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"Aerodynamic Relations with Variable Specific Heats"

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SUMMARY AND CONCLUSIONS

It has been common practice to base thermodynamic calculations concerning the flow through ducts and across shock waves on formulae which have been derived under the assumption that the specific heats of air are constant. For Mach Numbers greater than two, the range between the static and stagnation values of the temperature is large, and the variation in the specific heats of air between these temperatures cannot be neglected. In this memorandum, the influence of variable specific heats on certain flow relations has been investigated.

For isentropic flow, it has been found that the error in the ratio of the static value to the stagnation value of the pressure, density, and temperature calculated on the basis of a specific heat ratio of 1.4, is less than 1% for flow Mach Numbers that do not exceed two. For Mach Numbers greater than two, the errors rise rapidly. For a Mach Number of six, the errors in the pressure, density, and temperature ratios are 26%, 34% and 13% respectively. Values of the ratios versus Mach Number are plotted on Figures 2, 3 and 4. Errors versus Mach Number are plotted on Figure 5.

For flow through a normal shock wave, the ratios of the upstream to the downstream values of the pressure, density, temperature, and stagnation pressure have been plotted versus Mach Number on Figures 6, 7, 8, and 9. On Figure 10, the Mach Number downstream of a normal shock has been plotted

versus the Mach Number upstream. On Figure 11, the error arising as a consequence of holding the specific heat ratio constant at the value 1.4 is plotted versus Mach Number. For Mach Numbers less than two, the use of a constant value for the specific heat ratio leads to an error of less than 1% in the quantities considered. At an upstream Mach Number of six, the errors in the downstream values of the pressure, density, temperature, and stagnation pressure ratios are 3%, 16%, 11%, and 24% respectively. The error in the downstream value of the Mach Number is 5%.

The calculations for flow through a shock wave have been made for an upstream static temperature of 520°R. At lower temperatures, the errors resulting from the use of a constant specific heat ratio are smaller. Calculations at a Mach Number of six using an upstream static temperature of 393°R (stratosphere temperature) have been made and this one point is plotted on Figures 6 through 10.

SYMBOLS

a = Local speed of sound, ft per sec

c_p = Specific heat at constant pressure, Btu per lb per degree

c_v = Specific heat at constant volume, Btu per lb per degree

h = Enthalpy, Btu per lb mass; OR, ft lb per slug

p = Pressure, lb per sq ft

u = Velocity, ft per sec

v = Specific volume, cu ft per slug

M = Mach Number

R = The gas constant (1715 ft lb per slug per degree Rankine)

T = Absolute temperature, degrees Rankine

$\beta = u_2 + \frac{RT_2}{u_2}$, ft per sec

$\gamma = \frac{c_p}{c_v}$ = Ratio of specific heats

$\psi = u_1 + \frac{RT_1}{u_1}$, ft per sec

ρ = Density, slug per cu ft

Subscripts:

0 = Stagnation value

1 = Position immediately upstream of shock

2 = Position immediately downstream of shock

s = Arbitrary standard value

Superscripts:

* = Critical value ($M = 1$)

0 = Stagnation value

INTRODUCTION

In aerodynamic calculations at low Mach Numbers the temperature variations are sufficiently small that the specific heat ratio $\gamma = \frac{c_p}{c_v}$ can be considered constant. This permits the quantities $\frac{p}{p_0}$, $\frac{\rho}{\rho_0}$, and $\frac{T}{T_0}$ for isentropic flow and the quantities $\frac{p_2}{p_1}$, $\frac{\rho_2}{\rho_1}$, $\frac{T_2}{T_1}$ and $\frac{p_2^0}{p_1^0}$ for shock flow to be expressed conveniently in terms of the flow Mach Number. At high Mach Numbers γ cannot be considered constant and the validity of formulae based on this assumption must be investigated.

In Reference 1, values of the specific heats of air at vanishing pressure, computed from experimentally determined data on nitrogen and oxygen, are listed for temperatures between 200°R and 6600°R. Values of the enthalpy computed from these specific heats are shown to be accurate within close limits, even when the pressures are very high. Values of γ have also been computed from these data and are listed in Table 3 of Reference 1. On Figure 1 of this report, the important parameter (γ - 1) is plotted versus temperature. The variation of this factor in the temperature range between 200°R and 6600°R is 42%. It is interesting to note that at 6400°R, the value of γ computed in Reference 1 is 1.280, which is close to the value 1.286 predicted by Kinetic Theory for gas molecules with seven degrees of freedom. The subject of variation in specific heats of gases with temperature is discussed in References 2 and 3.

From the enthalpy tables of Reference 1, it is a simple matter to compute the isentropic flow relations versus Mach Number. The flow through a normal shock is more involved and a graphical solution to the equations is employed.

At present, the very high stagnation temperatures resulting from supersonic flow at Mach Numbers between three and six are encountered only in free flight. Therefore calculations have been based on a temperature of the undisturbed flow of 520°R at all Mach Numbers. It should be noted that in flight at high altitudes the undisturbed flow temperature is lower and errors in the formulae based on constant γ may be expected to be smaller. One calculation of the parameters has been made at a Mach Number of six and an undisturbed flow temperature of 393°R to illustrate this point.

It is also pointed out that the temperature rise through a plane, or conical, oblique shock is smaller than the rise through a normal shock under the same conditions. Therefore, the errors indicated in this report serve as an upper limit on the errors one may expect if γ is treated as a constant in calculating the parameters downstream of any shock wave.

DISCUSSION

I. Isentropic Flow

The energy equation for a gas can be written:

(1) $$\frac{u^2}{2} + \int_0^T c_p dT = \int_0^{T_0} c_p dT$$

If the specific heats are considered constant, the energy equation reduces to:

$$\frac{u^2}{2} + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} = \frac{\gamma}{\gamma - 1} \frac{p_0}{\rho_0}$$

Assuming further that the flow is isentropic, and employing the relation $\frac{p}{\rho^\gamma}$ equals a constant, the static to stagnation ratios are easily derived. For $\gamma = 1.4$

(2) $$\frac{p}{p_0} = (1 + 0.2M^2)^{-3.5}$$

(3) $$\frac{\rho}{\rho_0} = (1 + 0.2M^2)^{-2.5}$$

(4) $$\frac{T}{T_0} = (1 + 0.2M^2)^{-1}$$

For high Mach Numbers, Equations 2, 3, and 4 are invalid because of the large variation in γ, and air tables similar to those of Reference 1 must be employed. In the air tables, the values of enthalpy $\int_{T_s}^T c_p dT$, pressure ratio $\frac{p}{p_s}$, and specific volume ratio $\frac{v}{v_s}$, are tabulated for values of the temperature between 300°R and 6500°R. From these figures it is a simple matter to determine the ratios $\frac{p}{p_0}$, $\frac{\rho}{\rho_0}$, and $\frac{T}{T_0}$, versus stream

Mach Number for any given value of stream temperature. This has been done for a stream temperature of 520°R and the results have been plotted on Figures 2, 3, and 4. A sample calculation is given in the Appendix. For comparison, the relations for a constant γ given by Equations 2, 3, and 4 are plotted on the same figures. The correction factors which must be applied to Equations 2, 3, and 4 are plotted versus Mach Number on Figure 5.

II. Flow Through a Normal Shock

The relations across a normal shock can be found by combining the three equations expressing the conservation of mass, momentum, and energy and the equation of state, in the proper manner. These equations are listed below. The subscripts 1 and 2 refer to positions immediately upstream and downstream of the shock, respectively.

