THE UNIVERSITY OF MICHIGAN RESEARCH INSTITUTE
ANN ARBOR

SYNTHESIS OF R-L-C NETWORKS BY DISCRETE TSCHEBYSCHEFF APPROXIMATIONS IN THE TIME DOMAIN

Technical Report No. 107

Electronic Defense Group
Department of Electrical Engineering

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Project 2899

TASK ORDER NO. EDG-1
CONTRACT NO. DA-36-039 sc-78283
SIGNAL CORPS, DEPARTMENT OF THE ARMY
DEPARTMENT OF ARMY PROJECT NO. 3-99-04-106

Submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy in The University of Michigan

April 1960

ACKNOWLEDGEMENTS

The author wishes to express his sincere appreciation to the Chairman of his Doctoral Committee, Professor C. B. Sharpe for his advice and counsel during this investigation. He is also grateful to the other members of the Doctoral Committee for their helpful comments concerning the material. He feels especially indebted to Professor N. R. Scott for his guidance and suggestions.

Further, the author wishes to acknowledge his gratitude to Professor A. B. Macnee, whose lectures on Network Synthesis aroused the author's interest in the subject. Thanks are also expressed to Dr. B. F. Barton for his assistance.

Finally, the author wishes to thank Mr. R. E. Graham and Mrs. P. E. Stohrer for their preparation of this paper for publication. The author's wife typed the complete rough draft of the manuscript and offered many valuable suggestions. For this, and for her continued encouragement the author is deeply grateful.

A part of the research reported in this paper was supported by the U. S. Army Signal Corps.

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LIST OF SYMBOLS

Symbol	Description	First Used on Page
A_k	Residue of H*(s) at the pole $s=s_k$	7
B_o	Constant term in Eq. (A2)	137
B_k	Coefficient of $e^{s_k t}$ in Eq. (A2)	137
C_k	Coefficient of $e^{jk\omega t}$ in Eq. (3.1), coefficient of $\varphi_k(t)$ in Eq. (3.11)	10
D(s)	Denominator polynomial of H*(s) in Eq. (3.10)	13
$[E_\sigma]$	Reference composed of n+1 hyperplanes selected in the n dimensional Euclidean space R^n	36
$E_1, E_2, \ldots E_p$	p hyperplanes in the n dimensional Euclidean space R^n	36
G*(s)	An approximation to H(s) defined in Eq. (3.5)	13
H_{mn}	mth row, nth column Padé table approximant to H(s)	20
H(s)	Prescribed system function	6
H*(s)	System function of the network N	6
I_1	Quantity defined in Eq. (4.18)	30
I_2	Quantity defined in Eq. (4.19)	30
I_3	Quantity defined in Eq. (A15)	140
J(z)	Function defined by Eq. 3.23	18
K(s)	Laplace transform of k(t)	18
N	Network to be synthesized	6
N(s)	Numerator polynomial of H*(s) in Eq. (3.10)	13

Symbol	Description	First Used on Page
$P(r_1,\ldots,r_n)$	A point with coordinates $r_1, r_2, \ldots, r_n$ in the n dimensional space R^n	35
$P_m(y)$	$P_m(y) = \sum_{k=0}^{m} r_k y^{m-k}$	28
$Q_1(s)$	Laplace transform of an approximation to a pulse	11
$Q_2(s)$	Laplace transform of an approximation to a delayed pulse	12
R^n	n dimensional Euclidean space	36
T	Period of $h_p(t)$ in Eq. (3.1)	10
T-point	Center of a reference	41
T_k point	Center of the kth reference	50
T'-point	A point with a maximum error equal to that of the T-point	41
a_ℓ	Real part of A_ℓ, where A_ℓ is complex	60
$a'_{\ell m}$	Coefficient of a_ℓ in Eq. (4.71)	60
b_ℓ	Imaginary part of A_ℓ, where A_ℓ is complex	60
b_{km}	$b_{km} = e^{s_k t_m}$	60
$b'_{\ell m}$	Coefficient of b_ℓ in Eq. (4.71)	60
c_k	Coefficient of $\delta(t-kd)$ in Eq. (B4)	142
$\hat{c}_k$	Coefficient of r_k in Eq. (4.47)	58
d	Uniform time increment between given points of data	25
d_k	Coefficient of s^k in D(s) in Eq. (3.10)	13
e	Base of the natural logarithm system: e = 2.71828182 ...	6

Symbol	Description	First Used on page
$e_i(t)$	Arbitrary input function	8
$e_k(t)$	Quantity defined in Eqs. (3.12)	15
$e_o(t)$	Response of N to $e_i(t)$	8
$\bar{e}_o(t)$	Approximation to $e_o(t)$	142
f_o	$f_o = \lim_{s \to \infty} \frac{K(s)}{s}$	18
f_k	$(k = 1,2,\ldots)$ Coefficients in continued fraction of Eq. (3.24)	19
f_{kj}	Coefficient of $e_j(t)$ in the kth sequence of differential sums in Eqs. (3.12)	15
g_k	Coefficient of s^k in $G^*(s)$	13
h_m	$h_m = h(t_m)$	25
$h(t)$	Prescribed response to unit impulse function $\delta(t)$	7
$h_p(t)$	Periodic repetition of $h(t)$	10
$h^*(t)$	Response of N to unit impulse function $\delta(t)$	6
j	$j = \sqrt{-1}$	10
k_m	$k_m = k(t_m)$	138
$k(t)$	Prescribed response to unit step function	137
$k^*(t)$	Response of N to the unit step function	137
$\ell^*(t)$	$\ell^*(t) = k^*(t) - B_o$	138
m_k	kth moment of $h(t)$ around the origin	16
n_k	Coefficient of s^k in $N(s)$ in Eq. (3.10)	13

Symbol	Description	First Used on Page
r_m	$r_m = (-1)^m \sum_{\substack{k=1 \\ k_{v+1} > k_v}}^{n} y_{k_1} y_{k_2} \cdots y_{k_m}$	27
s_k	kth pole of $H^*(s)$	7
t	Independent variable; time in seconds	6
t_m	Abscissa of the mth point of data	25
w_k	Mapping of y_k in w-plane	57
x_v	Vector normal to the hyperplane E_v in the Euclidean space R^n	36
y_k	$y_k = e^{s_k d}$	26
z_{kv}	$z_{kv} = A_k e^{s_k t_v} \quad (1 \leqq v \leqq q)$	26
α_ℓ	Re s_ℓ , where s_ℓ is complex	60
β_ℓ	Re (s_ℓ), where s_ℓ is complex	60
γ_k	Im (s_ℓ), where s_ℓ is complex	53
δ_k	Re (y_k) for complex y_k	53
$\delta(t)$	Im (y_k) for complex y_k	6
ϵ	The error for the center of a reference	40
ϵ_v	$(v = 1, 2, \ldots, p)$, the error of the vth equation	35
$\bar{\epsilon}_v$	The error at the vth pole location	61
$\hat{\epsilon}_k$	Error for the center of kth reference	49
ϵ'_σ	Error of T'-point respective to E_σ	41

Symbol	Description	First Used on Page
λ_σ	Coefficient of x_σ in Eq. (4.29)	36
μ_k	Coefficient of x_k in Eq. (4.37)	44
π	3.14159265...	10
σ	As a subscript denotes a choice of n+1 numbers from the series 1, 2, ..., p	36
$\varphi_k(t)$	(k = 1,2,...,n) kth orthonormal function in Eq. (3.11)	14
$\psi(x)$	Function defined in Eq. (3.19)	18
ω	$\omega = \frac{2\pi}{T}$	10
Δ_{v+m}	$(v + m = 1, 2, \ldots, q)\ k_{v+m+1} - k_{v+m}$	139
Δt	Width of an approximating rectangle in Fig. 3.4	12
$\sum'_{(n+1)}$	Summation containing only the selected n+1 terms of the reference	36

ABSTRACT

The purpose of this thesis is to develop a new method for synthesizing an R-L-C network, which when excited by a prescribed unit impulse input will have a prescribed output function as its response. A synthesis problem generally requires the solution of two problems: (1) the approximation problem; (2) the realization problem. In this study attention is focused on the approximation problem. In particular, a method is developed for approximating a prescribed impulse response by a function which can be an impulse response of a realizable R-L-C network. The more general problem of obtaining a network with a prescribed response to an arbitrary input (i.e., input different from the unit impulse) can be reduced to an equivalent one with a "prescribed" impulse response by known approximation techniques, and can thus be solved by the method presented.

The method proposed here is a numerical approximation process, which yields an impulse response function approximating the prescribed one. The error of the approximation, which is defined as the difference between the approximate impulse response and the prescribed one, is minimized in the Tschebyscheff sense. The approximating impulse response is such that its Laplace transform is an R-L-C network function. Thus one can find a network having an impulse response approximating the prescribed one.

In this work the approximate impulse response function is represented as a sum of exponential functions of a form $A_k e^{s_k t}$ ($k = 1, 2, \ldots, n$) where s_k [Re $(s_k) < 0$, $k = 1, 2, \ldots, n$] is the position of the kth pole of the approximating function, and the coefficient A_k ($k = 1, 2, \ldots, n$) is the residue of the approximating function at the pole s_k. The number n denotes the number of terms in the approximating impulse response function.

Since an efficient approximation process requires optimizations of both the pole positions and the residues, two such optimizations are made here. Both these optimizations are made through the application of the discrete Tschebyscheff approximation theory, and yield, as stated above, an error of approximation which is a minimum in the Tschebyscheff sense.

The final part of this investigation consists of presentation of two examples illustrating the approximation process.

CHAPTER I

INTRODUCTION

The advent of radar, automatic control mechanisms, electronic computers, and other new devices has made it necessary to provide means for the synthesis of networks meeting prescribed time requirements. This type of synthesis is commonly designated as "time-domain synthesis" or "transient-response synthesis." The problem of synthesis in the time domain consists of prescribing an electrical input (usually voltage or current input) and an electrical output (commonly voltage or current output), where both these quantities are functions of time. The solution to the problem consists of finding a network which, when excited by the prescribed input, will yield the prescribed output.

A solution to a problem in network synthesis is rarely an exact one. The limitations of physical realizability, and of the modest number of elements which may be employed in a practical network, stringently limit the results which can be obtained using synthesis. A given synthesis problem may reduce to the selection of the simplest network meeting prescribed requirements, or, perhaps to the determination of the network of restricted complexity which would best fulfill the requirements. This means that, in general, the network which is found will, when excited by the prescribed input, yield an output different from the prescribed one. The difference between the two outputs constitutes an error. In the normal case it is desired to find a network which will minimize the error between these two outputs in some sense.

In network synthesis, the three most commonly employed approximations are Taylor, least-mean-square error, and Tschebyscheff. The

Taylor approximation provides the best approximation to some given function at a single point. At that point the error and as many of its derivatives as possible equal zero. An error which is zero at this point and which increases slowly in the immediate vicinity results. Generally the price paid for this desirable behavior is a much larger error away from the zero-error point. The least-mean-square-error approximation minimizes the mean-square error. The parameters in the approximating function are varied, and that set which minimizes the mean-square error is chosen as the solution.

The Tschebyscheff approximation minimizes the magnitude of the maximum error. The Tschebyscheff approximation is one of the most desired types of approximation in the area of network synthesis. This approximation makes a highly efficient use of circuit elements. No approximation procedure minimizing the error in the time domain has been offered in the past, probably because this type of approximation gives rise to more complicated mathematical expressions than the more-commonly-employed least-mean-square-error type of approximation.

The problem of synthesis of networks from prescribed time requirements is a relatively important one in the general area of network synthesis. Consequently, a considerable effort was expended on this problem in the last decade, resulting in several excellent contributions. In particular, the application of orthogonal functions, computers, Fourier series theory, numerical methods, and time-moment matching have provided useful approaches to this problem.

A number of useful methods have been proposed for the solution of the problem of synthesis in the time domain. However, each method that has been proposed suffers from either (1) restriction on the classes of functions that can be approximated, (2) nonphysical realizability of the resulting function (in general), (3) unsatisfactory control of the approximation error, or (4) inefficient use of circuit elements.

It is the aim of this investigation to propose a general approximation procedure which will yield a system function resulting in a network realizable with R-L-C elements. The method of approximation is a numerical one, employing discrete Tschebyscheff approximations. The error between the desired time response and the obtained time response is a minimum in the Tschebyscheff sense. It is felt that the proposed method largely overcomes the outlined four limitations. The proposed method also provides an approximation procedure minimizing the error in the time domain in the Tschebyscheff sense, thus providing a desirable type of error control. The theoretical development and the practical application of this approximation method are the purposes of this dissertation.

In the method proposed in this investigation, attention is focused on the problem of finding a network from the prescribed response to unit impulse. It is shown (Appendix B) that a problem in which an arbitrary input and a prescribed response are given, can be reduced to one of prescribed impulse response. The Laplace transform of the impulse response is the system function. It follows from network theory considerations that the system function will result in a realizable R-L-C network, if the impulse response is a sum of n exponential functions, of a form $A_k e^{s_k t}$ $(k = 1, 2, \ldots, n)$. s_k $(k = 1, 2, \ldots, n)$ is the location of the kth pole of the system function, and the coefficient A_k $(k = 1, 2, \ldots, n)$ is the residue of the system function at the pole s_k. The number n denotes the number of terms in the impulse response and in the system function.

n is also directly proportional to the minimum number of elements in the network realizing the system function. Hence, an efficient approximation process is one which yields a tolerable approximation error with a minimum number of terms for the impulse response. To obtain such an approximation, two optimizations must be made, namely, an optimization of pole positions and an optimization of residues.

In the proposed method two such optimizations are made. The solution for optimum pole positions is developed. Once these pole positions are determined, an optimization of residues takes place. Both these optimizations are made in the Tschebyscheff sense through the application of discrete Tschebyscheff approximation theory. Thus, the approximate impulse response obtained results in a network which yields a tolerable approximation error with a minimum number of elements. To say this in other words, it is believed that the errors obtained using the method to be presented will for a given complexity of approximating function, generally be smaller than those obtained by previous approaches.

The proposed method also largely overcomes the drawback of restriction on the classes of functions that can be approximated. The numerical process culminates in the system function and places no restrictions on types of function which can be approximated, provided, of course, that one does not demand properties not obtainable with R-L-C networks. Also, the drawback of non-physical realizability is overcome. The approximation error, being Tschebyscheff, is rather satisfactorily controlled.

The author believes the contributions of this dissertation as providing a satisfactory solution to the problem of synthesis of networks for prescribed time requirements. A summary of the more important results includes: (1) application of the discrete Tschebyscheff approximation theory to network problems; (2) development of a general solution to the problem of approximation of networks in the time domain; (3) development of a general numerical method of approximation, optimizing both pole locations and residues; and (4) detailed investigation of the errors of approximation both in the pole determination and in the residue determination.

In the second chapter of this work the problem of synthesis of networks in the time domain is stated in detail. In the succeeding chapter the state of the art is reviewed, and some of the contributions to the problem are outlined. In the fourth chapter, the impulse-response approximation problem, i.e., synthesis of a network from the prescribed impulse response, is stated and solved. Two examples illustrating the approximation process are worked out in detail. In Appendix A, the problem of synthesis from prescribed step response is solved, and it is shown that the methods of Chapter IV can be applied to yield the desired network. The arbitrary input problem is treated in Appendix B, where it is shown that this problem can be reduced to the one treated in Chapter IV, i.e., synthesis of networks from prescribed impulse response requirements.

CHAPTER II

STATEMENT OF THE PROBLEM

The synthesis problem is essentially an input-output problem; i.e., it is in general desired to find a network that will produce a prescribed response to a specified excitation. The specifications of the input and of the output yield the system function H(s). In general, there may exist no R-L-C network with H(s) as the system function. A network N can be found, however, which when excited by the specified input will produce an output approximating the prescribed one. For present purposes the network N which is to be found will be characterized by its system function H*(s) such that:

$$H^*(s) = L[h^*(t)] = \int_0^\infty h^*(t)e^{-st}\,dt, \qquad (2.1)$$

where h*(t) is the response of N to the unit impulse function δ(t), as shown in Fig. 2.1.

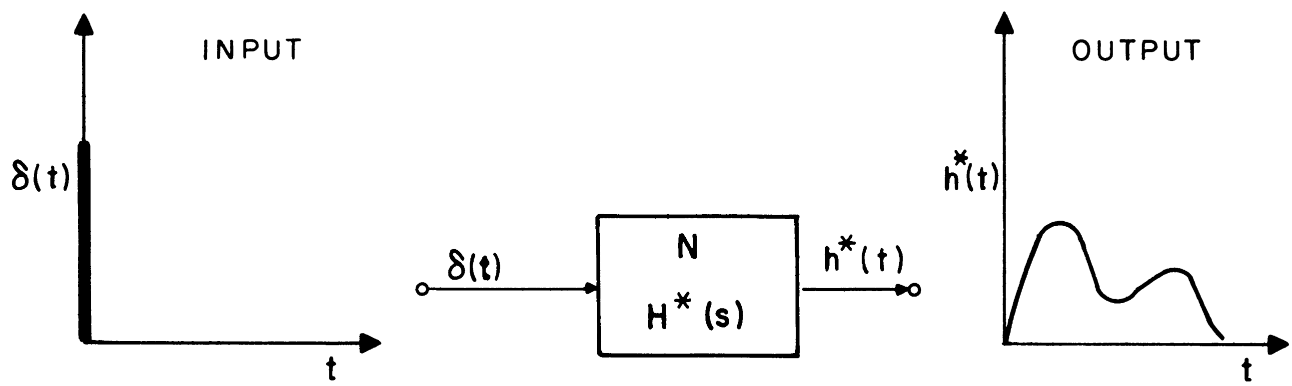

FIG. 2.1 TIME-DOMAIN SYNTHESIS REQUIREMENTS

It is assumed here that N is a finite, linear, passive, lumped-parameter, bilateral electrical network. Then $H^*(s)$ can be written:

$$H^*(s) = \sum_{k=1}^{n} \frac{A_k}{s-s_k} \qquad (2.2)$$

where s_k $(k = 1, 2, \ldots, n)$ are the poles of $H^*(s)$ and A_k is the residue at the pole $s = s_k$. Complex poles occur in conjugate pairs. If it is assumed that there are no coincident poles, $h^*(t)$, the inverse Laplace transform of $H^*(s)$, is then

$$h^*(t) = \sum_{k=1}^{n} A_k e^{s_k t} \; . \qquad (2.3)$$

Stability requires that

$$\text{Re } (s_k) \leq 0 \qquad (k = 1, 2, \ldots, n) \; . \qquad (2.4)$$

Since coincident poles give rise to $te^{s_k t}$, $t^2 e^{s_k t}$, ..., $t^{k-1} e^{s_k t}$, for a kth-order pole, they must not occur with zero real part.

The statement of an approximation problem consists of prescribing the impulse response $h(t)$ with tolerances on the allowable error. One is to find a system function $H^*(s)$, as given in Eq. (2.2), with the constraint of Eq. (2.4). It is also expected that the distance† from $h(t)$ to $h^*(t)$ is minimized in some sense (i.e., the error is minimized in some sense). In Fig. 2.2 the above requirement is portrayed in the form of a block diagram.

In the following chapters a method will be developed for the synthesis of an approximated system N when $h(t)$ is prescribed. The system N is restricted to be an interconnection of R-L-C elements

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† Distance between x and y $\triangleq d(xy) \triangleq$ a number which provides a measure of the disparity between x and y (Frechet, 1906).

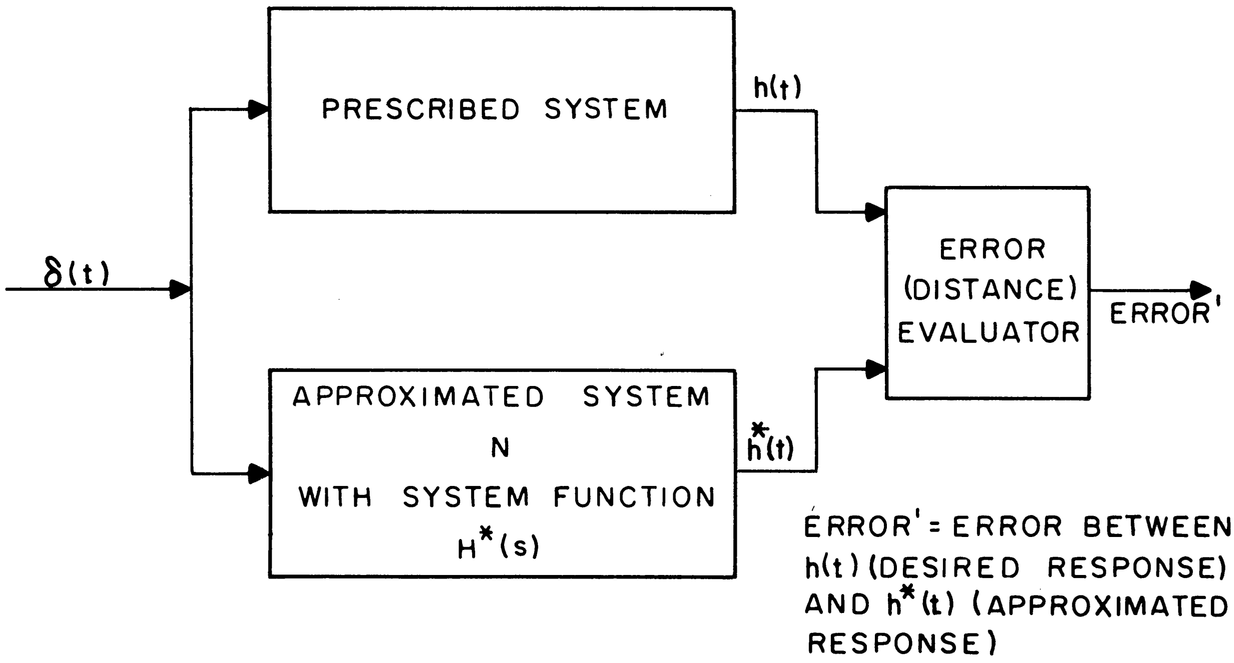

FIG. 2.2 THE APPROXIMATION PROCESS

(i.e., an R-L-C network). The error (distance) between h(t) and h*(t) will be defined as h*(t)-h(t). A network N will be found, such that max $|h^*(t) - h(t)|$ will be a minimum. The error, hence, will be minimized in the Tschebyscheff sense.

It is to be noted that most problems for prescribed time response include specifications of a particular input $e_i(t)$ and of the corresponding response $e_o(t)$ rather than the impulse response h(t). However, a number of techniques are available for reduction of such input conditions to an equivalent prescribed impulse response h(t) [31]. One such technique is discussed in Appendix B.

CHAPTER III

STATUS OF THE ART

Recent contributions to the time-domain synthesis problem have resulted from exploiting several ideas, of which the following are noteworthy:

3.1 Fourier Series Approach [8, 24]

This method is based upon the following idea: if h(t), the prescribed impulse response, were to repeat periodically, it could then be approximated by a finite trigonometric polynomial to any required error tolerance. A corresponding Laplace transform could then be obtained at once, and since the approximation takes place in the time domain, time-domain error is controlled. Also, the methods of Fourier series are well known and understood. In effect, if the function shown in Fig. 3.1 is the desired impulse response of the sought network N,

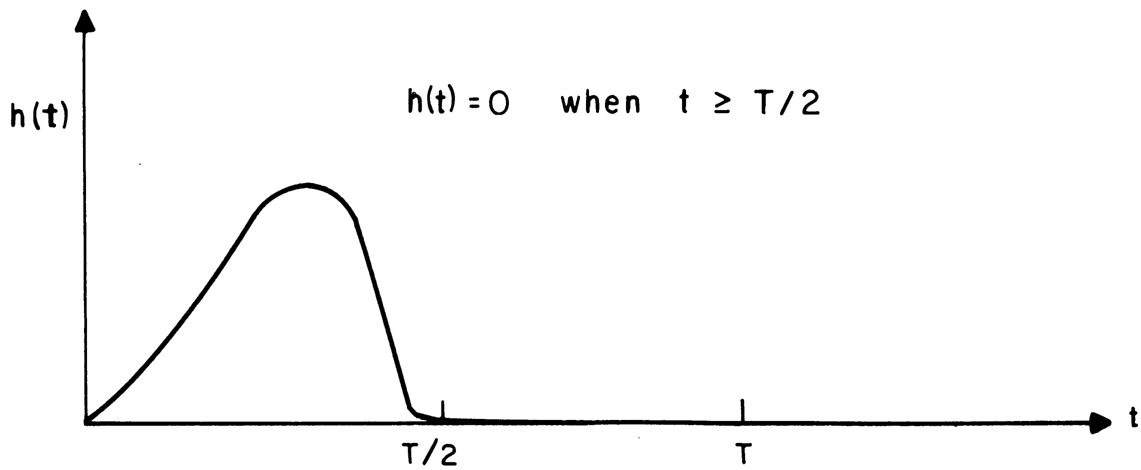

FIG. 3.1 DESIRED IMPULSE RESPONSE OF N

and if the periodic function $h_p(t)$ is as shown in Fig. 3.2,

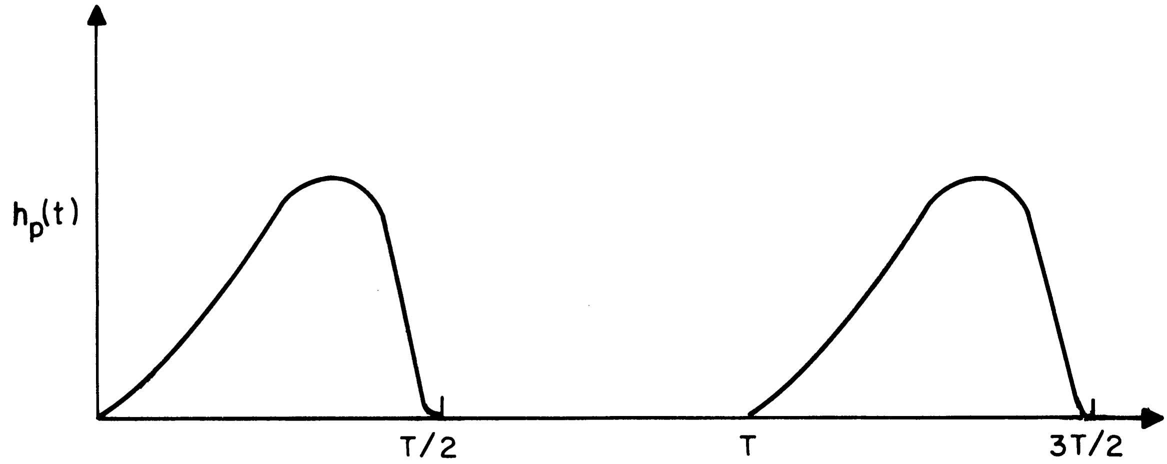

FIG. 3.2 PERIODIC REPETITION OF h(t)

then evidently

$$h_p(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega t}, \quad \omega = \frac{2\pi}{T} . \qquad (3.1)$$

If $h_p(t) = 0$ for $t < 0$

then

$$h(t) = [h_p(t)-h_p(t-T)] .$$

$h_p(t)$ can then be approximated by trigonometric functions resulting in an approximate network.

This method yields excellent results for certain types of waveforms. Among its drawbacks is the fact that synthesis by this technique often requires a non-positive real admittance (and hence it is not realizable with passive elements only), and a modification has to take place. It is also believed that this technique is not very efficient in terms of degree of approximation for a utilized number of elements.

3.2 Impulse Method of Approximation [7,8,26]

This method is based upon the idea that an arbitrary function can be related to a train of impulses. If the desired impulse response h(t) is given, an approximate response h*(t) is obtained as a sequence of v-1 curves, each of which is given by an (m-1)-degree polynomial. This concept is illustrated in Fig. 3.3.

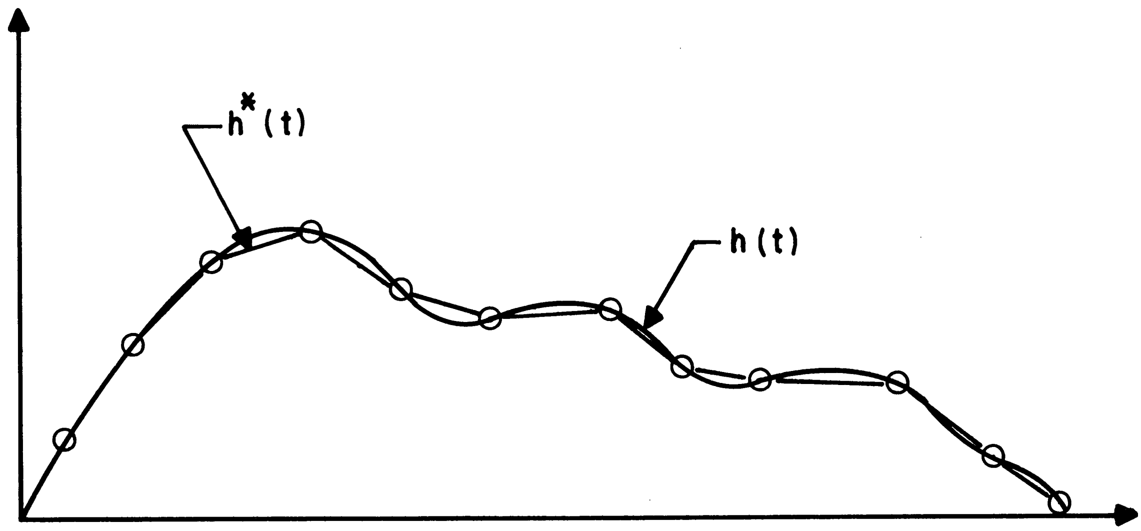

FIG. 3.3 APPROXIMATION OF h(t) BY h*(t)

If h*(t) is then differentiated m times, it will yield a sequence of v impulses which, in turn, are approximated by some reasonable facsimile. A suggestive approximation for a delayed impulse is a delayed pulse. A good approximation, evidently, requires narrow pulse width, and if the Laplace transform of one such pulse is approximated by an expression of the form

$$Q_1(s) = \sum_{v=1}^{n} \frac{A_v}{s-s_v} , \qquad (3.2)$$

then a pulse occuring t seconds earlier would have a Laplace transform†:

$$Q_2(s) = \sum_{v=1}^{n} \frac{A_v e^{s_v t}}{s-s_v} \quad . \qquad (3.3)$$

This method yields good results in many cases; however, one of its drawbacks lies in the fact that two approximations are required—one in the approximation of h(t) by a sequence of curves and the other in the approximation of the delayed impulse by $Q_2(s)$—thus increasing the final error.

3.3 Numerical Calculations by Time Series [1,14,15]

A representative method in this class is the one due to Ba Hli [1]. If h(t) is approximated by rectangles as shown in Fig. 3.4,

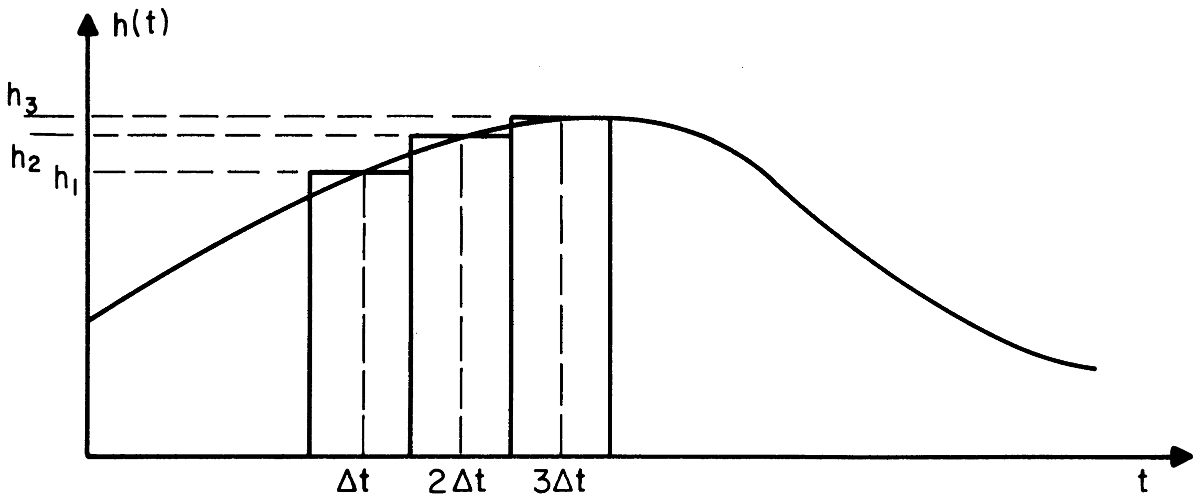

FIG. 3.4 APPROXIMATION OF h(t) BY RECTANGLES

then, since

$$H(s) = \int_0^\infty h(t)e^{-st}dt \qquad (3.4)$$

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† Note that if $L^{-1}[Q_1(s)] = \sum_{v=1}^{n} A_v e^{s_v t} = q_1(t)$,

then $q_2(t) = q_1(t+\tau) = \sum_{v=1}^{n} (A_v e^{s_v \tau}) e^{s_v t}$,

hence $Q_2(s)$ follows as given in Eq. (3.3).

can be approximated by G*(s) given by:

$$G^*(s) = h_1 \Delta t\, e^{-s\Delta t} + h_2 \Delta t\, e^{-2s\Delta t} + \ldots \quad , \tag{3.5}$$

consequently,

$$G^*(s) \approx \sum_{v=1}^{n} h_v \Delta t\, e^{-vs\Delta t} \quad . \tag{3.6}$$

Expansion of $e^{-s\Delta t}$ in a power series yields

$$e^{-s\Delta t} = 1 - \frac{s\Delta t}{1!} + \frac{(s\Delta t)^2}{2!} + \ldots \quad , \tag{3.7}$$

an expansion valid for all finite s.

