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STUDIES IN RADAR CROSS SECTIONS - XLVI
THE CONVERGENCE OF LOW FREQUENCY EXPANSIONS IN SCALAR SCATTERING BY SPHEROIDS

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XLVI	"The Convergence of Low Frequency Expansions in Scalar Scattering by Spheroids", T. B. A. Senior (3648-4-T, August 1961), AF19(604)-6655. UNCLASSIFIED. 143 pgs.

PREFACE

This is the forty-sixth in a series of reports growing out of the study of radar cross sections at The Radiation Laboratory of The University of Michigan. Titles of the reports already published or presently in process of publication are listed on the preceding pages.

When the study was first begun, the primary aim was to show that radar cross sections can be determined theoretically, the results being in good agreement with experiment. It is believed that by and large this aim has been achieved.

In continuing this study, the objective is to determine means for computing the radar cross section of objects in a variety of different environments. This has led to an extension of the investigation to include not only the standard boundary-value problems, but also such topics as the emission and propagation of electromagnetic and acoustic waves, and phenomena connected with ionized media.

Associated with the theoretical work is an experimental program which embraces (a) measurement of antennas and radar scatterers in order to verify data determined theoretically; (b) investigation of antenna behavior and cross section problems not amenable to theoretical solution; (c) problems associated with the design and development of microwave absorbers; and (d) low and high density ionization phenomena.

K. M. Siegel

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SUMMARY

For the scalar problem of the diffraction of a plane wave by a spheroid the exact solution is known, and at low frequencies the expression for the far field amplitude can be expanded in a series of increasing positive powers of ka, where k is the wave number and 2a is the interfocal distance. This is the Rayleigh series, and is convergent for sufficiently small values of ka.

In order to determine the range of frequencies for which this expansion is applicable an essential factor is the radius of convergence, and the paper is devoted entirely to the calculation of this quantity. Attention is concentrated on the case in which the plane wave is incident nose-on, and the radius of convergence is obtained as a function of the length-to-width ratio for prolate and oblate spheroids, hard as well as soft. For other angles of incidence it can be shown that the radius is not greater than this, and in most instances it would appear to be the same.

I

INTRODUCTION

In recent years it has become increasingly apparent that one of the most difficult problems in diffraction theory is the quantitative description of the scattered field at wavelengths which are comparable with the effective dimensions of the body. Even in those cases where an exact solution of the scattering problem exists in the form of a wave function expansion, a large number of terms are necessary to calculate the rapid variation as a function of frequency which is often typical of the resonance region. Moreover, there are as yet no approximate methods specifically designed for treating this frequency range, and for this reason attempts have been made to push the high and low frequency approximations as far as possible in the hope of narrowing the gap. In particular, some success has been achieved in applying high frequency techniques such as the geometrical theory of diffraction even when the wavelength is as large as a typical dimension of the body.

At the other end of the frequency spectrum the scattered field can be expanded as a series of increasing positive powers of ka, where k is the wave number and a is a dimension representative of the body. This is the so-called Rayleigh series and for sufficiently small values of ka (that is, for sufficiently low frequencies) the expansion can be shown to be convergent. Nevertheless, for any body the radius of convergence is almost certainly finite and sets an upper limit on the portion of the resonance region to which the Rayleigh series is applicable. It is obviously desirable to know this radius before attempting to predict the resonant behavior using a finite number of terms in the low frequency expansion.

The present paper is concerned with scalar scattering by prolate and oblate spheroids with particular reference to the convergence of the Rayleigh series. In §2 the series for the sphere is considered briefly and this serves to illustrate the methods which are available for assessing the convergence. For the more general problem of the spheroid the expression for the far field amplitude is derived in §3 and this is followed (§4) by an analysis of the case in which the ellipticity is small (almost spherical bodies). The next two sections are devoted to the problem of an oblate spheroid whose ellipticity is unity (a disc) or only slightly different from unity (almost a disc), and in §7 the convergence for oblate spheroids of intermediate ellipticity is determined. In §8 the convergence is calculated for a prolate spheroid whose ellipticity is near to unity (a body which is almost a vanishing rod), and a study of the intermediate ellipticities then completes the discussion of the prolate spheroid.

In all cases the bodies considered are either soft or hard (Dirichlet or Neumann boundary condition respectively at the surface), and the resulting values for the radius of convergence are displayed in §9. It must not be assumed, however, that these values would also be applicable if the boundary condition differed from the above.

II

THE SPHERE

A limiting case of both the prolate and oblate spheroids is the sphere and it is convenient to begin by considering the Rayleigh series for this more simple body.

In view of the symmetry possessed by the sphere it is sufficient to take a field which is incident in the direction of the negative z axis of a Cartesian co-ordinate system (x, y, z), and if the field is also assumed to be a plane wave, it can be written as

$$V^i = e^{-ikz} \tag{1}$$

where the time factor $e^{-i\omega t}$ has been suppressed. In terms of spherical polar co-ordinates (R, θ, ϕ) with

$$x = R \sin\theta \cos\phi, \qquad y = R \sin\theta \sin\phi, \qquad z = R\cos\theta,$$

V^i has the expansion

$$V^i = \sum_{n=0}^{\infty} (-i)^n (2n+1)\, j_n(kR)\, P_n(\cos\theta) \tag{2}$$

and if the scattered field is written similarly as

$$V^s = \sum_{n=0}^{\infty} (-i)^n (2n+1)\, A_n\, h_n(kR)\, P_n(\cos\theta), \tag{3}$$

application of the boundary condition at R = a gives

$$A_n = - \frac{\Delta\, j_n(\rho)}{\Delta\, h_n(\rho)} \tag{4}$$

with $\rho = ka$. Δ is either unity or $\partial/\partial\rho$ depending on whether the body is soft (Dirichlet boundary condition at the surface) or hard (Neumann condition) respectively.

If each Hankel function of argument kR is replaced by the first term in its asymptotic expansion for large kR, the coefficient of $\frac{e^{ikR}}{kR}$ in the scattered field is the far field amplitude, and from equation (3) its expression is seen to be

$$f(\cos\theta) = 2i \sum_{n=0}^{\infty} (-1)^{n+1} \left(n + \frac{1}{2}\right) A_n P_n(\cos\theta). \tag{5}$$

This series is absolutely convergent for all (real) values of ρ. Moreover, within some neighbourhood of the origin in the complex ρ plane, each A_n $(n = 0, 1, 2, \ldots)$ can be expanded in a series of positive (integral) powers of ρ and is therefore an analytic function of ρ within the region. It then follows that if the terms in (5) are re-arranged, an expansion for $f(\cos\theta)$ is obtained which includes only positive powers of ρ and is convergent for values of ρ inside the smallest circle of convergence of the individual A_n. This is the Rayleigh series, and for the sphere the terms up to and including ρ^6 have been given by Senior (1960a).

The calculation of the least circle of convergence is a trivial matter. From equation (4) it is apparent that the only singularities of A_n are poles at the zeros of the spherical Hankel function (or its derivative), and the location of these zeros is such that the singularity nearest to the origin is provided by one of the smaller values of n. Using the expressions for the $h_n(\rho)$ it is found that whereas $h_o(\rho)$ has no zero, both $\frac{\partial}{\partial\rho} h_o(\rho)$ and $h_1(\rho)$ have a zero at $\rho = - i$, and the zeros of $\frac{\partial}{\partial\rho} h_1(\rho)$ and $h_2(\rho)$ are $(-i \pm 1)$ and $\frac{1}{2}(-3i \pm \sqrt{3})$ respectively. Accordingly, A_1 is infinite on the unit circle and since all the other coefficients are regular inside, the entire Rayleigh expansion must converge for $|\rho| < 1$. The fact that a singularity exists for which $|\rho| = 1$ shows that the expansion does not converge outside this region, and in consequence the Rayleigh series for both hard and soft spheres converges only for

$$ka < 1. \tag{6}$$

The above method is based upon the location of the smallest singularities in the complex ρ plane, and for this purpose it is essential to have available the exact expressions for the individual A_n. If these are not known, or if their complication is such that the location of the singularities is not practicable, the method is no longer applicable, and it is necessary to employ a more intuitive approach. The one which has proved most valuable involves a comparison of the numerical coefficients in the low frequency expansions for the A_n, and although the method cannot be regarded as rigorous, it is generally sufficient to indicate the radius of convergence with a reasonable degree of certainty. The new technique will be illustrated with reference to the sphere.

The starting point for the analysis is the expansion of the A_n in the form

$$A_n = \rho^m a_o^n \left\{ 1 + \sum_{r=1}^{\infty} \left(\frac{-i\rho}{a_r^n} \right)^r \right\} \tag{7}$$

where m is some integer and the a_r^n are independent of ρ. In general, the a_r^n will be complex and for most bodies only a small number of them will be known. For the sphere, on the other hand, it is a straight forward matter to determine as many of the a_r^n as are desired by inserting the series developments of the Bessel functions into equation (4), and in Tables I and II the values are given for $r < 11$ and $n = 0, 1$ and 2. If the moduli $|a_r^n|$ are now plotted sequentially ($r = 1, 2, 3, \ldots.$) for each n, the curves shown in Figures I and II are obtained. In reality, each curve only has a meaning when r is an integer, but it is tempting to join the discrete points by the continuous curves shown; an infinity then implies that the corresponding power of ρ has zero coefficients.

These curves confirm in a striking manner the radii of convergence previously found. When $n = 1$ (soft) and $n = 0$ (hard) the curves are asymptotic to the value unity, and this represents the smallest radius of convergence (and hence the radius of convergence of the low frequency expansion for f) in accordance with equation (6). The fact that for the soft sphere A_o has no singularity in the finite portion of the ρ plane is reflected in the upward trend of the corresponding curve in Figure I.

TABLE I. CONVERGENCE COEFFICIENTS FOR SOFT SPHERE

	n = 0		n = 1		n = 2	
	$(a_r^o)^{-r}$	$\|a_r^o\|$	$(a_r^1)^{-r}$	$\|a_r^1\|$	$(a_r^2)^{-r}$	$\|a_r^2\|$
1	1	1	0	∞	0	∞
2	$\frac{2}{3}$	1.2247	$\frac{3}{5}$	1.2910	$\frac{5}{21}$	2.0494
3	$\frac{1}{3}$	1.4422	$-\frac{1}{3}$	1.4422	0	∞
4	$\frac{2}{15}$	1.6548	$\frac{3}{7}$	1.2359	0	∞
5	$\frac{2}{45}$	1.8639	$-\frac{2}{5}$	1.2011	$\frac{1}{45}$	2.1411
6	$\frac{4}{315}$	2.0703	$\frac{11}{27}$	1.1614	$-\frac{5}{297}$	1.9753
7	$\frac{1}{315}$	2.2746	$-\frac{71}{175}$	1.1376	$\frac{2}{189}$	1.9151
8	$\frac{2}{2835}$	2.4771	$\frac{67}{165}$	1.1193	$-\frac{5}{1053}$	1.9518
9	$\frac{2}{14175}$	2.6781	$-\frac{1151}{2835}$	1.1054	$\frac{5}{3969}$	2.0999
10	$\frac{4}{155925}$	2.8780	$\frac{8313}{20475}$	1.0943	$\frac{2}{6075}$	2.2297
11	$\frac{2}{467775}$	3.0767	$-\frac{29543}{72765}$	1.0853	$-\frac{2}{2673}$	1.9239

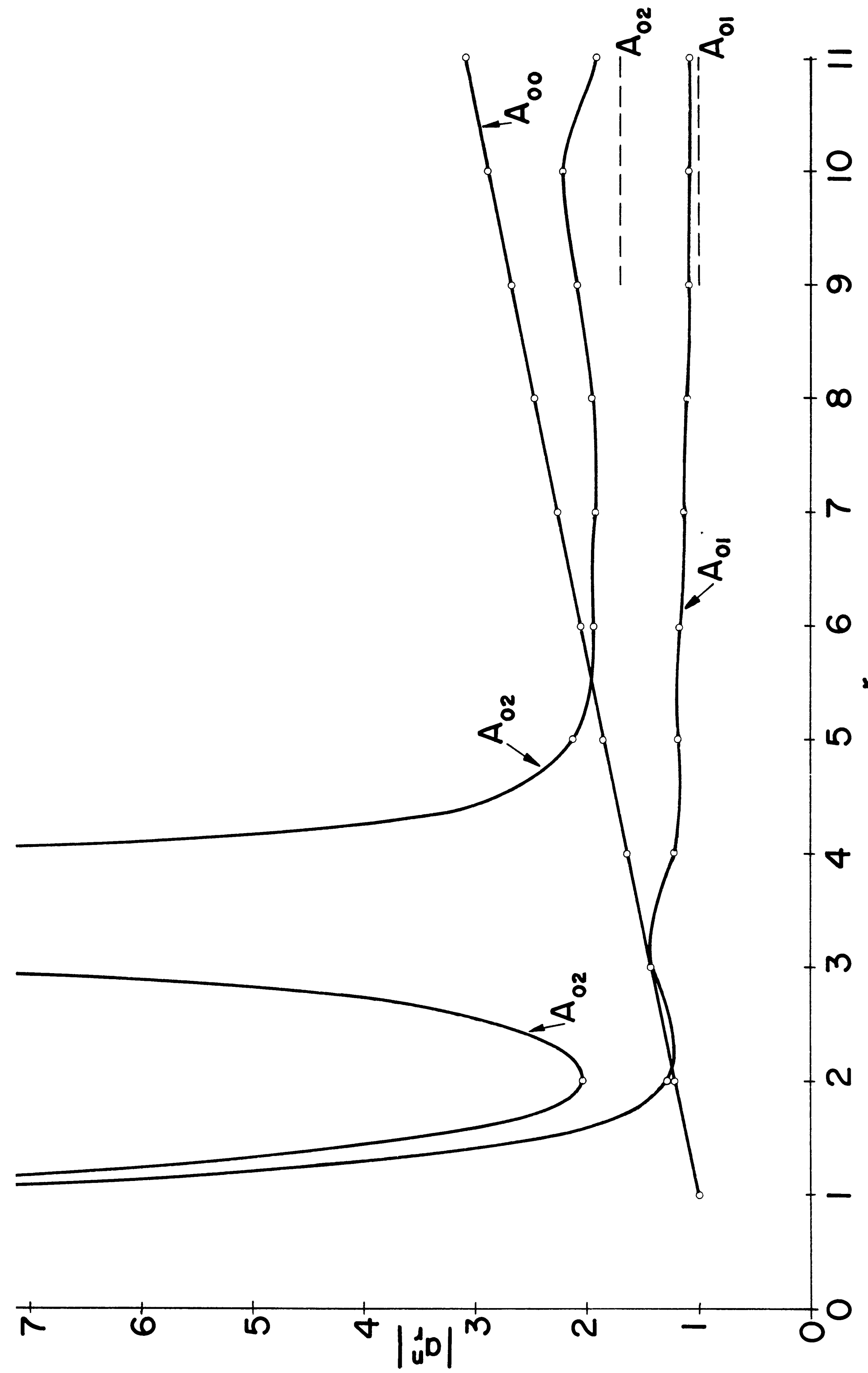

FIGURE I. CONVERGENCE COEFFICIENTS FOR SOFT SPHERE

TABLE II. CONVERGENCE COEFFICIENTS FOR HARD SPHERE

	n = 0		n = 1		n = 2	
	$(a_r^o)^{-r}$	$\lvert a_r^o \rvert$	$(a_r^1)^{-r}$	$\lvert a_r^1 \rvert$	$(a_r^2)^{-r}$	$\lvert a_r^2 \rvert$
1	0	∞	0	∞	0	∞
2	$\frac{3}{5}$	1.2910	$\frac{3}{10}$	1.8258	$\frac{25}{126}$	2.2450
3	$-\frac{1}{3}$	1.4422	$\frac{1}{6}$	1.8171	0	∞
4	$\frac{3}{7}$	1.2359	$-\frac{3}{28}$	1.7479	$\frac{5}{162}$	2.3858
5	$-\frac{2}{5}$	1.2011	$\frac{1}{10}$	1.5849	$-\frac{2}{135}$	2.3220
6	$\frac{11}{27}$	1.1614	$-\frac{5}{216}$	1.8732	$\frac{185}{16038}$	2.1038
7	$-\frac{71}{175}$	1.1376	$-\frac{29}{1400}$	1.7400	$-\frac{10}{1701}$	2.0829
8	$\frac{67}{165}$	1.1193	$\frac{89}{2640}$	1.5277	$\frac{250}{85293}$	2.0731
9	$-\frac{1151}{2835}$	1.1054	$-\frac{523}{22680}$	1.5202	$-\frac{107}{71442}$	2.0599
10	$\frac{8313}{20475}$	1.0943	$\frac{1367}{218400}$	1.6609	$\frac{2557}{2952450}$	2.0242
11	$-\frac{29543}{72765}$	1.0853	$\frac{4918}{931392}$	1.6107	$-\frac{793}{1515591}$	1.9875

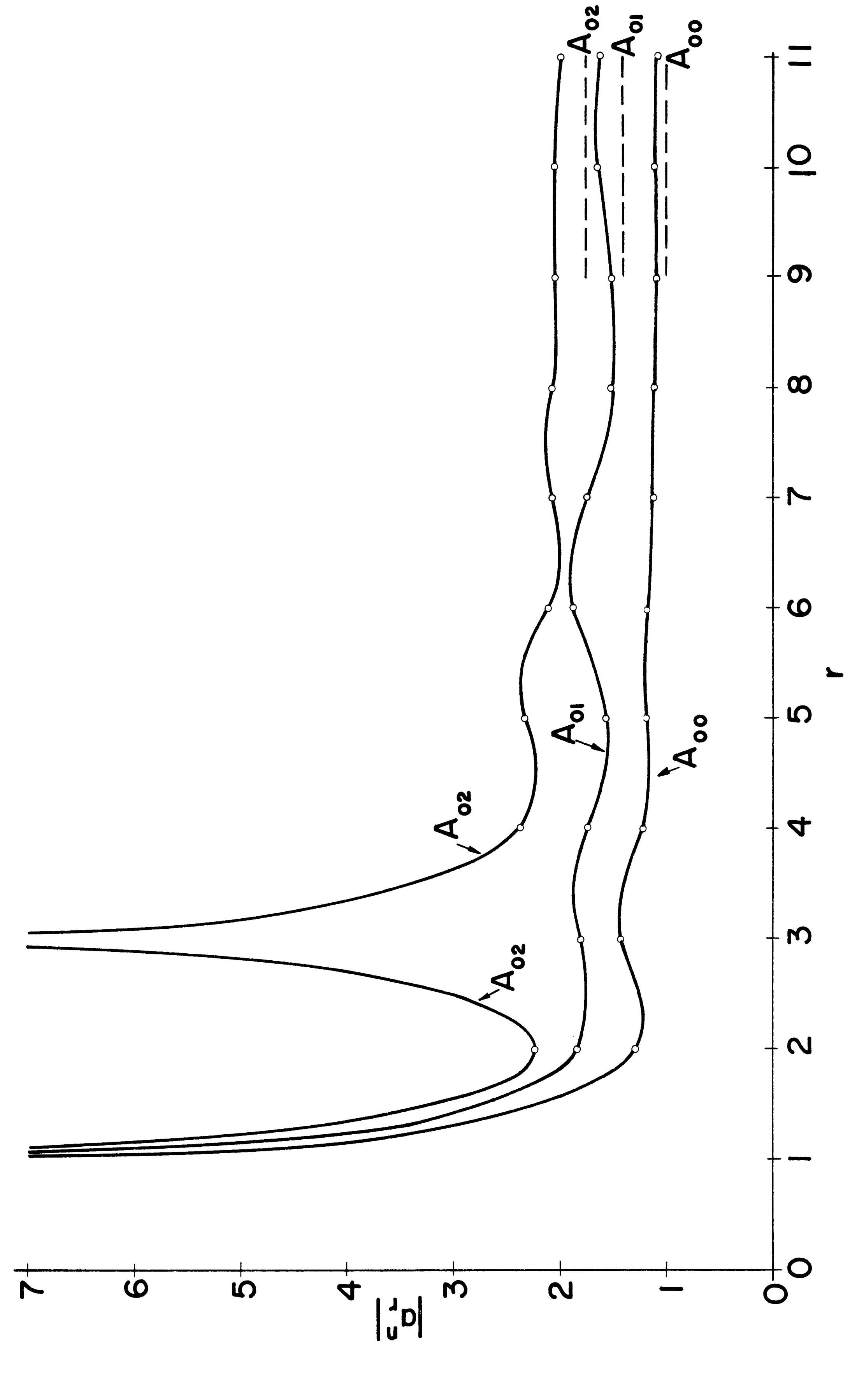

FIGURE II. CONVERGENCE COEFFICIENTS FOR HARD SPHERE

III

THE GENERAL SPHEROID

Having obtained the radius of convergence of the low frequency expansions for the sphere, we now turn our attention to the more general problem of the spheroid and seek to determine the corresponding limits on the convergence using the methods previously described. For this purpose a necessary preliminary is the derivation of an expression for the scattering function, and since the solutions for prolate and oblate spheroids can be deduced from one another by a trivial change of parameter, the analysis will be given for only the first type of body.

Consider, therefore, a prolate spheroid which is defined in terms of the prolate spheroidal co-ordinates (ξ, η, ϕ) by the equation $\xi = \xi_o$. Incident upon the body is a plane wave travelling in the x z plane of a Cartesian co-ordinate system (x, y, z) where

$$x = a\left[(1-\eta^2)(\xi^2-1)\right]^{1/2} \cos\phi, \quad y = a\left[(1-\eta^2)(\xi^2-1)\right]^{1/2} \sin\phi$$

$$z = a\eta\xi,$$

with 2a as the interfocal distance, and if the direction of incidence makes an angle ζ with the positive z axis, the field can be written as

$$V^i = e^{ik(x \sin\zeta + z\cos\zeta)} \tag{8}$$

The incident field can also be expressed as a sum over angular and radial spheroidal functions, and by postulating a similar expression for the scattered field V^s the unknown amplitude factor can be determined from the boundary condition at the surface. The details of the analysis are given in Senior (1960a), and it is there shown that

$$V^s = 2 \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} (2-\delta_{om}) i^n \frac{S_{mn}(c, \cos \zeta)}{N_{mn}(c)} A_{mn} R^{(3)}_{mn}(c, \xi) S_{mn}(c, \eta) \cos m\phi \tag{9}$$

with

$$A_{mn} = - \frac{\Delta R^{(1)}_{mn}(c, \xi_o)}{\Delta R^{(3)}_{mn}(c, \xi_o)} \tag{10}$$

where now Δ is unity or $\partial/\partial\xi_o$ depending on whether the body is soft or hard respectively, and c = ka. The notation is that of Flammer (1957) and the reader is referred to this book for the definitions of the symbols and functions here used.

In the far field $c\xi \sim kR$, where R is the distance from the center of the spheroid, and since

$$R^{(3)}_{mn}(c, \xi) \sim \frac{1}{c\xi} e^{i\left\{c\xi - \frac{1}{2}(n+1)\pi\right\}}$$

as $c\xi \rightarrow \infty$, the far field amplitude is

$$f(\eta, \zeta) = -2i \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} (2-\delta_{om}) \frac{S_{mn}(c, \cos \zeta)}{N_{mn}(c)} A_{mn} S_{mn}(c, \eta) \cos m\phi. \tag{11}$$

For sufficiently small values of c (that is, for sufficiently low frequencies) all the terms in (11) can be expanded in positive integral powers of c, and by re-arranging the resulting series the Rayleigh expansion is then obtained.

Of the various factors which make up the individual terms in (11), the angular functions are free of singularities, whereas A_{mn} (see equation 10) has poles at the zeros of the denominator. In addition, of course, there is the possibility of singularities arising from the vanishing of the normalization constant N_{mn} for some (complex) c, but if such a singularity were the nearest to the origin of the c plane, the convergence of the Rayleigh series would be independent of the precise shape of the spheroid. This is intuitively unlikely, and it will be shown later that any singularities of N_{mn} can be discounted as far as the convergence of the series is concerned. The radius of convergence is then specified by the coefficient A_{mn} whose pole has the smallest modulus, and is given by the smallest zero of the radial functions $\Delta R_{mn}^{(3)}$

In the particular case $\zeta = \pi$ (incidence along the negative z axis) the expression for the far field amplitude can be simplified by observing that

$$S_{mn}(c, -1) = 0$$

unless m = 0. The summation over m now contributes only the first term and accordingly

$$f(\eta, \pi) = -2i \sum_{n=0}^{\infty} \frac{S_{on}(c, -1)}{N_{on}(c)} A_{on} S_{on}(c, \eta). \tag{12}$$

Using this equation the Rayleigh series for both soft and hard prolate spheroids have been computed up to and including terms in c^6, and the results are given in Senior (1960a).

An interesting feature of equation (12) is that all the functions which can affect the convergence of the low frequency expansion also appear in the expression for $f(\eta, \zeta)$, $\zeta \neq \pi$, and it therefore follows that the radius of convergence when $\zeta \neq \pi$ cannot be greater than it is for incidence along the z axis. Since this conclusion also holds for an oblate spheroid (as can be seen by replacing c by $-ic$ and ξ_o by $i\xi_o$ in (11) and (12)), it is clear that the convergence of the Rayleigh series is not directly related to the radius of curvature at the 'point' at which the incident field strikes the body. This rules out one of the ways in which the convergence could depend on ζ and ξ_o.

Because of the complication involved, a detailed study of the dependence of the radius of convergence on ζ has not been pursued, and the subsequent analysis is confined to the case $\zeta = \pi$. The study is then aimed at an investigation of the way in which the convergence varies with ξ_o, and accordingly the results which are found represent only the upper bound on the convergence when $\zeta \neq \pi$. Nevertheless, preliminary calculations do suggest that this upper bound is in fact the actual radius of convergence, implying that the convergence is independent of the angle ζ at which the field is incident.

When $\zeta = \pi$ the far field amplitude is given by equation(12) and in the following sections the radius of convergence of the expansion for $f(\eta, \pi)$ is determined for certain selected values of ξ_o. From these results it is possible to infer the convergence for all ξ_o, and to this end we start by considering the problem of an almost spherical body.

IV

THE SPHEROID OF SMALL ELLIPTICITY

The ellipticity of the (prolate) spheroid $\xi = \xi_o$ is*

$$e = 1/\xi_o$$

and if $\xi_o \rightarrow \infty$, $c \rightarrow 0$ in such a way that $c\xi_o$ tends to a finit limit ρ, the spheroid degenerates into a sphere of radius ρ/k. The amplitude function defined in equation (12) then reduces to the sphere amplitude shown in equation (5), and this fact is most clearly seen by expanding the radial spheroidal functions in terms of the spherical Bessel functions.

From Flammer (equation 4.1.15) we have

$$c_o^{on}(c)\, R_{on}^{(1)}(c, \xi) = \sum_{r=0,1}^{\infty}{}' \, i^{r-n}\, d_r^{on}(c)\, j_n(c\xi) \tag{13}$$

where $c_o^{on}(c) = \sum_{r=0,1}^{\infty}{}' \, d_r^{on}(c)$ and the summation extends over even or odd values of n according as n is even or odd (denoted by a prime attached to the summation sign). Similarly,

$$c_o^{on}(c)\, R_{on}^{(3)}(c, \xi) = \sum_{r=0,1}^{\infty}{}' \, i^{r-n}\, d_r^{on}(c)\, h_n(c\xi) \tag{14}$$

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*The analogous formula for an oblate spheroid is $e = \dfrac{1}{\sqrt{\xi_o^2 + 1}}$.

and since

$$d_r^{on}(c) = \delta(r-n) + 0(c^2)$$

for small c, it follows that as $c \to 0$ the spheroidal amplitude coefficient A_{on} reduces to*

$$-\frac{\Delta\, j_n(c\xi)}{\Delta\, h_n(c\xi)}$$

in agreement with the spherical coefficient (4). Morever,

$$S_{on}(c,\eta) = \sum_{r=0,1}^{\infty}{}' \, d_r^{on}(c)\, P_n(\cos\theta)$$

$$\longrightarrow P_n(\cos\theta)$$

and

$$N_{on}(c) = 2 \sum_{r=0,1}^{\infty}{}' \frac{\left\{d_o^{on}(c)\right\}^2}{2r+1}$$

$$\longrightarrow \frac{2}{2n+1},$$

which completes the reduction of equation (12) to equation (5).

A consequence of this identification is that in the sphere limit the convergence of the Rayleigh series for the soft body is determined by the smallest zero of $R_{o1}^{(3)}(c, \xi)$ as a function of $c\xi$, and for the hard body by the smallest zero of $\frac{\partial}{\partial\xi} R_{oo}^{(3)}(c, \xi)$. This suggests that when the ellipticity is small but not zero a possible method for assessing the convergence is to expand

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*From henceforth we shall omit the suffix 'o' from the radial variable specifying the spheroid.

$R^{(3)}_{on}(c,\xi)$ in terms of the spherical Hankel functions, and calculate the perturbation of the zeros of $h_n(c\xi)$ by retaining only the leading powers of e and c. Although it would appear that two small but unrelated quantities are now involved, this is in fact not so, and the requirement that the spheroid goes over into a sphere of radius ρ as $c \rightarrow 0$ implies that

$$e = c/\rho$$

for all values of c under discussion.

