THE UNIVERSITY OF MICHIGAN
College of Engineering
Department of Mechanical Engineering
Cavitation and Multiphase Flow Laboratory

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# THERMALLY CONTROLLED SPHERICALLY SYMMETRIC BUBBLE GROWTH; UNIQUENESS OF THE SIMILARITY SOLUTION

T. C. Scott
B. R. Hao
F. G. Hammitt

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## ABSTRACT

The uniqueness of the similar solution of the problem of spherically symmetric thermally controlled bubble growth is demonstrated using a generalized group-theoretic analysis. The inability of similar solution methods to account for viscous dissipation is also shown.

# THERMALLY CONTROLLED SPHERICALLY SYMMETRIC BUBBLE GROWTH; UNIQUENESS OF THE SIMILARITY SOLUTION

## INTRODUCTION

The problem of thermally controlled spherically symmetric bubble growth in an infinite liquid has been extensively treated by Scriven [2]* and further examined by many others. By neglecting viscous dissipation, the solution to the above described problem follows from the introduction of the transformation

$$\eta = \frac{r}{\sqrt{t}} \qquad f(\eta) = \frac{T}{T_\infty} \tag{1}$$

by which the energy equation is transformed to an ordinary differential equation. Under this transformation, the bubble radius is required to vary as

$$R = \Phi\sqrt{t} \tag{2}$$

$\Phi$ being a suitable constant

In problems such as this, the selection of the proper transformation is often arbitrary. That is; the one that works becomes the correct one.

Any given transformation such as equation 1 must satisfy both the governing equation and the boundary conditions.

*Numbers in brackets [] refer to references at the end of this paper

From physical reasoning, there should be only one way in which the thermally controlled bubble can grow.

It is possible to derive the transformation of equation 1 from purely mathematical reasoning and to show its uniqueness. The particular method employed follows from Lie group theory and has been developed by Na [1] and applied to problems such as the boundary layer equations which also yield to similarity transformations.

For thermally controlled bubble growth, the governing equations are continuity and energy. Following Scriven, they may be written as

$$u r^2 = \epsilon R^2 \frac{dR}{dt} \tag{3}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r} = \frac{\alpha}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial T}{\partial r}\right] + \frac{2\mu}{\rho_\ell c_p}\left[\frac{\partial u}{\partial r}\right]^2 + \frac{4\mu}{\rho_\ell c_p}\left[\frac{u}{r}\right]^2 \tag{4}$$

For completeness, viscous dissipation has been retained in the energy equation.

A general solution to the above equations is now desired. In particular, the question of solution by similarity methods and the number of transformation groups possible is to be investigated.

## SOLUTION

By substituting equation 3 into equation 4 and introducing the dimensionless parameters

$$\bar{R} = R/R_0 \;;\quad \bar{r} = r/R_0 \;;\quad \bar{t} = t\alpha/R_0^2 \;;\quad \theta = T/T_\infty \;;\quad F = \frac{12\mu\alpha}{T_\infty \rho_\ell c_p R_0^2}$$

the governing equation becomes:

$$\frac{\partial \theta}{\partial \bar{t}} + \epsilon \left[ \frac{\bar{R}}{\bar{r}} \right]^2 \frac{d\bar{R}}{d\bar{t}} \frac{\partial \theta}{\partial \bar{r}} = \frac{1}{\bar{r}^2} \frac{\partial}{\partial \bar{r}} \left[ \bar{r}^2 \frac{\partial \theta}{\partial \bar{r}} \right] + \epsilon^2 F \frac{\bar{R}^4}{\bar{r}^6} \left[ \frac{d\bar{R}}{d\bar{t}} \right]^2 \qquad (5)$$

The search for all possible similarity transformations now proceeds following the methods outlined by Na [1].

