THE UNIVERSITY OF MICHIGAN

AFCRC-TN-58-351
ASTIA Document No. AD-160791

STUDIES IN RADAR CROSS SECTIONS XXX

The Theory of Scalar Diffraction with Application to the Prolate Spheroid

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The University of Michigan Report No. 2591-4-T

Scientific Report No. 4
on
Air Force Contract AF-19(604)-1949

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August 1958

The research reported in this document has been sponsored by the Air Force Cambridge Research Center, Air Research and Development Command

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PREFACE

This paper is the thirtieth in a series growing out of studies of radar cross sections at The University of Michigan Research Institute. The primary aims of this program are:

1. To show that radar cross sections can be determined analytically.

2. A. To obtain means for computing the radiation patterns from antennas by approximate techniques which determine the pattern to the accuracy required in military problems but which do not require the exact solutions.

   B. To obtain means for computing the radar cross sections of various objects of military interest.

   (Since 2A and 2B are interrelated by the reciprocity theorem it is necessary to solve only one of these problems.)

3. To demonstrate that these theoretical cross sections and theoretically determined radiation patterns are in agreement with experimentally determined ones.

Intermediate objectives are:

1. A. To compute the exact theoretical cross sections of various simple bodies by solution of the appropriate boundary-value problems arising from electromagnetic theory.

   B. To compute the exact radiation patterns from infinitesimal sources on the surfaces of simple shapes by the solution of appropriate boundary-value problems arising from electromagnetic theory.

   (Since 1A and 1B are interrelated by the reciprocity theorem it is necessary to solve only one of these problems.)

2. To examine the various approximations possible in this problem and to determine the limits of their validity and utility.

3. To find means of combining the simple-body solutions in order to determine the cross sections of composite bodies.

4. To tabulate various formulas and functions necessary to enable such computations to be done quickly for arbitrary objects.

5. To collect, summarize, and evaluate existing experimental data.

K.M. Siegel

SUMMARY

It is shown that the inhomogeneous wave equation, $\nabla^2 u - u_{tt} = \rho\, e^{i\omega t}$, with prescribed initial conditions and subject to outer boundary conditions on a smooth closed surface, has a solution of the form $v(x,t)\, e^{i\omega t}$, and that $\lim_{T\to\infty} T^{-1} \int_0^T v(x,t)\,dt$ is given by the formula $\phi(x) = \lim_{s\to o} \phi(x,s)$, where $\phi(x,s)$ is the solution of $\nabla^2\phi + (\omega - is)^2 \phi = \rho$, subject to the same boundary conditions as u, and which is square-integrable in the exterior region. The theory of analytic resolvents of Sims and Phillips is applied to obtain a representation of the solution of this equation as a contour integral. In the case of the sphere, this integral is the same as the representation obtained by others by use of the Watson transform. In the case of the axially symmetric scattering of a plane wave by a prolate spheroid, the integral can be evaluated, in the shadow region, as a residue series which can be summed, following Franz, as a sum of "creeping waves". The computations are performed only for the surface distribution; the methods of Langer are exploited for the asymptotic evaluation, for large ω, of the residues. The attenuation of the "creeping waves" is shown to be proportional to $\omega^{1/3} R^{-2/3}$, where R is the radius of curvature at the tip; this contradicts several previously held theories.

INTRODUCTION

In a series of papers (Refs. 3, 9, 10), Franz and Depperman showed how one could obtain representations of the solution of problems involving diffraction by cylinders and spheres which converged much more rapidly, for large frequencies, than does the classical Mie series*. Their methods involved, on the one hand, an investigation of the integral equation for the field (scalar or vector), and on the other hand, the direct application of the Watson transform to the Mie series. The first technique had the advantage of not depending, in the general nature of its conclusions, upon the shape of the diffracting body, and gave results, in the penumbra region, consistent with those earlier published by Fock (Refs. 5, 6, 7, 8).

The second technique, despite its specialization to the sphere and cylinder, was especially interesting because, aside from physical considerations, it exhibited an expansion of the solution in terms of the "radial part" of the operator $\nabla^2 + k^2$. Marcuvitz had previously obtained a similar result, through direct consideration of the one dimensional "radial" operator (Ref. 17), and independently, Imai obtained a similar result using similar methods (Ref. 13). More recently, Felsen has used like techniques to solve problems which separate in spherical coordinate systems (Ref. 4).

In the present paper, the analogue of the Franz residue series (Ref. 9) will be derived for the axially symmetric scalar scattering of a plane wave by a prolate spheroid. The method is apparently new, but it is a refinement of the idea of

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* Throughout this paper, the phrase "Mie series" will be used to describe expansions in series of spherical harmonics.

Marcuvitz. An outline of the method is given in Section 5 and in Section 6 it is applied to the sphere, for the sake of comparison with known results. In Section 7 and Section 8 the work on the prolate spheroid is carried out.

As will be seen, the method consists of obtaining a contour integral representation of the solution of the scalar wave equation, in which the wave number is given a small negative imaginary part, eventually to be allowed to converge to zero. The introduction of this imaginary part permits a rigourous application of the work of Sims (Ref. 19), and Phillips (Ref. 18), described in Section 4, to secure the contour integral. It also permits a rigourous application of the work of Langer (Refs. 15 and 16), in the asymptotic representation, for large wave numbers, of the residues of the integrand. This is done in the Appendix, which was written by N. D. Kazarinoff.

The introduction of this imaginary part is discussed in Sections 1, 2, and 3. The situation is not simple enough to be described in a few words, but roughly, it is shown that the method is merely the correct use of the Laplace transform in solving the time dependent wave equation.

The technique employed here is applicable to any boundary surface which is a level surface in a coordinate system in which the scalar wave equation separates. Further, it is easy to verify, since all such systems are known, that the evaluation of the residues invariably involves the study of a second order differential equation at a simple turning point. This introduces circular functions of order 1/3, and is the basis for what has come to be known as the "Fock" approximation. A survey of this approximation has been written by R. F. Goodrich (Ref. 11), whose helpful comments during the preparation of this manuscript are gratefully acknowledged.

I

THE CLASSICAL PROBLEM

The classical approach to scalar diffraction problems is to solve, if possible, the scalar wave equation:

$$\left\{\nabla^2 + \omega^2\right\} \quad \phi = \rho(x) . \tag{1.1}$$

In Equation (1.1), $\rho(x)$ represents a spacial distribution of radiating sources, ω is the frequency of the radiation, and ϕ satisfies an homogeneous boundary condition at the surface of the scattering body. As a mathematical problem, this is not well-set; the introduction of the radiation condition satisfies the dual purpose of guaranteeing that the solution behave, asymptotically, as an outgoing wave, and that the solution be unique.

The apparently ad hoc introduction of the radiation condition is a disturbing aspect of the theory, since it violates the principle that the correct mathematical representation of a physical problem should be a well-set mathematical problem. To appreciate the difficulty it is necessary to examine the genesis of Equation (1.1). This equation is obtained by assuming that the time dependent distribution, $u(x,t)$, which is a solution of the equation:

$$\nabla^2 u - u_{tt} = \rho(x) e^{i\omega t}, \tag{1.2}$$

can be represented in the form:

$$u(x,t) = \phi(x) e^{i\omega t} . \tag{1.3}$$

There are two objections to be raised. First, Equation (1.2), as it stands, is not a well-set problem. It does not become well-set until, for example, the values of u(x, t) and its time derivative are specified for some value of t. Second, once this specification has been made, the representation (1.3) may no longer be justified. This can be verified very easily for the one-dimensional problem, for which the solution can be computed exactly.

Thus is exhibited the paradox: Starting with the well-set physical problem - namely, a distribution known to satisfy the inhomogeneous wave equation (1.2), and having, together with its first time derivative, a specified value at some fixed time, the derivation of the scalar wave equation (1.1) cannot, in general, be justified. On the other hand, the Equation (1.1), with the radiation condition imposed, is known to give satisfactory results in the theory of diffraction.

II

THE REFORMULATION

It is possible to resolve the paradox. First, to be definite, let it be assumed that $u(x,t)$ satisfies the Equation (1.2), and that at time $t = 0$, $u(x,t)$ satisfies the initial condition:

$$u(x,0) = f(x) \; ; \quad u_t(x,0) = g(x) \; . \tag{2.1}$$

Further, let it be assumed that on the boundary of a smooth, closed surface, B, the normal derivative is zero, i.e.:

$$u_n(x,t) \Big]_{x \in B} = 0 \; . \tag{2.2}$$

Let if finally be assumed that $f(x)$, $g(x)$, and $\rho(x)$ vanish outside a bounded region. If the Kirchoff integral equation, which involves the use of retarded potentials, is set up, this last condition is seen to imply that for a fixed value of t, $u(x,t)$ vanishes for $|x| > |x_o| + t$; the $|x_o|$ just introduced is the radius of a sphere exterior to which $f(x)$, $g(x)$ and $\rho(x)$ vanish.