Now,

$$\begin{aligned} G^*(s) = {} & h_1\Delta t \left(1 - \frac{s\Delta t}{1!} + \frac{(s\Delta t)^2}{2!} - \ldots\right) \\ & + h_2\Delta t \left(1 - \frac{2s\Delta t}{1!} + \frac{(2s\Delta t)^2}{2!} - \ldots\right) \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + h_n\Delta t \left(1 - \frac{ns\Delta t}{1!} + \frac{(ns\Delta t)^2}{2!} - \ldots\right) . \end{aligned} \tag{3.8}$$

Collecting terms,

$$G^*(s) = \sum_{v=1}^{n} h_v\Delta t - \frac{s\Delta t}{1!} \sum_{v=1}^{n} v h_v \Delta t + \ldots \quad , \tag{3.9}$$

or

$$G^*(s) = g_o + g_1 s + g_2 s^2 + \ldots \quad .$$

This last expression is then approximated at the origin as a ratio of two polynomials; i.e.,

$$G^*(s) \approx \frac{N(s)}{D(s)} = \frac{n_o + n_1 s + \ldots + n_m s^m}{d_o + d_1 s + \ldots + d_n s^n} \quad . \tag{3.10}$$

d_o is usually chosen to be one and the coefficients n_k and a_k are found through solution of a system of equations.

This method yields adequate results for many applications not requiring a great degree of accuracy. It is seen that a number of approximations has taken place, thus adding to the overall error. In particular, experience reveals that functions which are not monotonically rising and then monotonically falling are not well approximated by this procedure.

3.4 Approximation by Means of Least-Square Criteria [6,11]

This approach consists of approximating the impulse response h(t) as the sum of orthogonal functions; i.e.,

$$h^*(t) = \sum_{k=1}^{n} C_k \varphi_k(t) \quad , \qquad (3.11)$$

where $[\varphi_1(t), \varphi_2(t) \ldots, \varphi_n(t)]$ form an orthonormal set, and C_k is chosen so as to minimize the least-square error between h(t) and h*(t). In particular, $\varphi_k(t)$ is a sum of decaying exponentials and exponentially-damped sinusoids.

This method, advanced by W. L. Kautz [11], has also been studied by E. G. Gilbert [6]. Gilbert has shown that analog-computer circuits can be implemented to yield the desired constants C_k. However, in this method the approximation is essentially in terms of coefficients C_k (which form the residues) only, and by and large the poles are assumed arbitrarily. Even though two methods are suggested by Kautz for locating pole positions, these methods leave something to be desired. In the first method $H(s) = L[h(t)]$ is found and expanded in a power series which is then expanded into a continued fraction. Termination of this continued fraction after several divisions will yield a rational fraction. The roots of the denominator polynomial are suggested as pole positions.

This method, even though straight-forward enough, requires that:

1. h(t) be given in an analytical form;
2. h(t) be L-transformable;
3. The roots in the denominator polynomial be in the left-hand plane.

These conditions are not met in general.

The second method described by Kautz for locating the poles is the Prony method. It consists of finding a linear, constant-coefficient differential equation, which is nearly satisfied by h(t). Such an equation can be solved through substitution of $h(t) = e^{st}$ into it, and it leads to a characteristic polynomial in s, the roots of which are the "natural resonances" of its solution - hence related approximately to the "natural resonances" of h(t). This polynomial is obtained by forming the sequence of differential sums:

$$e_o(t) = h(t)$$

$$e_1(t) = f_{1o}\, e_o(t) + h'(t)$$

$$\cdots\cdots\cdots \tag{3.12}$$

$$e_n(t) = f_{no}\, e_o(t) + f_{n1} e_1(t) + \ldots f_{n,n-1}\, e_{n-1}(t) + h^{(n)}(t)$$

The solution of the characteristic equation for e_n is used to determine the n poles. The coefficients f_{kj} are computed from the integrals

$$f_{kj} = \frac{{}_o\int^{\infty} f^{(k)}(t) e_j(t) dt}{-{}_o\int^{\infty} e_j^2(t) dt} \, . \tag{3.13}$$

Thus, this method requires the availability of the numerator and denominator integrals in addition to a considerable amount of computational work. A more detailed treatment of Prony's method is found in the classical work of Prony [10].

3.5 Synthesis Through Matching of Time Moments [10,22]

This approach is based upon the expansion of the impulse response into time moments. This method has the advantage that moments are easily obtained from a graphical presentation of the impulse response and that the moment coefficients are simply related to the transfer function of the network. Since

$$H(s) = \int_0^\infty h(t)e^{-st}dt \quad , \tag{3.14}$$

expansion of e^{-st} into power series yields

$$H(s) = \int_0^\infty h(t)dt - \frac{s}{1!}\int_0^\infty t h(t)dt + \frac{s^2}{2!}\int_0^\infty t^2 h(t)dt - \ldots$$

$$= m_0 - m_1 s + m_2 s^2 - \ldots \quad , \tag{3.16}$$

where

$$m_k = \frac{1}{k!}\int_0^\infty t^k h(t)dt \tag{3.17}$$

is the kth moment of the impulse-response function h(t) around the origin. Then, in particular:

m_0 = area under the impulse function,

$\frac{m_1}{m_0}$ = center of gravity,

$\frac{2m_2}{m_0}$ = moment of inertia about the line t = 0,

and so on. Therefore, the coefficients can be identified with time moments.

The power series is then approximated by a continuous-fraction expansion, as described in the previous section†. In effect, H(s) is equated to a rational function, and the fraction is cleared by multiplying both sides by the denominator polynomial. Matching coefficients of equal powers of s yields the coefficients of the numerator and denominator polynomials. Some of the advantages of this method were stated earlier. Its drawbacks are: (1) the error of the approximation is not predictable in advance; (2) moments exist only for certain classes of functions which are sufficiently bounded in amplitude and time; (3) there is no guarantee that zeros of the denominator will lie in the left-hand plane. In general, the method does not work well for functions with oscillatory terms, such as the one illustrated in Fig. 3.5.

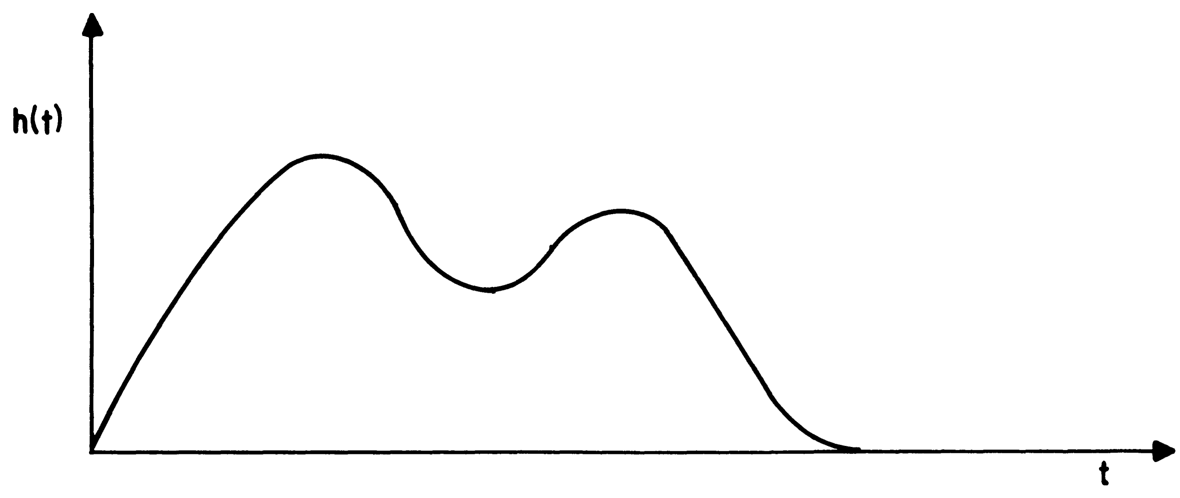

FIG. 3.5 IMPULSE RESPONSE WITH OSCILLATORY TERMS

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† This method of approximation of a power series by a rational function is described in detail in [20].

3.6 Continued-Fraction Expansion [13] and Padé [9,15,19,25,27] Approximants

The method of continued-fraction expansion has been described by Nadler [17]. It is not a time-domain synthesis method in the sense of the statement of the problem, as stated in Chapter II. This approach consists of finding the L-transform of the step response, approximating it in the s-domain, and obtaining the inverse L-transform of the approximated function. The details of this method follow:

Let K(s) be the Laplace transform of the prescribed response of the sought network N to the unit step function. Then,

$$K(s) = \frac{1}{s} H(s). \tag{3.18}$$

Cauer [2] has shown that if K(s) is positive-real and regular, then

$$K(s) = s[f_o + \int_o^\infty \frac{d\psi(x)}{s^2+x}] \tag{3.19}$$

where

$$f_o = \lim_{s\to\infty} \frac{K(s)}{s} \tag{3.20}$$

and

$$d\psi(x) = \frac{\text{Re}[K(j\sqrt{x})]}{\pi\sqrt{x}} \tag{3.21}$$

Stieltes [2] has shown that if $\psi(x)$ is an increasing function with infinitely many poles, and if the integrals

$$\int_o^\infty (-x)^{k-1} dx(x) \quad (k = 1, 2, 3, \ldots) \tag{3.22}$$

all exist, then one can represent the integral

$$J(x) = \int_o^\infty \frac{d\psi(x)}{z+x} \tag{3.23}$$

by a continued fraction of the form:

$$J(z) = \cfrac{1}{f_1 z + \cfrac{1}{f_2 + \cfrac{1}{f_3 z + \cfrac{1}{f_4 + \cdots}}}} \qquad (3.24)$$

with an infinite number of terms.

It can be seen that the right-hand side of Eq. (3.19) can be expanded into a continued fraction similar to Eq. (3.24).

Then

$$K(s) = sf_o + \cfrac{s}{f_1 s^2 + \cfrac{1}{f_2 + \cfrac{1}{f_3 s^2 + \cfrac{1}{f_4 + \cdots}}}}, \qquad (3.25)$$

or

$$K(s) = sf_o + \cfrac{1}{f_1 s + \cfrac{1}{f_2 s + \cfrac{1}{f_3 s + \cfrac{1}{f_4 s + \cdots}}}}, \qquad (3.26)$$

One can now terminate the continued fraction in Eq. (3.26) and equate the coefficients f_k to those in the power series expansion for K(s). The terminated continued fraction in addition to the sf_o term, represents, then, a rational function approximation to K(s).

The Padé [17] method of approximation consists of listing various rational fraction approximations in a double-entry table. Hence, a function H(s) is approximated as a ratio of two polynomials, N(s) and D(s) [i.e., $H^*(s) = \frac{N(s)}{D(s)}$]. Now, if N(s) is an mth-degree polynomial and D(s) an nth-degree polynomial, then N(s) has m+1 coefficients and D(s) has n+1 coefficients. The rational function $\frac{N(s)}{D(s)}$ has, however, only m+n+1 independent coefficients. Hence, if one equates $\frac{N(s)}{D(s)}$ to the power series of H(s) he can determine the coefficients of N(s) and D(s) so that $H(s)\cdot D(s)-N(s)$ has s^{m+n+1} as the lowest power of s [i.e., coefficients of $s^{m+n+1-k}$ ($k = 1, 2, \ldots, m+n$) are all zero]. These various rational function approximants to H(s) can then be tabulated in a double-entry table, as shown in Table I

n ↓ \ m →			
H_{oo}	H_{o1}	H_{o2}	...
H_{1o}	H_{11}	H_{12}	...
H_{2o}	H_{21}	H_{22}	...

where

$$H_{mn} = \frac{N(s)}{D(s)} = \frac{\sum_{k=0}^{m} a_k s^{m-k}}{\sum_{k=0}^{n} b_k s^{n-k}} \qquad (3.27)$$

TABLE I PADE TABLE FOR H(s)

Teasdale [25] suggests for better approximation in the time domain the employment of an "indirect Padé approximant." A method for obtaining this approximant is shown in the block diagram of Fig. 3.6.

This approach, excellent for s-plane approximations, suffers in the time domain from the shortcomings mentioned in Section 3.4.

In this chapter, several contributions have been reviewed. Other approaches have made use of Laguerre's functions [13,18] and of analog computers [6,12,18]. These will not be discussed here.

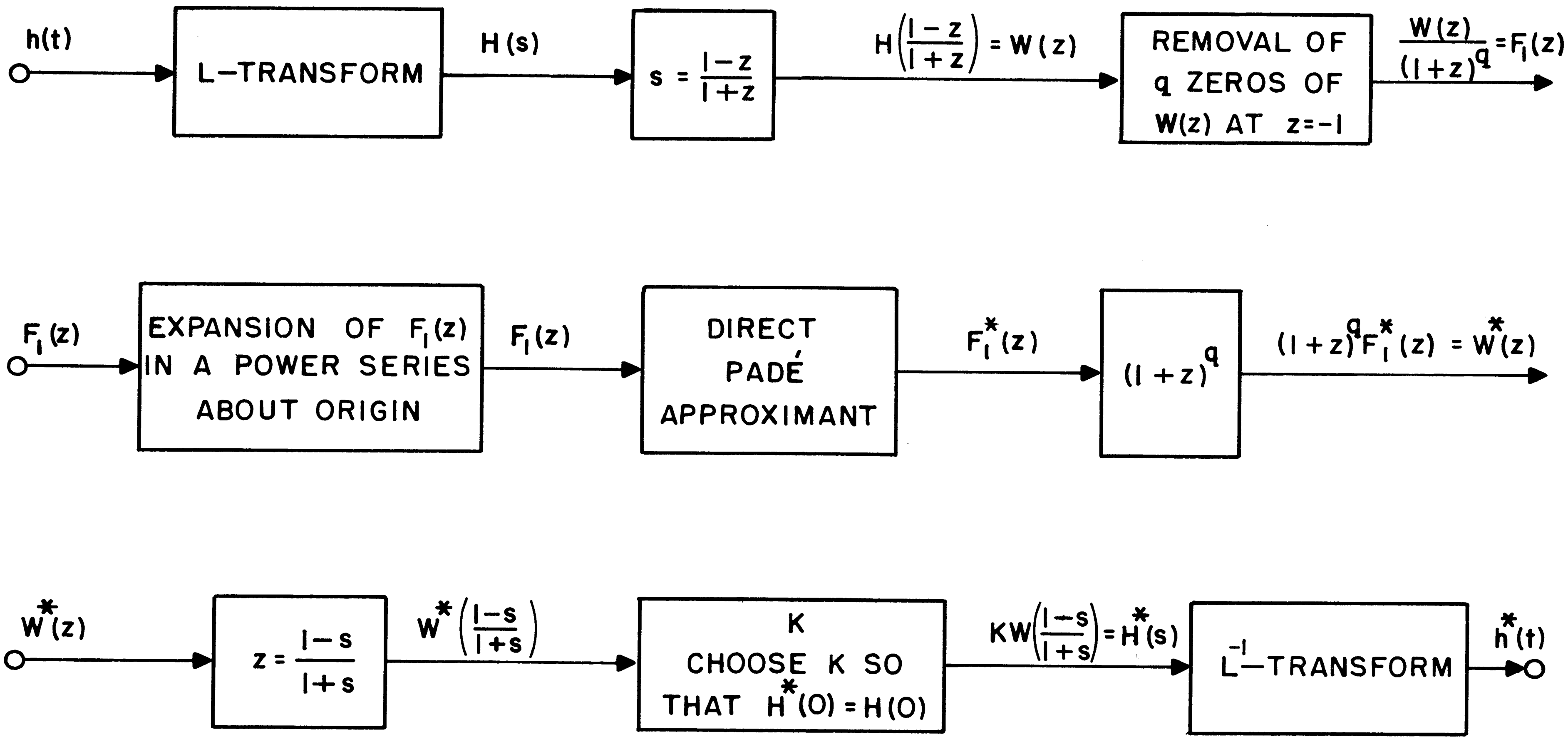


FIG. 3.6 BLOCK DIAGRAM SHOWING THE PROCESS OF OBTAINING THE "INDIRECT PADÉ APPROXIMANT"

CHAPTER IV

THE IMPULSE-RESPONSE-APPROXIMATION PROBLEM

4.1 Introduction

In the second chapter the approximation problem has been stated, and in the third chapter methods of attack have been reviewed. In this chapter an alternative method of attack is developed, employing the concept of Tschebyscheff discrete approximations.

The problem considered here is one of determining an R-L-C network with an impulse response $h*(t)$ which approximates a prescribed impulse response $h(t)$. Since one knows how to synthesize a network function of the form

$$H*(s) = \sum_{k=1}^{n} \frac{A_k}{s-s_k} \tag{4.1}$$

with

$$Re(s_k) \leq 0 \quad (k = 1, 2, \ldots, n) , \tag{4.2}$$

the problem will be solved if the approximate impulse response $h*(t)$ can be found which is given by

$$h*(t) = \sum_{k=1}^{n} A_k e^{s_k t} \quad . \tag{4.3}$$

with†

$$Re(s_k) \leq 0 \quad (k = 1, 2, \ldots, n) . \tag{4.4}$$

The problem can be considered to be two-fold, its part being:

(a) Determination of pole locations, i.e., the determination of exponents of the approximating function in Eq. (4.3), and

(b) Determination of residues, i.e., the determination of coefficients A_k in Eq. (4.3).

† Note that this expansion assumes no coincident poles.

Both determinations are to be made so as to minimize the approximation error in some sense.

In this chapter the parts (a) and (b) of the problem are solved. The approximation error is minimized in the Tschebyscheff sense. A brief summary of the contents of each section is given below.

In Section 4.2 of this chapter the problem of determination of pole locations is solved. This is accomplished by a reduction of the problem to an overdetermined system of linear equations. It is shown that the solution of the overdetermined system will yield a set of coefficients from which the desired poles can be obtained.

In the same section the theory of discrete Tschebyscheff approximations is reviewed. It is shown that this theory can be applied to solve the overdetermined system of equations. The formulas for optimum poles (in Tschebyscheff sense) are developed, and are listed in Table III.

In Section 4.3 the problem of determination of residues is solved. It is shown that this problem can likewise be reduced to an overdetermined system of linear equations, the solution of which yields the residues.

In Section 4.4 the errors of the approximation are discussed. A relationship between the Tschebyscheff error in the pole determination and the final approximation is derived.

In Section 4.5 two examples are worked out to illustrate the developed method. The results lead to realizable networks which are shown at the end of each example. Also a comparison is made between the desired impulse response, and the obtained one for each example.

In section 4.6 discussion of the suggested method is made, and the drawn conclusions are stated. A summary of the process is offered at the end of the section.

4.2 The Determination of Pole Locations

In this section the problem of determination of pole locations is solved. It is shown that this problem can be reduced to an over-determined system of equations. The theory of discrete Tschebyscheff approximations is reviewed, and it is shown that with this theory, formulas for optimum poles (in Tschebyscheff sense) can be developed.

4.2.1 Reduction of the Problem to an Overdetermined System of Linear Equations.

In this section the problem of determination of pole locations is reduced to the problem of an overdetermined system of linear equations. The prescribed impulse response is assumed to be given in form of q ordinates of h(t), denoted as h_m $(m = 1, 2, \ldots, q)$, at uniform time intervals. A set of n coefficients $r_1, r_2, \ldots, r_n$ is introduced from which the desired poles $s_1, s_2, \ldots, s_n$ can be obtained. The relationship between r_k $(k = 1, 2, \ldots, n)$ and h_m $(m = 1, 2, \ldots, q)$ is stated in Theorem 1. Theorem 1 in essence, is found in the literature [21,29,30], the proof of the theorem, however, is the author's. By Theorem 1 one obtains a set of linear equations for r_k $(k = 1, 2, \ldots, n)$ which are in general overdetermined (i.e., the number of equations exceeds the number of unknowns).

The method developed for the determination of pole locations is similar to the Prony method [10]. However, the discrete Tschebyscheff approximation concept is employed to solve the overdetermined system of equations.

In general, h(t) will be given in the form of an equation, a graph, or a set of data. From any of these three forms one can obtain data for equal increments of time. If there are q points of data, and if the first point of data is given at t_1 [i.e., $h(t_1)$ is given], and if the increment is d, then the m ordinate of h(t) is given by

$$h_m = h(t_m) \quad (m = 1, 2, \ldots, q). \qquad (4.5)$$

When v is an integer less than or equal to q, t_v is given by

$$t_v = t_1 + (v-1)d \qquad (v \leq q). \qquad (4.6)$$

Equating h_v to $h^*(t_v)$, one obtains

$$h_v = \sum_{k=1}^{n} A_k e^{s_k t_v} . \qquad (4.7)$$

A sketch of h(t) and the corresponding ordinates of h(t) for equal time increments are shown in Fig. 4.1.

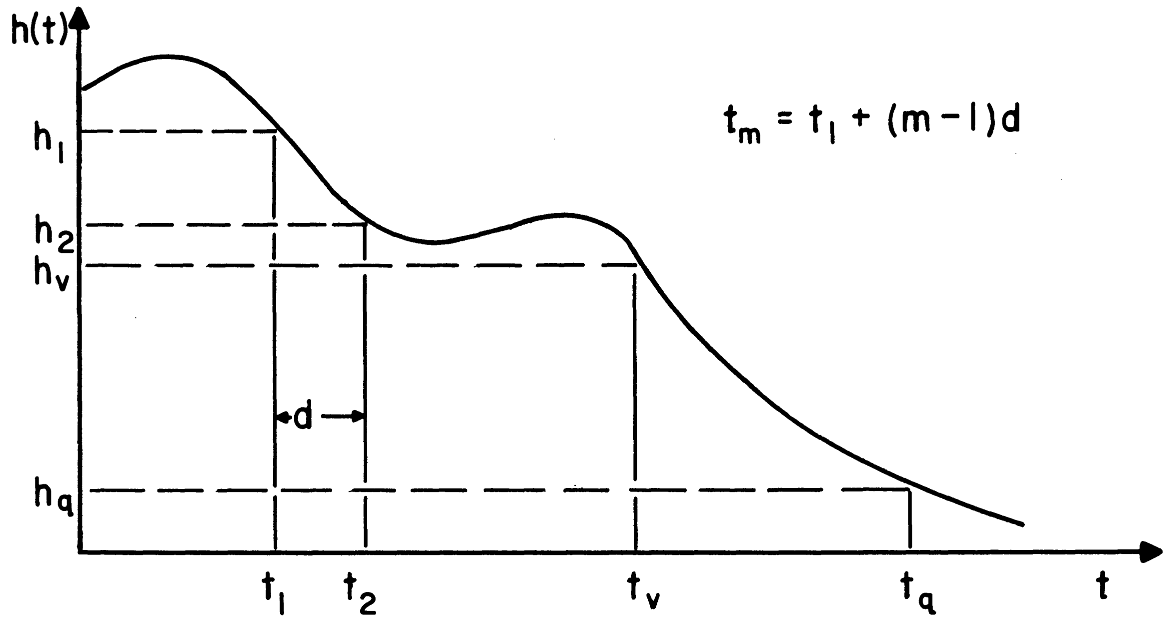


FIG. 4.1 h(t) AND CORRESPONDING ORDINATES FOR EQUAL TIME INCREMENTS

Let

$$e^{s_k d} = y_k \quad (k = 1, 2, \ldots, n) \qquad (4.8)$$

and

$$A_k^{s_k t_v} = z_{kv} \quad (k = 1, 2, \ldots, n; \; 1 \leq v \leq q) \qquad (4.9)$$

Then

$$h_v = \sum_{k=1}^{n} z_{kv}$$

$$h_{v+1} = \sum_{k=1}^{n} z_k y_k$$

$$h_{v+2} = \sum_{k=1}^{n} z_k y_k^2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$h_{v+i} = \sum_{k=1}^{n} z_k y_k^i \quad . \qquad (4.10)$$

Since only q points of data are given, then $v+i \leq q$ in Eq. (4.10).

Assume that n+1 equations of the same type as Eq. (4.10) for the n unknowns z_{kv} $(k = 1, 2, \ldots, n; 1 \leq v \leq q)$ were used. The n+1 equations for the n unknowns z_{kv} can be satisfiéd simultaneously if and only if the determinant of coefficients of z_{kv} is zero. This will yield a relationship between y_k $(k = 1, 2, \ldots, n)$ and h_{v+i} $(i = 0, 1, \ldots, n)$.

This relationship can alternatively be obtained in the following manner:

Let

$$r_1 = -\sum_{k=1}^{n} y_k$$

$$r_2 = \sum_{\substack{k_1=1 \\ n \geq k_2 > k_1}}^{n} y_{k_1} y_{k_2}$$

$$r_3 = \sum_{\substack{k_1=1 \\ n \geq k_3 > k_2 > k_1}}^{n} y_{k_1} y_{k_2} y_{k_3}$$

....

$$r_n = (-1)^n \prod_{k=1}^{n} y_k \qquad (4.11)$$

Hence r_m is the coefficient of y^{n-m} in an algebraic equation of nth degree, whose roots are y_k (k = 1, 2, ..., n). The coefficient of the leading term (i.e., the coefficient of y^n) is r_o (i.e., unity, since $r_o = 1$). If one multiplies the first n+1 equations in Eqs. (4.10) and (4.11) by one another so that h_{v+i} is multiplied by r_{n-i} (i = 0, 1, ..., n), then,

$$r_n h_v = (-1)^n \prod_{k=1}^{n} y_k \sum_{k=1}^{n} z_{kv}$$

$$r_{n-1} h_{v+1} = (-1)^{n-1} \sum_{\substack{k=1 \\ n \geq k_{v+1} > k_v}}^{n} y_{k_1} y_{k_2} \cdots y_{k_{n-1}} \sum_{k=1}^{n} z_{kv} y_k$$

....

$$r_1 h_{v+n-1} = -\sum_{k=1}^{n} y_k \sum_{k=1}^{n} z_{kv} y_k^{n-1}$$

$$h_{v+n} = \sum_{k=1}^{n} z_{kv} y_k^n \quad . \tag{4.12}$$

From Eqs. (4.12) one can obtain the following theorem.

Theorem 1

If Eqs. (4.12) are valid for all sets z_{kv} ($k = 1, 2, \ldots, n$; $1 \leq v \leq q - n$) then

$$\sum_{k=0}^{n} r_{n-k} h_{v+k} = 0. \qquad (1 \leq v \leq q - n) \tag{4.13}$$

Proof:

Assume that Eq. (4.13) is valid for sets z_{kv} ($k = 1, 2, \ldots, m$; $1 \leq v \leq q - n$) for a particular number of poles, say m. Then,

$$\sum_{k=0}^{m} r_{m-k} h_{v+k} = 0 \qquad (1 \leq v \leq q - n).$$

where $r_o = 1$, and r_k ($k = 1, 2, \ldots, m$) are functions of y_k ($k = 1, 2, \ldots, m$) as defined in Eqs. (4.11).

It will be shown that if Eq. (4.13) is valid for $n = m$ then also

$$\sum_{k=0}^{m+1} r'_{m+1-k} h_{v+k} = 0 \quad (1 \leq v \leq q - n)$$

where $r'_o = 1$ and r'_k ($k = 1, 2, \ldots, m+1$) are functions of y_k ($k = 1, 2, \ldots, m+1$) as defined in Eqs. (4.11).

It will be shown, moreover, that Eq. (4.13) is satisfied for $n = 1$. This will then complete the proof of Theorem 1.

The coefficients r_k ($k = 1, 2, \ldots, m$) are functions of y_k ($k = 1, 2, \ldots, m$) defined in Eqs. (4.11). $r_o = 0$. The m values y_k (i.e., $y_1, y_2, \ldots, y_m$) are the zeros of $P_m(y)$, an algebraic equation of mth degree.

If $$P_m(y) = y^m + r_1 y^{m-1} + \dots + r_m , \qquad (4.14)$$

let $P_{m+1}(y)$ be defined as

$$P_{m+1}(y) = P_m(y)(y-y_{m+1}) .$$

Then $y_1, y_2, \dots, y_{m+1}$ are the zeros of $P_{m+1}(y)$.

Let r denote the functions in Eqs. (4.11) of m roots ($y_1, y_2, \dots, y_m$) and r' denote the functions in Eqs. (4.11) of m+1 roots ($y_1, y_2, \dots, y_{m+1}$). Then,

$$\begin{aligned}
r'_o &= 1 && && = r_o \\
r'_1 &= -\sum_{k=1}^{m} y_k && -y_{m+1} && = r_1 - y_{m+1} \\
r'_2 &= \sum_{\substack{k_1=1 \\ n\geq k_2>k_1}}^{m} y_{k_1} y_{k_2} && -y_{m+1}\sum_{k=1}^{m} y_k && = r_2 - y_{m+1} r_1 \\
r'_3 &= -\sum_{\substack{k=1 \\ n\geq k_3>k_2>k_1}}^{m} y_{k_1} y_{k_2} y_{k_3} && -y_{m+1}\sum_{\substack{k=1 \\ k_2>k_1}}^{m} y_{k_1} y_{k_2} && = r_3 - y_{m+1} r_2 \qquad (4.15) \\
&\dots\dots && && \\
r'_m &= && && = r_m - y_{m+1} r_{m-1} \\
r'_{m+1} &= && && = -y_{m+1} r_m \;\dagger
\end{aligned}$$

---

† This last relationship is not apparent but follows if one considers both cases, m even and m odd. For m even: $r_m = \prod_{k=1}^{m} y_k$, hence $r'_{m+1} = -y_{m+1} r_m$. For m odd $r_m = -\prod_{k=1}^{m} y_k$ and $r'_{m+1} = -y_{m+1} r_m$.

Then
$$\sum_{k=0}^{m+1} r'_{(m+1)-k}\, h_{v+k} =$$

$$= r'_{m+1}\, h_v + r'_m\, h_{v+1} + \cdots + r'_1\, h_{v+m} + h_{v+m+1}. \qquad (4.16)$$

Substitution of Eqs. (4.15) into Eq.(4.16) yields,

$$\sum_{k=0}^{m+1} r'_{(m+1)-k}\, h_{v+k} =$$

$$= (-y_{m+1}r_m)\, h_v + (r_m - y_{m+1}r_{m-1})\, h_{v+1} + \cdots$$

$$+ (r_2 - y_{m+1}r_1)\, h_{v+m-1} + (r_1 - y_{m+1})\, h_{v+m} + h_{v+m+1}$$

$$= (-y_{m+1})(r_m h_v + r_{m-1}h_{v+1} + \cdots r_1 h_{v+m-1} + h_{v+m})$$

$$+ r_m h_{v+1} + \cdots + r_2 h_{v+m-1} + r_1 h_{v+m} + r_o h_{v+m+1}, \qquad (4.17)$$

or

$$\sum_{k=0}^{m+1} r'_{(m+1)-k}\, h_{v+k} = I_1 + I_2 =$$

$$= -y_{m+1} \sum_{k=0}^{m} r_{m-k} h_{v+k} + \sum_{k=0}^{m} r_{m-k}\, h_{(v+1)+k}. \qquad (4.18)$$

Now I_1, the first expression on the right side of Eq. (4.18), is zero by the assumption. I_2, the second expression, is equal to:

$$I_2 = r_m \sum_{k=1}^{m} (z_{kv}y_k) + r_{m-1} \sum_{k=1}^{m} (z_{kv}y_k + \cdots$$

$$+ r_1 \sum_{k=1}^{m} (z_{kv}y_k)y_k^{m-1} + \sum_{k=1}^{m} (z_{kv}y_k)y_k^{m}. \qquad (4.19)$$

Let $z_{kv}y_k = z'_{kv}$

and $h'_v = \sum_{k=1}^{m} z'_{kv}$.

Then

$$I_2 = \sum_{k=0}^{m} r_{m-k} h'_{v+k} \qquad (4.20)$$

is zero, since by the assumption Eq. (4.14) is valid for all sets z_{kv} ($k = 1, 2, \ldots, n$; $1 \leq v \leq q-n$), hence also for $z'_{kv} = z_{kv} y_k$ ($k = 1, 2, \ldots, n$; $1 \leq v \leq q-n$). Consequently, it was shown that if:

$$\sum_{k=0}^{m} r_{m-k} h_{v+k} = 0, \qquad (1 \leq v \leq q-n) \qquad (4.21)$$

then also

$$\sum_{k=0}^{m+1} r'_{(m+1)-k} h_{v+k} = 0 \qquad (1 \leq v \leq q-n) \qquad (4.22)$$

where

$$r'_k = r_k - y_{m+1} r_{k-1} \qquad (k = 0, 1, \ldots, m+1) \qquad (4.23)$$

($r_{-1} = r_{m+1} = 0$).

Therefore, it was proved that if Eq. (4.13), is valid for the m roots of $P_m(y)$, then it is also valid for the m+1 roots of $P_{m+1}(y)$. It will be shown now that Eq. (4.13) is valid for n = 1, which will complete the proof of Theorem 1.

Expanding Eq. (4.10) for n = 1 yields

$$r_1 h_v = -y_1 z_{1v}$$

$$h_{v+1} = y_1 z_{1v} \qquad (4.24)$$

Hence

$$\sum_{k=0}^{1} r_{1-k} h_{v+k} = 0. \qquad (1 \leq v \leq q-n) \qquad (4.25)$$

This completes the proof of Theorem 1.

Equation (4.13) gives a relation between (h_v, h_{v+1}, $\ldots$, h_{v+n}) and (r_o, r_1, $\ldots$, r_n). Through an increase in the index v, from v to v+1, Eq. (4.13) will relate $(h_{v+1}, h_{v+2}, \ldots, h_{v+n+1})$ with $(r_o, r_1, \ldots, r_n)$. If one has q equally-spaced values of h(t) (e.g., $h_1, h_2, \ldots, h_q$), then Eq. (4.13) will give rise to q-n equations. If q-n = p, one can write in general

$$\sum_{k=0}^{n} r_{n-k} h_{v+k} = 0 \quad (v = 1, 2, \ldots, p) \qquad (4.26)$$

This allows one to differentiate among three cases.

