THE UNIVERSITY OF MICHIGAN
COMPUTING RESEARCH LABORATORY

ON THE RELATIONSHIPS BETWEEN SCOTT DOMAINS, SYNCHRONIZATION TREES, AND METRIC SPACES.

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CRL-TR-25-83

JULY 1983

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I. Introduction

Scott's theory of Information Systems [S] is intended to provide an easy way to define partial order structures (domains) for denotational semantics. This paper illustrates the new method by considering a simple modal logic, due to Hennessy and Milner [HM], as an example of an information system. The models of formulas in this logic are the rigid synchronization trees of Milner [M]. We characterize the domain defined by the Hennessy-Milner information system as the complete partial order of synchronization forests: nonempty closed sets of synchronization trees. "Closed" means closed with respect to a natural metric distance on synchronization trees, first defined by de Bakker and Zucker [BaZ] and characterized by Golson and Rounds [GR].

After notational preliminaries and background results, Section III treats the Hennessy-Milner information system. The background results [BR], [GR] are used as lemmas in the characterization of the partial order. The next part (Section IV) shows how to use metric space methods to extend certain natural tree operations to forests. These operations become sup-continuous when so extended, and therefore can be used to provide a denotational semantics for concurrency which allows the full power of least-fixed point methods for recursion (Section V).

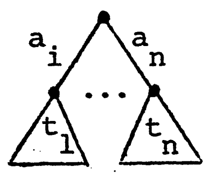
From the results in this paper, we conclude that the Information System approach to denotational semantics shows real promise. We began by investigating the Hennessy-Milner information system as a way to construct complete partial orders for synchronization trees. The result was a surprisingly natural construction of a complete partial order which we might not have found without the tools provided by information systems.

II. Notation and previous work

Definition 2.1. Let Σ be a finite alphabet. A Σ-tree is a tree graph on a nonempty finite or countable set of nodes, with arc labels from Σ. No ordering on the arcs leaving a node is presumed, and more than one such arc may have the same label. Nodes are unlabeled, although leaf nodes are considered to be the one-node tree NIL.

Σ-trees are the rigid synchronization trees of Milner [M]. They correspond to 'unfoldings' of state graphs for nondeterministic transition systems. Milner develops an algebraic system based on these trees and their generalizations, suitable for a semantics for communicating systems. Our purpose here is to show a way of associating a Scott order structure (domain) to Σ-trees. First we need to recall some definitions.

Notation. Suppose t is the Σ-tree represented by



$(a_i \in \Sigma)$

We then write $t = \Sigma a_i t_i$. The same notation will suffice for a tree with countably many arcs from the root. For each $a \in \Sigma$ we define the binary transition relation $\xrightarrow{a}$ on the set T_Σ of all Σ-trees by

$$t \xrightarrow{a} u$$

iff $t = \Sigma a_i t_i$, $a = a_i$ for some i, and $u = t_i$.

<u>Definition</u> 2.2 (weak observational equivalence).

Let a series W_k $(k \geq 0)$ of equivalence relations on T_Σ be given as follows:

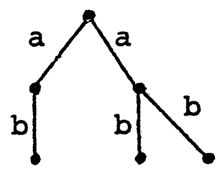
$t\ W_0\ u$ always;

$t\ W_{k+1}\ u \iff (\forall t', a)[t \xrightarrow{a} t' \Rightarrow (\exists u')(u \xrightarrow{a} u' \text{ and } u'\ W_k\ t')]$

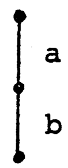
and vice versa.

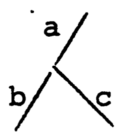
The weak observational equivalence W on T_Σ is given by $t\ W\ u \iff (\forall k \geq 0)(t\ W_k\ u)$.

<u>Examples</u>. Let $\Sigma=\{a,b,c\}$

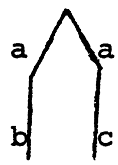


W





$\neg W_2$



Hennessy-Milner logic (HML) first appeared in [HM] as a language for describing Σ-trees (or transition systems).

<u>Definition</u> 2.3. HML is the least class of formulas containing the Boolean constants tt and ff, and closed under Boolean connectives $\wedge$, $\vee$, $\neg$ and under application of the (unary) modal operators $\langle a \rangle$ for each $a \in \Sigma$.

<u>Examples</u>. $\langle a \rangle (\langle b \rangle tt \wedge \langle c \rangle tt)$

$\neg \langle a \rangle (\langle b \rangle tt \vee \langle c \rangle tt)$

<u>Definition</u> 2.4. (Semantics of HML). Let $\varphi \in$ HML. $t \in T_\Sigma$ <u>satisfies</u> φ $(t \models \varphi)$ iff one of the following inductive cases holds:

(i) $\varphi = tt \iff t \in T_\Sigma$

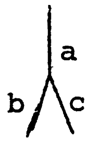
(ii) $\varphi = \psi \wedge \theta \iff t \models \varphi$ and $t \models \theta$

(iii) $\varphi = \neg\psi \iff$ not $(t \models \psi)$

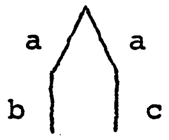
(iv) $\varphi = \psi \vee \theta \iff t \models \varphi$ or $t \models \varphi$

(v) $\varphi = \langle a \rangle \theta \iff (\exists u)(t \xrightarrow{a} u \wedge u \models \theta)$.

Examples.



$\models \theta = \langle a \rangle (\langle b \rangle tt \wedge \langle c \rangle tt)$



$\models \neg \theta$.

Definition 2.5. The modal rank $r(\varphi)$ of a formula $\varphi \in$ HML is defined inductively

$r(tt)=r(ff)=0;$

$r(\varphi \vee \psi)=r(\varphi \wedge \psi)=\max\{r(\varphi), r(\psi)\};$

$r(\neg \varphi)=r(\varphi)$

$r(\langle a \rangle \varphi)=1+r(\varphi)$.

Let $HML_k = \{\varphi \in HML \mid r(\varphi) \leq k\}$.

Definition 2.6. (Elementary equivalence)

Define the equivalences E_k and E on T_Σ by

$t \; E_k \; u \iff \forall \varphi \in HML(_k) [t \models \varphi \iff u \models \varphi]$, and $t \; E \; u \iff (\forall k) \; t \; E_k \; U$.

Finally we have the notion of logical equivalence.

Definition 2.7. $\theta \equiv \psi$ iff $\forall t (t \models \varphi \iff t \models \psi)$.

The proofs of the following facts can be found in [BR] and [GR]. Note: $|\Sigma| < \infty$.

Lemma 2.1. For all $t, u \in T_\Sigma$, and $k \geq 0$, $t \; W_k \; u \iff t \; E_k \; u$.

Corollary: $W = E$.

Lemma 2.3. (Master formula theorem for HML.)

For each Σ-tree t and each $k \geq 0$ there is a formula $\varphi(k,t) \in HML_k$ such that

(i) $t \models \varphi(k,t)$

(ii) for all u, if $u \models \varphi(k,t)$ then $u \; W_k \; t$.