If $v(x,t)$ is defined to be

$$v(x,t) = e^{-i\omega t} u(x,t) \; , \tag{2.3}$$

it follows, from Equation (1.2), that

$$\nabla^2 v - \left[v_{tt} + 2\, i\omega v_t - \omega^2 v \right] = \rho(x) \; . \tag{2.4}$$

At this point, following the notation of Phillips (Ref. 12), the column vectors $y(x,t)$, $z(x,t)$ will be defined to be:

$$y(x,t) = \begin{pmatrix} v(x,t) \\ v_t(x,t) \end{pmatrix}; \qquad z(x,t) = \begin{pmatrix} u(x,t) \\ u_t(x,t) \end{pmatrix}. \tag{2.5}$$

Then, the following relationships hold

$$y(x,t) = e^{-i\omega t} \begin{pmatrix} 1 & 0 \\ -i\omega & 1 \end{pmatrix} z(x,t)\ ; \quad y_n(x,t) \Big]_{x \in B} = 0. \tag{2.6}$$

Now Equation (2.4) can be written in a more suggestive form:

$$\frac{d}{dt}\, y(x,t) = \mathfrak{M}\, y(x,t) + y_o(x)\ . \tag{2.7}$$

In Equation (2.7) $\mathfrak{M}$ and $y_o(x)$ are respectively an operational matrix and a column vector described by the equations:

$$\mathfrak{M} \equiv \begin{pmatrix} 0 & 1 \\ (\nabla^2 + \omega^2) & -2\,i\omega \end{pmatrix} \tag{2.8}$$

$$y_o(x) \equiv \begin{pmatrix} 0 \\ -\rho(x) \end{pmatrix} \tag{2.9}$$

$\mathfrak{M}$ is the infinitesimal generator of a semi-group. To see this, consider the homogeneous equation

$$\frac{d}{dt}\, z(x,t) = \mathfrak{N}\, z(x,t)\ . \tag{2.10}$$

In Equation (2.10), $\mathfrak{N}$ is the operational matrix:

$$\mathfrak{N} \equiv \begin{pmatrix} 0 & 1 \\ \nabla^2 & 0 \end{pmatrix} \tag{2.11}$$

This equation, subject to initial conditions and boundary conditions at B, is known to have a semigroup for its solution (Ref. 12). Following the notation of this reference, the solution of Equation (2.10) can be represented by the formula:

$$z(x,t) = \exp(\mathcal{N} t)\, z(x,0)\ . \qquad (2.12)$$

If y(x, t) is related to z(x, t) by the first equation in Equation (2.6), then

$$y(x,t) = e^{-i\omega t} \begin{pmatrix} 1 & 0 \\ -i\omega & 1 \end{pmatrix} \exp(\mathcal{N}\, t) \begin{pmatrix} 1 & 0 \\ i\omega & 1 \end{pmatrix} y(x,0)\ . \qquad (2.13)$$

If one computes $y(x, t_1 + t_2)$ from Equation (2.13), and makes use of the fact that $\mathcal{N}$ is the infinitesimal generator of a semigroup, one finds that Equation (2.13) represents a semigroup. Then, if one differentiates Equation (2.13) with respect to t, and takes the limit as $t \rightarrow 0$, the infinitesimal generator of this semigroup is seen to be $\mathcal{M}$.

Now, by using the method of variation of parameters, the solution of Equation (2.7) is seen to be

$$y(x,t) = \exp(\mathcal{M} t)\, y(x,0) + \left(\int_0^t \exp(\mathcal{M} t)\, dt \right) y_0(x). \qquad (2.14)$$

Until now, the class of functions to which the operators are being applied has been left unspecified. The only description of the functions which have come into the discussion has been that they vanish outside of bounded sets. To apply correctly the theory of semigroups, a complete space of functions should be provided; so the functions considered should be thought of as being embedded in a complete space; in particular, a space in which the functions vanish at ∞. If <u>in this space $\mathcal{M}$ has a bounded inverse</u>, it is possible to apply a Tauberian theorem (Ref. 12), and to demonstrate that the (C-1) limit of $\exp(\mathcal{M} t)$ is zero. That is,

$$\lim_{T \rightarrow \infty} T^{-1} \int_0^T \exp(\mathcal{M}\, t)\ dt = 0\ . \qquad (2.15)$$

If one then considers Equations (2.14) and (2.15), one obtains:

$$\lim_{T\to\infty} T^{-1} \int_0^T y(x,t)\,dt = \mathfrak{M}^{-1}\, y_o(x) \,. \tag{2.17}$$

If the left member of Equation (2.17) is designated to be the column vector

$$\tilde{y}(x) = \begin{pmatrix} \tilde{v}_1(x) \\ \tilde{v}_2(x) \end{pmatrix} , \tag{2.18}$$

then Equation (2.17) can be written as

$$\mathfrak{M}\,\tilde{y}(x) = y_o(x) \quad . \tag{2.19}$$

Taking into account Equations (2.8), (2.9), (2.18) and (2.19), it is found that

$$(\nabla^2 + \omega^2)\,\tilde{v}_1(x) = \rho(x) \;;\quad \tilde{v}_2(x) = 0 \,. \tag{2.20}$$

The preceding remarks can be summed up in the form of a theorem:

IF THE CLASS OF FUNCTIONS FROM WHICH THE INITIAL CONDITIONS AND SOURCE DISTRIBUTION ARE CHOSEN HAS A COMPLETION IN WHICH THE SCALAR WAVE EQUATION HAS A UNIQUE SOLUTION, IF $u(x,t)$ IS A SOLUTION OF THE WELL-SET PROBLEM, EQUATIONS (1.2) AND (2.1), AND IF $v(x,t) = e^{-i\omega t}\,u(x,t)$, THEN $\lim_{T\to\infty} T^{-1} \int_0^T v(x,t)\,dt$ EXISTS AND SATISFIES THE SCALAR WAVE EQUATION.

III

THE USE OF THE LAPLACE TRANSFORM

The preceding sections show that the scalar wave equation should be solved to find the (C-1) limit of $v(x,t)$. To show this, a Tauberian theorem has been used. But now an Abelian theorem can be applied to secure this (C-1) limit by means of the Laplace transform. If $V(x,s)$ is the Laplace transform of $v(x,t)$, and if (C-1) limit exists, then (Refs. 2 and 12),

$$\lim_{T\to\infty} T^{-1} \int_0^T v(x,t)\, dt = \lim_{s\to 0^+} s\, V(x,s) \quad . \tag{3.1}$$

Since it has been seen in Section II that the limit in the left member of Equation (3.1) is independent of the initial conditions, no generality is lost in assuming the initial conditions to vanish. But then, from Equation (2.4), it is found that if $\phi(x,s) = s\, V(x,s)$, then

$$\nabla^2 \phi + (\omega - is)^2 \phi = \rho(x) \ . \tag{3.2}$$

The condition mentioned above, that $\rho(x)$ vanishes outside a sphere of radius $|x_0|$, implies that $\phi(x,s)$ vanishes exponentially as $|x| \to \infty$. In fact,

$$\phi(x,s) = \int_{|x|-|x_0|}^{\infty} e^{-st}\, v(x,t)\, dt \ ; \tag{3.3}$$

since $v(x,t)$ is of exponential type, a fact which is assured by consideration of Equation (2.14), the stated result follows.

It is also true that Equation (3.2), together with the boundary condition at B and the imposition of the requirement that $|\phi|^2$ be integrable over the volume exterior to B, presents a problem whose solution is unique. For if

$$\nabla^2 \phi + (\omega - is)^2 \phi = 0 \,, \tag{3.4}$$

and the boundary and integrability conditions are satisfied by ϕ, then

$$\nabla \cdot \left\{ \phi \nabla \bar{\phi} - \bar{\phi} \nabla \phi \right\} + 4\, is\, \omega |\phi|^2 = 0 \,, \tag{3.5}$$

so that

$$4\, is\, \omega \int_{V_R} |\phi|^2 \, dx = \int_{\Sigma_r} (\phi \nabla \bar{\phi} - \bar{\phi} \nabla \phi) \cdot d\Sigma_R \,. \tag{3.6}$$

In Equation (3.6), Σ_R designates the surface of a sphere of radius R, and V_R is the volume interior to this sphere and exterior to B. But the integrability condition assures that $\phi'(x,s) = o\left(\frac{1}{|x|}\right)$, $\phi = o\left(\frac{1}{|x|^2}\right)$, so that the surface integral in Equation (3.6) vanishes as $R \to \infty$. This proves the stated result. Thus, the unique solution will not only satisfy the integrability condition, but, on the basis of the preceding result, will vanish exponentially as $|x| \to \infty$. In the applications, this will be seen to lead to a solution of the scalar wave equation which satisfies the radiation condition.

IV

THE THEORY OF SECOND ORDER DIFFERENTIAL OPERATORS

Before the preceding formulation is used to solve specific problems, certain mathematical results should be stated. The results to be used here are to be found in various places. The theory of singular self-adjoint second order differential operators is treated extensively in the monographs of Titchmarsh and of Stone (Refs. 20 and 21). The modified theory, in which the coefficients of the operator are permitted to be complex, has been studied by Phillips (Ref. 18) and Sims (Ref. 19). It is from these references that the remarks to follow derive their justification.

Let L be a formal differential operator, defined by the expression:

$$L\ y = -\left\{p(x)\ y'\right\}' + q(x)\ y \quad . \tag{4.1}$$

In Equation (4.1) the prime indicates differentiation with respect to x; the values of x lie on an interval $-\infty \leq a < x < b \leq +\infty$; p(x) is a real valued function and $p(x) > 0$ for all x on the interval $a < x < b$; q(x) is a complex valued function, and on the interval $\mathrm{Im}[q] \geq q_0$.

There exist integral operators, R_λ , called resolvents, defined and analytic in the half-plane $\mathrm{Im}[\lambda] < q_0$, having the property that if L y exists, and if y and L y are square integrable on the interval, then

$$R_\lambda (L\ y - \lambda\ y) = y \tag{4.2}$$

and

$$\int_{-i\delta-\infty}^{-i\delta+\infty} R_\lambda\ y\ d\lambda = -\pi i y\ , \text{ for } \mathrm{Im}[\lambda] < q_0 \quad , \tag{4.3}$$

both hold. In general, R_λ is not uniquely determined, depending upon "boundary conditions" at a and b, and upon the nature of q(x). In diffraction problems, only two cases are of importance.

Case I. This case is defined by the conditions:

$$-\infty < a;\ b = \infty;\ p(a) \neq 0;\ \text{the homogeneous equation } L\,y - \lambda\,y = 0, \tag{4.4}$$

$\mathrm{Im}\,[\lambda] < q_o$, has exactly one solution which is square integrable on $a < x < \infty$. In this case, if an homogeneous boundary condition is defined, and it is required that at x = a, $R_\lambda y$ satisfy this boundary condition and that $R_\lambda y$ be square-integrable on $a < x < \infty$, then $R_\lambda y$ is uniquely determined; in fact, it is described by an integral operator of the form:

$$R_\lambda y = \int_a^\infty G(x,x',\lambda)\, y(x')\, dx' . \tag{4.5}$$

The kernel in the integral, called the resolvent Green's function, is represented by a formula:

$$G(x,x',\lambda) = \frac{1}{p(x)\, W(y_1,y_2)} \begin{cases} y_1(x)\, y_2(x'), & \text{if } x < x' \\ y_1(x')\, y_2(x), & \text{if } x' < x . \end{cases} \tag{4.6}$$

In Equation (4.6), y_1 is the solution of $L(y) - \lambda y = 0$ which satisfies the boundary condition at x = a, and y_2 is the solution of the same equation which is square integrable; $W(y_1, y_2)$ is the Wronskian determinant of y_1 and y_2.