Let us therefore begin by considering the soft spheroid and attempt to locate the smallest zero of $R^{(3)}_{on}(c,\xi)$. From the recurrence relations defining the spheroidal coefficients d^{on}_r, we have

$$d^{on}_{n+2} = \alpha^n_{n+2} c^2 + 0(c^4)$$

$$d^{on}_{n} = 1 + \alpha^n_n c^2 + 0(c^4)$$

$$d^{on}_{n-2} = \alpha^n_{n-2} c^2 + 0(c^4)$$

and in general

$$d^{on}_r = 0(c^{|n-r|}),$$

where

$$\alpha^n_{n+2} = - \frac{(n+1)(n+2)}{2(2n+1)(2n+3)^2}$$

$$\alpha_n^n = \frac{2 \mp (2n-1)(2n+3)}{2(2n-1)^2(2n+3)^2} \qquad (\mp \text{ for n } \begin{matrix}\text{odd}\\ \text{even}\end{matrix})$$

and

$$\alpha_{n-2}^n = \frac{n(n-1)}{2(2n-1)^2(2n+1)}$$

with $\alpha_{n-2} \equiv 0$ if $n < 2$. Bearing in mind that the factor c_o^{on} is cancelled by a like factor in the expression $R_{on}^{(1)}(c, \xi)$ and can therefore play no part in the convergence analysis, equation (14) now gives

$$c_o^{on}(c)\, R_{on}^{(3)}(c,\xi) = (1 + \alpha_n^n c^2)\, h_n(c\xi) - c^2 \alpha_{n-2}^n h_{n-2}(c\xi)$$

$$-c^2 \alpha_{n+2}^n h_{n+2}(c\xi) + 0(c^4),$$

and since

$$h_{n-2}(c\xi) = -\left\{1 - \frac{(n+1)(2n-1)}{(c\xi)^2}\right\} h_n(c\xi) + \frac{2n-1}{c\xi} h'_n(c\xi)$$

$$h_{n+2}(c\xi) = -\left\{1 - \frac{n(2n+3)}{(c\xi)^2}\right\} h_n(c\xi) - \frac{2n+3}{c\xi} h'_n(c\xi)$$

where the prime attached to the Hankel function denotes differentiation with respect to the whole argument, we have

$$c_o^{on}(c)\, R_{on}^{(3)}(c,\xi) = \Big[1 + c^2 (\alpha_{n-2}^n + \alpha_n^n + \alpha_{n+2}^n) + e^2 \Big\{ (n+1)(2n-1)\alpha_{n-2}^n$$

$$+ n(2n+3)\,\alpha^n_{n+2}\Big\}\Big]\, h_n(c\xi) - ce\,\Big[(2n-1)\,\alpha^n_{n-2}$$

$$- (2n+3)\,\alpha^n_{n+2}\Big]\; h'_n(c\xi) + 0(c^4).$$

The zeros of $R^{(3)}_{on}(c,\xi)$ are therefore given by the roots of

$$h_n(\rho) = e^2\Big\{ (2n-1)\,\alpha^n_{n-2} - (2n+3)\,\alpha^n_{n+2}\Big\}\,\rho h'_n(\rho) + 0(e^4), \qquad (15)$$

and the smallest root in the complex ρ plane determines the smallest radius of convergence of the expansions for the individual A_{on}, and hence the radius of convergence of the Rayleigh series.

In passing it should be pointed out that to order e equation (15) is simply

$$h_n(\rho) = 0 \qquad (16)$$

and consequently a first order analysis will not reveal any change in the radius of convergence. This is otherwise obvious from the fact that the ratio of the major to the minor axes is $(1 - e^2)^{-1/2} = 1 + 0\,(e^2)$, which implies that the first order terms in e or c correspond only to a change in the radius of the sphere but not to any deformation of shape.

To obtain the solution of equation (15) it is assumed that

$$\rho = \rho_n + \tau e^2 + 0\,(e^4)$$

where ρ_n is a solution of (16). If this is inserted into (15) it follows immediately that

$$\tau = \rho_n \left\{ (2n-1)\, \alpha^n_{n-2} - (2n+3)\, \alpha^n_{n+2} \right\}$$

and since

$$(2n-1)\, \alpha^n_{n-2} - (2n+3)\, \alpha^n_{n+2} = \frac{1}{4} \left\{ 1 - \frac{1}{(2n-1)(2n+3)} \right\},$$

the perturbed root is therefore

$$\rho = \rho_n \left[1 + \frac{e^2}{4} \left\{ 1 - \frac{1}{(2n-1)(2n+3)} \right\} \right] + 0(e^4). \tag{17}$$

It will be observed that the terms in braces are real, so that the effect of the perturbation is merely to change the magnitude of the root whilst leaving its phase unchanged. Moreover, the terms are also positive for all integer n (including zero) and thus the magnitude of the perturbed root exceeds that of the unperturbed, showing that the effect of the perturbation is to increase the radius of convergence of the expansion for A_{on}. For a fixed ellipticity the percentage change increases with increasing $n \geqslant 1$.

To determine the smallest radius of convergence it is only necessary to recall that $|\rho_n|$ also increases with increasing $n \geqslant 1$, and whereas $\rho_o = \infty$, $\rho_1 = -i$. Accordingly, the smallest perturbed root is that corresponding to A_{o1}, and is

$$\rho = -i\,(1 + \frac{e^2}{5}) + 0(e^4). \tag{18}$$

The modulus of this represents the value of $c\xi$ specifying the radius of convergence and consequently for a soft prolate spheroid of small ellipticity the Rayleigh series converges for

$$c\xi < 1 + \frac{1}{5\xi^2} + 0(\xi^{-4}) . \tag{19}$$

When the spheroid is hard rather than soft a similar treatment is possible based upon the location of the roots of $\frac{\partial}{\partial\xi} R^{(3)}_{on}(c,\xi)$. By expanding this derivative in powers of c we have

$$\frac{1}{c} c^{on}_{o}(c) \frac{\partial}{\partial\xi} R^{(3)}_{on}(c,\xi) = (1 + \alpha^n_n c^2) h'_n(c\xi) - c^2 \alpha^n_{n-2} h'_{n-2}(c\xi)$$

$$- c^2 \alpha^n_{n+2} h'_{n+2}(c\xi) + 0(c^4),$$

and since

$$h'_{n-2}(c\xi) = -\left\{1 - \frac{(n-2)(2n-1)}{(c\xi)^2}\right\} h'_n(c\xi) - \frac{2n-1}{c\xi}\left\{1 - \frac{(n-2)(n+1)}{(c\xi)^2}\right\} h_n(c\xi)$$

$$h'_{n+2}(c\xi) = -\left\{1 - \frac{(n+3)(2n+3)}{(c\xi)^2}\right\} h'_n(c\xi) + \frac{2n+3}{c\xi}\left\{1 - \frac{n(n+3)}{(c\xi)^2}\right\} h_n(c\xi),$$

the zeros of $\frac{\partial}{\partial\xi} R^{(3)}_{on}(c,\xi)$ are the roots of the equation

$$\rho h'_n(\rho) = e^2 \left[(2n-1)\left\{(n-2)(n+1) - \rho^2\right\}\alpha^n_{n-2} - (2n+3)\left\{n(n+3) - \rho^2\right\}\alpha^n_{n+2}\right] h_n(\rho)$$

$$+ 0(e^4). \tag{20}$$

If it is now assumed that

$$\rho = \rho'_n + \tau' e^2 + 0(e^4)$$

where ρ'_n is a solution of the equation

$$h'_n(\rho) = 0,$$

the second derivative of the Hankel function can be eliminated using the relation

$$h''_n(\rho'_n) = -\left\{1 - \frac{n(n+1)}{\rho'^2_n}\right\} h_n(\rho'_n)$$

to give

$$\tau' = \frac{1}{\rho'_n}\left[(2n-1)\left\{\rho'^2_n - (n-2)(n+1)\right\}\alpha^n_{n-2} - (2n+3)\left\{\rho'^2_n - n(n+3)\right\}\alpha^n_{n+2}\right]\left\{1 - \frac{n(n+1)}{\rho'^2_n}\right\}^{-1}$$

Moreover,

$$(n-2)(n+1)(2n-1)\,\alpha^n_{n-2} - n(n+3)(2n+3)\,\alpha^n_{n+2} = \frac{n^2(n+1)^2}{(2n-1)(2n+3)}$$

and hence

$$\rho = \rho'_n\left[1 + \frac{e^2}{(2n-1)(2n+3)}\left\{n(n+1) - \frac{\rho'^2_n}{\rho'^2_n - n(n+1)}\right\}\right] + 0(e^4). \qquad (21)$$

In contrast to equation (17) the terms in braces involve both e and ρ'_n, and consequently the root differs from ρ'_n in phase as well as amplitude unless ρ'_n is either real or purely imaginary.

The final step is to insert the values of ρ'_n and select the root of smallest magnitude. Since

$$\rho'_o = -i$$

and

$$\rho'_1 = -i\sqrt{2}e^{\pm i\pi/4},$$

with ρ_n' increasing as n increases, it is clear that the smallest perturbed root is now produced by A_{oo}, and the fact that ρ_o' is purely imaginary then leads to a perturbed root which differs from the unperturbed root in amplitude alone. The root which therefore specifies the radius of convergence is

$$\rho = - i \left(1 + \frac{e^2}{3}\right) + 0(e^4) \tag{22}$$

(cf equation 18), implying that for a hard prolate spheroid of small ellipticity the Rayleigh series converges for

$$c\xi < + \frac{1}{3\xi^2} + 0(\xi^{-4}). \tag{23}$$

Providing $\xi \neq \infty$, this exceeds the radius for the soft body.

Using the above results it is a trivial matter to deduce the radii of convergence for the oblate spheroids. The oblate coefficients A_{on} differ from the prolate coefficients only in having c replaced by -ic and ξ by $i\xi$ and since the ρ_n and ρ_n' are unaffected by this transformation, the formulae for the perturbed zeros can be obtained from (17) and (21) by changing the sign of e^2. It now follows that for the soft oblate spheroid of small ellipticity the Rayleigh series converges for

$$c\xi < 1 - \frac{1}{5\xi^2} + 0(\xi^{-4}) \tag{24}$$

(cf equation 19), and for the hard oblate spheroid, when

$$c\xi < 1 - \frac{1}{3\xi^2} + 0(\xi^{-4}) \tag{25}$$

(cf equation 23).

It will be observed that both radii are less than unity and whereas the deformation of a sphere into a prolate spheroid served to increase the range of $c\xi$ for which the Rayleigh series converge, the reverse is true when the spheroid is oblate. On the other hand, it should be noted that $c\xi$ is the semi-major axis only when the spheroid is prolate and since it is more natural to express the convergence criteria in terms of the maximum dimension of the body, the above limits on the convergence for an oblate spheroid are better written as

$$c(\xi^2 + 1)^{1/2} < 1 + \frac{3}{10\xi^2} + 0(\xi^{-4}) \qquad \text{(soft)} \tag{26}$$

$$c(\xi^2 + 1)^{1/2} < 1 + \frac{1}{6\xi^2} + 0(\xi^{-4}) \qquad \text{(hard)} \tag{27}$$

When expressed in this manner it is seen that any deformation of a sphere into a spheroid produces an increase in the radius of convergence.

V

THE DISC

A part from the sphere the only other spheroidal body for which precise results are easily obtained is the disc. This is the limiting case of an oblate spheroid as $\xi \to 0$ (or $e \to 1$), and has the advantage that the amplitude coefficients A_{on} are such as to permit a direct location of the singularities in the complex c plane. In addition, the disc can be treated by methods other than those involving spheroidal functions, and in recent years several integral equation techniques have become available. These are particularly suited to the derivation of the low frequency expansion and since a significant number of terms can be obtained without undue effort (Bazer and Brown, 1959), the approximate radius of convergence can be inferred using the intuitive argument described in § 2. This provides a check upon the conclusions reached from a study of the singularities of the A_{on}.

In seeking to calculate the radius of convergence it is convenient to begin with the rigorous method, and for this it is necessary to have an exact expression for the A_{on} when $\xi = 0$. Bearing in mind that oblate spheroidal coordinates are now required, the amplitude coefficient A_{on} is defined as

$$A_{on} = - \frac{\Delta R^{(1)}_{on}(-ic, i\xi)}{\Delta R^{(3)}_{on}(-ic, i\xi)} \tag{28}$$

(see equation 10), and by making the transformation $c \to -ic$ and $\xi \to i\xi$ in equation (13) an expansion for $R^{(1)}_{on}(-ic, i\xi)$ is obtained in the form

$$R^{(1)}_{on}(-ic, i\xi) = \frac{1}{c_o^{on}(-ic)} \sum_{r=0,1}^{\infty}{}' \, i^{r-n} (-ic)\, j_r(c\xi). \tag{29}$$

But in the limit $\xi = 0$,

$$j_r(c\xi) = \delta(r)$$

$$\frac{\partial}{\partial \xi} j_r(c\xi) = \frac{1}{3}\, \delta(r-1)$$

and hence for even values of n

$$R^{(1)}_{on}(-ic, i0) = (-i)^n \frac{d_o^{on}(-ic)}{c_o^{on}(-ic)} \tag{30}$$

$$\left[\frac{\partial}{\partial \xi} R^{(1)}_{on}(-ic, i\xi)\right]_{\xi=0} = 0 \tag{31}$$

whilst for odd values of n

$$R^{(1)}_{on}(-ic, i0) = 0 \tag{32}$$

$$\left[\frac{\partial}{\partial \xi} R^{(1)}_{on}(-ic, i\xi)\right]_{\xi=0} = \frac{1}{3}(-i)^{n-1} \frac{d_1^{on}(-ic)}{c_o^{on}(-ic)}\,. \tag{33}$$

Accordingly, for the soft disc the A_{on} are identically zero for all odd values of n, and it is only necessary to consider the form of the coefficients when n is even. And similarly, for the hard disc the expansion for the far field amplitude is confined to odd values of n, so that in this case only the A_{on} for odd n have to be considered.

For the radial functions of the third kind the expansion in terms of spherical Hankel functions deduced from equation (14) is not appropriate to the determination of the functions in the limit $\xi = 0$, and an alternative expansion is desirable. From Flammer (equation 4.4.19) we have

$$R^{(3)}_{on}(-ic, i\xi) = R^{(1)}_{on}(-ic, i\xi)\left\{1 + i\,Q^{*}_{on}(-ic)\left(\tan^{-1}\xi - \frac{\pi}{2}\right)\right\} + i\,g_{on}(-ic, i\xi) \qquad (34)$$

where

$$Q^{*}_{on}(-ic) = \frac{1}{c}\left\{\frac{n!}{2^n\,\frac{n}{2}!\,\frac{n}{2}!\,d^{on}_{o}(-ic)}\right\}^2 \qquad \text{n even} \qquad (35)$$

$$= -\frac{1}{c}\left\{\frac{3\,(n+1)!}{2^n\,\frac{n-1}{2}!\,\frac{n+1}{2}!\,c\,d^{on}_{1}(-ic)}\right\}^2 \qquad \text{n odd} \qquad (36)$$

and

$$g_{on}(-ic, i\xi) \quad \sum_{r=0}^{\infty} B^{on}_{2r}\,\xi^{2r+1} \qquad \text{n even} \qquad (37)$$

$$= \sum_{r=0}^{\infty} B^{on}_{2r}\,\xi^{2r} \qquad \text{n odd} \qquad (38)$$

Since $g_{on}(-ic, i0) = 0$ when n is even, as is the ξ - derivative when n is odd, an expression for the B_{2r}^{on} is not required at this stage, and using the values found for the radial functions of the first kind it follows immediately that for n even

$$R_{on}^{(3)}(-ic, i0) = R_{on}^{(1)}(-ic, i0)\left\{1 - i\,\frac{\pi}{2}\,Q_{on}^{*}(-ic)\right\} \qquad (39)$$

and for n odd

$$\left[\frac{\partial}{\partial\xi}\,R_{on}^{(3)}(-ic, i\xi)\right]_{\xi=0} = \left[\frac{\partial}{\partial\xi}\,R_{on}^{(1)}(-ic, i\xi)\right]_{\xi=0}\left\{1 - i\,\frac{\pi}{2}\,Q_{on}^{*}(-ic)\right\}. \qquad (40)$$

Hence

$$A_{on} = -\left\{1 - i\,\frac{\pi}{2}\,Q_{on}^{*}(-ic)\right\}^{-1} \qquad (41)$$

for all n, with the even values applying to the soft body and the odd values to the hard.

The singularities of A_{on} are therefore given by the equation

$$i\,\frac{\pi}{2}\,Q_{on}^{*}(-ic) = 1 \qquad (42)$$

and substituting the expression for $Q_{on}^{*}(-ic)$, this becomes

$$d_{o}^{on}(r) = \pm\,\frac{n!}{2^{n}\,\frac{n}{2}!\,\frac{n}{2}!}\sqrt{\frac{\pi}{2r}} \qquad (43)$$

for n even, and

$$d_{1}^{on}(r) = \pm\,\frac{3(n+1)!}{2^{n}\,\frac{n-1}{2}!\,\frac{n+1}{2}!}\,\frac{1}{r}\sqrt{\frac{\pi}{2r}} \qquad (44)$$

for n odd, where for convenience the variable r has been introduced in place of -ic.

The problem of finding the radius of convergence of the Rayleigh series is now equivalent to the solution of equations (43) and (44), with the required radius specified by the root of smallest magnitude.

Before attempting the solution it is desirable to give some thought to the values of n which may provide this root. In the first place, we remark that for both the soft and hard discs the values differ from those found for the sphere. Thus, for a soft sphere the amplitude coefficient A_{oo} has no singularities in the finite portion of the $c\xi$ plane and the smallest singularity belongs to A_{o1}, whereas for the soft disc n is limited to even values. This suggests that for some particular ellipticity two of the amplitude coefficients must have (smallest) singularities which are equal in magnitude, and the same sort of transition also occurs for the hard bodies. Accordingly, the sphere results give no direct indication of the values of n which must be considered.

On the other hand it is relatively easy to determine the roots of (43) and (44) when n is sufficiently large. Taking for example equation (43), the spheroidal coefficient $d_o^{on}(r)$ can be represented by the first term of its expansion in powers of r providing n is large and $r << 4n\sqrt{n}$, and from the recurrence relations defining the d_o^{on} it can be shown that

$$d_o^{on}(r) \sim \left(\frac{r}{2}\right)^n \frac{1}{2n+1}\left(\frac{\frac{n-1}{2}!}{n-\frac{1}{2}!}\right)^2 .$$

Substituting into (43) then gives

$$\left(\frac{r}{2}\right)^{n+\frac{1}{2}} \sim \pm \frac{\sqrt{\pi}\, n!\, \left(n-\frac{1}{2}\right)!\, \left(n+\frac{1}{2}\right)!}{2^n \, \frac{n}{2}!\, \frac{n}{2}!\, \frac{n-1}{2}!\, \frac{n-1}{2}!} \tag{45}$$

and for large values of n the right hand side is asymptotic to

$$\pm \frac{1}{\sqrt{e}} \left(\frac{2n}{e}\right)^{n+\frac{1}{2}} .$$

Hence

$$r \sim \frac{4n}{e} \exp\left\{\frac{2im\pi - 1}{2n+1}\right\} \tag{46}$$

($m = 0, 1, 2, \ldots, 2n$), and for n sufficiently large the roots are 2n+1 in number and spaced equally around a circle of radius $\frac{4n}{e}$ in the complex r plane. It will be observed that this radius is proportional to n and such that the assumed representation of $d_o^{on}(r)$ is valid. Since it is the smallest root which is required out of the totality of roots of (43) for even n, it is now apparent that only the lower values have to be examined in detail, with the probability that the lowest value (i.e. $n=0$) will provide the root of smallest magnitude.

For equation (44) the analysis is similar in all respects and the fact that the difference between the right hand sides of (43) and (44) is precisely compensated by a like difference in the formula for d_o^{on} and d_1^{on} means that the asymptotic behaviour of the roots is also given by (46). Once again it is expected that the lowest value of n will produce the smallest root, but since n is confined to odd values for the hard disc, the appropriate value here is n = 1.

In Figure III the roots obtained directly from equation (45) are compared with the asymptotic form for large n, and in interpreting the graph it must be remembered that the even integers refer to equation (43) and the odd integers to equation (44).

Unfortunately, for the smaller values of n not even the formula given in equation (45) is sufficiently accurate for our purposes, and whilst both (45) and (46) would suggest that of the 2n+1 roots one of them is always real, this is true only in an asymptotic sense, and for all finite n the smallest root of each equation has an imaginary part which cannot be ignored. This is clearly seen by a study of the tabulated values of $d_o^{on}(r)$ and $d_1^{on}(r)$ for real r (see, for example, Flammer), and from these it would appear that when n is even

$$d_o^{on}(r) \sim \frac{n}{2^n \frac{n}{2}! \frac{n}{2}!} \sqrt{\frac{\pi}{2r}}$$

and when n is odd

$$d_1^{on}(r) \sim \frac{3(n+1)!}{2^n \frac{n-1}{2}! \frac{n+1}{2}!} \frac{1}{r} \sqrt{\frac{\pi}{2r}}$$

as r tends to infinity through real values. Indeed, the accuracy of these representations is such that for $r > 6 \cdot 5$, $d_o^{oo}(r)$ differs from $\sqrt{\frac{\pi}{2r}}$ by less than one unit in the fifth significant figure. Since $d_o^{on}(r)$ and $d_1^{on}(r)$ are non-negative for real r, it

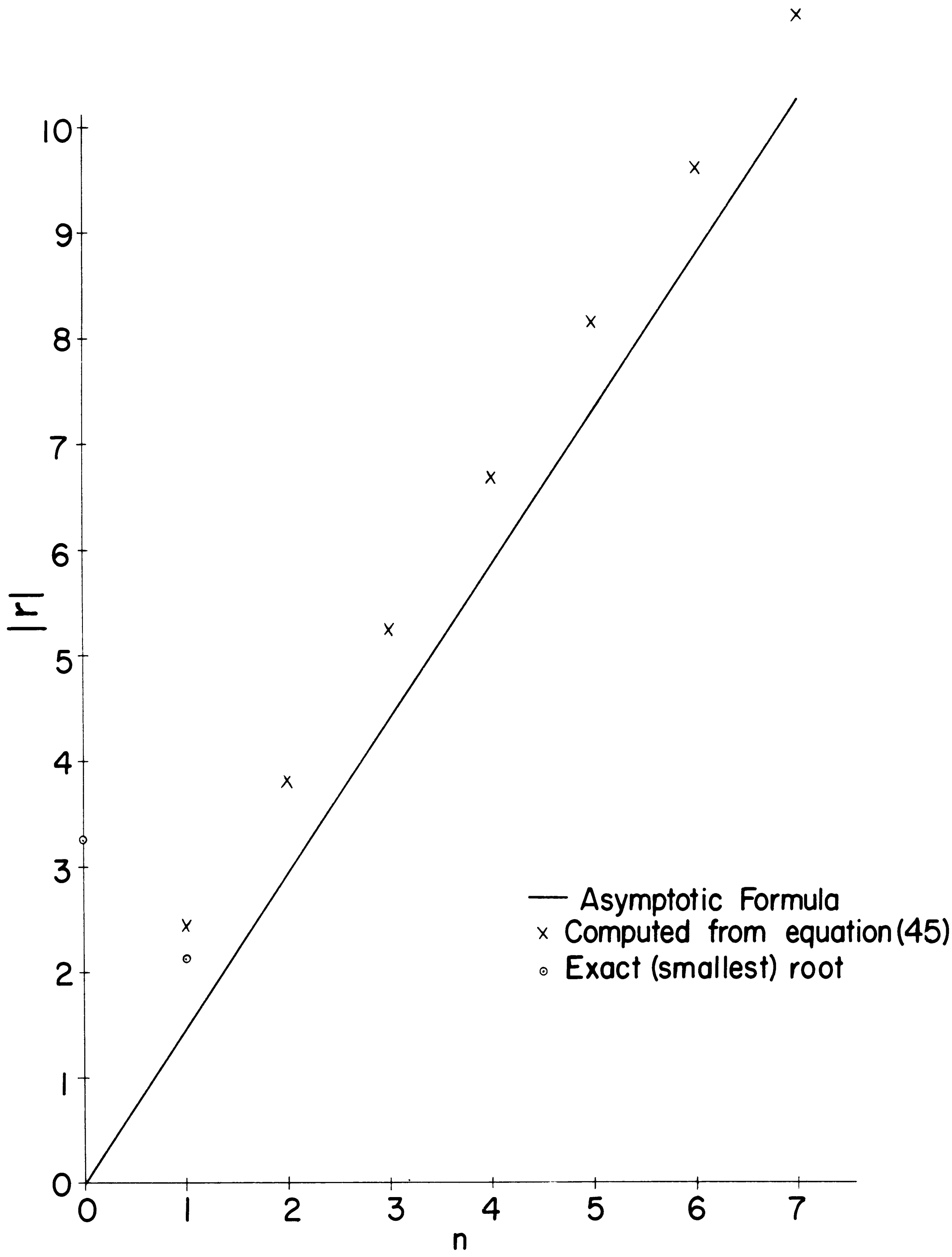

FIGURE III. THE SOLUTION OF EQUATION (45) now follows that equations (43) and (44) can have no real roots in the finite portion of the r plane, and this immediately rules out the possibility of using tabulated values for the spheroidal coefficients in the solution of the equations. We are therefore compelled to rely on the series developments of $d_o^{on}(r)$ and $d_1^{on}(r)$ in powers of c, with enough terms included to ensure an accurate determination of the roots, and we shall begin by examining the problem of the soft disc, for which n is restricted to even values.

When n is even the relevant equation is (43) and taking first the case n = 0, the equation from which to calculate the singularities of A_{oo} is

$$d_o^{oo}(r) = \pm \sqrt{\frac{\pi}{2r}} \ . \qquad (47)$$

In Appendix A it is shown that

$$d_o^{oo}(r) = 1 - \alpha_1^o r^2 + \alpha_2^o r^4 - \alpha_3^o r^6 + \alpha_4^o r^8 + \alpha_5^o r^{10} + 0(r^{12})$$

where

$$\alpha_1^o = 5 \cdot 555555556 \times 10^{-2}$$

$$\alpha_2^o = 4 \cdot 135802469 \times 10^{-3}$$

$$\alpha_3^o = 2 \cdot 125500406 \times 10^{-4}$$

$$\alpha_4^o = 3 \cdot 903827192 \times 10^{-6}$$

$$\alpha_5^o = 3 \cdot 787031007 \times 10^{-7},$$

and writing $r = se^{i\theta}$, (47) splits up into the two real equations

$$1 - \alpha_1^o s^2 \cos 2\theta + \alpha_2^o s^4 \cos 4\theta - \alpha_3^o s^6 \cos 6\theta + \alpha_4^o \cos 8\theta + \alpha_5^o s^{10} \cos 10\theta$$

$$+ 0(s^{12}) = \pm \sqrt{\frac{\pi}{2s}} \cos \frac{\theta}{2}, \qquad (48)$$

$$\alpha_1^o s^2 \sin 2\theta - \alpha_2^o s^4 \sin 4\theta + \alpha_3^o s^6 \sin 6\theta - \alpha_4^o s^8 \sin 8\theta - \alpha_5^o s^{10} \sin 10\theta$$

$$+ 0(s^{12}) = \pm \sqrt{\frac{\pi}{2s}} \sin \frac{\theta}{2}, \qquad (49)$$

in which either the upper or the lower signs must be taken in conjunction with one another.

A casual examination of (48) and (49) shows that if a root of these equations exists for some particular value of θ, a further root can be obtained by changing θ into $2\pi - \theta$, and consequently it is sufficient to confine attention to the range $0 < \theta < \pi$. In addition, the magnitude of the coefficients is such that for the equations with the upper signs the smallest root will almost certainly occur within the range $0 < \theta < \pi/2$, and for the equations with the lower signs the corresponding range is $\pi/2 < \theta < \pi$.

After a few trial calculations it is found that the smallest root has θ approximately 8^o and a magnitude somewhat greater than 3·2. Unfortunately, for values of s as large as this the convergence of the series for $d_o^{oo}(r)$ is extremely slow, and even the terms is given in (48) and (49) are insufficient to determine the root with reasonable accuracy. Although the labour involved in deriving additional terms in the expansion for d_o^{oo} is such that it seems unprofitable to pursue the matter further, it is possible to use the tabulated values of $d_o^{oo}(r)$ for real r (see, for example, Flammer) to estimate the coefficient of c^{12}, and this turns out to be negative and of order $2 \cdot 7 \times 10^{-8}$. With this additional coefficient available, the calculation of the smallest root can be refined to give

$$r = 3 \cdot 25 \exp\left\{i\, 0 \cdot 044\, \pi\right\},$$

and greater accuracy is only possible by including still more terms in the expansion for d_o^{oo}.