Lemma 2.4. (Compactness theorem for HML.)

Let $\Gamma \subseteq$ HML. If every finite subset of Γ has a tree model then so does Γ.

Let Mod(Γ) be the set of tree models of Γ; that is,

$$\{t \mid (\forall \varphi \varepsilon \Gamma)(t \models \varphi)\}.$$

The compactness theorem states $((\forall$ F finite $\subseteq \Gamma)$ Mod(F)$\neq$0) implies Mod(Γ)$\neq$0 . We note that 2.4 holds even for infinite Σ.

Finally, we recall some facts about the "Golson metric" d_W on T_Σ.

Definition 2.8.

Let $e_W(t,u)=\max\{k \mid t\ W_k\ u\}$ with $e_W(t,u)=\infty$ if t W u.

The pseudo-metric distance $d_W(t,u)$ is then given by $2^{-e_W(t,u)}$, where $2^{-\infty}=0$.

d_W is actually an ultrametric on T_Σ. We have

$$d_W(t,v) \leq \max(d_W(t,u),\ d_W(u,v)).$$

Further, $d_W(t,u)=0 \Leftrightarrow$ t W u, so that d_W is a metric on T_Σ/W.

Definition 2.9. The k-section $t^{(k)}$ of a tree t is defined to be the set of nodes at distance k or less from the root, together with the relevant arcs. The 0-section is then just the one-node root.

Lemma 2.5. $t\ W_k\ u \Leftrightarrow t^{(k)}\ W\ u^{(k)}$

Lemma 2.6. $d_W(t^{(k)},t) \to 0$ as $k \to \infty$.

Lemma 2.7. $\langle T_\Sigma/W,\ d_W\rangle$ is a compact metric space.

Recall that a compact space is one where every covering by open sets has a finite subcovering. For a metric space it is equivalent to saying that every infinite sequence has a convergent subsequence.

III. Information systems and Hennessy-Milner logic

First we recall the general definition of information system from Scott [S].

Definition 3.1. An information system is a structure $<D, \varphi_0, \text{Con}, \vdash>$ where D is a set of 'propositions', $\varphi_0 \varepsilon D$ is the least informative proposition, Con is a collection of finite subsets of D (the finite consistent sets), and $\vdash$ is the entailment relation, a subset of Con x D. The following axioms hold:

(i) $\Gamma \varepsilon \text{Con} \wedge \Delta \subseteq \Gamma \Rightarrow \Delta \varepsilon \text{Con}$;

(ii) $\{\varphi\} \varepsilon \text{Con}$ for all $\varphi \varepsilon D$;

(iii) $\Gamma \vdash \varphi \wedge \Gamma \varepsilon \text{Con} \Rightarrow \Gamma \cup \{\varphi\} \varepsilon \text{Con}$;

(iv) $\Gamma \varepsilon \text{Con} \Rightarrow \Gamma \vdash \varphi_0$

(v) $\Gamma \varepsilon \text{Con} \wedge \varphi \varepsilon \Gamma \Rightarrow \Gamma \vdash \varphi$

(vi) $\Gamma \vdash \varphi$ for all $\varphi \varepsilon \Delta$ and $\Delta \vdash \theta \Rightarrow \Gamma \vdash \theta$.

An information system is a way of giving 'facts', expressed in D, about abstract structures. The more 'facts' we know, the more 'well-defined' the structure becomes. We can express these notions purely in terms of D itself using sets of propositions.

Definition 3.2. The ideal elements defined by the information system D are those subsets Γ of D satisfying

(i) Γ is consistent: Every finite $\Delta \subseteq \Gamma$ is a member of Con;

(ii) Γ is deductively closed: $\Delta \subseteq \Gamma$, $\Delta \varepsilon \text{Con}$, and $\Delta \vdash \theta \Rightarrow \theta \varepsilon \Gamma$.

An element Γ is total if it is maximal with respect to the inclusion ordering on PD, and otherwise partial.

Lemma 3.1. The ideal elements I_D of an information system D form a complete partial order under ordinary inclusion.

For our purposes, all we need to know about complete partial orders is that every chain Γ_i has a supremum $\cup\Gamma_i$. In this case the union of the Γ_i sets is the obvious supremum. It can be shown that the cpo's defined by information systems are exactly the <u>consistently complete, algebraic</u> cpo's. See [S] for details.

<u>HML as an information system.</u>

HML provides a natural example of an information system describing sets of Σ-trees.

<u>Definition</u> 3.3. The HML system is given by:

$D=\{\theta\varepsilon HML \mid (\exists t)(t \models \theta)\}$;

$Con=\{\Delta \subseteq HML \mid \Delta \text{ finite} \wedge Mod(\Delta) \neq \emptyset\}$;

$\varphi_0=tt$;

$\Delta \vdash \theta \iff (\forall t)[(\forall\psi\varepsilon\Delta)(t \models \psi) \Rightarrow t \models \theta]$.

(Alternatively, if $\Delta=\{\theta_1,\ldots,\theta_n\}$, $\theta\wedge\ldots\wedge\theta_n \to \theta$ is valid.)

We want to characterize the abstract cpo $\langle I_{HML}, \subseteq\rangle$ in terms of Σ-trees. To do this we need two simple definitions.

<u>Definition</u> 3.3. Let E be an equivalence relation in T_Σ. A set $K \subseteq T_\Sigma$ is E-closed iff u ε K and t E u imples t ε K.

<u>Definition</u> 3.4. $K \subseteq T_\Sigma$ is metric-closed iff t_i ε K and $d_W(t_i,t)\to 0$ imply $t\ W\ \bar{t}$ for some $\bar{t}$ ε K.

Let $P_c(T_\Sigma)$ be the set of all nonempty W-closed, metric-closed subsets of T_Σ. (The elements of $P_c(T_\Sigma)$ are called Σ-forests.)

<u>Theorem</u> 3.2. $\langle I_{HML}, \subseteq\rangle$ is isomorphic to $\langle P_c(T_\Sigma), \supseteq\rangle$.

<u>Proof.</u> Consider the map $\Gamma \to Mod(\Gamma)$ from I_{HML} to $P(T_\Sigma)$. We verify that Mod is the required isomorphism.

(i) $\Gamma \subseteq \Gamma' \iff Mod(\Gamma) \supseteq Mod(\Gamma')$.

The => direction is trivial. Consider the reverse implication. Suppose $Mod(\Gamma) \supseteq Mod(\Gamma')$ and let $\theta \in \Gamma$. By 3.2, we need only find a finite $\Delta \subseteq \Gamma'$ such that $\Delta \vdash \theta$. Suppose this is not the case: for every finite subset Δ of Γ', we have $\neg(\Delta \vdash \theta)$. Let $\Delta = \{\theta_1, \ldots, \theta_n\}$: we have that $\Delta \cup \{\neg\theta\}$ has a tree model, because $\neg(\theta_1 \wedge .. \wedge \theta_n \to \theta)$ is satisfiable. So every finite subset of $\Gamma' \cup \{\neg\theta\}$ has a tree model, and therefore by compactness $\Gamma' \cup \{\neg\theta\}$ has a tree model. But if $t \in Mod(\Gamma')$, $t \in Mod(\Gamma)$ by hypothesis. However, $\theta \in \Gamma$, so $t \models \theta$ and $t \models \neg\theta$, a contradiction.