Case II. This case is defined by the conditions:

$$-\infty < a < b < \infty;\quad p(a) = p(b) = 0;\quad \text{a and b are regular singular points for the differential equation } L\,y - \lambda\,y = 0 . \tag{4.7}$$

Here, aside from the condition that $R_\lambda y$ be square integrable, no boundary conditions are needed in order to specify R_λ. R_λ again has a representation of the form (4.5) (in which, of course, "∞" is replaced by "b"). In this representation, the kernel is again described by Equation (4.6), except that the y_1 and y_2 which appear in Equation (4.6) are now, respectively, the solutions of $L\,y - \lambda\,y = 0$ which are regular at a and at b.

V

OUTLINE OF GENERAL METHOD

It is now possible to outline a procedure by means of which one can solve diffraction problems in the case where the surface B is a level surface for a coordinate system in which the scalar wave equation is separable. Roughly speaking, the three coordinates (ξ_1, ξ_2, ξ_3) can be thought of as respectively the "radial", the "angular", and the "axial" variable. The axial variable, ξ_3, has a range, $0 \leq \xi_3 \leq 2\pi$, and the solution is expected to be periodic in ξ_3. Hence, a preliminary Fourier decomposition is applied, reducing the problem to one in which only the coordinates ξ_1 and ξ_2 appear. The differential operators involving ξ_1 and ξ_2 are respectively in Cases I and II of the preceding section (it is to be remembered that the equation to be solved is Equation (1.1), in which "ω" has been replaced by "$\omega - i s$"). It is then possible to apply the resolvents of these differential operators to both sides of Equation (1.1) in such a way that there exists a path, Γ, in the λ plane such that when the equation which is secured by applying the resolvents is integrated along Γ, an integral representation for the solution of Equation (1.1) is obtained. This integral representation can be evaluated by means of a residue series. The singularities of the integrand occur both above and below the path Γ. These correspond, respectively, to singularities caused by poles of the resolvents of the radial and angular differential operator. The problem then reduces to the evaluation of this integral.

To illustrate, these methods shall be applied first to the case in which B is a sphere, and second to the case in which B is a prolate spheroid. In the first case, the well-known result, obtained by other methods (Ref. 9), and which leads to the residue series which is called the "creeping wave" approximation is obtained. In the second case, an analogous, but apparently new, representation is secured, and by application of the asymptotic theory of second order differential equations, an analogous residue series shall be given.

VI

THE SPHERE

The source distribution will be assumed to be axially symmetric. This simply eliminates an inessential complication. The equation to be solved is:

$$(r^2 \phi_r)_r + k^2 r^2 \phi + \frac{1}{\sin\theta} (\sin\theta\, \phi_\theta)_\theta = r^2 \rho(r,\theta) . \qquad (6.1)$$

In Equation (6.1), $k = \omega - i s$; the range of r is $0 < a \leq r < \infty$; the range of θ is $0 < \theta < \pi$; and the solution is required to obey the boundary condition $\phi_r \big|_{r=a} = 0$.

The differential operator,

$$L_r \phi = -(r^2 \phi_r)_r - k^2 r^2 \phi , \quad a \leq r < \infty , \qquad (6.2)$$

will be studied. When regarded as an ordinary differential operator, this is clearly of the form (4.1), and $\mathrm{Im}[q] \geq 2\omega s \; a^2$. The equation

$$L_r \phi - \lambda \phi = 0 \qquad (6.3)$$

is easily found to have two solutions,

$$\frac{w_1(r,\lambda)}{r} \sim \frac{1}{r} e^{ikr} ; \qquad \frac{w_2(r,\lambda)}{r} \sim \frac{1}{r} e^{-ikr} . \qquad (6.4)$$

In Equation (6.4), as elsewhere, the symbol $\sim$ is used to designate an asymptotic representation. If the variable with respect to which the representation is asymptotic is not apparent, this will be explicitly stated. In the present case, r, not k, is this variable.

These two solutions are linearly independent; the first is not square integrable on $a \leq r < \infty$; the second is. Hence, the condition for Case I of Section IV applies. The resolvent Green's function is described by the formula

$$G(r, r', \lambda) = \frac{1}{-a^2 \phi_1(a, \lambda)\, \phi_2'(a, \lambda)} \begin{cases} \phi_1(r)\, \phi_2(r') & (r < r') \\ \phi_1(r')\, \phi_2(r) & (r > r') \ . \end{cases} \tag{6.5}$$

In Equation (6.5), the ϕ_1 and ϕ_2 are defined as:

$$\begin{cases} \phi_1(r, \lambda) = \left[a\, w_2'(a, \lambda) - w_2(a, \lambda)\right] \dfrac{w_1(r, \lambda)}{r} - \left[a\, w_1'(a, \lambda) - \omega_1(a, \lambda)\right] \dfrac{w_2(r, \lambda)}{r} \\ \phi_2(r, \lambda) = \dfrac{w_2(r, \lambda)}{r} \ . \end{cases} \tag{6.6}$$

For later use it should be observed that

$$\phi_2(r, \lambda) = k \sqrt{\frac{\pi}{2}}\; e^{-\pi \frac{i}{4}}\; e^{-\frac{i\mu\pi}{2}} \left[\frac{H_\mu^{(2)}(kr)}{\sqrt{kr}} \right]. \tag{6.7}$$

In Equation (6.7), $\mu^2 = 1/4 - \lambda$, and $H_\mu^{(2)}$ is the Hankel function (Ref. 1). This follows from the observation that $\frac{\phi_2}{\sqrt{r}}$ is a circular function of order μ which is asymptotic to $\frac{e^{-ikr}}{r}$. Using Equation (6.5), the resolvent, R_λ is defined by Equation (4.5).

Next, it is necessary to consider the differential operator L_θ, defined by the formula:

$$L_\theta \phi = -\frac{1}{\sin\theta} (\sin\theta\, \phi_\theta)_\theta, \quad 0 < \theta < \pi \ . \tag{6.8}$$

If the substitution $\eta = \cos\theta$ is made, and L_θ is considered to be an ordinary differential operator in η , it can be seen that condition (4.7) is satisfied. The resolvent, R_ζ , is given by the formula

$$\tilde{R}_\zeta \, y = \int_0^\pi H(\theta, \theta', \zeta) \, y(\theta') \sin\theta' \, d\theta' \; . \tag{6.9}$$

In Equation (6.9), $H(\theta,\theta',\zeta)$ is defined by the formula: (6.10)

$$H(\theta,\theta',\zeta) = \frac{\pi}{2 \sin \pi\nu} \begin{cases} P_\nu(-\cos\theta) \, P_\nu(\cos\theta') & (\theta' < \theta) \\ P_\nu(-\cos\theta') \, P_\nu(\cos\theta) & (\theta' > \theta) \; . \end{cases}$$

In Equation (6.10), ν is given by the relation

$$\nu(\nu + 1) = \zeta , \tag{6.11}$$

and P_ν is the Legendre function.

Now, Equation (6.1) can be written in the following form:

$$- L_r \phi - L_\theta \phi = r^2 \rho(r,\theta) \; . \tag{6.12}$$

If $\lambda = i\delta + \ell$, $0 < \delta < 2\omega s a^2$, and the integral operator R_λ is applied to both members of Equation (6.12), there is obtained the equation:

$$-\phi - \left\{ L_\theta R_\lambda \phi + \lambda R_\lambda \phi \right\} = R_\lambda \left[r^2 \rho(r,\theta) \right] . \tag{6.13}$$

This follows from two facts: first, Formula (4.2) is applicable; second, that a priori, it is known that ϕ decreases exponentially in r; hence, the integral describing $R_\lambda \phi$ converges uniformly with respect to θ, so the operators L_θ and R_λ can be commuted.

If now the operator $\tilde{R}_{-\lambda}$ is applied to all members of Equation (6.13), there is obtained the formula:

$$- \tilde{R}_{-\lambda} \phi - R_\lambda \phi = \tilde{R}_{-\lambda} R_\lambda \left[r^2 \rho(r,\theta) \right] \; . \tag{6.14}$$

If all members of Equation (6.14) are then integrated with respect to ℓ, ℓ

going from $-\infty$ to $+\infty$, consideration of Equation (4.3) shows that

$$\phi(r,\theta) = \frac{1}{2\pi i} \int_{i\delta-\infty}^{i\delta+\infty} \tilde{R}_{-\lambda} R_{\lambda} \left[r^2 \rho(r,\theta) \right] d\lambda =$$

$$= \frac{1}{2\pi i} \int_{i\delta-\infty}^{i\delta+\infty} \int_0^{\pi} \int_a^{\infty} H(\theta,\theta',-\lambda)\, G(r,r',\lambda)\, \rho(r',\theta')\, r'^2 \sin\theta' \, dr' d\theta' d\lambda \, . \tag{6.15}$$

If it is recalled that axial symmetry has been assumed, the iterated integral with respect to r' and θ' can be thought of as a volume integral. Hence if $\rho(r,\theta)$ is the distribution corresponding to a point source located at $r = R$, $\theta = 0$ the corresponding $\phi(r,\theta)$ is given by

$$\phi(r,\theta) = \frac{1}{2\pi i} \int_{i\delta-\infty}^{i\delta+\infty} H(\theta,0,-\lambda)\, G(r,R,\lambda)\, d\lambda \, . \tag{6.16}$$

If Equation (6.16) is evaluated and s is allowed to approach 0, the resulting expression will be the Green's function for a sphere. If this evaluation is performed by summing over the residues in the lower half-plane, the Mie series results.

In general, it is easier to perform the computation by setting r = a, finding the pressure distribution on the boundary of the sphere, and then solving the scalar wave equation by the conventional double-layer technique of potential theory. If this is done, taking into account Equations (6.16), (6.5), (6.7), (6.10), and (6.11) the following representation is obtained:

$$\phi(a,\theta) = -\frac{1}{\pi i} \frac{e^{\frac{\pi i}{4}} \sqrt{\pi/2}}{k^2 a^2} \int_{c_\mu} \frac{\mu \, P_{\mu-1/2}(-\cos\theta)\, e^{\frac{\pi i \mu}{2}} \, d\mu}{\cos \pi\mu \left[\frac{H_\mu^{(2)}(z)}{z} \right]'_{z=ka}} \tag{6.17}$$

where c_μ is the contour in the plane indicated in Figure 1.