Case 1: $n > p$ - undetermined system.

Case 2: $n = p$ - determined system.

Case 3: $n < p$ - overdetermined system.

Case 1 implies that there are more terms chosen for the approximating function, Eq. (4.3) than justified by the available points of data. If n-p = m, then there are m conditions that can be fulfilled arbitrarily. Since, in general, an economy of elements is desired, and these are directly related to the number of terms in Eq. (4.3), Case 1 is of little practical interest, and will be commonly reduced to Case 2 [which can be simply accomplished by requiring the Eq. (4.3) to have p terms only].

Case 2 will theoretically occur whenever essentially no approximation error can be tolerated at the given points. Case 2 provides the theoretical optimum, or best approximation, for a given number of points. Since the elements used for synthesis are not ideal, it is of little value to talk about zero synthesis error.† One may wonder whether with fewer elements [i.e., fewer terms in Eq. (4.3)] one may not have at times a smaller synthesis error than one would

† By synthesis error the overall error is meant, i.e., the error due to approximation in addition to the error caused through use of physical elements.

obtain with more elements (even though the approximation error is reduced). Also, in general, for the sake of economy, one would like to have the maximum error that can be tolerated, since, by and large, a design allowing a greater error is less costly than one allowing a smaller error. All this reduces to the fact that one would like to design with as few elements as possible, and therefore, in the typical case, Case 3 is of most importance and interest.

Case 3 gives rise to an overdetermined system of equations. The characteristics of such a system are such that, in general, no equation will be solved exactly [i.e., the right side of Eqs. (4.26) will be different from zero]. Since this case is of most practical interest, a theory treating it will be developed in detail in the succeeding section of this work.

4.2.2 Solution of Overdetermined Systems of Equations by Means of Discrete Tschebyscheff Approximation. In this section the overdetermined system of equations obtained in Eqs. (4.26), will be solved by means of discrete Tschebyscheff approximations. The theory of discrete Tschebyscheff approximations to overdetermined systems will be reviewed. It will be shown that this theory provides a solution to Eqs. (4.26). The right side of Eqs. (4.26) will, in general, be different from zero, and comprises an error. This error will be minimized in Tschebyscheff sense.

The mathematical theory of the applications of discrete Tschebyscheff approximations to overdetermined systems has been treated by Vallee-Poussin [28]. More recent works in this area are due to Collatz [5] and Stiefel [23]. The following review of the theory of discrete Tschebyscheff approximations to overdetermined systems follows closely along lines of Stiefel [23]. The development is in a form suitable for solution of the problem stated.

4.2.2.1 The Theory of Tschebyscheff Approximations to Overdetermined Systems of Equations. The theory of discrete Tschebyscheff approximations offers a solution to an overdetermined system of equations. If there are p equations such as Eqs. (4.26), in n unknowns $r_1, r_2, \ldots, r_n$ and if $p > n$, then, in general, one can not satisfy all p equations simultaneously. Hence, any choice of values for $r_1, r_2, \ldots, r_n$, will cause an error different from zero on the right side of Eqs. (4.26). The discrete Tschebyscheff approximation theory provides a means for finding those values for the unknowns $r_1, r_2, \ldots, r_n$, which will minimize the magnitude of the maximum error on the right side of Eqs. (4.26).

The process of finding the desired values for $r_1, r_2, \ldots, r_n$ consists of a number of cycles. Each cycle is composed of the following four steps:

(1) A set of n+1 equations for the n unknowns $r_1, r_2, \ldots, r_n$ is selected out of the p given equations. This set is called a reference.

(2) The Tschebyscheff error for the selected reference is computed.

(3) A set of values $r_1, r_2, \ldots, r_n$ is obtained corresponding to the reference.

(4) Errors for the p equations are obtained.

These four steps complete the cycle.

If the error for any of the p equations does not exceed the reference error, the set of $r_1, r_2, \ldots, r_n$ computed in (3) is the desired one. If there is an equation which has an error larger than the reference error, then this equation must replace one of the equations of the reference. A replacement process is discussed which provides definite rules determining the equation to be replaced. The n remaining equations of the old reference and the new equation form a new reference, thus providing step (1) for a new cycle.

It is shown that the process is convergent and that regardless of the initial choice of reference the process terminates always with the same values for $r_1, r_2, \ldots, r_n$. The details of the theory will follow below.

Case 3 gives rise to an overdetermined system of equations for n unknowns $r_1, r_2, \ldots, r_n$ $(r_o = 1)$.

Hence,

$$h_v r_n + h_{v+1} r_{n-1} + \cdots + h_{v+n-1} r_1 + h_{v+n} = 0 \qquad (4.27)$$

$$(v = 1, 2, \ldots, p)$$

where

$$p > n.$$

One can interpret the system of equations in Eqs. (4.27) geometrically by considering every point P $(r_1, r_2, \ldots, r_n)$ to be a point in n-dimensional Euclidean space R^n. Since $p > n$, there is, in general, no point in R^n with coordinates which will satisfy Eqs. (4.27) for all v $(v = 1, 2, \ldots, p)$. If the coordinates of an arbitrary point P are substituted into Eqs. (4.27), then in general, there will be an error on the right side of some equations of Eqs. (4.27), rather than zero. If this error is denoted ϵ_v then,

$$\epsilon_v = h_v r_n + h_{v+1} r_{n-1} + \cdots + h_{v+n-1} r_1 + h_{v+n}$$

$$(v = 1, 2, \ldots, p) \qquad (4.28)$$

The Tschebyscheff approximation problem consists then of finding a point P such that

$$\text{Max } |\epsilon_v| \text{ is a minimum} \qquad (v = 1, 2, \ldots, p) .$$

The point P of the best approximation is named the T-point (the Tschebyscheff point).

Let us introduce the concept of a reference. The term reference shall denote the choice of (n+1) hyper-planes from the p given hyper-planes $E_1, E_2, \ldots, E_p$ of the Euclidean space R^n [i.e., choice of (n+1) equations from the p equations in Eqs. (4.27)]. The greek letter index σ will denote a choice of n+1 numbers from the series 1, 2, ..., p. The reference will be denoted as $[E_\sigma]$.

Let

$$x_v = (h_v, h_{v+1}, \ldots, h_{v+n-1}) \qquad (v = 1, 2, \ldots, p)$$

be the normal vectors of the hyper-planes. Since the space is an n-dimensional one, there are only n independent vectors. Hence, n+1 vectors are linearly dependent, and there exists a set of numbers λ such that:

$$\sum_{(n+1)}' \lambda_\sigma x_\sigma = 0 . \tag{4.29}$$

The sign $\sum_{(n+1)}'$ denotes that only the selected (n+1) terms are summed. Equation (4.29) gives the dependence condition between the normal vectors. In addition, $\lambda_\sigma \neq 0$ for all values of σ, since otherwise the Euclidean space could not be n-dimensional.

A point P is denoted a reference point if for its residues ϵ_σ either

$$\text{sgn } \epsilon_\sigma = \text{sgn } \lambda_\sigma \qquad \text{for all } \sigma$$

or

$$\text{sgn } \epsilon_\sigma = -\text{sgn } \lambda_\sigma \qquad \text{for all } \sigma. \tag{4.30}$$

This condition can be interpreted geometrically. In three dimensional Euclidean space R^3 it characterizes the points inside the volume formed by the reference planes. In the general case, Eq. (4.30) limits the magnitude of the error $|\epsilon_\sigma|$. Thus a reference point in R^n can be considered to be located "inside" the volume formed by the hyper-planes.

An example will be worked out to illustrate the concepts introduced above. Let the space be two dimensional (n = 2), hence the hyper-planes are straight lines. Let the following five equations be given (p = 5):

$$\begin{aligned}
E_1:&\quad .5r_2 + 3r_1 + 1.5 &= 0\\
E_2:&\quad 3r_2 + 1.5r_1 + .5 &= 0\\
E_3:&\quad 1.5r_2 + .5r_1 - .5 &= 0\\
E_4:&\quad .5r_2 - .5r_1 + 2 &= 0\\
E_5:&\quad -.5r_2 + 2r_1 + 4 &= 0 \ .
\end{aligned}$$

If the reference is composed of E_3, E_4, and E_5, then the vectors normal to the reference are:

$$\begin{aligned}
x_3 &= (1.5, .5)\\
x_4 &= (.5, -.5)\\
x_5 &= (-.5, 2) \ .
\end{aligned}$$

By Eq. (4.29), there exists a set of numbers λ such that

$$\sum_{(3)}{}' \lambda_\sigma x_\sigma = 0 \ .$$

Hence,

$$1.5\lambda_3 + .5\lambda_4 - .5\lambda_5 = 0$$

and

$$.5\lambda_3 - .5\lambda_4 + 2\lambda_5 = 0$$

Let $\lambda_3 = 1$, then $\lambda_4 = -\frac{13}{3}$, and $\lambda_5 = -\frac{4}{3}$.

These concepts are illustrated in Fig. 4.2. The vectors x_v ($v = 1, 2, \ldots, 5$) are orthogonal to their corresponding planes (i.e., x_σ is orthogonal to E_σ). $\sum_{(3)}' \lambda_\sigma x_\sigma$ is a sum of three vectors. Since, $\sum_{(3)}' \lambda_\sigma x_\sigma = 0$, these vectors form a triangle.

From Fig. 4.2, one notices that the point P(-2,-2) is located inside the triangle formed by the reference planes E_3, E_4, E_5. Hence, P is a reference point, and its errors must satisfy one of the two conditions of Eq. (4.30). Substitution of $r_2 = -2$ and $r_1 = -2$, into E_3, E_4, and E_5, yields $\epsilon_3 = -4.5$, $\epsilon_4 = 2$, $\epsilon_5 = 1$. Hence, the condition $\operatorname{sgn} \epsilon_\sigma = -\operatorname{sgn} \lambda_\sigma$ is satisfied for all σ ($\sigma = 3, 4, 5$).

By Eq. (4.28)

$$\epsilon_\sigma = h_\sigma r_n + h_{\sigma+1} r_{n-1} + \ldots + h_{\sigma+n-1} r_1 + h_{\sigma+n} .$$

Therefore,

$$\lambda_\sigma \epsilon_\sigma = \lambda_\sigma (h_\sigma r_n + h_{\sigma+1} r_{n-1} + \ldots + h_{\sigma+n-1} r_1 + h_{\sigma+n}) .$$

Since,

$$x_\sigma = (h_\sigma, h_{\sigma+1}, \ldots, h_{\sigma+n-1}) =$$

$$= h_\sigma r_n + h_{\sigma+1} r_{n-1} + \ldots + h_{\sigma+n-1} r_1 ,$$

hence

$$\lambda_\sigma \epsilon_\sigma = \lambda_\sigma x_\sigma + \lambda_\sigma h_{\sigma+n} .$$

Due to Eq. (4.29),

$$\sum_{(n+1)}' \lambda_\sigma \epsilon_\sigma = \sum_{(n+1)}' \lambda_\sigma h_{\sigma+n} . \qquad (4.31)$$

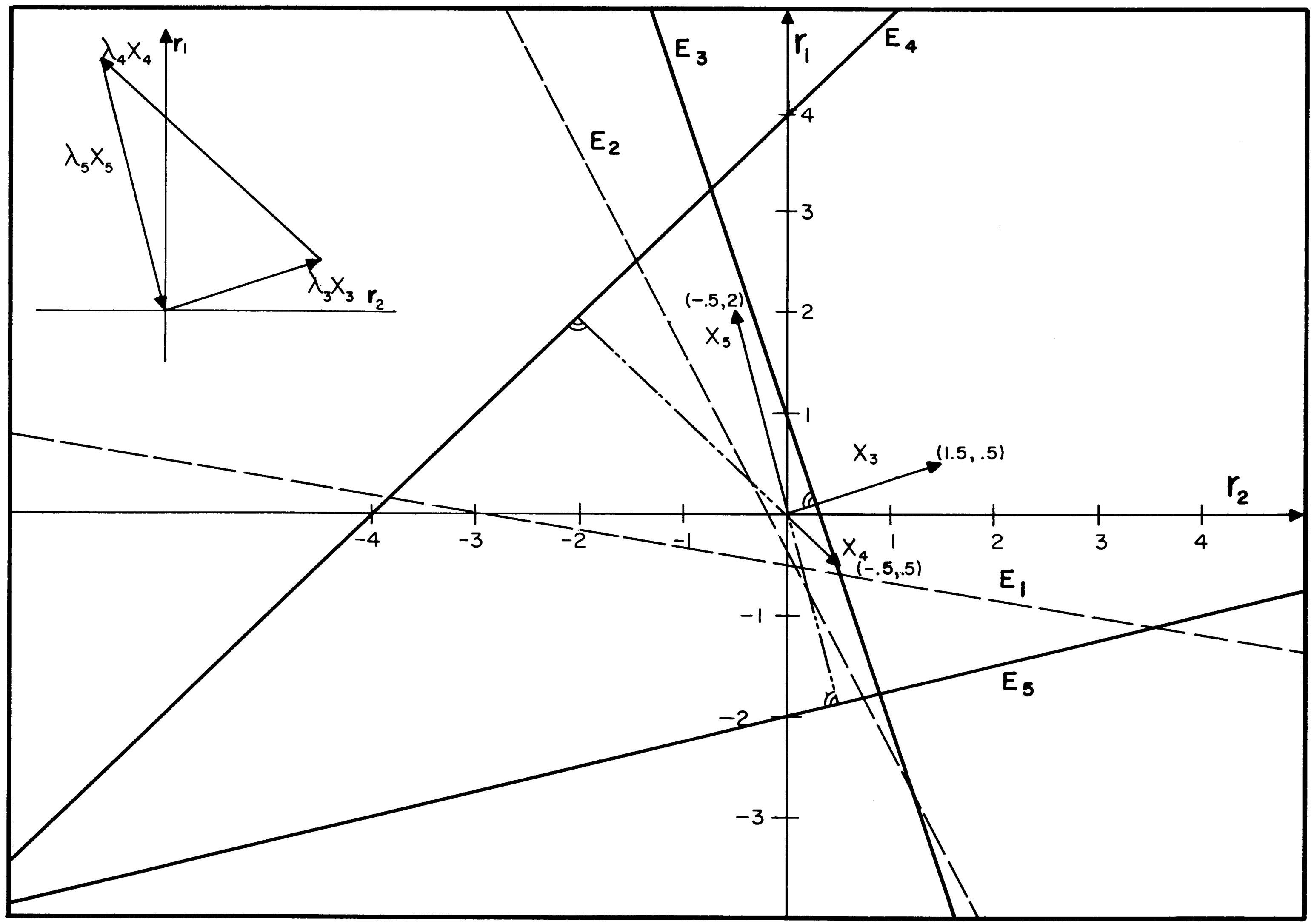


FIG. 4.2 GEOMETRICAL INTERPRETATION OF HYPER-PLANES AND NORMAL VECTORS

Because of the condition Eq. (4.30), either all $\lambda_\sigma \epsilon_\sigma$ are positive or all are negative.

Therefore,

$$\sum_{(n+1)}' |\lambda_\sigma| \cdot |\epsilon_\sigma| \quad = \quad \pm \sum_{(n+1)}' \lambda_\sigma \epsilon_\sigma \quad = \quad \pm \sum_{(n+1)}' \lambda_\sigma h_{\sigma+n} \qquad (4.32)$$

One shall denote by the term center of a reference that reference point all of which errors ϵ_σ have the same magnitude $|\epsilon|$. Hence,

$$\epsilon_\sigma \quad = \quad \epsilon(\mathrm{sgn}\lambda_\sigma) \quad . \qquad (4.33)$$

One can interpret $|\epsilon|$ to be a measure of distance necessary to bring the hyper-planes ϵ_σ to a mutual intersection. One can compute the error at the center from Eq. (4.31),

$$\sum_{(n+1)}' \lambda_\sigma \epsilon_\sigma \quad = \quad \sum_{(n+1)}' \lambda_\sigma h_{\sigma+n} \; ,$$

or

$$\epsilon \sum_{(n+1)}' \lambda_\sigma (\mathrm{sgn}\ \lambda_\sigma) \quad = \quad \sum_{(n+1)}' \lambda_\sigma h_{\sigma+n} \quad .$$

But

$$\lambda_\sigma (\mathrm{sgn}\ \lambda_\sigma) \quad = \quad |\lambda_\sigma| \; .$$

Hence,

$$\epsilon \quad = \quad \frac{\sum_{(n+1)} \lambda_\sigma n_{\sigma+n}}{\sum_{(n+1)}' |\lambda_\sigma|} \quad . \qquad (4.34)$$

By Eq. (4.32)

$$\sum_{(n+1)}' \lambda_\sigma h_{\sigma+n} \quad = \quad \pm \sum_{(n+1)}' |\lambda_\sigma| \cdot |\epsilon_\sigma| \quad .$$

Therefore

$$|\epsilon| \quad = \quad \frac{\sum_{(n+1)}' |\lambda_\sigma| \cdot |\epsilon_\sigma|}{\sum_{(n+1)}' |\lambda_\sigma|} \quad . \qquad (4.35)$$

From Eq. (4.35) it follows that

$$|\epsilon| \cdot \sum_{(n+1)}' |\lambda_\sigma| = \sum_{(n+1)}' |\lambda_\sigma| \cdot |\epsilon_\sigma| \ .$$

Let

$$\text{Min.} |\epsilon_\sigma| = |\epsilon_k| \ .$$

Then

$$|\epsilon| \cdot \sum_{(n+1)}' |\lambda_\sigma| \geqq |\epsilon_k| \cdot \sum_{(n+1)}' |\lambda_\sigma| \ .$$

Hence

$$|\epsilon| \geq \text{Min} \ |\epsilon_\sigma| \ .$$

Similarly

$$|\epsilon| \leq \text{Max} \ |\epsilon_\sigma| \ . \qquad (4.36)$$

This result is valid at every reference point and leads to the following theorem.

Theorem 2

The center of a reference has the property that its error satisfies Eq. (4.36) and is the T-point of the (n+1) reference hyper-planes.

One can prove the uniqueness of the T-point for a system in a general position (no hyper-planes are parallel to one another) by assuming that there exists another T-point, T', which has errors ϵ'_σ . By Eq. (4.36),

$$|\epsilon| = \text{Max} \ |\epsilon'_\sigma| \ .$$

The two T-points must satisfy the requirement that their maximum errors are equal. By Eq. (4.36)

$$|\epsilon| = \text{Max} \ |\epsilon'_\sigma| \ .$$

Then

$$|\epsilon'_\sigma| \leq |\epsilon| \ .$$

Let ϵ_σ denote the error of the T point relative to ϵ_σ. Therefore,

whenever, $\epsilon_\sigma \geq 0$ then $\epsilon_\sigma - \epsilon'_\sigma \geq 0$;

and

whenever $\epsilon_\sigma \leq 0$, then $\epsilon_\sigma - \epsilon'_\sigma \leq 0$.

If the first condition of Eq. (4.30) is satisfied, i.e., if

$$\operatorname{sgn} \epsilon_\sigma = \operatorname{sgn} \lambda_\sigma ,$$

then, whenever $\epsilon_\sigma \geq 0$, $\lambda_\sigma \geq 0$ and $\lambda_\sigma (\epsilon_\sigma - \epsilon'_\sigma) \geq 0$. If $\epsilon_\sigma \leq 0$, then $\lambda_\sigma \leq 0$, and $\lambda_\sigma(\epsilon_\sigma - \epsilon'_\sigma) \geq 0$. Hence, in both cases all $\lambda_\sigma(\epsilon_\sigma - \epsilon'_\sigma)$ will be positive.

If,

$$\operatorname{sgn} \epsilon_\sigma = - \operatorname{sgn} \lambda_\sigma ,$$

whenever $\epsilon_\sigma \geq 0$, then $\lambda_\sigma \leq 0$ and $\lambda_\sigma(\epsilon_\sigma - \epsilon'_\sigma) \leq 0$. If $\epsilon_\sigma \leq 0$, then $\lambda_\sigma \geq 0$, and $\lambda_\sigma(\epsilon_\sigma - \epsilon'_\sigma) \leq 0$. Thus, for the second sign condition (i.e., $\operatorname{sgn} \epsilon_\sigma = - \operatorname{sgn} \lambda_\sigma$), all $\lambda_\sigma(\epsilon_\sigma - \epsilon'_\sigma)$ will be negative.

The above shows that for any one of these four cases all the expressions $\lambda_\sigma(\epsilon_\sigma - \epsilon'_\sigma)$ will have same sign.

From Eq. (4.31),

$$\sum_{(n+1)}' \lambda_\sigma \epsilon_\sigma = \sum_{(n+1)}' \lambda_\sigma h_{\sigma+n} = \sum_{(n+1)}' \lambda_\sigma \epsilon'_\sigma .$$

Therefore,

$$\sum_{(n+1)}' \lambda_\sigma \epsilon_\sigma - \sum_{(n+1)}' \lambda_\sigma \epsilon'_\sigma = 0$$

and

$$\sum_{(n+1)}' \lambda_\sigma(\epsilon_\sigma - \epsilon'_\sigma) = 0 .$$

Since $\lambda_\sigma \neq 0$, and all terms in the summation have the same sign, it follows that $\epsilon_\sigma = \epsilon'_\sigma$. This shows that T and T' are the same point.

One sees, therefore, that the (n+1) hyper-planes in the R^n space which are in general position (no two planes are parallel to one another) have only one T-point, which is in the center. In the general case of (n+1) planes in R^n one could expect a convex polyhedron consisting of T-points.

It has been shown that, given a reference $[E_\sigma]$ consisting of (n+1) equations (i.e., hyper-planes), there exists a unique T-point (if the hyper-planes are in a general position) with its corresponding error ϵ. If the coordinates $r_1, r_2, \ldots, r_n$ of the T point are substituted into the remaining equations (other than the reference), one of the equations, E_i, may have an error whose magnitude exceeds $|\epsilon|$. Now, a new reference can be obtained consisting of E_i and n of the hyper-planes in the old reference. Hence, E_i "replaces" one of the hyper-planes of the reference. These ideas are expressed in the following theorem.

Theorem 3

Given a reference $[E_\sigma]$ and a corresponding reference point P, let E_i be an additional hyper-plane which is not in the reference. Then one can replace one of the (n+1) hyper-planes of $[E_\sigma]$ by E_i and obtain a new reference for which P is also a reference point.

Proof:

Assume that the given reference is $[E_\sigma]$ ($\sigma = 1, 2, \ldots, n+1$), and let E_{n+2} be the additional hyper-plane. Equation (4.29) yields

$$\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_{n+1} x_{n+1} = 0;$$

Since P is a reference point in the reference $[E_\sigma]$, hence, for its residues ϵ_σ the sign rules are valid. Consequently, either

$$\text{sgn } \epsilon_\sigma = \text{sgn } \lambda_\sigma \qquad (\sigma = 1, 2, \dots, n+1),$$

or

$$\text{sgn } \epsilon_\sigma = -\text{sgn } \lambda_\sigma .$$

Consider in the following the first of the two cases.

Since the system is n-dimensional, the unit vector x_{n+2} is linearly dependent upon the unit vectors in $[E_\sigma]$. Consequently, a relationship exists between the (n+1) unit vectors in $[E_\sigma]$ and x_{n+2}. Hence, numbers μ_k exist such that the relationship

$$\mu_1 x_1 + \mu_2 x_2 + \dots + \mu_{n+1} x_{n+1} + x_{n+2} = 0 \qquad (4.37)$$

can be assumed.

But

$$\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_{n+1} x_{n+1} = 0 .$$

Therefore:

$$\lambda_1 x_{n+2} + (\lambda_1 \mu_2 - \lambda_2 \mu_1) x_2 + \dots + (\lambda_1 \mu_{n+1} - \lambda_{n+1} \mu_1) x_{n+1} = 0$$

$$\lambda_2 x_{n+2} + (\lambda_2 \mu_1 - \lambda_1 \mu_2) x_1 + \dots + (\lambda_2 \mu_{n+1} - \lambda_{n+1} \mu_2) x_{n+1} = 0$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \qquad (4.38)$$

$$\lambda_{n+1} x_{n+2} + (\lambda_{n+1} \mu_1 - \lambda_1 \mu_{n+1}) x_1 + \dots + (\lambda_{n+1} \mu_n - \lambda_n \mu_{n+1}) x_n = 0$$

Case 1: The error ϵ_{n+2} of P relative to E_{n+2} is positive. Then, one shall replace that hyper-plane which is designated by the number given by

$$\text{Min}\ \frac{\mu_\sigma}{\lambda_\sigma} \quad (\sigma = 1, 2, \ldots, n+1).$$

One can note that no two quotients are equal. If this were not the case, if, say $\frac{\mu_\ell}{\lambda_\ell} = \frac{\mu_m}{\lambda_m}$, then $\mu_\ell\lambda_m - \mu_m\lambda_\ell = 0$, and one coefficient in the mth equation of Eq. (4.38) will vanish. But this will contradict the assumption that the system is n-dimensional (since if

$$\lambda_m x_{n+2} + a_1 x_1 + \ldots + a_{m-1}x_{m-1} + a_{m+1}x_{m+1} + \ldots$$

$$+ a_{\ell-1}x_{\ell-1} + a_{\ell+1}x_{\ell+1} + \ldots + a_{n+1}x_{n+1} = 0$$

the n-unit vectors are not linearly independent).

Assume that ℓ yields the minimum.

Then

$$\frac{\mu_\ell}{\lambda_\ell} < \frac{\mu_\sigma}{\lambda_\sigma} \quad (\sigma = 1,2,\ldots,\ell-1,\ell+1,\ldots,n+1)$$

Assume

$$\lambda_\ell > 0\ , \quad \lambda_k > 0$$

Then

$$\frac{\mu_\ell}{\lambda_\ell} < \frac{\mu_k}{\lambda_k}$$

$$\mu_\ell\lambda_k < \mu_k\lambda_\ell$$

$$\mu_k\lambda_\ell - \mu_\ell\lambda_k > 0 \quad \text{and } \lambda_\ell\lambda_k > 0\ .$$

Hence,

$$\text{sgn}\ (\lambda_\ell\mu_k - \mu_\ell\lambda_k) = \text{sgn}\ (\lambda_\ell\lambda_k)\ .$$

If

$$\lambda_\ell < 0, \quad \lambda_k > 0$$

then

$$\mu_\ell\lambda_k > \mu_k\lambda_\ell$$

$$\lambda_\ell\mu_k - \mu_\ell\lambda_k < 0 \quad \text{and } \lambda_\ell\lambda_k < 0\ .$$

Hence

$$\operatorname{sgn}(\lambda_\ell \mu_k - \mu_\ell \lambda_k) = \operatorname{sgn}(\lambda_\ell \lambda_k) .$$

It is obvious that the same is true for

$$\lambda_\ell > 0, \quad \lambda_k < 0 \text{ and } \lambda_\ell < 0, \quad \lambda_k < 0 .$$

Therefore,

$$\operatorname{sgn}(\lambda_\ell \mu_k - \mu_\ell \lambda_k) = \operatorname{sgn}(\lambda_\ell \lambda_k) .$$

Consider the ℓth equation in the set of Eqs. (4.38). This equation has the form

$$\lambda_\ell x_{n+2} + (\lambda_\ell \mu_1 - \lambda_1 \mu_\ell) x_1 + \dots + (\lambda_\ell \mu_{\ell-1} - \lambda_{\ell-1} \mu_\ell) x_{\ell-1} +$$

$$+ (\lambda_\ell \mu_{\ell+1} - \lambda_{\ell+1} \mu_\ell) x_{\ell+1} + \dots + (\lambda_\ell \mu_{n+1} - \lambda_{n+1} \mu_\ell) x_{n+1} = 0 .$$

Division of this equation by sgn λ_ℓ will yield the following relation for the signs:

$$+ , \operatorname{sgn} \lambda_1, \dots, \operatorname{sgn} \lambda_{\ell-1}, \operatorname{sgn} \lambda_{\ell+1}, \dots, \operatorname{sgn} \lambda_{n+1} .$$

But, by assumption,

$$\operatorname{sgn} \lambda_\sigma = \operatorname{sgn} \epsilon_\sigma$$

and

$$\epsilon_{n+2} \text{ relative to } E_{n+2} > 0 .$$

Hence the errors $\epsilon_{n+2}, \epsilon_1, \dots, \epsilon_{\ell-1}, \epsilon_{\ell+1}, \dots, \epsilon_{n+1}$ relative to $[E_\sigma]$ ($\sigma = 1, 2, \dots, \ell-1, \ell+1, \dots, n+2$) have the same sign as the corresponding λ. The new coefficients of Eq. (4.38) have the same signs as the corresponding ϵ_σ. The coefficient of x_{n+2} is positive, but, $\epsilon_{n+2} > 0$, hence P is a reference point and Theorem 3 is proved for this case.

Case 2: The error ϵ_{n+2} relative to E_{n+2} is negative. A discussion similar to the above will show that the plane with the number given by Max $\frac{\mu_\sigma}{\lambda_\sigma}$ must be replaced.

In the case that $\epsilon_{n+2} = 0$, P lies on E_{n+2} and one can consider it to be a special case of either case.

A consideration of the second sign rule (sgn ϵ_σ = -sgn λ_σ) will cause an exchange of max with min and vice versa.

The above gives rise to the following table for the number of the plane to be replaced.

TABLE II

RULES FOR THE HYPER-PLANE TO BE REPLACED

sgn ϵ_σ	ϵ_i †	Hyper-plane to Be Replaced
sgn λ_σ	> 0	Min $\frac{\mu_\sigma}{\lambda_\sigma}$
sgn λ_σ	< 0	Max $\frac{\mu_\sigma}{\lambda_\sigma}$
-sgn λ_σ	> 0	Max $\frac{\mu_\sigma}{\lambda_\sigma}$
-sgn λ_σ	< 0	Min $\frac{\mu_\sigma}{\lambda_\sigma}$

† relative to E_i.

An example will be worked out which will illustrate the replacement process. Let the space be 3-dimensional, hence $n = 3$. Let the following four equations be given:

$$E_1: \quad 2x + 3y + z = 1 \quad (I)$$

$$E_2: \quad x - 2y + 3z = -1 \quad (II)$$

$$E_3: \quad x + y - z = 5 \quad (III)$$

$$E_4: \quad x + 3y - 2z = 12 \quad (IV) \; .$$

Let the reference point be (2, 1, -3) and let E_5 be given by $-3x - 10y + 8z = -20$.

Then for $[E_\sigma]$, the normal vectors are

$$x_1 = (2, 3, 1)$$

$$x_2 = (1, -2, 3)$$

$$x_3 = (1, 1, -1)$$

$$x_4 = (1, 3, -2) \; .$$

From

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4 = 0 \; ,$$

one obtains

$$2\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$$

$$3\lambda_1 - 2\lambda_2 + \lambda_3 + 3\lambda_4 = 0$$

$$\lambda_1 + 3\lambda_2 - \lambda_3 - 2\lambda_4 = 0 \; .$$

Let $\lambda_1 = 5$, then $\lambda_2 = -7$, $\lambda_3 = 10$, $\lambda_4 = -13$.

Now

$$\epsilon_1 = 2(2) + 3(1) + 1(-3) - 1 = 3$$

$$\epsilon_2 = 1(2) - 2(1) + 3(-3) + 4 = -5$$

$$\epsilon_3 = 1(2) + 1(1) - 1(-3) - 5 = 1$$

$$\epsilon_4 = 1(2) + 3(1) - 2(-3) - 12 = -1 \; .$$

Therefore

$$\operatorname{sgn} \lambda_\sigma = \operatorname{sgn} \epsilon_\sigma \; .$$

Since

$$x_5 = (-3, -10, 8),$$

and by Eq. (4.37),

$$\mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3 + \mu_4 x_4 + x_5 = 0 ,$$

therefore,

$$\begin{aligned} 2\mu_1 + \mu_2 + \mu_3 + \mu_4 - 3 &= 0 \\ 3\mu_1 - 2\mu_2 + \mu_3 + 3\mu_4 - 10 &= 0 \\ \mu_1 + 3\mu_2 - \mu_3 - 2\mu_4 + 8 &= 0 . \end{aligned}$$

The last system of equations is undetermined. Since any exact solution will be satisfactory, one unknown can be assumed arbitrarily.

Let $\mu_4 = 0$, then $\mu_1 = 1$, $\mu_2 = -2$, $\mu_3 = 3$.

The error ϵ_5 is given by

$$\epsilon_5 = (-3)(2) - 10(1) + 8(-3) + 20 = -10 .$$

Since $\epsilon_i < 0$, sgn λ_σ = sgn ϵ_σ; therefore one must replace that equation which is designated by the number given by Max $\frac{\mu_\sigma}{\lambda_\sigma}$.

The $\frac{\mu_\sigma}{\lambda_\sigma}$ are

$$\frac{\mu_1}{\lambda_1} = \frac{1}{5}; \frac{\mu_2}{\lambda_2} = \frac{-2}{-7}; \frac{\mu_3}{\lambda_3} = \frac{3}{10}; \frac{\mu_4}{\lambda_4} = \frac{0}{-13} .$$

Since Max $\frac{\mu_\sigma}{\lambda_\sigma} = \frac{3}{10} = \frac{\mu_3}{\lambda_3}$, hence E_3 is to be replaced.