(ii) $Mod(\Gamma)$ is metrically closed and W-closed.

Certainly $Mod(\Gamma)$ is W-closed because of 2.1. Let $d_W(t_n, t) \to 0$ where $t_n \in Mod(\Gamma)$. Suppose that t is not W-equivalent to any u in $Mod(\Gamma)$. Then there is some $\theta \in \Gamma$ such that $t \models \neg\theta$. Let p be the modal rank of $\neg\theta$. Choose n such that $d_W(t_n, t) < 1/2^p$. Then $t_n \; W_p \; t$ so by 2.1, $t_n \; E_p \; t$. But $t_n \models \theta$ and $t \models \neg\theta$, a contradiction.

(iii) If K is W-closed and metrically closed, then $(\exists\Gamma)\; K = mod(\Gamma)$.

Let $\Gamma = \{\theta \mid (\forall t \in K)(t \models \theta)\}$.

It is easy to check that Γ is consistent and deductively closed. Certainly $K \subseteq Mod(\Gamma)$ by definition. We assert $Mod(\Gamma) \subseteq K$: To show this we let $t \in Mod(\Gamma)$ and construct a sequence $\langle s_n \rangle$ such that $d_W(s_n, t) \to 0$ and $s_n \in K$.

Fix $n \geq 0$. For each $s \in K$ let $\varphi(n,s)$ be the master formula $\in HML_n$ satisfied by s (Lemma 2.3). By lemma 2.2 there are only a finite number of logically distinct $\varphi(n,s)$ as s ranges over K.

Let $\varphi_n = \bigvee_{s \in K} \varphi(n,s)$. This is a finite disjunction, and $\forall s \in K$, $s \models \varphi_n$, so $\varphi_n \in \Gamma$. Since $t \in Mod(\Gamma)$ $t \models \varphi_n$. By definition of φ_n, $\exists s_n \in K$ such that $t \models \varphi(n, s_n)$. The s_n are the required sequence, since by 2.3 $t \; W_n \; s_n$ and by 2.6 and 2.7 $d(t, s_n) \to 0$.

Now (i) shows that the map $\Gamma \to \mathrm{Mod}(\Gamma)$ is one-one and order-preserving, and (ii) and (iii) show that it is onto. This completes the proof of Theorem 3.2.

As corollary, we have the following proposition.

Corollary 3.3. The maximal elements of $\langle P_c(T_\Sigma), \supseteq \rangle$ are the (equivalence classes of) singleton Σ-trees; the bottom element is the set T_Σ itself.

It is an instructive exercise to show this corollary directly from the definition of $\langle I_{HML}, \subseteq \rangle$.

IV. Forests and operations on forests

Definition 4.1. A Σ-forest is a metrically closed, W-closed subset of T_Σ.

We recall the notion of Hausdorff distance between closed subsets of a metric space.

Definition 4.2. Let $\langle X,d \rangle$ be a metric space, and Y,Z closed subsets of X.

$$d_H(Y,Z)=\max\{\sup_{y\in Y}\{d(y,Z)\},\ \sup_{z\in Z}\{d(z,Y)\}\}$$

where $d(y,Z)=\inf_{z\in Z}\{d(y,z)\}$.

Intuitively, $d(Y,Z)$ is the maximum distance any point in Y must travel to enter Z, or vice versa.

We would like to characterize d_H when d is the Golson metric d_W. To do this we extend the W_k and W relations to forests in the expected way.

Definition 4.3. Let H and H' be forests.

$H\ W_k\ H' \iff (\forall t\in H)(\exists t'\in H')\, t\ W_k\ t'$ and conversely.

$H\ W\ H' \iff (\forall t\in H)(\exists t'\in H')\, t\ W\ t'$ and conversely.

Lemma 4.1.

$$H\ W\ H' \iff (\forall k)(H\ W_k\ H').$$

Proof. (=>) is trivial. Let $H\ W_k\ H'$ for each k. If $t \varepsilon H$, then for each k there is a $t_k \varepsilon H'$ such that $t\ W_k\ t_k$. Thus $d_w(t_k,t)\to 0$, which implies $t \varepsilon H'$ because H' is a forest. Similarly $H' \subseteq H$, completing the proof.

Lemma 4.2. $H\ W_k\ H' \Leftrightarrow H^{(k)}\ W\ H'^{(k)}$ where $H^{(k)} = \{t^{(k)} | t \varepsilon H\}$ and $t^{(k)}$ is the k-section of t (Definition 2.9).

Proof. routine.

Definition 4.4. (Golson metric on forests)

$$d_w(H,H') = \frac{1}{2^{e_w(H,H')}}$$

where

$$e_w(H,H') = \max\{k \mid H\ W_k\ H'\}$$

with the usual proviso $1/\infty = 0$.

Lemma 4.3. d_w is a metric on forests.

Proof. routine.

Theorem 4.1. $d_w = d_H$.

Proof. $(d_H \leq d_w)$. Let $j = e_w(H,H')$, so $d_w(H,H') = 1/2^j$.

$\Rightarrow \forall t \varepsilon H\ \exists t'\ H' \varepsilon d_w(t,t') \leq \frac{1}{2^j}$

$\Rightarrow \inf_{t \varepsilon H} d_w(t,H') \leq 1/2^j$ for all $t \varepsilon H$

$\Rightarrow \sup_{t \varepsilon H} d_w(t,H') \leq 1/2^j$

Similarly

$$\sup_{t' \varepsilon H'} d_w(t',H) \leq 1/2^j$$

and the inequality holds.

Now we want $d_H \geq d_w$. Again let $d_w(H,H') = 1/2^j$ with j as above.

If $j = \infty$ there is nothing to show. Therefore we have $j < \infty$ and $H \neg W_{j+1} H'$.

$\Rightarrow (\exists t \varepsilon H)(\forall t' \varepsilon H') d_w(t,t') > \frac{1}{2^{j+1}}$

or the same assertion with the roles of H,H' reversed.

In the first case, which occurs without loss of generality, we have

$$(\exists t \varepsilon H)(\forall t' \varepsilon H')\ d_w(t,t') \geq 1/2^j$$

because d_w takes only discrete values. Therefore

$$\inf_{t' \varepsilon H'} d_w(t,t') = d_w(t,H') \geq 1/2^j$$

and so

$$\sup_{t \varepsilon H} d_w(t,H') \geq 1/2^j$$

and the inequality follows.

An extension theorem for compact metric spaces.

The space T_Σ of trees admits a number of operations suitable for defining semantics for concurrency. DeBakker and Zucker, in particular, consider the operations of "sum" - joining trees at the root;, "shuffle" - interleaving trees nondeterministically; and "composition" - grafting one tree to terminating nodes of another. They prove these operations to be continuous in the metric topology of T_Σ. We would like to extend these operators to forests in a manner analogous to extending string-valued functions to languages. We present a general theorem which allows this in any compact metric space. In what follows, X is such a compact space, and G, H, and K are closed (hence compact) nonempty subsets.