For the reasons stated previously it is to be expected that this is the smallest root of equation (43) for all (even) values of n, but in the interests of completeness the equation for n = 2 has also been investigated. We have

$$d_o^{o2}(r) = \frac{r^2}{45}\left\{1-\alpha_1^2\, r^2 - \alpha_2^2\, r^4 + \alpha_3^2\, r^6 + \alpha_4^2\, r^8 + 0(r^{10})\right\}$$

with

$$\alpha_1^2 = 5 \cdot 668934252 \times 10^{-3}$$

$$\alpha_2^2 = 2 \cdot 394192749 \times 10^{-3}$$

$$\alpha_3^2 = 5 \cdot 248224599 \times 10^{-5}$$

$$\alpha_4^2 = 8 \cdot 783406594 \times 10^{-6}$$

which can be inserted into equation (43) to give

$$1-\alpha_1^2 s^2 \cos 2\theta - \alpha_2^2 s^4 \cos 4\theta + \alpha_3^2 s^6 \cos 6\theta + \alpha_4^2 s^8 \cos 8\theta$$

$$+ 0(s^{10}) = \pm \frac{45}{2} \frac{1}{s^2} \sqrt{\frac{\pi}{2s}} \cos \frac{5\theta}{2},$$

$$\alpha_1^2 s^2 \sin 2\theta + \alpha_2^2 s^4 \sin 4\theta - \alpha_3^2 s^6 \sin 6\theta - \alpha_4^2 s^8 \sin 8\theta$$

$$+ 0(s^{10}) = \pm \frac{45}{2} \frac{1}{s^2} \sqrt{\frac{\pi}{2s}} \sin \frac{5\theta}{2},$$

and whilst the number of terms in d_o^{o2} is not sufficient to permit an actual calculation of the roots, it has been verified that no root exists whose magnitude is less than 3·5. We are therefore led to the conclusion that the smallest root of equation (43)(and hence, of (42) for n even) is provided by the case n = 0, and is

$$c = 3 \cdot 25 \exp\left\{ i\,(0 \cdot 5 \pm 0 \cdot 044)\,\pi \right\} \tag{50}$$

corresponding to a singularity of A_{oo}. The magnitude of this root represents the smallest radius of convergence of the expansions for the individual A_{on}, n even, and accordingly for a soft disc the Rayleigh series converges for

$$c < 3 \cdot 25. \tag{51}$$

In comparison with the above, the problem of a hard disc is relatively easy. When n is odd the relevant equation is (44), and taking first the case n = 1 the equation with which to calculate the singularities of A_{o1} is

$$d_1^{o1}(r) = \pm \frac{3}{2}\sqrt{\frac{\pi}{2r}} \ . \qquad (52)$$

As shown in Appendix A,

$$d_1^{o1}(r) = 1 - \alpha_1^1 r^2 + \alpha_2^1 r^4 - \alpha_3^1 r^6 + \alpha_4^1 r^8 + 0(r^{10})$$

where

$$\alpha_1^1 = 6 \times 10^{-2}$$

$$\alpha_2^1 = 2 \cdot 216326531 \times 10^{-3}$$

$$\alpha_3^1 = 4 \cdot 316150166 \times 10^{-5}$$

$$\alpha_4^1 = 2 \cdot 412282939 \times 10^{-7}$$

and equation (52) then splits into the two real equations

$$1 - \alpha_1^1 s^2 \cos 2\theta + \alpha_2^1 s^4 \cos 4\theta - \alpha_3^1 s^6 \cos 6\theta + \alpha_4^1 s^8 \cos 8\theta + 0(s^{10}) = \pm \frac{3}{s}\sqrt{\frac{\pi}{2s}} \cos\frac{3\theta}{2} \ , \qquad (53)$$

$$\alpha_1^1 s^2 \sin 2\theta - \alpha_2^1 s^4 \sin 4\theta + \alpha_3^1 s^6 \sin 6\theta - \alpha_4^1 s^8 \sin 8\theta + 0(s^{10}) = \pm \frac{3}{s}\sqrt{\frac{\pi}{2s}} \sin\frac{3\theta}{2} \ . \qquad (54)$$

Once again it is sufficient to restrict attention to $0 < \theta < \pi$, and the magnitude of the α_1^1 are such that the smallest root almost certainly lies within the ranges $0 < \theta < \pi/3$ or $\pi/2 < \theta < 2\pi/3$ depending on whether the upper or lower signs are chosen. It is now a straight-forward matter to show that

$$r = 2\cdot 1255 \ \exp\left\{ i\ 0\cdot 62456\ \pi \right\}$$

and because the modulus is so much smaller than that found in the case n = 0, even the fewer terms shown in (53) and (54) give the root to a high degree of accuracy.

For completeness the corresponding equation for n = 3 has also been investigated, and since

$$d_1^{o3}(r) = \frac{3}{175} r^2 \left\{ 1 + \alpha_1^3 r^2 - \alpha_2^3 r^4 - \alpha_3^3 r^6 + \alpha_4^3 r^8 + 0\,(r^{10}) \right\}$$

with

$$\alpha_1^3 = 8\cdot 888888889 \times 10^{-3}$$
$$\alpha_2^3 = 2\cdot 504052873 \times 10^{-4}$$
$$\alpha_3^3 = 1\cdot 111387425 \times 10^{-5}$$
$$\alpha_4^3 = 1\cdot 961434362 \times 10^{-7} ,$$

the equation specifying the singularities of A_{o3} can be written as

$$1 + \alpha_1^3 s^2 \cos 2\theta - \alpha_2^3 s^4 \cos 4\theta - \alpha_3^3 s^6 \cos 6\theta + \alpha_4^3 s^8 \cos 8\theta + 0(s^{10}) = \pm \frac{525}{2 s^3} \sqrt{\frac{\pi}{2s}} \cos \frac{7\theta}{2}$$

$$- \alpha_1^3 s^2 \sin 2\theta + \alpha_2^3 s^4 \sin 4\theta + \alpha_3^3 s^6 \sin 6\theta - \alpha_4^3 s^8 \sin 8\theta + 0(s^{10}) = \pm \frac{525}{2 s^3} \sqrt{\frac{\pi}{2 s}} \sin \frac{7\theta}{2} .$$

Since the number of terms is insufficient, a precise root has not been obtained, but it has been verified that no root exists whose amplitude is less than 4·5. Bearing in mind the asymptotic behaviour of the roots as a function of n, it is now concluded that the smallest root of equation (44) (and hence, of equation (42) for n odd) is provided by the case n = 1, and is

$$c = 2\cdot 1255 \exp\left\{ i\,(0\cdot 5 \pm 0\cdot 62456)\,\pi \right\}, \tag{55}$$

corresponding to a singularity of A_{01}. Accordingly, for a hard disc the Rayleigh series converges for

$$c < 2\cdot 1255\ , \tag{56}$$

which is significantly smaller than the radius of convergence for the soft disc.

As a final check, the radii of convergence have also been determined from the actual coefficients in the Rayleigh series using the alternative approach referred to in §2, and the details are given in Appendix B. Such a check is desirable in view of the fact that in applying the rigorous method there is always the possibility that a dominant singularity may have been overlooked, and it is therefore pleasing to find that the results agree with those obtained above.

VI

THE OBLATE SPHEROID OF ELLIPTICITY NEAR TO UNITY

It is convenient to follow up the discussion of the disc by considering the problem of the oblate spheroid which is almost a disc, and for which the ellipticity is almost unity. This implies that

$$0 < \xi << 1 ,$$

and by obtaining the expansions of the amplitude coefficients A_{on} in terms of ξ, it is possible to investigate the perturbation of the singularities consequent upon the presence of the non-zero parameter ξ. It is then a trivial matter to deduce the changes in the radii of convergence.

The expression for the A_{on} is shown in equation (28), and taking first the radial function $R_{on}^{(1)}(-ic, i\xi)$, the Bessel functions in equation (29) can be replaced by the leading terms in their series expansions for small $c\xi$ to give

$$R_{on}^{(1)}(-ic, i\xi) = (-i)^n \frac{d_o^{on}(-ic)}{c_o^{on}(-ic)} \left\{ 1 - \frac{(c\xi)^2}{6} \left(1 + \frac{2}{5} \frac{d_2^{on}}{d_o^{on}} \right) + 0(\xi^4) \right\}, \quad (57)$$

$$\frac{\partial}{\partial \xi} R_{on}^{(1)}(-ic, i\xi) = (-i)^{n-2} \frac{c^2 \xi}{3} \frac{d_o^{on}(-ic)}{c_o^{on}(-ic)} \left(1 + \frac{2}{5} \frac{d_2^{on}}{d_o^{on}} \right) + 0(\xi^3) \quad (58)$$

when n is even, and

$$R^{(1)}_{on}(-ic, i\xi) = (-i)^{n-1}\ \frac{c\xi}{3}\ \frac{d^{on}_1(-ic)}{c^{on}_o(-ic)} + 0(\xi^3), \tag{59}$$

$$\frac{\partial}{\partial\xi} R^{(1)}_{on}(-ic, i\xi) = (-i)^{n-1}\ \frac{c}{3}\ \frac{d^{on}_1(-ic)}{c^{on}_o(-ic)} + 0(\xi^2) \tag{60}$$

when n is odd.

For the radial functions of the third kind we have from equation (34)

$$R^{(3)}_{on}(-ic, i\xi) = R^{(1)}_{on}(-ic, i\xi)\left\{1 - i\,\frac{\pi}{2}\,Q^{*}_{on}(-ic)\left(1 - \frac{2\xi}{\pi} + 0(\xi^3)\right)\right\} + i\,g_{on}(-ic, i\xi)$$

where

$$g_{on}(-ic, i\xi) = \xi\, B^{on}_o + 0(\xi^3)$$

for n even, and

$$g_{on}(-ic, i\xi) = B^{on}_o + \xi^2 B^{on}_2 + 0(\xi^4)$$

for n odd. Moreover, from Flammer (equations 4.4.25 to 4.4.27)

$$B^{on}_o = i^n\ \frac{1}{c}\ \frac{c^{on}_o(-ic)}{d^{on}_o(-ic)}\left[1 - c\,Q^{*}_{on}(-ic)\left\{\frac{d^{on}_o(-ic)}{c^{on}_o(-ic)}\right\}^2\right]$$

for n even;

$$B^{on}_o = 3\,i^{n+1}\ \frac{1}{c^2}\ \frac{c^{on}_o(-ic)}{d^{on}_1(-ic)}\ ,$$

$$B_2^{on} = \frac{1}{2}\lambda_{on}(-ic)\,B_o^{on} - \frac{c}{3}\,i^{n-1}\,Q_{on}^{*}(-ic)\,\frac{d_1^{on}(-ic)}{c_o^{on}(-ic)}$$

for n odd, where $\lambda_{on}(-ic)$ is the eigenvalue, and consequently when n is even

$$\frac{R_{on}^{(1)}(-ic,i\xi)}{R_{on}^{(3)}(-ic,i\xi)} = \left\{1 - i\,\frac{\pi}{2}\,Q_{on}^{*}(-ic)\left(1-\frac{2\xi}{\pi}\right) + \xi\,i^{n+1}B_o^{on}\,\frac{c_o^{on}(-ic)}{d_o^{on}(-ic)} + 0(\xi^3)\right\}^{-1},$$

$$\frac{\frac{\partial}{\partial\xi}R_{on}^{(1)}(-ic,i\xi)}{\frac{\partial}{\partial\xi}R_{on}^{(3)}(-ic,i\xi)} = \frac{ic^2\xi}{3}\left(1+\frac{2}{5}\,\frac{d_2^{on}}{d_o^{on}}\right)\left[Q_{on}^{*}(-ic) + i^n\,B_o^{on}\,\frac{c_o^{on}(-ic)}{d_o^{on}(-ic)}\right.$$

$$\left. + \frac{ic^2\xi}{3}\left(1+\frac{2}{5}\,\frac{d_2^{on}}{d_o^{on}}\right)\left\{1 - i\,\frac{\pi}{2}\,Q_{on}^{*}(-ic)\right\} + 0(\xi^2)\right]^{-1},$$

and when n is odd

$$\frac{R_{on}^{(1)}(-ic,i\xi)}{R_{on}^{(3)}(-ic,i\xi)} = i^{n-2}\,\frac{c\xi}{3}\,\frac{d_1^{on}(-ic)}{c_o^{on}(-ic)}\left[B_o^{on} + \frac{c\xi}{3}\,i^{n-2}\,\frac{d_1^{on}(-ic)}{c_o^{on}(-ic)}\,\cdot\right.$$

$$\left.\left\{1 - i\,\frac{\pi}{2}\,Q_{on}^{*}(-ic)\right\} + 0(\xi^2)\right]^{-1},$$

$$\frac{\frac{\partial}{\partial\xi}R_{on}^{(1)}(-ic,i\xi)}{\frac{\partial}{\partial\xi}R_{on}^{(3)}(-ic,i\xi)} = \left[1 - i\,\frac{\pi}{2}\,Q_{on}^{*}(-ic) + 2i\xi\left\{Q_{on}^{*}(-ic) + 3i^{n-1}B_2^{on}\,\frac{c^{on}(-ic)}{c\,d_1^{on}(-ic)}\right\}\right.$$

$$\left. + 0(\xi^2)\right]^{-1}.$$

It now follows that for a soft spheroid of ellipticity ~1 the amplitude coefficient A_{on} has the form

$$A_{on} = - \left[1 - i\,\frac{\pi}{2} Q^{*}_{on}(-ic) + i\,\xi\,\frac{1}{c} \left\{ \frac{c_{o}^{on}(-ic)}{d_{o}^{on}(-ic)} \right\}^{2} + 0(\xi^{3}) \right]^{1} \tag{61}$$

for even values of n, and in the disc limit this reduces to the expression given in equation (41). When n is odd

$$A_{on} = - \frac{i c^{3} \xi}{9} \left\{ \frac{d_{1}^{on}(-ic)}{c_{o}^{on}(-ic)} \right\}^{2} \left[1 + \frac{i c^{3} \xi}{9} \left\{ \frac{d_{1}^{on}(-ic)}{c_{o}^{on}(-ic)} \right\} \left\{ 1 - i\,\frac{\pi}{2}\, Q^{*}_{on}(-ic) \right\} + 0(\xi^{2}) \right]^{-1} \tag{62}$$

and is zero when $\xi = 0$, which agrees with the original finding that only even values of n contribute towards the soft disc solution. Similarly, for a hard spheroid of ellipticity near to unity

$$A_{on} = - \frac{i c^{3} \xi}{3} \left\{ \frac{d_{o}^{on}(-ic)}{c_{o}^{on}(-ic)} \right\}^{2} \left(1 + \frac{2}{5}\,\frac{d_{2}^{on}}{d_{o}^{on}} \right) \left[1 + \frac{i c^{3} \xi}{3} \left\{ \frac{c_{o}^{on}(-ic)}{d_{o}^{on}(-ic)} \right\}^{2} \cdot \left(1 + \frac{2}{5}\,\frac{d_{2}^{on}}{d_{o}^{on}} \right) \left\{ 1 - i\,\frac{\pi}{2}\, Q^{*}_{on}(-ic) \right\} + 0(\xi^{2}) \right]^{-1} \tag{63}$$

for n even and

$$A_{on} = - \left[1 - i\,\frac{\pi}{2} Q^{*}_{on}(-ic) + \frac{9\,i\,\xi\,\lambda_{on}(-ic)}{c^{3}} \left\{ \frac{c_{o}^{on}(-ic)}{d_{1}^{on}(-ic)} \right\}^{2} + 0(\xi^{2}) \right]^{-1} \tag{64}$$

for n odd, and here again the limiting values are in agreement with those employed in §5.

In order to find the radii of convergence for the individual A_{on}, and hence the radius of convergence of the Rayleigh series, it is necessary to locate the smallest singularities of the A_{on} in the complex c plane. Taking first the case of the soft body, the singularities for even values of n are given by the roots of the equation.

$$1 - i\frac{\pi}{2} Q^{*}_{on}(-ic) + \frac{i\xi}{c}\left\{\frac{c_o^{on}(-ic)}{d_o^{on}(-ic)}\right\}^2 + 0(\xi^3) = 0, \tag{65}$$

and as such can be obtained by a perturbation analysis based on the singularities for a disc. If $c = c_n$ is a solution of (65) with $\xi = 0$, and if

$$c = c_n + \tau_n \xi + 0(\xi^2) \tag{66}$$

is the corresponding solution of (65) with the term in ξ retained, substitution of (66) into (65) gives immediately

$$\tau_n = - i \frac{c_o^{on}(-ic)}{d_o^{on}(-ic)}\left\{1 + c_n \frac{\frac{\partial}{\partial c_n} d_o^{on}(-ic_n)}{d_o^{on}(-ic_n)}\right\}^{-1},$$

and consequently the roots of (65) are

$$c = c_n\left[1 - \frac{i\xi}{c_n}\frac{c_o^{on}(-ic)}{d_o^{on}(-ic)}\left\{1 + 2c_n \frac{\frac{\partial}{\partial c_n} d_o^{on}(-ic_n)}{d_o^{on}(-ic_n}\right\}^{-1}\right] + 0(\xi^2). \tag{67}$$

In view of the restriction to small values of ξ, the smallest root is almost certainly produced by the smallest c_n, and from the results in §5 it is known that for n even

$$c_o < c_n , \quad n = 2, 4, 6, \ldots\ldots$$

The value of c_o is shown in equation (50) and using now the expansions for $d_o^{oo}(t)$ and $c_o^{oo}(t) / d_o^{oo}(t)$ (see Appendix A) with t replaced by $-ic_o$, the perturbation of the singularity can be computed. From equation (67) it is found that

$$c = c_o \left\{ 1 - \xi \, (1{\cdot}027 + 0{\cdot}1992 \, i) \right\} + 0(\xi^2) \tag{68}$$

and thus the effect of the non-zero parameter ξ is to modify c_o in both phase and amplitude. Of most importance, however, is the reduction in the amplitude, implying a reduction in the radius of convergence, and from (68) the actual radius is

$$3{\cdot}25 \, (1 - 1{\cdot}027 \, \xi) . \tag{69}$$

This formula is valid as long as terms in ξ^2 are negligible, or until another singularity becomes smaller in magnitude, and bearing in mind that for even values of $n > 0$ the c_n exceed c_o by 0·25 at the very least, it seems reasonable to regard (69) as holding for ξ as large as 0·1 or even 0·2.

All this is based on the assumption that the coefficients A_{on} for odd n have no singularities which are smaller in magnitude that the one whose expression is given in (68). If such a singularity were dominant, the radius of convergence of the Rayleigh series would change discontinuously in the limit as the spheroid became a disc, and although this is intuitively unlikely, it is necessary to investigate these other singularities to make sure that they do not include the smallest.

From equation (63) it is clear that the only singularities of A_{on} for odd n are those which correspond to the vanishing* of $c_o^{on}(-ic)$, and using the expansions derived in Appendix A it has been verified that the smallest root of the equation for n = 1, namely

$$d_1^{o1}(-ic) + d_3^{o1}(-ic) + d_5^{o1}(-ic) + \ldots\ldots = 0,$$

is approximately $5{\cdot}2 \exp\left\{ i(0{\cdot}5 \pm 0{\cdot}083)\pi \right\}$. Since this exceeds c_o in magnitude and the singularities for higher (odd) values of n are even larger, it follows that for the soft body the singularities of the odd coefficients have no effect on the convergence, and accordingly the radius of convergence is as shown in (69).

Turning now to the case of the hard body, it is convenient to consider first the coefficients A_{on} for odd n, the singularities of which are given by the equation

$$1 - i\frac{\pi}{2} Q_{on}^{*}(-ic) + \frac{9i\xi\lambda_{on}(-ic)}{c^3} \left\{ \frac{c_o^{on}(-ic)}{d_1^{on}(-ic)} \right\}^2 + 0(\xi^2) = 0. \qquad (70)$$

---

*In passing we note the interesting fact that any singularity of A_{on} for n even or odd is simultaneously a zero of the coefficient of ξ in the denominator for n odd or even respectively.

Here again a perturbation analysis is applicable, and writing

$$c = c_n + \tau'_n \xi + 0(\xi^2)$$

where c_n is defined as before, we have

$$\tau'_n = 3i\lambda_{on}(-ic_n)\left\{\frac{c_o^{on}(-ic_n)}{c_n d_1^{on}(-ic_n)}\right\}^2 \left\{1 + \frac{2}{3} c_n \frac{\frac{\partial}{\partial c_n} d_1^{on}(-ic_n)}{d_1^{on}(-ic_n)}\right\}^{-1}$$

and consequently the roots of equation (70) are

$$c = c_n\left[1 + \frac{3i\xi\lambda_{on}(-ic_n)}{c_n}\left\{\frac{c_o^{on}(-ic_n)}{c_n d_1^{on}(-ic_n)}\right\}^2 \left\{1 + \frac{2}{3} c_n \frac{\frac{\partial}{\partial c_n} d_1^{on}(-ic_n)}{d_1^{on}(-ic_n)}\right\}^{-1}\right] + 0(\xi^2).$$

When n is odd the c_n of smallest magnitude is provided by n = 1 and if the value for c_1 shown in equation (55) is inserted into the expansions for d_1^{o1} and c_o^{o1}, the corresponding singularity of A_{o1} is found to be

$$c = c_1\left\{1 - \xi(0{\cdot}2615 - 0{\cdot}5719i)\right\} + 0(\xi^2). \qquad (72)$$

It will be observed that as a result of the finite ellipticity c_1 is modified in both phase and amplitude, but since the amplitude is decreased, a deformation of a disc serves to decrease the radius of convergence. Providing terms in ξ^2 are negligible, the actual radius is

$$2{\cdot}1255\,(1 - 0{\cdot}2615\,\xi), \qquad (73)$$

and it only remains to verify that no coefficient A_{on} for even values of n has a smaller singularity to assert that (73) is the radius of convergence of the complete Rayleigh series.

The singularities of the A_{on} when n is even are given by the zeros of $c_o^{on}(-c)$, and the smallest root of the equation for n = 0, namely

$$d_o^{oo}(-ic) + d_2^{oo}(-ic) + d_4^{oo}(-ic) + \ldots\ldots = 0$$

is approximately $4{\cdot}1 \exp\left\{ i(0{\cdot}5 \pm 0{\cdot}05)\pi \right\}$. This is greater than c_1 in magnitude and since the singularities for higher (even) values of n are still larger, the even coefficients can be discounted as far as the overall convergence is concerned. It follows that for the hard spheroid of ellipticity almost unity the radius of convergence is as shown in (73).

In keeping with the form of presentation used in §4, it is convenient to express the above results in terms of the maximum dimension of the body. For an oblate spheroid the semi-major axis is $c(\xi^2 + 1)^{1/2}$, which differs from c only by terms $0(\xi^2)$ for small ξ, and consequently the radii shown in (69) and (73) are equivalent to the following convergence criteria:

$$c(\xi^2 + 1)^{1/2} < 3{\cdot}25\,(1 - 1{\cdot}027\,\xi) + 0(\xi^2) \qquad \text{(soft)} \qquad (74)$$

$$c(\xi^2 + 1)^{1/2} < 2{\cdot}1255(1 - 0{\cdot}2615\,\xi) + 0(\xi^2) \qquad \text{(hard)} \qquad (75)$$

It will be appreciated that these two equations summarize the conclusions of §5 as well as of the present section, and it is of interest to note that whilst the soft body gives the larger radius of convergence, its convergence decreases more rapidly with increasing ξ. This is eminently reasonable in view of the fact that both soft and hard spheres have radii of convergence equal to unity.

VII

THE OBLATE SPHEROID OF INTERMEDIATE ELLIPTICITY

In order to complete the discussion of the oblate spheroid it is necessary to consider the convergence of the Rayleigh series when the ellipticity is neither small nor near to unity. Unfortunately this is a difficult task and whilst it is possible to obtain several different integral expressions for $R^{(3)}_{on}(-ix, i\xi)$ (see, for example, Flammer equation 5.4.1), no expansions are available by means of which the zeros can be determined analytically. Moreover, the zeros almost certainly correspond to complex values of c^2, so that any attempt to discover them by purely numerical means (i.e. by computing an integral expression for a variety of c and ξ) would be an extremely laborious undertaking.

Nevertheless, it is important to have some estimate of the convergence for these ellipticities, and this is particularly true in view of the change in the dominant singularity which takes place somewhere within the range $0 < e < 1$. Thus, for a soft spheroid, the coefficient A_{o1} specifies the convergence as long as the ellipticity is small, but by the time that the body has become disc-like ($e \sim 1$) the coefficient A_{oo} has taken over; and with the hard spheroid the behaviour is just the opposite. As a consequence, the 'curve' giving the radius of convergence as a function of ξ may well possess an abrupt change of slope for some value of the ellipticity, and the nature of the 'transition' is therefore a problem of some interest.

In the absence of any other method with which to explore this region, it is necessary to rely on a numerical comparison of the coefficients in the expansion of the A_{on}, with the hope that sufficient terms can be included to give a reliable estimate of the convergence.

For the values of ξ under consideration it is convenient to write the ratio of the radial functions as

$$\frac{R_{on}^{(1)}(-ic, i\xi)}{R_{on}^{(3)}(-ic, i\xi)} = \left[1 + Q_{on}^{*}(-ic) \frac{\sum_{r=0,1}^{\infty}{}' d_r^{on}(-ic) Q_r(i\xi) + \sum_{r=2,1}^{\infty}{}' d_{\rho/r}^{on}(-ic) P_{r-1}(i\xi)}{\sum_{r=0,1}^{\infty}{}' d_r^{on}(-ic)\, P_r(i\xi)} \right]^{-1} \tag{76}$$

(see Flammer, equations 4.2.3 and 4.2.7), where $Q_{on}^{*}(-ic)$ is as defined in equations (35) and (36), and $P_r(i\xi)$ and $Q_r(i\xi)$ are the Legendre functions of the first and second kinds respectively. Attention will be directed only at the cases n = 0 and n = 1, and since the solutions for the hard body can be obtained from those for the soft by differentiating the functions of the radial variable, it is sufficient to write down the expansions for the soft body alone.

Taking first the coefficient A_{o1}, we have

$$Q_{o1}^{*}(-ic) = -\frac{1}{c} \left\{ \frac{3}{c d_1^{o1}(-ic)} \right\}^2 \tag{77}$$

so that*

*Unless otherwise stated, the argument of the Legendre functions is $i\xi$.

$$A_{o1} = \frac{c^3}{9}(d_1^{o1})^2 \frac{P_1}{Q_1} \left[-\frac{c^3}{9}(d_1^{o1})^2 \frac{P_1}{Q_1} + \frac{\sum_{r=1}^{\infty} \frac{d_r^{o1}}{d_1^{o1}} \frac{Q_r}{Q_1} + \sum_{r=1}^{\infty} \frac{d_{\rho/r}^{o1}}{d_1^{o1}} \frac{P_{r-1}}{Q_1}}{\sum_{r=1}^{\infty} \frac{d_r^{o1}}{d_1^{o1}} \frac{P_r}{P_1}} \right]^{-1} \tag{78}$$

and the expansions in Appendix A can now be used with t replaced by -ic to give

$$A_{o1} = \frac{c^3}{9}(d_2^{o1}) \frac{P_1}{Q_1} \left[\sum_{r=0}^{\infty} (ic)^r B_r \right]^{-1} \tag{79}$$

where

$$B_0 = 1$$

$$B_1 = 0$$

$$B_2 = \frac{1}{2.3} \frac{P_0}{Q_1} + \frac{1}{5^2}\left(\frac{P_3}{P_1} - \frac{Q_3}{Q_1}\right)$$

$$B_3 = -\frac{i}{3^2} \frac{P_1}{Q_1}$$

$$B_4 = \frac{1}{5^2} B_2 \frac{P_3}{P_1} - \frac{1}{3^2.5} \frac{P_0}{Q_1} - \frac{1}{2^2.3^2} \frac{P_2}{Q_1} + \frac{2}{3^2.5^4}\left(\frac{P_3}{P_1} - \frac{Q_3}{Q_1}\right) - \frac{1}{3^2.5.7^2}\left(\frac{P_5}{P_1} - \frac{Q_5}{Q_1}\right)$$

$$B_5 = \frac{i}{3.5^2} \frac{P_1}{Q_1}$$

$$B_6 = \frac{1}{5^2}\left(B_4 + \frac{2}{3^2 5^2} B_2\right)\frac{P_3}{P_1} - \frac{1}{3^2 . 5 . 7^2} B_2 \frac{P_5}{P_1} + \frac{53}{2^2 . 3^2 . 5^3 . 7}\frac{P_0}{Q_1}$$

$$+ \frac{1}{3^2 . 5 . 7}\frac{P_2}{Q_1} + \frac{1}{2 . 3^3 . 5 . 7}\frac{P_4}{Q_1} - \frac{229}{3^2 . 3^5 . 7^2 . 11}\left(\frac{P_3}{P_1} - \frac{Q_3}{Q_1}\right)$$

$$- \frac{4}{3^2 . 5^2 . 7^2 . 13}\left(\frac{P_5}{P_1} - \frac{Q_5}{Q_1}\right) + \frac{1}{3^4 . 5 . 7 . 11 . 13}\left(\frac{P_7}{P_1} - \frac{Q_7}{Q_1}\right)$$

$$B_7 = - \frac{82\,i}{3 . 5^4 . 7^2}\frac{P_1}{Q_1}$$

$$B_8 = \frac{1}{5^2}\left(B_6 + \frac{2}{3^2 5^2} B_4 - \frac{229}{3^2 . 5^3 . 7^2 . 11} B_2\right)\frac{P_3}{P_1}$$

$$- \frac{1}{3^2 . 5 . 7^2}\left(B_4 + \frac{4}{5^2 . 13} B_2\right)\frac{P_5}{P_1}$$

$$+ \frac{1}{3^4 . 5 . 7 . 11 . 13} B_2 \frac{P_7}{P_1} - \frac{11}{2 . 3 . 5^5 . 7}\frac{P_0}{Q_1} - \frac{2}{3^4 . 5^3}\frac{P_2}{Q_1} - \frac{1}{3^2 . 5^2 . 7 . 11}\frac{P_4}{Q_1}$$

$$- \frac{1}{2^3 . 3^4 . 5 . 7 . 11}\frac{P_6}{Q_1} - \frac{12542}{3^2 . 5^8 . 7^2 . 11 . 13}\left(\frac{P_3}{P_1} - \frac{Q_3}{Q_1}\right) + \frac{386}{3^4 . 5^4 . 7^4 . 13}\left(\frac{P_5}{P_1} - \frac{Q_5}{Q_1}\right)$$

$$+ \frac{2}{3^3 . 5^3 . 7 . 11 . 13 . 17}\left(\frac{P_7}{P_1} - \frac{Q_7}{Q_1}\right) - \frac{1}{3^3 . 5^2 . 7 . 11^2 . 13 . 17}\left(\frac{P_9}{P_1} - \frac{Q_9}{Q_1}\right)$$

$$B_9 = \frac{3121\, i}{3^6 \cdot 5^6 \cdot 7} \; \frac{P_1}{Q_1} .$$

Hence

$$A_{o1} = \frac{c^3}{9} (d_1^{\,o1})^2 \; \frac{P_1}{Q_1} \sum_{r=0}^{\infty} (ic)^r \, \alpha_r \tag{80}$$

where the α_r are given by the equation

$$\sum_{s=0}^{r} \alpha_s \, B_{r-s} = 0 , \tag{81}$$

$r = 1, 2, 3, \ldots\ldots$, with $\alpha_o \equiv 1$.