The above example illustrates the process of finding the equation to be replaced. With the aid of this process the problem of an overdetermined system of equations can be solved. If p equations in n unknowns are given one will choose an arbitrary reference $[E_\sigma]$ and compute its center T_1, and its error $\hat{\epsilon}_1$ (note that the error of T_1 relative to any equation in $[E_\sigma]$ is $\hat{\epsilon}_1$). Then all the errors ϵ_i of T_1 relative to the remaining equations are calculated. If all these errors ϵ_i satisfy the condition $|\epsilon_i| \leq |\hat{\epsilon}_1|$, one has the solution of the problem and the process terminates. If there exists one equation for which $|\epsilon_i| > |\epsilon|$, the above-illustrated replacement process should be applied. One equation is replaced by the new one, thereby forming a new reference. The new center T_2, and the new error $\hat{\epsilon}_2$ are then computed. It is to be noted that for the second reference the absolute value of the error is larger than previously (i.e., $|\hat{\epsilon}_2| > |\hat{\epsilon}_1|$). The process is now repeated. It is to be observed that the error increases monotonically (i.e., $|\hat{\epsilon}_g| > |\hat{\epsilon}_{g-1}|$), hence, this insures that the process will terminate after a finite number of steps, since, under the condition on errors of monotonic growth, the same reference cannot be used twice. When the process terminates, one will have the final reference center T and the final error ϵ. Then obviously

$$|\epsilon| \geq |\epsilon_v| \quad (v = 1, 2, \ldots, p), \qquad (4.39)$$

and ϵ_v is taken relative to its corresponding hyperplanes. Obviously any point P has, by Eq. (4.36), errors ϵ'_σ relative to the last reference such that

$$|\epsilon| \leq \text{Max } |\epsilon'_\sigma| \quad .$$

But

$$\text{Max } |\epsilon'_\sigma| \leq \text{Max } |\epsilon'_v| \quad (v = 1, 2, \ldots, p) \quad .$$

Therefore

$$|\epsilon| \leq \text{Max } |\epsilon'_v| \quad (v = 1, 2, \ldots, p) \,. \qquad (4.40)$$

This can be stated as a Theorem.

Theorem 4

The last reference center is the T-point of the p-equations.

It is to be noted that the last center has the property that the magnitude of any of its p errors is at most equal to the magnitude of ϵ, the error of the last ereference.

The uniqueness of the T-point can be shown by assuming the existence of another T-point T', with error ϵ'.

Then

$$\text{Max } |\epsilon_k'| = |\epsilon| \quad (k = 1, 2, \ldots, p) .$$

But for any point P,

$$\text{Max } |\epsilon_\sigma| \geq |\epsilon|$$

where σ denotes the equations of the last reference (with center at T), and

$$\text{Max } |\epsilon_k'| \geq \text{Max } |\epsilon_\sigma'| \quad (k = 1, 2, \ldots, p).$$

Therefore,

$$\text{Max } |\epsilon_\sigma| \geq \text{Max } |\epsilon_\sigma'| ,$$

and T' is also a T-point of the last reference. But uniqueness of the T-point in a reference was proved already. Hence T' and T are the same point.

The above shows that the replacement process may be initiated with any reference and will always yield the same T-point. It also shows that the last error $|\epsilon|$ is the sought approximation error.

4.2.2.2 Application to Solution of Overdetermined Systems of Equations. The theory of the preceding section can be applied to find those values for the unknowns $r_1, r_2, \ldots, r_n$ which will minimize the magnitude of the maximum error on the right side of Eqs. (4.26). The magnitude of the maximum error, corresponding to $r_1, r_2, \ldots, r_n$ can also be found.

Apparently, the coordinates of the T-point, in the Euclidean space R^n, have the desired property. Hence the coordinates of the T-point are the desired values for $r_1, r_2, \ldots, r_n$.

The coordinates of the T-point can be determined from the last reference. By Eqs. (4.28),

$$\epsilon_v = h_v r_n + h_{v+1} r_{n-1} + \cdots + h_{v+n-1} r_1 + h_{v+n}$$

$$(v = 1, 2, \ldots, p).$$

For the last reference the error ϵ can be determined from Eq. (4.34) as

$$\epsilon = \frac{\sum\limits_{(n+1)}' \lambda_\sigma h_{\sigma+n}}{\sum\limits_{(n+1)}' |\lambda_\sigma|} ,$$

where σ represents the last reference. The error of any equation of the reference is given by Eq. (4.33) as

$$\epsilon_\sigma = \epsilon(\text{sgn}\ \lambda_\sigma) .$$

Therefore,

$$h_\sigma r_n + h_{\sigma+1} r_{n-1} + \cdots + h_{\sigma+n-1} r_1 + h_{\sigma+n} - \epsilon\ \text{sgn}\ \lambda_\sigma = 0, \quad (4.41)$$

where σ represents the n+1 equations of the last reference. n of the n+1 equations of the reference will be used to find $r_1, r_2, \ldots, r_n$.

The magnitude of maximum error in Eqs. (4.28) is $|\epsilon|$. Hence, one has found the desired values for $|\epsilon|$ and $r_1, r_2, \ldots, r_n$, thus solving the overdetermined system of equations.

4.2.3 Formulas for Optimum Poles. With the method of the previous section by approximate values for the functions of the roots $r_1, r_2, \ldots, r_n$ were found. By the fundamental theorem of algebra the roots of

$$y^n + r_1 y^{n-1} + \dots + r_n = 0 \tag{4.42}$$

are

$$y_1, y_2, \dots, y_n .$$

From these values one can find the exponents of the approximating function, which are the pole positions.

By Eq. (4.8),

$$e^{s_k d} = y_k \quad (k = 1, 2, \dots, n).$$

Hence,

$$s_k d = \ell n \; y_k ,$$

or

$$s_k = \frac{1}{d} \ell n \; y_k , \tag{4.43}$$

which yields the desired pole positions.

If y_k is a complex root, then, since the algebraic equation has real coefficients, there must exist a conjugate complex root. Consequently, if:

$$y_k = \gamma_k + j\delta_k = \sqrt{\gamma_k^2 + \delta_k^2} \; e^{j(\text{arc tan} \frac{\delta_k}{\gamma_k})}$$

and

$$y_{k+1} = \gamma_k - j\delta_k = \sqrt{\gamma_k^2 + \delta_k^2} \; e^{j(\text{arc tan} \frac{-\delta_k}{\gamma_k})} ,$$

then

$$s_k = \frac{1}{d} (\ell n \sqrt{\gamma_k^2 + \delta_k^2} + j \text{ arc tan} \frac{\delta_k}{\gamma_k}) =$$

$$= \frac{1}{2d} \ell n \; (\gamma_k^2 + \delta_k^2) + j \frac{1}{d} \text{ arc tan} \frac{\delta_k}{\gamma_k} . \tag{4.44}$$

Similarly,

$$s_{k+1} = \frac{1}{d} (\ell n \sqrt{\gamma_k^2 + \delta_k^2} + j \text{ arc tan} \frac{-\delta_k}{\gamma_k})$$

$$= \frac{1}{2d} \ell n(\gamma_k^2 + \delta_k^2) - j \frac{1}{d} \text{ arc tan} \frac{\delta_k}{\gamma_k} .$$

It is to be noted that the principal value of the arc tan has been listed in Eqs. (4.44). One could also use for the imaginary part of Eq. (4.44) the expression $\frac{1}{d}$ (arc tan $\frac{\delta_k}{\gamma_k}$ + $2\ell\pi$),

where

$$\ell = 0, \pm 1, \pm 2, \ldots \quad .$$

The approximation apparently will be unaffected when the conjugate complex poles are moved vertically a distance $\pm \frac{2\ell\pi}{d}$.

If y_k is a negative root then the pole is again complex, as it is shown below.

$$y_k = -|y_k| = |y_k| \, e^{j\pi}$$

$$s_k = \frac{1}{d} (\ell n \, |y_k| - j\pi) \quad .$$

Since also $-|y_k| = |y_k| e^{-j\pi}$, one can have also the representation

$$s_k = \frac{1}{d} [\ell n \, |y_k| - j\pi] \; .$$

Therefore, in case of negative y_k one can use the conjugate complex pair in the approximating function.†

If

$$y_k = 0 \; ,$$

then

$$s_k = -\infty$$

and

$$A_k e^{-s_k t} = 0 \quad .$$

Hence, in this case the term containing this pole can be eliminated from the approximating function. These results are listed in Table III.

---

† It is to be noted that here, too, the principal value has been used. In general, $s_k = \frac{1}{d} \ell n \, |y_k| + j \frac{1}{d} (2\ell + 1)\pi$ ($\ell = 0, \pm 1, \pm 2, \ldots$).

TABLE III

FORMULAS FOR POLES

y_k	s_k	Comments
> 0	$\frac{1}{d} \ln y_k$	
< 0	$\frac{1}{d} \ln\lvert y_k\rvert + j \frac{1}{d} \pi$	introduce an additional pole at $\frac{1}{d} \ln\lvert y_k\rvert - j\pi$
$= 0$	$- \infty$	eliminate the term containing this pole from the approximating function
$y_k + j\delta_k$	$\frac{1}{2d} \ln(\gamma_k^2 + \delta_k^2) + j \frac{1}{d} \arctan \frac{\delta_k}{\gamma_k}$	

The requirement of stability demands that Re $(s_k) \leqq 0$. This requirement restricts (y_k) to the unit circle. In general, if the ordinates h_v were taken to include the decaying part of h(t), then this requirement should reflect itself in decaying exponentials.† If this is not the case, one

† The requirement of stability demands that $\int_0^\infty |h(\tau)| d\tau$ be finite. This follows from: $|e_o(t)| = | \int_0^\infty h(\tau) e_1 (t-\tau) d\tau|$ [where $e_o(t)$ is the response, and $e_i(t)$ is the input]; then $|e_o(t)| \leq \int_0^\infty |h(\tau)| |e_i(t-\tau)| d\tau \leq M \int_0^\infty |h(\tau)| d\tau$, where M is the maximum of $e_i(t)$. Now, if for bounded $e_i(t)$, $e_o(t)$ is to be bounded, then $\int_0^\infty |h|(\tau)| d\tau < \infty$.

can always go back and include more decaying terms (more equations for decaying ordinates). Also, in many instances it has been found that if there were a pole with a positive real part, the residue proved to be very small, thus providing a negligible contribution to the approximating function and enabling one to neglect this pole altogether.

It is of interest to know whether any poles in the right-hand plane are to be expected, prior to the solution of the algebraic equation for y_k $(k = 1, 2, \ldots, n)$, which is probably the most tedious step in the process. Since whenever all y_k are in the unit circle all poles will be in the left-hand plane, it is sufficient to determine whether all y_k are inside the unit circle. This can be done by transforming the interior of the unit circle in the y-plane into the left-hand w-plane, and applying the Routh's criteria [3] to the transformed polynomial, to determine whether it is a Hurwitz polynomial. The Routh's criteria will also reveal which constants r_k $(k = 1, 2, \ldots, n)$ contribute to the instability, and by how much one has to adjust them in order to bring the poles to the left-hand plane (although the algebraic expressions are difficult to handle).

A transformation which maps the interior of the unit circle into the left-hand plane is the bilinear fractional transformation [4]. In particular, one can map the unit circle in the y-plane into the imaginary axis in the w-plane as shown in Fig. 4.3.

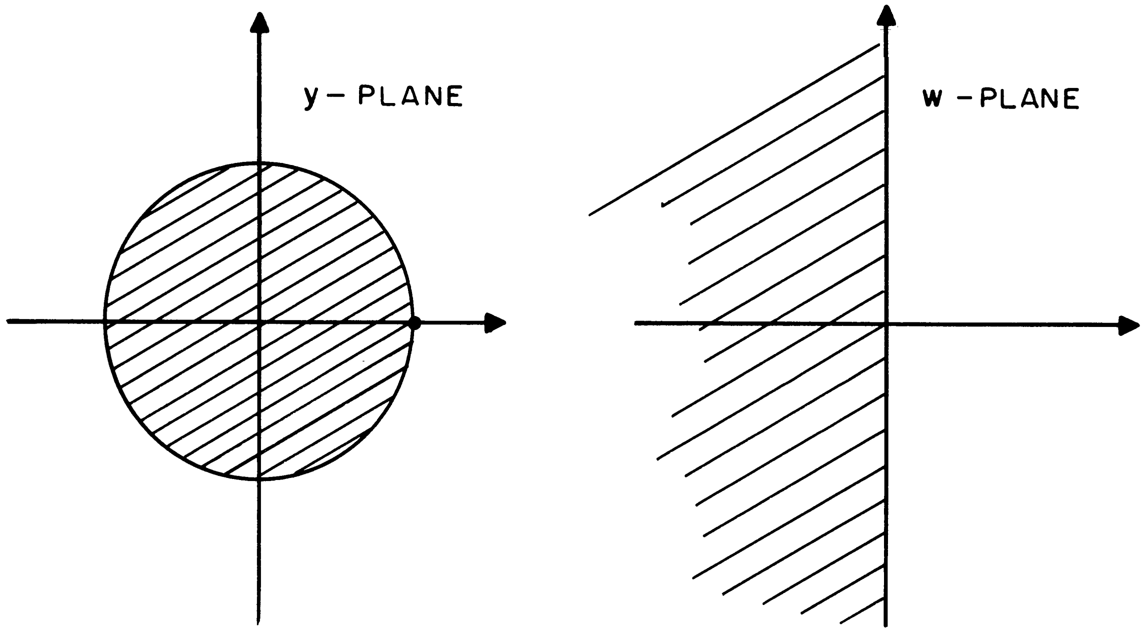


FIG. 4.3 TRANSFORMATION OF THE INTERIOR OF THE UNIT CIRCLE INTO THE LEFT-HAND PLANE

If one requires that when

$$y_1 = j, \qquad w_1 = j;$$
$$y_2 = -j, \qquad w_2 = -j;$$
$$y_3 = 1, \qquad w_3 = 0.$$

Then from

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(y - y_1)(y_2 - y_3)}{(y - y_3)(y_2 - y_1)} ,$$

one obtains

$$\frac{(w - j)(-j)}{w(-j - j)} = \frac{(y - j)(-j - 1)}{(y - 1)(-j - j)} .$$

Hence,

$$w = \frac{y - 1}{y + 1} \quad (y \neq -1) \tag{4.45}$$

is the desired transformation. That this transformation maps the interior of the unit circle into the left-hand plane is seen from the fact that the point $y = 0$ maps into $w = -1$, i.e., the left-hand plane.

Solving for y one obtains

$$y = \frac{1 + w}{1 - w} \quad (w \neq 1) \quad . \tag{4.46}$$

If Eq. (4.46) is substituted into Eq. (4.42) the result is:

$$(1 + w)^n + r_1(1 + w)^{n-1}(1 - w) + \ldots + r_n(1 - w)^n = 0 \quad . \tag{4.47}$$

If one denotes the coefficient of r_k by $\hat{c}_k$ then,

$$\hat{c}_k = (1 + w)^{n-k} (1 - w)^k$$

$$= \left[\sum_{j=0}^{n-k} \binom{n-k}{j} w^{n-k-j} \right] \left[\sum_{j=0}^{k} \binom{k}{j} w^{k-j} \right] \quad (-1)^k$$

$$= \left[\binom{n-k}{0} w^{n-k} + \binom{n-k}{1} w^{n-k-1} + \ldots + 1 \right] \left[\binom{k}{0} w^k - \binom{k}{1} w^{k-1} + \ldots + (-1)^k \right] (-1)^k$$

$$= (-1)^k \sum_{j=0}^{n} \sum_{m=0}^{j} w^{n-j} \binom{k}{m} \binom{n-k}{j-m} (-1)^m \, .$$

Therefore, the coefficient of w^{n-j} in $\hat{c}_k$ is

$$(-1)^k \sum_{m=0}^{j} \binom{k}{m} \binom{n-k}{j-m} (-1)^m \, ,$$

and, consequently, the coefficient of w^{n-j} in Eq. (4.47) is

$$\sum_{k=0}^{n} (-1)^k r_k \left[\sum_{m=0}^{j} \binom{k}{m} \binom{n-k}{j-m} (-1)^m \right] ,$$

where $r_o = 1$.

Therefore, Eq. (4.47) can be written as:

$$\sum_{j=0}^{n} w^{n-j} \left\{ \sum_{k=0}^{n} (-1)^k r_k \left[\sum_{m=0}^{j} \binom{k}{m} \binom{n-k}{j-m} (-1)^m \right] \right\} = 0 \quad . \tag{4.48}$$

If the Routh's test is applied to Eq. (4.48), one will determine whether any y_k $(k = 1, 2, \ldots, n)$ lie outside the unit circle.

4.3 The Determination of Residues

In the previous section the poles have been determined by the method of discrete Tschebyscheff approximations to an overdetermined system of equations. It will be shown here that the same method can be applied to determine residues.

By Eq. (4.5) and Eq. (4.7)

$$h_m = h(t_m) = \sum_{k=1}^{n} A_k e^{s_k t_m} \quad .$$

Let

$$e^{s_k t_m} = b_{km} , \tag{4.49}$$

then

$$h_m = \sum_{k=1}^{n} b_{km} A_k \quad (m = 1, 2, \ldots, q) \quad . \tag{4.50}$$

Equation (4.50) represents a system of q equations. Since $q > n$, the system is overdetermined and can be solved by means of the replacement process. The resulting residues will then minimize the error in $h(t)$ in the Tschebyscheff sense at the t_m $(m = 1, 2, \ldots, q)$ points of the time intervals. Since the approximating functions have exponentially-decaying envelopes, it can be argued that for sufficiently small intervals, a good approximation at the interval points will yield a good approximation between the interval points. The application of this method shows that in general good approximations are obtained, and that the Tschebyscheff error $|\epsilon|$ (which is the error of the last reference) is a meaningful indicator of the overall maximum error to be expected.

If $s_\ell = \alpha_\ell + j\beta_\ell$ is a complex pole, then let $s_{\ell+1} = \alpha_\ell - j\beta_\ell = \bar{s}_\ell$. Hence $b_{\ell m} = \bar{b}_{\ell+1,m}$ and $A_\ell = \bar{A}_{\ell+1}$. If $A_\ell = a_\ell + jb_\ell$ then

$$b_{\ell m}A_\ell + b_{\ell+1,m}A_{\ell+1} = (a_\ell + jb_\ell)\, e^{s_\ell t_m} + (a_\ell - jb_\ell)\, e^{\bar{s}_\ell t_m} =$$

$$= 2e^{\alpha_\ell t_m}(a_\ell \cos \beta_\ell t_m - b_\ell \sin \beta_\ell t_m) = a'_{\ell m} a_\ell + b'_{\ell m} b_\ell \qquad (4.51)$$

where

$$a'_{\ell m} = 2e^{\alpha_\ell t_m} \cos \beta_\ell t_m$$

and

$$b'_{\ell m} = -2e^{\alpha_\ell t_m} \sin \beta_\ell t_m \, . \qquad (4.52)$$

It is somewhat simpler to solve equations for a_ℓ and b_ℓ rather than A_ℓ and $A_{\ell+1}$. Hence, if there are n poles, 2w of which are complex, then one may use the form

$$h_m = \sum_{k=1}^{n-2w} b_{km}A_k + \sum_{\ell=1}^{w} (a'_{\ell m} a_\ell + b'_{\ell + m} b_\ell) \qquad (4.53)$$

where

$$b_{km} = e^{s_k t_m}$$

$$a'_{\ell m} = 2e^{\alpha_\ell t_m} \cos \beta_\ell t_m$$

$$b'_{\ell + m} = -2e^{\alpha_\ell t_m} \sin \beta_\ell t_m \, . \qquad (4.54)$$

It should be noted that if desired the residues may also be determined by means of the least-square-error criteria [6,11]. The poles can be determined as before.

4.4 The Discussion of Errors in the Approximation Process

The purpose of this section is to discuss the errors of the approximation process. A comparison will be made between an approximation optimizing residues only and an approximation optimizing both poles and residues. A rough relationship will be developed between the errors in determination of pole locations and the errors at residues. It is to be noted that the final Tschebyscheff residue error is the error of the approximation.

From Eqs. (4.7),

$$h_m = \sum_{k=1}^{n} A_k e^{s_k t_m} \quad (m = 1, 2, \ldots, q). \quad (4.55)$$

Equations (4.55) form an overdetermined set and are, therefore, not satisfied exactly. The error of the approximation at the q points is:

$$\epsilon_m = h_m - \sum_{k=1}^{n} A_k e^{s_k t_m} \quad (m = 1, 2, \ldots, q) \quad (4.56)$$

Hence,

$$h_m - \epsilon_m = \sum_{k=1}^{n} A_k e^{s_k t_m} \quad (m = 1, 2, \ldots, q). \quad (4.57)$$

In Chapter IV, the relationship between r_k $(k = 0, 1, \ldots, n)$ and h_v $(v = 1, 2, \ldots, p)$ was derived as Theorem 1 ,

$$\sum_{k=0}^{n} r_{n-k} h_{v+k} = 0 \quad (v = 1, 2, \ldots, p) . \quad (4.58)$$

The right side of Eqs. (4.58) is, in general, different from zero, since Eqs. (4.55) are not satisfied exactly. If the error on the right side of Eqs. (4.58) is denoted by $\bar{\epsilon}_v$, then,

$$\bar{\epsilon}_v = \sum_{k=0}^{n} r_{n-k} h_{v+k} \quad (v = 1, 2, \ldots, p). \quad (4.59)$$

Equations (4.57) are satisfied exactly. Hence, if $(h_i - \epsilon_i)$ is substituted for h_i in Eqs. (4.58), the right sides of Eqs. (4.58) will be exactly zero.

Therefore,

$$\sum_{k=0}^{n} r_{n-k} (h_{v+k} - \epsilon_{v+k}) = 0 \quad (v = 1, 2, \ldots, p). \quad (4.60)$$

From Eqs. (4.60) it follows that,

$$\sum_{k=0}^{n} r_{n-k} h_{v+k} = \sum_{k=0}^{n} r_{n-k} \epsilon_{v+k} \quad (v = 1, 2, \ldots, p). \quad (4.61)$$

But by Eq. (4.59), the left side of Eqs. (4.61) is $\bar{\epsilon}_v$. Hence

$$\bar{\epsilon}_v = \sum_{k=0}^{n} r_{n-k} \epsilon_{v+k} \quad (v = 1, 2, \ldots, p). \quad (4.62)$$

Expansion of Eqs. (4.62) yields ($r_o = 1$),

$$\bar{\epsilon}_1 = r_n \epsilon_1 + r_{n-1} \epsilon_2 + \ldots + \epsilon_{n+1}$$

$$\bar{\epsilon}_2 = r_n \epsilon_2 + r_{n-1} \epsilon_3 + \ldots + \epsilon_{n+2}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\bar{\epsilon}_p = r_n \epsilon_p + r_{n-1} \epsilon_{p+1} + \ldots + e_{p+n} . \quad (4.63)$$

The above equations relate errors in the pole locations ($\bar{\epsilon}_v$) with the final errors (ϵ_m). It is seen from Eqs. (4.63) that if $\bar{\epsilon}_v$ ($v = 1, 2, \ldots, p$) and r_k ($k = 1, 2, \ldots, n$) are known, Eqs. (4.63) form an underdetermined set for the ($q = p+n$) unknowns ϵ_m ($m = 1, 2, \ldots, p+n$). Thus, there are infinitely many sets of values for ϵ_m ($m = 1, 2, \ldots, p+n$) which will satisfy Eqs. (4.63). The approximation process will permit one to find that set of values for ϵ_m (m = 1, 2, ..., p+n), which will be a minimum in the Tschebyscheff sense. However, sets of values for $\bar{\epsilon}_v$ (v = 1, 2, ..., p) and r_k (k = 1, 2, ..., n) different from the previous ones, may produce a different Tschebyscheff minimum set ϵ_m(m = 1, 2, ..., p+n). It is desired, of course, to determine those values for r_k and $\bar{\epsilon}_v$, which will yield the minimal set ϵ_m. However, the $\bar{\epsilon}_v$ (v = 1, 2, ..., p) are not independent, but are given by Eq. (4.59). Hence, one has freedom only in the choice of values for r_k (k = 1, 2, ..., n).

It follows from Eqs. (4.63) that an optimum choice for the r_k will be one which will cause all $\bar{\epsilon}_v$ to be zero. If all $\bar{\epsilon}_v$ are zero, all ϵ_m (m = 1, 2, ..., p+n) will be zero, too, thus yielding zero error. But, whenever,$p > n$ there does not exist, in general, a set of r_k (k = 1, 2, ..., n) which causes all $\bar{\epsilon}_v$ (v = 1, 2, ..., p) to be zero. Hence, the best choice is to select r_k (k = 1, 2, ..., n) so as to minimize $\bar{\epsilon}_v$ (v = 1, 2, ..., p). This has been done. In the process developed in Section 4.2, pole locations were selected which minimize $\bar{\epsilon}_v$ in the Tschebyscheff sense.

It is of interest to compare an approximation procedure which will optimize residues only with an approximation procedure optimizing both pole locations and residues. One may inquire, for example, how many terms must the approximation function have (i.e.; how large is n?) in both cases in order to produce zero error at the interval points t_m (m = 1, 2, ..., q). The difference between the number of terms in the approximating function (which is proportional to the number of elements in the network) required for both cases will provide a measure of comparison.

It follows from Eqs. (4.56) that for arbitrary pole positions $n = q$ is needed to cause all the ϵ_m $(m = 1, 2, \ldots, q)$ to be zero. However, if both pole positions and residues are optimized, Eqs. (4.63) indicate that $n = p$ is sufficient for all $\bar{\epsilon}_v$ $(v = 1, 2, \ldots, p)$ to be zero, and then all ϵ_m $(m = 1, 2, \ldots, q)$ can be made zero also. Since $p = q-n$, hence $n = \frac{q}{2}$ in the latter case, thus requiring only half as many terms as in an approximation optimizing the residues only.

A rough relationship will now be developed between the maximum $|\bar{\epsilon}_v|$ $(v = 1, 2, \ldots, p)$ and the maximum $|\epsilon_m|$ $(m = 1, 2, \ldots, q)$. This relationship will enable one to estimate the expected maximum error in the early stages of the computational process.

If the ℓth equation in Eqs. (4.63) is squared, one obtains,

$$\begin{aligned}\bar{\epsilon}_\ell^2 = {} & r_n^2\epsilon_\ell^2 + r_{n-1}^2\epsilon_{\ell+1} + \cdots + \epsilon_{\ell+n}^2 + 2r_n\epsilon_\ell\,(r_{n-1}\epsilon_{\ell+1} + \cdots + \epsilon_{\ell+n}) \\ & + 2r_{n-1}\epsilon_{\ell+1}(r_{n-2}\epsilon_{\ell+2} + \cdots + \epsilon_{\ell+n}) \\ & \cdots\cdots\cdots\cdots\cdots\cdots \\ & + 2r_1\epsilon_{\ell+n-1}\epsilon_{\ell+n} \quad . \end{aligned} \qquad (4.64)$$

If all equations in Eqs. (4.63) are squared and added, then,

$$\sum_{v=1}^{p} \bar{\epsilon}_v = r_n^2 \sum_{v=1}^{p} \epsilon_v^2 + r_{n-1}^2 \sum_{v=1}^{p} \epsilon_{v+1}^2 + \cdots + \sum_{v=1}^{p} \epsilon_{v+n}^2 \qquad (4.65)$$

$$+ \text{(cross-product terms)} \quad .$$

Since, in the cross-product terms, all ϵ_m $(m = 1, 2, \ldots, q)$ are equally likely to be positive as negative (since ϵ_m are points on the Tschebyscheff error curve), it can be argued that the cross-product terms contribute a small amount to the right side of Eq. (4.65). Then,

$$\sum_{v=1}^{p} \bar{\epsilon}_v^2 \approx r_n^2 \sum_{v=1}^{p} \epsilon_v^2 + r_{n-1}^2 \sum_{v=1}^{p} \epsilon_{v+1}^2 + \ldots + \sum_{v=1}^{p} \epsilon_{v+n}^2 \quad . \qquad (4.66)$$

Similarly, it can be argued that for sufficiently large p

$$\sum_{v=1}^{p} \epsilon_v^2 \approx \sum_{v=1}^{p} \epsilon_{v+k}^2 \quad (k = 1, 2, \ldots, n). \qquad (4.67)$$

The above means that it does not matter too much whether the squares of the p errors are summed from ϵ_1 to ϵ_p or from ϵ_{k+1} to ϵ_{p+k}.

If Eqs. (4.67) are substituted into Eq. (4.66) one obtains:

$$\sum_{v=1}^{p} \bar{\epsilon}_v^2 \approx (r_n^2 + r_{n-1}^2 + \ldots + 1) \sum_{v=1}^{p} \epsilon_v^2 \quad . \qquad (4.68)$$

Taking square roots on both sides of Eq. (4.68) yields,

$$\left(\sum_{v=1}^{p} \bar{\epsilon}_v^2 \right)^{\frac{1}{2}} \approx (r_n^2 + r_{n-1}^2 + \ldots + 1)^{\frac{1}{2}} \left(\sum_{v=1}^{p} \epsilon_v^2 \right)^{\frac{1}{2}} \quad . \qquad (4.69)$$

It can be observed, that if $\bar{\epsilon}_v$ vs. v is plotted, and a curve is drawn between the points, such a curve would be similar to a curve obtained from a plot of ϵ_v vs. v. Each curve will have n+1 ripples, but the curves will differ in amplitude. The amplitude of the $\bar{\epsilon}_v$ curve will be $\bar{\epsilon}_{max}$, and the amplitude of the ϵ_v curve will be ϵ_{max} . Thus, if $\left(\sum_{v=1}^{p} \bar{\epsilon}_v^2 \right)^{\frac{1}{2}}$ is equal to $\bar{\epsilon}_{max}$ times some constant k, then $\left(\sum_{v=1}^{p} \epsilon_v^2 \right)^{\frac{1}{2}}$ will be approximately equal to ϵ_{max} times the same constant k. This result can be expressed as,

$$\bar{\epsilon}_{max} \approx \epsilon_{max} \sqrt{r_n^2 + r_{n+1}^2 + \ldots + 1} \quad . \qquad (4.70)$$

Therefore,

$$\epsilon_{max} \approx \frac{\bar{\epsilon}_{max}}{\sqrt{r_n^2 + r_{n-1}^2 + \ldots + 1}} \quad . \qquad (4.71)$$

Equation (4.71) relates the error at the pole positions ($\bar{\epsilon}_{max}$) with the final error (ϵ_{max}). It can be obverved that there is a degree of proportionality between the maximum error in the pole locations and the final maximum error. Since at end of the first cycle of computation the maximum error obtained is larger than $\bar{\epsilon}_{max}$ one has an upper-bound estimate on the expected approximation error quite soon in the process. It should be noted, however, that Eq. (4.71) provides a rough estimate only, since several assumptions were made in its derivation. In a particular case, these assumptions may not be met; hence the final error may differ substantially from the one predicted by Eq. (4.71).

In summary, it has been shown that there are definite relationships between the errors in pole locations and the approximation errors (residue errors). Also, it has been shown (as was expected) that an optimization both of pole locations and of residues will produce better results (i.e, smaller error) than an optimization of residues only. Also, a rough relationship was developed between the maximum error in pole locations and the final maximum error.

4.5 Applications

In the preceeding sections of this chapter the impulse-response-approximation problem has been stated and solved. The proposed method consists first of determination of poles, and then of determination of residues. In this section two examples will be worked to illustrate the suggested process.

4.5.1 Determination of a Network with Impulse Response $\frac{1}{(1+t)^2}$.

In the first example considered, h(t) is given as

$$h(t) = \frac{1}{(1+t)^2} \qquad (4.72)$$

It is desired to find an R-L-C network, N, having an output voltage h(t) given by Eq. (4.72), when excited by an impulse-voltage input $\delta(t)$.

A plot of h(t) vs. t is presented in Fig. 4.4. A choice of the interval spacing d, and the number of interval points q, must be made. The choice of d is dictated by the behavior of h(t). Since h*(t), the approximation to h(t), is a sum of decaying exponentials and of decaying sinusoids, d must be selected in such a way as to prevent the error between the inverval points from exceeding the error at the interval points. As a simple guide for selecting d, one may imagine h(t) as being replaced by a series of straight line segments whose end-points are d seconds apart, and coincide with h(t). (See Fig. 3.3).

The time interval of approximation (i.e., the time necessary for $|h(t)|$ to reach a small fraction of its maximum magnitude) is equal to (q-1)d. Hence both d and q are obtained through an examination of the plot of h(t). An examination of Fig. 4.4 indicates that a choice of 0.5 for the interval spacing d, and the choice of q = 9 interval points are reasonable ones. These choices result in the values of h_m (m = 1, 2, ..., 9) at interval points listed in Table IV.