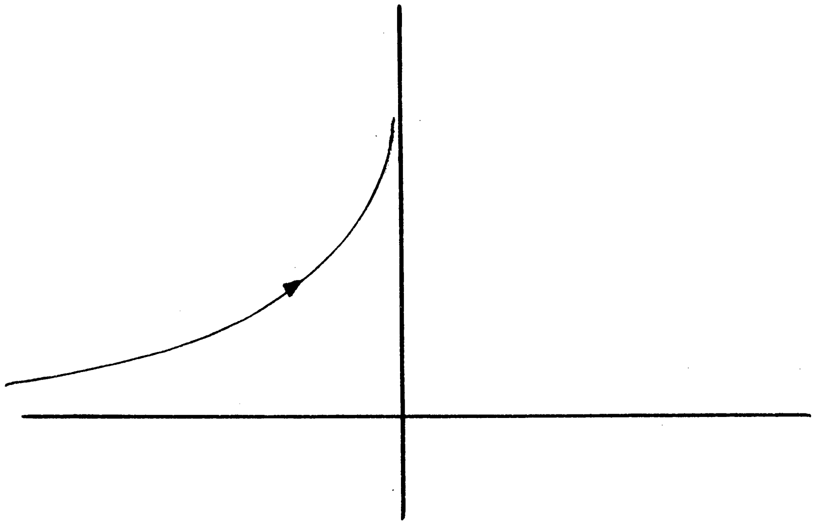

Figure 1

Equation (6.17) is identical to the result obtained in References 9 and 3 by applying the Watson transform to the Mie series. The apparent discrepancy in path (in Ref. 9 and in Ref. 3 the path is in the first quadrant) is accounted for by the use of $H^{(2)}$ instead of $H^{(1)}$.

VII

THE PROLATE SPHEROID

With modifications, the technique of Section 6 can be duplicated to secure a representation of the surface distribution in the case where B is a prolate spheroid. Here again, axial symmetry will be assumed. The prolate spheroidal coordinates (ξ, η) will be used; ξ is a parameter defining an ellipsoid of revolution about its major axis, with semi-focal distance c and eccentricity $\frac{1}{\xi}$. As ξ takes on all values $1 < \xi < \infty$, a family of confocal ellipsoids is generated. As η ranges over the interval $-1 < \eta < 1$, the associated coordinate surfaces form the family of confocal hyperboloids orthogonal to the members of the first family. In this coordinate system the equation to be solved becomes:

$$\left\{(\xi^2 - 1)\,\phi_\xi\right\}_\xi + \gamma^2(\xi^2 - 1)\,\phi + \left\{(1 - \eta^2)\,\phi_\eta\right\}_\eta + \gamma^2(1 - \eta^2)\,\phi = \tag{7.1}$$
$$= J(c, \xi, \eta)\,\rho \quad .$$

In Equation (7.1), $\gamma = kc = (\omega - i\,s)c$, and $J(c, \xi, \eta)$ is the volume Jacobian. If radial symmetry were not assumed, the term:

$$-\,m^2\left[\frac{1}{\xi^2 - 1} + \frac{1}{1 - \eta^2}\right]\phi \tag{7.2}$$

would have to be added to the left member of Equation (7.1), and the ρ in the right member of Equation (7.1) would be replaced by the m^{th} Fourier coefficient of ρ. The solution would then be not ϕ, but the m^{th} Fourier coefficient of ϕ (with respect to the axial variable. This is a complication which, even in the idealized case of diffraction of a plane incident wave, creates difficulties that are not present in the sphere problem. The question will be considered in another paper.

The boundary condition is given by the equation:

$$\phi_{\xi}\Big]_{\xi=\xi_0} = 0 \; . \tag{7.3}$$

In Equation (7.3), $\xi_0 > 1$ is the reciprocal of the eccentricity of the particular spheroid under consideration. The differential operator L_{ξ}, defined by the formula

$$L_{\xi}\;\phi = -\left\{(\xi^2-1)\,\phi_{\xi}\right\}_{\xi} - \gamma^2(\xi^2-1)\,\phi \; , \tag{7.4}$$

is introduced. This is an operator of the type considered in Section 4, and if $\xi \geq \xi_0$, $\mathrm{Im}\,[q] \geq 2\omega s\, c^2(\xi_0^2 - 1)$. To construct the resolvent, it is necessary to study the homogeneous equation $L_{\xi}\,\phi - \lambda\phi = 0$, when $\mathrm{Im}\,\lambda < 2\omega s\, c^2(\xi_0^2 - 1)$.

If w is related to ϕ by the equation

$$w = (\xi^2 - 1)^{1/2}\,\phi \; , \tag{7.5}$$

then the homogeneous equation for ϕ is transformed into an equation for w. This equation is:

$$w_{\xi\xi} + \left\{\gamma^2 + \frac{1}{\xi^2 - 1} + \frac{1}{(\xi^2-1)^2}\right\} w = 0 \; . \tag{7.6}$$

Equation (7.6) has two linearly independent solutions, $w_1(\xi,\lambda)$ and $w_2(\xi,\lambda)$ which are, for large ξ, asymptotic respectively to $e^{i\gamma\xi}$ and $e^{-i\gamma\xi}$. Only the second of the corresponding $\phi(\xi,\lambda)$ is square integrable on the interval $\xi_0 \leq \xi < \infty$. Thus, Case I of Section 4 applies.

The functions $\phi_1(\xi,\lambda)$, $\phi_2(\xi,\lambda)$ are defined by the formulas:

$$\phi_1(\xi,\lambda) = (\xi^2-1)^{-1/2} \left\{ \left[(\xi_o^2 - 1)\, w_2'(\xi_o,\lambda) - \xi_o\, w_2(\xi_o,\lambda) \right] w_1(\xi,\lambda) - \right.$$

$$\left. - \left[(\xi_o^2 - 1)\, w_1'(\xi_o.\lambda) - \xi_o\, w_1(\xi_o,\lambda) \right] w_2(\xi,\lambda) \right\} \qquad (7.7)$$

$$\phi_2(\xi,\lambda) = (\xi^2-1)^{-1/2}\, w_2(\xi,\lambda) \quad .$$

Then, $\phi_1(\xi,\lambda)$ satisfies the boundary condition (7.3), and the resolvent Green's function is computed to be:

$$G(\xi, \xi',\lambda) = \frac{1}{2\,i\,\gamma\left[(\xi_o^2-1)w_2'(\xi_o,\lambda) - \xi_o w_2(\xi_o,\lambda)\right]} \begin{cases} \phi_1(\xi,\lambda)\phi_2(\xi',\lambda) & (\xi<\xi') \\ \phi_1(\xi',\lambda)\phi_2(\xi,\lambda) & (\xi>\xi') \end{cases} .$$

(7.*0

It should also be noted that

$$\phi_1(\xi_o,\lambda) = -2\,i\,\gamma\ (\xi_o^2 - 1)^{1/2} \quad , \qquad (7.9)$$

and

$$(\xi_o^2-1)\, w_2'(\xi_o,\lambda) - \xi_o\, w_2(\xi_o,\lambda) = (\xi_o^2 - 1)^{3/2}\ \phi_2'(\xi_o,\lambda) \; . \qquad (7.10)$$

The resolvent R_λ is then given by the formula

$$R_\lambda y = \int_{\xi_o}^{\infty} G(\xi, \xi',\lambda)\, y(\xi')\, d\xi' \; . \qquad (7.11)$$

The differential operator L_η is defined to be:

$$L_\eta\ \phi = - \left\{ (1-\eta^2)\,\phi_\eta \right\}_\eta - \gamma^2(1-\eta^2)\,\phi, \ -1 < \eta < 1. \qquad (7.12)$$

This, again, is an operator of the type considered in Section 4, where $Im\,[q] > 0$ on the interval $-1 < \eta < 1$. Case II of Section 4 applies; thus if

$\psi_2(\eta, \zeta)$ is the solution of the homogeneous equation $L_\eta \psi - \zeta \psi = 0$ which is regular at $\eta = 1$, and normalized so that $\psi_2(1, \zeta) = 1$, and if $\psi_1(\eta, \zeta) = \psi_2(-\eta, \zeta)$, then the resolvent Green's function for the operator L_η is:

$$G(\eta, \eta', \zeta) = \frac{1}{(1-\eta^2) W(\psi_1, \psi_2)} \begin{cases} \psi_1(\eta)\, \psi_2(\eta'), & (\eta < \eta') \\ \psi_1(\eta')\, \psi_2(\eta), & (\eta > \eta') \end{cases}, \tag{7.13}$$

and the resolvent, $\tilde{R}_\zeta$, is:

$$\tilde{R}_\zeta \; y = \int_{-1}^{1} \tilde{G}(\eta, \eta', \zeta)\, y(\eta')\, d\eta' . \tag{7.14}$$

Equation (7.1) can be rewritten in the form

$$-L_\xi \; \phi - L_\eta \; \phi = J \rho . \tag{7.15}$$

Let Γ be a path in the complex λ plane defined by the conditions:

$$\lambda = i\,\delta + \ell, \quad 0 < \delta < 2\omega s c^2 (\xi_o^2 - 1) . \tag{7.16}$$

Γ is oriented in such a fashion that increasing ℓ corresponds to positive orientation. As in Section 6, successive application of R_λ and $\tilde{R}_{-\lambda}$ to (7.15) yields:

$$-\tilde{R}_{-\lambda} \phi - R_\lambda \phi = \tilde{R}_{-\lambda} R_\lambda \left[J \rho \right]. \tag{7.17}$$

If Equation (4.3) is considered, Equation (7.17) implies that

$$\phi(\xi, \eta) = \frac{1}{2\pi i} \int_{\Gamma} \tilde{R}_{-\lambda} R_\lambda \left[J \rho \right] d\lambda. \tag{7.18}$$

The integrand of the right member of Equation (7.18) has the value:

$$\int_{-1}^{1} \int_{0}^{\infty} \tilde{G}(\eta, \eta', -\lambda)\, G(\xi, \xi', \lambda)\, J(c, \eta', \xi')\, \rho(\xi', \eta')\, d\xi'\, d\eta' . \tag{7.19}$$

Because the assumption of axial symmetry has been made, Equation (7.19) can be regarded as a volume integral. If $\rho(\xi, \eta)$ is a distribution corresponding to a point source at $\xi = \Xi$, $\eta = 1$, Equation (7.19) becomes:

$$\widetilde{R}_{-\lambda} R_{\lambda} \left[J \rho \right] = \widetilde{G}(\eta, 1, -\lambda) \, G(\xi, \Xi, \lambda) \, . \qquad (7.20)$$

In this case, $\phi(\xi, \eta)$ has the integral representation:

$$\phi(\xi, \eta) = \frac{1}{2\pi i} \int_{\Gamma} \widetilde{G}(\eta, 1, -\lambda) \, G(\xi, \Xi, \lambda) \, d\lambda \, . \qquad (7.21)$$

This, when $s \rightarrow 0$, is the Green's function for the prolate spheroid relative to a point on the axis of symmetry.