By making the following definitions

$$\phi = \epsilon \bar{R}^2 \frac{d\bar{R}}{d\bar{t}} \; ; \quad p = \frac{\partial \theta}{\partial \bar{t}} \; ; \quad q = \frac{\partial \theta}{\partial \bar{r}}$$

$$p_{12} = \frac{\partial^2 \theta}{\partial \bar{t} \partial \bar{r}} \; ; \quad p_{11} = \frac{\partial^2 \theta}{\partial \bar{t}^2} \; ; \quad p_{22} = \frac{\partial^2 \theta}{\partial \bar{r}^2}$$

equation 5 may be written as:

$$G(p, q, p_{11}, p_{12}, p_{22}, \theta, \bar{r}, \bar{t}, \phi) = p_{22} + \left[ \frac{2}{\bar{r}} - \frac{\phi}{\bar{r}^2} \right] q - p + F \frac{\phi^2}{\bar{r}^6} = 0 \qquad (6)$$

The requirement that equation 6 be invariant under a given transformation is

$$\xi \frac{\partial G}{\partial \bar{t}} + \psi \frac{\partial G}{\partial \bar{r}} + \zeta \frac{\partial G}{\partial \theta} + \pi_1 \frac{\partial G}{\partial p} + \pi_2 \frac{\partial G}{\partial q} + \pi_{11} \frac{\partial G}{\partial p_{11}} + \pi_{12} \frac{\partial G}{\partial p_{12}} + \pi_{22} \frac{\partial G}{\partial p_{22}} = 0 \qquad (7)$$

where

$$\xi = \frac{\partial W}{\partial p} \; ; \quad \psi = \frac{\partial W}{\partial q} \; ; \quad \zeta = p \frac{\partial W}{\partial p} + q \frac{\partial W}{\partial q} - W$$

$$-\pi_1 = \frac{\partial W}{\partial \bar{t}} + p \frac{\partial W}{\partial \theta} \; ; \quad -\pi_2 = \frac{\partial W}{\partial \bar{r}} + q \frac{\partial W}{\partial \theta}$$

$$-\pi_{22} = \frac{\partial^2 W}{\partial \bar{r}^2} + 2q \frac{\partial^2 W}{\partial \bar{r} \partial \theta} + q^2 \frac{\partial^2 W}{\partial \theta^2} + 2p_{12} \left[ \frac{\partial^2 W}{\partial \bar{r} \partial p} + q \frac{\partial^2 W}{\partial \theta \partial p} \right] + p_{22} \frac{\partial W}{\partial \theta}$$
$$+ 2p_{22} \left[ \frac{\partial^2 W}{\partial \bar{r} \partial q} + q \frac{\partial^2 W}{\partial \theta \partial q} \right] + p_{12}^2 \frac{\partial^2 W}{\partial p^2} + 2p_{12} p_{22} \frac{\partial^2 W}{\partial p \partial q} + p_{22}^2 \frac{\partial^2 W}{\partial q^2}$$

and $W(\bar{t},\bar{r},\theta,P,q)$ is the characteristic function sought. With W known, the similarity transformations follow from the absolute invariants which are given by the solution of

$$\frac{d\bar{t}}{\xi} = \frac{d\bar{r}}{\psi} = \frac{d\theta}{\zeta} \tag{8}$$

Taking the indicated derivatives of equation 6 and substituting them into equation 7 results in:

$$A_0 + A_1 P_{12} + A_2 P_{12}^2 = 0 \tag{9}$$

where

$$\begin{aligned} A_0 = {} & \left[ -\frac{\phi' q}{\bar{r}^2} + 2\bar{r}\,\frac{\phi\phi'}{\bar{r}^6} \right]\frac{\partial W}{\partial P} + \left[ -\frac{2}{\bar{r}^2} + \frac{2\phi}{\bar{r}^3} \right] q \frac{\partial W}{\partial q} - 6\bar{r}\,\frac{\phi^2}{\bar{r}^7}\frac{\partial W}{\partial q} + \frac{\partial W}{\partial t} \\ & + P\frac{\partial W}{\partial \theta} - \left[ \frac{2}{\bar{r}} - \frac{\phi}{\bar{r}^2} \right]\frac{\partial W}{\partial \bar{r}} - q\left[ \frac{2}{\bar{r}} - \frac{\phi}{\bar{r}^2} \right]\frac{\partial W}{\partial \theta} - \frac{\partial^2 W}{\partial \bar{r}^2} - 2q\frac{\partial^2 W}{\partial \bar{r}\partial\theta} \\ & - q^2\frac{\partial^2 W}{\partial \theta^2} - 2P_{22}\left[ \frac{\partial^2 W}{\partial \bar{r}\partial q} + q\frac{\partial^2 W}{\partial\theta\partial q} \right] - P_{22}^2\frac{\partial^2 W}{\partial q^2} - P_{22}\frac{\partial W}{\partial \theta} \end{aligned} \tag{10}$$