(5) $$\rho_1 u_1 = \rho_2 u_2$$ conservation of mass

(6) $$p_1 - p_2 = \rho_2 u_2^2 - \rho_1 u_1^2$$ conservation of momentum

(7) $$\frac{u_1^2}{2} + h_1 = \frac{u_2^2}{2} + h_2 = h_o$$ conservation of energy

(8) $$p = \rho RT$$ equation of state

If γ is considered constant, conservation of energy can be reduced to:

(7A) $$\frac{u_1^2}{2} + \frac{a_1^2}{\gamma - 1} = \frac{u_2^2}{2} + \frac{a_2^2}{\gamma - 1} = \frac{\gamma + 1}{2(\gamma - 1)}\, a^{*2}$$

The relations across a normal shock, using Equations 5, 6, 7A, and 8 and the value $\gamma = 1.4$, are:

(9) $$\frac{p_2}{p_1} = 1.165\ M_1^2 - .167$$

(10) $$\frac{T_2}{T_1} = \left[2 + 0.4\ M_1^2\right]\left[\frac{2.8\ M_1^2 - 0.4}{5.76\ M_1^2}\right]$$

(11) $$\frac{p_2^0}{p_1^0} = \left[\frac{2.4}{2.8\ M_1^2 - 0.4}\right]^{2.5}\left[\frac{2.4\ M_1^2}{2 + 0.4\ M_1^2}\right]^{3.5}$$

(12) $$M_2^2 = \frac{5 + M_1^2}{7\ M_1^2 - 1}$$

When γ is considered to be a function of temperature, Equation 7A is not available. Air tables must be used in its place. Equations 5, 6, and 8 do not involve the specific heats. The four unknowns p_2, ρ_2, T_2, and u_2 occur in these three equations and from them u_2 may be solved as a function of T_2 in the following manner:

Substituting (8) in (6):

$$\rho_2 RT_2 = \rho_1 u_1^2 - \rho_2 u_2^2 + \rho_1 RT_1$$

which becomes, after making use of (5) and simplifying:

(13) $$u_2 + \frac{RT_2}{u_2} = u_1 + \frac{RT_1}{u_1}$$

Another value of u_2 is found from Equation 7

(14) $$u_2 = \sqrt{2(h_o - h_2)}$$

Substituting this in Equation 13 yields:

(15) $$\sqrt{2(h_o - h_2)} + \frac{RT_2}{\sqrt{2(h_o - h_2)}} = u_1 + \frac{RT_1}{u_1}$$

From Equation 15 and the air tables (which give h as a function of T), the value of the downstream static temperature T_2 may be solved for any set of upstream conditions u_1 and T_1. A graphical solution shows two values of T_2 that satisfy Equation 15. One solution indicates that $T_2 = T_1$ and corresponds to the trivial case of a discontinuity of zero intensity.

After T_2 has been determined for the specific upstream conditions u_1 and T_1, h_2 can be found from the air tables and u_2 can be computed from Equation 14. Then $\frac{T_2}{T_1}$ and $\frac{u_2}{u_1}$ are known. $\frac{\rho_2}{\rho_1}$ can be computed from the continuity relation: $\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2}$ and $\frac{p_2}{p_1}$ from the equation of state:

$$\frac{p_2}{p_1} = \frac{\rho_2}{\rho_1} \cdot \frac{T_2}{T_1}$$

a_2 can be determined for any value of T_2 from the air tables, and M_2 computed from the definition $M_2 = \frac{u_2}{a_2}$. This leaves only the ratio of the stagnation pressures, which is computed from the following expression:

$$\frac{p_2^0}{p_1^0} = \frac{p_2^0}{p_2} \cdot \frac{p_2}{p_1} \cdot \frac{p_1}{p_1^0}$$

$\frac{p_1}{p_1^0}$ is read from the curves of Part I of this discussion. $\frac{p_2^0}{p_2}$ can be computed from Equation 2 because the Mach Number downstream of the shock is less than unity. However, in this equation, care must be taken to use a value of γ corresponding to the stagnation temperature of the flow.

The ratios $\frac{p_2}{p_1}$, $\frac{\rho_2}{\rho_1}$, $\frac{T_2}{T_1}$ and $\frac{p_2^0}{p_1^0}$, computed in the manner described above, have been plotted versus Mach Number on Figures 6, 7, 8, and 9 respectively. On Figure 10, M_2 has been plotted versus M_1. On the same figures, these quantities as computed from Equations 9 through 12 are also plotted for purpose of comparison. The error which arises as a result of holding γ equal to 1.4 is plotted versus Mach Number on Figure 11.

REFERENCES

(1) Keenan and Kaye - Thermodynamic Properties of Air, John Wiley & Sons, Inc., New York, 1945, First Edition

(2) Loeb - Kinetic Theory of Gases, McGraw-Hill Book Company, Inc., New York, 1927, First Edition

(3) Kennard - Kinetic Theory of Gases, McGraw-Hill Book Company, Inc., New York, 1938, First Edition

APPENDIX

I. Isentropic Flow Calculations

As an example, the flow properties are calculated for a Mach Number of six and a static temperature of 520°R.

From the air tables (Ref. 1), corresponding to T = 520°R:

(1) $a = 1118$ ft per sec

(2) $(h - h_s) = 28.77$ Btu per lb mass

(3) $\frac{p}{p_s} = 2.504$

(4) $\frac{v}{v_s} = \frac{1}{\frac{\rho}{\rho_s}} = 5192$

Then $u = Ma = (6)\cdot(1118) = 6708$ ft per sec, and from the energy equation:

$$(h_o - h_s) = (h - h_s) + \frac{u^2}{2}$$

$$= 28.77 + \frac{(6708)^2}{2} \cdot \frac{40}{10^6}$$

(40×10^{-6} converts ft lb per slug to Btu per lb mass)

$$= 928.715 \text{ Btu per lb mass}$$

Again from the air tables (Ref. 1), corresponding to

$h_o - h_s = 928.715$ Btu per lb mass

$T_o = 3785°R$

$\frac{p_o}{p_s} = 5345.0$

$\frac{v_o}{v_s} = \frac{1}{\frac{\rho_o}{\rho_s}} = 17.70$

Then:

$$\frac{T}{T_0} = \frac{520}{3785} = .1374$$

$$\frac{p}{p_0} = \frac{p}{p_s} \cdot \frac{p_s}{p_0} = (2.504) \cdot \frac{1}{5345.0} = .004685$$

$$\frac{\rho}{\rho_0} = \frac{v_0}{v} = \frac{v_s}{v} \cdot \frac{v_0}{v_s} = \frac{1}{5192} \cdot (17.70) = .003409$$

Assuming γ to be constant, the expressions for $\frac{T}{T_0}$, $\frac{p}{p_0}$, and $\frac{\rho}{\rho_0}$ may be obtained analytically. Thus:

$$\frac{p}{p_0} = \left[1 + \frac{\gamma - 1}{2} M^2\right]^{\frac{-\gamma}{\gamma - 1}}$$

$$= .000630 \text{ for } \gamma = 1.40$$

The error due to assuming γ constant is therefore:

$$\Delta \left(\frac{p}{p_0}\right) = .000630 - .000468 = .000162$$

The per cent error being:

$$\frac{\Delta \left(\frac{p}{p_0}\right) \times 100}{\frac{p}{p_0}} = \frac{.0162}{.000630} = 25.7$$ Based on constant specific heat ratio determination of $\frac{p}{p_0}$

$$= \frac{.0162}{.0004685} = 34.6$$ Based on variable specific heat ratio determination of $\frac{p}{p_0}$

II. Normal Shock Wave Calculations

As an example of these calculations the flow parameters are computed for an upstream Mach Number of six and an upstream static temperature of 520°R. Equation 15 is repeated below.

$$(15) \qquad \sqrt{2(h_o - h_2)} + \frac{RT_2}{\sqrt{2(h_o - h_2)}} = u_1 + \frac{RT_1}{u_1}$$

or by defining $\beta = \sqrt{2(h_o - h_2)} + \frac{RT_2}{\sqrt{2(h_o - h_2)}}$

and $\psi = u_1 + \frac{RT_1}{u_1}$

$$(15a) \qquad \beta = \psi$$

This gives T_2 and h_2, ($h_2 = f(T_2)$), expressed only in terms of the initial conditions upstream of the shock wave. This equation is solved by plotting β against T_2 for a specific u_1 and T_1, and observing the values of T_2 for which $\beta = \psi$. See Figure 12. Intersections may be noted for values of T_2 of 520°R and 3703°R. The first intersection represents the trivial case of a discontinuity of zero intensity. The remainder of the quantities may be obtained as follows:

(1) $u_2 = \sqrt{2(h_o - h_2)}$

$= \sqrt{2(928.8 - 904.43)\,\frac{10^6}{40}}$

$= 1103.8$ ft per sec

where h_2 = 904.43 Btu per lb mass from Reference 1, corresponding to T_2 = 3703.5°R.