VIII

EVALUATION OF THE INTEGRAL

In this section, the representation (7.21) will be used to obtain the surface distribution induced on a prolate spheroid by a plane wave whose front is orthogonal to the axis of symmetry. To prepare the problem, first Equation (7.21) will be rewritten in terms of the functions defined in Section 7, and for $\xi = \xi_o$. Taking into consideration Equations (7.21), (7.13), (7.8), (7.9) and (7.10), it is possible to rewrite Equation (7.21) as:

$$\phi(\xi_o, \eta) = -\frac{1}{2\pi i} \int_{\Gamma} \frac{\Psi_1(\eta, -\lambda)\, \phi_2(\Xi, \lambda)\, d\lambda}{(1-\eta^2)(\xi_o^{\,2}-1)\, \phi_2'(\xi_o, \lambda)\, W(\Psi_1, \Psi_2)} . \tag{8.1}$$

Now $\phi_2(\Xi, \lambda)$ is asymptotic, for large Ξ, to $(\Xi^2 - 1)^{-1/2} e^{-i\gamma\Xi}$. Then, the standard physical considerations lead to the definition of the "plane wave" solution to be:

$$\phi_p(\xi_o, \eta) = -\frac{1}{2\pi i} \int_{\Gamma} \frac{\Psi_1(\eta, -\lambda)\, d\lambda}{(1-\eta^2)(\xi_o^{\,2}-1)\, \phi_2'(\xi_o, \lambda)\, W(\Psi_1, \Psi_2)} . \tag{8.2}$$

A word of caution is necessary. The desired solution of the exact problem is obtained by evaluating Equation (8.1), letting $s \to 0$, and then letting $\Xi \to \infty$. In deriving Equation (8.2) from Equation (8.1), it is necessary to show that

$$\lim_{s \to 0} \phi_p(\xi_o, \eta) = \lim_{\Xi \to \infty} (\Xi^2 - 1)^{1/2} e^{ic\omega\Xi} \lim_{s \to 0} \phi(\xi_o, \eta). \tag{8.3}$$

This is, in fact true, but the argument will not be given; now to evaluate the integral in Equation (8.2) one can use the residue series in the lower half-plane, corresponding to the zeros of $W(\Psi_1, \Psi_2)$. This choice is simply the analogue of finding the Mie series, and on the basis of classical Sturm-Liouville theory, can be shown to converge and to give a representation of $\phi_p(\xi_o, \eta)$. The proof of this depends on the fact that the differential operator, L_η, on the interval $-1 < \eta < 1$, and for γ real, is a self-adjoint operator with a pure point spectrum (Refs. 20 and 21). However, even for γ real, the operator L_ξ does not fit into the classical Sturm-Liouville theory. The reason for this is that the requirement that the resolvent be asymptotic to $(\xi^2-1)^{-1/2} e^{-i\gamma\xi}$ for γ complex, becomes the equivalent of imposing the radiation condition for γ real. But the operator, L_ξ, subject to the radiation condition, is not self-adjoint. Therefore, whether the integral (8.2) can be evaluated as a residue series in the upper half-plane is a question which must be settled by considerations removed from Sturm-Liouville theory. The work now follows very closely that of Franz (Ref. 9). Let attention be focused on the two equations which must be studied:

$$\left\{(1-\eta^2)\, y_\eta\right\}_\eta + \left\{\gamma^2(1-\eta^2) - \lambda\right\} y = 0 \tag{8.4}$$

and

$$\left\{(\xi^2-1)\, y_\xi\right\}_\xi + \left\{\gamma^2(\xi^2-1) + \lambda\right\} y = 0 . \tag{8.5}$$

If $|\lambda| \gg \gamma^2$, the second bracket in Equation (8.4) can be replaced by the single term, $-\lambda$. This is justified by the restriction of η to the interval $-1 < \eta < 1$, so that any asymptotic estimate can be made uniformly with respect to η. But, this replacement having been made, Equation (8.4) is just the Legendre equation of order ν, $\nu(\nu+1) = -\lambda$, and the resolvent Green's function is the same as that obtained in the case of the sphere.

Care must be taken with Equation (8.5). Let $\tilde{y}(\xi)$ be defined by the relation:

$$y(\xi) = (\xi^2 - 1)^{-1/4} \tilde{y}(\xi) \quad . \tag{8.6}$$

Then, if $y(\xi)$ is a solution of Equation (8.5), $\tilde{y}(\xi)$ satisfies the equation:

$$(\xi^2-1)\tilde{y}_{\xi\xi} + \xi \tilde{y}_\xi + \left\{ \gamma^2(\xi^2-1) - \frac{1}{4(\xi^2-1)} + (\lambda - \frac{1}{4}) \right\} \tilde{y} = 0 \, . \tag{8.7}$$

Let z be the new variable defined by

$$z = \frac{1}{2} \left\{ (\xi^2 - 1)^{1/2} + \xi \right\}. \tag{8.8}$$

Then, Equation (8.7) becomes:

$$z^2 \tilde{y}_{zz} + z\tilde{y} + \left\{ \gamma^2 z^2 - \left[\frac{\gamma^2}{2} + \frac{1}{4} - \lambda \right] + \frac{1}{4(z + \frac{1}{4z})^2} \right\} \tilde{y} = 0 \, . \tag{8.9}$$

If $\mu^2 = \frac{\gamma^2}{2} + \frac{1}{4} - \lambda$, Equation (8.9) resembles the Bessel equation of order μ, and if the last term in the bracket is suppressed, the solutions of Equation (8.9) are precisely the circular functions $C_\mu(\gamma z)$. The use of these circular functions is valid for either fixed λ and large z or fixed z and large λ, γ^2 being held fixed. Hence, the identification with the function which, except for factors, behaves for large ξ as $e^{-i\gamma\xi}$ can be made, and the resolvent Green's function, although not the same, even for large μ, as that for the sphere, is similar to that for the sphere, the difference being established explicitly by means (8.6) and (8.8). Hence, for $|\lambda| \gg \gamma^2$, the integrand resembles that of the sphere case sufficiently closely that the argument of Franz can be applied to show that in the illuminated region; i.e., for positive values of η, the residue series diverges, whereas, for negative values of η, the series converges, and represents the solution. The evaluation of the integral consists, then, of two parts. First the residues in the upper half-plane will be computed for values of λ which are comparable to γ^2. Then, the residue series will be exhibited as a double series, which, if the order of summation is interchanged, can be interpreted as a series of "creeping waves". It will be seen that if the first "creeping wave" is deleted from this series, that the remaining series converges even in the illuminated region. And finally, it will be shown that the terms which have been removed can be re-evaluated by the stationary phase technique, and shown to correspond to the optical contribution.

The residues which will be of significance in the computation are those which occur at values of λ which are comparable to γ^2. More precisely, these values of λ satisfy the relation (A.17), given in the appendix. The immediate task is to refine this estimate. It should be observed, before doing this, that the values of λ for which $\phi_2'(\xi_0, \lambda)$ vanish are in the upper half-plane. This is a consequence of the material in Section IV.

Now, consideration of formulas (A.12), (A.13), (A.6), (A.7) and (A.4) of the appendix leads to the conclusion that λ will be a zero (comparable to γ^2) of $\phi_2'(\xi_0, \lambda)$, if and only if

$$\lambda = (1 - x_1^2)\,\gamma^2 \,, \tag{8.10}$$

$$\int_{x_1}^{\xi_0} \sqrt{\frac{x^2 - x_1^2}{x^2 - 1}}\, dx = \frac{\alpha_n}{\gamma}\left[1 + 0\left(\frac{1}{\gamma}\right)\right] \tag{8.11}$$

where α_n is a root of the equation

$$\frac{d}{d\alpha} H^{(2)}_{1/3} (\alpha) = 0 . \tag{8.12}$$

The left member of Equation (8.11), when expanded in powers of $(x_1 - \xi_0)$, becomes

$$- i \sqrt{\frac{2 \xi_o}{\xi_o^2 - 1}} (x_1 - \xi_o)^{3/2} \left[1 + \dots \right] . \tag{8.13}$$

In Equation (8.13), the dots indicate higher powers of $(x_1 - \xi_o)$, and the branch of $(x_1 - \xi_o)^{3/2}$ will remain unspecified. Making use of Equations (8.11) and (8.13) there is obtained the relationship:

$$(x_1 - \xi_o)^3 \left[1 + \dots \right] = - \left(\frac{\xi_o^2 - 1}{2 \xi_o} \right) \left(\frac{\alpha_n}{\gamma}\right)^2 \left[1 + 0 \left(\frac{1}{\gamma}\right) \right] \tag{8.14}$$

from which is derived the formula:

$$(x_1 - \xi_o) = e^{-\frac{\pi i}{3}} \left(\frac{\xi_o^2 - 1}{2 \xi_o} \right)^{1/3} \left(\frac{\alpha_n}{\gamma}\right)^{2/3} \left[1 + 0 \left(\frac{1}{\gamma}\right) \right] . \tag{8.15}$$

Here, again the ambiguity of the branch will remain unresolved. By replacing x_1 in Equation (8.10) by its value given in Equation (8.15), one obtains:

$$\lambda = (1 - \xi_o^2) \gamma^2 \left\{ 1 + e^{-\frac{\pi i}{3}} \left(\frac{2 \xi_o \alpha_n}{(\xi_o^2 - 1) \gamma} \right)^{2/3} + 0 \left(\frac{1}{\gamma^{\frac{4}{3}}} \right) \right\} . \tag{8.16}$$