$$A_1 = \left[ \frac{\partial^2 W}{\partial \bar{r}\partial P} + q\frac{\partial^2 W}{\partial\theta\partial P} \right] + P_{22}\frac{\partial^2 W}{\partial P\partial q} \tag{11}$$

$$A_2 = \frac{\partial^2 W}{\partial P^2} \tag{12}$$

and

$$\phi' = \frac{d\phi}{d\bar{t}} \tag{13}$$

Since W is independent of $P_{12}$, the coefficients of equation 9 are set equal to zero. Equation 12 then indicates that W is linear in P so that

$$W(\bar{t},\bar{r},\theta,P,q) = W_1(\bar{t},\bar{r},\theta,q) + PW_2(\bar{t},\bar{r},\theta,q) \tag{14}$$

Substituting this relation into equation 11 set equal to zero along with

$$P_{22} = -\left[\frac{2}{\bar{r}} - \frac{\phi}{\bar{r}^2}\right] q + P - F\frac{\phi^2}{\bar{r}^6} \qquad (15)$$

from equation 6 results in:

$$\frac{\partial W_2}{\partial \bar{r}} + q\frac{\partial W_2}{\partial \theta} + \left\{P - \left[\frac{2}{\bar{r}} - \frac{\phi}{\bar{r}^2}\right] q - F\frac{\phi^2}{\bar{r}^6}\right\}\frac{\partial W_2}{\partial q} = 0 \qquad (16)$$

Since $W_2$ is independent of P, setting coefficients of like powers of P equal to zero indicates that $W_2$ is a function of $\bar{t}$ alone. Thus:

$$W = W_1(\bar{t}, \bar{r}, \theta, q) + P W_2(\bar{t}) \qquad (17)$$

Substituting this relation into equation 10 along with equation 15 and setting the result equal to zero yields:

$$B_0 + B_1 P + B_2 P^2 = 0 \qquad (18)$$

where

$$\begin{aligned} B_0 = {} & \left\{-\frac{\phi' q}{\bar{r}^2} + 2F\frac{\phi\phi'}{\bar{r}^6}\right\} W_2 - \left[\frac{2}{\bar{r}^2} - \frac{2\phi}{\bar{r}^3}\right] q\frac{\partial W_1}{\partial q} - 6F\frac{\phi^2}{\bar{r}^7}\frac{\partial W_1}{\partial q} \\ & + \frac{\partial W_1}{\partial \bar{t}} - \left[\frac{2}{\bar{r}} - \frac{\phi}{\bar{r}^2}\right]\frac{\partial W_1}{\partial \bar{r}} - \frac{\partial^2 W_1}{\partial \bar{r}^2} - 2q\frac{\partial^2 W_1}{\partial \bar{r}\partial\theta} - q^2\frac{\partial^2 W_1}{\partial \theta^2} \\ & + F\frac{\phi^2}{\bar{r}^6}\frac{\partial W_1}{\partial \theta} + 2\left[\left\{\frac{2}{\bar{r}} - \frac{\phi}{\bar{r}^2}\right\} q + F\frac{\phi^2}{\bar{r}^6}\right]\left[\frac{\partial^2 W_1}{\partial \bar{r}\partial q} + q\frac{\partial^2 W_1}{\partial \theta\partial q}\right] \\ & - \left\{\left[\frac{2}{\bar{r}} - \frac{\phi}{\bar{r}^2}\right]^2 q^2 + F^2\frac{\phi^4}{\bar{r}^{12}} - 2qF\frac{\phi^2}{\bar{r}^2}\left[\frac{2}{\bar{r}} - \frac{\phi}{\bar{r}^2}\right]\right\}\frac{\partial^2 W_1}{\partial q^2} \end{aligned} \qquad (19)$$

$$B_1 = \frac{dW_2}{d\bar{t}} - 2\frac{\partial^2 W_1}{\partial \bar{r}\partial q} - 2q\frac{\partial^2 W_1}{\partial \theta\partial q} + 2q\left[\frac{2}{\bar{r}} - \frac{\phi}{\bar{r}^2}\right]\frac{\partial^2 W_1}{\partial q^2} + 2F\frac{\phi^2}{\bar{r}^6}\frac{\partial^2 W_1}{\partial q^2} \qquad (20)$$

$$B_2 = \frac{\partial^2 W_1}{\partial q^2} \qquad (21)$$

Since $W_1$ and $W_2$ are independent of P, each of these coefficients may be set equal to zero.

