' consisting of the nodes $n_1, n_2, \ldots, n_\ell$ and branches $d_1, d_2, \ldots, d_\ell$, with d_k directed from n_k to n_{k+1}, $1 \leq k \leq \ell-1$, and d_ℓ directed from n_ℓ to n_1. Choose a proper execution of $\mathcal{E}$, and consider the linear form

$$F = q(1,N) + \frac{\bar{W}_1}{\bar{U}_2} q(2,N) + \frac{\bar{W}_1 \bar{W}_2}{\bar{U}_2 \bar{U}_3} q(3,N) + \ldots + \frac{\bar{W}_1 \bar{W}_2 \ldots \bar{W}_{\ell-1}}{\bar{U}_2 \bar{U}_3 \ldots \bar{U}_\ell} q(\ell,N).$$

Now $q(k,N) = A_k + \sum_{1}^{x(k,N)} U_k(n) - \sum_{1}^{x(k+1,N)} W_k(n)$. By applying the inequalities (8) to this equation, we obtain

$$q(k,N) \substack{\leq \\ \geq} A_k + \bar{U}_k\ x(k,N) - \bar{W}_k\ x(k+1,N) \pm \{\bar{U}_k(p_k-1) + \bar{W}_k(q_k-1)\}.$$

Substitute for $q(k,N)$, $1 \leq k \leq \ell$, in F yields

$$F \substack{\leq \\ \geq} \bar{U}_1(\frac{g-1}{g})x(1,N) + A_1 + \frac{\bar{W}_1}{\bar{U}_2} A_2 + \ldots + \frac{\bar{W}_1\bar{W}_2 \cdots \bar{W}_{\ell-1}}{\bar{U}_2\bar{U}_3 \cdots \bar{U}_\ell} A_\ell$$

$$\pm \{[\bar{U}_1(p_1-1) + \bar{W}_1(q_1-1)] + \frac{\bar{W}_1}{\bar{U}_2} [\bar{U}_2(p_2-1) + \bar{W}_2(q_2-1)] + \ldots$$

$$+ \frac{\bar{W}_1\bar{W}_2 \cdots \bar{W}_{\ell-1}}{\bar{U}_2\bar{U}_3 \cdots \bar{U}_\ell} [\bar{U}_\ell(p_\ell-1) + \bar{W}_\ell(q_\ell-1)]\}.$$

Suppose g=1. Then by taking the $\leq$ case in the above inequality, we see that F is bounded above by a constant, and since each q(k,N) is bounded below by zero, a uniform upper bound on each of the quantities q(k,N) must exist.

Suppose $g > 1$. Then the coefficient of x(1,N) is positive. Since G' is nonterminating, x(1,N) grows without limit in any proper execution of G. By choosing the $\geq$ case in the above inequality, we see that F must also grow without limit showing that some q(k,N) must tend to infinity.

8. Conclusion

An analysis has been presented of determination for computation graphs, and of termination and queueing in the case that $\{U_p(n)\}$ and $\{W_p(n)\}$ are periodic. An open problem is the analysis of termination and queueing in the case that $\{U_p(n)\}$ and $\{W_p(n)\}$ are not periodic. However, such an analysis would appear to be of little practical value since most arithmetic calculations which one would wish to effect by a computation graph seem to be iterative.

Many other problem areas immediately suggest themselves. Among these are:

(i) the synthesis of a computation graph given a computation to be performed

(ii) the determination of a proper execution which minimizes the time for a computation

(iii) programming techniques

(iv) the allocation of storage registers for data queues and their utilization so as to minimize the number required.

References

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