(2) $\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{6708}{1103.8} = 6.0772$

(3) $\frac{T_2}{T_1} = \frac{3703.5}{520} = 7.1221$

(4) $\frac{p_2}{p_1} = \frac{\rho_2}{\rho_1} \cdot \frac{T_2}{T_1} = (6.0772)(7.1221)$

$= 43.282$

(5) $$M_2 = \frac{u_2}{a_2} = \frac{1103.8}{2872}$$

$$= .3843$$

(6) $$\frac{p_2^{\,0}}{p_1^{\,0}} = \frac{p_2^{\,0}}{p_2} \cdot \frac{p_2}{p_1} \cdot \frac{p_1}{p_1^{\,0}} = (1.0997)(43.282)(.004685)$$

$$= .0223$$

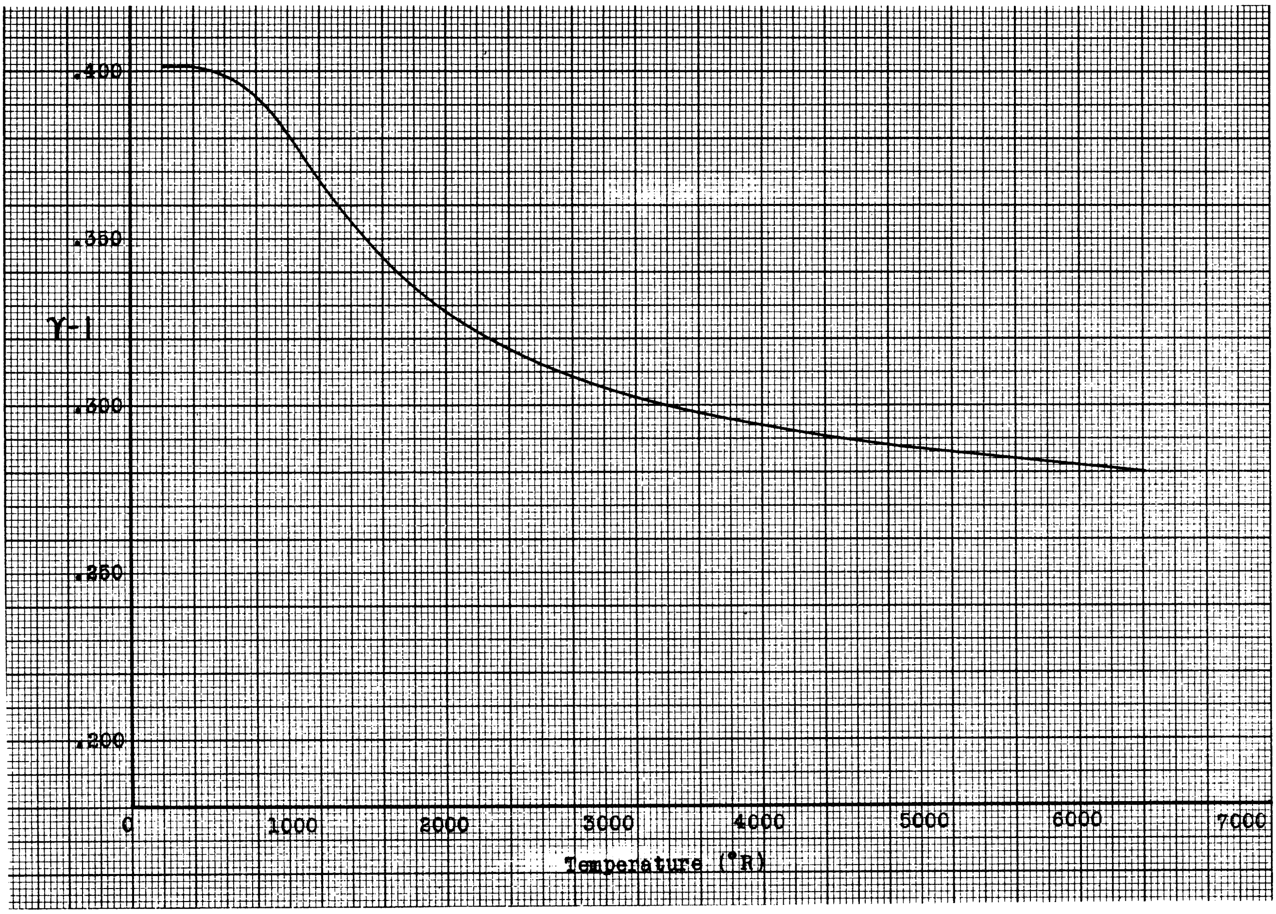

Figure No. 1

VARIATION OF (γ - 1) WITH TEMPERATURE