For $0 \leqslant r \leqslant 9$ the α_r have been computed for a sequence of values of ξ spanning the range from a disc ($\xi = 0$) to a point at which the convergence shown in equation (26) can be assumed to be applicable. The values of the Legendre functions were obtained from the N. B. S. Tables (1945), reinforced where necessary by direct calculation of the functions from their formulae, and the results are shown in Table III. It was then a trivial matter to determine the convergence coefficients $|a_r^{\,1}|$ defined as

$$|a_r^{\,1}| = |\alpha_r|^{-1/r}$$

and these are also tabulated together with the values of $(\xi^2 + 1)^{1/2} \, |a_r^{\,1}|$ indicating the convergence measured in terms of the semi-major axis of the spheroid. This

TABLE III. CONVERGENCE COEFFICIENTS FOR A_{01} (SOFT)

	$\xi = 1.2$			$\xi = 0.6$			$\xi = 0.4$		
r	α_r	$\|a_r^1\|$	$(\xi^2+1)^{1/2}\|a_r^1\|$	α_r	$\|a_r^1\|$	$(\xi^2+1)^{1/2}\|a_r^1\|$	α_r	$\|a_r^1\|$	$(\xi^2+1)^{1/2}\|a_r^1\|$
2	1.20254	0.91190	1.4244	5.23870×10^{-1}	1.3816	1.6112	3.81764×10^{-1}	1.6184	1.7431
3	8.01696×10^{-1}	1.0765	1.6815	1.74623×10^{-1}	1.7891	2.0864	8.48364×10^{-2}	2.2758	2.4511
4	1.52720	0.89955	1.4051	2.45103×10^{-1}	1.4212	1.6574	1.15639×10^{-1}	1.7148	1.8469
5	1.83195	0.88597	1.3839	1.62005×10^{-1}	1.4391	1.6783	5.45946×10^{-2}	1.7888	1.9266
6	2.54840	0.85565	1.3366	1.43436×10^{-1}	1.3822	1.6119	4.11153×10^{-2}	1.7022	1.8333
7	3.38311	0.84020	1.3124	1.12973×10^{-1}	1.3655	1.5924	2.48936×10^{-2}	1.6948	1.8254
8	4.58890	0.82658	1.2912	8.22250×10^{-2}	1.3665	1.5936	1.59641×10^{-2}	1.6773	1.8065
9	6.12600	0.81770	1.2773	7.51219×10^{-2}	1.3333	1.5549	1.05680×10^{-2}	1.6579	1.7856

	$\xi = 0.3$			$\xi = 0.2$			$\xi = 0.1$		
r	α_r	$\|a_r^1\|$	$(\xi^2+1)^{1/2}\|a_r^1\|$	α_r	$\|a_r^1\|$	$(\xi^2+1)^{1/2}\|a_r^1\|$	α_r	$\|a_r^1\|$	$(\xi^2+1)^{1/2}\|a_r^1\|$
2	3.24571×10^{-1}	1.7553	1.8326	2.75740×10^{-1}	1.9044	1.9421	2.34498×10^{-1}	2.0651	2.0754
3	5.40952×10^{-2}	2.6441	2.7605	3.06378×10^{-2}	3.1958	3.2591	1.30276×10^{-2}	4.2499	4.2711
4	7.71550×10^{-2}	1.8974	1.9809	5.05071×10^{-2}	2.1094	2.1512	3.24774×10^{-2}	2.3556	2.3673
5	2.86240×10^{-2}	2.0355	2.1251	1.32196×10^{-2}	2.3755	2.4225	4.54659×10^{-3}	2.9408	2.9555
6	2.02689×10^{-2}	1.9151	1.9994	9.28174×10^{-3}	2.1814	2.2246	3.85537×10^{-3}	2.5254	2.5380
7	1.02668×10^{-2}	1.9234	2.0081	3.64290×10^{-3}	2.2303	2.2745	9.34045×10^{-4}	2.7090	2.7225
8	7.43396×10^{-3}	1.8455	1.9268	1.90913×10^{-3}	2.1872	2.2305	4.73578×10^{-4}	2.6036	2.6166
9	3.33953×10^{-3}	1.8843	1.9673	8.67370×10^{-4}	2.1888	2.2321	1.53654×10^{-4}	2.6529	2.6661

	$\xi = 0.05$			$\xi = 0.01$			$\xi = 0.0$		
r	α_r	$\|a_r^1\|$	$(\xi^2+1)^{1/2}\|a_r^1\|$	α_r	$\|a_r^1\|$	$(\xi^2+1)^{1/2}\|a_r^1\|$	α_r	$\|a_r^1\|$	$(\xi^2+1)^{1/2}\|a_r^1\|$
2	2.16460×10^{-1}	2.1494	2.1521	2.03171×10^{-1}	2.2185	2.2186	2.00000×10^{-1}	2.2361	2.2361
3	6.01278×10^{-3}	5.4993	5.5062	1.12873×10^{-3}	9.6044	9.6049	0	∞	∞
4	2.58893×10^{-2}	2.4930	2.4961	2.15442×10^{-2}	2.6102	2.6103	2.05714×10^{-2}	2.6405	2.6405
5	1.88152×10^{-3}	3.5083	3.5127	3.23203×10^{-4}	4.9901	4.9903	0	∞	∞
6	2.37246×10^{-3}	2.7382	2.7416	1.56921×10^{-2}	2.9335	2.9336	1.41037×10^{-3}	2.9861	2.9861
7	3.28993×10^{-4}	3.1444	3.1483	4.92560×10^{-5}	4.1244	4.1246	0	∞	∞
8	2.00580×10^{-4}	2.8958	2.9024	8.81922×10^{-5}	3.2123	3.2125	6.99710×10^{-5}	3.3066	3.3066
9	4.33278×10^{-5}	3.0535	3.0573	5.28150×10^{-6}	3.8580	3.8582	0	∞	∞

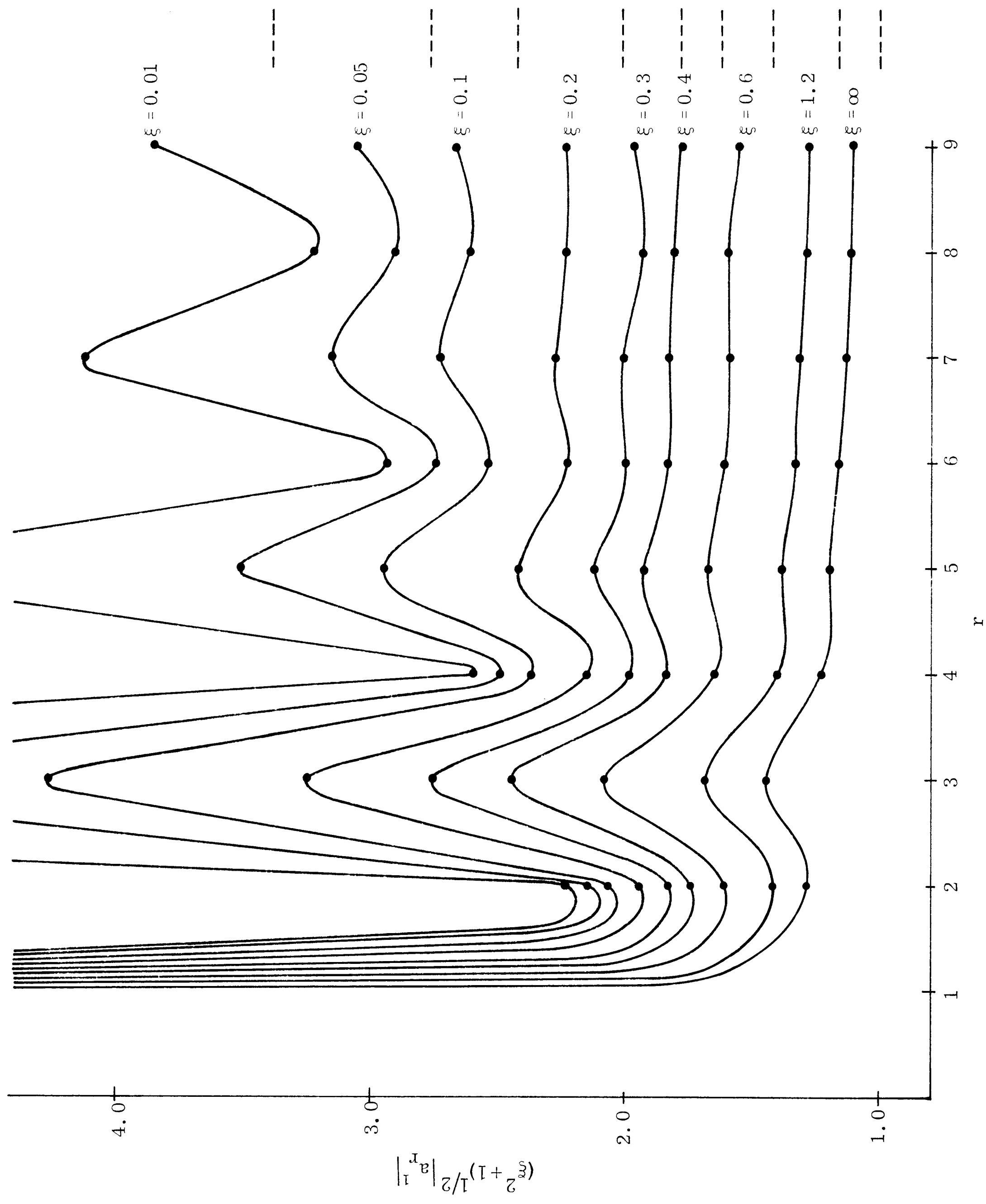

FIGURE IV. CONVERGENCE COEFFICIENTS FOR SOFT OBLATE A_{o1}

last set of data is plotted as a function of $(1 + 1/\xi^2)^{1/2}$ in Figure IV, and it will be observed that as ξ decreases the overall level of the curves increases accompanied by a build-up of oscillations. In the main, the maxima and minima occur at odd and even values of r respectively, and when ξ is less than 0.2 the first few minima for each curve increase in magnitude as r increases, whereas the maxima decrease. In other words, the amplitude of oscillations decreases with increasing r. As ξ becomes smaller, the maxima increase in size, and are infinite in the limiting case of the disc. Since an infinity corresponds to the absence of that power of c, the result implies that no odd powers of c occur in the expansion of A_{01} for a disc, a fact which is otherwise obvious when equation (80) is compared with (62). Indeed, when $\xi = 0$ the α_r are merely the coefficients of $(-t)^r$ in the expansion of $\left\{ c_o^{o1}(t) \right\}^{-1}$, and our previous consideration of this function showed it to be regular for $|t| < 5.2$ approximately.

To find the convergence of the expansion for A_{01} when ξ is not zero, we have to determine the limiting values of the quantities $(\xi^2 + 1)^{1/2} |a_r^1|$ as $r \to \infty$, and in practice the most convenient way of doing this is to express them as multiples of the corresponding convergence coefficients for the sphere (see Table I). Taking for example the case $\xi = 0.4$, the resulting ratios are

r = 4	1.4944
r = 5	1.6040
r = 6	1.5785

r = 7	1.6046
r = 8	1.6140
r = 9	1.6153

the limit of which is estimated to be 1.62. Bearing in mind that for the soft sphere the A_{01} converges when $c(\xi^2 + 1)^{1/2} < 1$, the radius of convergence for $\xi = 0.4$ is therefore 1.62.

This procedure proved effective for all except the smallest values of ξ, and here the interpretation of the ratios was made easier by the fact that the successive minima on each curve increase with increasing r, whilst the maxima decrease. Since the trend is relatively uniform, it is possible to obtain a reasonable estimate of the convergence even when the mean level of the curve is still increasing at r = 9, and the results are shown in Table IV.

TABLE IV. RADIUS OF CONVERGENCE FOR A_{01} (SOFT)

ξ	1.2	0.6	0.4	0.3	0.2	0.1	0.05	0.01	0
$c(\xi^2+1)^{1/2}$	1.16	1.42	1.62	1.78	2.01	2.42	2.76	3.38	5.2

It is believed that these are accurate to within 1% for the larger ξ, but the error could conceivably be as much as 5% for ξ as small as 0.01

For the hard body the expression for the amplitude coefficient A_{01} differs from that in equation (78) in having all functions of ξ replaced by their first derivatives, and consequently the expansion can be decuded from the above by the simple process of differentiating each Legendre function. In this instance, however, the α_r are required only to indicate the convergence for those values of ξ between the ranges for which either the near-disc formula (75) or the near-sphere formula (derivable from (21) with n = 1) is valid. Even then an accurate determination is unnecessary unless the curve of convergence against $(1 + 1/\xi^2)^{1/2}$ is found to dip below the one for A_{00}, and this is indeed fortunate in view of the almost random nature of the results.

The calculations have been carried out for ξ = 1.2, 0.6, 0.4 and 0.3, and the α_r are given in Table V together with the convergence coefficients $|a_r^1|$ and $(\xi^2 + 1)^{1/2} \; |a_r^1|$ deduced therefrom. The last of these represent the convergence measured in terms of the semi-major axis of the spheroid, and are plotted as a function of $(1 + 1/\xi^2)^{1/2}$ in Figure V. If anything, the curves are notable only for their lack of uniformity, and any attempt to deduce the ultimate level of each curve as $r \longrightarrow \infty$ is largely a matter of guesswork. Nevertheless, by concentrating on the minima and comparing these with the values for the sphere (see Table II), it is possible to come up with some approximate values for the radius of convergence, and these are shown in Table VI.

TABLE V. CONVERGENCE COEFFICIENTS FOR A_{o1} (HARD)

	$\xi = 1.2$			$\xi = 0.6$		
r	α_r	$\lvert a_r^1\rvert$	$(\xi^2+1)^{1/2}\lvert a_r^1\rvert$	α_r	$\lvert a_r^1\rvert$	$(\xi^2+1)^{1/2}\lvert a_r^1\rvert$
2	4.84690×10^{-1}	1.4364	2.2437	1.49754×10^{-1}	2.5841	3.0136
3	-5.47521×10^{-1}	1.2224	1.9094	-1.88580×10^{-1}	1.7438	2.0336
4	-4.36995×10^{-1}	1.2299	1.9212	-7.88932×10^{-2}	1.8869	2.2005
5	-4.65054×10^{-1}	1.1655	1.8206	-3.38517×10^{-2}	1.9683	2.2954
6	2.13025×10^{-2}	1.8993	2.9668	2.67026×10^{-2}	1.8291	2.1331
7	4.09194×10^{-1}	1.1362	1.7748	3.07891×10^{-2}	1.6442	1.9174
8	5.74060×10^{-1}	1.0718	1.6742	1.50781×10^{-2}	1.6893	1.9700
9	3.26687×10^{-1}	1.1324	1.7689	-2.42565×10^{-3}	1.9524	2.2769
	$\xi = 0.4$			$\xi = 0.3$		
r	α_r	$\lvert a_r^1\rvert$	$(\xi^2+1)^{1/2}\lvert a_r^1\rvert$	α_r	$\lvert a_r^1\rvert$	$(\xi^2+1)^{1/2}\lvert a_r^1\rvert$
2	8.15714×10^{-2}	3.5013	3.7710	5.48205×10^{-2}	4.2710	4.4591
3	-1.31421×10^{-1}	1.9669	2.1184	-1.10656×10^{-1}	2.0829	2.1746
4	-3.78142×10^{-2}	2.2677	2.4424	-2.43580×10^{-2}	2.5313	2.6428
5	-5.66945×10^{-3}	2.8138	3.0306	1.14628×10^{-3}	3.8738	4.0444
6	1.69996×10^{-2}	1.9721	2.1240	1.30692×10^{-2}	2.0604	2.1511
7	1.05818×10^{-2}	1.9152	2.1627	5.62521×10^{-3}	2.0961	2.1884
8	1.80478×10^{-3}	2.2027	2.3724	-2.50891×10^{-4}	2.8188	2.9429
9	-2.60133×10^{-3}	1.9373	2.0865	-1.90734×10^{-3}	2.0053	2.0936

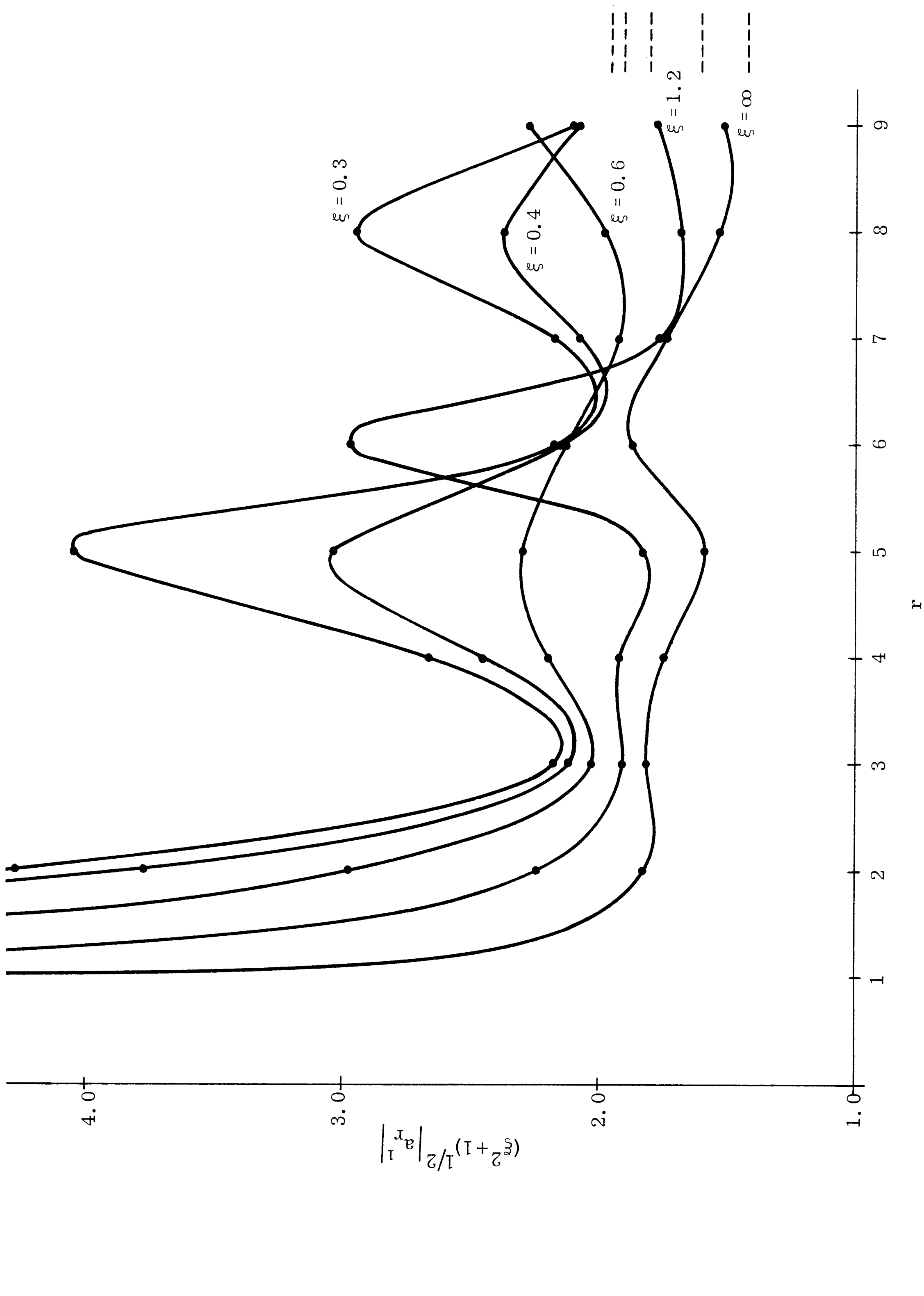

FIGURE V. CONVERGENCE COEFFICIENTS FOR HARD OBLATE A_{o1}

TABLE VI. RADIUS OF CONVERGENCE FOR A_{01} (HARD)

ξ	1.2	0.6	0.4	0.3
$c(\xi^2+1)^{1/2}$	1.6	1.8	1.9	1.9(5)

The accuracy of the above results is hard to estimate, but to judge from the way in which they agree with the formulae for $\xi \sim \infty$ and $\xi \sim 0$, they cannot be too much in error. On the other hand, it is comforting to find that the convergence of the Rayleigh series for a hard spheroid is not determined by them.

Turning now to the case n = 0, the expression for the amplitude coefficient A_{oo} for the soft body can be obtained from equation (76), and since

$$Q_{oo}^{*}(-ic) = \frac{1}{c\left\{d_o^{oo}(-ic)\right\}^2} \tag{82}$$

we have

$$A_{oo} = -c(d_o^{oo})^2 \frac{P_0}{Q_0}\left[c(d_o^{oo})^2\frac{P_0}{Q_0} + \frac{\sum_{r=0}^{\infty}\frac{d_r^{oo}}{d_o^{oo}}\frac{Q_r}{Q_0} + \sum_{r=2}^{\infty}\frac{d_{p/r}^{oo}}{d_o^{oo}}\frac{P_{r-1}}{Q_0}}{\sum_{r=0}^{\infty}\frac{d_r^{oo}}{d_o^{oo}}\frac{P_r}{P_0}}\right]^{-1}. \tag{83}$$

This can be written as

$$A_{oo} = - c(d_o^{oo})^2 \frac{P_0}{Q_0} \left[\sum_{r=0}^{\infty} (-ic)^r B_r \right]^{-1} \tag{84}$$

with

$$B_0 = 1$$

$$B_1 = i \frac{P_0}{Q_0}$$

$$B_2 = -\frac{1}{2}\frac{P_1}{Q_0} + \frac{1}{3^2}\left(\frac{P_2}{P_0} - \frac{Q_2}{Q_0}\right)$$

$$B_3 = -\frac{i}{3^2}\frac{P_0}{Q_0}$$

$$B_4 = \frac{1}{3^2} B_2 \frac{P_2}{P_0} + \frac{2}{3.5}\frac{P_1}{Q_2} + \frac{1}{2^2 3.5}\frac{P_3}{Q_0} - \frac{2}{3^4 7}\left(\frac{P_2}{P_0} - \frac{Q_2}{Q_0}\right)$$

$$- \frac{1}{3.5^2 7}\left(\frac{P_4}{P_0} - \frac{Q_4}{Q_0}\right)$$

$$B_5 = \frac{23\,i}{3^4 5^2}\frac{P_0}{Q_0}$$

$$B_6 = \frac{1}{3^2}\left(B_4 - \frac{2}{3^2 7} B_2\right)\frac{P_2}{P_0} - \frac{1}{3.5^2 7} B_2 \frac{P_4}{P_0} - \frac{5}{2^2 3^3 7}\frac{P_1}{Q_0} + \frac{1}{3^4 5}\frac{P_3}{Q_0}$$

$$+ \frac{1}{2.3^4 5.7}\frac{P_5}{Q_0} - \frac{13}{3^6 5^2 7}\left(\frac{P_2}{P_0} - \frac{Q_2}{Q_0}\right) + \frac{4}{3^3 5^2 7.11}\left(\frac{P_4}{P_0} - \frac{Q_4}{Q_0}\right)$$

$$+ \frac{1}{3^3 5.7^2 11}\left(\frac{P_6}{P_0} - \frac{Q_6}{Q_0}\right)$$

$$B_7 = -\frac{158i}{3^6 5.7^2}\frac{P_0}{Q_0}$$

$$B_8 = \frac{1}{3^2}\left(B_6 - \frac{2}{3^2 7}B_4 - \frac{13}{3^4 5^2 7}B_2\right)\frac{P_2}{P_0} - \frac{1}{3.5^2 7}\left(B_4 - \frac{4}{3^3 11}B_2\right)\frac{P_4}{P_0}$$

$$+ \frac{1}{3^3 5.7^2 11}B_2\frac{P_6}{P_0} + \frac{41}{2.3^5 5^2 7}\frac{P_1}{Q_0} + \frac{16}{3^5 5^2 11}\frac{P_3}{Q_0}$$

$$+ \frac{1}{3^4 5.7.13}\frac{P_5}{Q_0} + \frac{1}{2^3 3^3 5.7.11.13}\frac{P_7}{Q_0} + \frac{46}{3^8 5.7.11}\left(\frac{P_2}{P_0} - \frac{Q_2}{Q_0}\right)$$

$$+ \frac{2498}{3^5 5^4 7^2 11.13}\left(\frac{P_4}{P_0} - \frac{Q_4}{Q_0}\right) - \frac{2}{3^5 5^2 7^2 11}\left(\frac{P_6}{P_0} - \frac{Q_6}{Q_0}\right)$$

$$- \frac{1}{3^6 5^2 7.11.13}\left(\frac{P_8}{P_0} - \frac{Q_8}{Q_0}\right)$$

$$B_9 = \frac{199i}{3^8 5^4}\frac{P_0}{Q_0}$$

and hence

$$A_{oo} = -c\,(d_o^{oo})^2\,\frac{P_0}{Q_0}\sum_{r=0}^{\infty}(-ic)^r\,\alpha_r \qquad (85)$$

in which the α_r are related to the B_r by equation (81).

The coefficient A_{oo} for the soft body is analogous to the A_{o1} for the hard in that each provides the smallest radius of convergence for the appropriate disc (or near-disc), but neither is important when the spheroid is almost a sphere. Indeed, for the soft sphere the expansion of the coefficient A_{00} has an infinite radius of convergence, and consequently any calculations of the α_r based on the above formulae are concerned only with indicating the manner in which the radius approaches infinity for values of ξ greater than those for which the expression in equation (74) is applicable.

The calculations have therefore been limited to $\xi = 1.2,\ 0.6,\ 0.4$ and 0.3, and the corresponding α_r are shown in Table VII. Also listed are the convergence coefficients $\left|a_r^{o}\right|$ and $(\xi^2 + 1)^{1/2} \left|a_r^{o}\right|$, and the latter are plotted as a function of $(1 + 1/\xi^2)^{1/2}$ in Figure VI. It will be observed that none of the curves show any signs of turning over and are effectively straight lines as far out as the largest r considered. This almost certainly implies a radius of convergence in excess of 2.5, and even the possibility of an infinite value (as in the sphere limit, $\xi = \infty$) cannot be ruled out entirely. Under these circumstances it would be a risky undertaking to try to estimate the convergence, but it is clear that for $\xi > 0.3$ the radius is too large to play any role in the analysis.

When the body is hard the coefficient A_{00} differs in having all the functions of the radial variable replaced by their first derivatives, but if an attempt is made to differentiate the Legendre functions in equation (83) a difficulty arises owing to the occurrence of a factor $P_0'(i\xi) \equiv 0$ in the denominator, where the prime denotes $\partial/\partial\xi$.