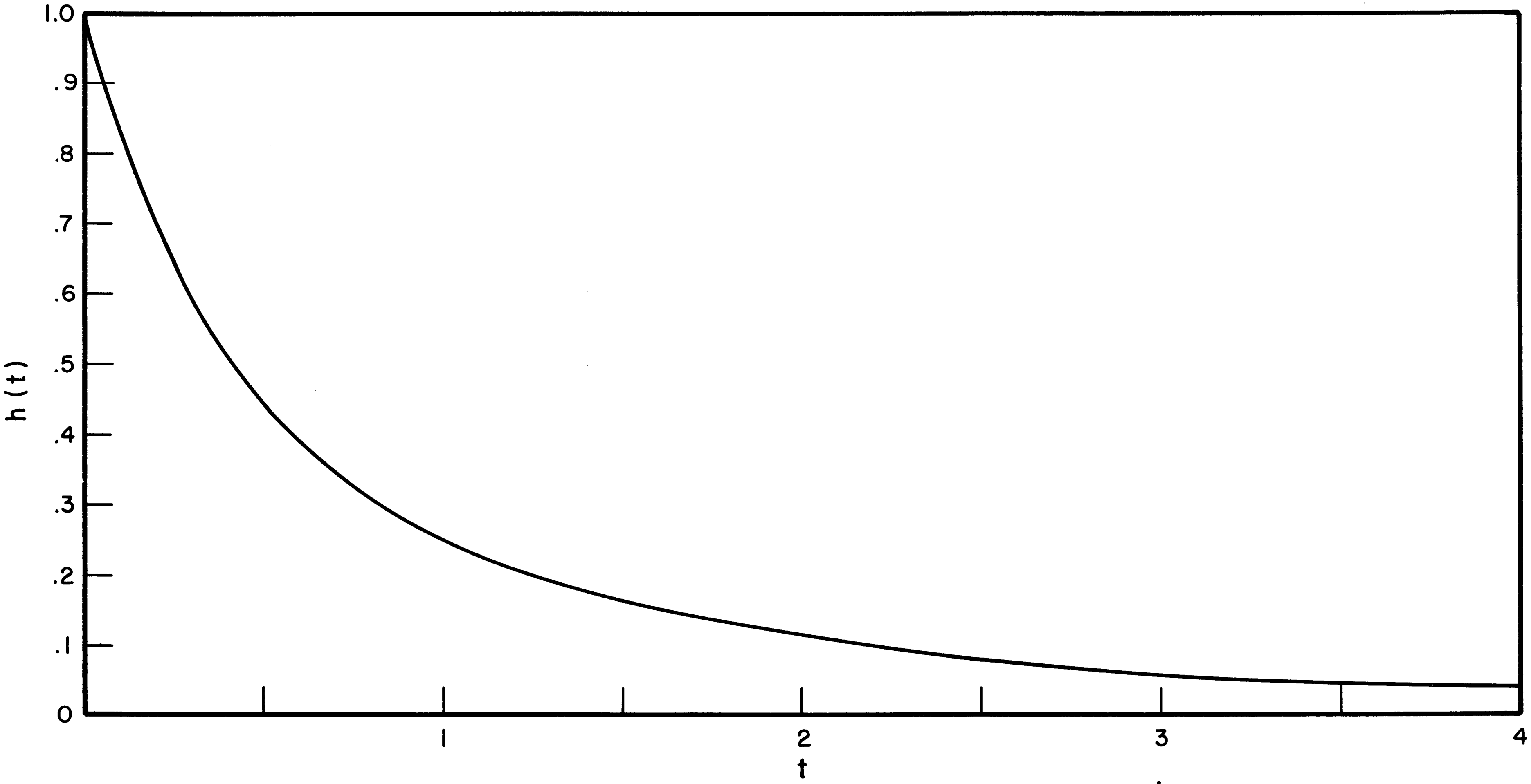


FIG. 4.4 IMPULSE RESPONSE $h(t) = \frac{1}{(1+t)^2}$

TABLE IV

VALUES OF h_m AT INTERVAL POINTS

m	t_m	h_m
1	0	1.0000
2	.5	.4450
3	1.0	.2500
4	1.5	.1600
5	2.0	.1110
6	2.5	.0817
7	3.0	.0625
8	3.5	.0494
9	4.0	.0400

Let one consider an approximation with one approximating term. Hence n = 1, and from Eq. (4.27) one obtains the following relationship:

$$h_v r_1 + h_{v+1} = 0 \quad (v = 1, 2, \ldots, 8). \qquad (4.73)$$

Substitution of values for h_v from Table IV yields,

$$\begin{aligned} r_1 + .445 &= 0 \\ .445r_1 + .25 &= 0 \\ .25r_1 + .16 &= 0 \\ .16r_1 + .111 &= 0 \\ .111r_1 + .0817 &= 0 \\ .0817r_1 + .0625 &= 0 \\ .0625r_1 + .0494 &= 0 \\ .0494r_1 + .04 &= 0 \; . \end{aligned} \qquad (4.74)$$

Choosing the first two equations in Eqs. (4.74), one obtains,

$$r_1 + .445 = 0$$

and
$$.445r_1 + .25 = 0 \ . \tag{4.75}$$

Then,
$$x_1 = (1)$$

and
$$x_2 = (.445) \ . \tag{4.76}$$

From Eq. (4.29),

$$\lambda_1 x_1 + \lambda_2 x_2 = 0 \ .$$

Hence,
$$\lambda_1 + .445\lambda_2 = 0 \ . \tag{4.77}$$

Let $\lambda_2 = 1$, then $\lambda_1 = -.445$.

Consequently,
$$\sum_{(2)}' |\lambda_\sigma| = 1.445$$

and
$$\sum_{(2)}' \lambda_\sigma h_{\sigma+1} = .052 \ .$$

From Eq. (4.34)
$$\epsilon = \frac{\sum_{(2)}' \lambda_\sigma h_{\sigma+1}}{\sum_{(2)}' |\lambda_\sigma|} = .036 \ . \tag{4.78}$$

By Eq. (4.33)

$$\epsilon_\sigma = \epsilon \ (\text{sgn } \lambda_\sigma) \ .$$

Therefore
$$\epsilon_1 = -.036$$

and
$$\epsilon_2 = .036 \ .$$

From Eq. (4.28)
$$r_1 + .445 + .036 = 0 \ . \tag{4.79}$$

Solving for r_1 yields,
$$r_1 = -.481 \ . \tag{4.80}$$

Now one can compute the errors in Eqs. (4.74). These are:

$$\epsilon_1 = -.036$$
$$\epsilon_2 = .036$$
$$\epsilon_3 = .04$$
$$\epsilon_4 = .034$$
$$\epsilon_5 = .0282$$
$$\epsilon_6 = .0232$$
$$\epsilon_7 = .01935$$
$$\epsilon_8 = .01624 \quad .$$

Since $|\epsilon_3| > |\epsilon|$, the replacement process must be undertaken.

Now,
$$x_3 = (.25) \; , \tag{4.82}$$

and by Eq. (4.37)

$$\mu_1 x_1 + \mu_2 x_2 + x_3 = 0 \; .$$

Therefore,

$$\mu_1 + .445\,\mu_2 + .250 = 0 \; . \tag{4.83}$$

Let $\mu_2 = 0$, then $\mu_1 = -.25$.

Since $\text{sgn}\,\epsilon_\sigma = \text{sgn}\,\lambda_\sigma$, and $\epsilon_i > 0$, from Table II, the equation which is designated by the number given by Min $\frac{\mu_\sigma}{\lambda_\sigma}$ must be replaced.

Now, $\frac{\mu_1}{\lambda_1} = \frac{-.25}{-.445}$, $\frac{\mu_2}{\lambda_2} = \frac{0}{1}$.

Since Min $\frac{\mu_\sigma}{\lambda_\sigma} = \frac{\mu_2}{\lambda_2}$, the second equation in Eqs. (4.75) must be replaced. The new equations are

$$r_1 + .445 = 0$$

and

$$.25 r_1 + .16 = 0 \; . \tag{4.84}$$

The equation for λ_σ is given by

$$\lambda_1 + .25\,\lambda_3 = 0 \ .$$

Let $\lambda_3 = 1$, then $\lambda_1 = -.25$.

Consequently,

$$\sum_{(2)}' |\lambda_\sigma| = 1.25 \ ,$$

and

$$\sum_{(2)}' \lambda_\sigma h_{\sigma+1} = .04875 \ .$$

Hence,

$$\epsilon = \frac{\sum_{(2)}' \lambda_\sigma h_{\sigma+1}}{\sum_{(2)}' |\lambda_\sigma|} = .039 \ . \qquad (4.85)$$

Then

$$\epsilon_1 = -.039$$

and

$$\epsilon_3 = .039 \ .$$

By Eq. (4.28)

$$r_1 + .445 + .039 = 0 \ .$$

Hence,

$$r_1 = -.484 \ . \qquad (4.86)$$

The errors for all the Eqs. (4.74) are now:

$$\epsilon_1 = -.039$$

$$\epsilon_2 = .035$$

$$\epsilon_3 = .039$$

$$\epsilon_4 = .0336$$

$$\epsilon_5 = .0241$$

$$\epsilon_6 = .0230$$

$$\epsilon_7 = .0192$$

$$\epsilon_8 = .0162 \ .$$

Since $|\epsilon_v| \leq |\epsilon|$, $(v = 1, \ldots, 8)$

$r_1 = -.484$ is the desired solution.

From Eq. (4.42),

$$y - .484 = 0. \qquad (4.88)$$

Therefore,

$$y_1 = .484 \qquad (4.89)$$

From Table III,

$$s_1 = \frac{1}{d} \ln y_1 = 2 \ln (.484) = 2 (-.725) = -1.45. \qquad (4.90)$$

The residue A_1 can be found from Eqs. (4.50). This equation requires the knowledge of the coefficients b_{km}, which are defined in Eq. (4.49) as $b_{km} = e^{s_k t_m}$. The computed values of b_{1m} are listed in Table V.

TABLE V

VALUES OF b_{1m} AT INTERVAL POINTS

m	t_m	b_{km}
1	0	1.0000
2	.5	.4840
3	1.0	.2346
4	1.5	.1132
5	2.0	.0550
6	2.5	.0265
7	3.0	.0129
8	3.5	.0062
9	4.0	.0030

From Eqs. (4.50) one has the following relationship:

$$h_m = b_{1m} A_1 \quad (m = 1, 2, \ldots, 9). \qquad (4.91)$$

Substituting values for h_m from Table IV, and values for b_{1m} from Table V, one obtains:

$$1 = A_1$$

$$.445 = .484\ A_1$$

$$.25 = .2346\ A_1$$

$$
\begin{aligned}
.16 &= .1132\ A_1 \\
.111 &= .055\ A_1 \\
.0817 &= .0265\ A_1 \\
.0625 &= .0129\ A_1 \\
.0494 &= .0062\ A_1 \\
.04 &= .003\ A_1 \quad .
\end{aligned}
\tag{4.92}
$$

Considering the first two equations in Eqs. (4.92),

$$
\begin{aligned}
A_1 - 1 &= 0 \\
.484\ A_1 - .445 &= 0.
\end{aligned}
\tag{4.93}
$$

Then,

$$
\begin{aligned}
x_1 &= (1) \\
x_2 &= (.484)
\end{aligned}
$$

and

$$\lambda_1 x_1 + \lambda_2 x_2 = 0 \ .$$

Hence,

$$\lambda_1 + .484\lambda_2 = 0 \ ;$$

$$\text{let } \lambda_2 = 1, \quad \text{then } \lambda_1 = -.484 \ .$$

Consequently,

$$\sum_{(2)}{}' |\lambda_\sigma| = 1.484$$

and

$$\sum_{(2)}{}' \lambda_\sigma(-h_\sigma) = (-.484)(-1) + (1)(-.445) = .039 \ .$$

Hence

$$\epsilon = \frac{\sum_{(2)}{}' \lambda_\sigma(-h_\sigma)}{\sum_{(2)}{}' |\lambda_\sigma|} = .0263. \tag{4.94}$$

Then,

$$
\begin{aligned}
\epsilon_1 &= \epsilon\ (\text{sgn } \lambda_1) = -.0263 \\
\epsilon_2 &= \epsilon\ (\text{sgn } \lambda_2) = .0263 \ .
\end{aligned}
$$

Therefore,

$$A_1 - 1 + .0263 = 0$$

or

$$A_1 = .9737 \quad (4.95)$$

The errors in Eqs. (4.92) are:

$$\epsilon_1 = -.0263$$
$$\epsilon_2 = .0263$$
$$\epsilon_3 = -.022$$
$$\epsilon_4 = -.0498$$
$$\epsilon_5 = -.0575$$
$$\epsilon_6 = -.0559$$
$$\epsilon_7 = -.04994$$
$$\epsilon_8 = -.0432$$
$$\epsilon_9 = -.037 \quad (4.96)$$

Since $|\epsilon_{max}| = |\epsilon_5| > |\epsilon|$, the replacement process must be undertaken.

Now,

$$x_5 = (.055) , \quad (4.97)$$

and, as before,

$$\mu_1 x_1 + \mu_2 x_2 + x_5 = 0 .$$

Therefore,

$$\mu_1 + .484\, \mu_2 + .055 = 0 . \quad (4.98)$$

Let $\mu_2 = 0$, then $\mu_1 = -.055$.

Since sgn ϵ_σ = sgn λ_σ, and $\epsilon_i < 0$, the equation which is designated by the number given by Max $\frac{\mu_\sigma}{\lambda_\sigma}$ must be replaced.

Now,

$$\frac{\mu_1}{\lambda_1} = \frac{-.055}{-.484} , \quad \frac{\mu_2}{\lambda_2} = \frac{0}{1} .$$

Since Max $\frac{\mu_\sigma}{\lambda_\sigma} = \frac{\mu_1}{\lambda_1}$, the first equation in Eqs. (4.92) must be replaced.

The new equations are:

$$.484\,A_1 - .445 = 0$$

and

$$.055\,A_1 - .111 = 0\,. \tag{4.99}$$

Now,

$$.484\,\lambda_2 + .055\lambda_5 = 0$$

Let $\lambda_5 = .484$, then $\lambda_2 = -.055$.

Consequently,

$$\sum_{(2)}{}' |\lambda_\sigma| = .539$$

and

$$\sum_{(2)}{}' \lambda_\sigma(-h_\sigma) = -.0291\,.$$

Then

$$\epsilon = \frac{-.0291}{.539} = -.054,$$

$$\epsilon_2 = \epsilon(\text{sgn}\,\lambda_2) = .054,$$

and

$$\epsilon_3 = \epsilon(\text{sgn}\,\lambda_5) = -.054.$$

Therefore,

$$.484\,A_1 - .445 - .054 = 0\,,$$

and

$$A_1 = 1.03\,. \tag{4.101}$$

The errors in Eqs. (4.92) are:

$$\begin{aligned} \epsilon_1 &= .03 \\ \epsilon_2 &= .054 \\ \epsilon_3 &= -.008 \\ \epsilon_4 &= -.043 \\ \epsilon_5 &= -.054 \\ \epsilon_6 &= -.053 \\ \epsilon_7 &= -.049 \\ \epsilon_8 &= -.043 \\ \epsilon_9 &= -.037\,. \end{aligned} \tag{4.102}$$

Since $|\epsilon_m| \leq |\epsilon|$ (m = 1, ..., 9), A_1 is the desired residue, and the one-term approximation is

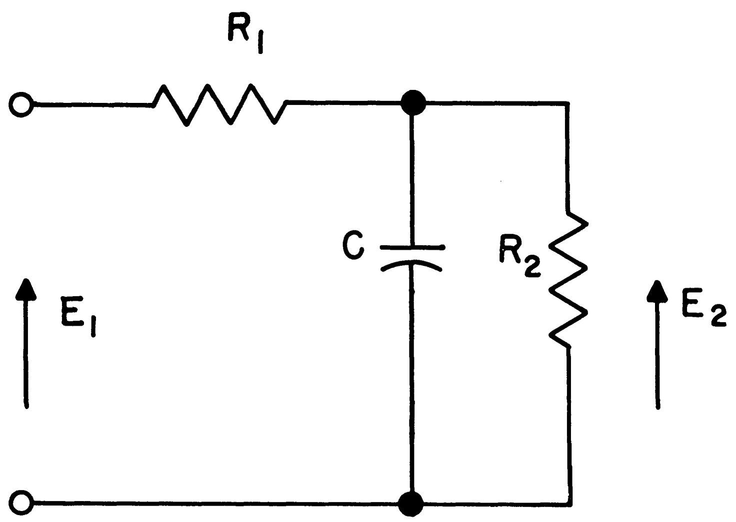
$$h^*(t) = 1.03 e^{-1.45t} . \qquad (4.103)$$

The Tschebyscheff error is .054.

The Laplace transform of Eq. (4.103) is

$$H^*(s) = 1.03 \frac{1}{s + 1.45} . \qquad (4.104)$$

A network with the voltage-transfer function of this equation is shown in Fig. 4.5.



$$H^*(s) = \frac{E_2}{E_1} = 1.03 \frac{1}{s + 1.45}$$

$$R_1 = .97\ \Omega$$
$$R_2 = 2.38\ \Omega$$
$$C = 1\ \text{fd}$$

FIG. 4.5 NETWORK REALIZING THE h*(t) OF EQ. (4.103)

The example will be now repeated, using the same interval spacing (d = 0.5) and the same number of interval points (q = 9), as before; but now a two-term approximation is sought. Hence, n = 2, and one obtains

$$h_v r_2 + h_{v+1} r_1 + h_{v+2} = 0 \qquad (4.105)$$

$$(v = 1, 2, \ldots, 7).$$

Substitution of values for h_v from Table IV yields:

$$\begin{aligned}
r_2 + .445r_1 + .25 &= 0\\
.445r_2 + .25r_1 + .16 &= 0\\
.25r_2 + .16r_1 + .111 &= 0\\
.16r_2 + .111r_1 + .0817 &= 0\\
.111r_2 + .0817r_1 + .0625 &= 0\\
.0817r_2 + .0625r_1 + .0494 &= 0\\
.0625r_2 + .0494r_1 + .04 &= 0 \ .
\end{aligned} \qquad (4.106)$$

Choosing the first three equations in Eqs. (4.106) as reference, one obtains

$$\begin{aligned}
r_2 + .445r_1 + .25 &= 0\\
.445r_2 + .25r_1 + .16 &= 0\\
.25r_2 + .16r_1 + .111 &= 0 \ .
\end{aligned} \qquad (4.107)$$

Then,

$$\begin{aligned}
x_1 &= (1, .445),\\
x_2 &= (.445, .25),\\
x_3 &= (.25, .16) \ .
\end{aligned} \qquad (4.108)$$

From Eq. (4.29),

$$\lambda_1 + .445\lambda_2 + .25\lambda_3 = 0$$

and

$$.445\lambda_1 + .25\lambda_2 + .16\lambda_3 = 0 \ .$$

Let

$$\lambda_3 = 1 \ .$$

Then

$$\lambda_1 = .17, \quad \lambda_2 = -.942 \ .$$

Hence,

$$\sum_{(3)}' |\lambda_\sigma| = 2.112$$

and

$$\sum_{(3)}' \lambda_\sigma h_{\sigma+2} = .0025 \quad .$$

Consequently,

$$\epsilon = \frac{\sum_{(3)}' \lambda_\sigma h_{\sigma+2}}{\sum_{(3)}' |\lambda_\sigma|} = .001183 \ . \qquad (4.109)$$

By Eq. (4.33),

$$\epsilon_\sigma = \epsilon \ (\text{sgn} \ \lambda_\sigma) \ .$$

Therefore,

$$\epsilon_1 = .001183$$
$$\epsilon_2 = -.001183$$
$$\epsilon_3 = .001183 \ .$$

By Eq. (4.28),

$$r_2 + .445 r_1 + .25 - .001183 = 0$$

$$.445 r_3 + .25 r_1 + .16 + .001183 = 0 \quad .$$

Hence,

$$r_1 = -.968$$
$$r_2 = .1822 \quad . \qquad (4.110)$$

Now one can compute the errors in Eqs. (4.106). These are:

$$\epsilon_1 = .0012$$
$$\epsilon_2 = -.0012$$
$$\epsilon_3 = .0012$$
$$\epsilon_4 = .0034$$
$$\epsilon_5 = .0037$$
$$\epsilon_6 = .0008$$
$$\epsilon_7 = .0036 \quad . \qquad (4.111)$$

Since $|\epsilon_5| > |\epsilon|$, the replacement process must be undertaken.

Now,
$$x_5 = (.111, .0817) \tag{4.112}$$

and by Eq. (4.37),

$$\mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3 + x_5 = 0 .$$

Therefore,

$$\mu_1 + .445\mu_2 + .25\mu_3 + .111 = 0$$

$$.445\mu_1 + .25\mu_2 + .16\mu_3 + .0817 = 0 . \tag{4.113}$$

Let $\mu_3 = 0$, then $\mu_1 = .165$, $\mu_2 = -.621$.

Since sgn ϵ_σ = sgn λ_σ and $\epsilon_i > 0$, hence from Table II, the equation which is designated by the number given by Min $\frac{\mu_\sigma}{\lambda_\sigma}$ must be replaced.

Now,

$$\frac{\mu_1}{\lambda_1} = \frac{.165}{.17} , \qquad \frac{\mu_2}{\lambda_2} = \frac{-.621}{-.942} , \qquad \frac{\mu_3}{\lambda_3} = \frac{0}{1} .$$

Min $\frac{\mu_\sigma}{\lambda_\sigma} = \frac{\mu_3}{\lambda_3}$; hence, the third equation in Eqs. (4.106) must be replaced.

The new equations are:

$$r_2 + .445r_1 + .25 = \quad = 0$$

$$.445r_2 + .25r_1 + .16 = 0$$

$$.111r_2 + .0817r_1 + .0625 = 0$$

The equations for λ_σ are:

$$\lambda_1 + .445\lambda_2 + .111\lambda_5 = 0$$

$$.445\lambda_1 + .25\lambda_2 + .0817\lambda_5 = 0 \tag{4.115}$$

Let $\lambda_5 = 1$, then $\lambda_1 = .1654$ and $\lambda_2 = -.621$. Consequently,

$$\sum_{(3)}' |\lambda_\sigma| = 1.7864$$

and

$$\sum_{(3)}' \lambda_\sigma h_{\sigma+2} = .00449 \ .$$

Hence,

$$\epsilon = \frac{\sum_{(3)}' \lambda_\sigma h_{\sigma+2}}{\sum_{(3)}' |\lambda_\sigma|} = .00251 \ . \qquad (4.116)$$

Then

$$\epsilon_1 = .00251$$

$$\epsilon_2 = -.00251$$

$$\epsilon_5 = .00251 \ .$$

By Eq. (4.28),

$$r_2 + .445 r_1 + .25 - .0025 = 0$$

and

$$.445 r_2 + .25 r_1 + .16 + .0025 = 0 \ . \qquad (4.117)$$

Hence,

$$r_1 = -1.007, \quad r_2 = .2009 \ . \qquad (4.118)$$

The errors in Eqs. (4.106) are:

$$\epsilon_1 = .0025$$

$$\epsilon_2 = -.0025$$

$$\epsilon_3 = 0$$

$$\epsilon_4 = .0020$$

$$\epsilon_5 = .0025$$

$$\epsilon_6 = .0028$$

$$\epsilon_7 = .0028 \ . \qquad (4.119)$$

Since $|\epsilon_6| > |\epsilon|$, the replacement process must take place.

Now,

$$x_6 = (.0817, \quad .0625) , \tag{4.120}$$

and by Eq. (4.37),

$$\begin{aligned} \mu_1 + .445\mu_2 + .111\mu_5 + .0817 &= 0 \\ .445\mu_1 + .25\mu_2 + .0817\mu_5 + .0625 &= 0 . \end{aligned} \tag{4.121}$$

Let $\mu_5 = 0$, then $\mu_1 = .1420$, $\mu_2 = -.0526$.

Since sgn ϵ_σ = sgn λ_σ and $\epsilon_i > 0$, hence from Table II, the equation designated by the number given by Min $\frac{\mu_\sigma}{\lambda_\sigma}$ must be replaced.

Now,

$$\frac{\mu_1}{\lambda_1} = \frac{.1420}{.1654} , \quad \frac{\mu_2}{\lambda_2} = \frac{-.5026}{-.621} , \quad \frac{\mu_5}{\lambda_5} = \frac{0}{1} .$$

Min $\frac{\mu_\sigma}{\lambda_\sigma} = \frac{\mu_5}{\lambda_5}$; hence the fifth equation in Eqs. (4.106) must be replaced.

The new equations are:

$$\begin{aligned} r_2 + .445r_1 + .25 &= 0 \\ .445r_2 + .25r_1 + .16 &= 0 \\ .0817r_2 + .0625r_1 + .0494 &= 0 . \end{aligned} \tag{4.122}$$

λ_σ are determined from:

$$\begin{aligned} \lambda_1 + .445\lambda_2 + .0817\lambda_6 &= 0 \\ .445 + .25\lambda_2 + .0625\lambda_6 &= 0 . \end{aligned} \tag{4.123}$$

Let $\lambda_6 = 1$, then $\lambda_1 = .1420$, $\lambda_2 = -.5026$.

Consequently,

$$\sum_{(3)}' |\lambda_\sigma| = 1.6446$$

and

$$\sum_{(3)}' \lambda_\sigma h_{\sigma+2} = .004484 .$$

Hence,

$$\epsilon = \frac{\sum_{(3)}' \lambda_\sigma h_{\sigma+2}}{\sum_{(3)}' |\lambda_\sigma|} = .00273 \quad . \tag{4.124}$$

Then,

$$\epsilon_1 = .00273$$
$$\epsilon_2 = -.00273$$
$$\epsilon_6 = .00273 \quad .$$

By Eq. (4.28):

$$r_2 + .445r_1 + .25 - .0027 = 0$$
$$.445r_2 + .25r_1 +.16 + .0027 = 0 \quad . \tag{4.125}$$

Therefore,

$$r_1 = -1.00128, \quad r_2 = .2033 \quad . \tag{4.126}$$

The errors in Eqs. (4.106) are:

$$\epsilon_1 = .0027$$
$$\epsilon_2 = -.0027$$
$$\epsilon_3 = -.0002$$
$$\epsilon_4 = .0018$$
$$\epsilon_5 = .0023$$
$$\epsilon_6 = .0027$$
$$\epsilon_7 = .0026 \quad . \tag{4.127}$$

Since $|\epsilon_k| \leq |\epsilon|$ $(k = 1, 2, \ldots, 7)$ $r_1 = -1.0128$, and $r_2 = .2033$ are the desired solutions.

From Eq. (4.42):

$$y^2 - 1.0128y + .2033 = 0 \quad . \tag{4.128}$$

Hence,

$$y_1 = .7369$$

$$y_2 = .2759 \quad . \tag{4.129}$$

From Table III,

$$s_1 = \frac{1}{d}\,\ell n y_1 = 2\ell n(.7369) = -.6106$$

$$s_2 = \frac{1}{d}\,\ell n y_2 = 2\ell n(.2759) = -2.5754 \quad . \tag{4.130}$$

The residues A_1 and A_2 can be found from Eq. (4.69). The coefficients b_{1k} and b_{2k} are found from:

$$b_{1m} = e^{s_1 t_m}$$

$$b_{2m} = e^{s_2 t_m} \quad (m = 1, 2, \ldots, 9). \tag{4.131}$$

The values of b_{1m} and b_{2m} are computed, and are listed in Table VI.

TABLE VI

VALUES OF b_{1m} AND b_{2m} AT INTERVAL POINTS

m	t_m	b_{1m}	b_{2m}
1	0	1.00000	1.00000
2	.5	.73690	.27590
3	1.0	.54302	.07612
4	1.5	.40015	.02100
5	2.0	.29487	.00579
6	2.5	.21729	.00160
7	3.0	.16012	.00044
8	3.5	.11799	.00012
9	4.0	.08695	.00003

From Eq. (4.50), one obtains the relationship,

$$h_m = b_{1m}A_1 + b_{2m}A_2 \quad (m = 1, 2, \ldots, 9). \tag{4.132}$$

Substituting values for h_m from Table IV, and values for b_{1m} and b_{2m} from Table VI, one obtains:

$$
\begin{aligned}
1.0000 &= A_1 + A_2 \\
.4450 &= .73690A_1 + .27590A_2 \\
.2500 &= .54302A_1 + .07612A_2 \\
.1600 &= .40015A_1 + .02100A_2 \\
.1110 &= .29487A_1 + .00579A_2 \\
.0817 &= .21729A_1 + .00160A_2 \\
.0625 &= .16012A_1 + .00044A_2 \\
.0494 &= .11799A_1 + .00012A_2 \\
.0400 &= .08695A_1 + .00003A_2 \ .
\end{aligned}
\tag{4.133}
$$

The first, sixth and ninth equations in Eqs. (4.133) are:

$$
\begin{aligned}
A_1 + A_2 &= 1 \\
.21729A_1 + .00160A_2 &= .0817 \\
.08695A_1 + .00003A_2 &= .0400 \ .
\end{aligned}
\tag{4.134}
$$

The equations for λ_σ are:

$$
\begin{aligned}
\lambda_1 + .21729\lambda_6 + .08695\lambda_9 &= 0 \\
\lambda_1 + .00160\lambda_6 + .00003\lambda_9 &= 0 \ .
\end{aligned}
\tag{4.135}
$$

Let $\lambda_9 = 1$, then $\lambda_1 = .00061$, $\lambda_6 = -.40299$.

Consequently,

$$\sum_{(3)}' |\lambda_\sigma| = 1.40360$$

and

$$\sum_{(3)}{}' \lambda_\sigma(-h_\sigma) = -.0076857 .$$

Hence,

$$\epsilon = \frac{\sum_{(3)}' \lambda_\sigma(-h_\sigma)}{\sum_{(3)}' |\lambda_\sigma|} = -.00548 . \tag{4.136}$$

Then,

$$\epsilon_1 = \epsilon \operatorname{sgn} \lambda_1 = -.00548$$

$$\epsilon_6 = \epsilon \operatorname{sgn} \lambda_6 = .00548$$

$$\epsilon_9 = \epsilon \operatorname{sgn} \lambda_9 = -.00548 .$$

A_1 and A_2 are determined from:

$$\begin{aligned} A_1 + A_2 - 1 + .00548 &= 0 \\ .21729A_1 + .00160A_2 - .0817 - .00548 &= 0 . \end{aligned} \tag{4.137}$$

Hence,

$$A_1 = .39681, \quad A_2 = .59771 . \tag{4.138}$$

The errors in Eqs. (4.133) are:

$$\begin{aligned} \epsilon_1 &= -.00548 \\ \epsilon_2 &= .01232 \\ \epsilon_3 &= .01097 \\ \epsilon_4 &= .01134 \\ \epsilon_5 &= .00947 \\ \epsilon_6 &= .00548 \\ \epsilon_7 &= .00130 \\ \epsilon_8 &= -.00251 \\ \epsilon_9 &= -.00548 . \end{aligned} \tag{4.139}$$

Since $|\epsilon_2| > |\epsilon|$, the replacement process must be undertaken.

Now,

$$x_2 = (.73690,\ .27590)\ , \qquad (4.140)$$

and by Eq. (4.37),

$$\mu_1 + .21729\epsilon_6 + .08695\mu_9 + .73690 = 0$$
$$\mu_1 + .00160\mu_6 + .00003\mu_9 + .27590 = 0\ . \qquad (4.141)$$

Let $\mu_9 = 0$, then $\mu_1 = -.27248$, $\mu_6 = -2.13733$.

Since sgn ϵ_σ = - sgn λ_σ and $\epsilon_i > 0$, hence from Table II, the equation which is designated by the number given by Max $\frac{\mu_\sigma}{\lambda_\sigma}$ must be replaced. Now,

$$\frac{\mu_1}{\lambda_1} = \frac{-.27248}{.00061}\ , \quad \frac{\mu_6}{\lambda_6} = \frac{-2.13733}{-.40299}\ , \quad \frac{\mu_9}{\lambda_9} = \frac{0}{1}\ .$$

Max $\frac{\mu_\sigma}{\lambda_\sigma} = \frac{\mu_6}{\lambda_6}$, hence the sixth equation in Eqs. (4.133) must be replaced.

The new equations are:

$$A_1 + A_2 = 1$$
$$.73690A_1 + .27590A_2 = .4450$$
$$.08695A_1 + .00003A_2 = .04000\ . \qquad (4.142)$$

The equations for λ_σ are:

$$\lambda_1 + .73690\lambda_2 + .08695\lambda_9 = 0$$
$$\lambda_1 + .27590\lambda_2 + .00003\lambda_9 = 0\ . \qquad (4.143)$$

Let $\lambda_9 = 1$, then $\lambda_1 = .05199$, $\lambda_2 = -.18855$.

Consequently,

$$\sum_{(3)}{}' |\lambda_\sigma| = 1.24054$$

and

$$\sum_{(3)}{}' \lambda_\sigma(-h_\sigma) = -.00808525\ .$$

Hence,

$$\epsilon = \frac{\sum_{(3)}' \lambda_\sigma(-h_\sigma)}{\sum_{(3)}' |\lambda_\sigma|} = -.00652 . \qquad (4.144)$$

Then,

$$\epsilon_1 = \epsilon \text{ sgn } \lambda_1 = -.00652$$

$$\epsilon_2 = \epsilon \text{ sgn } \lambda_2 = .00652$$

$$\epsilon_9 = \epsilon \text{ sgn } \lambda_9 = -.00652 .$$

The equations for A_1 and A_2 are:

$$A_1 + A_2 - 1 + .00652 = 0$$

$$.73690A_1 + .27590A_2 - .4450 - .00652 = 0 . \qquad (4.145)$$

Hence,

$$A_1 = .38485, \quad A_2 = .60863 . \qquad (4.146)$$

The errors in Eqs. (4.133) are:

$$\epsilon_1 = -.00652$$

$$\epsilon_2 = .00652$$

$$\epsilon_3 = .00531$$

$$\epsilon_4 = .00678$$

$$\epsilon_5 = .00600$$

$$\epsilon_6 = .00290$$

$$\epsilon_7 = -.00061$$

$$\epsilon_8 = -.00392$$

$$\epsilon_9 = -.00652 . \qquad (4.147)$$

Since $|\epsilon_4| > |\epsilon|$, the replacement process must take place. From Eqs. (4.133),

$$x_4 = (.40015, .02100) , \qquad (4.148)$$

and by Eq. (4.37),

$$\mu_1 + .73690\mu_2 + .08695\mu_9 + .40015 = 0$$
$$\mu_1 + .27590\mu_2 + .00003\mu_9 + .02100 = 0 . \qquad (4.149)$$

Let $\mu_9 = 0$, then $\mu_1 = .20591$, $\mu_2 = -.82245$.

Since sgn ϵ_σ = -sgn λ_σ and $\epsilon_i > 0$, hence from Table II, the equation which is designated by the number given by Max $\frac{\mu_\sigma}{\lambda_\sigma}$ must be replaced.

Now,

$$\frac{\mu_1}{\lambda_1} = \frac{.20591}{.05199}, \quad \frac{\mu_2}{\lambda_2} = \frac{-.82245}{-.18855}, \quad \frac{\mu_9}{\lambda_9} = \frac{0}{1} .$$

Max $\frac{\mu_\sigma}{\lambda_\sigma} = \frac{\mu_2}{\lambda_2}$, hence the second equation in Eqs. (4.133) must be replaced.