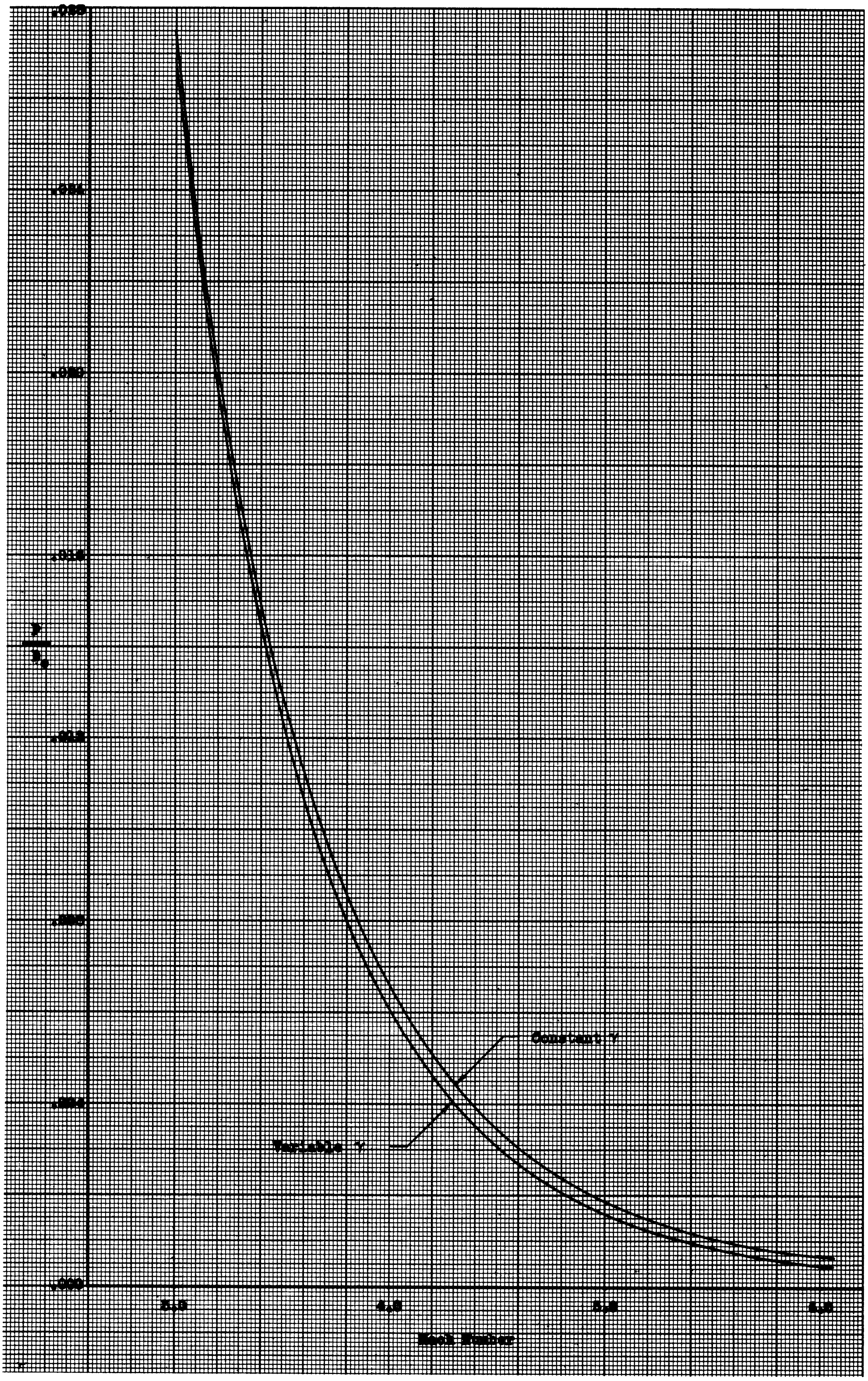

Figure No. 2

VARIATION OF $\frac{p}{p_0}$ WITH MACH NUMBER

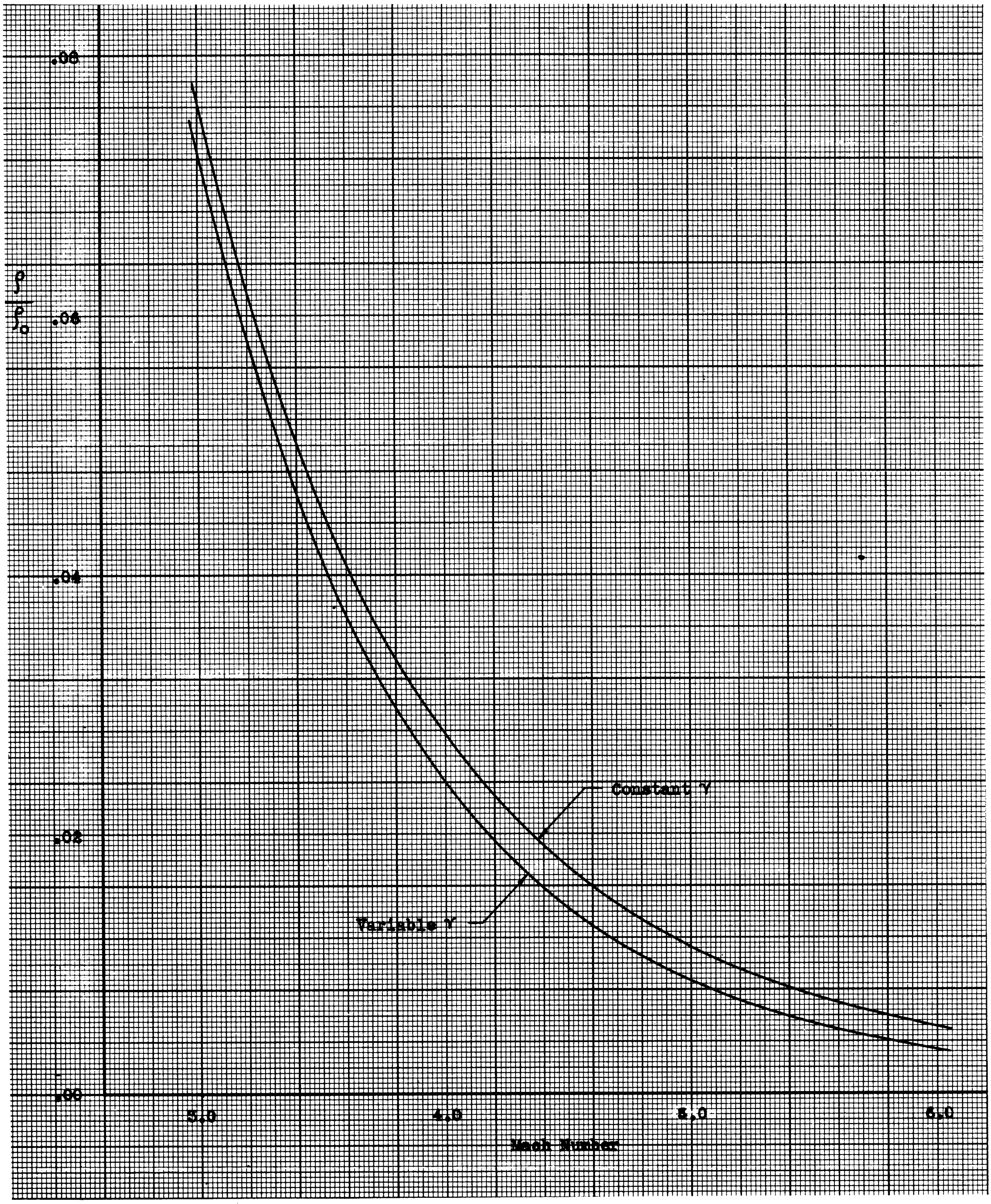

Figure No. 3

VARIATION OF $\frac{\rho}{\rho_0}$ WITH MACH NUMBER

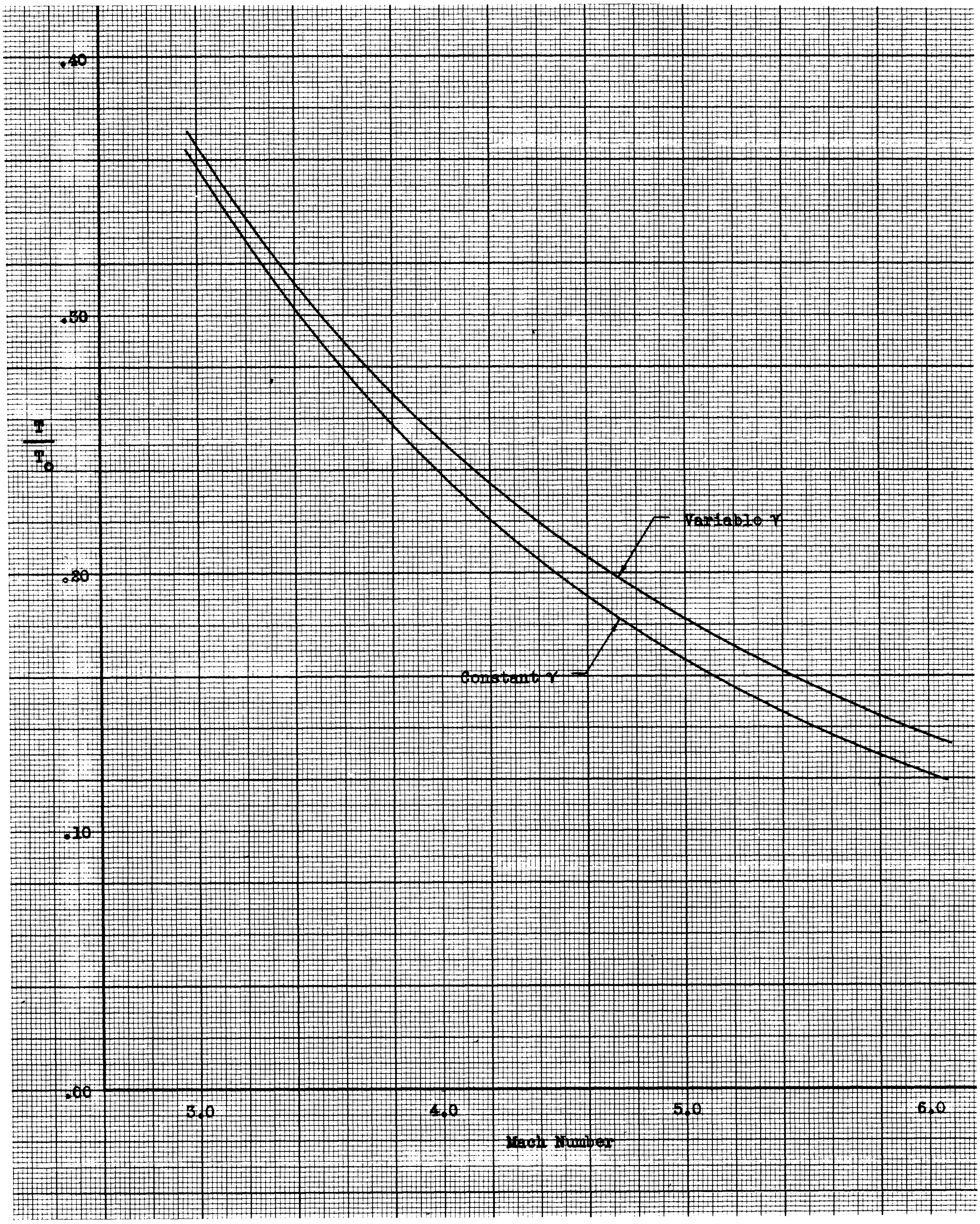

Figure No. 4