Equation 21 indicates that $W_1$ is linear in q

$$W_1(\bar{t}, \bar{r}, \theta, q) = W_{11}(\bar{t}, \bar{r}, \theta) + q W_{12}(\bar{t}, \bar{r}, \theta) \tag{22}$$

Substituting equation 22 into equation 20 and equating it with zero gives

$$\frac{dW_2}{d\bar{t}} - 2\frac{\partial W_{12}}{\partial \bar{r}} - 2q\frac{\partial W_{12}}{\partial \theta} = 0 \tag{23}$$

from which it follows that $W_{12}$ is independent of Θ and linear in $\bar{r}$.

$$W_{12}(\bar{t}, \bar{r}) = W_{121}(\bar{t}) + \bar{r} W_{122}(\bar{t}) \tag{24}$$

At this point, the characteristic function is given by

$$W = W_{11}(\bar{t}, \bar{r}, \theta) + q W_{121}(\bar{t}) + q\bar{r} W_{122}(\bar{t}) + P W_2(\bar{t}) \tag{25}$$

Substituting equation 24 into equation 22 and putting the result into equation 19 set equal to zero gives

$$D_0 + D_1 q + D_2 q^2 = 0 \tag{26}$$

where

$$D_0 = 2F\frac{\phi\phi'}{\bar{r}^6}W_2 - 6F\frac{\phi^2}{\bar{r}^7}W_{121} - 4F\frac{\phi^2}{\bar{r}^6}W_{122} + \frac{\partial W_{11}}{\partial t} - \left[\frac{2}{\bar{r}} - \frac{\phi}{\bar{r}^2}\right]\frac{\partial W_{11}}{\partial \bar{r}} - \frac{\partial^2 W_{11}}{\partial \bar{r}^2} + F\frac{\phi^2}{\bar{r}^6}\frac{\partial W_{11}}{\partial \theta} \tag{27}$$

$$D_1 = -\frac{\phi'}{\bar{r}^2}W_2 - \left[\frac{2}{\bar{r}^2} - \frac{2\phi}{\bar{r}^3}\right]W_{121} + \frac{dW_{121}}{d\bar{t}} + \bar{r}\frac{dW_{122}}{d\bar{t}} + \frac{\phi}{\bar{r}^2}W_{122} - 2\frac{\partial^2 W_{11}}{\partial \bar{r}\,\partial \theta} \tag{28}$$

$$D_2 = \frac{\partial^2 W_{11}}{\partial \theta^2} \tag{29}$$

And, since $W_2$, $W_{11}$, $W_{121}$, and $W_{122}$ are not functions of q, these coefficients may be set equal to zero.

Equation 29 then gives

$$W_{11}(\bar{t}, \bar{r}, \theta) = W_{111}(\bar{t}, \bar{r}) + \theta W_{112}(\bar{t}, \bar{r}) \tag{30}$$

which, when substituted into equation 28 set equal to zero, results in

$$2\bar{r}^3 \frac{\partial W_{112}}{\partial \bar{r}} = -\phi' \bar{r} W_2 - [2\bar{r} - 2\phi] W_{121} + \bar{r}^3 \frac{dW_{121}}{d\bar{t}} + \bar{r}^4 \frac{dW_{122}}{d\bar{t}} + \phi \bar{r} W_{122} \tag{31}$$

Since all functions on the right side of equation 31 are functions of $\bar{t}$ alone, $W_{112}$ is a quadratic in $\bar{r}$.

$$W_{112}(\bar{t}, \bar{r}) = W_{1121}(\bar{t}) + \bar{r} W_{1122}(\bar{t}) + \bar{r}^2 W_{1123}(\bar{t}) \tag{32}$$

