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A RECTIFICATION TRANSFORMATION FOR TILTED PHOTOGRAPHS

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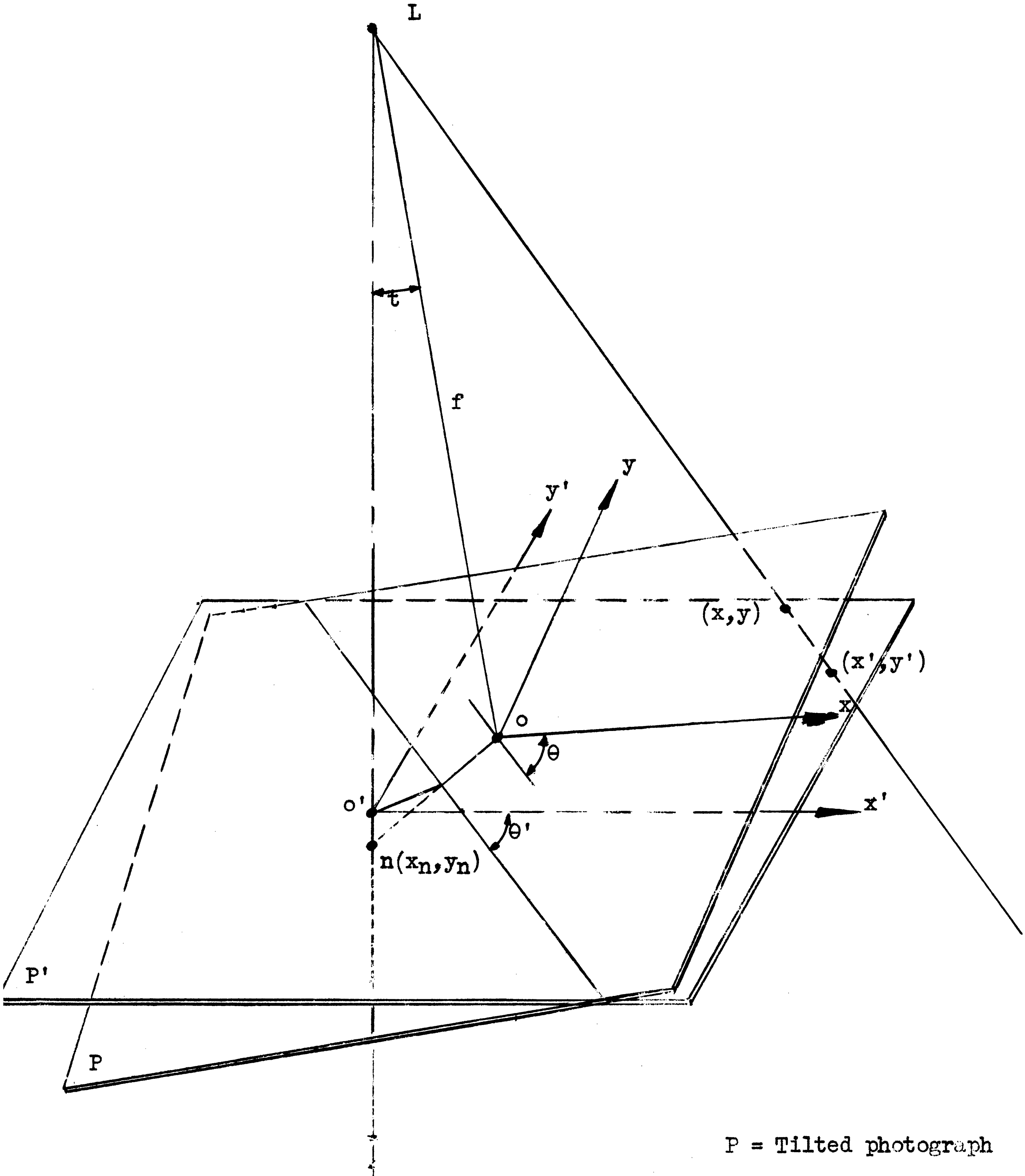
I. INTRODUCTION

The problem of the rectification of tilted aerial photographs is one of fundamental importance in photogrammetry. Practical solutions to this problem for the rapid and efficient construction of maps have been made, utilizing slotted templates, projectors, and various other mechanical and mechanical-graphical systems. A more accurate solution to the rectification problem is required for any precise rectification, which might be desirable in particular special applications or theoretical investigations. Derivations for the so-called tilt displacements and procedures for the calculation of image point coordinates on the equivalent vertical photographs are familiar to the photogrammetrist. However, these procedures are usually quite similar and require several rotations and translations of coordinates in their performance, with the result that the numerical calculations are rather complicated and lengthy and thus are very liable to mistakes by the computer. It was felt therefore that a definite need existed for a direct transformation which could be given by a single relation between the coordinates of the tilted plane, the coordinates on the equivalent vertical photograph, and the elements of exterior orientation. Such a transformation has been derived and is presented in this report.