Now if $\rho^2 = -\lambda$, the values $\left\{ \rho_n \right\}$, corresponding to the values of λ at which the residues must be taken is given by the formula

$$\rho_n = (\xi_o^2 - 1)^{1/2} \gamma \left\{ 1 + \frac{1}{2} e^{-\frac{\pi i}{3}} \left(\frac{2 \xi_o \alpha_n}{(\xi_o^2 - 1)} \right)^{2/3} + 0 \left(\frac{1}{\gamma^{\frac{4}{3}}} \right) \right\} . \tag{8.17}$$

There remains an ambiguity as to the branch of the cube root to be chosen. But if the real part of ρ_n is positive, it is clear, from the observation that λ lies in the upper half-plane, that the imaginary part of ρ_n must be negative. Now it is possible to write the values of the residues. Using formulas (A.21), (A.42) and (A.43) of the appendix, and using the asymptotic relation (A.17), the residues for values of λ comparable to γ^2 become

$$R_n = \frac{\Gamma(4/3)\, 3^{3/4} e^{\frac{7\pi i}{12}}}{2^{2/3}\, f'(\xi_o)} \left(\frac{\xi_o^2 - 1}{(1-\eta^2)(\xi_o^2 - \eta^2)} \right)^{1/4} \left(\frac{\xi_o^2}{\xi_o^2 - 1} \right)^{1/2} \frac{1}{\sqrt{\rho_n}} \cdot$$

$$\cdot \left(\frac{\xi_o^2 - 1}{\xi_o} \right)^{1/6} \left(\gamma \right)^{1/6} e^{\left(-i\rho_n (\xi_o^2 - 1)^{-1/2} \cdot f(\xi_o)\right)} \frac{\cos\left[\rho_n \Phi(-\eta) - \frac{\pi}{4}\right]}{\cos\left[2\rho_n \Phi(o)\right]} \tag{8.18}$$

where

$$\Phi(x) = -\frac{1}{(\xi_o^2 - 1)^{1/2}} \int_x^1 \left(\frac{\xi_o^2 - z^2}{1 - z^2} \right)^{1/2} dz \, . \tag{8.19}$$

To evaluate the functions, $f(\xi_o)$, $f'(\xi_o)$, let it first be observed that

$$f(\xi_o) = \lim_{x \to \infty} \left\{ \int_{\xi_o}^{x} \sqrt{\frac{x^2 - \xi_o^2}{x^2 - 1}}\, dx - x \right\} . \tag{8.20}$$

If, in Equation (8.20) $z = \frac{\xi_o}{x}$ is chosen as the new variable, then

$$f(\xi_o) = \lim_{z \to 0} \left\{ -\xi_o + \xi_o^2 \int_z^1 \frac{1}{z^2} \sqrt{\frac{1 - z^2}{\xi_o^2 - z^2}}\, dz - \xi_o \int_z^1 \frac{1}{z^2}\, dz \right\} =$$

$$= \lim_{z \to 0} \left\{ \xi_o^2 \int_z^1 \frac{1}{z^2} \sqrt{\frac{1 - z^2}{\xi_o^2 - z^2}}\, dz - \frac{\xi_o}{z} \right\} . \tag{8.21}$$

Integration by parts can then be performed; this results in the formula:

$$f(\xi_o) = \lim_{z\to 0} \frac{\sqrt{(1-z^2)(\xi_o^2-z^2)}}{z} - \xi_o - \int_z^1 \sqrt{\frac{\xi_o^2 - z^2}{1-z^2}}\, dz =$$

$$= - \int_0^1 \sqrt{\frac{\xi_o^2 - z^2}{1-z^2}}\, dz = (\xi_o^2 - 1)^{1/2}\ \Phi(o) \tag{8.22}$$

where Φ has been defined by Equation (8.19). Also,

$$f'(\xi_o) = - \int_0^1 \frac{\xi_o}{\sqrt{(\xi_o^2-z^2)(1-z^2)}}\, dz\ . \tag{8.23}$$

These computations having been made, it is now possible, by introducing some new parameters, to rewrite the formulas (8.18) in a more suggestive form. But first it shall be supposed that the limit of the residues as $s \to 0$ has been taken. In this case, γ can be replaced by $\varepsilon a\omega$, where $\varepsilon = \frac{1}{\xi_o}$ is the eccentricity of the ellipse, and $(1 - \varepsilon^2)^{1/2}\, a = b$, the semiminor axis.

Further, let $d(\eta)$ and $d^*(\eta)$ be defined by the formulas:

$$d(\eta) = b \left\{ \Phi(-\eta) - \Phi(o) \right\}; \quad d^*(\eta) = -b \left\{ \Phi(-\eta) + \Phi(o) \right\} . \tag{8.24}$$

The quantities $d(\eta)$ and $d^*(\eta)$ have the following geometric interpretation. Let a point on the ellipsoid be designated by its coordinate, η . Then $d(\eta)$ is the length of the shortest geodesic arc from the point to the shadow boundary; i.e., the curve $\eta = 0$. If $-1 < \eta < 0$, $d(\eta) > 0$; if $0 < \eta < 1$, $d(\eta) > 0$. $d^*(\eta)$ is the length of that geodesic arc from the point to the shadow boundary which passes through $\eta = -1$. (See Figure 2).

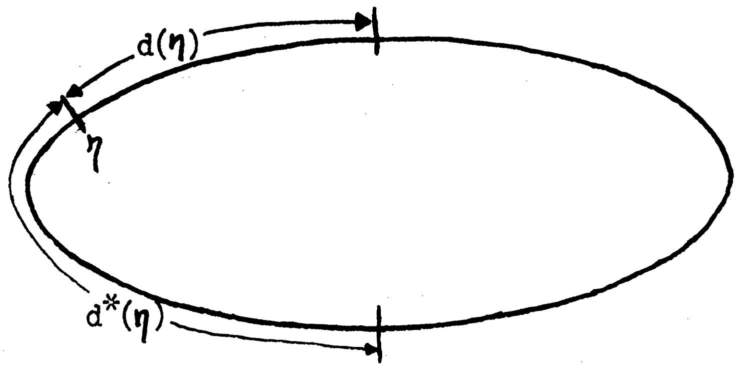

Figure 2

Also, let

$$\nu_n = -\frac{\rho_n}{b} = -\omega\left\{1 - \frac{e^{-\frac{\pi i}{3}}}{2^{1/3}}\left(\frac{\alpha_n}{\frac{b^2}{a}\,\omega}\right)^{2/3} + 0\left(\frac{1}{\omega^{4/3}}\right)\right\}. \qquad (8.25)$$

The second equality in Equation (8.25) is obtained from Equation (8.17). The imaginary part of ν_n is positive. Finally, observe that if L is the length of the circumference of the ellipse in Figure 2, then

$$4b\;\Phi(o) = -L\;. \qquad (8.26)$$

If the $\sqrt{\rho_n}$ which appears in Equation (8.18) is replaced by $(\xi_o^2-1)^{1/2}\,\gamma$, and if the results and formulas (8.22) - (8.26) are used, one obtains:

$$\lim_{s\to 0} R_n = -\frac{\Gamma(4/3)\,(9/2)^{2/3}\,e^{\frac{7\pi i}{12}}}{F(\varepsilon)}\left\{\frac{1}{(1-\eta^2)(1-\varepsilon^2\eta^2)}\right\}^{1/4}\frac{1}{(1-\varepsilon^2)^{5/6}(a\omega)^{1/3}}\cdot$$

$$\cdot\left\{\frac{e^{\frac{\pi i}{4}}\,e^{i\,\nu_n d(\eta)} + e^{-\frac{\pi i}{4}}\,e^{i\,\nu_n d^*(\eta)}}{1+e^{i\,\nu_n L}}\right\}. \qquad (8.27)$$

In Equation (8.27), $F(\varepsilon)$ is the complete elliptic integral of the first kind. If the residues in Franz's paper are specialized to the case of a plane wave, and if the formula (8.27) is specialized by allowing $\varepsilon \to 0$, the two results coincide. It is clear that for $d(\eta) > 0$, the series of residues converges, and from the remarks at the beginning of this section, represents the solution. If, however, $d(\eta) < 0$, this residue series has large early terms. The situation is dealt with in the following way:

Let the right member of Equation (8.27) be designated by:

$$A(\varepsilon) \left\{ \frac{e^{\frac{\pi i}{4}} e^{i\nu_n d(\eta)} + e^{-\frac{\pi i}{4}} e^{i\nu_n d^*(\eta)}}{1 + e^{i\nu_n L}} \right\} . \tag{8.28}$$

In the shadow, $d(\eta) > 0$, and the residue series can be summed by the "creeping wave" representation

$$\sum_n R_n = A(\varepsilon) \sum_{\ell=0}^{\infty} (-1)^{\ell} \left\{ \sum_n \begin{matrix} e^{\frac{\pi i}{4}} e^{i\nu_n (d(\eta) + \ell L)} \\ + e^{-\frac{\pi i}{4}} e^{i\nu_n (d^*(\eta) + \ell L)} \end{matrix} \right\} . \tag{8.29}$$

If one rewrites this last summation as

$$\sum_n R_n = A(\varepsilon) \left\{ \sum_n e^{\frac{\pi i}{4}} e^{i\nu_n d(\eta)} \right\} + \sum{}^{*} \tag{8.30}$$

$\sum^{*}$ represents a series which converges even in the illuminated region. The first member on the right in Equation (8.30), however, is the integral

$$- \frac{1}{2\pi i} \int_{\Gamma} \sqrt{\frac{\pi}{2\rho}} \; \frac{e^{\frac{\pi i}{4}} e^{-\frac{i\rho}{b} d(\eta)}}{(\xi_0^2 - 1)\, \phi_2'(\xi_0, \lambda)} \, d\lambda . \tag{8.31}$$

Now an examination of Equation (8.9) reveals that if the combination $\gamma^2 z^2 - \mu^2$ is large in comparison to $(z + \frac{1}{4z})^{-2}$, the Hankel function representation for $\phi_2'(\xi_0, \lambda)$ can be used. In this case, the integral (8.31) can be evaluated by a stationary phase technique, making use of the Langer asymptotic representations of the Hankel functions. The computation follows that of Franz. The stationary phase point occurs for the value of λ which causes the equality

$$- \frac{d(\eta)}{b} = \alpha \, , \; \cos \alpha = \frac{\mu}{\gamma z} \tag{8.32}$$

to be true. But Equation (8.32) is inconsistent with the assumption that $\gamma^2 z^2 - \mu^2$ be large unless α is larger than zero, or that $|d(\eta)|$ be large. This means that this saddle point evaluation is legitimate only in the neighborhood of the specular point of the spheroid; as a matter of fact, the evaluation of the integral (8.31) by the saddle point method gives precisely the geometric optic solution.