Substituting this relation back into equation 31 gives:

$$-2W_{1122} - 4\bar{r} W_{1123} - \frac{\phi'}{\bar{r}^2} W_2 - \left[\frac{2}{\bar{r}^2} - \frac{2\phi}{\bar{r}^3}\right] W_{121} + \frac{dW_{121}}{d\bar{t}} + \bar{r} \frac{dW_{122}}{d\bar{t}} + \frac{\phi}{\bar{r}^2} W_{122} = 0 \tag{33}$$

Turning to the remaining coefficient of q (equation 27), and setting it equal to zero results in

$$E_0 + E_1 \theta = 0 \tag{34}$$

where

$$E_0 = 2F \frac{\phi\phi'}{\bar{r}^6} W_2 - 6F \frac{\phi^2}{\bar{r}^7} W_{121} - 4F \frac{\phi^2}{\bar{r}^6} W_{121} + \frac{\partial W_{111}}{\partial \bar{t}} - \left[\frac{2}{\bar{r}} - \frac{\phi}{\bar{r}^2}\right] \frac{\partial W_{111}}{\partial \bar{r}} - \frac{\partial W_{111}}{\partial \bar{r}} + F \frac{\phi^2}{\bar{r}^6} \left[W_{1121} + \bar{r} W_{1122} + \bar{r}^2 W_{1123}\right] \tag{35}$$

$$E_1 = \frac{dW_{1121}}{d\bar{t}} + \bar{r} \frac{dW_{1122}}{d\bar{t}} + \bar{r}^2 \frac{dW_{1123}}{d\bar{t}} - \left[\frac{2}{\bar{r}} - \frac{\phi}{\bar{r}^2}\right]\left[W_{1122} + 2\bar{r} W_{1123}\right] - 2W_{1123} \tag{36}$$

All of the functions in equations 35 and 36 are independent of $\Theta$ so that they may be set equal to zero. Also, since all of the functions in equation 36 are independent of $\bar{r}$, setting coefficients of like powers of $\bar{r}$ equal to zero yields:

$$2\phi w_{121} = 0 \tag{37}$$

$$\phi w_{122} - \phi' w_2 - 2 w_{121} = 0 \tag{38}$$

$$-2 w_{1122} + \frac{d w_{121}}{d\bar{t}} = 0 \tag{39}$$

$$-4 w_{1123} + \frac{d w_{122}}{d\bar{t}} = 0 \tag{40}$$

The same situation applies to equation 33 resulting in:

$$\phi w_{1122} = 0 \tag{41}$$

$$-2 w_{1122} + 2\phi w_{1123} = 0 \tag{42}$$

$$\frac{d w_{1121}}{d\bar{t}} - 6 w_{1123} = 0 \tag{43}$$

$$\frac{d w_{1122}}{d\bar{t}} = 0 \tag{44}$$

$$\frac{d w_{1123}}{d\bar{t}} = 0 \tag{45}$$

The characteristic function may now be found by solving equations 37 - 45 and equation 35 set equal to zero. Thus, from equations 39, 40, 43, 44, and 45

$$w_{1122} = c_2 \tag{46}$$

$$w_{1123} = c_3 \tag{47}$$

$$w_{1121} = 6 c_3 \bar{t} + c_1 \tag{48}$$

$$w_{122} = 4 c_3 \bar{t} + c_4 \tag{49}$$

$$W_{121} = 2c_2\bar{t} + c_5 \tag{50}$$

And, equations 37, 38, 41, and 42 also require

$$\phi\left[2c_2\bar{t} + c_5\right] = 0 \tag{51}$$

$$\phi\left[4c_3\bar{t} + c_4\right] - \phi' W_2 - 2\left[2c_2\bar{t} + c_5\right] = 0 \tag{52}$$

$$\phi c_2 = 0 \tag{53}$$

$$c_2 = \phi c_3 \tag{54}$$

while substitution of these relations into equation 35 equated to zero gives:

$$2F\frac{\phi\phi'}{\bar{F}^6}W_2 - \left[\frac{2}{\bar{F}} - \frac{\phi}{\bar{F}^2}\right]\frac{\partial W_{111}}{\partial \bar{F}} - 6F\frac{\phi^2}{\bar{F}^7}\left[2c_2\bar{t} + c_5\right] + \frac{\partial W_{111}}{\partial \bar{t}}$$
$$-4F\frac{\phi^2}{\bar{F}^6}\left[2c_2\bar{t} + c_5\right] - \frac{\partial W_{111}}{\partial \bar{F}} + F\frac{\phi^2}{\bar{F}^6}\left[6c_3\bar{t} + c_1 + \bar{F}c_2 + \bar{F}^2c_3\right] = 0 \tag{55}$$

From equations 51 and 52, $C_2 = C_5 = 0$ (i.e. $W_{121} = 0$).