Lemma 4.4. If $H,K \subseteq X$, then there is a $\bar{k} \varepsilon K$ such that $d(K,H)=d(\bar{k},H)$. Also, for any $k \varepsilon K$, there is an $\bar{h} \varepsilon H$ such that $d(k,H)=d(k,\bar{h})$.

Proof. We show only the second result; the first is proved similarly. Recall that $\lambda y.\ d(x,y)$ is a continuous function of y for fixed x. (So also is $\lambda x.\ d(x,H)$.) Since $d(k,H)=\inf_{h \varepsilon H} d(k,h)$, we can find a sequence $h_i \ \varepsilon\ H$ such that $d(k,h_i)$ approaches $d(k,H)$. Since H is compact the h_i have a convergent subsequence h_{i_j} such that $h_{i_j} \to \bar{h} \in H$ as $j \to \infty$. By continuity of d, we know $\lim_{j\to\infty} d(k,h_{i_j}) = d(k,\bar{h})$. But $\lim_{j\to\infty} d(k,h_{i_j}) = \lim_{i\to\infty} d(k,h_i) = d(k,H)$.

<u>Lemma</u> 4.5. Let $\langle K_i \rangle$ be a decreasing chain of nonempty closed subsets of X and let $k_i \in K_i$ for each i. Then there is a convergent subsequence $k_{i_j} \to \bar{k}$ with $\bar{k} \in \bigcap_k K_i$.

<u>Proof</u>. By compactness there is a $\bar{k}$ and subsequence $k_{i_j} \to \bar{k}$ as $j \to \infty$. We need $\bar{k} \in K_i$ for each i. It is enough to show $\bar{k} \in K_{i_j}$ for each j. Pick such a j; then since $k_{i_p} \in K_{i_j}$ for all $p \geq j$, we have $\bar{k} \in K_{i_j}$ because K_{i_j} is closed.

Finally, we need a lemma on Hausdorff distances.

<u>Lemma</u> 4.6. Let H_i be a decreasing sequence of nonempty closed subsets of X. Let $H = \bigcap_i H_i$. Then $d(H_i,H) \to 0$.

<u>Proof</u>. We show $\sup_{h\in H_i} \{d(h,H)\} \to 0$ as $i \to \infty$, from which the result follows. By Lemma 4.4, choose $h_i \in H_i$ such that $\sup_{h\in H_i} d(h,H) = d(h_i,H)$. By 4.5 the sequence $\langle h_i \rangle$ has a convergent subsequence $h_{i_j} \to \bar{h} \in H$. Now let $\varepsilon > 0$. Choose i_k such that $d(h_{i_k},\bar{h}) < \varepsilon$. Then for any $j \geq i_k$,

$$\begin{aligned} \sup_{h\in H_j} d(h,H) &\leq \sup_{h\in H_{i_k}} d(h,H) \\ &= d(h_{i_k},H) \\ &\leq d(h_{i_k},\bar{h}) < \varepsilon. \end{aligned}$$

<u>Definition</u> 4.5. Let $f: X \to Y$. The <u>direct</u> <u>image</u> function is the map $f[\ ]: PX \to PY$ given by $f[K] = \{f(k) \mid k \in K\}$.

<u>Theorem</u> 4.2. Let f be a function from a compact metric space X into a metric space Y. The following are equivalent:

(i) f is continuous;

(ii) $f[\bigcap_i H_i] = \bigcap_i f[H_i]$ for all decreasing sequences $\langle H_i \rangle$ of closed nonempty subsets of X; and f[H] is closed for all closed H;

(iii) $f[\cdot]$ is continuous in the Hausdorff metric.

Proof. We show (i)=>(iii)=>(ii)=>(i).

(i)=>(iii). Let $\varepsilon > 0$. We will find a δ such that

$$\forall H,H': \ d(H,H')<\delta \Rightarrow d(f[H],f[H']) \leq \varepsilon.$$

(i.e. f[ ] is uniformly continuous.) Since X is compact and (i) holds, we know f enjoys this property already. Choose δ such that

$$\forall h,h' (d(h,h')<\delta \Rightarrow d(f(h),f(h')) < \varepsilon).$$

Suppose $d(H,H')<\delta$. Then

$$\sup_{h\varepsilon H} d(h,H')<\delta$$

$$\Rightarrow \forall h\varepsilon H \ d(h,H')<\delta$$

$$\Rightarrow \forall h\varepsilon H[\inf_{h'\varepsilon H'} d(h,h') < \delta]$$

$$\Rightarrow \forall h\varepsilon H \ \exists h'\varepsilon H' (d(h,h') < \delta) \text{ (by 4.4)}$$

$$\Rightarrow \forall h\varepsilon H \ \exists h'\varepsilon H' (d(f(h),f(h')) < \varepsilon \text{ (by unif. cont.)}$$

$$\Rightarrow \forall h\varepsilon H \ \inf_{h'\varepsilon H'} d(f(h),f(h')) < \varepsilon$$

$$= \sup_{h\varepsilon H} d(f(h),f[H']) \leq \varepsilon.$$

Similarly

$$\sup_{h'\varepsilon H'} d(f(h'),f[H]) \leq \varepsilon$$

and (iii) follows.

(iii) => (ii).

Let $\langle H_i \rangle$ be a decreasing sequence of closed subsets of X and let $H=\bigcap_i H_i$. Then by 4.6 $d(H_i,H)\to 0$ as $i\to\infty$. By (iii), $d(f[H_i],f[H])\to 0$. But $f[H_i]$ is again a decreasing sequence of closed sets, so $d(f[H_i],\bigcap_i f[H_i])\to 0$ by 4.6. This proves (ii) because limits are unique.

(ii) => (i). Let $d(t_n,t)\to 0$; prove using (ii) that $d(f(t_n),f(t))\to 0$.

Define for $k\geq 0$

$$S_k=\{u \mid d(t,u)\leq\frac{1}{k+1}\}$$

Then S_k are a decreasing sequence and $\bigcap_k S_k=\{t\}$. By (ii) we have

$$f[\bigcap_k S_k] = f[\{t\}] = \bigcap_k f[S_k].$$

Again by 4.6, since $f[S_k]$ decrease, $d(f[S_k], \bigcap_k f[S_k])\to 0$.

Let $\varepsilon > 0$. Choose K such that

$$d(f[S_k], \bigcap_k f[S_k]) < \varepsilon.$$

Now since $t_n \to t$, there is an N_K such that $\forall n \geq N_K$, $t_n \varepsilon S_K$ and $f(t_n)\varepsilon F[S_K]$.

Thus for $n \geq N_K$,

$$\begin{aligned} d(f(t_n),f(t)) &= d(f(t_n),f[\{t\}]) \\ &= d(f(t_n), \bigcap_k f[S_k]) \text{ by (ii)} \\ &\leq d(f[S_K], \bigcap_k f[S_k]) \leq \varepsilon. \end{aligned}$$

This completes the proof of Theorem 4.2. (Note: (i) => (ii) is well-known; see [K], p. 414.)