It is not clear what happens in the region between the shadow and the optical regions. If z, and consequently ξ_0, is very large, one sees that the Hankel function representation, can be used for the function $\phi_2(\xi_0, \lambda)$ even in the neighborhood of the stationary phase point, and that a Fock type approximation can be used (Ref. 11). But large ξ_0 corresponds to small eccentricity, and this might have been predicted in advance. The question which for the present must remain unanswered is whether, in the so-called transition region, the denominator of the integrand in Equation (8.31) can be represented as a 1/3-order Hankel function of an argument which depends on the parameters γ, λ, and z. If this can be accomplished, the Fock approximation will apply.

This completes the discussion of the evaluation of the integral (8.2). A final observation, however, should be made. In the shadow region, the representation (8.29) seems to describe a physical picture in which surface waves, generated at the shadow boundary, propagate with an attenuation factor which is proportional to $\omega^{1/3} \frac{1}{R^{2/3}}$, where R is the radius of curvature at the tip and not at the shadow boundary. This appears to contradict the work of Keller (Ref. 14), and of Fock. It seems, on the basis of the local theory of Fock, that what is happening here is that only in the transition region is the attenuation of the surface waves dependent upon the local radius of curvature, whereas in the shadow and optical region, one can merely say that the surface waves behave as if they had been generated at the shadow boundary, with an attenuation which depends upon the radius of curvature at the tip.

APPENDIX

(by N. D. Kazarinoff)

The problem is to evaluate the residues of the function

$$\frac{\psi_2(-\eta, -\lambda)}{(\xi_o^2 - 1)(1 - \eta^2)\, W(\psi_1, \psi_2)\, \phi_2'(\xi_o, \lambda)}$$

at the values of λ in the upper half λ-plane which make this function (considered as an analytic function of λ) singular. Although an exact representation of these residues is not to be expected, the application of the asymptotic theory of second order differential equations makes possible an asymptotic representation of the residues with respect to large values of $|\gamma|$. The theory of Section IV guarantees that these singularities will occur only for the values of λ which annul $\phi_2'(\xi_o, \lambda)$. The program to be followed is:

i) Obtain an asymptotic representation for the zeros of $\phi_2'(\xi_o, \lambda)$.

ii) At these zeros, obtain an asymptotic representation for $\frac{d}{d\lambda}\phi_2'(\xi_o, \lambda)$.

iii) At these zeros, obtain an asymptotic representation for $\frac{\psi_2(-\eta, -\lambda)}{W(\psi_1, \psi_2)}$.

In carrying out (i), only those zeros which are comparable to γ^2 will be found. If $|\lambda| << |\gamma|^2$, it will be seen that no zeros exist; the zeros, if any, which occur for $|\lambda| >> |\gamma|^2$ will simply not be considered.

For convenience, in this appendix, a change of notation will be made: x will be the independent variable and y will be the dependent variable. The homogeneous equation to be considered is:

$$\{(x^2 - 1)y'\}' + \{\gamma^2(x^2 - 1) + \lambda\}y = 0, \quad x \geq 1 + \varepsilon , \quad \mathrm{Im}(\gamma) < 0. \tag{A.1}$$

Let

$$y = (x^2 - 1)^{-1/2} w . \tag{A.2}$$

Then

$$\omega'' + \left\{\gamma^2 + \frac{\lambda}{x^2 - 1} + \frac{1}{(x^2 - 1)^2}\right\} \omega = 0 . \tag{A.3}$$

For $|\lambda| << |\gamma|^2$, (A.3) has linearly independent solutions $w_j \sim e^{\pm i\gamma x}$, $j = 1, 2$. (Whenever this notation appears, the upper sign is to be used when j = 1, the lower sign when j = 2.) Thus $y_j = (x^2-1)^{-1/2} w_j = (x^2-1)^{-1/2} e^{\pm i\gamma x} \left[1 + \frac{0(1)}{\gamma}\right]$ are the independent solutions of (A.1). But $y_j' = \left(\pm i\gamma - \frac{x}{x^2-1}\right)w_j$. Hence $y_j' = 0$ implies that $\frac{x}{x^2-1} = 0(|\gamma|)$. But if $|\gamma|$ is large, and $x \geq 1 + \varepsilon$, this does not happen. Therefore, for $|\lambda| << |\gamma|^2$, y_j' has no zeros.

Since it is now to be assumed that λ is comparable to γ^2, the auxiliary parameter $x_1 \geq 1 + \varepsilon$ is introduced so that

$$(x_1^2 - 1)\gamma^2 + \lambda = 0 . \tag{A.4}$$

For this configuration of x_1, γ, λ, (A.3) exhibits a simple turning point at x_1; and eliminating λ, one finds that

$$w'' + \left\{\gamma^2 \frac{x^2 - x_1^2}{x^2 - 1} + \frac{1}{(x^2 - 1)^2}\right\} w = 0. \tag{A.5}$$

The behavior of the solution of (A.5) which $\sim e^{-i\gamma x}$ as $x \to \infty$ will now be determined. In the notation of Langer (Refs. 15, 16), let

$$\phi^2 = \frac{x^2 - x_1^2}{x^2 - 1} , \tag{A.6}$$

where ϕ is the root which is positive for $x > x_1$.

Also define

$$\Phi(x) = \int_{x_1}^{x} \phi(t)\, dt, \quad \text{and} \quad \xi = \gamma \Phi . \tag{A.7}$$

Then introduce the function:

$$\Psi(x) = \Phi^{1/6} \phi^{-1/2}, \qquad \Psi(x_1) = \lim_{x \to x_1} \Psi(x) . \tag{A.8}$$

With this definition, $\Psi(x)$ is regular at x_1.

The equation

$$v'' + \left(\gamma^2\phi^2 - \frac{\Psi''}{\Psi}\right) v = 0 \tag{A.9}$$

has solutions:

$$v_j = \Psi \xi^{1/3} J_{\pm 1/3}(\xi) , \quad j = 1,2 . \tag{A.10}$$

The general theory (Ref. 14) guarantees that on the half line $x \geqslant 1 + \varepsilon$, the solution which is sought has the following representation:

$$\begin{aligned} w &= e^{i\gamma f(x_1)} \phi^{-1/2} e^{-i\xi} \left[1 + \frac{B(x,\gamma)}{\gamma} + \frac{B(\xi)}{\xi}\right] \\ w' &= -i\gamma \phi^{1/2} e^{i\gamma f(x_1)} e^{-i\xi} \left[1 + \frac{B(x,\gamma)}{\gamma} + \frac{B(\xi)}{\xi}\right] \end{aligned} \qquad (|\xi| > N) \tag{A.11}$$

and

$$\begin{aligned} w &= \gamma^{1/6} e^{i\gamma f(x_1)} \left[\gamma_1 v_1 + \gamma_2 v_2 + \frac{B(x,\gamma)}{\gamma}\right] \\ w' &= \gamma^{1/6} e^{i\gamma f(x_1)} \left[\gamma_1 v_1' + \gamma_2 v_2' + B(x,\gamma)\right] \end{aligned} \qquad (|\xi| < N). \tag{A.12}$$

In these formulas, B is a generic notation for a bounded function of its arguments,

$$\gamma_1 = i\sqrt{\frac{2\pi}{3}}\, e^{-\frac{5\pi i}{12}}, \qquad \gamma_2 = -i\sqrt{\frac{2\pi}{3}}\, e^{-\frac{\pi i}{12}}, \tag{A.13}$$

and

$$f(x_1) = \lim_{x \to \infty} \Phi(x) - x . \tag{A.14}$$

To find the zeros of $y' = \left[(x^2-1)^{-1/2}\, w\right]'$, it is necessary to find the zeros of

$$w' - \frac{x}{x^2-1}\, w . \tag{A.15}$$

If $|\xi| > N$, (A.15) has zeros only if

$$x = -i\,\gamma\,(x^2-1)^{1/2}\,(x^2-x_1^2)^{1/2} + O\left(\frac{1}{|\gamma|}\right) . \tag{A.16}$$

This results from considering Equations (A.15), (A.11) and (A.6). If $x > x_1$, both square roots which appear in (A.16) are real; hence x must have an imaginary part, which is impossible. If $x < x_1$, $(x^2-x_1^2)^{1/2}$ is pure imaginary, and again x must have an imaginary part, since γ does. Therefore, for unbounded $|\xi|$, if $|\gamma|$ is large and γ has an imaginary part, y' has no zeros. On the other hand, if $|\xi| < N$, the defining relationship, (A.7), for ξ, shows that $x - x_1 \sim N\,\gamma^{-2/3}$.

Thus, if λ obeys (A.4), the only zeros of $y'(x,\lambda)$ are, for large values of γ, asymptotic to x_1. From this, and the facts developed previously, it is then possible to infer that if $y'(x_0,\lambda) = 0$, then, for large $|\gamma|$, λ obeys the asymptotic relationship:

$$\lambda = (1-x_0^2)\,\gamma^2\left[1 + O\left(|\gamma|^{-2/3}\right)\right] . \tag{A.17}$$

Thus, to compute $\frac{\partial}{\partial\lambda}\, y'(x_0, \gamma, \lambda)$ at the value of λ for which $y'(x_0, \gamma, \lambda) = 0$, it is sufficient to compute

$$\frac{\partial y'}{\partial \lambda} = \frac{\partial y'}{\partial x_o} \frac{\partial x_o}{\partial \lambda} , \tag{A.18}$$

where y' means $\left.\frac{dy}{dx}\right|_{x=x_o}$, and $\frac{\partial x_o}{\partial \lambda}$ is asymptotically computed from (A.17). The representations (A.12) are used to evaluate y and y', because $\xi(x_o) = 0$. [x_1 is now replaced by x_o] .