Thus, equation 52 gives

$$\phi c_4 = \phi' W_2 \tag{56}$$

so that $W_2$ must be linear in $\bar{t}$.

$$W_2 = c_6\bar{t} + c_7$$

and

$$\frac{\phi'}{\phi} = \frac{c_4}{c_6\bar{t} + c_7}$$

which integrates to

$$\phi = b\left[c_6\bar{t} + c_7\right]^{\frac{c_4}{c_6}} \tag{57}$$

where b is the integration constant. Furthermore, since

$$\phi = \epsilon\bar{R}^2\frac{d\bar{R}}{d\bar{t}}$$

equation 57 yields the restriction on $\bar{R}$.

$$\frac{\epsilon}{3}\bar{R}^3 = \frac{b}{c_6}\left[c_6\bar{t} + c_7\right]^{\frac{c_4+c_6}{c_6}}\left[\frac{c_6}{c_4+c_6}\right] + c$$

Or,

$$\bar{R}^3 = a\left[c_6\bar{t} + c_7\right]^{\frac{c_4+c_6}{c_6}} + C \tag{58}$$

From the restrictions on $\phi$ and $W_2$, equation 55 becomes

$$\left[c_1 + \frac{2c_6^2}{c_4}\right]\frac{Fb^2}{\bar{F}^6}\left[c_6\bar{t}+c_7\right]^{\frac{2c_4}{c_6}} + \frac{\partial W_{111}}{\partial \bar{t}} + \left\{\frac{b\left[c_6\bar{t}+c_7\right]^{\frac{c_4}{c_6}}}{\bar{F}^2} - \frac{2}{\bar{F}} - 1\right\}\frac{\partial W_{111}}{\partial \bar{F}} = 0 \tag{59}$$

The characteristic function is thus given by

$$W = W_{111}(\bar{t},\bar{F}) + \theta\left[6c_3\bar{t}+c_1\right] + \theta\bar{F}^2c_3 + 2\bar{F}\left[4c_3\bar{t}+c_4\right] + P\left[c_6\bar{t}+c_7\right] \tag{60}$$

From which the absolute invariants are given by

$$\frac{d\bar{t}}{\xi} = \frac{d\bar{F}}{\psi} = \frac{d\theta}{\mathcal{L}} \tag{61}$$

Substituting equation 60 into the relations given for $\xi$, $\psi$, and $\mathcal{L}$ yields:

$$\frac{d\bar{t}}{c_6\bar{t}+c_7} = \frac{d\bar{F}}{\bar{F}\left[4c_3\bar{t}+c_4\right]} = \frac{-d\theta}{W_{111} - \theta\left[6c_3\bar{t}+c_3\bar{F}^2+c_1\right]} \tag{62}$$

## PARTICULAR SOLUTIONS

Consider the case when the bubble growth rate is finite, ( $\phi \neq 0$ ). Then, equation 54 requires that $C_3 = 0$ and equation 58 requires that $C_6$ be finite. In this case, equation 62 becomes

$$\frac{d\bar{t}}{c_6\bar{t}+c_7} = \frac{d\bar{F}}{c_4\bar{F}} = \frac{d\theta}{c_1\theta - W_{111}} \tag{63}$$

The first absolute invariant follows from the solution of

$$\frac{c_4\,dt}{c_6\bar{t}+c_7} = \frac{d\bar{F}}{\bar{F}}$$

Or,

$$\eta = \text{CONSTANT} = \frac{r}{\{C_6\bar{t}+C_7\}^{\gamma}} \quad ; \quad \left(\gamma = \frac{C_4}{C_6}\right) \qquad (64)$$