The new equations are:

$$A_1 + A_2 = 1$$
$$.40015A_1 + .02100A_2 = .1600$$
$$.08695A_1 + .00003A_2 = .04000 \qquad (4.150)$$

The equations for λ_σ are:

$$\lambda_1 + .40015\lambda_4 + .08695\lambda_9 = 0$$
$$\lambda_1 + .02100\lambda_4 + .00003\lambda_9 = 0 . \qquad (4.151)$$

Let $\lambda_9 = 1$, then $\lambda_1 = .00478$, $\lambda_4 = -.22925$.

Therefore,

$$\sum_{(3)}' |\lambda_\sigma| = 1.23403$$

and

$$\sum_{(3)}' \lambda_\sigma(-h_\sigma) = -.00810 .$$

Hence,

$$\epsilon = \frac{\sum\limits_{(3)}' \lambda_\sigma(-h_\sigma)}{\sum\limits_{(3)}' |\lambda_\sigma|} = -.00656 \; . \qquad (4.152)$$

Then,

$$\epsilon_1 = \epsilon \operatorname{sgn} \lambda_1 = -.00656$$
$$\epsilon_4 = \epsilon \operatorname{sgn} \lambda_4 = .00656$$
$$\epsilon_9 = \epsilon \operatorname{sgn} \lambda_9 = -.00656 \; .$$

A_1 and A_2 are determined from:

$$A_1 + A_2 - 1 + .00656 = 0$$
$$.40015A_1 + .02100A_2 - .1600 - .00656 = 0 \; . \qquad (4.153)$$

Hence,

$$A_1 = .38427, \quad A_2 = .60917 \; . \qquad (4.154)$$

The errors in Eqs. (4.133) are:

$$\epsilon_1 = -.00656$$
$$\epsilon_2 = .00424$$
$$\epsilon_3 = .00504$$
$$\epsilon_4 = .00656$$
$$\epsilon_5 = .00584$$
$$\epsilon_6 = .00277$$
$$\epsilon_7 = -.00070$$
$$\epsilon_8 = -.00399$$
$$\epsilon_9 = -.00656 \qquad (4.155)$$

Since $|\epsilon_m| \leq |\epsilon|$ $(m = 1, 2, \ldots, 9)$, A_1 and A_2 are the desired residues, and the two-term approximation is:

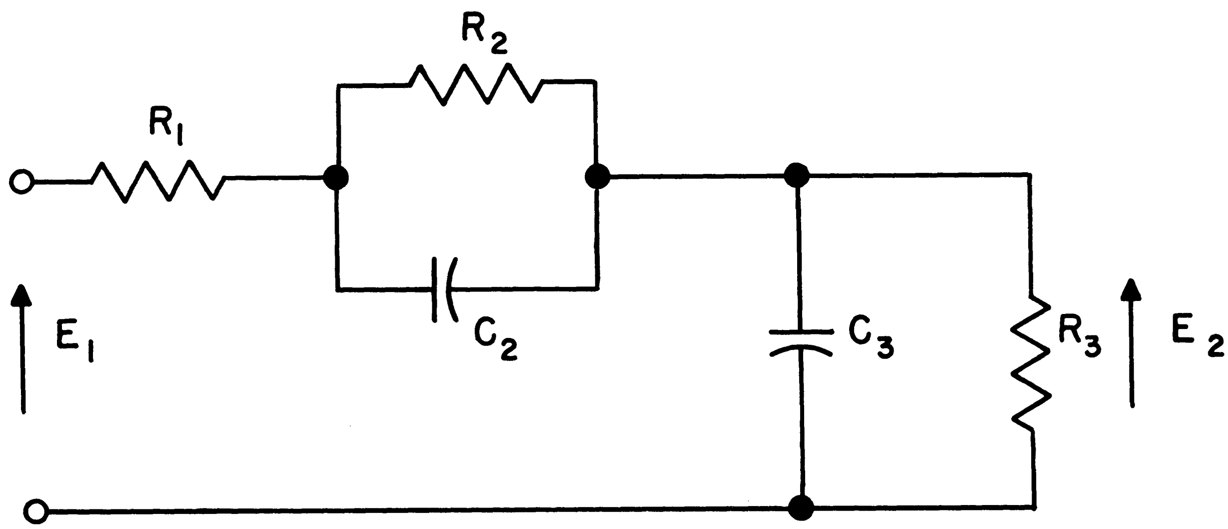
$$h^*(t) = .3843e^{-.6106t} + .6092\, e^{-2.5754t}. \qquad (4.156)$$

The Tschebyscheff error is .00656 only, as compared with .054 for the one-term approximation.

The Laplace transform of Eq. (4.169) is:

$$H^*(s) = \frac{.3843}{s + .6106} + \frac{.6092}{s + 2.5754} \tag{4.157}$$

A network realizing H*(s) as a voltage ratio is shown in Fig. 4.6.



$$H^*(s) = \frac{E_2}{E_1} = \frac{.9935\ (s + 1.3706)}{(s + .6106)(s + 2.5754)}$$

$R_1 = .5000\ \Omega$

$R_2 = .2631\ \Omega$

$R_3 = 4.9297\ \Omega$

$C_2 = 2.7731$ fd

$C_3 = 2.0132$ fd

FIG. 4.6 NETWORK REALIZING THE h*(t) OF EQ. (4.156)

Plots of h(t) and h*(t) vs. t for one-term and two-term approximations,are shown in Fig. 4.7. A plot of h*(t) for the two-term approximation agrees so closely with h(t) that in Fig. 4.7 it appears to coincide with h(t). The plots of [h*(t) - h(t)] vs. t for these two cases appear in Fig. 4.8.

4.5.2 Determination of a Network with Impulse Response $t \cdot e^{-t^2}$.

As the second example, let

$$h(t) = t \cdot e^{-t^2} . \tag{4.158}$$

It is desired to find an R-L-C network N, having an output voltage h(t) given by Eq. (4.158) where excited by an impulse-current input δ(t).

A plot of h(t) vs. t is shown in Fig. 4.9. An examination of this figure indicates that a choice of 0.2 second for the interval spacing d, and the choice of q = 16 points are reasonable ones. These choices result in the values of h_m at interval points listed in Table VII.

TABLE VII

VALUES OF h_m AT INTERVAL POINTS

m	t_m	h_m
1	0	0
2	.2	.1922
3	.4	.3408
4	.6	.4187
5	.8	.4219
6	1.0	.3679
7	1.2	.2843
8	1.4	.1973
9	1.6	.1237
10	1.8	.0706
11	2.0	.0366
12	2.2	.0158
13	2.4	.0051
14	2.6	.003
15	2.8	.0011
16	3.0	.0003

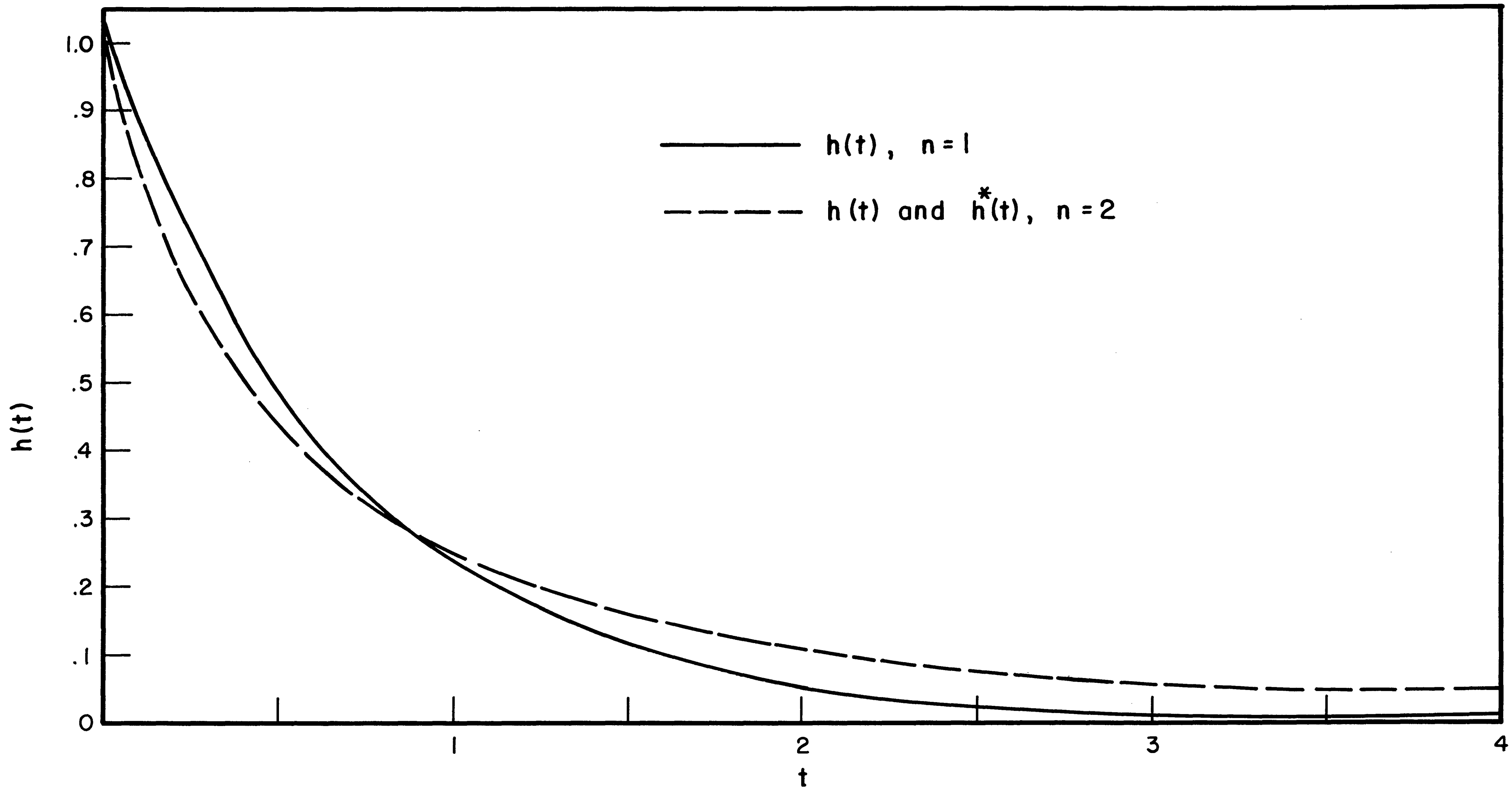


FIG. 4.7 $h(t)$ AND $h^*(t)$ FOR ONE-TERM AND TWO-TERM APPROXIMATIONS. (EXAMPLE OF SEC. 4.5.1)

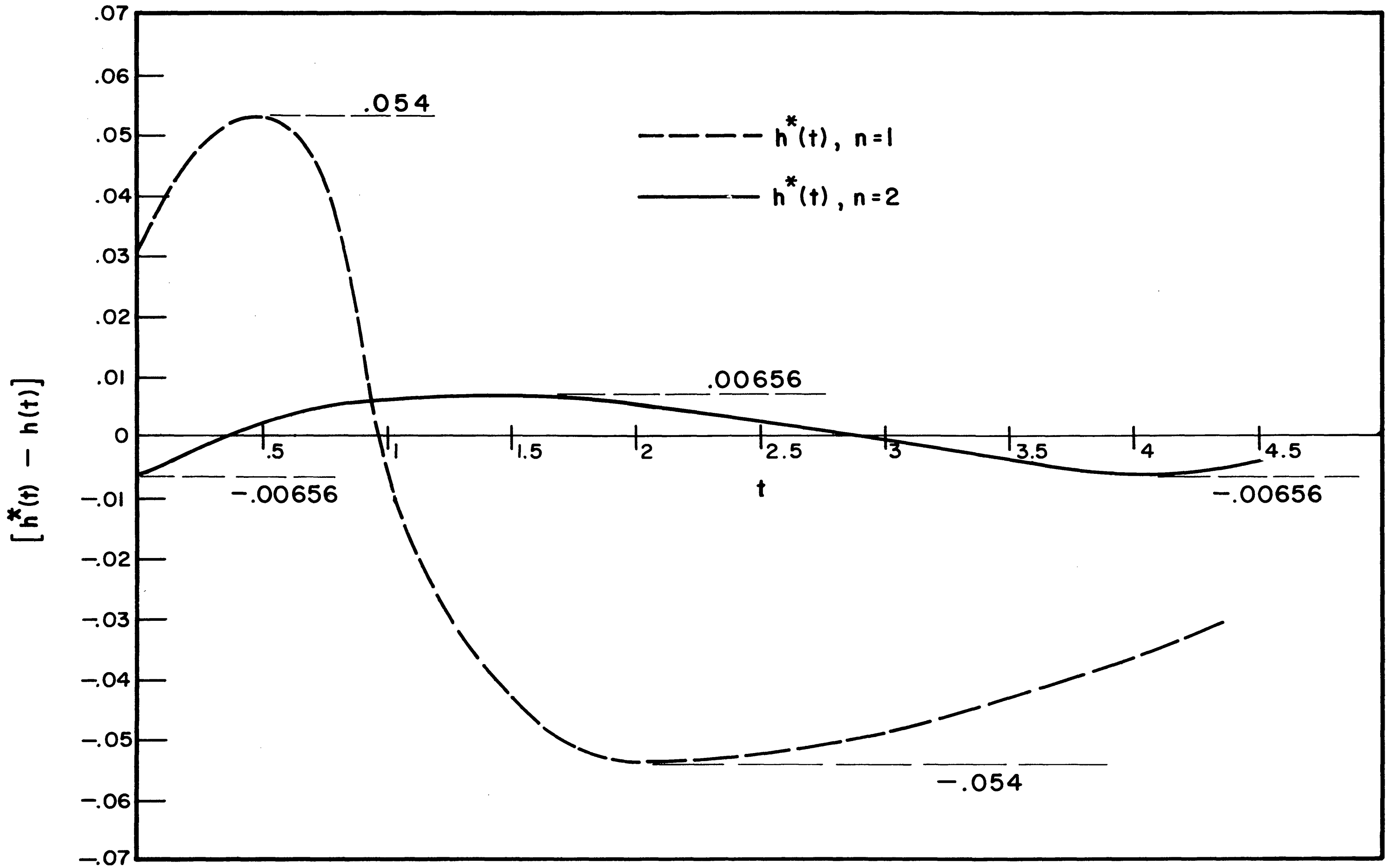


FIG. 4.8 $[h^*(t)-h(t)]$ FOR ONE-TERM AND TWO-TERM APPROXIMATIONS(EXAMPLE OF SEC. 4.51)

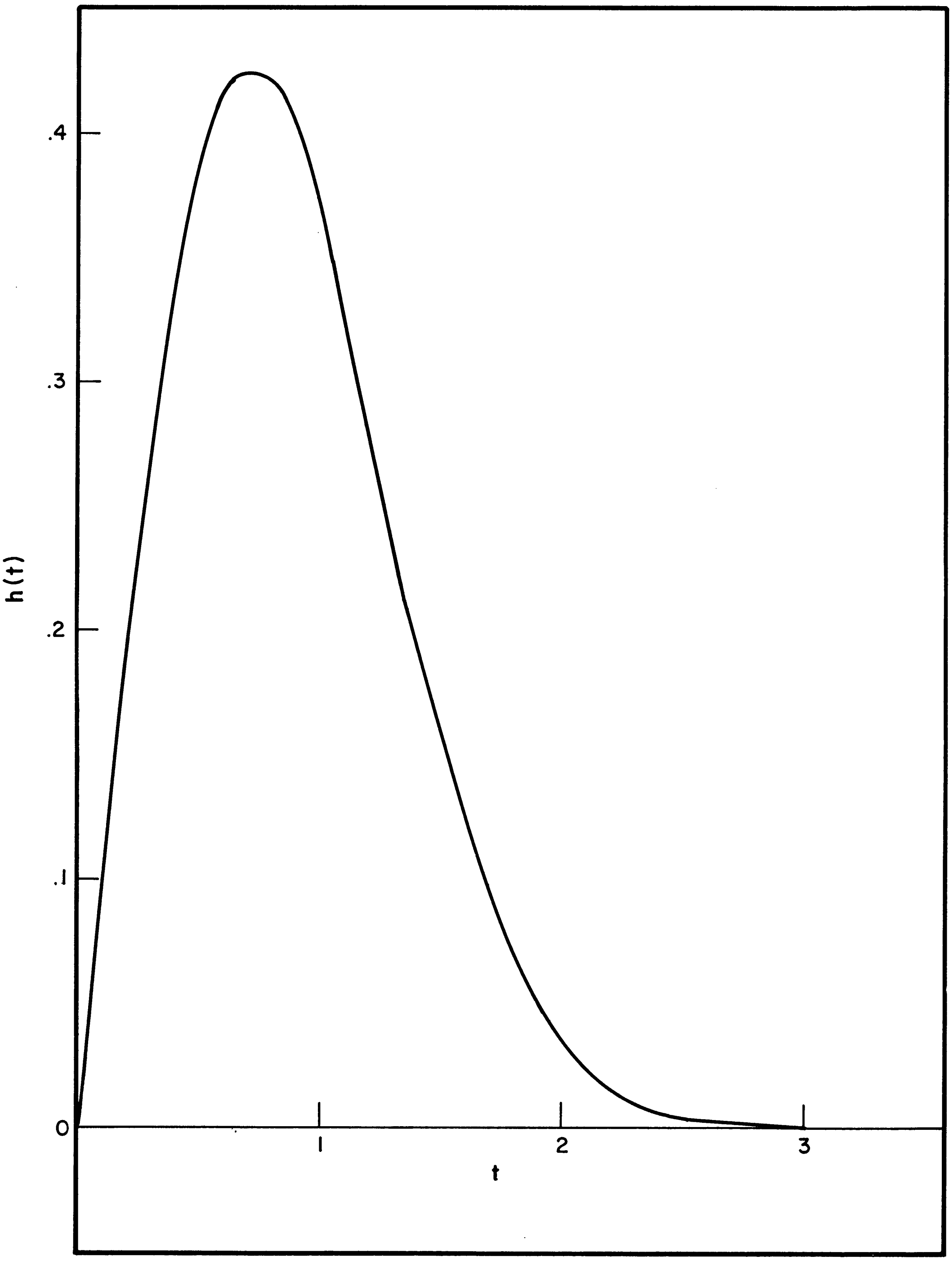


FIG. 4.9 IMPULSE RESPONSE $h(t) = t e^{-t^2}$

Let one consider an approximation with three approximating terms. Hence n = 3, and from Eq. (4.27) one obtains the relationship:

$$h_v r_3 + h_{v+1} r_2 + h_{v+2} r_1 + h_{v+3} = 0 \qquad (4.159)$$

$$(v = 1,\ 2,\ \ldots,\ 13).$$

Substitution of values for h_v from Table VII yields:

$$\begin{aligned}
.1922r_2 + .3408r_1 + .4187 &= 0 \\
.1922r_3 + .3408r_2 + .4187r_1 + .4219 &= 0 \\
.3408r_3 + .4187r_2 + .4219r_1 + .3679 &= 0 \\
.4187r_3 + .4219r_2 + .3679r_1 + .2843 &= 0 \\
.4219r_3 + .3679r_2 + .2843r_1 + .1973 &= 0 \\
.3769r_3 + .2843r_2 + .1973r_1 + .1237 &= 0 \\
.2843r_3 + .1973r_2 + .1237r_1 + .0706 &= 0 \\
.1973r_3 + .1237r_2 + .0706r_1 + .0366 &= 0 \\
.1237r_3 + .0706r_2 + .0366r_1 + .0158 &= 0 \\
.0706r_3 + .0366r_2 + .0158r_1 + .0051 &= 0 \\
.0366r_3 + .0158r_2 + .0051r_1 + .003 &= 0 \\
.0158r_3 + .0051r_2 + .003r_1 + .0011 &= 0 \\
.0051r_3 + .003r_2 + .0011r_1 + .0003 &= 0\ .
\end{aligned} \qquad (4.160)$$

Choosing the first, fourth, ninth, and last equations as reference, one obtains:

$$\begin{aligned}
.1922r_2 + .3408r_1 + .4187 &= 0 \\
.4187r_3 + .4219r_2 + .3679r_1 + .2843 &= 0 \\
.1237r_3 + .0706r_2 + .0366r_1 + .0158 &= 0 \\
.0051r_3 + .003r_2 + .0011r_1 + .0003 &= 0\ .
\end{aligned} \qquad (4.161)$$

Then,

$$x_1 = (0, .1922, .3408)$$

$$x_4 = (.4187, .4219, .3679)$$

$$x_9 = (.1237, .0706, .0366)$$

$$x_{13} = (.0051, .003, .0011) . \qquad (4.162)$$

From Eq. (4.29)

$$\lambda_1 x_1 + \lambda_4 x_4 + \lambda_9 x_9 + \lambda_{13} x_{13} = 0 .$$

Hence,

$$.4187\lambda_4 + .1237\lambda_9 + .0051\lambda_{13} = 0$$

$$.1922\lambda_1 + .4219\lambda_4 + .0706\lambda_9 + .003\lambda_{13} == 0$$

$$.3408\lambda_1 + .3679\lambda_4 + .0366\lambda_9 + .0011\lambda_{13} = 0 . \qquad (4.163)$$

Let $\lambda_{13} = 1$ then $\lambda_1 = .006253$, $\lambda_4 = -.007057$, $\lambda_9 = -.017345$.

Consequently,

$$\sum_{(4)}{}' |\lambda_\sigma| = 1.030655$$

and

$$\sum_{(4)}{}' \lambda_\sigma h_{\sigma+3} = .00637775.$$

Then

$$\epsilon = \frac{\sum_{(4)}{}' \lambda_\sigma h_{\sigma+3}}{\sum_{(4)}{}' |\lambda_\sigma|} = .000619 . \qquad (4.164)$$

By Eq. (4.33),

$$\epsilon_\sigma = \epsilon \, (\text{sgn } \lambda_\sigma) .$$

Therefore,

$$\epsilon_1 = .000619$$

$$\epsilon_4 = -.000619$$

$$\epsilon_9 = -.000619$$

$$\epsilon_{13} = .000619 .$$

By Eq. (4.28),

$$.1922r_2 + .3408r_1 + .4187 - .000619 = 0$$

$$.4187r_3 + .4219r_2 + .3679r_1 + .2843 + .000619 = 0$$

$$.0051r_3 + .003r_2 + .0011r_1 + .0003 - .000619 = 0 . \quad (4.165)$$

Hence,

$$r_1 = -2.097{,}922, \quad r_2 = 1.544{,}698, \quad r_3 = -.393{,}604. \quad (4.166)$$

Now one can compute the errors in Eqs. (4.160). These are:

$$\epsilon_1 = .000619$$
$$\epsilon_2 = -.005718$$
$$\epsilon_3 = -.004588$$
$$\epsilon_4 = -.000619$$
$$\epsilon_5 = .003094$$
$$\epsilon_6 = .004131$$
$$\epsilon_7 = .003954$$
$$\epsilon_8 = .001908$$
$$\epsilon_9 = -.000617$$
$$\epsilon_{10} = .000700$$
$$\epsilon_{11} = .002301$$
$$\epsilon_{12} = -.003535$$
$$\epsilon_{13} = .000619 . \quad (4.167)$$

Since $|\epsilon_2| > |\epsilon|$, the replacement process must be undertaken.

Now,

$$x_2 = (.1922, .3408, .4187) , \quad (4.168)$$

and by Eq. (4.37),

$$\mu_1 x_1 + \mu_4 x_4 + \mu_9 x_9 + \mu_{13} x_{13} + x_2 = 0 .$$

Therefore,

$$
\begin{aligned}
.4187\mu_4 + .1237\mu_9 + .0051\mu_{13} + .1922 &= 0 \\
.1922\mu_1 + .4219\mu_4 + .0706\mu_9 + .003\mu_{13} + .3408 &= 0 \\
.3408\mu_1 + .3679\mu_4 + .0366\mu_9 + .0011\mu_{13} + .4187 &= 0 \ . \quad (4.169)
\end{aligned}
$$

Let $\mu_4 = 0$, then $\mu_1 = -1.1638$, $\mu_9 = 1.8791$, $\mu_{13} = -83.3232$.

Since sgn ϵ_σ = sgn λ_σ, and $\epsilon_i < 0$, from Table II, the equation which is designated by the number given by Max $\frac{\mu_\sigma}{\lambda_\sigma}$ must be replaced.

Now,

$$
\frac{\mu_1}{\lambda_1} = \frac{-1.1638}{.006253}, \quad \frac{\mu_4}{\lambda_4} = \frac{0}{-.007057},
$$

$$
\frac{\mu_9}{\lambda_9} = \frac{1.8791}{-.017345}, \quad \frac{\mu_{13}}{\lambda_{13}} = \frac{-83.3232}{1} \ .
$$

Max $\frac{\mu_\sigma}{\lambda_\sigma} = \frac{\mu_4}{\lambda_4}$, hence the fourth equation in Eqs. (4.160) must be replaced.

The new equations are:

$$
\begin{aligned}
.1922r_2 + .3408r_1 + .4187 &= 0 \\
.1922r_3 + .3408r_2 + .4187r_1 + .4219 &= 0 \\
.1237r_3 + .0706r_2 + .0366r_1 + .0158 &= 0 \\
.0051r_3 + .003r_2 + .0011r_1 + .0003 &= 0 \ . \quad (4.170)
\end{aligned}
$$

Then,

$$
\begin{aligned}
x_1 &= (0, .1922, .3408) \\
x_2 &= (.1922, .3408, .4187) \\
x_9 &= (.1237, .0706, .0366) \\
x_{13} &= (.0051, .003, .0011) \ . \quad (4.171)
\end{aligned}
$$

The equations for λ are:

$$.1922\lambda_2 + .1237\lambda_9 + .0051\lambda_{13} = 0$$

$$.1922\lambda_1 + .3408\lambda_2 + .0706\lambda_9 + .003\lambda_{13} = 0$$

$$.3408\lambda_1 + .4187\lambda_2 + .0366\lambda_9 + .0011\lambda_{13} = 0 \,. \qquad (4.172)$$

Let $\lambda_{13} = 1$, then $\lambda_1 = .0137585$, $\lambda_2 = -.0118285$, $\lambda_9 = -.0228503$.

Consequently,

$$\sum_{(4)}{}' |\lambda_\sigma| = 1.048437$$

and

$$\sum_{(4)}{}' \lambda_\sigma h_{\sigma+3} = .000709205 \,.$$

Then

$$\epsilon = \frac{\sum_{(4)}{}' \lambda_\sigma h_{\sigma+3}}{\sum_{(4)}{}' |\lambda_\sigma|} = .000676 \,. \qquad (4.173)$$

The errors for the reference are:

$$\epsilon_1 = .000676$$

$$\epsilon_2 = -.000676$$

$$\epsilon_9 = -.000676$$

$$\epsilon_{13} = .000676 \,.$$

By Eq. (4.28),

$$.1922r_2 + .3408r_1 + .4187 - .000676 = 0$$

$$.1922r_3 + .3408r_2 + .4187r_1 + .4219 + .000676 = 0$$

$$.1237r_3 + .0706r_2 + .0366r_1 + .0158 + .000676 = 0 \,. \qquad (4.174)$$

Hence,

$$r_1 = -2.203555; \quad r_2 = 1.732297; \quad r_3 = -.469897 \,. \qquad (4.175)$$

The errors in Eqs. (4.160) are now computed and are listed below:

$$
\begin{aligned}
\epsilon_1 &= .000676 \\
\epsilon_2 &= -.000676 \\
\epsilon_3 &= .003392 \\
\epsilon_4 &= .007722 \\
\epsilon_5 &= .009892 \\
\epsilon_6 &= .008556 \\
\epsilon_7 &= .006211 \\
\epsilon_8 &= .002603 \\
\epsilon_9 &= -.000676 \\
\epsilon_{10} &= .000511 \\
\epsilon_{11} &= .001934 \\
\epsilon_{12} &= -.004100 \\
\epsilon_{13} &= .000676 \quad .
\end{aligned} \tag{4.176}
$$

Since $|\epsilon_5| > |\epsilon|$, the replacement process must again be undertaken. Now,

$$x_5 = (.4219,\ .3679,\ .2843) \ . \tag{4.177}$$

Hence, by Eq. (4.37),

$$
\begin{aligned}
.1922\mu_2 + .1237\mu_9 + .0051\mu_{13} + .4219 &= 0 \\
.1922\mu_1 + .3408\mu_2 + .0706\mu_9 + .0030\mu_{13} + .3679 &= 0 \\
.3408\mu_1 + .4187\mu_2 + .0366\mu_9 + .0011\mu_{13} + .2843 &= 0 \ .
\end{aligned} \tag{4.178}
$$

Let $\mu_2 = 0$, then $\mu_1 = -.607373$, $\mu_9 = 1.380154$, $\mu_{13} = -116.200940$.

Since sgn ϵ_σ = sgn λ_σ and $\epsilon_i > 0$, from Table II, the equation which is designated by the number given by Min $\frac{\mu_\sigma}{\lambda_\sigma}$ must be replaced.

Now,

$$\frac{\mu_1}{\lambda_1} = -\frac{-.607373}{.0137585}, \qquad \frac{\mu_2}{\lambda_2} = \frac{0}{-.0118285},$$

$$\frac{\mu_9}{\lambda_9} = \frac{1.380154}{-.0228503}, \qquad \frac{\mu_{13}}{\lambda_{13}} = \frac{-116.200940}{1}.$$

Since $\text{Min}\,\frac{\mu_\sigma}{\lambda_\sigma} = \frac{\mu_{13}}{\lambda_{13}}$, the 13th equation in Eqs. (4.160) must be replaced.

The new equations are:

$$\begin{aligned} .1922r_2 + .3408r_1 + .4187 &= 0 \\ .1922r_3 + .3408r_2 + .4187r_1 + .4219 &= 0 \\ .4219r_3 + .3679r_2 + .2843r_1 + .1973 &= 0 \\ .1237r_3 + .0706r_2 + .0366r_1 + .0158 &= 0\,. \end{aligned} \tag{4.179}$$

Then

$$\begin{aligned} x_1 &= (0,\ .1922,\ .3408) \\ x_2 &= (.1922,\ .3408,\ .4187) \\ x_5 &= (.4219,\ .3679,\ .2843) \\ x_9 &= (.1237,\ .0706,\ .0366)\,. \end{aligned} \tag{4.180}$$

The equations for λ are:

$$\begin{aligned} .1922\lambda_2 + .4219\lambda_5 + .1237\lambda_9 &= 0 \\ .1922\lambda_1 + .3408\lambda_2 + .3679\lambda_5 + .0706\lambda_9 &= 0 \\ .3408\lambda_1 + .4187\lambda_2 + .2843\lambda_5 + .0366\lambda_9 &= 0\,. \end{aligned} \tag{4.181}$$

Let $\lambda_9 = 1$, then $\lambda_1 = -.777570$, $\lambda_2 = 1.078037$, $\lambda_5 = -.784306$.

Consequently,

$$\sum_{(4)}' |\lambda_\sigma| = 3.639913$$

and

$$\sum_{(4)}' \lambda_\sigma h_{\sigma+3} = -.009688\ .$$

Then

$$\epsilon = \frac{\sum'_{(4)} \lambda_\sigma h_{\sigma+3}}{\sum'_{(4)} |\lambda_\sigma|} = -.002662 \qquad (4.182)$$

and

$$\epsilon_1 = .002662$$
$$\epsilon_2 = -.002662$$
$$\epsilon_5 = .002662$$
$$\epsilon_9 = -.002662 .$$

By Eq. (4.28),

$$.1922r_2 + .3408r_1 + .4187 - .002662 = 0$$
$$.1922r_3 + .3408r_2 + .4187r_1 + .4219 + .002662 = 0$$
$$.4219r_3 + .3679r_2 + .2843r_1 + .1973 - .002662 = 0 . \qquad (4.183)$$

Hence, $r_1 = -2.176675, \quad r_2 = 1.694968, \quad r_3 = -.472597 . \qquad (4.184)$

The errors in Eqs. (4.160) are:

$$\epsilon_1 = .002662$$
$$\epsilon_2 = -.002662$$
$$\epsilon_3 = -.001817$$
$$\epsilon_4 = .000732$$
$$\epsilon_5 = .002661$$
$$\epsilon_6 = .002253$$
$$\epsilon_7 = .001403$$
$$\epsilon_8 = -.000649$$
$$\epsilon_9 = -.002662$$
$$\epsilon_{10} = --.000621$$
$$\epsilon_{11} = .001382$$
$$\epsilon_{12} = -.004253$$
$$\epsilon_{13} = .000580 . \qquad (4.185)$$

Since $|\epsilon_{12}| > |\epsilon|$, one of the equations in Eqs. (4.179) must be replaced.

Now,
$$x_{12} = (.0158, .0051, .0030) . \quad (4.186)$$

Hence, by Eq. (4.37),

$$\begin{aligned} .1922\mu_2 + .4219\mu_5 + .1237\mu_9 + .0158 &= 0 \\ .1922\mu_1 + .3408\mu_2 + .3769\mu_5 + .0706\mu_9 + .0051 &= 0 \\ .3408\mu_1 + .4187\mu_2 + .2843\mu_5 + .0366\mu_9 + .0030 &= 0 . \quad (4.187) \end{aligned}$$

Let $\mu_9 = 0$, then $\mu_1 = -.343950$, $\mu_2 = .431783$, and $\mu_5 = -.234152$.

Since $\text{sgn}\ \epsilon_\sigma = -\text{sgn}\ \lambda_\sigma$ and $\epsilon_i < 0$, from Table II, the equation which is designated by the number given by Min $\frac{\mu_\sigma}{\lambda_\sigma}$ must be replaced.