In order to apply the results of Theorem 4.2 we present several tree operators. These can be used to define the semantics for appropriate combinations of processes, as in [BaZ]. We call on the lemmas of [BaZ], in fact, to establish that certain operators are metrically continuous. Theorem 4.2 then applies to show that the extended versions are sup-continuous. We give the versions for finite trees first.

Definition 4.7.1 (alternative choice.)

Let s and t be trees. Denote by s+t the result of joining s and t at the root.

4.7.2 Δ-Synchronized shuffle.

Suppose $\Delta \subseteq \Sigma$. We want a tree operation which matches two trees (glües them together) at points Δ and otherwise interleaves events outside Δ. The appropriate definition is inductive.

Define NIL $\underset{\Delta}{||}$ s = s $\underset{\Delta}{||}$ NIL = s;

$$(\sum a_i s_i \underset{\Delta}{||} \sum b_j t_j) = \sum_{a_i = b_j \varepsilon \Delta} a_i (s_i \underset{\Delta}{||} t_j)$$

$$+ \sum_{a_i \notin \Delta} a_i (s_i \underset{\Delta}{||} \sum b_j t_j)$$

$$+ \sum_{b_j \notin \Delta} b_j (\sum a_i s_i \underset{\Delta}{||} t_j).$$

4.7.3. Sequential Composition

We would like to model two kinds of stopped processes: one which can continue (successful termination) and one which cannot (failed termination). In order to do this we introduce a new sort of nullary tree besides NIL, which we call END. Technically speaking, we should like W-equivalence to distinguish these two trees, and the HML formulas to distinguish them as well. We declare NIL and END to be W_0-inequivalent, and we introduce elementary propositional variables FAILED and DONE into HML, with the definitions

$$t \models \text{FAILED} \iff t = \text{NIL}.$$

$$t \models \text{DONE} \iff t = \text{END}.$$

Then one can check that the lemmas of Section II still hold. The base clause of Definition 4.7.2 should be amended to read END||s = s||END = s. Also in 4.7.1 we put NIL+END = END. We proceed with the definition of $t \circ u$ for finite t,u.

$$\text{NIL} \circ t = \text{NIL};$$

$$\text{END} \circ t = t;$$

$$(\Sigma a_i s_i) \circ t = \Sigma a_i (s_i \circ t).$$

4.7.4. Renaming

We wish to rename events so, for example, they are removed from a communication alphabet. This is done by considering a function $h: \Sigma \to \Sigma$.

Define

$h(End)=End;$

$h(Nil)=Nil;$

$h(\Sigma a_i t_i)=\Sigma h(a_i)h(t_i).$

Now we extend the above definitions to infinite trees. There is no problem with (1). For the others, we take Cauchy limits as in [BaZ].

4.8.2 $s||t = \lim_{k\to\infty} s^{(k)} \underset{\Delta}{||} t^{(k)}$

where $s^{(k)}$ is the k-section of s (Definition 2.9).

4.8.3 $s\circ t = \lim_{k\to\infty} s^{(k)} \circ t$

4.8.4 $h(t) = \lim_{k\to\infty} h(t^{(k)})$.

We need to check that the above limits exist in the metric sense.

Lemma 4.7.

Let s,s',t,t' be finite trees and $M = \max(d(s,s'),d(t,t'))$. Then

(1) $d(s+t, s'+t') \leq M$;

(2) $d(s \underset{\Delta}{||} t, s' \underset{\Delta}{||} t') \leq M$,

(3) $d(s\circ t, s'\circ t') \leq M$;

and

(4) $d(h(s), h(s')) \leq d(s,s')$.

Proof. deBakker and Zucker show (1) and (3). The also prove (2) when $\Delta=\emptyset$. We include a proof for arbitrary $\Delta \subsetneq \Sigma$. We assume all trees are non nullary, as these cases are all easy.

Let $e(s,s') = \max\{k \mid s\ W_k\ s'\}$. We will show (dropping the Δ subscript)

$$e_W(s||t,s'||t') \geq \min(e_W(s,s'),e_W(t,t')).$$

Now let $s = \Sigma a_i s_i$

$s' = \Sigma c_i s_i'$

$t = \Sigma b_j t_j$

$t' = \Sigma d_j t_j'$.

Then $s \underset{\Delta}{||} t = u_1+u_2+u_3$ where

$$u_1 = \sum_{a_i=b_j\varepsilon\Delta} a_i(s_i \; || \; t_j)$$

$$u_2 = \sum_{a_i\not\varepsilon\Delta} a_i(s_i \; || \; t)$$

$$u_3 = \sum_{b_j\not\varepsilon\Delta} b_j(s \; || \; t_j)$$

Similar equations obtain for $s' \; || \; t' = u_1'+u_2'+u_3'$.

Let $e_w(s,s')=p$ and suppose (w.l.o.g.) $p = \min(e_w(s,s'),e_w(t,t'))$.

We prove $e_w(u_i,u_i') \geq p$ for i=1,2,3. Then by the result for +,

$$e_w(u_1+u_2+u_3,\; u_1+u_2+u_3) \geq \min_i\{e_w(u_i,u_i')\} \geq p.$$

We proceed inductively on the maximum height of the trees s,s',t,t'. The height zero cases are straightfoward.

Estimating $e_w(u_1,u_1')$: Let $u_1 \xrightarrow{a_i} s_i \; || \; t_j$. Then $s \xrightarrow{a_i} s_i$, and $t \xrightarrow{a_i} t_j$ for some $a_i\varepsilon\Delta$. Now $p=e_w(s,s')$. If p=0 there is nothing to show. So we may assume $(\exists_i)$ $s' \xrightarrow{a_i} s_i'$ with $e_w(s_i',s_i) = p-1$.

Since $a_i\varepsilon\Delta$, and $e_w(t,t') \geq p$, we know $(\exists j)$ $t' \xrightarrow{a_i} t'$ with $e_w(t_j,t_j') \geq p-1$. Now $u_1' \xrightarrow{a_i} s_i' \; || \; t_j'$ and by induction hypothesis,

$$e_w(s_i' \; || \; t_j',\; s_i \; || \; t_j) \geq \min(e_w(s_i \; || \; s_i'),\; e_w(t_j \; || \; t_j')) \geq p-1.$$

Similarly if

$u_1' \xrightarrow{a_i} s_i' \; || \; t_j'$ we deduce

that there are s_i and t_j such that

$u_1 \xrightarrow{a_i} s_i \; || \; t_j$

and $e_w(s_i' \; || \; t_j',\; s_i \; || \; t_j) \geq p-1$.

Therefore by definition of W, $e_w(u_1,u_1') \geq p$.

We need only consider $e_w(u_2,u_2')$ as the proof for u_3,u_3' is the same.