Now

$$y'(x_o) = \gamma^{1/6} e^{i\gamma f(x_o)} \Big\{ (x_o^2-1)^{-1/2} \left[\gamma_1 v_1' + \gamma_2 v_2' + B(x,\gamma) \right. -$$

$$\left. - x_o(x_o^2-1)^{-3/2} \quad \gamma_1 v_1 + \gamma_2 v_2 + \frac{B(x,\gamma)}{\gamma}\right]\Big\} . \tag{A.19}$$

If the derivative $\frac{\partial y'(x_o)}{\partial x_o}$ is computed, because of the presence of the γ in the exponential, only the term

$$i \; \gamma^{7/6} f'(x_o) \; e^{i\gamma f(x_o)} \Big\{ \qquad \Big\} \tag{A.20}$$

need be considered. In (A.20) the expression in the brackets is the same as in (A.19). But if the defining relationships for v_j are studied, one finds that in the bracketed expression, the term involving $v_2'(x_o)$ has the factor $\gamma^{2/3}$, and the other terms have lower powers of γ as factors. Hence, for the asymptotic evaluation, one finds that

$$\frac{\partial y'}{\partial \lambda} \sim - \frac{2^{1/6} \, 3^{-4/3} \sqrt{\pi} \, e^{-\frac{\pi i}{12}} \, e^{i\gamma f(x_o)} \, f'(x_o)}{\Gamma(4/3) \quad (x_o^2 - 1)^{1/2} \quad x_o} \left(\frac{x_o}{x_o^2 - 1}\right)^{-1/6} \gamma^{-1/6} .$$

Referring to the integrand described at the beginning of this appendix, it is seen that the contribution of the factor

$$\frac{1}{(\xi_o^2 - 1)\,\phi_2'(\xi_o, \lambda)}$$

to the residues at the zeros of $\phi_2'(\xi_0, \lambda)$ is, for large $|\gamma|$, asymptotic to

$$-\frac{\Gamma(4/3)\,3^{4/3}\,e^{\frac{\pi i}{12}}}{2^{1/6}\sqrt{\pi}}\left(\frac{x_0}{(x_0^2-1)^{1/2}}\right)\frac{e^{-i\gamma f(x_0)}}{f'(x_0)}\left(\frac{x_0^2-1}{x_0}\right)^{1/6}\gamma^{1/6} \qquad \text{(A.21)}$$

To find the contribution of the η term to the residue, the equation which must be studied is

$$\left\{(1-x^2)\,y'\right\}' + \left\{\gamma^2(1-x^2) - \lambda\right\} y = 0, \qquad -1 < x < 1. \qquad \text{(A.22)}$$

If

$$y = (1-x^2)^{-1/2}\,w\;, \qquad \text{(A.23)}$$

then

$$w'' + \left\{\gamma^2 - \frac{\lambda}{1-x^2} + \frac{1}{(1-x^2)^2}\right\} w = 0. \qquad \text{(A.24)}$$

Using the relationship (A.17), and setting $\lambda = -\rho^2$, one can write (A.24) in the form:

$$w'' + \left\{\rho^2\left[\frac{x_0^2 - x^2}{(x_0^2-1)(1-x^2)}\right] + \frac{B(x,\rho)}{(1-x^2)^2}\right\} w = 0\,. \qquad \text{(A.25)}$$

Let

$$\phi^2 = \frac{x_0^2 - x^2}{(x_0^2-1)(1-x^2)}, \quad \phi \text{ being the positive root,} \qquad \text{(A.26)}$$

$$\Phi(x) = \int_1^x \phi(t)\,\partial t\;, \quad \xi = \rho\,\Phi\;, \text{ and } \Psi = \left[\phi\,\Phi\right]^{-1/2}, \qquad \text{(A.27)}$$

where Ψ is defined to be continuous at $x = 1$.

If

$$v = \Psi\,\xi\,C_0(\xi)\;, \qquad \text{(A.28)}$$

where $C_0(\xi)$ is any circular function, then v satisfies the equation

$$v'' + \left[\rho^2\phi^2 + \frac{1}{(1-x^2)^2} + k\right] v = 0\;, \qquad \text{(A.29)}$$

where k is a function of x which is $O\left(\frac{1}{1-x}\right)$ on the closed interval $[-1, 1]$. Hence for large $|\rho|$, the solutions of (A.22) behave asymptotically on the interval $-1 < x < 1$ like functions of the form (A.28). In particular, if

$$v(x) = \sqrt{\xi}\, J_o(\xi), \tag{A.30}$$

then the function

$$y(\eta) = (1 - \eta^2)^{-1/2} \quad v(\eta) \tag{A.31}$$

has the same asymptotic behavior, for large $|\rho|$, as the solution of $(L\eta + \lambda)\psi$ which is regular at x = 1.

Now it is easy to verify that if $y(\eta)$ is given by)A.31), then

$$\lim_{\eta \to 1} y(\eta) = i\rho, \tag{A..32}$$

and thus, for large $|\rho|$, the function $\psi_2(x, -\lambda)$ has the asymptotic representation

$$\psi_2(x, -\lambda) \sim \frac{1}{i\rho} (1 - x^2)^{-1/2} \sqrt{\xi}\, J_o(\xi) = \frac{1}{i\rho} (1 - x^2)^{-1/2} \quad v(x) \cdot$$

$$\psi_2'(x, -\lambda) \sim \frac{1}{i\rho} \left\{ (1 - x^2)^{-1/2} v'(x) + x (1 - x^2)^{-3/2} \quad v(x) \right\} \tag{A.33}$$

and $\psi_1(x, -\lambda) = \psi_2(-x, -\lambda)$ satisfies the relations

$$\psi_1(x, -\lambda) \sim \frac{1}{i\rho} \left\{ (1 - x^2)^{-1/2} \quad v(-x) \right\}$$

$$\psi_1'(x, \lambda) \sim \frac{1}{i\rho} \left\{ -(1 - x^2)^{-1/2} \quad v'(-x) + x (1 - x^2)^{-3/2} \quad v(x) \right\} . \tag{A.34}$$

Using (A.33) and (A.34), one obtains:

$$(1 - x^2)\, W(\psi_1, \psi_2) \sim \frac{v'(-x)\, v(x) + v'(x)\, v(-x)}{\rho^2} \quad . \tag{A.35}$$

and since this is independent of x,

$$(1 - x^2)\, W(\Psi_1, \Psi_2) \sim \frac{2\, v(o)\, v'(o)}{\rho^2} \quad . \tag{A.36}$$

Now,

$$2\, v(x)\, v'(x) = \frac{d}{dx}\left[\ v(x)\ \right]^2 = \frac{d}{dx}\left[\Psi^2 \xi^2 J_o^2(\xi)\right] = \tag{A.37}$$

$$= \rho^2 \frac{d}{dx}\left[\frac{\Phi}{\phi}\ J_o^2(\xi)\right] = \rho^2 \left\{2\ \Phi\ p\, J_o(\xi) J_o'(\xi) + \frac{\phi^2 - \Phi\phi'}{\phi^2}\ J_o^2(\xi)\right\} .$$

Since, if any closed subinterval of the open interval (-1, 1) is to be considered, ξ is large for large ρ, and the representations:

$$J_o(\xi) \sim \left(\frac{2}{\pi\xi}\right)^{1/2} \cos\left(\xi - \frac{\pi}{4}\right),\quad J_o'(\xi) \sim -\left(\frac{2}{\pi\xi}\right)^{1/2} \sin\left(\xi - \frac{\pi}{4}\right) \tag{A.38}$$

can be used.

Therefore, using the highest power of ρ that is present in (A.37), one finds that

$$2\, v(o)\, v'(o) \sim 2\rho^3\ \Phi(o)\, J_o(\rho\ \Phi(o))\ J_o'(\rho\ \Psi(o))$$

$$\sim -\frac{4}{\pi}\ \rho^2 \cos\left(\rho\ \Phi(o) - \frac{\pi}{4}\right) \sin\left(\rho\ \Phi(o) - \frac{\pi}{4}\right) \tag{A.39}$$

$$\sim \frac{2}{\pi}\ \rho^2\ \cos(2\rho\ \Phi(o)) \quad .$$

Thus,

$$(1 - x^2)\, W(\Psi_1, \Psi_2) \sim \frac{2}{\pi}\ \cos\ (2\rho\ \Phi(o)) \quad . \tag{A.40}$$

Also,

$$\Psi_2(-x, -\lambda) \sim \frac{1}{i\rho}\ (1 - x^2)^{-1/2}\ \Psi\ \xi\, J_o(\xi) \tag{A.41}$$

$$\sim \frac{1}{i}\left(\frac{x_o^2 - 1}{(x_o^2 - x^2)(1 - x^2)}\right)^{1/4} \sqrt{\frac{2}{\pi\rho}}\ \ \cos\left(\rho\ \Phi(-x) - \frac{\pi}{4}\right) \quad .$$

Consequently, if $\{\rho_n\}$ is the set of values of complex values corresponding to those zeros $\{\lambda_n\}$ of $\phi_2^1(\xi_0, \lambda)$ for which $\lambda_n \sim (1 - x_o^2)\gamma^2$, then the corresponding contribution to the residue by the factor

$$\frac{\Psi_2(-\eta, -\lambda)}{(1-\eta^2)\, W(\Psi_1, \Psi_2)}$$

is asymptotically, (A.42)

$$\left.\frac{\Psi_2(-\eta, -\lambda)}{(1-\eta^2)\, W(\Psi_1, \Psi_2)}\right|_{\lambda=\lambda_n} \sim -i\sqrt{\frac{\pi}{2\rho_n}} \left(\frac{(\xi_o^2 - 1)}{(1-\eta^2)(\xi_o^2 - \eta^2)}\right)^{1/4} \frac{\cos\left(\rho_n \Phi(-\eta) - \frac{\pi}{4}\right)}{\cos\left(2\rho_n \Phi(o)\right)}$$

where,

$$\Phi(x) = -\frac{1}{(\xi_o^2 - 1)^{1/2}} \int_x^1 \left(\frac{\xi_o^2 - t^2}{1 - t^2}\right)^{1/2} dt =$$

(A.43)

$$= \frac{\xi_o}{(\xi_o^2 - 1)^{1/2}} \left\{ E\left(\frac{1}{\xi_o}, \sin^{-1} x\right) - E\left(\frac{1}{\xi_o}, \frac{\pi}{2}\right)\right\}.$$

Here, E(k, z) is the elliptic function of the second kind.

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