TABLE VII. CONVERGENCE COEFFICIENTS FOR A_{oo} (SOFT)

	ξ = 1.2			ξ = 0.6		
r	α_r	$\|a_r^o\|$	$(\xi^2+1)^{1/2}\|a_r^o\|$	α_r	$\|a_r^o\|$	$(\xi^2+1)^{1/2}\|a_r^o\|$
1	1.43939	0.6947	1.0852	9.70516×10^{-1}	1.0304	1.2016
2	1.49609	0.8176	1.2771	7.47798×10^{-1}	1.1564	1.3486
3	1.16479	0.9504	1.4846	4.29535×10^{-1}	1.3254	1.5457
4	7.42627×10^{-1}	1.0772	1.6826	2.06231×10^{-1}	1.4839	1.7305
5	4.02260×10^{-1}	1.1998	1.8741	8.51713×10^{-2}	1.6366	1.9086
6	1.89628×10^{-1}	1.3193	2.0608	3.67042×10^{-2}	1.7870	2.0840
7	7.91543×10^{-1}	1.4367	2.2442	9.80720×10^{-3}	1.9361	2.2579
8	2.95953×10^{-2}	1.5527	2.4254	2.79074×10^{-3}	2.0859	2.4326
9	9.99243×10^{-3}	1.6682	2.6059	7.08031×10^{-4}	2.2387	2.6107
	ξ = 0.4			ξ = 0.3		
r	α_r	$\|a_r^o\|$	$(\xi^2+1)^{1/2}\|a_r^o\|$	α_r	$\|a_r^o\|$	$(\xi^2+1)^{1/2}\|a_r^o\|$
1	8.40131×10^{-1}	1.1903	1.2820	7.81653×10^{-1}	1.2793	1.3356
2	5.93803×10^{-1}	1.2977	1.3977	5.32816×10^{-1}	1.3700	1.4303
3	3.11415×10^{-1}	1.4753	1.5889	2.68529×10^{-1}	1.5500	1.6182
4	1.37798×10^{-1}	1.6413	1.7677	1.14727×10^{-1}	1.7182	1.7939
5	5.27307×10^{-2}	1.8013	1.9401	4.25182×10^{-2}	1.8806	1.9634
6	1.75955×10^{-2}	1.9068	2.1199	1.37261×10^{-2}	2.0437	2.1337
7	5.19818×10^{-3}	2.1199	2.2832	3.92111×10^{-3}	2.2070	2.3042
8	1.36373×10^{-3}	2.2812	2.4569	9.93377×10^{-4}	2.3733	2.4778
9	3.16332×10^{-4}	2.4484	2.6370	2.21463×10^{-4}	2.5473	2.6595

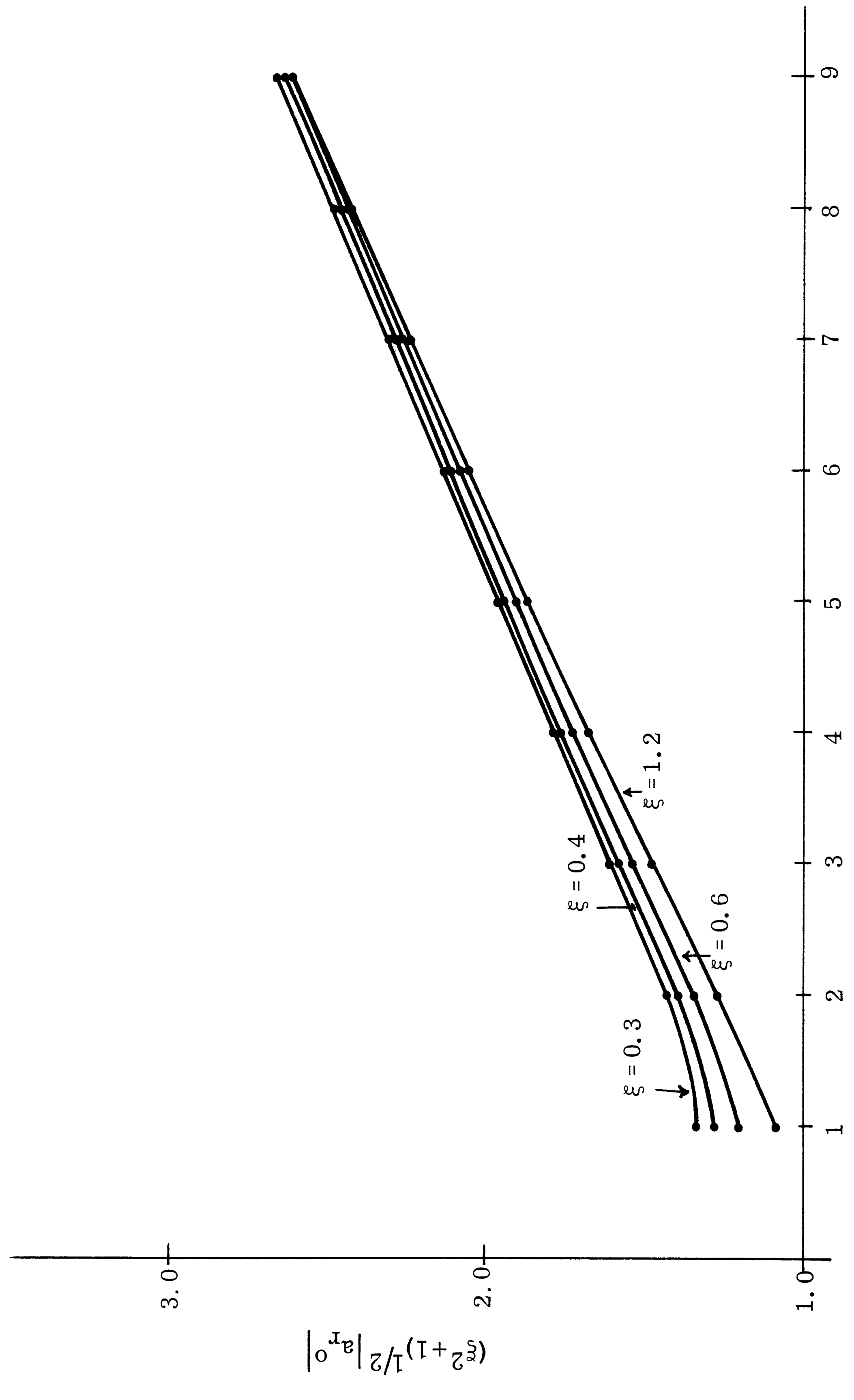

FIGURE VI. CONVERGENCE COEFFICIENTS FOR SOFT OBLATE A_{oo}

The difficulty is most easily overcome by multiplying through by $\frac{P_2}{P_0}$ prior to the differentiation, and the expression for A_{oo} in the hard case is then

$$A_{oo} = -c(d_o^{oo})^2 \frac{P_2'}{Q_0'} \left[c(d_o^{oo})^2 + \frac{\sum_{r=0}^{\infty} \frac{d_r^{oo}}{d_o^{oo}} \frac{Q_r'}{Q_0'} + \sum_{r=2}^{\infty} \frac{d_{p/r}^{oo}}{d_o^{oo}} \frac{P_{r-1}'}{Q_0'}}{\sum_{r=2}^{\infty} \frac{d_r^{oo}}{d_o^{oo}} \frac{P_r'}{P_2'}} \right]^{-1} \tag{86}$$

which can be written in the form

$$A_{oo} = -\frac{c^3}{9}(d_o^{oo})^2 \frac{P_2'}{Q_0'} \left[\sum_{r=0}^{\infty} (ic)^r B_r \right]^{-1} \tag{87}$$

with

$$B_0 = 1$$

$$B_1 = 0$$

$$B_2 = \frac{1}{7}\left(\frac{2}{3^2} + \frac{3}{5^2}\frac{P_4'}{P_2'}\right) - \frac{1}{3^2}\frac{Q_2'}{Q_0'} - \frac{1}{2}\frac{P_1'}{Q_0'}$$

$$B_3 = \frac{i}{3^2}\frac{P_2'}{Q_0'}$$

$$B_4 = \frac{1}{7}B_2\left(\frac{2}{3^2} + \frac{3}{5^2}\frac{P_4'}{P_2'}\right) + \frac{1}{3.5.7}\left(\frac{13}{3.3^3 5} - \frac{4}{5.11}\frac{P_4'}{P_2'} - \frac{1}{7.11}\frac{P_6'}{P_2'}\right)$$

$$+ \frac{1}{3.^2 7}\left(\frac{2}{3^2}\frac{Q_2'}{Q_0'} + \frac{3}{5^2}\frac{Q_4'}{Q_0'}\right) + \frac{1}{3.5}\left(\frac{P_1'}{Q_0'} + \frac{1}{2^2}\frac{P_3'}{Q_0'}\right)$$

$$B_5 = -\frac{i}{3^4}\frac{P_2'}{Q_0'}$$

$$B_6 = \frac{1}{7} B_4\left(\frac{2}{3^2} + \frac{3}{5^2}\frac{P_4'}{P_2'}\right) + \frac{1}{3.5.7} B_2 \left(\frac{13}{3.^3 5} - \frac{4}{5.11}\frac{P_4'}{P_2'} - \frac{1}{7.11}\frac{P_6'}{P_2'}\right)$$

$$- \frac{1}{3.^3 5.7.11}\left(\frac{46}{3^3} + \frac{2498}{5.^3 7.13}\frac{P_4'}{P_2'} - \frac{2}{5.7}\frac{P_6'}{P_2'} - \frac{1}{3.5.13}\frac{P_8'}{P_2'}\right)$$

$$+ \frac{1}{3.^3 5.7}\left(\frac{13}{3.^3 5}\frac{Q_2'}{Q_0'} - \frac{4}{5.11}\frac{Q_4'}{Q_0'} - \frac{1}{7.11}\frac{Q_6'}{Q_0'}\right)$$

$$- \frac{1}{3^3}\left(\frac{5}{2.^2 7}\frac{P_1'}{Q_0'} + \frac{1}{3.5}\frac{P_3'}{Q_0'} + \frac{1}{2.3.5.7}\frac{P_5'}{Q_0'}\right)$$

$$B_7 = \frac{23i}{3.^6 5^2}\frac{P_2'}{Q_0'} .$$

It will be noted that as a consequence of the differentiation two terms of the series have been 'lost', and the expansions given in Appendix A now serve to determine the B_r up to and including B_7.

From equation (87) it follows immediately that

$$A_{oo} = -\frac{c^3}{9}\left(d_o^{oo}\right)^2 \frac{P_2'}{Q_0'} \sum_{r=0}^{\infty} (ic)^r \alpha_r \tag{88}$$

where the α_r are related to the B_r by equation (81). These have been computed for ξ = 1.2, 0.6, 0.4, 0.3, 0.2, 0.1 and 0.05, and the values are given in Table VIII. Also listed are the convergence coefficients $|a_r^o|$ and $(\xi^2+1)^{1/2}\ |a_r^o|$, and the latter are plotted as functions of $(1+1/\xi^2)^{1/2}$ in Figure VII. The regularity is at once apparent, and by comparing the coefficients with those for a sphere it is possible to estimate the convergence with a reasonable degree of accuracy notwithstanding the smaller number of the $|a_r^o|$ available. The results are shown in Table IX.

TABLE IX. RADIUS OF CONVERGENCE FOR A_{oo} (HARD)

ξ	1.2	0.6	0.4	0.3	0.2	0.1	0.05
$c(\xi^2+1)^{1/2}$	1.08	1.21	1.32	1.43	1.56	1.82	2.10

This completes the discussion of the oblate spheroid, and in combination with the formulae of §§4, 5 and 6 the radii of curvature obtained above are sufficient to specify the convergence of the Rayleigh series for both hard and soft spheroids of any ellipticity. Rather than summarize the data here, however, we shall now go on to consider the prolate spheroid and reserve the presentation of the final results for §9.

TABLE VIII. CONVERGENCE COEFFICIENTS FOR A_{oo} (HARD)

r	$\xi = 1.2$ α_r	$\lvert a_r^o \rvert$	$(\xi^2+1)^{1/2}\lvert a_r^o\rvert$	$\xi = 0.6$ α_r	$\lvert a_r^o \rvert$	$(\xi^2+1)^{1/2}\lvert a_r^o\rvert$	$\xi = 0.4$ α_r	$\lvert a_r^o \rvert$	$(\xi^2+1)^{1/2}\lvert a_r^o\rvert$
2	1.35095	0.8604	1.3439	6.65151×10^{-1}	1.2261	1.4299	5.15654×10^{-1}	1.3926	1.4999
3	9.76000×10^{-1}	1.0081	1.5747	2.72000×10^{-1}	1.5434	1.7999	1.54667×10^{-1}	1.8630	2.0065
4	1.95652	0.8453	1.3205	4.12168×10^{-1}	1.2481	1.4555	2.22677×10^{-1}	1.4557	1.5678
5	2.52862	0.8307	1.2975	3.31620×10^{-1}	1.2470	1.4542	1.42324×10^{-1}	1.4769	1.5907
6	3.74135	0.8026	1.2537	3.29480×10^{-1}	1.2033	1.4033	1.20057×10^{-1}	1.4238	1.5335
7	2.65924	0.7876	1.2303	3.07444×10^{-1}	1.1835	1.3802	9.40407×10^{-2}	1.4018	1.5097

r	$\xi = 0.3$ α_r	$\lvert a_r^o \rvert$	$(\xi^2+1)^{1/2}\lvert a_r^o\rvert$	$\xi = 0.2$ α_r	$\lvert a_r^o \rvert$	$(\xi^2+1)^{1/2}\lvert a_r^o\rvert$	$\xi = 0.1$ α_r	$\lvert a_r^o \rvert$	$(\xi^2+1)^{1/2}\lvert a_r^o\rvert$
2	4.52337×10^{-1}	1.4869	1.5524	3.94778×10^{-1}	1.5916	1.6231	3.41084×10^{-1}	1.7123	1.7208
3	1.09000×10^{-1}	2.0934	2.1856	6.93334×10^{-2}	2.4342	2.4824	3.36667×10^{-2}	3.0969	3.1123
4	1.58031×10^{-1}	1.5860	1.6558	1.07294×10^{-1}	1.7473	1.7819	6.69587×10^{-2}	1.9658	1.9756
5	8.64984×10^{-2}	1.6316	1.7034	4.70389×10^{-2}	1.8429	1.8794	1.92256×10^{-2}	2.2041	2.2151
6	6.63515×10^{-2}	1.5717	1.6409	3.22539×10^{-2}	1.7724	1.8075	1.31998×10^{-2}	2.1079	2.1184
7	1.70346×10^{-2}	1.5476	1.6157	2.03886×10^{-2}	1.7439	1.7784	6.25585×10^{-3}	2.0645	2.1748

r	$\xi = 0.05$ α_r	$\lvert a_r^o \rvert$	$(\xi^2+1)^{1/2}\lvert a_r^o\rvert$
2	3.14966×10^{-1}	1.7818	1.7840
3	1.67083×10^{-2}	3.9116	3.9165
4	4.98593×10^{-2}	2.1162	2.1188
5	8.66864×10^{-3}	2.5847	2.5519
6	4.65610×10^{-3}	2.4472	2.4503
7	2.34398×10^{-3}	2.3753	2.3783

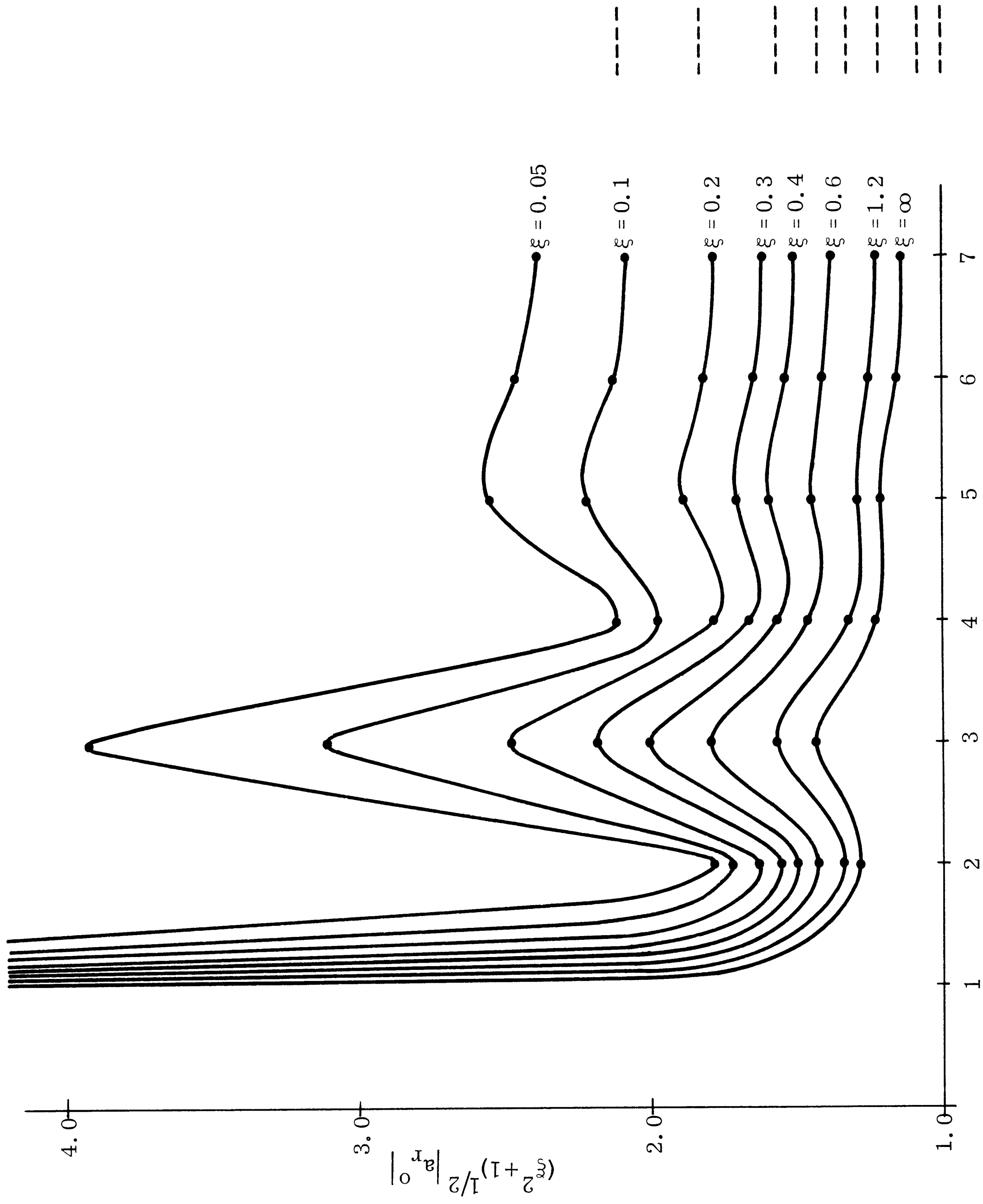

FIGURE VII. CONVERGENCE COEFFICIENTS FOR HARD OBLATE A_{00}

VIII

PROLATE SPHEROIDS

For a prolate spheroid, ξ is confined to the range $1 \leqslant \xi \leqslant \infty$ and the lower limit ($\xi = 1$) represents a rod of zero thickness joining the foci of the coordinate system, whilst the upper value ($\xi = \infty$) again corresponds to a sphere. In spite of the change in the coordinates, some of the analysis for the oblate spheroid is immediately applicable here. In particular, when the ellipticity is small (ξ large compared with unity) the convergence of the Rayleigh series has already been determined (§4), and if the body is soft the series converges for

$$c\xi < 1 + \frac{1}{5\xi^2} + 0(\xi^{-4}), \tag{19}$$

whereas for the hard body the criterion is

$$c\xi < 1 + \frac{1}{3\xi^2} + 0(\xi^{-4}), \tag{20}$$

where $c\xi$ is now the semi-major axis of the spheroid.

At the other extreme ($\xi \sim 1$) the spheroid approaches a 'vanishing' rod and since the volume of this is zero it is not surprising to find that each term in the expansion of the far field amplitude is zero in the limit $\xi = 1$, leading to a null Rayleigh series. On the other hand, for all $\xi \neq 1$, no matter how close to unity, the Rayleigh series exists, and it is therefore meaningful to consider the convergence as $\xi \to 1$.

When $\epsilon = \xi - 1$ is small, the most convenient representation of the radial functions is

$$R^{(1)}_{on}(c,\xi) = \frac{1}{k^{(1)}_{on}(c)} \sum_{r=0}^{\infty} (-1)^r c^{on}_{2r}(c)(\xi^2-1)^r \qquad \text{(n even)} \tag{89}$$

$$= \frac{\xi}{k^{(1)}_{on}(c)} \sum_{r=0}^{\infty} (-1)^r c^{on}_{2r}(c)(\xi^2-1)^r \qquad \text{(n odd)} \tag{90}$$

(Flammer, equations 4.4.1a and b), both of which are equivalent to

$$R^{(1)}_{on}(c,\xi) = \frac{c^{on}_{o}(c)}{k^{(1)}_{on}(c)} + 0(\epsilon) \tag{91}$$

for small ϵ. Here,

$$\frac{c^{on}_{o}(c)}{k^{(1)}_{on}(c)} = \frac{2^n \, \frac{n}{2}! \, \frac{n}{2}!}{n!} \, d^{on}_{o}(c) \qquad \text{(n even)}$$

$$= \frac{2^n \, \frac{n-1}{2}! \, \frac{n+1}{2}!}{3(n+1)!} \, c \, d^{on}_{1}(c) \qquad \text{(n odd)}$$

and is finite for all finite c. In addition,

$$R^{(3)}_{on}(c,\xi) = R^{(1)}_{on}\left[1 - \frac{i}{2c}\left\{\frac{k^{(1)}_{on}(c)}{c^{on}_{o}(c)}\right\}^2 \log\frac{\xi+1}{\xi-1}\right] + i\, g_{on}(c,\xi) \tag{92}$$

(Flammer, equations 4.4.9 and 4.4.6) where

$$g_{on}(c,\xi) = \xi \sum_{r=0}^{\infty} b_r^{on}(c)\,(\xi^2-1)^r \qquad \text{(n even)}$$

$$= \sum_{r=0}^{\infty} b_r^{on}(c)\,(\xi^2-1)^r \qquad \text{(n odd)}$$

and both of these imply

$$g_{on}(c,\xi) = b_o^{on}(c) + 0(\epsilon) \tag{93}$$

when ϵ is small. Accordingly, for a soft spheroid of ellipticity almost equal to unity

$$A_{on} = -\left[1 - \frac{i}{2c}\left\{\frac{k_{on}^{(1)}(c)}{c_o^{on}(c)}\right\}^2 \log\frac{\xi+1}{\xi-1} + i\,b_o^{on}(c)\,\frac{k_{on}^{(1)}(c)}{c_o^{on}(c)} + 0(\epsilon)\right]^{-1} \tag{94}$$

and the singularities are given by

$$\log\frac{2}{\epsilon} - 2c\,b_o^{on}(c)\,\frac{c_o^{on}(c)}{k_{on}^{(1)}(c)} + 2ic\left\{\frac{c_o^{on}(c)}{k_{on}^{(1)}(c)}\right\}^2 + 0(\epsilon) \doteq 0. \tag{95}$$

The first term in this equation is infinite in the limit $\epsilon = 0$, whereas the third term does not contain ϵ and is finite for all finite c. It is therefore apparent that when $\epsilon = 0$ the only possibility of obtaining a finite root is to have the second term become infinite, and such an infinity must then arise from the factor $b_o^{on}(c)$. The expression for this gives

$$c\,b_o^{on}(c)\,\frac{c_o^{on}(c)}{k_{on}^{(1)}(c)} = \frac{X_{on}(c)}{c_o^{on}(c)} \tag{96}$$

where

$$X_{on}(c) = \sum_{r=0}^{\infty} d_{2r}^{on} \sum_{s=0}^{r-1} \frac{4r-4s-1}{(2r-s)(2s+1)} - \sum_{r=1}^{\infty} d_{\rho/2r}^{on} \qquad \text{(n even)}$$

$$= \sum_{r=0}^{\infty} d_{2r+1}^{on} \sum_{s=0}^{r} \frac{4r-4s+1}{2r-s+1} - \sum_{r=1}^{\infty} d_{\rho/2r-1}^{on} \qquad \text{(n odd)}$$

(see Flammer, equations 4.4.16a and b), and though $X_{on}(c)$ is finite, $b_o^{on}(c)$ is infinite at all zeros of $c_o^{on}(c)$. As shown in § 6, the zero for which $|c|$ is smallest is provided by n = 0 and is approximately

$$c = 4 \cdot 1 \exp\left\{\pm\ i\ 0 \cdot 05\,\pi\right\}$$

which now represents the limiting value of the smallest root of (95). Consequently when $\epsilon \rightarrow 0$ the radius of convergence of the Rayleigh series for the soft body approaches 4·1, and the criterion for convergence is therefore

$$c\,\xi < 4 \cdot 1\ . \qquad (97)$$

It is of interest to note that as in the case of a soft disc (an oblate spheroid of ellipticity e = 1) the convergence is dictated by the amplitude coefficient A_{oo}, and for the next coefficient, A_{o1}, the radius is approximately 5·2, corresponding to the smallest zero of $c_o^{o1}(c)$.

When ϵ is small but not zero, some idea of the way in which the convergence varies as a function of ϵ can be obtained from a perturbation analysis applied to equation (95). For this purpose let

$$c = x_n + f(\epsilon)$$

be a root of the equation when $\epsilon \neq 0$, where $f(\epsilon)$ tends to zero with ϵ and x_n is a zero of $c_o^{on}(c)$. If this is substituted into (95) bearing in mind that the first two terms are the dominant ones, we have approximately

$$\log \frac{2}{\epsilon} - \frac{2}{f(\epsilon)} \left[\frac{X_{on}(c)}{\partial/\partial c \; c_o^{on}(c)} \right]_{c = x_n} = 0$$

giving

$$f(\epsilon) = \frac{\tau_n X_n}{\log \frac{2}{\epsilon}}$$

where

$$\tau_n = 2 \left[\frac{X_{on}(c)}{c \; \partial/\partial c \; c_o^{on}(c)} \right]_{c = x_n} .$$

Hence,

$$c = x_n \left(1 + \frac{\tau_n}{\log \frac{2}{\epsilon}}\right) \tag{98}$$

for small ϵ, and as required the second term is zero in the limit $\epsilon = 0$.

Into (98) we now insert the value of x_n corresponding to the smallest zero of $c_o^{oo}(c)$. This is

$$x_o = 4 \cdot 1 \; \exp \left\{ \pm \, i \; 0 \cdot 05 \; \pi \right\} \tag{99}$$

and consequently for $\epsilon \neq 0$ the radius of convergence for A_{oo} is

$$c\xi = |x_o| \, (1+\epsilon) \left| 1 + \frac{\tau_o}{\log \frac{2}{\epsilon}} \right| . \tag{100}$$

If ϵ is sufficiently small this is also the radius of convergence of the Rayleigh series, and under these circumstances equation (100) can be written as

$$c\xi = 4\cdot 1 \left(1 + \frac{1}{\log \frac{2}{\epsilon}} \, R_e \, \tau_o \right) \tag{101}$$

where R_e denotes the real part.

Unfortunately the expansions in Appendix A do not contain enough terms to enable us to calculate τ_o with the accuracy desired, but using the terms which are available it is found that

$$R_e \tau_o = - 1\cdot 5 .$$

Although it would be unwise to rely on the second figure, the result does give some indication of the convergence for $\epsilon \neq 0$, and accordingly for a soft spheroid of ellipticity almost equal to unity the convergence criterion is taken as

$$c\xi < 4\cdot 1 \left\{ 1 - 1\cdot 5 \left(\log \frac{\xi + 1}{\xi - 1} \right)^{-1} \right\} . \tag{102}$$

It will be observed that the radius is always less than 4· 1 and approaches the limiting value extremely slowly. Indeed, for $\xi = 1 + 10^{-10}$ the right hand side of (102) is still only 3· 8.

In the case of the hard body the analysis is similar to the above in most respects. For the radial functions of the first kind we have from (89) and (90)

$$\frac{\partial}{\partial \xi} R^{(1)}_{on}(c,\xi) = -2 \frac{c^{on}_{2}(c)}{k^{(1)}_{on}(c)} + 0(\epsilon) \qquad \text{(n even)} \qquad (103)$$

$$= \frac{1}{k^{(1)}_{on}(c)} \left\{ c^{on}_{o}(c) - 2\, c^{on}_{2}(c) \right\} + 0(\epsilon) \qquad \text{(n odd)} \qquad (104)$$

and since

$$c_2 = -\frac{1}{4}(\lambda_{on} - c^2)\, c^{on}_{o}(c) \qquad \text{(n even)}$$

$$= -\frac{1}{4}(\lambda_{on} - c^2 - 2)\, c^{on}_{o}(c) \qquad \text{(n odd)}$$

(Flammer, equations 3.2.11a and b), where λ_{on} is the eigenvalue, (103) and (104) are both equivalent to

$$\frac{\partial}{\partial \xi} R^{(1)}_{on}(c,\xi) = \frac{1}{2}(\lambda_{on} - c^2) \frac{c^{on}_{o}(c)}{k^{(1)}_{on}(c)} + 0(\epsilon). \qquad (105)$$

Also,

$$\frac{\partial}{\partial \xi} g_{on}(c,\xi) = b^{on}_{o}(c) + 2\, b^{on}_{1}(c) + 0(\epsilon) \qquad \text{(n even)}$$

$$= 2\, b^{on}_{1}(c) + 0(\epsilon) \qquad \text{(n odd)}$$

but here again the differences are compensated by differences in $b^{on}_{1}(c)$ for n even and odd, and inserting the expresions for $b^{on}_{1}(c)$ (Flammer, equations 4.4.6a and b) we have

$$\frac{\partial}{\partial \xi} g_{on}(c, \xi) = \frac{1}{2} (\lambda_{on} - c^2) \left\{ b_o^{on}(c) - \frac{1}{c} \frac{k_{on}^{(1)}(c)}{c_o^{on}(c)} \right\} + 0(\epsilon) \tag{106}$$

for all n. Hence

$$A_{on} = \frac{\frac{ic}{2}(\lambda_{on} - c^2) \left\{ \frac{c_o^{on}(c)}{k_{on}^{(1)}(c)} \right\}^2}{\frac{1}{\xi^2 - 1} - \frac{1}{4}(\lambda_{on} - c^2) \log \frac{\xi + 1}{\xi - 1} + \frac{1}{2}(\lambda_{on} - c^2) \left\{ c\, b_o^{on}(c) \frac{c_o^{on}(c)}{k_{on}^{(1)}(c)} - 1 \right\} + 0(\epsilon)}$$

the singularities of which are given by

$$\frac{1}{\epsilon} - \frac{1}{2}(\lambda_{on} - c^2) \log \frac{2}{\epsilon} + (\lambda_{on} - c^2) \left\{ c\, b_o^{on}(c) \frac{c_o^{on}(c)}{k_{on}^{(1)}(c)} - 1 \right\} + 0(\epsilon) = 0 \; . \tag{108}$$

The first two terms of this are infinite when c=0, but owing to the markedly different rates at which $1/\epsilon$ and $\log \frac{2}{\epsilon}$ approach infinity as $\epsilon \rightarrow 0$, it is obvious that the only possibility of having a finite root is to have the third term become infinite, and this again leads us to the zeros of $c_o^{on}(c)$. Since the smallest zero is provided by $c_o^{oo}(c)$ and has a magnitude 4·1, the radius of convergence of the Rayleigh series for the hard spheroid must tend to 4·1 as $\epsilon \rightarrow 0$, and the convergence criterion (97) therefore applies to both soft and hard bodies. In each case the convergence is dictated by the amplitude coefficient A_{oo}, and for the coefficient A_{o1} the radius is approximately 5·2.