Let $u_2 \xrightarrow{a_i} s_i \,||\, t$, where $a_i \notin \Delta$. Then as before, $s \xrightarrow{a_i} s_i$, and $s' \xrightarrow{a_i} s_i'$ for some i, where

$$e_w(s_i,s_i') = p-1.$$

Now in this case $u_2' = \Sigma a_i(s_i' \,||\, t')$, and $u_2' \xrightarrow{a_i} s_i' \,||\, t'$. Similarly if $u_2' \xrightarrow{a_i} s_i' \,||\, t'$ we have $u_2 \xrightarrow{a_i} s_i \,||\, t$ for some i. By I.H.

$$e_w(s_i \,||\, t,\ s_i' \,||\, t') \geq p-1.$$

Therefore $e_w(u_2,u_2') \geq p$, as desired.

The proof of (2) is now complete. We leave (4) as an easy exercise, and now the lemma is finished.

<u>Corollary</u>. The sequences used in Definition 4.8 are convergent (to an equivalence class in the pseudo-metric).

<u>Lemma</u> 4.8. The inequalities of 4.7 hold for infinite trees.

<u>Proof</u>. We show only (2) as the proof works the same way for the other cases. Consider first a special case: $d_w(s,s')=0$. Then we must show $d_w(s \,||\, t,\ s' \,||\, t') \leq d_w(t,t')$. (Incidentally, this shows that W is a congruence with respect to ||.)

We know for each k : $d(s^{(k)} \,||\, t^{(k)}, s'^{(k)} \,||\, t'^{(k)}) \leq d(t^{(k)},t'^{(k)})$ because $s^{(k)}\ W\ s'^{(k)}$ for each k. As $k \to \infty$ $d(t^{(k)},t'^{(k)}) \to d(t,t')$, and the LHS $\to d(s \,||\, t,\ s' \,||\, t')$. Therefore the inequality holds.

We can now assume $s \neg W s'$, and $t \neg W t'$. Let $\varepsilon > 0$. Choose k so large that $d(s^k,s'^{(k)}) = d(s,s')$, $d(t^{(k)},t'^{(k)}) = d(t,t')$, $d(s^k \,||\, t) \leq \varepsilon/2$, and $d(s'^k \,||\, t'^k,\ s' \,||\, t') \leq \varepsilon/2$. Then by the triangle inequality

$$d(s||\,t,\ s'||\,t') \leq d(s||\,t,\ s^k||\,t^k) + d(s'^k||\,t'^k,\ s'||\,t') + d(s^k||\,t^k,\ s'^k||\,t'^k)$$

$$\leq \varepsilon/2 + \varepsilon/2 + \max d(s^k,s'^k),d(t^k,t'^k)) = \varepsilon + \max(d(s,s'),\ d(t,t'))$$

But ε was arbitrary so the result holds.

Corollary. The operations of +, $\underset{\Delta}{||}$, $\circ$, and h are continuous (jointly).

Corollary. The same operations are sup-continuous on forests and Hausdorff-continuous, when we pass to direct images.

Unfortunately, the sequential composition operator is not quite the right one to lift to forests. That is,

$$G\circ H = \{t\circ u \mid t\varepsilon G,\ u\varepsilon H\}$$

requires that the same tree u be 'grafted' wherever END occurs in t. We want a freer choice than this, which will allow substitution of any tree from H for END nodes in t.

We give a special definition for the operator $G\circ H$. We start with

Definition 4.9. Let G be a forest and t be a finite tree. The forest $t\circ G$ is given recursively:

$NIL\circ G = \{NIL\}$;

$END\circ G = G$;

$(\Sigma a_i t_i)\circ G = \{\Sigma a_i u_i \mid u_i \varepsilon t_i \circ G\}$.

It is easy to check that $t\circ G$ is closed for each t. Now let

$$t\circ G = \lim_{k\to\infty} t^{(k)}\circ G$$

for infinite t. Again we need to check that this limit exists.

Lemma 4.9. For finite t,u: $t\ W_k\ u \Rightarrow t\circ G\ W_k\ u\circ G$

where W_k on the right is from Definition 4.3.

Proof.

We use induction on k. By Definition 4.3 we need to check that if $t\ W_{k+1}\ u$, then for all $w\ \varepsilon\ t\circ G$ there is a $z\ \varepsilon\ u\circ G$ such that $w\ W_{k+1}\ z$. Let $w=\Sigma a_i w_i\ \varepsilon\ t\circ G$, and $w \xrightarrow{a_i} w_i$. By definition $w_i\ \varepsilon\ t_i\circ G$ where $t = \Sigma a_i t_i$. Therefore $t \xrightarrow{a_i} t_i$ and since $t\ W_{k+1}\ u$, we have $u \xrightarrow{a_i} u_i$ for some $u_i\ W_k\ t_i$. If $w = \Sigma a_i w_i$ let $z = \Sigma a_i z_i$ for each such a_i, where $z_i \in u_i \circ G$. By IH, and the fact that $u_i\ W_k\ t_i$, we have $z_i\ W_k\ w_i$. Then clearly $z\ W_{k+1}\ w$ as desired. The reverse inclusion is similar.

Corollary. $d(t \circ G, u \circ G) \leq d(t,u)$ where the Hausdorff metric is used on the left, and t,u are finite.

Proof. Apply Theorem 4.1 and the definitions at the beginning of this section.

Corollary. The limit $t^{(k)} \circ G$ exists as $k \to \infty$.

Definition 4.10. Let $G, H \in P_c(T_\Sigma)$.

$$H \circ G = \bigcup_{t \in H} t \circ G$$

Our objective is to prove that $\lambda GH(H \circ G)$ is sup-continuous in G and in H. This will be established via a series of independently interesting lemmas. We begin with

Lemma 4.10. The map $\lambda t.t \circ G$ is metrically continuous in t.

Proof. By the triangle inequality, for any $k \geq 0$,

$$\begin{aligned} d(t \circ G, u \circ G) &\leq d(t \circ G, t^{(k)} \circ G) \\ &+ d(t^{(k)} \circ G, u^{(k)} G) \\ &+ d(u^{(k)} \circ G, u \circ G) \end{aligned}$$

Let $\varepsilon > 0$. Choose k large enough so that

(1) $d(t \circ G, t^{(k)} \circ G) < \varepsilon/3$

(2) $d(t^{(k)} \circ G, u^{(k)} \circ G) \leq d(t^{(k)}, u^{(k)}) \leq d(t,u) + \varepsilon/3$

(3) $d(u^{(k)} \circ G, u \circ G) \leq \varepsilon/3$

Then $d(t \circ G, u \circ G) \leq d(t,u) + \varepsilon \Rightarrow d(t \circ G, u \circ G) \leq d(t,u)$. The result follows directly.

Now we would like to establish

Lemma 4.11. The map $\lambda G.t \circ G$ is sup-continuous in G.

We begin with 4.11 for finite t. We must prove

$$(*)\quad t\circ\bigcap_i G_i = \bigcap_i (t\circ G_i).$$

This is clear if t = end or t = nil. Also, the left hand side is clearly included in the right. So let $t = \Sigma a_j t_j$, and let w be any tree in the right hand set. Then $w = \sum_j a_j w_j$ where for each i, $w_j \in t_j\circ G_i$, or $w_j \in \bigcap_i t_j\circ G_i$. By induction, $w_j \in t_j\circ\bigcap_i G_i$. Thus $w \in t\circ\bigcap_i G_i$.