Now,

$$\frac{\mu_1}{\lambda_1} = \frac{-.343950}{-.777570}, \quad \frac{\mu_2}{\lambda_2} = \frac{.431783}{1.078037},$$

$$\frac{\mu_5}{\lambda_5} = \frac{-.243152}{-.784306}, \quad \frac{\mu_9}{\lambda_9} = \frac{0}{1} .$$

Since Min $\frac{\mu_\sigma}{\lambda_\sigma} = \frac{\mu_9}{\lambda_9}$, the ninth equation in Eqs. (4.160) must be replaced.

The new equations are:

$$\begin{aligned} .1922r_2 + .3408r_1 + .4187 &= 0 \\ .1922r_3 + .3408r_2 + .4187r_1 + .4219 &= 0 \\ .4219r_3 + .3679r_2 + .2843r_1 + .1973 &= 0 \\ .0158r_3 + .0051r_2 + .003r_1 + .0011 &= 0 . \quad (4.188) \end{aligned}$$

Then

$$\begin{aligned} x_1 &= (0, .1922, .3408) \\ x_2 &= (.1922, 3408, .4187) \\ x_5 &= (.4219, .3679, .2843) \\ x_{12} &= (.0158, .0051, .0030) . \quad (4.189) \end{aligned}$$

The equations for λ are, therefore:

$$
\begin{aligned}
.1922\lambda_2 + .4219\lambda_5 + .0158\lambda_{12} &= 0 \\
.1922\lambda_1 + .3408\lambda_2 + .3679\lambda_5 + .0051\lambda_{12} &= 0 \\
.3408\lambda_1 + .4187\lambda_2 + .2843\lambda_5 + .0030\lambda_{12} &= 0. \qquad (4.190)
\end{aligned}
$$

Let $\lambda_{12} = 1$, then $\lambda_1 = -.343950$, $\lambda_2 = .431783$, $\lambda_5 = -.234152$.

Consequently,

$$\sum_{(4)}{}' |\lambda_\sigma| = 2.009885$$

and

$$\sum_{(4)}{}' \lambda_\sigma h_{\sigma+3} = -.006940807 \ .$$

Hence,

$$\epsilon = \frac{\sum_{(4)}{}' \lambda_\sigma h_{\sigma+3}}{\sum_{(4)}{}' |\lambda_\sigma|} = -.003453 \qquad (4.191)$$

and

$$
\begin{aligned}
\epsilon_1 &= .003453 \\
\epsilon_2 &= -.003453 \\
\epsilon_5 &= .003453 \\
\epsilon_{12} &= -.003453 \ .
\end{aligned}
$$

By Eq. (4.28),

$$
\begin{aligned}
.1922r_2 + .3408r_1 + .4187 - .003453 &= 0 \\
.1922r_3 + .3408r_2 + .4187r_1 + .4219 + .003453 &= 0 \\
.4219r_3 + .3679r_2 + .2843r_1 + .1973 - .003453 &= 0. \qquad (4.192)
\end{aligned}
$$

Hence,

$$r_1 = -2.072670, \quad r_2 = 1.514667, \quad r_3 = -.383583. \quad (4.193)$$

The errors in Eqs. (4.160) are found to be:

$$\begin{aligned}
\epsilon_1 &= .003453 \\
\epsilon_2 &= -.003453 \\
\epsilon_3 &= -.003093 \\
\epsilon_4 &= .000197 \\
\epsilon_5 &= .003452 \\
\epsilon_6 &= .004262 \\
\epsilon_7 &= .004002 \\
\epsilon_8 &= .001953 \\
\epsilon_9 &= -.000573 \\
\epsilon_{10} &= .000708 \\
\epsilon_{11} &= .002322 \\
\epsilon_{12} &= -.003454 \\
\epsilon_{13} &= .000608 \;. \qquad (4.194)
\end{aligned}$$

Since $|\epsilon_6| > |\epsilon|$, the replacement process must again be undertaken.

Now,

$$x_6 = (.3679, .2843, .1973) \;. \qquad (4.195)$$

Hence, by Eq. (4.37),,

$$\begin{aligned}
.1922\mu_2 + .4219\mu_5 + .0158\mu_{12} + .3679 &= 0 \\
.1922\mu_1 + .3408\mu_2 + .3679\mu_5 + .0051\mu_{12} + .2843 &= 0 \\
.3408\mu_1 + .4187\mu_2 + .2843\mu_5 + .0030\mu_{12} + .1973 &= 0 \;. \qquad (4.196)
\end{aligned}$$

Let $\mu_2 = 0$, then $\mu_1 = .107547$, $\mu_5 = -.803644$, $\mu_{12} = -1.825459$.

Since $\text{sgn}\ \epsilon_\sigma = -\text{sgn}\ \lambda_\sigma$ and $\epsilon_i > 0$, from Table II, the equation which is designated by the number given by Max $\frac{\mu_\sigma}{\lambda_\sigma}$ must be replaced.

Now,

$$\frac{\mu_1}{\lambda_1} = \frac{.107547}{-.343950}, \qquad \frac{\mu_2}{\lambda_2} = \frac{0}{.431783},$$

$$\frac{\mu_5}{\lambda_5} = \frac{-.803644}{-.234152}, \qquad \frac{\mu_{12}}{\lambda_{12}} = \frac{-1.825459}{1}.$$

$\text{Max}\ \frac{\mu_\sigma}{\lambda_\sigma} = \frac{\mu_5}{\lambda_5}$, hence the fifth equation in Eqs. (4.160) must be replaced.

The new equations are:

$$\begin{aligned}
.1922r_2 + .3408r_1 + .4187 &= 0 \\
.1922r_3 + .3408r_2 + .4187r_1 + .4219 &= 0 \\
.3679r_3 + .2843r_2 + .1973r_1 + .1237 &= 0 \\
.0158r_3 + .0051r_2 + .0030r_1 + .0011 &= 0. \qquad (4.197)
\end{aligned}$$

then

$$\begin{aligned}
x_1 &= (0,\ .1922,\ .3408) \\
x_2 &= (.1922,\ .3408,\ .4187) \\
x_6 &= (.3679,\ .2843,\ .1973) \\
x_{12} &= (.0158,\ .0051,\ .0030) \qquad (4.198)
\end{aligned}$$

The equations for λ are, therefore:

$$\begin{aligned}
.\quad .1922\lambda_2 + .3679\lambda_6 + .0158\lambda_{12} &= 0 \\
.1922\lambda_1 + .3408\lambda_2 + .2843\lambda_6 + .0051\lambda_{12} &= 0 \\
.3408\lambda_1 + .4187\lambda_2 + .1973\lambda_6 + .0030\lambda_{12} &= 0\ . \qquad (4.199)
\end{aligned}$$

Let $\lambda_{12} = 1$, then $\lambda_1 = -.244985$, $\lambda_2 = .281867$, $\lambda_6 = -.190201$.

Consequently,

$$\sum_{(4)}{}' |\lambda_\sigma| = 1.717053$$

and

$$\sum_{(4)}{}' \lambda_\sigma h_{\sigma+3} = -.006083\ .$$

Hence

$$\epsilon = \frac{\sum'_{(4)} \lambda_\sigma h_{\sigma+3}}{\sum'_{(4)} |\lambda_\sigma|} = -.003543 . \qquad (4.200)$$

The errors for the reference are:

$$\epsilon_1 = .003543$$
$$\epsilon_2 = -.003543$$
$$\epsilon_6 = .003543$$
$$\epsilon_{12} = -.003543 .$$

By Eq. (4.28),

$$.1922r_2 + .3408r_1 + .4187 - .003543 = 0$$
$$.1922r_3 + .3408r_2 + .4187r_1 + .4219 + .003543 = 0$$
$$.3679r_3 + .2843r_2 + .1973r_1 + .1237 - .003543 = 0 \qquad (4.201)$$

Hence,

$$r_1 = -2.080391 \quad r_2 = 1.528826 \quad r_3 = -.392337. \qquad (4.202)$$

The errors in Eqs. (4.160) are now:

$$\epsilon_1 = .003543$$
$$\epsilon_2 = -.003543$$
$$\epsilon_3 = -.003406$$
$$\epsilon_4 = -.000336$$
$$\epsilon_5 = .002773$$
$$\epsilon_6 = .003543$$
$$\epsilon_7 = .003352$$
$$\epsilon_8 = .001432$$
$$\epsilon_9 = -.000939$$
$$\epsilon_{10} = .000486$$
$$\epsilon_{11} = .002186$$

$$\epsilon_{12} = -.003543$$

$$\epsilon_{13} = .000597 \quad .$$

Since $|\epsilon_v| \leq |\epsilon|$ $(v = 1, \ldots, 13)$,

$$r_1 = -2.080391, \quad r_2 = 1.528826, \quad r_3 = -.392337,$$

are the desired solutions.

From Eq. (4.42)

$$P(y) = y^3 - 2.080391y^2 + 1.528826y - .392337 = 0 \qquad (4.204)$$

is the equation for y_k $(k = 1, 2, 3)$.

Prior to solving Eq. (4.204) it may be of value to best it using Eq. (4.48) to determine whether all y_k lie inside the unit circle. By Eq. (4.48),

$$\sum_{j=0}^{3} w^{3-j} \left\{ \sum_{k=0}^{3} (-1)^k r_k \left[\sum_{m=0}^{j} \binom{k}{m} \binom{3-k}{j-m} (-1)^m \right] \right\} = 0 . \qquad (4.205)$$

Expanding,

$$w^3(1 - r_1 + r_2 - r_3) + w_2(3 - r_1 - r_2 + 3r_3) +$$

$$w(3 + r_1 - r_2 - 3r_3) + (1 + r_1 + r_2 + r_3) = 0 .$$

Therefore,

$$P(w) = 5.001554\, w^3 + 2.374554\, w^2 + .567794\, w + .056098 = 0. \qquad (4.206)$$

Application of the Routh's test to P(w) shows that the roots are in the left-hand plane; hence, the roots of P(y) are inside the unit circle, as required.

Solving for the roots of P(y), one obtains:

$$y_1 = .683177$$

$$y_2 = .698607 + j\,.293651$$

$$y_3 = .698607 - j\,.293651 \quad . \qquad (4.207)$$

The poles from Table III are:

$$s_1 = \frac{1}{(.2)} \ln (.683177) = -1.905$$

$$s_2 = \frac{1}{(.4)} \ln (.574283) + j \frac{1}{(.2)} \text{arc tan} \frac{.293651}{.698607}$$

$$= -1.3866 + j\,1.98959$$

$$s_3 = -13866 - j\,1.98959 \quad . \qquad (4.208)$$

With the poles determined, one can now determine the residues. From Eq. (4.54) one can determine the coefficients b_{1m}, a'_{2m} and b'_{2m} from

$$b_{1m} = e^{s_1 t_m}$$

$$a'_{2m} = 2e^{\alpha_2 t_m} \cos \beta_2 t_m$$

$$b'_{2m} = -2e^{\alpha_2 t_m} \sin \beta_2 t_m \qquad (4.209)$$

$$(m = 1, 2, \ldots, 16).$$

The values of b_{1m}, a'_{2m}, b'_{2m} are computed and are listed in Table VIII.

TABLE VIII

VALUES OF b_{1m}, a'_{2m} AND b'_{2m} AT INTERVAL POINTS

m	t_m	b_{1m}	a'_{2m}	b'_{2m}
1	0	1.0000	2.0000	0
2	.2	.6832	1.3972	-.5873
3	.4	.4667	.8036	-.8206
4	.6	.3189	.3205	-.8093
5	.8	.2178	-.0138	-.6594
6	1.0	.1488	-.2033	-.4566
7	1.2	.1017	-.2761	-.2593
8	1.4	.0695	-.2784	-.1036
9	1.6	.0475	-.2173	.0091
10	1.8	.0324	-.1492	.0702
11	2.0	.0221	-.0836	.0928
12	2.2	.0151	-.0311	.0894
13	2.4	.0103	.0045	.0716
14	2.6	.0071	.0242	.0487
15	2.8	.0048	.0312	.0269
16	3.0	.0033	.0297	.0097

From Eq. (4.53) one obtains the relationship

$$h_m = b_{1m}A_1 + a'_{2m}\,a + b'_{2m}b \qquad (4.210)$$

$$(m = 1, 2, \ldots, 16).$$

Subsituting values for h_m from Table VII, and values for b_{1m}, a'_{2m} and b'_{2m} from Table VIII, one obtains:

$$0 = A_1 + 2a$$

$$.1922 = .6832A_1 + 1.3972a - .5873b$$

$$.3408 = .4667A_1 + .8036a - .8206b$$

$$.4187 = .3189A_1 + .3205a - .8093b$$

$$.4219 = .2178A_1 - .0138a - .6594b$$

$$.3679 = .1488A_1 - .2033a - .4566b$$

$$.2843 \quad .1017A_1 - .2761a - .2593b$$

$$
\begin{aligned}
.1973 &= .0695A_1 - .2784a - .1036b \\
.1237 &= .0475A_1 - .2173a + .0091b \\
.0706 &= .0324A_1 - .1492a + .0702b \\
.0366 &= .0221A_1 - .0836a + .0928b \\
.0158 &= .0151A_1 - .0311a + .0894b \\
.0051 &= .0103A_1 + .0045a + .0716b \\
.0030 &= .0071A_1 + .0242a + .0487b \\
.0011 &= .0048A_1 + .0312a + .0269b \\
.0003 &= .0033A_1 + .0297a + .0097b \; . \qquad (4.211)
\end{aligned}
$$

Choosing the first, fourth, sixth, and eleventh equations as reference, one obtains:

$$
\begin{aligned}
A_1 + 2a &= 0 \\
.3189A_1 + .3205a - .8093b - .4187 &= 0 \\
.1488A_1 - .2033a - .4566b - .3679 &= 0 \\
.0221A_1 - .0836a - .0928b - .0366 &= 0 \; . \qquad (4.212)
\end{aligned}
$$

Then,

$$
\begin{aligned}
x_1 &= (1, 2, 0) \\
x_4 &= (.3189, .3205, -.8093) \\
x_6 &= (.1488, -.2033, -.4566) \\
x_{11} &= (.0221, -.0836, .0928) \; . \qquad (4.213)
\end{aligned}
$$

The equations for λ are:

$$
\begin{aligned}
\lambda_1 + .3189\lambda_4 + .1488\lambda_6 + .0221\lambda_{11} &= 0 \\
2\lambda_1 + .3205\lambda_4 - .2033\lambda_6 + .0836\lambda_{11} &= 0 \\
- .8093\lambda_4 - .4566\lambda_6 + .0928\lambda_{11} &= 0 \; . \qquad (4.214)
\end{aligned}
$$

Let $\lambda_{11} = 1$, then $\lambda_1 = -.074541$, $\lambda_4 = .402446$, $\lambda_6 = -.510074$.

Then, $$\sum_{(4)}{}' |\lambda_\sigma| = 1.987061$$

and

$$\sum_{(4)}{}' \lambda_\sigma(-h_\sigma) = -.0174479 \ .$$

Hence,

$$\epsilon = \frac{\sum_{(4)}{}' \lambda_\sigma(-h_\sigma)}{\sum_{(4)}{}' |\lambda_\sigma|} = -.008781 \ . \qquad (4.215)$$

Since

$$\epsilon_\sigma = (\mathrm{sgn}\ \lambda_\sigma) \ ,$$

therefore,

$$\epsilon_1 = .008781$$
$$\epsilon_4 = -.008781$$
$$\epsilon_6 = .008781$$
$$\epsilon_{11} = -.008781 \ .$$

Consequently,

$$\begin{aligned} A_1 + 2a \qquad\qquad\qquad\qquad - .008781 &= 0 \\ .3189A_1 + .3205a - .8093b - .4187 + .008781 &= 0 \\ .1488A_1 - .2033a - .4566b - .3679 - .008781 &= 0 \ , \end{aligned} \qquad (4.216)$$

yielding

$$A_1 = .913966, \quad a = -.452592, \quad b = -.325604 \ . \qquad (4.217)$$

The errors in Eqs. (4.211) are:

$$\epsilon_1 = .008782$$
$$\epsilon_2 = -.008913$$
$$\epsilon_3 = -.010764$$
$$\epsilon_4 = -.008781$$
$$\epsilon_5 = -.001889$$
$$\epsilon_6 = .008781$$

$$\epsilon_7 = .018040$$
$$\epsilon_8 = .025955$$
$$\epsilon_9 = .015099$$
$$\epsilon_{10} = .003682$$
$$\epsilon_{11} = -.008781$$
$$\epsilon_{12} = -.017032$$
$$\epsilon_{13} = -.021036$$
$$\epsilon_{14} = -.023320$$
$$\epsilon_{15} = -.019593$$
$$\epsilon_{16} = -.013884 \ . \qquad (4.218)$$

Since $|\epsilon_8| > |\epsilon|$, the replacement process must be undertaken.

Now,

$$x_8 = (.0695, -.2784, -.1036) \ , \qquad (4.219)$$

and, by Eq. (4.37), the equations for μ are:

$$\begin{aligned} \mu_1 + .3189\mu_4 + .1488\mu_6 + .0221\mu_{11} + .0695 &= 0 \\ 2\mu_1 + .3205\mu_4 - .2033\mu_6 - .0836\mu_{11} - .2784 &= 0 \\ - .8093\mu_4 - .4566\mu_6 + .0928\mu_{11} - .1036 &= 0 \ . \end{aligned} \qquad (4.220)$$

Let $\mu_{11} = 0$, then $\mu_1 = -.065105$, $\mu_4 = .532406$, $\mu_6 = -1.170558$.

Since $\operatorname{sgn} \epsilon_\sigma = -\operatorname{sgn} \lambda_\sigma$ and $\epsilon_i > 0$, hence from Table II, the equation which is designated by the number given by Max $\frac{\mu_\sigma}{\lambda_\sigma}$ must be replaced.

Now,

$$\frac{\mu_1}{\lambda_1} = \frac{-.065105}{-.074541} \ , \qquad \frac{\mu_4}{\lambda_4} = \frac{.532406}{.402446} \ ,$$

$$\frac{\mu_6}{\lambda_6} = \frac{-1.170558}{-.510074}, \qquad \frac{\mu_{11}}{\mu_{11}} = \frac{0}{1} .$$

Max $\frac{\mu_\sigma}{\lambda_\sigma} = \frac{\mu_6}{\mu_6}$, thus the sixth equation in Eq. (4.211) must be replaced.

The new equations are:

$$\begin{aligned}
A_1 + 2a \qquad\qquad\qquad &= 0 \\
.3189A_1 + .3205a - .8093b - .4187 &= 0 \\
.0695A_1 - .2784a - .1036b - .1973 &= 0 \\
.0221A_1 - .0836a + .0928b - .0366 &= 0 .
\end{aligned} \tag{4.221}$$

Then,

$$\begin{aligned}
x_1 &= (1, 2, 0) \\
x_4 &= (.3189, .3205, -.8093) \\
x_8 &= (.0695, -.2784, -.1036) \\
x_{11} &= (.0221, -.0836, .0928) .
\end{aligned} \tag{4.222}$$

The λ are determined from

$$\begin{aligned}
\lambda_1 + .3189\lambda_4 + .0695\lambda_8 + .0221\lambda_{11} &= 0 \\
2\lambda_1 + .3205\lambda_4 - .2784\lambda_8 - .0836\lambda_{11} &= 0 \\
- .8093\lambda_4 - .1036\lambda_8 + .0928\lambda_{11} &= 0 .
\end{aligned} \tag{4.223}$$

Let $\lambda_{11} = 1$, then $\lambda_1 = -.046171$, $\lambda_4 = .170449$, $\lambda_8 = -.435753$.

Then

$$\sum_{(4)}' |\lambda_\sigma| = 1.652373$$

and

$$\sum_{(4)}' \lambda(-h_\sigma) = -.0219929 .$$

Consequently,

$$\epsilon = \frac{\sum_{(4)}' \lambda_\sigma(-h_\sigma)}{\sum_{(4)}' |\lambda_\sigma|} = - .013310 . \tag{4.224}$$

Since

$$\epsilon_\sigma = \epsilon\ (\text{sgn}\ \lambda_\sigma)\ ,$$

therefore,

$$\epsilon_1 = .013310$$

$$\epsilon_4 = -.013310$$

$$\epsilon_8 = .013310$$

$$\epsilon_{11} = -.013310\ .$$

By Eq. (4.28),

$$\begin{aligned} A_1 + 2a \qquad\qquad\qquad\qquad - .013310 &= 0 \\ .3189A_1 + .3205a - .8093b - .4187 + .013310 &= 0 \\ .0695A_1 - .2784a - .1036b - .1973 - .013310 &= 0\ . \end{aligned} \tag{4.225}$$

Hence,

$$A_1 = .853762, \quad a = -.420226, \quad b = -.330914. \tag{4.226}$$

The errors in Eq. (4.211) are:

$$\epsilon_1 = .013310$$

$$\epsilon_2 = -.001704$$

$$\epsilon_3 = -.008495$$

$$\epsilon_4 = -.013309$$

$$\epsilon_5 = -.011947$$

$$\epsilon_6 = -.004333$$

$$\epsilon_7 = .004358$$

$$\epsilon_8 = .013310$$

$$\epsilon_9 = .005157$$

$$\epsilon_{10} = -.003471$$

$$\epsilon_{11} = -.013309$$

$$\epsilon_{12} = -.019423$$

$$\epsilon_{13} = -.021891$$

$$\epsilon_{14} = -.023223$$

$$\epsilon_{15} = -.019015$$

$$\epsilon_{16} = -.013173 \ . \qquad (4.227)$$

Since $|\epsilon_{14}| > |\epsilon|$, the replacement process must take place.

Now, $$x_{14} = (.0071, .0242, .0487) \ , \qquad (4.228)$$

and, by Eq. (4.37), the μ are determined from

$$\begin{aligned} \mu_1 + .3189\mu_4 + .0695\mu_8 + .0221\mu_{11} + .0071 &= 0 \\ 2\mu_1 + .3205\mu_4 - .2784\mu_8 - .0836\mu_{11} + .0242 &= 0 \\ - .8093\mu_4 - .1036\mu_8 + .0928\mu_{11} + .0487 &= 0 \ . \end{aligned} \qquad (4.229)$$

Let $\mu_{11} = 0$, then $\mu_1 = -.027440$, $\mu_4 = .055043$, $\mu_8 = .040093$.

Since sgn $\epsilon_\sigma = -$sgn λ_σ and $\epsilon_i < 0$, hence from Table II, the equation which is designated by the number given by Min $\frac{\mu_\sigma}{\lambda_\sigma}$ must be replaced.

Now,

$$\frac{\mu_1}{\lambda_1} = \frac{-.027440}{-.046171} , \qquad \frac{\mu_4}{\lambda_4} = \frac{.055043}{.0170449} ,$$

$$\frac{\mu_8}{\lambda_8} = \frac{.040093}{-.435753} , \qquad \frac{\mu_{11}}{\lambda_{11}} = \frac{0}{1} .$$

Min $\frac{\mu_\sigma}{\lambda_\sigma} = \frac{\mu_8}{\lambda_8}$, hence, the eighth equation in Eqs. (4.211) must be replaced.

The new equations are:

$$\begin{aligned} A_1 + 2a &= 0 \\ .3189A_1 + .3205a - .8093b - .4187 &= 0 \\ .0221A_1 - .0836a + .0928b - .0366 &= 0 \\ .0071A_1 + .0242a + .0487b - .0030 &= 0 \, . \end{aligned} \tag{4.230}$$

Then,

$$\begin{aligned} x_1 &= (1, 2, 0) \\ x_4 &= (.3189, .3205, -.8093) \\ x_{11} &= (.0221, -.0836, .0928) \\ x_{14} &= (.0071, .0242, .0487) \, . \end{aligned} \tag{4.231}$$

The equations for λ are:

$$\begin{aligned} \lambda_1 + .3189\lambda_4 + .0221\lambda_{11} + .0071\lambda_{14} &= 0 \\ 2\lambda_1 + .3205\lambda_4 - .0836\lambda_{11} + .0242\lambda_{14} &= 0 \\ - .8093\lambda_4 + .0928\lambda_{11} + .0487\lambda_{14} &= 0 \, . \end{aligned} \tag{4.232}$$

Let $\lambda_{14} = 1$, then $\lambda_1 = -.023041$, $\lambda_4 = .053825$, $\lambda_{11} = -.055387$.

Then,

$$\sum_{(4)}{}' |\lambda_\sigma| = 1.132253$$

and

$$\sum_{(4)}{}' \lambda_\sigma(-h_\sigma) = -.0235094 \, .$$

Hence,

$$\epsilon = \frac{\sum_{(4)}{}' \lambda_\sigma(-h_\sigma)}{\sum_{(4)}{}' |\lambda_\sigma|} = -.020763 \, . \tag{4.233}$$

Since

$$\epsilon_\sigma = \epsilon \, (\text{sgn} \, \lambda_\sigma) \, ,$$

therefore,

$$\epsilon_1 = .020763$$
$$\epsilon_4 = -.020763$$
$$\epsilon_{11} = .020763$$
$$\epsilon_{14} = -.020763 .$$

By Eq. (4.28),

$$\begin{aligned} A_1 + 2a &= 0 \\ .3189A_1 + .3205a - .8093b - .4187 + .020763 &= 0 \\ .0221A_1 - .0836a + .0928b - .0366 - .020763 &= 0 . \end{aligned} \quad (4.234)$$

Hence,

$$A_1 = 1.260533, \quad a = -.619885, \quad b = -.240487. \quad (4.235)$$

The errors in Eqs. (4.211) are:

$$\epsilon_1 = .020763$$
$$\epsilon_2 = -.055869$$
$$\epsilon_3 = -.053305$$
$$\epsilon_4 = -.020763$$
$$\epsilon_5 = .019776$$
$$\epsilon_6 = .055496$$
$$\epsilon_7 = .077405$$
$$\epsilon_8 = .087797$$
$$\epsilon_9 = .068688$$
$$\epsilon_{10} = ..045846$$
$$\epsilon_{11} = .020763$$
$$\epsilon_{12} = .001013$$
$$\epsilon_{13} = -.012125$$
$$\epsilon_{14} = -.020763$$

$$\epsilon_{15} = -.020859$$

$$\epsilon_{16} = -.016884 \ . \qquad (4.236)$$

Since $|\epsilon_8| > |\epsilon|$, the replacement process must proceed.

Now,

$$x_8 = (.0695, -.2784, -.1036) \ , \qquad (4.237)$$

and, by Eq. (4.37), the equations for μ are:

$$\begin{aligned} \mu_1 + .3189\mu_4 + .0221\mu_{11} + .0071\mu_{14} + .0695 &= 0 \\ 2\mu_1 + .3205\mu_4 - .0836\mu_{11} + .0242\mu_{14} - .2784 &= 0 \\ - .8093\mu_4 + .0928\mu_{11} + .0487\mu_{14} - .1036 &= 0 \ . \end{aligned} \qquad (4.238)$$

Let $\mu_{14} = 0$, then $\mu_1 = .105957$, $\mu_4 = -.391159$, $\mu_{11} = -2.294879$.

Since $\text{sgn}\ \epsilon_\sigma = -\text{sgn}\ \lambda_\sigma$ and $\epsilon_i > 0$, hence, from Table II, the equation which is designated by the number given by Max $\frac{\mu_\sigma}{\lambda_\sigma}$ must be replaced.

Now,

$$\frac{\mu_1}{\lambda_1} = \frac{.105957}{-.023041} \qquad \frac{\mu_4}{\lambda_4} = \frac{-.391159}{.053825}$$

$$\frac{\mu_{11}}{\lambda_{11}} = \frac{-2.294879}{-.055387} \qquad \frac{\mu_{14}}{\lambda_{14}} = \frac{0}{1} \ .$$

Since Max $\frac{\mu_\sigma}{\lambda_\sigma} = \frac{\mu_{11}}{\lambda_{11}}$, the eleventh equation in Eqs. (4.211) must be replaced.

The new equations for the reference are:

$$\begin{aligned} A_1 + 2a &= 0 \\ .3189A_1 + .3205a - .8093b - .4187 &= 0 \\ .0695A_1 - .2784a - .1036b - .1973 &= 0 \\ .0071A_1 + .0242a + .0487b - .0030 &= 0 \ . \end{aligned} \qquad (4.239)$$

Then,

$$x_1 = (1, 2, 0)$$
$$x_4 = (.3189, .3205, -.8093)$$
$$x_8 = (.0695, -.2784, -.1036)$$
$$x_{14} = (.0071, .0242, .0487) \ . \qquad (4.240)$$

The equations for λ are:

$$\lambda_1 + .3189\lambda_4 + .0695\lambda_8 + .0071\lambda_{14} = 0$$
$$2\lambda_1 + .3205\lambda_4 - .2784\lambda_8 + .0242\lambda_{14} = 0$$
$$- .8093\lambda_4 - .1036\lambda_8 + .0487\lambda_{14} = 0 \ . \qquad (4.241)$$

Let $\lambda_{14} = 1$, then $\lambda_1 = -.025598$, $\lambda_4 = .063265$, $\lambda_8 = -.024135$.

Then,

$$\sum_{(4)}' |\lambda_\sigma| = 1.112998$$

and

$$\sum_{(4)}' \lambda_\sigma(-h_\sigma) = -.0247272 \ .$$

Consequently,

$$\epsilon = \frac{\sum_{(4)}' \lambda_\sigma(-h_\sigma)}{\sum_{(4)}' |\lambda_\sigma|} = -.022217 \ . \qquad (4.242)$$

Since

$$\epsilon_\sigma = \epsilon \ (\text{sgn } \lambda_\sigma) \ ,$$

therefore,

$$\epsilon_1 = .022217$$
$$\epsilon_4 = -.022217$$
$$\epsilon_8 = .022217$$
$$\epsilon_{14} = -.022217 \ .$$

By Eq. (4.28):

$$A_1 + 2a - .022217 = 0$$
$$.3189A_1 + .3205a - .8093b - .4187 + .022217 = 0$$
$$.0695A_1 - .2784a - .1036b - .1973 - .022217 = 0 \quad . \qquad (4.243)$$

Hence,

$$A_1 = .914645, \quad a = -.446214, \quad b = -.306209 \quad .(4.244)$$

The errors in Eqs. (4.211) are:

$$\epsilon_1 = .022217$$
$$\epsilon_2 = -.010928$$
$$\epsilon_3 = -.021238$$
$$\epsilon_4 = -.022216$$
$$\epsilon_5 = -.014618$$
$$\epsilon_6 = -.001270$$
$$\epsilon_7 = .011319$$
$$\epsilon_8 = .022217$$
$$\epsilon_9 = .013921$$
$$\epsilon_{10} = .004114$$
$$\epsilon_{11} = -.007499$$
$$\epsilon_{12} = -.015487$$
$$\epsilon_{13} = -.019612$$
$$\epsilon_{14} = -.022217$$
$$\epsilon_{15} = -.018869$$
$$\epsilon_{16} = -.013504 \quad . \qquad (4.245)$$

Since $|\epsilon_m| \leq |\epsilon|$ $(m = 1, \ldots, 16)$ A_1, a, b, are the desired values, and the three-term approximation is:

$$h^*(t) = .914645\, e^{-1.905t} + (-.446214 - j.306209)\, e^{(-1.3866 + j1.98959)t} + (-.446214 + j.306209)\, e^{(-1.3866 - j1.98959)t}, \qquad (4.246)$$

or

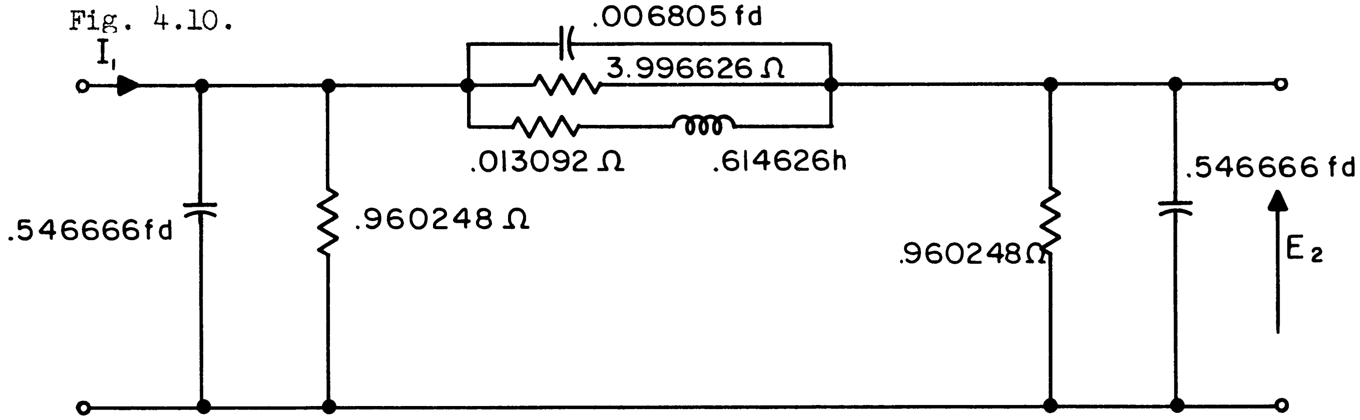
$$h^*(t) = .914645\, e^{-1.905t} + (-.446214)(2e^{-13866t} \cos 1.9859t) + (-306209)(-2e^{-1.3866t} \sin 1.98959t). \qquad (4.247)$$

The Tschebyscheff error is .022217.

The Laplace transform of Eq. (4.224) is:

$$H^*(s) = \frac{(.022217)(s^2 + 36.7901s + 239.8808)}{(s + 1.905)(s^2 + 2.7732s + 5.866458)}. \qquad (4.248)$$

A network realizing H*(s) as a transfer impedance is shown in Fig. 4.10.



$$H^*(s) = z_{12}(s) = \frac{E_2}{I_1} = \frac{(.022217)(s^2 + 36.7901s + 239.8808}{(s + 1.905)(s^2 + 2.7732s + 5.866458)}$$

FIG. 4.10 NETWORK REALIZING THE h*(t) OF EQ. (4.247)

Plots of $h(t)$ and $h^*(t)$ vs. t are shown in Fig. 4.11. The Tschebyscheff error is, in accordance with Eq. (4.242), .022217. The plot of $[h^*(t) - h(t)]$ vs. t is shown in Fig. 4.12.