VARIATION OF $\frac{T}{T_0}$ WITH MACH NUMBER

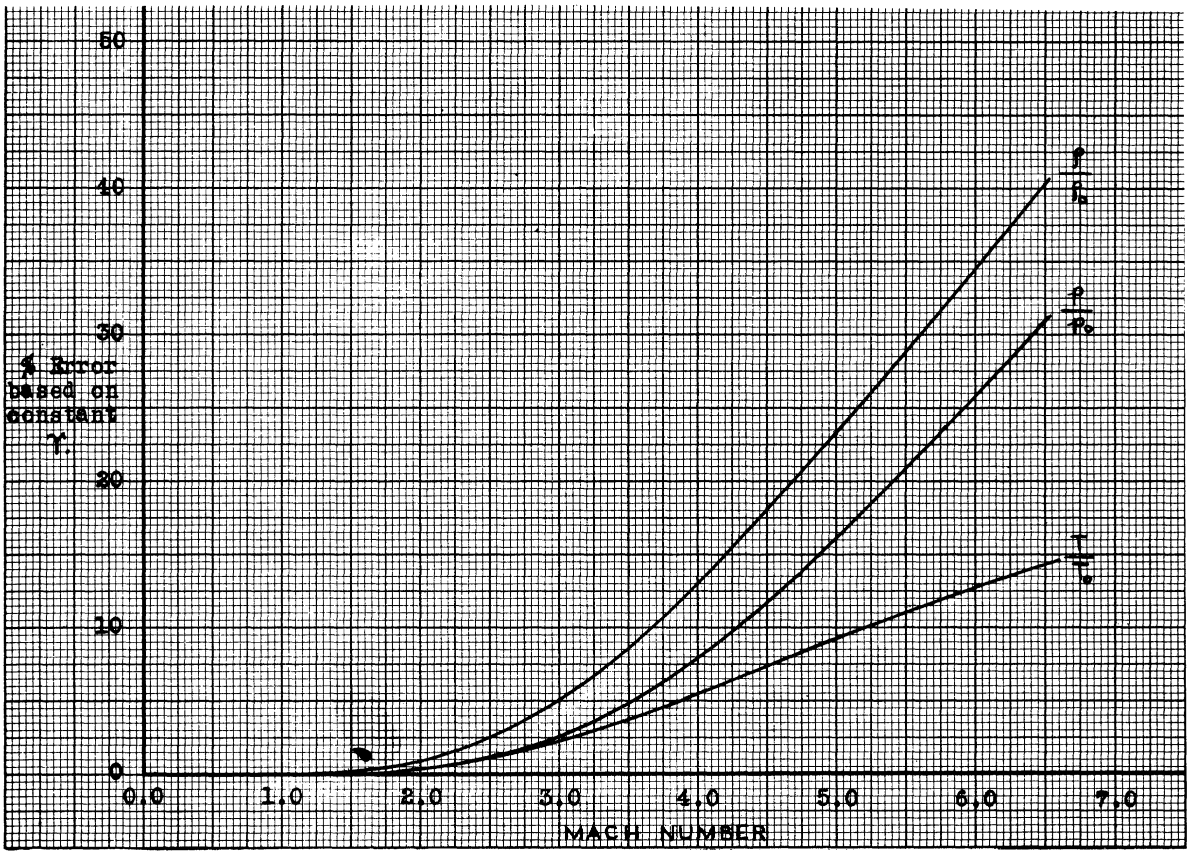

Figure No. 5

ERRORS; ISENTROPIC FLOW, γ CONSTANT

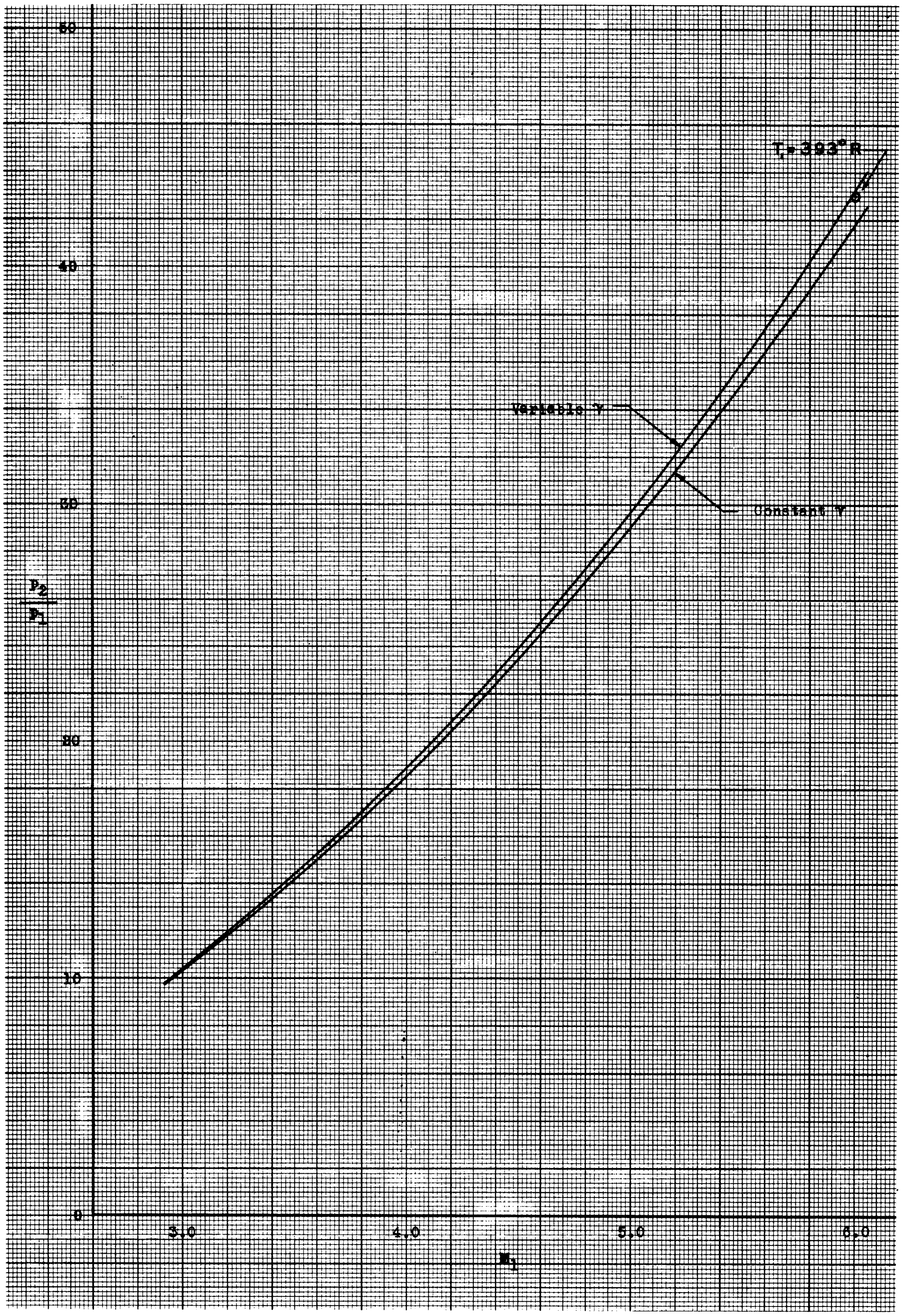

Figure No. 6

VARIATION OF $\frac{p_2}{p_1}$ WITH M_1

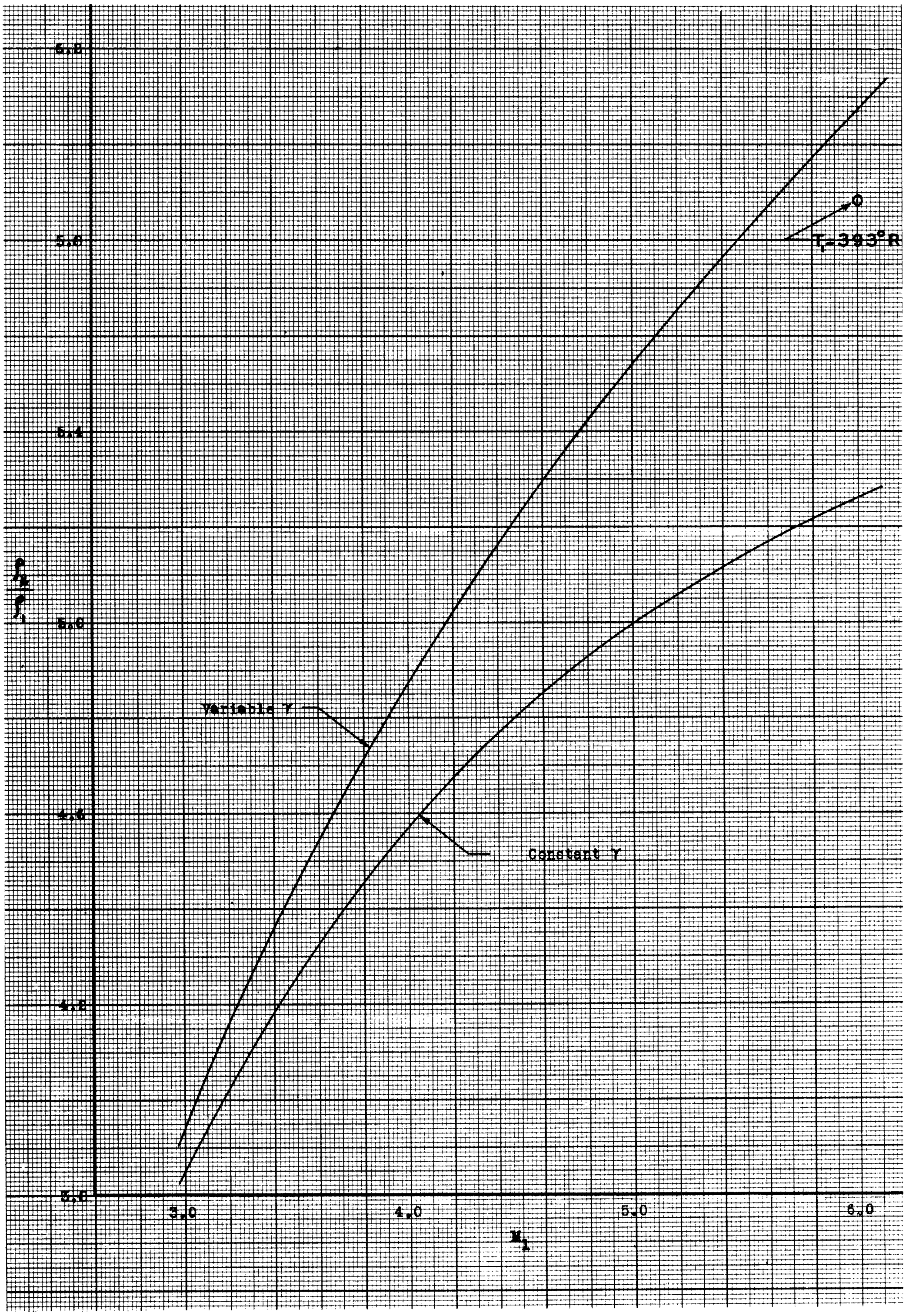

Figure No. 7