We would like to extend (*) to every t. This can be done by direct calculation using the definitions, but the proof can also be given using the theorems and lemmas of previous sections.

<u>Lemma</u> 4.12. For finite t, arbitrary closed G and $n \geq 0$

$$(t\circ G)^{(n)} = (t^{(n)}\circ G)^{(n)}$$

<u>Proof</u>. This will be established by induction on n. First we give an inductive definition of the n-section of a finite tree t.

Case (1) n=0.

$(\text{end})^0=\text{end}$; $(\text{nil})^0=\text{nil}$; $(\Sigma a_i t_i)^0=\text{nil}$.

Case (2) n>0.

$(\text{end})^n=\text{end}$; $(\text{nil})^n=\text{nil}$; $(\Sigma a_i t_i)^n=\Sigma a_i t_i^{n-1}$.

When n=0, the lemma is clear. Assume it for all values < n, and consider the case n.

$(\text{end}\circ G)^n=G^n=(\text{end}^n\circ G)^n$

$(\text{nil}\circ G)^n=\{\text{nil}\}=(\text{nil}^n\circ G)^n$

$$
\begin{aligned}
((\Sigma a_i t_i)\circ G)^n &= \{(\Sigma a_i u_i)^n \mid u_i \varepsilon t_i\circ G\}\\
&= \{\Sigma a_i u_i^{n-1} \mid u_i \varepsilon t_i\circ G\}\\
&= \{\Sigma a_i u_i^{n-1} \mid u_i \varepsilon t_i^{n-1}\circ G\} \text{ by I.H.}\\
&= \{(\Sigma a_i u_i)^n \mid u_i \varepsilon t_i^{n-1}\circ G\}\\
&= \{\Sigma a_i u_i \mid u_i \varepsilon t_i^{n-1}\circ G\}^n\\
&= \{(\Sigma a_i t_i^{n-1})\circ G\}^n\\
&= ((\Sigma a_i t_i)^n\circ G)^n \text{ as desired.}
\end{aligned}
$$

Now to establish 4.12 for infinite t, we state a well-known property of metrically continuous functions.

Lemma 4.13. Let f and g be continuous functions from a metric space X to a metric space Y. If f and g agree on a dense subset of X then f and g coincide.

As a corollary, we get

Lemma 4.14. The conclusion of 4.12 holds for infinite t.

Proof. The map $t \to t \circ G$ is metrically continuous. (Lemma 4.10) The map $F \to F^{(n)}$, where F is a closed set and $F^{(n)} = \{t^{(n)} \mid t \in F\}$ is also metrically continuous (proof: $\lambda t.t^{(n)}$ is obviously m.c., so $\lambda F.F^{(n)}$ is both m.c. and sup.c. by Theorem 4.2.). Composing these, we get a m.c. map $t \to (t \circ G)^{(n)}$. Similarly $t \to (t^{(n)} \circ G)^{(n)}$ is m.c., and by 4.12 these maps agree on the dense subset of finite trees.

Now we can prove the conclusion (*) of 4.11 for arbitrary t. Let G_i be a decreasing sequence of nonempty closed sets. We claim for each i

$$d(t \circ G_i, t \circ \bigcap_i G_i) \leq d(t \circ G_i, \bigcap_i t \circ G_i).$$

We know that the sequence $\langle t \circ G_i \rangle$ is also closed and decreasing. By Lemma 4.6, $d(t \circ G_i) \to 0$ as $i \to \infty$. But we also have, by the inequality above, $d(t \circ G_i, t \circ \bigcap G_i) \to 0$. Therefore, by uniqueness of limits, $t \circ \bigcap G_i = \bigcap_i t \circ G_i$.

Proof of claim. For any $k \geq 0$

$$d(t \circ G_i, t \circ \bigcap G_i) \leq d((t \circ G_i)^k, (t \circ \bigcap G_i)^k)$$

by properties of W-equivalence on forests. Let $F_i = t \circ G_i$.

Then $d(F_i^k, (t \circ \bigcap G_i)^k)$

$= d(F_i^k, (t^k \circ \bigcap G_i)^k)$ by 4.14

$= d(F_i^k, (\bigcap_i t^k \circ G_i)^k)$ by 4.11 for finite trees

$= d(F_i^k, \bigcap_i (t^k \circ G_i)^k)$ by sup-ctn. of $\lambda F.F^{(k)}$

$= d(F_i^k, \bigcap_i (t \circ G_i)^k)$ by 4.14.

Now as $k\to\infty$, $F^k\to F$, and d is continuous, so for any ε, we can choose a k such that

$$d((F_i)^k,(\bigcap_i t\circ G_i)^k) \le d(F_i,\bigcap_i t\circ G_i)+\varepsilon.$$

$$\text{Thus } d(F_i,t\circ \bigcap_i G_i) \le d(F_i,\bigcap_i t\circ G_i)+\varepsilon.$$

The inequality follows since ε was arbitrary. Lemma 4.11 is thus proved for all cases. We are almost ready for

Theorem 4.3. The map $\lambda GH.H\circ G$ is sup-continuous.

$$\text{Recall } H\circ G = \bigcup_{t\varepsilon H} t\circ G.$$

Theorem 4.3 will be a consequence of

Lemma 4.15. Let f be a metrically continuous function from the compact space X to $P_c(Y)$ (with the Hausdorff metric). Define for closed H:

$$\hat{f}[H] = \bigcup_{x\varepsilon H} f(x).$$

Then $\hat{f}$: $P_C(X)\to P_C(Y)$ is sup-continuous.

Proof. We leave to the reader the proof that $\hat{f}[H]$ is closed. We then must show

$$\hat{f}[\bigcap H_i] = \bigcap_i \hat{f}[H_i]$$

The inclusion of the left side in the right is obvious. Let $y \varepsilon \bigcap_i \hat{f}[H_i]$. Then for each i, $\exists x_i \varepsilon H_i$ such that $f(x_i)=y$. The x_i have a convergent subsequence $x_{i_k}\to\overline{x}$, and $\overline{x} \varepsilon \bigcap_i H_i$ by Lemma 4.5. We want $y\varepsilon f(\overline{x})$. But $d(f(x_{i_k}), f(\overline{x}))\to 0$ as $k\to\infty$, and $d(y,f(\overline{x})) \le d(f(x_{i_k}), f(\overline{x}))$ for each k. So $d(y,f(\overline{x}))=0$. Thus $y \varepsilon f(\overline{x})$ since $f(\overline{x})$ is closed.

As a corollary:

Theorem 4.3. λGH. $H\circ G$ is sup-continuous in both arguments.