II. BASIC PROPERTIES

The basic geometrical properties of the definition of an equivalent vertical photograph should be clearly in mind. This geometry is sketched in Fig. 1. The definition of the equivalent vertical photograph is such that the angles Θ and Θ' are equivalent. They are the angle between the direction of the axis of tilt and the x- and x'-axis, respectively. This condition, together with a knowledge of the exterior orientation of the tilted photograph (in this case the coordinates of the nadir point, x_n and y_n, as determined in the tilted plane), makes possible a derivation by the methods of solid analytic geometry and vector analysis. This derivation is given in the appendix, and the resulting rectification transformation is given below.



P = Tilted photograph

P' = Equivalent vertical photograph

Fig. 1
Geometry of Equivalent Vertical Photograph

III. THE TRANSFORMATION

The coordinates x' , y' of a point in the equivalent vertical photograph, corresponding to a point x, y in the tilted plane are given by the following relations,

$$x' = \frac{f\left\{(xy_n^2 - yx_ny_n)\sqrt{x_n^2 + y_n^2 + f^2} + f\,x_n\left[x_nx + y_ny - (x_n^2 + y_n^2)\right]\right\}}{(x_n^2 + y_n^2)(x_nx + y_ny + f^2)},$$

$$y' = \frac{f\left\{(yx_n^2 - xx_ny_n)\sqrt{x_n^2 + y_n^2 + f^2} + f\,y_n\left[x_nx + y_ny - (x_n^2 + y_n^2)\right]\right\}}{(x_n^2 + y_n^2)(x_nx + y_ny + f^2)}, \qquad (1)$$

where

f = the camera focal length
x, y = the coordinates of any point in the tilted plane
x', y' = the coordinates of the corresponding point in the vertical photograph
x_n, y_n = the coordinates of the nadir point in the tilted plane

IV. DISCUSSION

The rectification transformation presented in Equation (1) affords a very simple means for the determination of the positions of points in the equivalent vertical photograph. The computation is straight forward and is readily adapted to machine calculation. The form of the transformation presented in Equation (1) is not necessarily the most convenient for computation, but was chosen for the compactness of presentation.

It is of interest to note that the transformation is symmetric, that is, if,

$$x' = F(x,y,x_n,y_n,f) \;, \qquad y' = G(x,y,x_n,y_n,f) \;, \qquad (2)$$

then the reverse transformation of points from the equivalent vertical photograph to the tilted plane will be,

$$x = F(x',y',x_n',y_n',f) \;, \qquad y = G(x',y',x_n',y_n',f) \;. \qquad (3)$$

It is also clearly seen that when x = y = 0 in Equation (1),

$$x' = x'_n = -x_n ,$$
$$y' = y'_n = -y_n . \qquad (4)$$

It is further noted that the transformation in Equation (1)may also be written in the form

$$x' = \frac{x\left[\frac{f\,y_n^2\sqrt{x_n^2+y_n^2+f^2} + f^2\,x_n^2}{x_n^2+y_n^2}\right] + y\left[\frac{f^2\,x_n y_n - f\,x_n y_n\sqrt{x_n^2+y_n^2+f^2}}{x_n^2+y_n^2}\right] - f^2\,x_n}{x_n x + y_n y + f^2} , \qquad (5)$$

$$y' = \frac{x\left[\frac{f^2\,x_n y_n - f\,x_n y_n\sqrt{x_n^2+y_n^2+f^2}}{x_n^2+y_n^2}\right] + y\left[\frac{f\,x_n^2\sqrt{x_n^2+y_n^2+f^2} + f^2\,y_n^2}{x_n^2+y_n^2}\right] - f^2 y_n}{x_n x + y_n y + f^2} ,$$

which is a linear fractional or bilinear transformation. Such transformations are characteristic of colinear projections of the type represented by the rectification problem. Studies of the properties of the tilt transformation are thus simplified because the properties of the linear fractional transformations have been widely investigated by mathematicians. Further investigations of the transformation will be of considerable interest, e.g., expansion for very small tilts.