When ϵ is small but not zero, the variation in the radius of convergence as a function of ϵ can be determined in the same manner as for a soft spheroid. Bearing in mind that in equation (108) the dominant terms for small ϵ are the first and third, the root which corresponds to $c = x_n$ when $\epsilon \neq 0$ is

$$c = x_n \,(1 + \epsilon \tau_n') \tag{109}$$

where

$$\tau_n' = - \left[\frac{\left\{ \lambda_{on}(c) - c^2 \right\} X_{on}(c)}{c\, \partial / \partial c \; c_o^{on}(c)} \right]_{c = x_n}$$

and the smallest of these is obtained by taking n = 0 with x_o as shown in equation (99). The radius of convergence for A_{oo} is therefore

$$c\,\xi = 4{\cdot}1 \,(1 + \epsilon \; R_e \tau_o') \tag{110}$$

(cf equation 101), which is also the radius of convergence for the Rayleigh series providing ϵ is sufficiently small.

Using the formulae in Appendix A, τ_o' has been computed and its real part found to be

$$R_e \, \tau_o' = - \; 4{\cdot}2 \; .$$

Although the accuracy of this leaves much to be desired, the value is as good as can be obtained without including a significantly larger number of terms in the expansions (particularly for the $d_{\rho/r}$), and in consequence for a hard spheroid of ellipticity almost equal to unity the convergence criterion will be taken as

$$c\xi < 4\cdot 1\left\{1 - 4\cdot 2\,(\xi - 1)\right\}. \tag{111}$$

It will be observed that the radius differs negligibly from 4·1 for $\epsilon < 10^{-3}$, and as such the result is in marked constrast to that for a soft spheroid.

In order to bridge the gap between the ranges of ξ for which the criteria (19) and (23), (102) and (111) are applicable, it is necessary to resort to a numerical comparison of the actual coefficients in the expansions for the A_{on}. This is a simple matter for the hard spheroid, and since the amplitude A_{oo} specifies the convergence of the Rayleigh series as $\xi \sim \infty$ and $\xi \sim 1$, it is not surprising to find that this is true for all ξ. What is more, as ξ decreases from infinity the radius of convergence rapidly assumes the value indicated in (111).

For the soft spheroid, on the other hand, the convergence is determined by the amplitude A_{o1} for large ξ but by A_{oo} in the limit as $\xi \to 1$. and the value of ξ at which A_{oo} takes over differs from unity by an extremely small amount. In consequence there is a wide range of ξ for which no formula is available for calculating the convergence, and here the numerical approval is indispensable. We shall therefore begin by considering the case of the soft spheroid.

For a soft oblate spheroid the expansion of the amplitude coefficient A_{o1} is given in equation (80) and from this the analogous result for a prolate spheroid can be obtained by changing c into ic and ξ into $-i\xi$. The expansion then proceeds in powers of (-c) and the coefficients have been calculated for a sequence of ξ ranging from 1·7 down to 1·00001. These are listed in Table X together with the convergence coefficients $|a_r^1|$ and $\xi\,|a_r^1|$ which can be deduced therefrom, and the last-named are also plotted in Figure VIII. The regularity of the curves is at once apparent and it is interesting to compare them with the curves for the oblate coefficient A_{o1} (see Figure IV). As the oblate ξ increases from zero the oscillations rapidly die down, and this process continues systematically as ξ passes through infinity and then decreases through prolate values to unity. Such a correspondence between the two sets of curves is typical of the amplitude coefficients A_{oo} and A_{o1} for both the soft and hard bodies, and may well be true of the A_{on} in general.

The radius of convergence for a given ξ is represented by the limit of the curve $\xi\,|a_r^1|$ against r as $r \longrightarrow \infty$, and notwithstanding the fact that r = 9 is the largest value for which numerical data is available, it is possible to estimate the limit by comparing the $\xi\,|a_r^1|$ with the convergence coefficients for the sphere ($\xi = \infty$). The radii of convergence obtained in this manner are shown in Table II, and it is believed that they are in error by no more than 2 %.

TABLE XI. RADIUS OF CONVERGENCE FOR A_{01} (SOFT)

ξ	1.7	1.5	1.2	1.1	1.05	1.01	1.001	1.0001	1.00001
$c\xi$	1.09	1.12	1.24	1.35	1.47	1.72	1.98	2.15	2.31

TABLE X. CONVERGENCE COEFFICIENTS FOR A_{01} (SOFT)

r	ξ = 1.7			ξ = 1.5			ξ = 1.2		
	α_r	$\lvert a_r^1\rvert$	$\xi\lvert a_r^1\rvert$	α_r	$\lvert a_r^1\rvert$	$\xi\lvert a_r^1\rvert$	α_r	$\lvert a_r^1\rvert$	$\xi\lvert a_r^1\rvert$
2	-1.35650	0.85860	1.4596	-9.65819×10^{-1}	1.0175	1.5263	-4.55854×10^{-1}	1.4811	1.7773
3	$i1.28114$	0.92073	1.5652	$i8.04850\times10^{-1}$	1.0751	1.6126	$i3.03903\times10^{-1}$	1.4874	1.7849
4	2.41991	0.80177	1.3630	1.27498	0.94107	1.4116	3.27106×10^{-1}	1.3223	1.5867
5	$-i3.62948$	0.77274	1.3137	$-i1.65126$	0.90456	1.3568	$-i3.13539\times10^{-1}$	1.2611	1.5133
6	-5.74246	0.74728	1.2704	-2.23329	0.87467	1.3120	-3.05722×10^{-1}	1.2184	1.4620
7	$i8.98530$	0.73077	1.2423	$i2.99614$	0.85491	1.2824	$i2.97658\times10^{-1}$	1.1890	1.4268
8	1.40864×10	0.71846	1.2214	4.02445	0.84026	1.2604	2.89798×10^{-1}	1.1675	1.4009
9	$-i2.20773\times10$	0.70904	1.2054	$-i5.40489$	0.82905	1.2436	$-i2.82153\times10^{-1}$	1.1510	1.3811

r	ξ = 1.1			ξ = 1.05			ξ = 1.01		
	α_r	$\lvert a_r^1\rvert$	$\xi\lvert a_r^1\rvert$	α_r	$\lvert a_r^1\rvert$	$\xi\lvert a_r^1\rvert$	α_r	$\lvert a_r^1\rvert$	$\xi\lvert a_r^1\rvert$
2	-2.96522×10^{-1}	1.8364	2.0201	-2.10609×10^{-1}	2.1790	2.2880	-1.19178×10^{-1}	2.8967	2.9257
3	$i1.81208\times10^{-1}$	1.7672	1.9439	$i1.22855\times10^{-1}$	2.0116	2.1121	$i6.68718\times10^{-2}$	2.4637	2.4883
4	1.57735×10^{-1}	1.5870	1.7455	9.13523×10^{-2}	1.8190	1.9099	3.96746×10^{-2}	2.2406	2.2630
5	$-i1.29209\times10^{-1}$	1.5057	1.6563	$-i6.64917\times10^{-2}$	1.7197	1.8057	$-i2.39639\times10^{-2}$	2.1091	2.1302
6	-1.06255×10^{-1}	1.4530	1.5983	-4.82729×10^{-2}	1.6572	1.7401	-1.44379×10^{-2}	2.0265	2.0468
7	$i8.74495\times10^{-2}$	1.4164	1.5580	$i3.50924\times10^{-2}$	1.6137	1.6944	$i8.70589\times10^{-2}$	1.9693	1.9890
8	7.19562×10^{-2}	1.3895	1.5285	2.55083×10^{-2}	1.5819	1.6609	5.25093×10^{-2}	1.9274	1.9467
9	$-i5.92080\times10^{-2}$	1.3690	1.5059	$-i1.85407\times10^{-2}$	1.5575	1.6354	$-i3.16664\times10^{-2}$	1.8955	1.9144

r	ξ = 1.001			ξ = 1.0001			ξ = 1.00001		
	α_r	$\lvert a_r^1\rvert$	$\xi\lvert a_r^1\rvert$	α_r	$\lvert a_r^1\rvert$	$\xi\lvert a_r^1\rvert$	α_r	$\lvert a_r^1\rvert$	$\xi\lvert a_r^1\rvert$
2	-7.13139×10^{-2}	3.7447	3.7484	-5.06039×10^{-2}	4.4454	4.4458	-3.91919×10^{-2}	5.0513	5.0513
3	$i3.96585\times10^{-2}$	2.9324	2.9353	$i2.81161\times10^{-2}$	3.2886	3.2890	$i2.17735\times10^{-2}$	3.5812	3.5812
4	2.01798×10^{-2}	2.6532	2.6559	1.32610×10^{-2}	2.9468	2.9471	9.82237×10^{-3}	3.1765	3.1765
5	$-i1.04154\times10^{-2}$	2.4915	2.4940	$-i6.21950\times10^{-3}$	2.7622	2.7624	$-i4.31950\times10^{-3}$	2.9711	2.9711
6	-5.39444×10^{-3}	2.3879	2.3902	-2.92904×10^{-3}	2.6437	2.6440	-1.90090×10^{-3}	2.8412	2.8412
7	$i2.79962\times10^{-3}$	2.3158	2.3181	$i1.38501\times10^{-3}$	2.5607	2.5610	$i8.40880\times10^{-4}$	2.7500	2.7500
8	1.45428×10^{-3}	2.2629	2.2652	6.56370×10^{-4}	2.4995	2.4998	3.73397×10^{-4}	2.6821	2.6821
9	$-i7.55322\times10^{-3}$	2.2227	2.2249	$-i3.11101\times10^{-4}$	2.4529	2.4531	$-i1.65941\times10^{-4}$	2.6303	2.6303

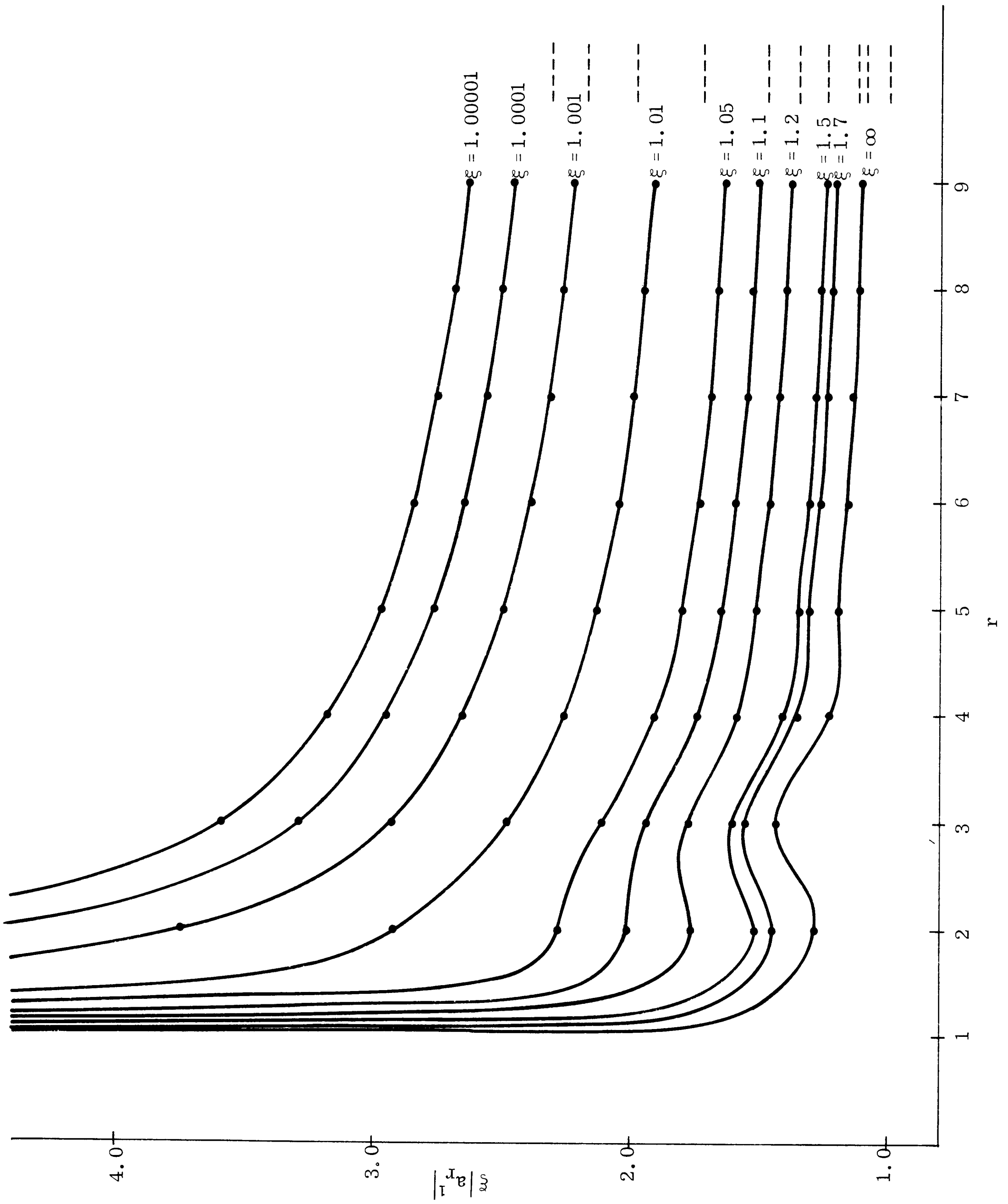

FIGURE VIII. CONVERGENCE COEFFICIENTS FOR SOFT PROLATE A_{o1}

Even when ξ is as small as $1 \cdot 00001$ the radius of convergence is still only $2 \cdot 3$, and this is far short of the value $5 \cdot 2$ (corresponding to the smallest zero of c_o^{o1}) which is reached when $\xi = 1$. It is also much less than the radius for A_{oo} and in order to determine the convergence for A_{o1} out to at least the neighbourhood of the cross-over point there are two possible methods of attack. The first of these would appear to be the most logical and is based on the formula (98). By taking $n = 1$ and $x_1 = 5 \cdot 2 \exp\left\{ \pm i\, 0 \cdot 083\, \pi \right\}$, the formula gives immediately the decrease in the radius of convergence as a function of ξ in terms of the factor τ_1, but unfortunately the expansions in Appendix A are not sufficient to compute $X_{o1}(c)$ when $|c|$ is as large as $5 \cdot 2$. In addition, there is reason to doubt the validity of (98) unless $\xi - 1$ is vanishingly small (less than, perhaps, 10^{-50}). We shall have more to say about this in a moment.

The second method is merely to extend Table X to smaller values of ξ and in this connection the computation can be simplified somewhat by analyzing the behaviour of the α_r as $\xi \rightarrow 1$. From the expressions for the Legendre functions $P_n(\xi)$ and $Q_n(\xi)$, we have

$$P_n(\xi) = 1 + 0(\epsilon)$$

$$Q_n(\xi) = \frac{1}{\delta} + \sum_{r=1}^{n} \frac{1}{r} + 0(\epsilon)$$

for small ϵ where $\delta = 2/\log \frac{2}{\epsilon}$, and hence

$$B_2 = \frac{1}{5}\delta + 0(\epsilon\,\delta)$$

$$B_3 = -\frac{i}{3^2}\delta + 0(\epsilon\,\delta)$$

$$B_4 = -\frac{37}{5^3\,7}\delta + 0(\epsilon\,\delta)$$

$$B_5 = \frac{i}{3.\,5^2}\delta + 0(\epsilon\,\delta)$$

$$B_6 = \frac{6484}{3^4\,5^5\,7}\delta + 0(\epsilon\,\delta)$$

$$B_7 = -i\,\frac{82}{3.\,5^4.\,7^2}\delta + 0(\epsilon\,\delta)$$

$$B_8 = -\frac{312401}{3^3\,5^6\,7^3.\,11}\delta + 0(\epsilon\,\delta)$$

$$B_9 = i\,\frac{3121}{3^6\,5^6\,7}\delta + 0(\epsilon\,\delta)\,.$$

The coefficients α_r are therefore

$$\alpha_2 = -\frac{1}{5}\delta + 0(\epsilon\,\delta)$$

$$\alpha_3 = \frac{i}{3^2}\delta + 0(\epsilon\,\delta)$$

$$\alpha_4 = \frac{37}{5^3 7}\delta + \frac{1}{5^2}\delta^2 + 0(\epsilon\,\delta)$$

$$\alpha_5 = -\frac{i}{3.\,5^2}\delta - i\,\frac{2}{3^2\,5}\delta^2 + 0(\epsilon\,\delta)$$

$$\alpha_6 = -\frac{6484}{3^4\,5^5\,7}\delta - \frac{10369}{3^4\,5^4\,7}\delta^2 - \frac{1}{5^3}\delta^3 + 0(\epsilon\,\delta)$$

$$\alpha_7 = i \frac{82}{3\,5^4 7^2} \delta + i \frac{116}{3^2 5^3 7} \delta^2 + i \frac{1}{3.5^2} \delta^3 + 0(\epsilon \delta)$$

$$\alpha_8 = \frac{312401}{3^3 5^6 7^3 11} \delta + \frac{77083}{3^4 5^5 7^2} \delta^2 + \frac{7372}{3^3 5^5 7} \delta^3 + \frac{1}{5^4} \delta^4 + 0(\epsilon \delta)$$

$$\alpha_9 = - i \frac{3121}{3^6 5^6 7} \delta - i \frac{256502}{3^6 5^5 7^2} \delta^2 - i \frac{5492}{3^6 5^3 7} \delta^3 - i \frac{4}{3^2 5^3} \delta^4 + 0(\epsilon \delta)$$

from which the convergence coefficients $|\alpha_r^1|$ can be determined as before. By comparing the $|a_r^1|$ for $\epsilon = 10^{-5}$ with the values shown in Table X it is found that the error produced by the terms in $\epsilon \delta$ in less than 5% for $r > 3$ and decreases rapidly with ϵ.

Using the above expressions the α_r have been computed for $\xi = 1 + 10^{-2m}$, m = 3, 4(2) 12 (4) 20, and these, together with the $|a_r^1|$, are shown in Table XII. The corresponding radii of convergence are given in Table XIII, and though it is difficult to estimate their accuracy, it is believed that the error is not more than 5 % at the very most.

TABLE XIII. RADIUS OF CONVERGENCE FOR A_{o1} (SOFT)

ϵ	10^{-6}	10^{-8}	10^{-12}	10^{-16}	10^{-20}	10^{-24}	10^{-32}	10^{-40}
$c\xi$	2.45	2.62	2.90	3.12	3.30	3.44	3.66	3.81

TABLE XII. CONVERGENCE COEFFICIENTS FOR A_{o1} (SOFT)

	$\epsilon = 10^{-6}$		$\epsilon = 10^{-8}$		$\epsilon = 10^{-12}$		$\epsilon = 10^{-16}$	
r	α_r	$\lvert a_r^1 \rvert$	α_r	$\lvert a_r^1 \rvert$	α_r	$\lvert a_r^1 \rvert$	α_r	$\lvert a_r^1 \rvert$
2	-2.75697×10^{-2}	6.0226	-2.09273×10^{-2}	6.9126	-1.41222×10^{-2}	8.4149	-1.06569×10^{-3}	9.6869
3	$i1.53165 \times 10^{-2}$	4.0267	$i1.16263 \times 10^{-2}$	4.4142	$i7.84568 \times 10^{-3}$	5.0326	$i5.92048 \times 10^{-3}$	5.5277
4	6.58912×10^{-3}	3.5099	4.86257×10^{-3}	3.7869	3.18528×10^{-3}	4.2093	2.36673×10^{-3}	4.5338
5	$-i2.68253 \times 10^{-3}$	3.2681	$-i1.88176 \times 10^{-3}$	3.5082	$-i1.16308 \times 10^{-3}$	3.8626	$-i8.36645 \times 10^{-4}$	4.1256
6	-1.08140×10^{-3}	3.1213	-7.12431×10^{-4}	3.3461	-4.07099×10^{-4}	3.6733	-2.79274×10^{-4}	3.9114
7	$i4.37865 \times 10^{-4}$	3.0186	$i2.69942 \times 10^{-4}$	3.2346	$i1.41159 \times 10^{-4}$	3.5485	$i9.13964 \times 10^{-5}$	3.7758
8	1.78423×10^{-4}	2.9415	1.03072×10^{-4}	3.1503	4.92791×10^{-5}	3.4548	3.00042×10^{-5}	3.6758
9	$-i7.28965 \times 10^{-5}$	2.8820	$-i3.95440 \times 10^{-5}$	3.0847	$-i1.73403 \times 10^{-5}$	3.3806	$-i9.94091 \times 10^{-6}$	3.5962

	$\epsilon = 10^{-20}$		$\epsilon = 10^{-24}$		$\epsilon = 10^{-32}$		$\epsilon = 10^{-40}$	
r	α_r	$\lvert a_r^1 \rvert$	α_r	$\lvert a_r^1 \rvert$	α_r	$\lvert a_r^1 \rvert$	α_r	$\lvert a_r^1 \rvert$
2	-8.55709×10^{-3}	10.810	-7.14858×10^{-3}	11.828	-5.37809×10^{-3}	13.636	-4.31051×10^{-3}	15.231
3	$i4.75394 \times 10^{-3}$	5.9472	$i3.97143 \times 10^{-3}$	6.3147	$i2.98783 \times 10^{-3}$	6.9430	$i2.39473 \times 10^{-3}$	7.4745
4	1.88244×10^{-3}	4.8009	1.56252×10^{-3}	5.0295	1.16601×10^{-3}	5.4115	9.29945×10^{-4}	5.7265
5	$-i6.51833 \times 10^{-4}$	4.3368	$-i5.33352 \times 10^{-4}$	4.5144	$-i3.90677 \times 10^{-4}$	4.8044	$-i3.08012 \times 10^{-4}$	5.0383
6	-2.10759×10^{-4}	4.0992	-1.68544×10^{-4}	4.2548	-1.19716×10^{-4}	4.5044	-9.32618×10^{-5}	4.6958
7	$i6.61960 \times 10^{-5}$	3.9539	$i5.13286 \times 10^{-5}$	4.1003	$i3.49108 \times 10^{-5}$	4.3323	$i2.62118 \times 10^{-5}$	4.5133
8	2.07570×10^{-5}	3.8491	1.55274×10^{-5}	3.9913	1.01505×10^{-5}	4.2091	7.24215×10^{-6}	4.3906
9	$-i6.56738 \times 10^{-6}$	3.7657	$-i4.73362 \times 10^{-6}$	3.9053	$-i2.88338 \times 10^{-6}$	4.1264	$-i2.00485 \times 10^{-6}$	4.2963

From a study of these results it is seen that the radius is still increasing even at the smallest value of ϵ, a fact which is otherwise obvious from the form of the expressions for the α_r. On the other hand, not all of the change in the α_r is automatically reflected in an increase in the radius, and for sufficiently small δ the changes in the α_r for r less than some fixed number have no effect on the convergence. This is most easily seen by dividing the α_r by δ and examining the convergence indicated by the remaining coefficients. As r increases, the contribution of the higher powers of δ becomes more important due to their relatively larger coefficients, and even for $r \leqslant 9$ the terms in δ^2 may still dominate when $\epsilon = 10^{-40}$. Thus, for $r = 9$ the ratio of the δ^2 and δ contributions is $1 \cdot 3$ when $\epsilon = 10^{-40}$ and does not fall to $0 \cdot 1$ until $\epsilon = 10^{-507}$!

Under these circumstances it is questionable whether a formula such as (98) in which terms in δ^2 are neglected can be expected to hold unless ϵ is extremely small, and it is therefore not too surprising to find that even for $\epsilon = 10^{-40}$ the rate at which the radius increases is not yet consistent with a formula of the type

$$5 \cdot 2 \, (1 - \tau\delta).$$

Ultimately, however, the radius must assume this dependence on δ, but it may well be necessary for ϵ to be appreciably smaller than the values considered in Tables XII and XIII, and possibly of order 10^{-500}. To pursue the analysis of the convergence to such values would be a trifle academic.

In view of the above results it is clear that the radius of convergence for the amplitude coefficient A_{oo} is of no concern until ϵ becomes extremely small (less than, perhaps, 10^{-70}) and one may hope that the criterion (102) is then applicable. Nevertheless, in the interests of completeness we have computed the coefficients in the expansion of A_{oo} in powers of (-ic) for $\xi = 1 \cdot 7,\ 1 \cdot 5,\ 1 \cdot 2$ and $1 \cdot 1$, and the results are displayed in Table XIV. The convergence coefficients $\xi | a_r^o |$ are plotted in Figure IX, and the extent to which the curves resemble those for the oblate body (see Figure VI) is quite striking. In the prolate case, however, the curves turn over somewhat sooner and minima occur for $r \leqslant 9$ with the smaller values of ξ. Although it is impossible to obtain any reliable estimates of the convergence from these curves, the radius would appear to be of order 4 for $\xi = 1 \cdot 2$ and $1 \cdot 1$.

For the hard body, the problem of finding the radius of convergence of the Rayleigh series is more straight forward. We have already seen that when ξ is large or near to unity the convergence is determined by the amplitude coefficient A_{oo} and as $\xi \rightarrow 1$ the radius rapidly approaches its limiting value $4 \cdot 1$. Since the corresponding limit for A_{o1} is $5 \cdot 2$ and is approached at a comparable rate, it is natural to expect that the coefficient A_{oo} will specify the convergence for all ξ, and this is indeed the case.

TABLE XIV. CONVERGENCE COEFFICIENTS FOR A_{oo} (SOFT)

r	$\xi = 1.7$			$\xi = 1.5$		
r	α_r	$\lvert a_r^o \rvert$	$\xi \lvert a_r^o \rvert$	α_r	$\lvert a_r^o \rvert$	$\xi \lvert a_r^o \rvert$
1	-i1.48156	0.6750	1.1474	-i1.24267	0.8047	1.2071
2	- 1.35548	0.8589	1.4602	-9.22893×10^{-1}	1.0409	1.5614
3	$i9.28991 \times 10^{-1}$	1.0249	1.7423	$i5.12812 \times 10^{-1}$	1.2493	1.8740
4	4.94757×10^{-1}	1.1924	2.0270	2.18152×10^{-1}	1.4632	2.1948
5	$-i2.11560 \times 10^{-1}$	1.3643	2.3193	$-i7.25571 \times 10^{-2}$	1.6899	2.5349
6	-7.39309×10^{-2}	1.5436	2.6241	-1.88449×10^{-2}	1.9385	2.9078
7	$i2.10389 \times 10^{-2}$	1.7361	2.9513	$i3.55228 \times 10^{-3}$	2.2384	3.3575
8	4.72423×10^{-3}	1.9530	3.3202	3.39789×10^{-4}	2.7139	4.0709
9	$-i7.41712 \times 10^{-4}$	2.2272	3.7862	$-i6.30718 \times 10^{-5}$	2.9288	4.3932
	$\xi = 1.2$			$\xi = 1.1$		
r	α_r	$\lvert a_r^o \rvert$	$\xi \lvert a_r^o \rvert$	α_r	$\lvert a_r^o \rvert$	$\xi \lvert a_r^o \rvert$
1	$-i8.34063 \times 10^{-1}$	1.1990	1.4387	$-i6.56918 \times 10^{-1}$	1.5223	1.6745
2	-3.62036×10^{-1}	1.6620	1.9944	-1.90671×10^{-1}	2.2901	2.5191
3	$i1.16371 \times 10^{-1}$	2.0483	2.4579	$i4.00143 \times 10^{-2}$	2.9237	3.2160
4	2.62887×10^{-2}	2.4835	2.9802	4.91704×10^{-3}	3.7764	4.1540
5	$-i3.37091 \times 10^{-3}$	3.1221	3.7466	$-i3.95666 \times 10^{-4}$	4.7922	5.2714
6	-7.91241×10^{-5}	4.8263	5.7916	-2.60938×10^{-4}	3.9559	4.3515
7	$i2.04309 \times 10^{-4}$	3.3659	4.0391	$i8.06818 \times 10^{-5}$	3.8437	4.2280
8	6.22227×10^{-5}	3.3555	4.0266	1.15107×10^{-5}	4.1435	4.5578
9	$-i9.79192 \times 10^{-6}$	3.6022	4.3227	$-i7.58100 \times 10^{-7}$	4.7866	5.2653

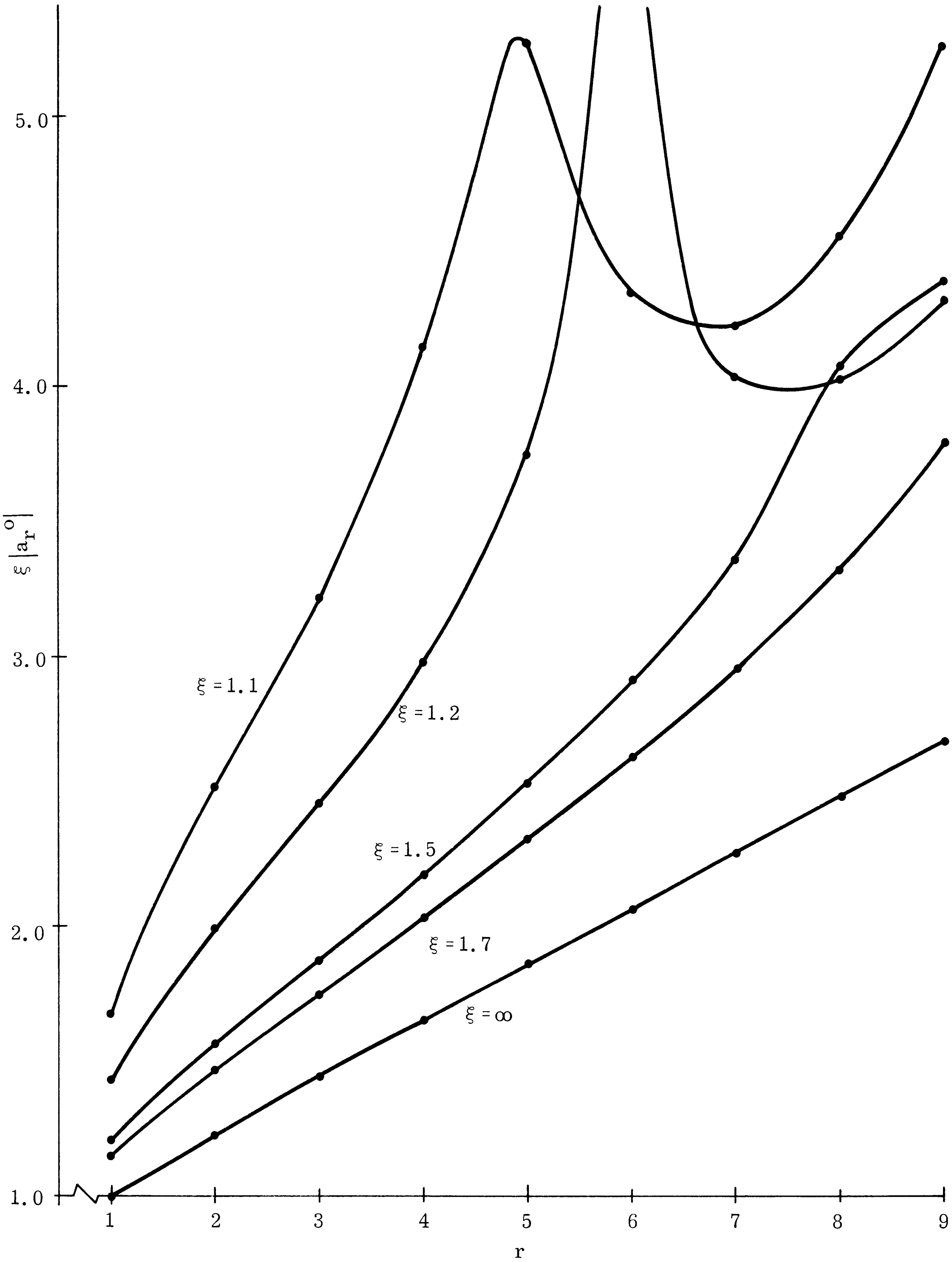

FIGURE IX. CONVERGENCE COEFFICIENTS FOR SOFT PROLATE A_{oo}

The expansion of the oblate coefficient A_{oo} is given in equation (28), and by changing c into ic and ξ into $-i\xi$ the analogous result for the prolate coefficient is obtained. The expansion then proceeds in powers of (-c) and the coefficients α_r have been computed for $\xi = 1{\cdot}7$, $1{\cdot}5$, $1{\cdot}2$, $1{\cdot}1$, $1{\cdot}05$ and $1{\cdot}01$. These are listed in Table XV, together with the convergence coefficients $|a_r^o|$ and $\xi\,|a_r^o|$ deduced therefrom. The last named are plotted in Figure X. Once again the similarity of the curves for the prolate and oblate coefficients (Figure VII) is apparent, and by comparing the coefficients with those for the sphere ($\xi=\infty$) the radius of convergence has been estimated as shown in Table XVI.