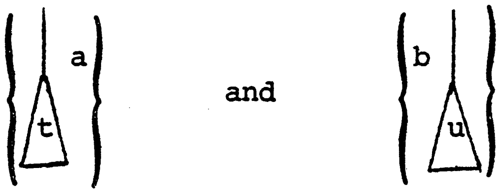
The final and most trivial operation on closed sets is that of union:

$$\lambda FG.(F\bigcup G).$$

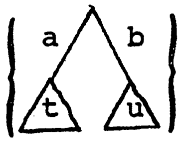
Clearly this is a sup-continuous operation by the distributive law.

V. Applications and conclusions.

The operators of Section IV can be used to give a denotational semantics to a CSP-like language, with two versions of nondeterminism. The operator □ of Dijkstra will be interpreted as + on forests. When applied to singleton forests of the form

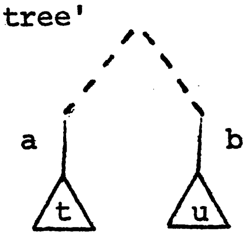


it produces

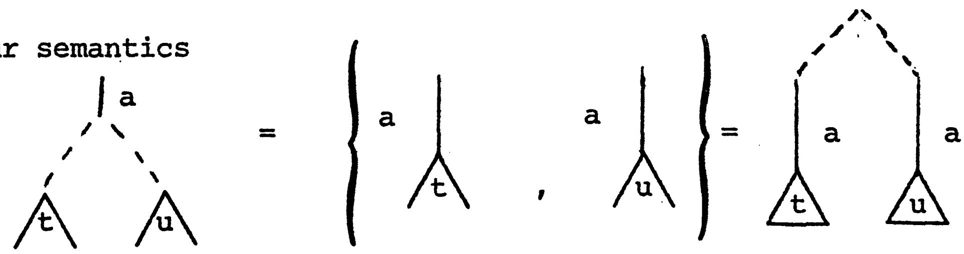


This is exactly the guarded command construct from Dijkstra. On the other hand, the operator ⊓ will be interpreted as union of forests. When applied to the above two singletons, it produces

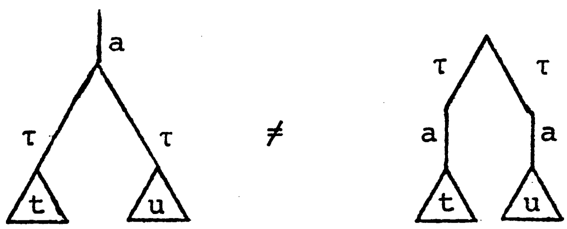
which can be thought of as a 'tree'



In a sense, the ⊓ operator is introducing a hidden choice as in [HBR]. It should not be identified with Milner's use of silent transitions, however. For example in our semantics



but for Milner



where τ is the silent transition.

Consider the CSP-like language L:

<S> := <stmtvariable>$|$skip$|$fail$|a\to S|S_1;S_2|S_1 \Box S_2|S_1 \sqcap S_2|S_1 \|_{\Delta} S_2|h(S)|\mu xS$

<stmtvar> := $x|y|z$ ...

where in the construct $a\to S$ we let $a\in\Sigma$ range over atomic events, and in $h(S)$ we specify a renaming $h:\Sigma\to\Sigma$. S stands for a statement expression with possible free occurrences of statement variables x,y,z.. . We interpret L in the space of maps $[Env \to P_c(T_\Sigma)]$, where

$$Env = [Stmvar \to P_c(T_\Sigma)]$$

is the set of environments = assignments of 'processes' to free statement variables. Let the variable ρ range over environments.

We have the semantic map $M:L\to[Env\to P_c(T_\Sigma)]$

$M [\![x]\!]\rho = \rho(x)$ for x a stmtvar

$M [\![\text{skip}]\!]\rho = \{end\}$

$M [\![\text{fail}]\!]\rho = \{nil\}$

$M [\![a\to S]\!]\rho = (a\cdot end)\circ[M [\![S]\!]\rho]$

$M [\![S_1;S_2]\!]\rho = M [\![S_1]\!]\rho \circ M [\![S_2]\!]\rho$

$M [\![S_1 \Box S_2]\!]\rho = M [\![S_1]\!]\rho + M [\![S_2]\!]\rho$

$M [\![S_1 \sqcap S_2]\!]\rho = M [\![S_1]\!]\rho \cup M [\![S_2]\!]\rho$

$M [\![S_1 \|_{\Delta} S_2]\!]\rho = M [\![S_1]\!] \|_{\Delta} M [\![S_2]\!]\rho$

$M [\![h(S)]\!]\rho = h[M [\![S]\!]\rho]$

$M [\![\mu xS]\!]$ = least fixpoint of ϕ

where ϕ is the map $\lambda K.M [\![S]\!]\rho(K/x)$ and $\rho(K/x)$ is the same as ρ except x is assigned the forest K.

Example: Consider

$$\mu x(a\to x \Box b\to x)$$

This can be given by the familiar fixpoint formula

$$\bigcap_{n \geq 0} \varphi^{n}(\bot)$$

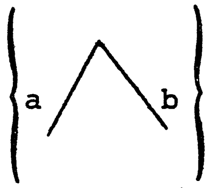
where $\bot = T_{\Sigma}$, and

$$\varphi^{0}(\bot) = \bot$$

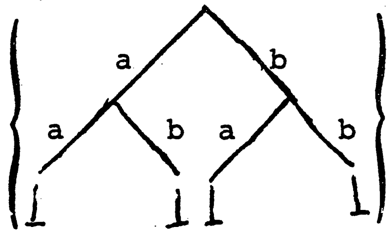
$$\varphi^{n+1}(\bot) = a \circ \varphi^{n}(\bot) + b \circ \varphi^{n}(\bot)$$

Pictorially $\varphi_0(\bot) = \bot$

$\varphi_1(\bot) =$



$\varphi_2(\bot) =$



etc.

Thus the least fixpoint is the singleton set consisting of the full infinite binary tree over $\{a,b\}$.

Example. Two more syntactic constructs can be introduced into L by definition. Let S be a statement expression not containing a free occurrence of the statement variable x.

Then

$$S^{\omega} =_{def} \mu x(s;x)$$

$$S^{*} =_{def} \mu x((s;x) \sqcap \underline{skip}).$$

In conclusion, we have shown how the information system approach leads to a natural cpo for the semantics of concurrency. This cpo is closely related to the metric spaces introduced in [BaZ], and even more closely to the structures explored in [BaBKM], where the cpo of closed languages (sets of strings) is used as a linear time semantic structure. The work of [BK] on projective limits gives another approach, although in this work no use is made of cpo methods.

It should be remarked that this theory is also connected closely to the work of Courcelle [C]. The contrast here is that we work with unordered trees and trees with infinite branching. However the metric space (T_Σ, d) can probably be obtained as a quotient of the free F-magma studied in [C]. Also in this paper compactness plays a key role for the definition of operators on our space. The space $P_c(T_\Sigma)$ seems a natural candidate for further study; one interesting problem is to relax the hypothesis of finite Σ and still obtain sup-continuity properties.

Acknowledgments

Most of this work grew from discussions with W. Golson and S. Brookes. Thanks also to the members of the Theory Seminar at Ann Arbor: A. Blass, P. Hinman, and Y. Gurevich.

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