In conclusion, the tilt transformation that has been presented here will greatly facilitate theoretical investigations by this project. .It should also be a definite contribution to theoretical photogrammetry, since it affords for the first time (to the author's knowledge) a simple, direct, and accurate method for the rectification of a tilted photograph after the orientation data have been determined.

APPENDIX

DERIVATION OF RECTIFICATION TRANSFORMATION

To simplify the following mathematical treatment a system of reduced coordinates will be introduced. These reduced coordinates are defined such that for a point x,y in the tilted plane the reduced coordinates of the point will be,

$$\xi = \frac{x}{f}, \qquad \eta = \frac{y}{f}, \tag{6}$$

where f is the camera focal length. The quantities $\xi_n = \frac{x_n}{f}$ and $\eta_n = \frac{y_n}{f}$ follow directly from this definition, as do the coordinates of the point corresponding to ξ, η in the vertical plane, namely $\xi' = \frac{x'}{f}$, $\eta' = \frac{y'}{f}$. With the introduction of the reduced coordinates, the tilted plane P and the equivalent vertical photograph P' may be represented as planes tangent to a unit sphere S at points O and O', respectively, as shown in Fig. 2. The equation for the vertical line in this system is given by

$$\frac{x}{\xi_n} = \frac{y}{\eta_n} = z, \tag{7}$$

and it pierces the tilted plane P, z = 1, in the nadir point $n(\xi_n, \eta_n, 1)$. The axis of tilt is the line aib, with i the isocenter. The line a' o' b' is a line parallel to the tilt axis aib and in the vertical plane P'. It is clear from Fig. 2 that if the distance d' = o' p' is known, then the required coordinates of $p'(\xi', \eta')$ in the vertical plane will be

$$\xi' = d' \sin \phi, \qquad \eta' = d' \cos \phi, \tag{8}$$

where $\phi = \mu - \theta$. Knowing the points $a\left(\frac{\sec t - 1}{\xi_n}, 0, 1\right)$, $b\left(0, \frac{\sec t - 1}{\eta_n}, 1\right)$, and the piercing points $O'(\xi_n \cos t, \eta_n \cos t, \cos t)$ and $p'\left(\frac{\xi \sec t}{\xi_n \xi + \eta_n \eta + 1}, \frac{\eta \sec t}{\xi_n \xi + \eta_n \eta + 1}, \frac{\sec t}{\xi_n \xi + \eta_n \eta + 1}\right)$, the angles μ and θ may be found by analytic geometry,