VARIATION OF $\frac{\rho_2}{\rho_1}$ WITH M_1

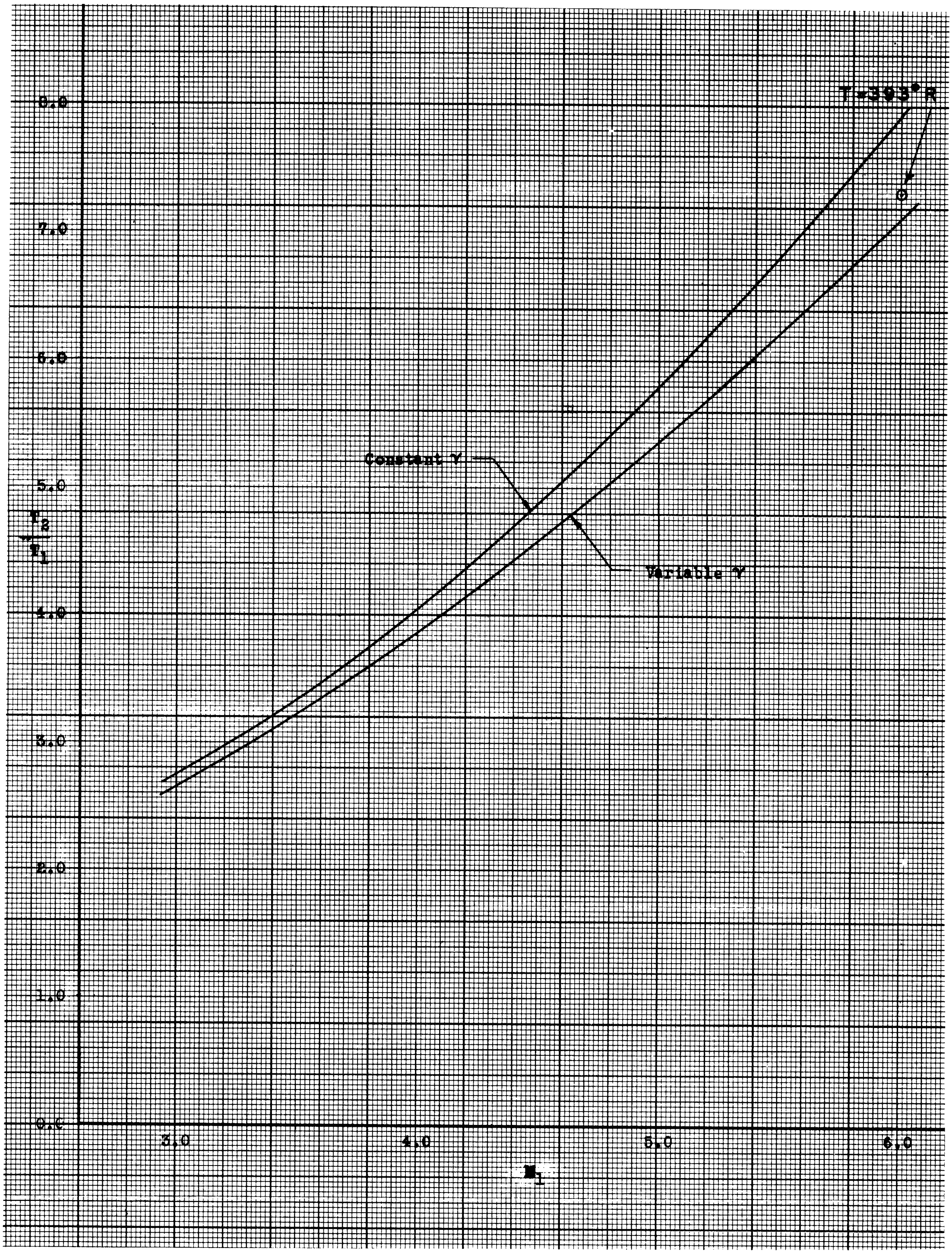

Figure No. 8

VARIATION OF $\frac{T_2}{T_1}$ WITH M_1

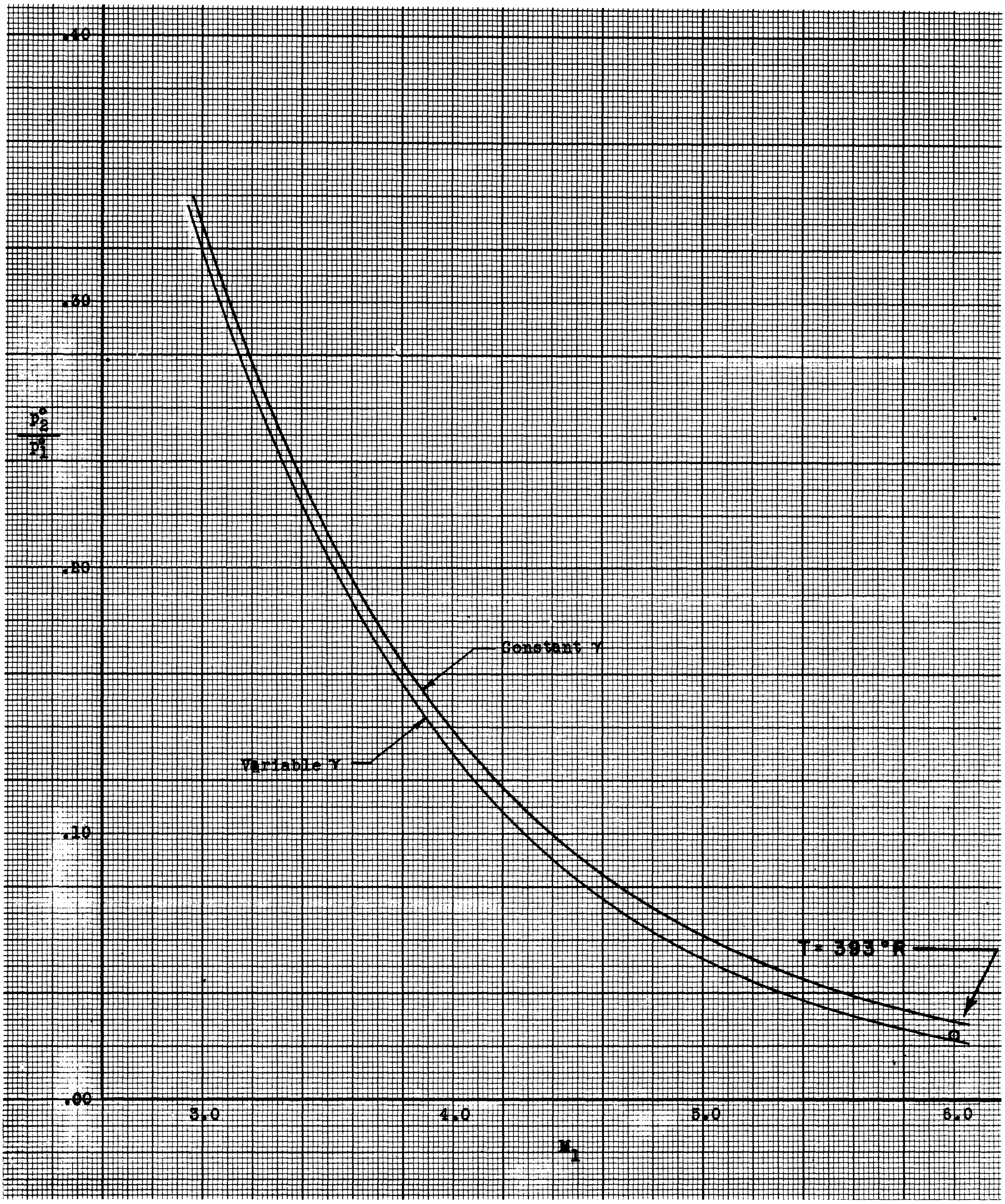

Figure No. 9

VARIATION OF $\frac{p_2^0}{p_1^0}$ WITH M_1

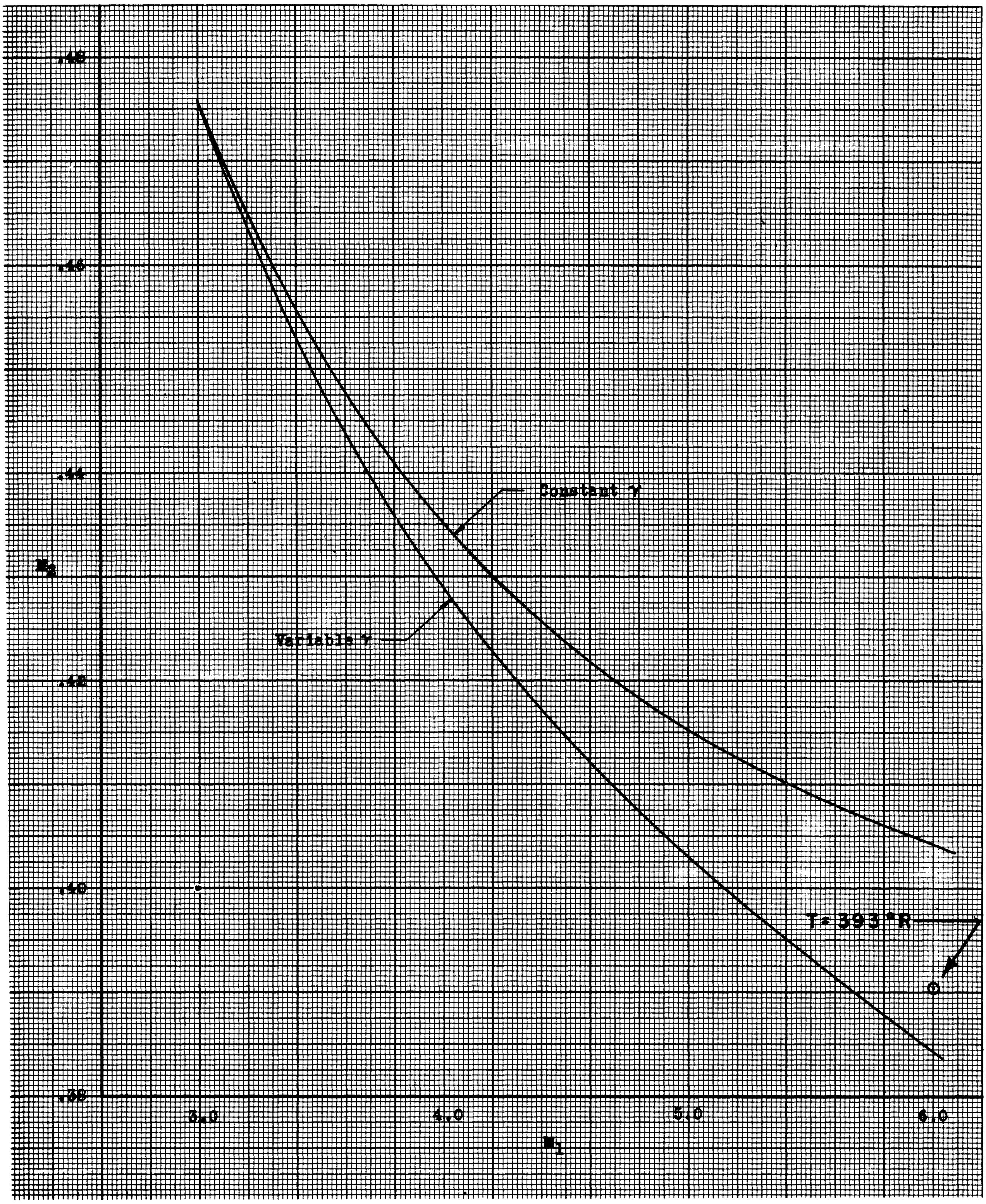