4.6 Conclusion and Summary

In the preceding sections of this chapter, the problem of approximation of the impulse response $h(t)$ by a function $h^*(t)$, whose Laplace transform is a network function has been solved. The process developed yields such a function that Max $|h(t) - h^*(t)|$ is minimized at the interval points. Since the terms of the approximating function are well-behaved functions themselves, with exponentially-decaying envelopes, it can be argued that for sufficiently small intervals a good approximation at the interval points will yield a good approximation between the interval points. The application of this method has shown that good approximations are to be expected and that the Tschebyscheff error for the residues is a meaningful indicator of the overall maximum error to be expected.

The amount of numerical work involved in obtaining $h^*(t)$ is governed by two numbers, q and n. q is the number of equispaced points [at which h_m $(m = 1, 2, \ldots, q)$ is known] considered, and n is the number of terms in $h^*(t)$. It has been observed that the amount of computational work varies roughly linearly with q, but goes up roughly with the square of n (i.e., for a given q, the amount of work for $n = 4$ is roughly $\frac{16}{9}$ times the amount for $n = 3$). This rough estimate of computational work to be expected should enable one to decide at which point it is advisable to utilize automatic computers for the calculation of $h^*(t)$.

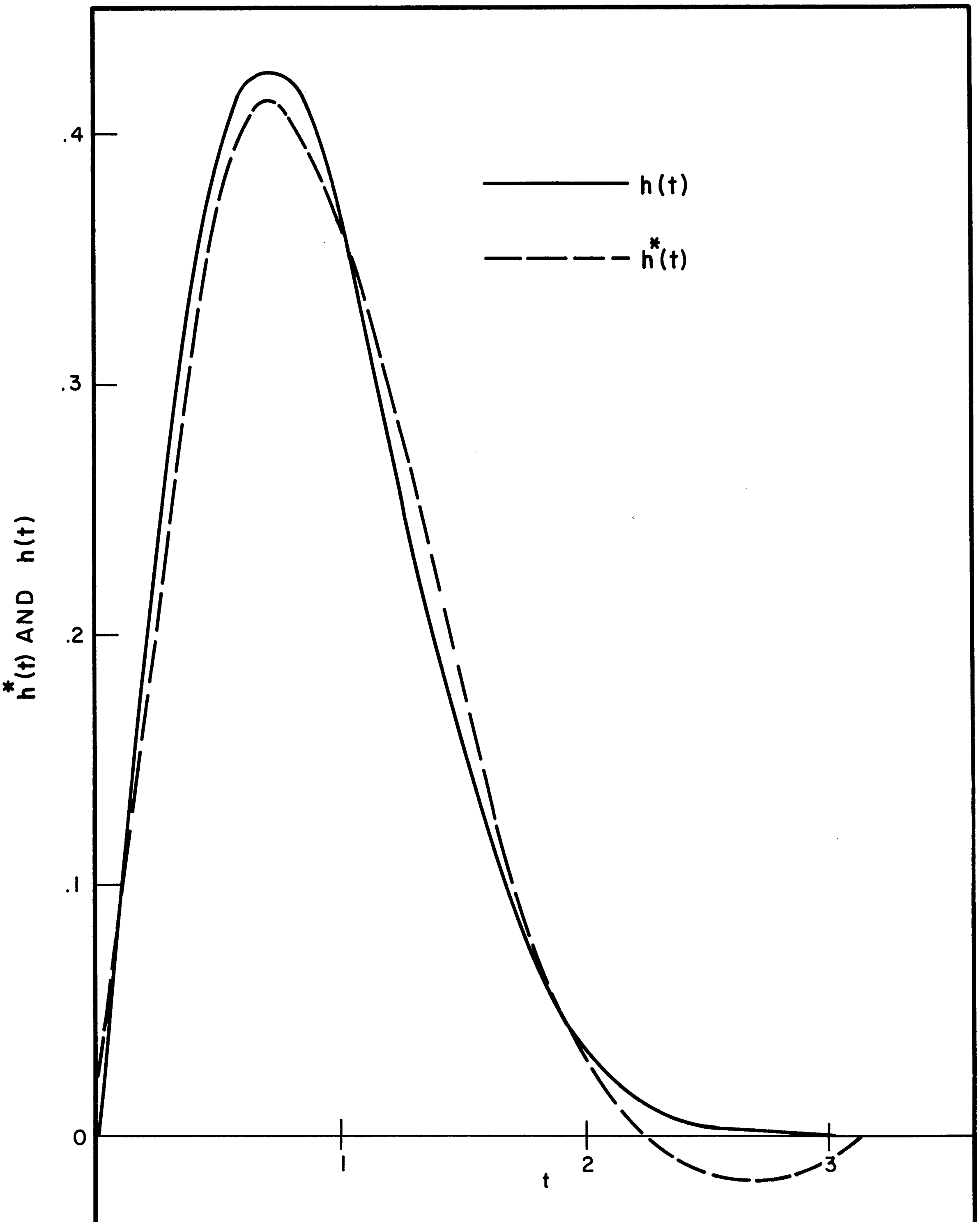


FIG. 4.11 PLOT OF $h^*(t)$ AND $h(t)$
(EXAMPLE OF SEC. 4.5.2)

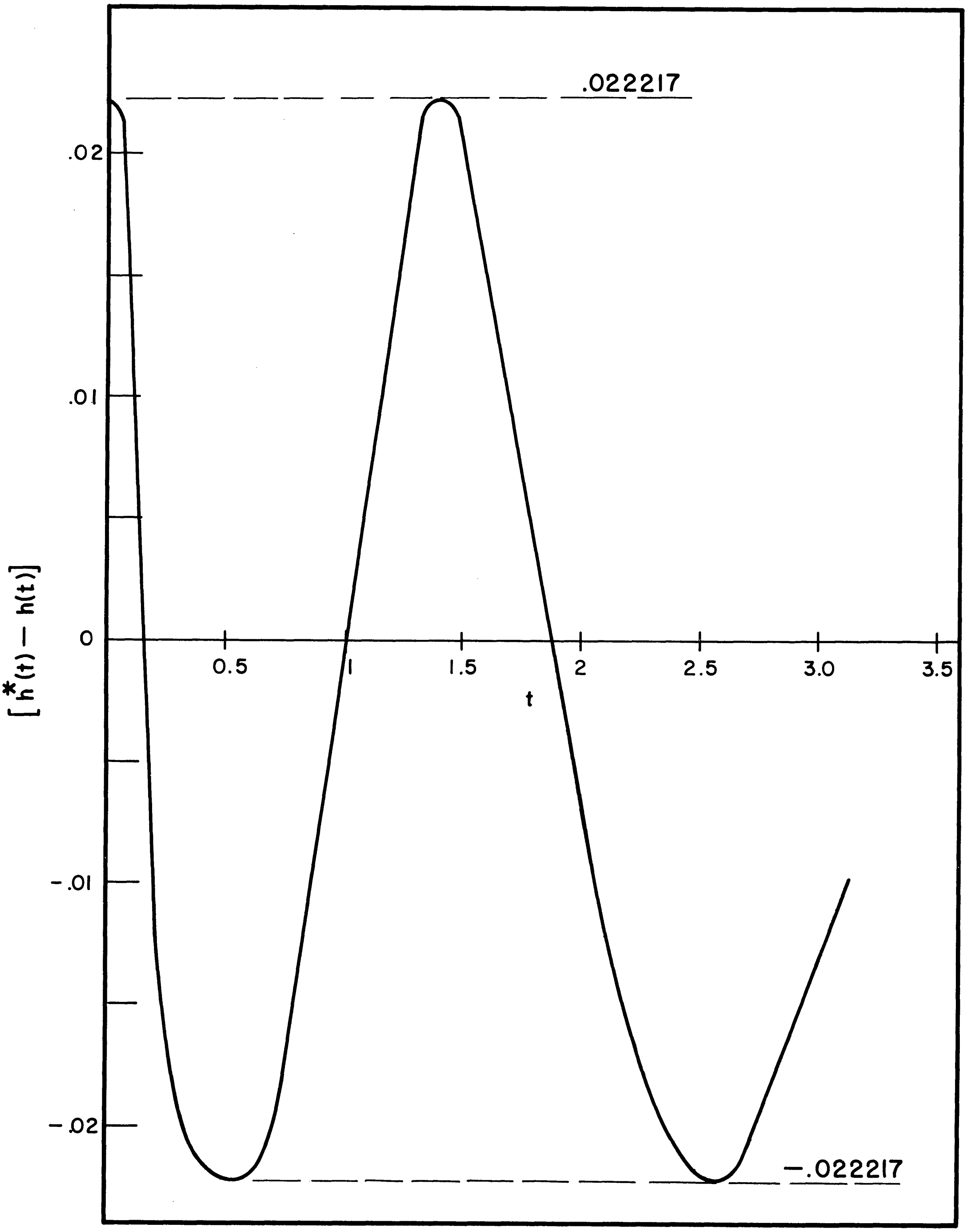


FIG. 4.12 PLOT OF $[h^*(t) - h(t)]$ (EXAMPLE OF SEC. 4.5.2)

The choice of a number for q is dictated by two considerations. These are: (1) the length of the time interval of interest, and (2) the behavior of h(t). If h(t) decays slowly, many points must be considered and q will be large. Similarly, if h(t) displays wild variations (i.e., the derivative of h(t) changes sign often) then q will also be large. Of course, if h(t) varies rather wildly, one may "smooth" h(t) first, and then approximate. Fortunately, the impulse responses demanded in practice from R-L-C networks are relatively well behaved. However, it may be of theoretical interest to consider an approximation to a wildly varying function by the proposed method. Such a function will demand a choice of a large number for q. The number chosen for q should be no less than the time interval in which approximation takes place divided by the smallest time interval between a relative maximum and a relative minimum of h(t). A good choice for q is a matter of judgement.

The choice of a number for n is dictated by two conflicting requirements. These are: (1) the magnitude of the approximation error and (2) the complexity of the resulting network. An increase in n reduces the error but does increase the complexity of the network (i.e., increases the number of network elements). Therefore, the choice of n will be determined either by the maximum error that can be tolerated, or by the maximum allowable complexity of the desired network, or both. Hence, the choice of n is a matter of engineering judgement and should not be prescribed without the knowledge of specific requirements.

In many application, it is desired to solve one of two possible problems: (1) the maximum allowable error is prescribed, and one wants the simplest network function which will satisfy the requirements on the error, and (2) the maximum number of elements in the network is prescribed, and one desirs a network which mimimizes the error.

The first problem requires the determination of the minimum n which will satisfy the error requirements. This requires a selection of n, and by the methods of Section 4.4, after some computational work, one can determine the magnitude of the expected final error. However, the precise value of the final error is determined with nearly the final computation. Hence, it is certainly conceivable that at the end of the computational process one may discover that either (1) the allowable error has been exceeded, or (2) the final error is sufficiently below the allowable error to question whether a choice of a smaller number for n would have been more appropriate. In both these cases one may have to repeat the computational process with a different choice for n. A remedy for these two possibilities would be a straightforward relationship between the final error and n. Unfortunately, such a relationship is not apparent. Fortunately, however, rarely if ever does one have to carry the process to near completion before discovering that a better choice for n was indicated. The Tschebyscheff error of every cycle conveys a more precise knowledge of the expected final error than the knowledge available at a previous cycle.

The second problem can be solved in a straightforward manner. n is approximately equal to twice the number of elements. Hence, n is prescribed and one can solve the approximation problem.

The approximation procedure will be now summarized. Prior to approximation, the preseribed impulse response h(t) should be subjected to a preliminary simplification. If h(t) has exponential terms, these terms can be subtracted from h(t) and the remainder approximated. The subtracted terms are then added to the approximated function obtained. Also, a replacement of t by a linear function of t [if t is replaced by a(t+b), then b represents a delay, and a represents change of time scale] may give rise to simplifications. The approximation obtained is then modified accordingly.

In some instances, it is simpler to approximate the differential or the integral of h(t) rather than h(t) itself. (The system function obtained is then multiplied or divided by s). Of course, then, h*(t) will not approximate h(t) in the Tschebyscheff sense.

As a result of the preliminary simplifications, the prescribed h(t) is at the start of the approximation process in its simplest form. The interval of approximation is now selected. After the choice of the numbers n and q has been made p and d are computed and the values for h_m (m = 1, 2, ..., q) are determined. The remaining steps of the process are outlined below:

1. Obtain p equations from $\sum_{k=0}^{n} h_{v+k} r_{n-k} = 0)(v = 1, 2, \ldots, p)$.
2. Select $[E_\sigma]$.
3. Find the n+1 x_σ from: $x_\sigma = (h_\sigma, h_{\sigma+1}, \ldots, h_{\sigma+n-1})$.
4. Find λ_σ from $\sum_{n+1}' \lambda_\sigma x_\sigma = 0$ (one of the λ is arbitrary, but $\lambda \neq 0$).
5. Compute
$$\epsilon = \frac{\sum_{(n+1)}' \lambda_\sigma h_{\sigma+n}}{\sum_{(n+1)}' |\lambda_\sigma|} .$$

6. Find the n+1 ϵ_σ from $\epsilon_\sigma = \epsilon \; (\mathrm{sgn}\; \lambda_\sigma)$.

7. Find r_k (k = 1, 2, ..., n) from n equations

$$\sum_{k=0}^{n} h_{\sigma+k} \, r_{n-k} - \epsilon \; (\mathrm{sgn}\; \lambda_\sigma) \;\; = \;\; 0 \; .$$

8. Compute ϵ_v (v = 1, 2, ..., p) from

$$\epsilon_v \;\; = \;\; \sum_{k=0}^{n} h_{v+k} r_{n-k} \;\; (\epsilon_v \text{ for } v = \sigma \text{ were already computed in 6}).$$

9. Is $|\epsilon_v| \leq |\epsilon_\sigma|$? If Yes, next step is 17. If No, next step is 10.

10. Find i from $|\epsilon_i| = |\epsilon_v|_{max}$ (v = 1, 2, ..., p). (If more than one max, choose any max).

11. Find $x_i = (h_i, h_{i+1}, \ldots, h_{i+n-1})$.

12. Find μ_σ from $\sum'_{(n+1)} \mu_\sigma x_\sigma + x_i = 0$ (one of the μ is arbitrary).

13. Compute $\dfrac{\mu_\sigma}{\lambda_\sigma}$.

14. Find from Table III the hyper-plane E_ℓ which is to be replaced by E_i.

15. Replace E_ℓ by E_i, forming a new reference.

16. Repeat from step 3.

17. Apply Routh's test to:

$$\sum_{j=0}^{n} w^{n-j} \left\{ \sum_{k=0}^{n} (-1)^k r_k \left[\sum_{m=0}^{j} \binom{k}{m} \binom{n-k}{j-m} (-1)^m \right] \right\} \;\; = \;\; 0$$

$$(r_0 \;=\; 1).$$

Are all roots in the left-hand plane? If Yes next step is 20. If No next step is 18.

18. Increase the interval of approximation, choosing a new q ($q_{new} > q_{old}$; $p_{new} > _{old}$).

19. Repeat from step 1.

20. Solve $\sum_{k=0}^{n} r_{n-k}\, y^k = 0 \quad (r_o = 1)$.

21. Compute poles from Table III.

22. Find b_{km}, $a'_{\ell m}$, $b'_{\ell m}$ from

$$b_{km} = e^{s_k t_m} \qquad (k = 1, 2, \ldots, n-2w)$$

$$a'_{\ell m} = 2e^{\alpha_\ell t_m} \cos \beta_\ell t_m \quad (k = 1, 2, \ldots, w)$$

$$b'_{\ell m} = -2e^{\alpha_\ell t_m} \sin \beta_\ell t_m \quad ,$$

where s_k are real poles, $s_\ell = \alpha_\ell + j\beta_\ell$ are complex poles. 2w is the number of complex poles.

23. Obtain q equations from

$$\sum_{k=1}^{n-2w} b_{km} A_k + \sum_{\ell=1}^{w} (a'_{\ell m} a_\ell + b'_{\ell m} b_\ell) - h_m \quad (m = 1, 2, \ldots, q).$$

24. Select $[E_\sigma]$.

25. Find the n+1 x_σ from

$$x_\sigma = (b_{1\sigma}, b_{2\sigma}, \ldots, b_{(n-2w)\sigma}, a'_{1\sigma}, a'_{2\sigma}, \ldots a'_{w\sigma}, b'_{1\sigma}, b'_{2\sigma}, \ldots, b'_{w\sigma}).$$

26. Find λ_σ from $\sum'_{(n+1)} \lambda_\sigma x_\sigma = 0$

(one of the λ is arbitrary, but $\lambda \neq 0$).

27. Compute

$$\epsilon = \frac{\sum'_{(n+1)} \lambda_\sigma(-h_\sigma)}{\sum'_{(n+1)} |\lambda_\sigma|} \quad .$$

28. Find the n+1 ϵ_σ from $\epsilon_\sigma = \epsilon\ (\mathrm{sgn}\ \lambda_\sigma)$.

29. Find A_k $(k = 1, 2, \ldots, n-2w)$, a_ℓ, b_ℓ $(\ell = 1, 2, \ldots, w)$

from n equations

$$\sum_{k=1}^{n-2w} b_{k\sigma}A_k + \sum_{\ell=1}^{w} (a'_{\ell\sigma}a_\ell + b'_{\ell\sigma}b_\ell) - h_\sigma - \epsilon(\mathrm{sgn}\ \lambda_\sigma) = 0$$

30. Compute ϵ_m $(m = 1, 2, \ldots, q)$ from

$$\epsilon = \sum_{k=1}^{n-2w} b_{km}A_k + \sum_{\ell=1}^{w} (a'_{\ell m}a_\ell + b'_{\ell m}b_\ell) - h_m \quad .$$

31. Is $|\epsilon_m| \leq |\epsilon_\sigma|$? If Yes next step is 39. If No next step is 32.

32. Find i from $|\epsilon_i| = |\epsilon_m|_{max}$ $(m = 1, 2, \ldots, q)$.

33. Find $x_i = (b_{1i}, \ldots, b_{(n-2w)i}, a'_{1i}, \ldots, a'_{wi}, b'_{1i}, \ldots, b'_{wi})$.

34. Find μ_σ from $\sum'_{(n+1)} \mu_\sigma x_\sigma + x_i = 0$. (One of the μ is arbitrary).

35. Compute $\dfrac{\mu_\sigma}{\lambda_\sigma}$.

36. Find from Table III the hyper-plane E_ℓ which is to be replaced by E_i.

37. Replace E_ℓ by E_i, forming a new reference $[E_\sigma]$.

38. Repeat from step 25.

39. From $h^*(t) = \sum_{k=1}^{n-2w} A_k e^{s_k t} + \sum_{\ell=1}^{w} a_\ell(2e^{\alpha_\ell t}\cos\beta_\ell t) + b_\ell(-2e^{\alpha_\ell t}\sin\beta_\ell t)$.

40. Find $H^*(s) = L\ [h^*(t)]$.

$H^*(s)$ is the system function of the desired network N.

This outline summarizes the process of approximation of $h(t)$ by $h^*(t)$, from which $H^*(s)$ can be obtained. $H^*(s)$ can now be synthesized as the desired transfer function, thus yielding the desired network N. The last ϵ (computed in 27), is the Tschebyscheff error of the approximation.

CHAPTER V

CONCLUSION

The goal of the preceeding chapters has been to develop a theory of synthesizing R-L-C networks meeting to a Tschebyscheff approximation in the time domain, prescribed input - output requirements. The method which is developed is a numerical one, and, hence, permits a prescription of the input - output relationship in either the form of an equation or in the form of data. The approximation process is discussed in detail for a prescribed impulse response. Consideration is given to the more general problem of obtaining a network with a prescribed response to an arbitrary input. It is shown in Appendix B that this problem can be reduced to an equivalent problem of obtaining a network having a prescribed impulse response.

The approximation process developed in this dissertation yields an impulse response function approximating the prescribed one. The approximating impulse response is the inverse Laplace transform of an R-L-C <u>network</u> function. In this way one may determine the impulse response of a realizable network which approximates the prescribed impulse response in the Tschebyscheff sense.

In the opinion of the author, the chief contributions of this investigation are as follows:

1. Application of the discrete Tschebyscheff approximation theory to network problems.
2. Development of a general solution to the problem of network approximation in the time domain.

3. Development of a general numerical method of approximation of a prescribed impulse response by the impulse response of a realizable R-L-C network. The error of the approximation of the realizable impulse response is minimized through optimization of both its pole locations and its residues.
4. A detailed investigation of the effect on the approximation of both the error in the pole determination and the error in the residue determination.

The above results are encouraging. However, it is noted that the calculations tend to become lengthy when the number of terms in the approximating function is large. In addition, as might be expected, a large number of points must be considered when the prescribed time response varies wildly (i.e., when the derivative of the time response changes sign often). Consequently, occasions may arise when the numerical calculations can advantageously be programmed on a digital computer.

A number of topics meriting further study has arisen during this investigation. In particular, it would be desirable to extend the method developed so as to permit a Tschebyscheff approximation with certain constraints. Such constraints might, for example, involve a restriction of the poles to the negative real axis (thus yielding an R-C network), or a restriction of the poles to the $j\omega$ axis (thus yielding a lossless network). Many other constraints dictated by practical considerations might be considered.

The method of discrete Tschebyscheff approximations can be employed advantageously whenever a problem can be reduced to an overdetermined system of linear equations. It is believed that some network problems in the frequency domain also show promise of solution by this approach.

APPENDIX A

Step Input Problem

In many applications a network N is desired which will provide a prescribed response k(t) to a unit step input. This problem can be reduced to the one discussed in Chapter IV, i.e., the synthesis of a network with prescribed impulse response, through differentiation of k(t) and equating it to h(t). Hence,

$$h(t) = k'(t) \quad . \tag{A1}$$

With h(t) determined, the methods of Chapter IV can be applied, producing the desired network N. However, in cases where k(t) is not differentiable without an error [for example, when k(t) is given as numerical data], it is more accurate to approximate k(t) by k*(t) and then to differentiate to obtain h*(t). Such an approach will be outlined in this appendix.

The requirement of stability demands h(t) to approach zero for sufficiently large t. This requires k(t) to approach a constant for large t. Hence, if k*(t), an approximation to k(t), is represented as

$$k^*(t) = B_o + \sum_{k=1}^{n} B_k e^{s_k t} \tag{A2}$$

$$(\mathrm{Re}\ s_k \leq 0) \ ,$$

then

$$h^*(t) = k^{*\prime}(t) = \sum_{k=1}^{n} (s_k B_k) e^{s_k t} = \sum_{k=1}^{n} A_k e^{s_k t} \tag{A3}$$

$$(\mathrm{Re}\ s_k \leq 0)$$

where $A_k = s_k B_k$, has the required form.

If B_o is known (which is the usual case) one can form $\ell^*(t) = k^*(t) - B_o$; then,

$$\ell^*(t) = \sum_{k=1}^{n} B_k e^{s_k t} \quad (\text{Re } s_k \leq 0) \ . \qquad (A4)$$

$\ell^*(t)$ can now be determined by the methods of Chapter IV, yielding $h^*(t)$ given by:

$$h^*(t) = \ell^{*\prime}(t) = \sum_{k=1}^{n} A_k e^{s_k t} \qquad (A5)$$

$$(\text{Re } s_k \leq 0),$$

from which the desired network can be synthesized.

If B_o is not known in advance to sufficient accuracy, and can not be subtracted, one can form a set of equations at equally-spaced intervals in a manner similar to Eq. (4.5) and Eq. (4.7). If the interval spacing is d, and if there are q points of data, then

$$k_m = k(t_m) = B_o + \sum_{k=1}^{n} B_k e^{s_k t_m} . \quad (m = 1, 2, \ldots, q) \ (A6)$$

By Eq. (4.8),

$$e^{s_k d} = y_k \qquad (A7)$$

and by Eq. (4.9)

$$B_k e^{s_k t_v} = z_{kv} . \qquad (A8)$$

Then,

$$k_v = B_o + \sum_{k=1}^{n} z_{kv}$$

$$k_{v+1} = B_o + \sum_{k=1}^{n} z_{kv} y_k$$

$$k_{v+2} = B_o + \sum_{k=1}^{n} z_{kv} y_k^2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \tag{A9}$$

$$k_{v+i} = B_o + \sum_{k=1}^{n} z_{kv} y_k^i \quad (v+i = 1, 2, \ldots, q)$$

If one takes the difference of two subsequently following equations, one obtains:

$$k_{v+1} - k_v = \sum_{k=1}^{n} z_{kv}(y_k - 1)$$

$$k_{v+2} - k_{v+1} = \sum_{k=1}^{n} z_{kv}(y_k - 1) y_k$$

$$k_{v+3} - k_{v+2} = \sum_{k=1}^{n} z_{kv}(y_k - 1) y_k^2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$k_{v+i} - k_{v+i-1} = \sum_{k=1}^{n} z_{kv}(y_k - 1) y_k^{i-1} . \tag{A10}$$

Let

$$\Delta_{v+m} = k_{v+m+1} - k_{v+m} , \tag{A11}$$

then,

$$\Delta_v = \sum_{k=1}^{n} z_{kv} y_k - \sum_{k=1}^{n} z_{kv}$$

$$\Delta_{v+1} = \sum_{k=1}^{n} z_{kv} y_k^2 - \sum_{k=1}^{n} z_{kv} y_k \quad (v+i = 1, 2, \ldots, q-1)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\Delta_{v+i-1} = \sum_{k=1}^{n} z_{kv} y_k^{n+1} - \sum_{k=1}^{n} z_{kv} y_k^n . \tag{A12}$$

Introducing the functions r_k $(k = 0, 1, \ldots, n)$ defined in Eqs. (4.11), one obtains:

$$r_n\Delta_v + r_{n-1}\Delta_{v+1} + \cdots \Delta_{v+n} = I_3 - I_2 , \qquad (A13)$$

where

$$I_2 = r_n \sum_{k=1}^{n}(z_{kv}y_k) + r_{n-1} \sum_{k=1}^{n}(z_{kv}y_k)y_k + \cdots + \sum_{k=1}^{n}(z_{kv}y_k)y_k^n \qquad (A14)$$

and

$$I_3 = r_n \sum_{k=1}^{n} z_{kv} + r_{n-1} \sum_{k=1}^{n} z_{kv}y_k + \cdots + \sum_{k=1}^{n} z_{kv}y_k^n . \qquad (A15)$$

But it was shown in the proof of Theorem 1 that

$$I_3 = I_2 = 0 . \qquad (A16)$$

Hence,

$$\sum_{k=0}^{n} r_{n-k} \Delta_{v+k} = 0 \quad (v+k = 1, 2, \ldots, q-1). \qquad (A17)$$

Therefore, Eqs. (A17) form an overdetermined system which can be solved by the methods of Chapter IV, yielding $h^*(t)$, and thus the desired network N. It should be noted that now the system has only p-1 equations rather than p (p=q-n) equations, since the number of equations was reduced by one through the difference-taking process.

APPENDIX B

Arbitrary Input Problem

It was pointed out in Chapter II that most problems for prescribed time-response include specifications of a particular input $e_i(t)$ and of the corresponding response $e_o(t)$, rather than the impulse response $h(t)$. A number of techniques are available, however, for the reduction of such input conditions to the equivalent $h(t)$ desired. One technique, a slight modification of an approach advanced by E. A. Guillemin[8] will be presented in this appendix.

The general relationship between $e_i(t)$, $e_o(t)$ and $h(t)$ is given by

$$e_o(t) = \int_0^t e_i(x)h(t-x)dx = \int_0^t e_i(t-x)h(x)\,dx \;. \quad \text{(B1)}$$

If $e_i(t)$ or $h(t)$ is replaced by its kth derivative, then $e_o(t)$ becomes replaced by its kth derivative. If $e_i(t)$ is differentiated k times and $h(t)$ integrated k times, then $e_o(t)$ is unaffected. It follows that if $e_i(t)$ is differentiated k times and $h(t)$ is differentiated m times, then $e_o(t)$ becomes replaced by its (k+m)th derivative. This relationship can be stated as

$$e_o^{(k+m)}(t) = \int_0^t e_i^{(k)}(x)h^{(m)}(t-x)\,dx \;. \quad \text{(B2)}$$

Equation (B2) can be considered to be a generalization of Eq. (B1). In particular, if only $e_i(t)$ is differentiated k times (i.e., m = 0),

$$e_o^{(k)}(t) = \int_0^t e_i^{(k)}(x)h(t-x)\,dx \;. \quad \text{(B3)}$$

If $e_o(t)$ is approximated by $\bar{e}_o(t)$, which is a sequence of q curves, each of which is given by a (k-1)-degree polynomial, then $\bar{e}_o^{(k)}(t)$ can be represented by a sequence of impulses. If a uniform time increment d is chosen between approximating curves, then all impulses are uniformly spaced. Then $\bar{e}_o^{(k)}(t)$ can be represented as

$$\bar{e}_o^{(k)}(t) = \sum_{m=0}^{q} c_m \delta(t - md) , \tag{B4}$$

where $\delta(t)$ is the unit impulse.

From Eq. (B3) one obtains:

$$\bar{e}_o^{(k)}(t) \cong c_o h(t) + c_1 h(t-d) + \ldots + c_q h\ (t-qd) . \tag{B5}$$

Since

$$h(t) = 0 \qquad \text{for } t < 0 ,$$

therefore,

$$\begin{aligned} \bar{e}_o^{(k)}(0) &\cong c_o h(0) \\ \bar{e}_o^{(k)}(d) &\cong c_o h(d) + c_1 h(0) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \bar{e}_o^{(k)}(qd) &\cong c_o h(qd) + c_1 h[(q-1)d] + \ldots c_q h(0). \end{aligned} \tag{B6}$$

Since $e_o(t)$ is known, $\bar{e}^{(k)}(md)$ $(m = 0, 1, \ldots, q)$ can be determined. The coefficients c_m $(m = 0, 1, \ldots, q)$ can be determined from Eq. (B4). Thus one has in Eq. (B6) a set of q+1 equations for the q+1 unknowns, h(0), h(d), ..., h(qd). The solutions of Eqs. (B6) yield, therefore, the values of the impulse response at the interval points, thus providing a starting point for the impulse-response approximation problem.

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ELECTRONIC DEFENSE GROUP

Page	Line	Read	For
ix	6	$\bar{e}_i(t)$	$\bar{e}_o(t)$
		$e_i(t)$	$e_o(t)$
4	20	functions	function
12	1	τ	t
	Eq. (3.3)	τ	t
16	2	[21]	[10]
18	1	[17]	[13]
	1	16	15
	Eq. (3.22)	$d\psi(x)$	$dx(x)$
	Eq. (3.23)	$J(z)$	$J(x)$
19	14	[19]	[17]
24	24	[21]	[10]
26	4	$A_k e^{s_k t_v}$	$A_k^{\ s k t_v}$
	8	z_{kv}	z_k
	9	z_{kv}	z_k
	10	z_{kv}	z_k
27	Eq. (4.11)	$r_3 = -\sum_{k_1=1}^{n} y_{k_1} y_{k_2} y_{k_3}$	$r_3 = \sum_{k_1=1}^{n} y_{k_1} y_{k_2} y_{k_3}$
28	21	$r_o = 1$	$r_o = 0$
30	Eq. (4.19)	$+r_{m-1} \sum_{k=1}^{m} (z_{kv} y_k) y_k + \dots$	$+r_{m-1} \sum_{k=1}^{m} (z_{kv} y_k + \dots$
39	Fig. 4.2	(.5, -.5)	(-.5, .5)

Page	Line	Read	For
40	Eq. (4.34)	$\sum'_{(n+1)} \lambda_\sigma h_{\sigma+n}$	$\sum_{(n+1)} \lambda_\sigma h_{\sigma+n}$
44	3	errors	residues
52	22	the	by
54	11	+	-
55	Table III	$\gamma_k+j\delta_k$	$y_k+j\delta_k$
58	9	$\left[\sum_{j=0}^{k} (-1)^j \binom{k}{j} w^{k-j}\right](-1)^k$	$\left[\sum_{j=0}^{k} \binom{k}{j} w^{k-j}\right](-1)^k$
64	Eq. (4.64)	$+r_{n-1}^2 \epsilon_{\ell+1}^2 +\cdots$	$+r_{n-1}^2 \epsilon_{\ell+1}$
	Eq. (4.65)	$\bar{\epsilon}_v^2$	$\bar{\epsilon}_v$
123	12	Z_{12}	z_{12}
	12	parenthesis missing	
127	15	judgment	judgement
	24	judgment	judgement
128	3	desires	desirs
130	Step 14	Table II	Table III
131	Step 22	$(\ell = 1, 2, \ldots, w)$	$(k = 1, 2, \ldots, w)$
	Step 23	$0 = \sum_{k=1}^{n-2w} b_{km} A_k \cdots$	$\sum_{k=1}^{n-2w} b_{km} A_k \cdots$
132	Step 36	Table II	Table III
	Step 39	Form	From
142	1	$e_i(t)$	$e_o(t)$
	1	$\bar{e}_i(t)$	$\bar{e}_o(t)$
	Eq. (B4)	$\bar{e}_i^{(k)}(t)$	$\bar{e}_o^{(k)}(t)$

Page	Line	Read	For
142	Eq. (B5)	$e_o^{(k)}(t)$	$\bar{e}_o^{(k)}(t)$
	Eq. (B6)	$e_o^{(k)}$	$\bar{e}_o^{(k)}$
	17	$e_o^{(k)}(md)$	$\bar{e}_o^{(k)}(md)$
145	Ref. 31	Vol. PGAC-1	Vol. and PGAC-1