TABLE XVI. RADIUS OF CONVERGENCE FOR A_{oo} (HARD)

ξ	1·7	1·5	1·2	1·1	1·05	1·01
$c\xi$	1·16	1·22	1·48	1·76	2·10	2·95

The error in these values is probably less than 2%, but as ξ decreases the less regular nature of the curves may increase the error to some extent. In particular, for $\xi = 1{\cdot}01$ the radius could be out by as much as 5%, and for $\xi = 1{\cdot}001$ the $\xi\,|a_r^o|$ with $r \leqslant 7$ are too scattered to give any reliable indication of the convergence.

TABLE XV. CONVERGENCE COEFFICIENTS FOR A_{oo} (HARD)

r	$\xi = 1.7$			$\xi = 1.5$			$\xi = 1.2$		
	α_r	a_r^o	$\xi\|a_r^o\|$	α_r	$\|a_r^o\|$	$\xi\|a_r^o\|$	α_r	$\|a_r^o\|$	$\xi\|a_r^o\|$
2	-1.20466	0.9111	1.5489	-8.14060×10^{-1}	1.1083	1.6625	-3.06126×10^{-1}	1.8074	2.1689
3	i 1.07100	0.9774	1.6616	$i\,6.25000 \times 10^{-1}$	1.1696	1.7544	$i\,1.76000 \times 10^{-1}$	1.7844	2.1413
4	1.90860	0.8508	1.4463	9.10846×10^{-1}	1.0236	1.5354	1.58776×10^{-1}	1.5842	1.9010
5	-i 2.69938	0.8199	1.3938	-i1.08702	0.9835	1.4752	$-i1.27312 \times 10^{-1}$	1.5102	1.8122
6	-4.02697	0.7928	1.3478	-1.35410	0.9507	1.4261	-1.05323×10^{-1}	1.4552	1.7462
7	i 5.94133	0.7753	1.3179	i 1.67290	0.9291	1.3937	$i\,8.63545 \times 10^{-2}$	1.4189	1.7027

r	$\xi = 1.1$			$\xi = 1.05$			$\xi = 1.01$		
	α_r	$\|a_r^o\|$	$\xi\|a_r^o\|$	α_r	$\|a_r^o\|$	$\xi\|a_r^o\|$	α_r	$\|a_r^o\|$	$\xi\|a_r^o\|$
2	-1.50992×10^{-1}	2.5735	2.8308	-7.17233×10^{-2}	3.7340	3.9207	-1.08151×10^{-3}	30.4078	30.7119
3	$i\,7.70001 \times 10^{-2}$	2.3506	2.5856	$i\,3.58750 \times 10^{-2}$	3.0321	3.1837	$i\,6.76700 \times 10^{-3}$	5.2869	5.3398
4	5.33776×10^{-2}	2.0805	2.2885	2.28932×10^{-2}	2.5708	2.6994	1.03552×10^{-2}	3.1348	3.1661
5	$-i3.18084 \times 10^{-2}$	1.9929	2.1922	$-i9.13226 \times 10^{-3}$	2.5579	2.6858	$-i7.66526 \times 10^{-4}$	4.1985	4.2405
6	-2.07826×10^{-2}	1.9071	2.0979	-4.77564×10^{-3}	2.4369	2.5587	-5.91949×10^{-4}	3.4511	3.4856
7	$i\,1.34339 \times 10^{-2}$	1.8510	2.0361	$i\,2.80640 \times 10^{-3}$	2.3150	2.4308	$i\,2.18641 \times 10^{-4}$	3.3335	3.3668

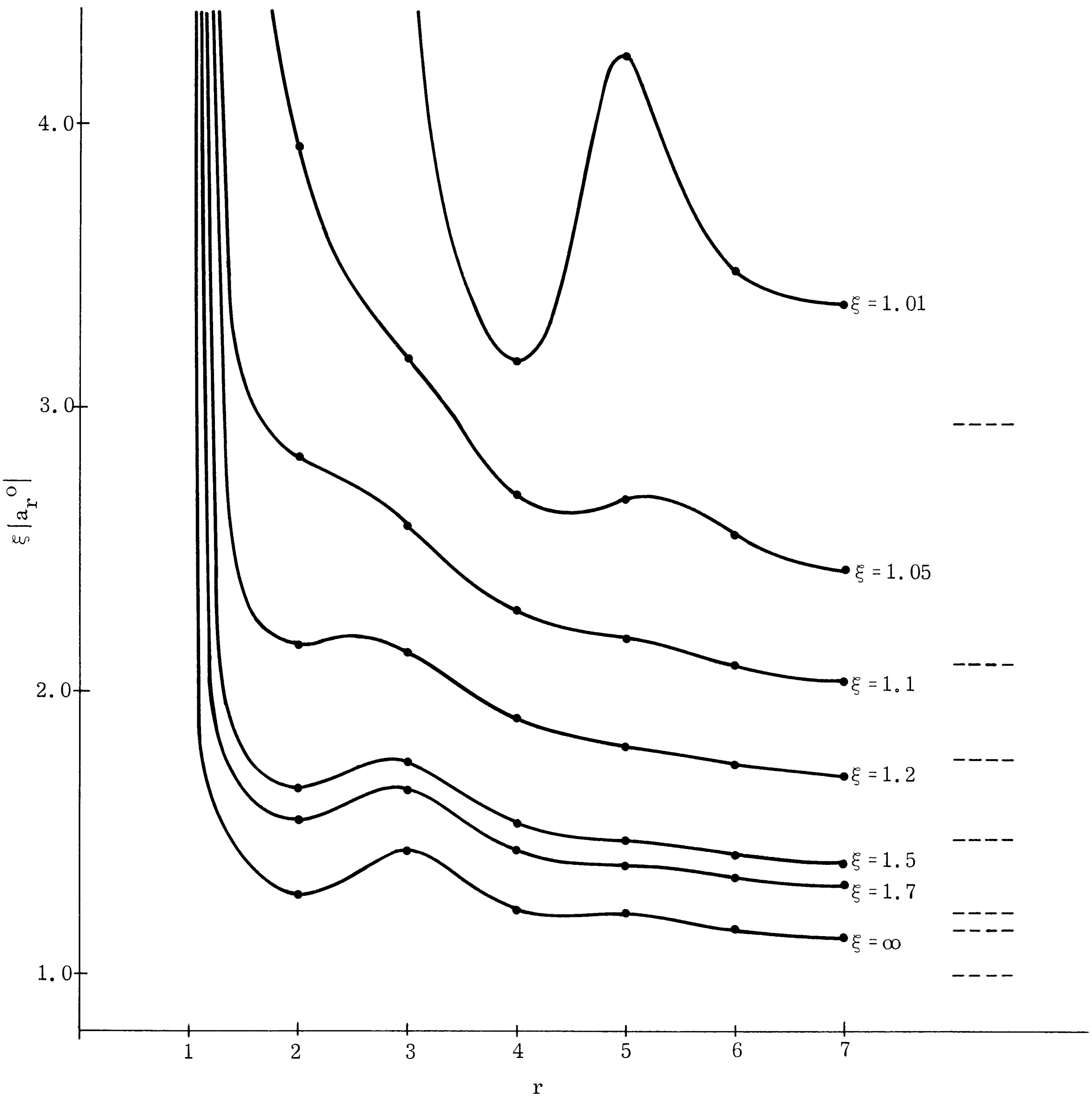

FIGURE X. CONVERGENCE COEFFICIENTS FOR HARD PROLATE A_{oo}

To complete the discussion of the hard spheroid it is only necessary to consider the convergence of the expansion for A_{o1} and verify that the radius is nowhere less than the radius for A_{oo}. This is known to be true for ξ large or near to unity, and for the intermediate range the analysis in §7 can be used to compute the coefficients α_r in the expansion of A_{o1} in powers of (-c). The results are given in Table XVII and the convergence coefficients $\xi \left| a_r^1 \right|$ are plotted in Figure XI. As in the case of the oblate spheroid (cf Figure V), the curves are characterized by an irregular set of peaks which make difficult any accurate estimate of the convergence, but by comparing the levels of the minima with those for the sphere curve, the values shown in Table XVIII have been deduced.

TABLE XVIII. RADIUS OF CONVERGENCE FOR A_{o1} (HARD)

ξ	1·7	1·5	1·2	1·1
$c\xi$	1·56	1·62	1·85	2·09

Although the errors associated with these results are impossible to assess, the radii are in good agreement with the formula for large ξ, and their trend is not inconsistent with the general formula for the convergence when ξ is close to unity.

TABLE XVII. CONVERGENCE COEFFICIENTS FOR A_{o1} (HARD)

r	$\xi = 1.7$			$\xi = 1.5$		
r	α_r	$\|a_r^1\|$	$\xi\|a_r^1\|$	α_r	$\|a_r^1\|$	$\xi\|a_r^1\|$
2	-8.01283×10^{-1}	1.1171	1.8991	-6.07163×10^{-1}	1.2834	1.9250
3	$-i4.94909 \times 10^{-1}$	1.2642	2.1492	$-i2.81094 \times 10^{-1}$	1.5266	2.2899
4	-5.27131×10^{-1}	1.1736	1.9951	-2.61935×10^{-1}	1.3978	2.0967
5	$i8.52514 \times 10^{-1}$	1.0324	1.7551	$i3.75071 \times 10^{-1}$	1.2167	1.8250
6	4.96922×10^{-1}	1.1236	1.9102	2.09175×10^{-1}	1.2979	1.9469
7	$i1.04854 \times 10^{-1}$	1.3801	2.3462	$i4.13844 \times 10^{-4}$	3.0431	4.5646
8	7.49680×10^{-1}	1.0367	1.7623	1.73763×10^{-1}	1.2445	1.8668
9	$-i1.04910$	0.9947	1.6910	$-i2.31610 \times 10^{-1}$	1.1765	1.7647

r	$\xi = 1.2$			$\xi = 1.1$		
r	α_r	$\|a_r^1\|$	$\xi\|a_r^1\|$	α_r	$\|a_r^1\|$	$\xi\|a_r^1\|$
2	-3.56897×10^{-1}	1.6739	2.0087	-2.81934×10^{-1}	1.8833	2.0717
3	$-i7.27010 \times 10^{-2}$	2.3960	2.8752	$-i2.99021 \times 10^{-2}$	3.2218	3.5440
4	-5.33162×10^{-2}	2.0811	2.4973	-2.06402×10^{-2}	2.6383	2.9021
5	$i6.06176 \times 10^{-2}$	1.7518	2.1021	$i2.04491 \times 10^{-2}$	2.1770	2.3947
6	3.19757×10^{-2}	1.7750	2.1300	1.06186×10^{-2}	2.1330	2.3463
7	$-i8.31925 \times 10^{-3}$	1.9821	2.3785	$-i3.40594 \times 10^{-3}$	2.2518	2.4770
8	4.15955×10^{-3}	1.9844	2.3812	-1.41363×10^{-4}	3.0284	3.3312
9	$-i7.17694 \times 10^{-3}$	1.7307	2.0769	$-i7.26718 \times 10^{-4}$	2.2322	2.4554

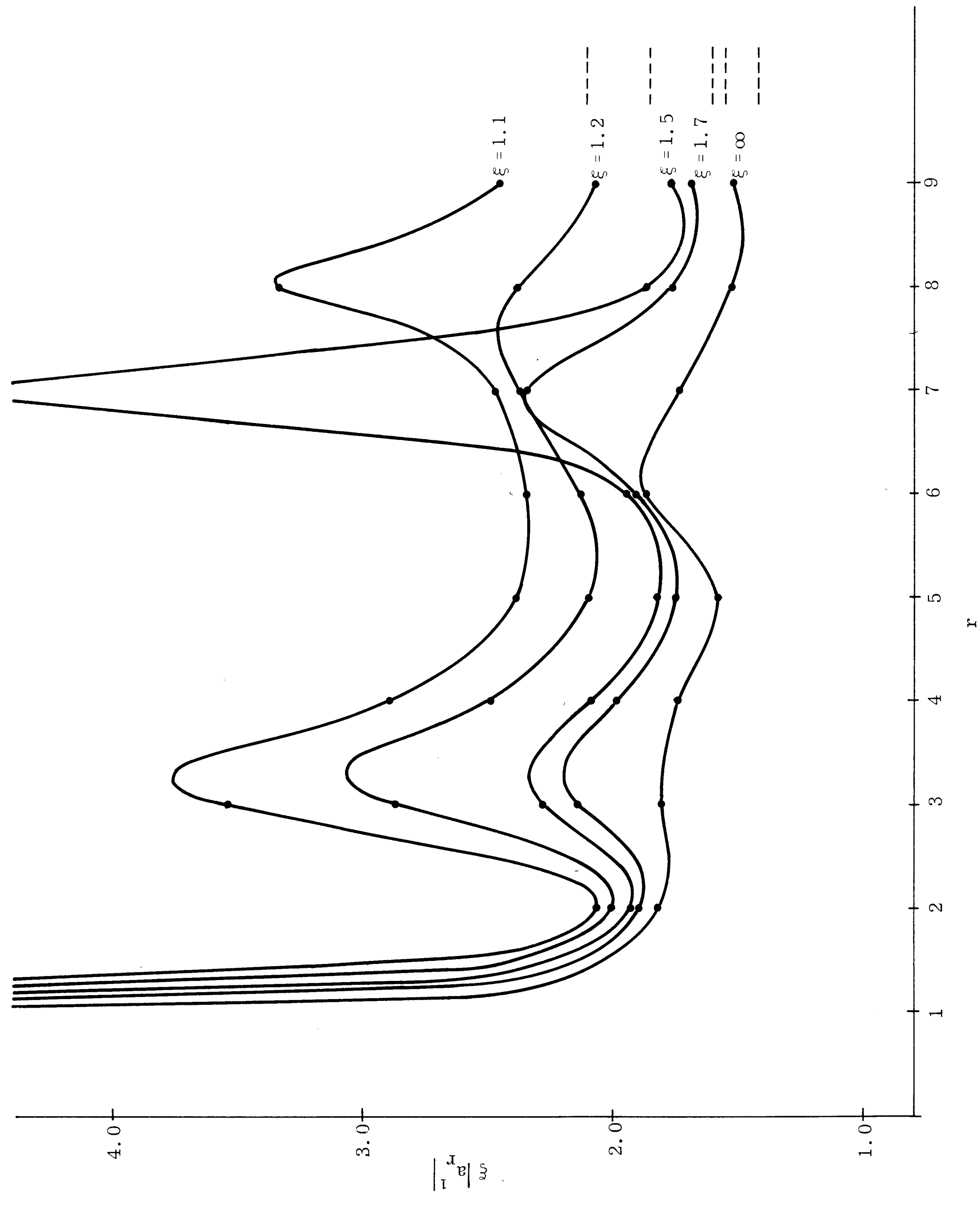

FIGURE XI. CONVERGENCE COEFFICIENTS FOR HARD PROLATE A_{o1}

IX

DISCUSSION OF RESULTS

The analysis in the previous sections is sufficient to specify the radius of convergence of the Rayleigh series for a spheriod when a plane wave is incident nose-on, and we shall now gather together the results of the various calculations. Before doing so, however, a few words are necessary about the style of presentation to be adopted.

The quantity of interest is, of course, the radius of convergence and this is inversely proportional to the wavelength. Since it is also dimensionless, it must be proportional to c, and the factor of proportionality is then at most a function of the spheroid's shape. From the physical point of view a convenient choice of factor is one which associates the radius with the semi-major axis of the spheroid and this has the advantage of being non-zero for all ellipticities. The radius of convergence is therefore defined as the limiting value of either $c\xi$ or $c(\xi^2+1)^{1/2}$ for which convergence exists, with the first and second applying to prolate and oblate spheroids respectively, and in any graphical presentation it is natural to choose this as the ordinate.

Having taken the frequency dependence into account via the choice of ordinate, it is desirable to have the abscissa independent of k (and hence c) and a function only of the variable ξ specifying the spheroid. In addition, the abscissae for prolate and oblate bodies should correspond to one another as much as possible, but because of the different ranges of the variable it is apparent that the mathematical description of the two scales will almost certainly differ. The formal analogy between the two types of body is one in which ξ^2 is replaced by $-\xi^2$, suggesting that the horizontal scale should be a function of ξ^2 rather than of ξ, and ultimately log $(1 \mp 1/\xi^2)^{1/2}$ was chosen as the function, where the upper and lower signs refer to prolate and oblate respectively. This satisfies all the required conditions, and since $(1 \mp 1/\xi^2)^{1/2} = \frac{w}{\ell}$ where w and ℓ are the maximum dimensions perpendicular and parallel to the direction of the incident field, the abscissa is continuous through the transition from prolate to oblate bodies.

Turning now to the actual results, it will be recalled that for both types of spheriod, hard as well as soft, the radius of convergence of the Rayleigh series is specified by the convergence of the expansions for one or other of the amplitude coefficients A_{oo} and A_{o1}. In Figure XII the radii of these expansions for a soft oblate spheriod are presented. When ξ^2 is much greater than unity, the convergence of the A_{o1} expansion can be obtained from the formula (26), and it will be observed that the curve goes smoothly into the values deduced from a numerical comparison of the coefficients in the expansion and listed in Table IV. The resulting graph is almost a straight line on the logarithmic plot, but ultimately a small amount of curvature becomes apparent ($\xi < 0.1$ approx) and the radius finally

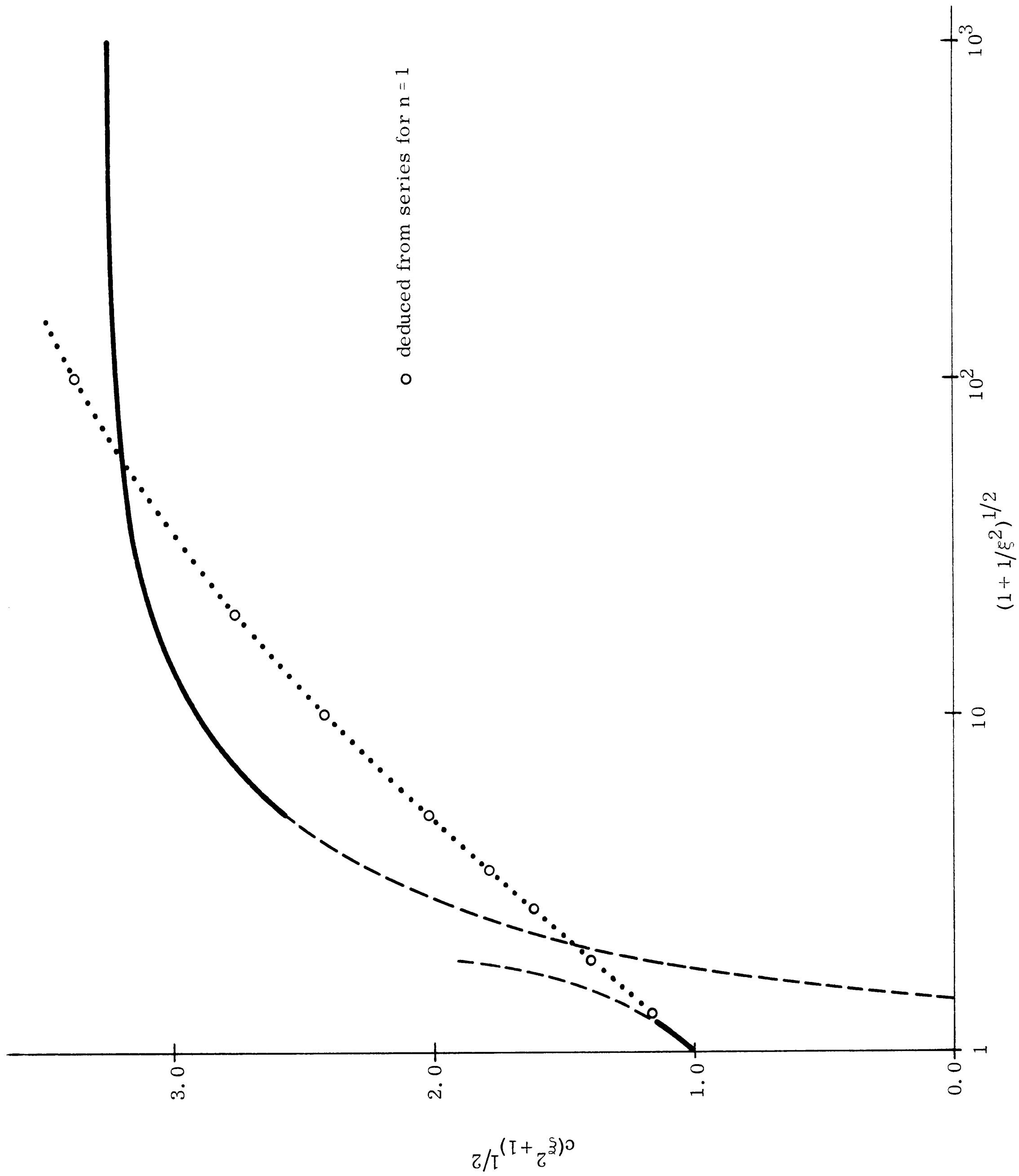

FIGURE XII. RADII OF CONVERGENCE FOR SOFT OBLATE SPHEROID reaches (at $\xi = 0$) the value 5.2 corresponding to the smallest zero of c_o^{01}.

For the amplitude coefficient A_{oo} the radius of convergence is infinite when $\xi = \infty$ (the sphere), and though it is presumably finite for all other ξ, the curves in Figure VI suggest that the radius is still in excess of 2.6 by the time ξ has decreased to 0.4 ($w = 2.69\,\ell$). If ξ is still smaller, say $\xi < 0.2$, the formula given in (74) becomes applicable, and this indicates a radius of convergence which increases slowly towards a maximum value of 3.25 at $\xi = 0$. It would therefore appear that the radius has a minimum for ξ somewhere near 0.3, but since the level is in excess of the corresponding radius for A_{o1}, it does not affect the overall convergence of the Rayleigh series. This last can be found by selecting the smaller of the radii for A_{oo} and A_{o1}, and accordingly the convergence of the Rayleigh series is determined by the convergence of A_{o1} out to the point at which the two curves cross. This occurs when $\xi = 0.017$ (i.e. $w = 59\,\ell$), and thereafter the radius for A_{oo} is the dominant one. The final curve showing the convergence of the Rayleigh series is given in Figure XVII, and the kink at $\xi = 0.017$ is at once apparent. It is not known whether this has any significance beyond the obvious one implied by the mathematics.

When the oblate spheroid is hard the results are even more detailed and are presented in Figure XIII. Taking first the amplitude coefficient A_{oo}, the radius of convergence is specified by (27) for $\xi^2 >> 1$ and is in excellent agreement with

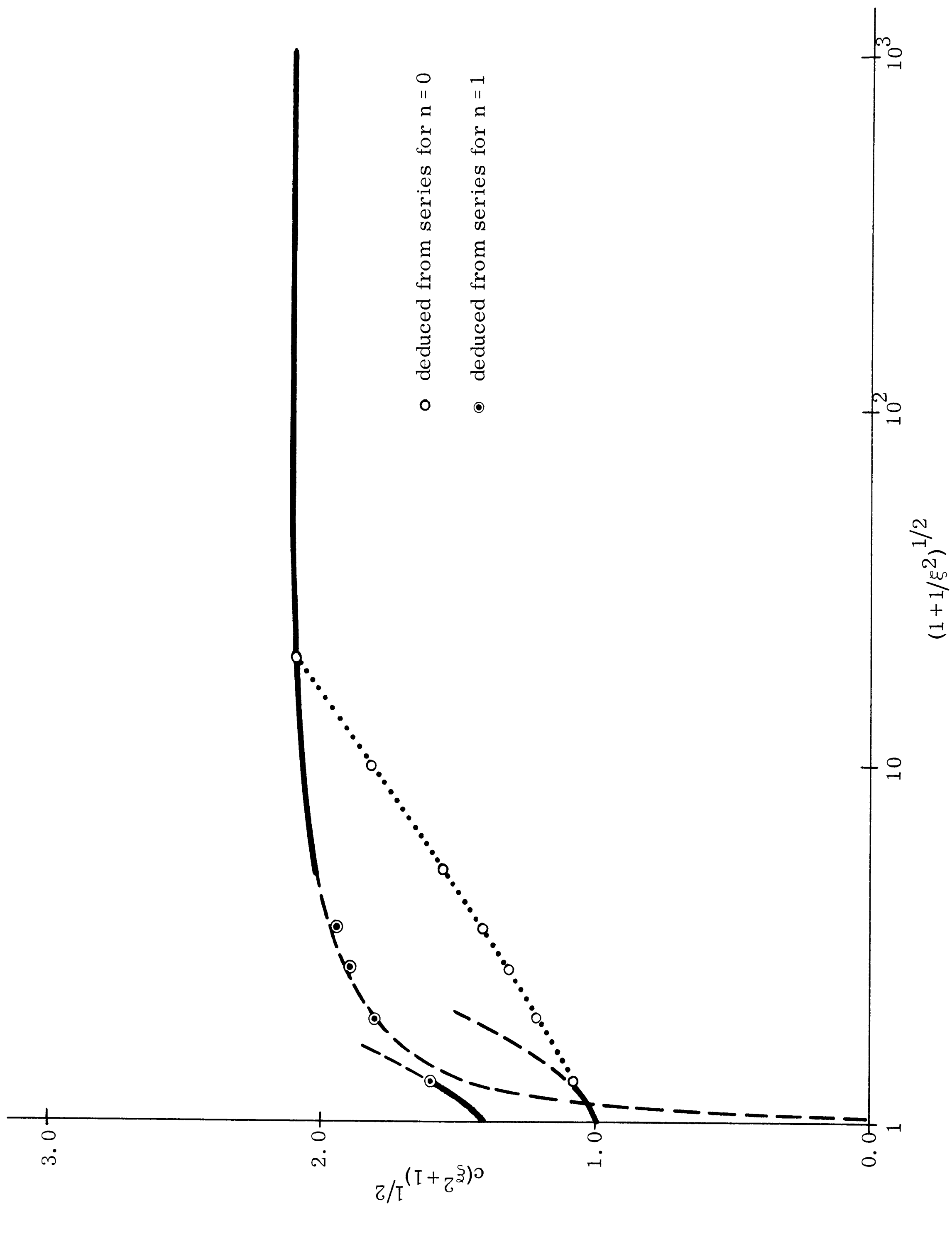

FIGURE XIII. RADII OF CONVERGENCE FOR HARD OBLATE SPHEROID the radii for smaller ξ deduced in §8 and listed in Table IX. The curve is concave upwards at least as far as $\frac{w}{\ell} = 20$, but the curvature is quite small and since the limiting radius for $\xi = 0$ is only 4.1 (corresponding to the zero of c_o^{oo}) it must reverse itself somewhere beyond the point $\xi = 0.05$.