Figure No. 10

VARIATION OF M_2 WITH M_1

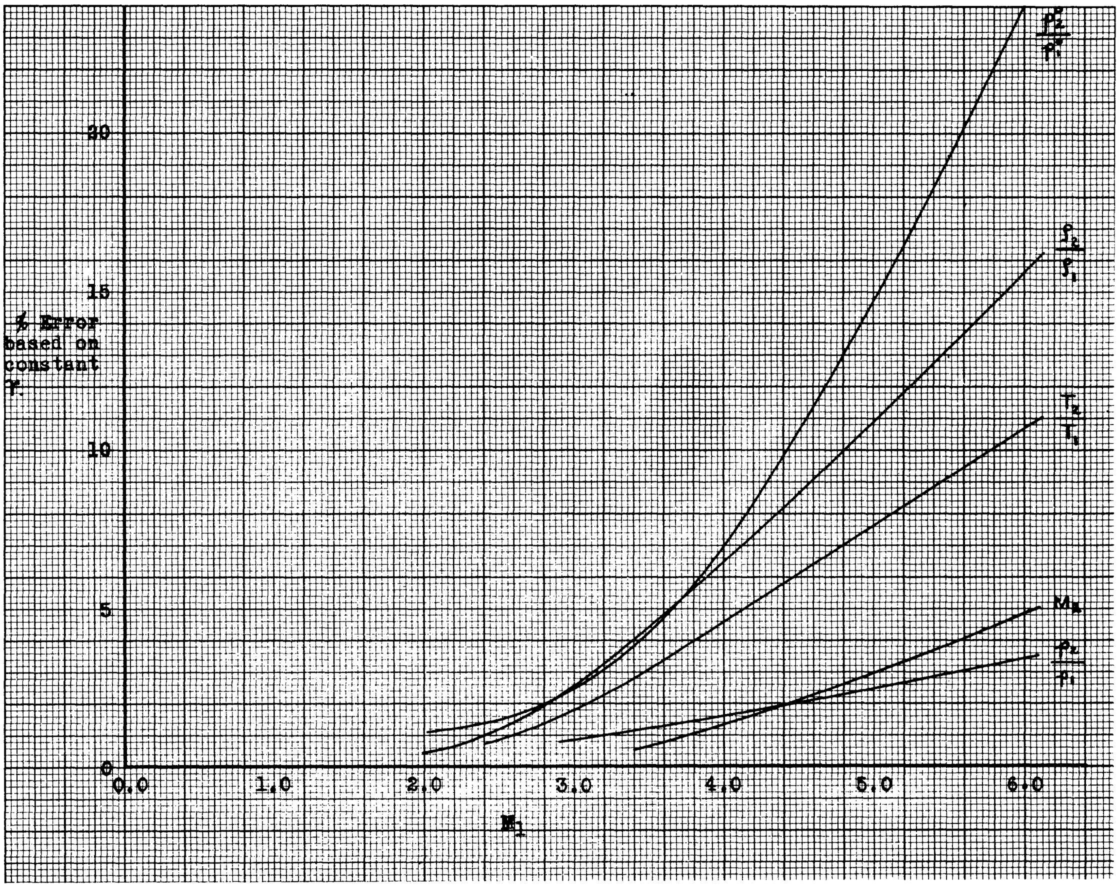

Figure No. 11

ERRORS; SHOCK FLOW, γ CONSTANT

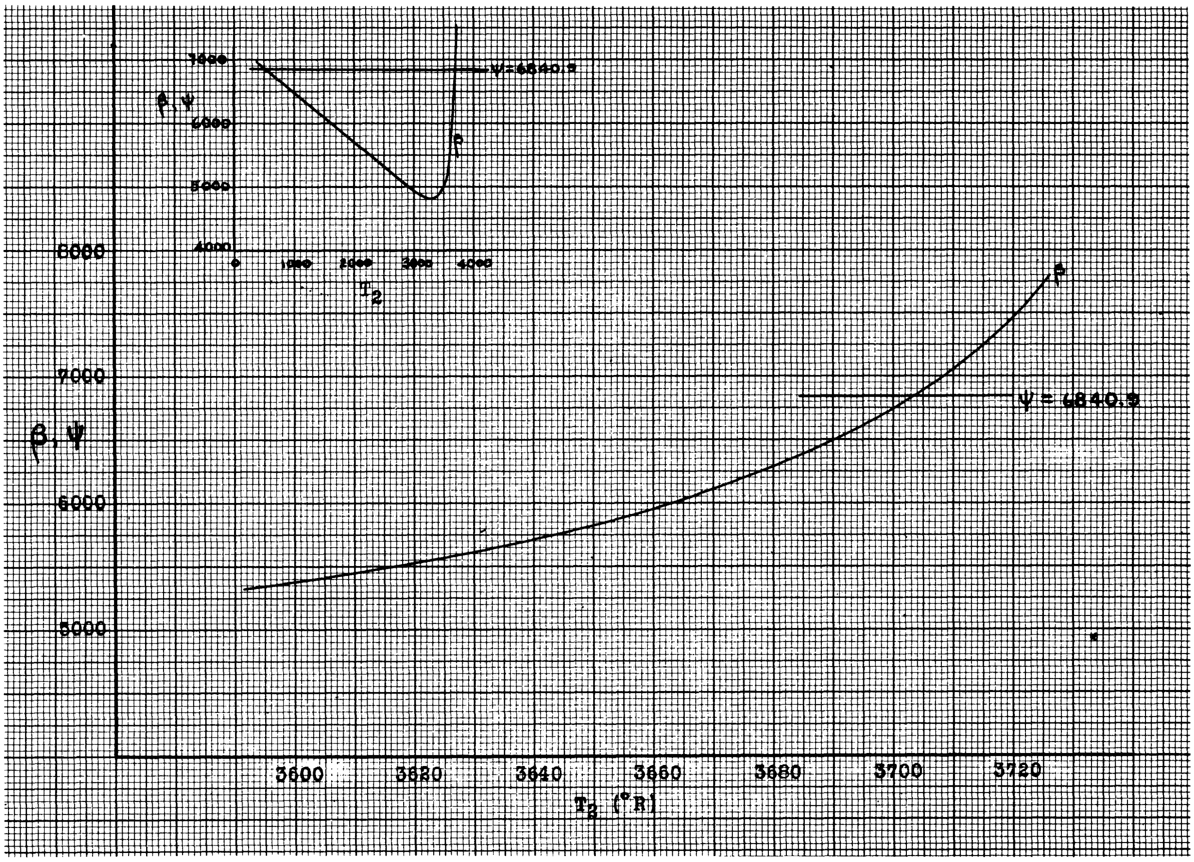

Figure No. 12

VARIATION OF β WITH T_2

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