There are now three choices for $W_{111}$; either $W_{111}$ is a constant or zero or it is a function of $\bar{r}$ and/or $\bar{t}$. By examining equation 59, one sees that the first two choices are impossible if this equation is to be satisfied. However, if viscous dissipation is ignored ( F = 0 ), the general case of $W_{111} = -C_8$ is valid. In this case, the second invariant is found from

$$\frac{d\bar{t}}{C_6\bar{t}+C_7} = \frac{d\theta}{C_1\theta+C_8} \qquad (65)$$

where $C_8 = -W_{111}$

Solution of equation 65 gives:

$$\frac{\theta + C_9}{\{C_6\bar{t}+C_7\}^{\beta}} = \text{CONSTANT} = f(\eta) \qquad \beta = C_1/C_6 \quad C_9 = C_8/C_1 \qquad (66)$$

With no viscous dissipation, the governing equation becomes

$$\frac{\partial\theta}{\partial\bar{t}} + \frac{\phi}{\bar{r}^2}\frac{\partial\theta}{\partial\bar{r}} = \frac{1}{\bar{r}^2}\frac{\partial}{\partial\bar{r}}\left[\bar{r}^2\frac{\partial\theta}{\partial\bar{r}}\right] \qquad (67)$$

Substitution of the two similarity variables defined above along with $\phi$ from equation 57 results in:

$$f'' + f'\left[\frac{2}{\eta} - \frac{b}{\eta^2} + \frac{\bar{r}^2 C_6\gamma}{\eta\{C_6\bar{t}+C_7\}}\right] - \frac{C_6\beta\bar{r}^2}{\{C_6\bar{t}+C_7\}}f = 0 \qquad (68)$$

from which $\gamma = 1/2$ is an obvious requirement so that

$$f'' + f'\left[\frac{2}{\eta} + \eta\frac{C_6}{2} - \frac{b}{\eta^2}\right] - C_6\beta\eta^2 f = 0 \qquad (69)$$

$$\left(f' = \frac{df}{d\eta}\right)$$

Equation 69 is the most general form of the transformed equation. The only transformation possible being the linear group. Therefore, without examining the particular boundary conditions, a similar solution requires that the bubble radius vary according to

$$\bar{R}^3 = a\left[c_6\bar{t} + c_7\right]^{3/2} + c \qquad (70)$$

A consideration of the boundary conditions of the problem itself, namely

$$\theta(\bar{r}, 0) = \theta(\infty, \bar{t}) = 1 \qquad (71)$$

$$\theta(\bar{R}, \bar{t}) = T_{SAT}/T_\infty \qquad (72)$$

shows that the dependent variable can be none other than

$$f(\eta) = \theta$$

This requires that $W_{111} = C_1 = 0$ and thus $\beta = 0$. Also, since $W_{111} = 0$ is incompatable with equation 59 unless $F = 0$, similar solutions are not possible unless viscous dissipation is neglected. Equation 71 also requires $C_7 = 0$ in order that it may transform to

$$\eta = \infty \; ; \quad f = 1$$

Finally, the heat balance at the bubble wall

$$\frac{\partial\theta(\bar{R}, \bar{t})}{\partial\bar{r}} = \frac{\alpha\rho_v\epsilon h_{fg}}{\lambda T_\infty}\frac{d\bar{R}}{d\bar{t}} \qquad (73)$$

may be transformed only if c = 0 in equation 70. The final form thus coincides with the results of Scriven.

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<u>NOMENCLATURE</u>

| | | |
|---|---|---|
| $a, b, c, C_1 \ldots C_9$ | = | constants |
| $C_p$ | = | specific heat |
| $h_{fg}$ | = | latent heat |
| $r$ | = | radial co-ordinate |
| $R$ | = | bubble radius |
| $R_o$ | = | reference bubble radius |
| $t$ | = | time |
| $T$ | = | temperature |
| $T_\infty$ | = | temperature far from the bubble |
| $T_{sat}$ | = | saturation temperature |
| $u$ | = | radial velocity |
| $f$ | = | dependent similarity variable |
| $\alpha$ | = | termal diffusivity |
| $\epsilon$ | = | $[\rho_l - \rho_v]/\rho_l$ |
| $\eta$ | = | independent similarity variable |
| $\lambda$ | = | thermal conductivity |
| $\mu$ | = | viscosity |
| $\rho_l, \rho_v$ | = | density of liquid and vapor |