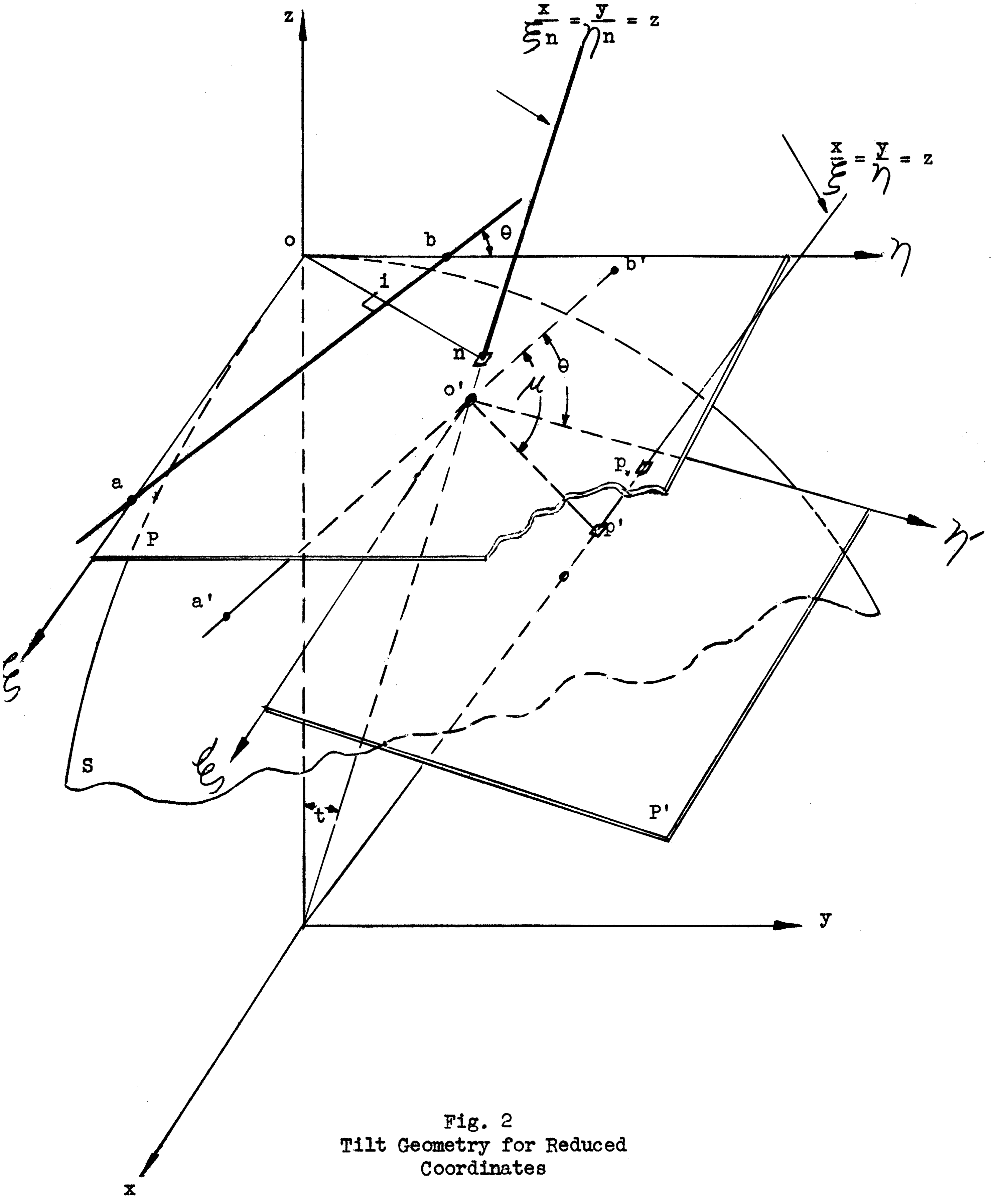


Fig. 2
Tilt Geometry for Reduced
Coordinates

$$\cos\theta = \frac{\xi_n}{\tan t}, \qquad \sin\theta = \frac{\eta_n}{\tan t}. \tag{9}$$

$$\cos\mu = \frac{(\eta\xi_n - \xi\eta_n)\sec t}{d'\tan t\,(\xi_n\xi + \eta_n\eta - 1)},$$

$$\sin\mu = \frac{\sqrt{d'^2\tan^2 t\,(\xi_n\xi + \eta_n\eta + 1)^2 - (\eta\xi_n - \xi\eta_n)^2\sec^2 t}}{d'\tan t\,(\xi_n\xi + \eta_n\eta + 1)}. \tag{10}$$

The magnitude of the distance d' may also be found and is given by,

$$d' = \sqrt{\frac{(\xi^2 + \eta^2 + 1)\sec^2 t}{(\xi_n\xi + \eta_n\eta + 1)^2} - 1}. \tag{11}$$

Applying the formulae for the sine and cosine of the differences of two angles, relations (9), (10), and (11) are substituted into relation (8), which yields after considerable algebraic reduction the required transformation,

$$\xi' = \frac{\sqrt{\xi_n^2 + \eta_n^2 + 1}\,(\eta\xi_n^2 - \eta\eta_n\xi_n) + \xi_n\left[\xi_n\xi + \eta_n\eta - (\xi_n^2 + \eta_n^2)\right]}{(\xi_n^2 + \eta_n^2)(\xi_n\xi + \eta_n\eta + 1)},$$

$$\eta' = \frac{\sqrt{\xi_n^2 + \eta_n^2 + 1}\,(\eta\xi_n^2 - \xi\xi_n\eta_n) + \eta_n\left[\xi_n\xi + \eta_n\eta - (\xi_n^2 + \eta_n^2)\right]}{(\xi_n^2 + \eta_n^2)(\xi_n\xi + \eta_n\eta + 1)}, \tag{12}$$

where $\tan^2 t$ and $\sec t$ have been replaced by their equivalent values, $\xi_n^2 + \eta_n^2$ and $\sqrt{\xi_n^2 + \eta_n^2 + 1}$, respectively. The change from reduced coordinates to plate coordinates to obtain Equation (1) is obvious.