For the coefficient A_{01} the radius is $\sqrt{2}$ when $\xi = \infty$ and providing $\xi^2 >> 1$ the variation as a function of ξ can be obtained from equation (21) with n = 1. The resulting formula is

$$c(\xi^2+1)^{1/2} = \sqrt{2}\,(1+\frac{1}{5\xi^2}) + O(\xi^{-4}) \tag{112}$$

and the curve is shown in Figure XIII. This can be assumed to cater for values of ξ greater than (about) 2 and for smaller ξ it is necessary to rely on the calculations in §8. Notwithstanding the difficulty in deducing radii from the convergence coefficients in Figure V, the estimates appear remarkably accurate and these bridge the gap between the regions for which the formulae for large and small ξ are applicable. When $\xi << 1$, (75) gives a radius of convergence which increases slowly with decreasing ξ, attaining a value 2.1255 at $\xi = 0$. The crossover point at which the radii for A_{00} and A_{01} are equal is 0.053 ($w = 19\ell$), and by selecting the smaller of the two at each value of ξ, the radius of convergence of the Rayleigh series is obtained. This is shown in Figure XVII and, like the curve for the soft oblate, is characterised by a kink at the point where the dominant coefficient changes. For all ellipticities, the radius for the hard body is less than that for the soft, and whereas with the former the dominant coefficient changes from A_{oo} to A_{o1} as ξ decreases, the reverse is true for the latter.

In the case of the prolate spheroid the results are not quite so complete, and this is particularly true when ξ is close to unity. From the physical standpoint it is not surprising to find that difficulties arise as $\xi \rightarrow 1$ since the entire Rayleigh series vanishes in the limit, but more important mathematically is the fact that the values of $|c|$ corresponding to the radii of convergence are of order 4 or greater, and for such large values the spheroidal function coefficients are extremely hard to compute. In addition, the rate at which the radii approach their values for $\xi = 1$ is so slow that any uncertainties in computation are reflected in errors over a wide range of ξ, and under these circumstances the numerical approach described in §2 is indispensable.

For the soft body the radii of convergence of A_{oo} and A_{o1} are plotted as functions of $(1 - 1/\xi^2)^{1/2}$ in Figures XIV and XV with the latter providing an abscissa which runs through 20 orders of magnitude. If ξ^2 is large compared with unity, the convergence of the expansion for A_{o1} is given in (19), and when ξ is such that the formula no longer holds (say, $\xi < 2$) the convergence can be found from Table XI. It will be observed that the values deduced by the numerical technique are in excellent agreement with (19). As ξ decreases, the radius continues to

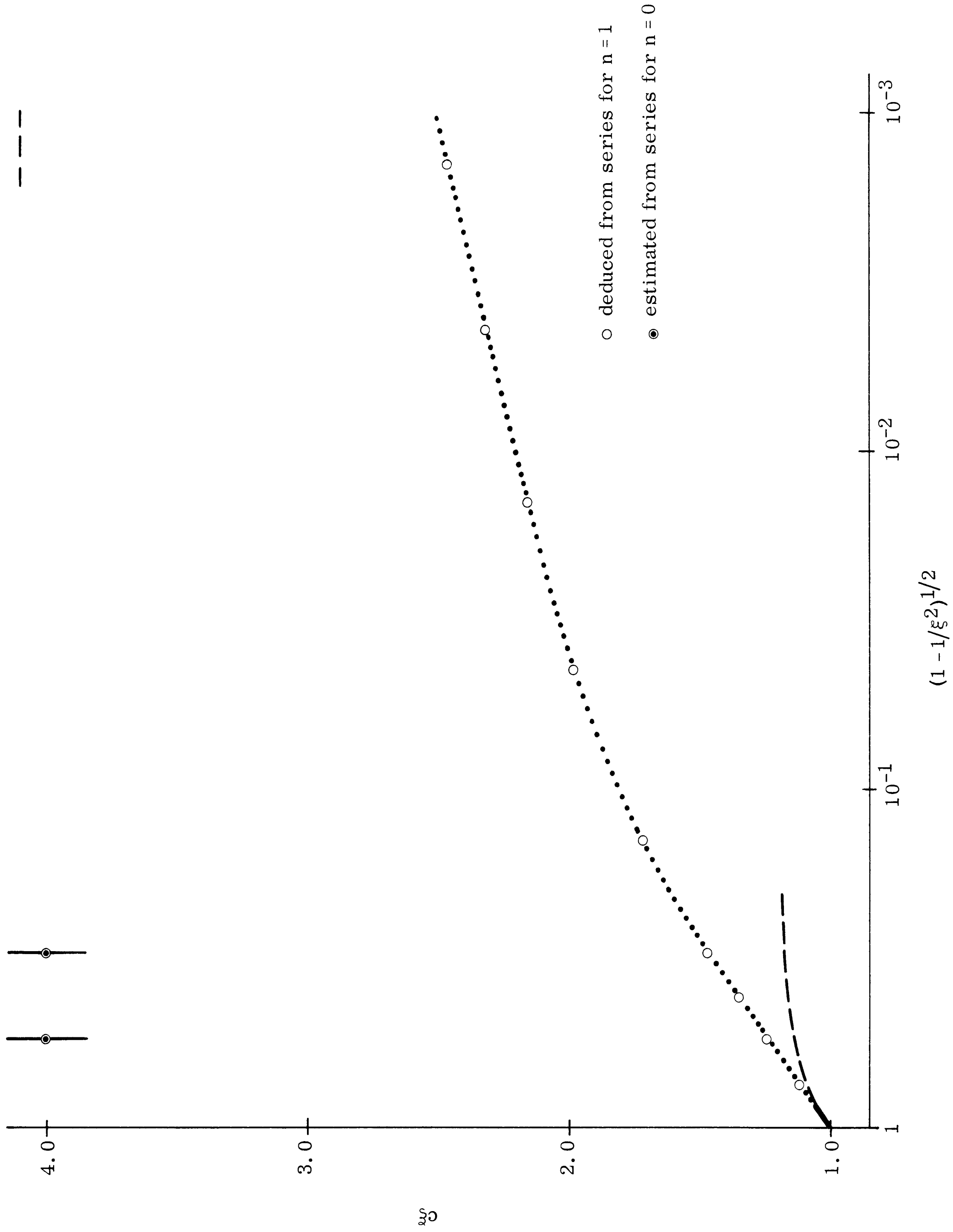

FIGURE XIV. RADII OF CONVERGENCE FOR SOFT PROLATE SPHEROID

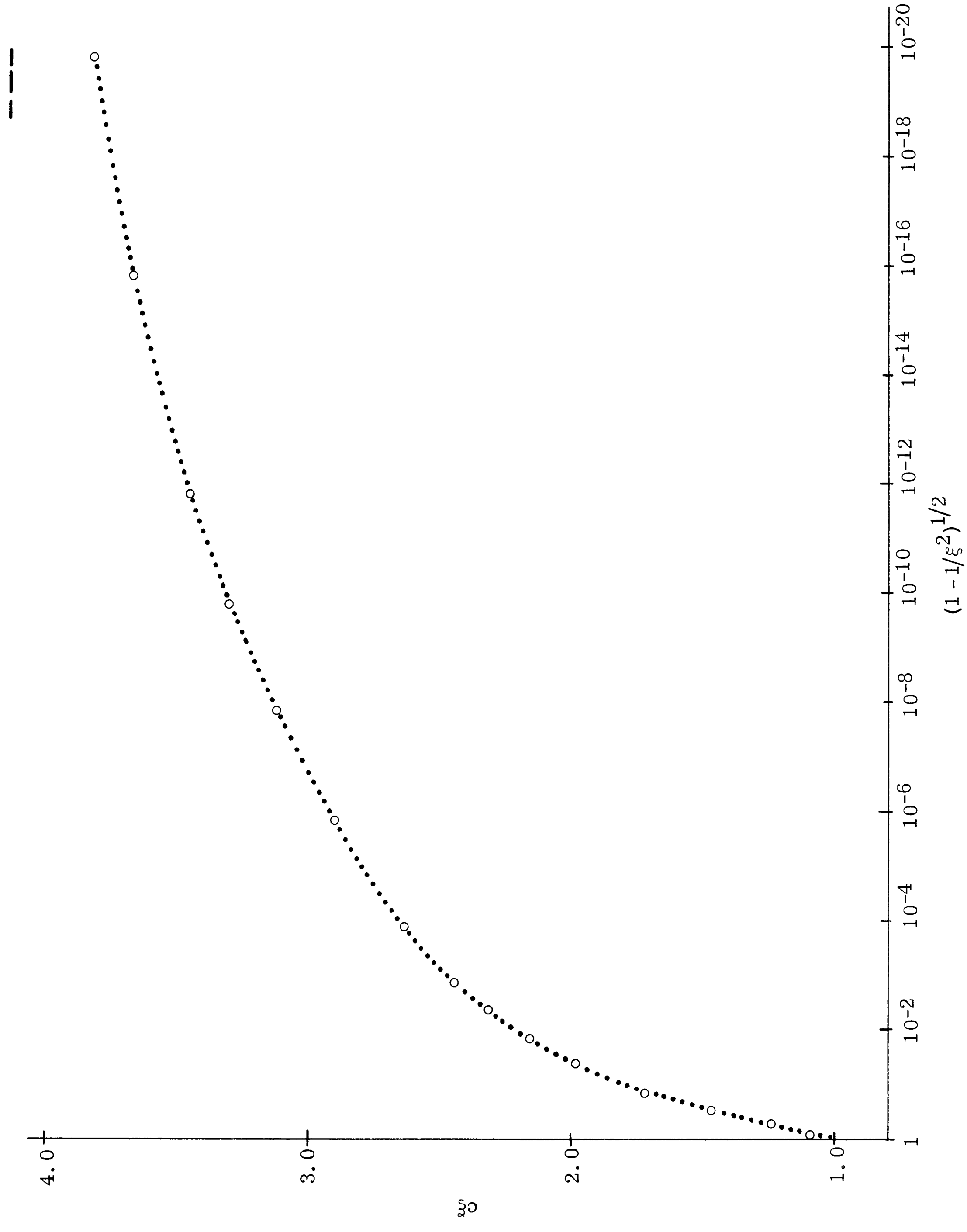

FIGURE XV. RADII OF CONVERGENCE FOR SOFT PROLATE SPHEROID increase but at a slower and slower rate (see Table XIII), and even when $\xi - 1$ has fallen to 10^{-40} (corresponding to $w/\ell = 10^{-20}$ in Figure XV) the radius is still far below the limiting value of 5.2 appropriate to the 'vanishing' rod ($\xi = 1$), and does not yet have the logarithmic dependence on ξ which equation (98) possesses. Ultimately, this dependence must obtain, but it probably does not do so before the radius of convergence of the amplitude coefficient A_{00} takes over.

For the coefficient A_{00} the radius is infinite when $\xi = \infty$ and of order 4 when $\xi = 1.2$ and 1.1. As $\xi \to 1$ the radius approaches its limiting value 4.1 according to formula (102), but here again $\xi - 1$ may have to be extremely small before (102) can be assumed to be applicable. Nevertheless, it is almost certain that the radius of convergence exceeds that for A_{01} until the latter crosses the 4.1 level, and some preliminary calculations suggest that this occurs when w/ℓ is approximately 10^{-37}. For all practical purposes, therefore, the convergence for the soft body is determined by the amplitude coefficient A_{01} alone, and the resulting radius is given in Figure XVII.

When the prolate spheroid is hard some of the above difficulties do not occur, but there is now a range of ξ which cannot be treated adequately by the numerical technique, and for which no formula for the convergence is applicable. Fortunately, however, the same amplitude coefficient specifies the radius at both ends of this range, and the end points of the two curves can be joined up without too much possibility of error.

The radii of convergence for A_{oo} and A_{o1} are shown plotted as functions of $(1 - 1/\xi^2)^{1/2}$ in Figure XVI, and taking first the amplitude coefficient A_{oo}, the radius is unity when $\xi = \infty$. Its value increases with decreasing ξ and is initially given by (23), but this no longer holds if ξ^2 is not large compared with unity, and thereafter the radius must be obtained by the numerical technique. It will be observed that the formula (23) goes over smoothly into the values listed in Table XVI. When ξ is near to unity the radius is given by (111), and although it would appear that this should be applicable for $\xi - 1$ as large as 10^{-2} (i.e. $w/\ell = 0.14$), the curve does not then join up with the tabulated values. The discrepancy, however, can probably be attributed to the computation of $R_e\, \tau_o'$. The number of terms used in the expansion for $X_{oo}(c)$ is insufficient to give a reliable determination of τ_o' for $|c|$ as large as 4.1, and from a consideration of the signs of the subsequent terms it can be shown that to include them would increase the magnitude of τ_o', possibly by as much as a factor 2. Such an increase would restore the agreement between the results of the two methods for calculating the convergence, and it is on this basis that the continuation of the curve shown in Figure XVI has been arrived at. For values of ξ less than (about) 10^{-4} (i.e. $w/\ell < 10^{-2}$) the change in $R_e \tau_o'$ has no significant effect on the convergence, and with further decrease of the radius remains constant and equal to 4.1.

For the amplitude coefficient A_{o1} the limiting value of the radius is 5.2 (corresponding to the smallest zero of c_o^{o1}) and is approached with a rapidity

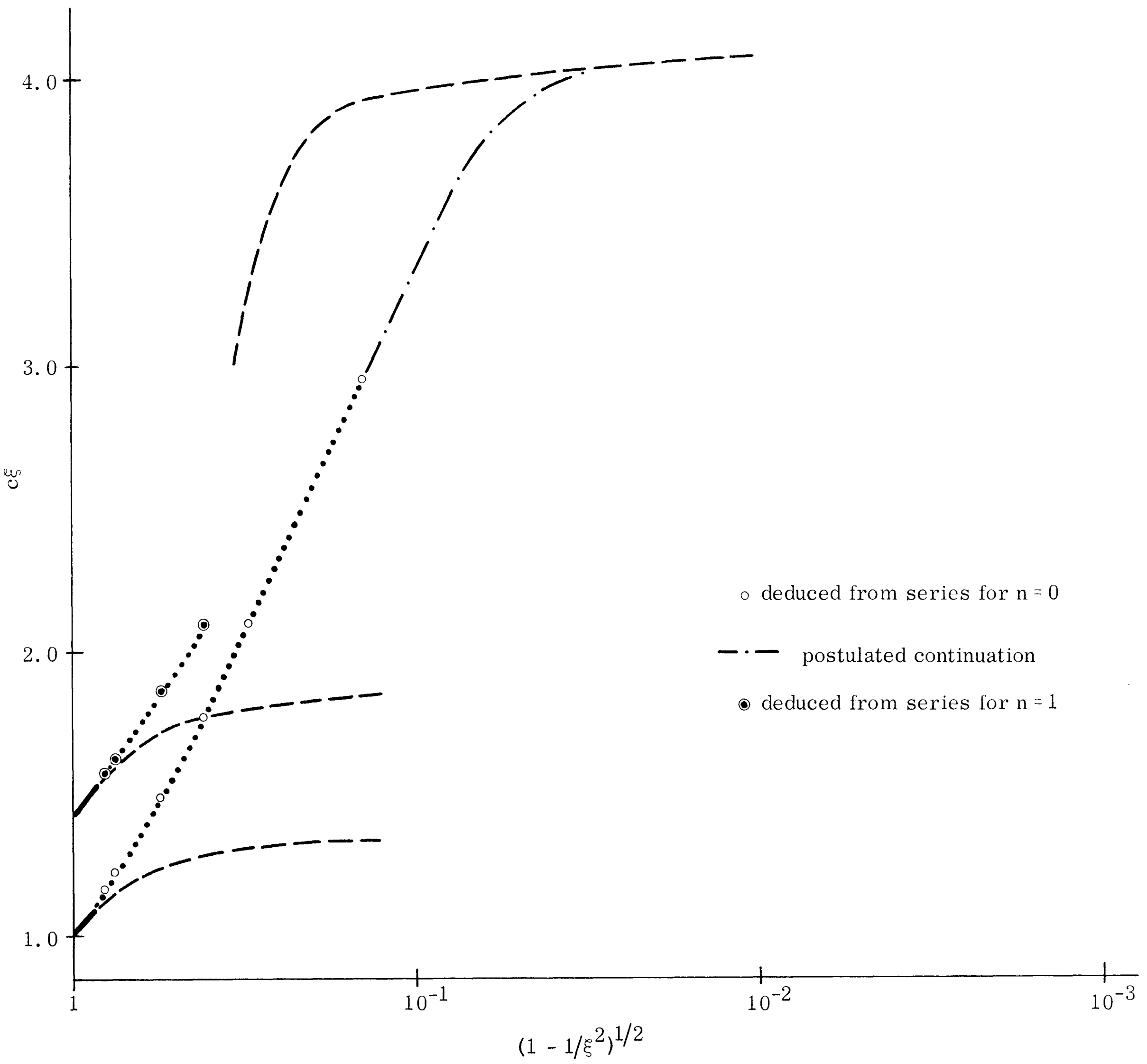

FIGURE XVI. RADIUS OF CONVERGENCE FOR HARD PROLATE SPHEROID comparable to that for A_{oo} as $\xi \to 1$. When ξ is large compared with unity the radius can be found from equation (21) with n = 1 (or, alternatively, from (112) by deducing $c\xi$ and changing the sign of ξ^2), and is consistent with the values for smaller ξ listed in Table XVIII, but for ϵ lying between 10^{-1} and (say) 10^{-2} no information is available. Nevertheless, it is unlikely that the radius is anywhere less than the radius for A_{oo} (if it were, there would be two hard prolate bodies for each of which A_{oo} and A_{o1} had the same convergence), and it seems probable that the curve for A_{o1} is more or less parallel to that for A_{o1}. The convergence of the Rayleigh series is then specified by the coefficient A_{oo} for all ellipticities.

The final results are shown in Figure XVII in which the radius of convergence for the soft body is represented by the solid line, and the radius for the hard body by the dashed line. All bodies, prolate as well as oblate, are encompassed by this graph and since the horizontal scale is logarithmic, the mid-point corresponds to the sphere ($w = \ell$), with the prolate bodies occupying the portion to the left ($w < \ell$), and the oblate bodies the portion to the right ($w > \ell$). Thus, for the oblate spheroids the radius of convergence for the soft body everywhere exceeds that for the hard, and with the prolate spheroids the reverse is true except in the limit of a 'vanishing' rod, where the two radii are equal. With this exception, the only case in which the two radii are equal is the transitional body, the sphere.

In Figure XVII the ordinate is k times the semi-major axis, and is therefore $c\xi$ or $c(\xi^2 + 1)^{1/2}$ depending on whether the spheroid is prolate or oblate

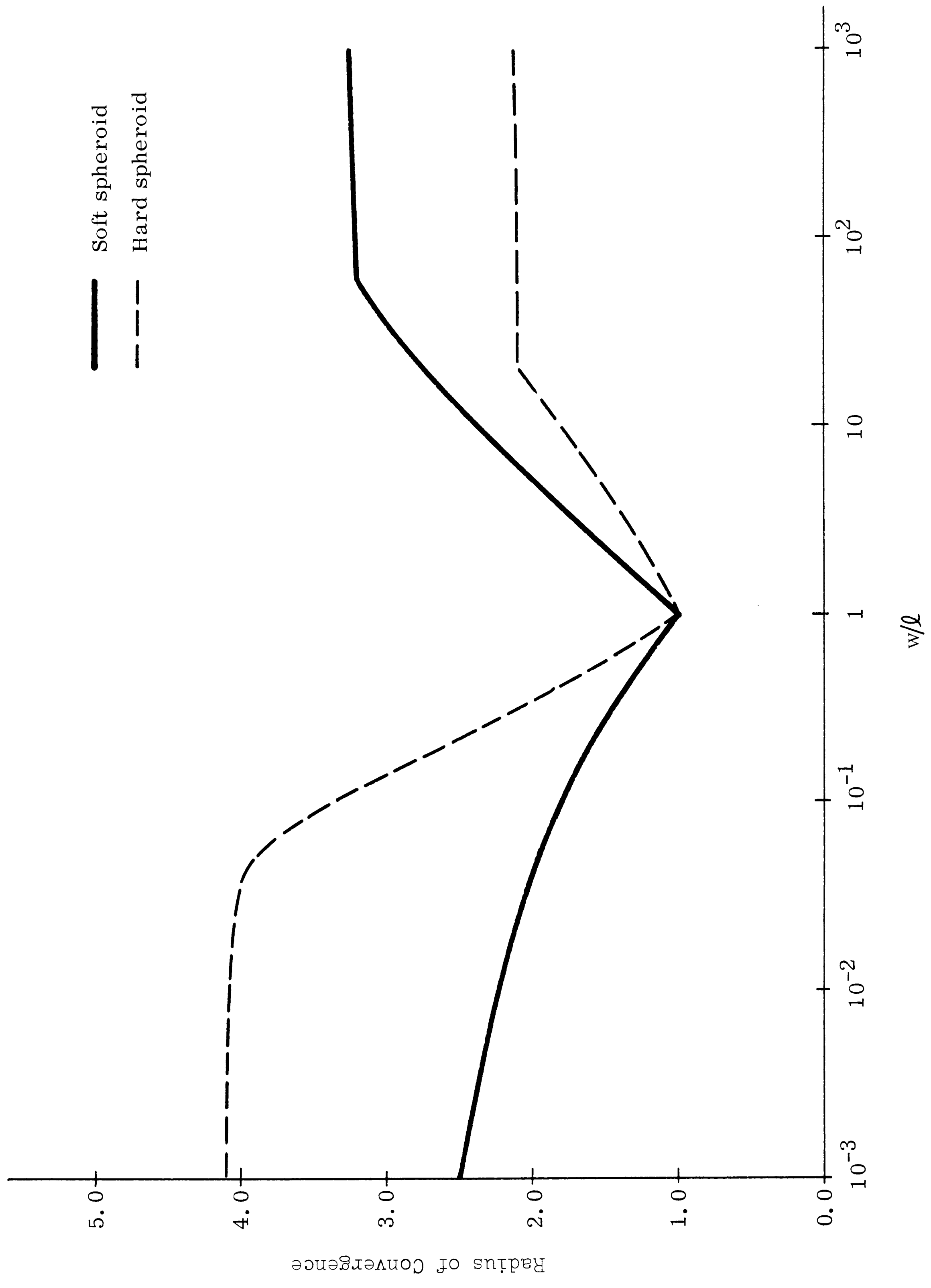

FIGURE XVII. RADIUS OF CONVERGENCE OF RAYLEIGH SERIES respectively. The discontinuity in slope at the mid-point occurs entirely as a consequence of this change in ordinate, and if it is taken as $c\xi$ (for example) throughout, the curves for both the hard and soft bodies are continuous in all their derivatives at $w = \ell$. On the other hand, for the oblate bodies the quantity which is plotted would then be zero in the limit of a disc, and would no longer provide a meaningful measure of the convergence of the Rayleigh series.

The radius of convergence is essentially the upper bound on the frequency for which the low frequency approximations are valid, and it will be seen from Figure XVII that of all the spheroidal bodies the sphere has the least radius. For a thin prolate spheroid, however, the radius can be as large as 4.1, and this is sufficient to include the first two minima in the pattern for the backscattering cross section as a function of $c\xi$ (see Siegel et al, 1956). With this body, therefore, it is possible to penetrate the 'resonance region' to a significant extent by using* low frequency techniques, whereas for a sphere the radius of convergence corresponds only to the first maximum in the pattern.

It may be desirable to end with a word of warning. All of the above analysis has been carried out for bodies which are hard or soft in the sense that a Newmann or Dirichlet boundary condition respectively is applicable at the surface, and if the

*However, as $c\xi$ approaches the radius of convergence, the number of terms which must be included in the low frequency expansion to get a reliable estimate for the field may become impossibly large.

boundary condition is other than one of these, the radii of convergence which have been found no longer apply. This is easily seen by considering a partially reflecting sphere whose boundary condition is such that*

$$(1 + i\Omega \frac{\partial}{\partial \rho})(V^i + V^s) = 0 \tag{113}$$

at the surface. For a sphere which is predominantly hard or soft, $\Omega >> 1$ or $<< 1$ respectively, and under the condition (112) the amplitude coefficients A_{on} are given by the formula

$$A_{on} = - \frac{j_n(\rho) + i\Omega \frac{\partial}{\partial \rho} j_n(\rho)}{h_n(\rho) + i\Omega \frac{\partial}{\partial \rho} h_n(\rho)} .$$

When n = 0 the denominator is simply

$$\frac{e^{ik\rho}}{\rho} (1 + i\Omega - \frac{\Omega}{\rho})$$

which has a zero at $\rho = \rho_o$ where

$$\rho_o = \frac{\Omega}{1 + i\Omega}$$

and the modulus of this is less than unity for all Ω, whether real or complex,

*This is the analogue of an impedance boundary condition in electromagnetic theory (see, for example, Senior 1960b).

providing $I_m \Omega < 1/2$. Indeed, $\rho_o \rightarrow 0$ as $\Omega \rightarrow 0$ indicating a radius of convergence for the exapnsion of A_{oo} which approaches zero with Ω. In the limit $\Omega = 0$, however, the above zero disappears and the radius for the entire Rayleigh series reverts to the value found in §2.

Such discontinuities are a direct consequence of the fact that under a mixed boundary condition the coefficients A_{on} have no expansions which are uniform in Ω. This is equally true for bodies other than the sphere, and it is to be expected that for a spheroid a similar behaviour will obtain. In general, the singularity provided by the 'joining' parameter Ω will be the dominant one, thereby producing a reduction in the radius of convergence. The convergence may then bear little resemblance to that for the corresponding 'perfect' body.

ACKNOWLEDGEMENTS

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APPENDIX A

SPHEROIDAL COEFFICIENT EXPANSIONS

Apart from the eigenvalues λ_{0n}, the most fundamental quantities are the spheroidal coefficients d_r^{0n}, and their expansions are involved in much of the preceding work. The derivation of the expansions is, in itself, not a trivial task, but if the corresponding terms in the expansion for λ_{0n} are known, the task is at least straightforward. Unfortunately, not all of these terms are known to the required accuracy, but by assuming in advance the form of the expansions for the d_r^{0n} it is possible to derive simultaneously the expansions for the spheroidal coefficients and the appropriate eigenvalue. The process will be illustrated in reference to the case n = 0.

The prolate spheroidal coefficients $d_r^{0n}(t)$ are defined by the recurrence relation

$$\frac{(r+2)(r+1)}{(2r+3)(2r+5)} t^2 d_{r+2}^{0n} + \left\{ r(r+1) - \lambda_{0n} + \frac{2r(r+1)-1}{(2r-1)(2r+3)} t^2 \right\} d_r^{0n}$$

$$+ \frac{r(r-1)}{(2r-3)(2r-1)} t^2 d_{r-2}^{0n} = 0 \qquad \text{(A.1)}$$

together with the normalizing conditions

$$\sum_{r=0}^{\infty}{}' \frac{(-1)^{r/2}\, r!}{2^r \frac{r}{2}!\, \frac{r}{2}!} d_r^{0n} = \frac{(-1)^{n/2}\, n!}{2^n \frac{n}{2}!\, \frac{n}{2}!} \tag{A.2}$$

for n even, and

$$\sum_{r=1}^{\infty}{}' \frac{(-1)^{\frac{r-1}{2}} (r+1)!}{2^r \frac{r-1}{2}!\, \frac{r+1}{2}!} d_r^{0n} = \frac{(-1)^{\frac{n-1}{2}} ((n+1)!}{2^n \frac{n-1}{2}!\, \frac{n+1}{2}!} \tag{A.3}$$

for n odd. From these equations it is apparent that all the expansions proceed in even powers of t and, in addition,

$$d_r^{0n} \longrightarrow \delta(r-n)$$

as $t \rightarrow 0$. When n = 0 it is therefore assumed that

$$d_r^{00} = t^r \left(D_r + t^2 D_r' + t^4 D_r'' + \dots\dots\right) d_0^{00} \tag{A.4}$$

with $D_0 = 1$ and $D_0', D_0'', \dots\dots = 0$. The coefficients $D_r, D_r', \dots\dots$ are independent of t.

If (A.4) is substituted into (A.1) with r = 0, we have immediately that

$$D_2 + t^2 D_2' + t^4 D_2'' + \dots = \frac{15}{2t^4}\left(\lambda_{00} - \frac{t^2}{3}\right) \tag{A.5}$$

and hence

$$\lambda_{00} = \frac{t^2}{3} + 0(t^4).$$

Knowing this one term we can now derive all the D_r and D_r' by merely inserting the expansions for d_r^{00} and d_{r-2}^{00} into the recurrence relation and equating the coefficients of t^r and t^{r-2}. It is found that

$$D_r = - \frac{r-1}{(r+1)(2r-3)(2r-1)} D_{r-2} \tag{A.6}$$

from which all the D_r can be calculated, and

$$D_r' = - \frac{r-1}{(r+1)(2r-3)(2r-1)} D'_{r-2} - \frac{1}{r(r+1)} \left\{ \frac{2r(r+1)-1}{(2r-1)(2r+3)} - \frac{1}{3} \right\} D_r$$

which can be shown to imply

$$D_r' = - \frac{r}{9(2r+3)} D_r , \tag{A.7}$$

thereby specifying the D_r'.

It will be observed that a knowledge of D_2 and D_2' specifies λ_{00} through t^6. Moreover, from the recurrence relation with r = 4,

$$D_4 + t^2 D_4' + t^4 D_4'' + \ldots.. = - \frac{21}{4t^4} \left\{ (6 - \lambda_{00} + \frac{11}{21} t^2)(D_2 + t^2 D_2' \right.$$

$$\left. + t^4 D_2'' + \ldots..) + \frac{2}{3} \right\} \tag{A.8}$$

and since D_4 and D_4' are known, D_2'' and D_2''' can be calculated, which in turn specify λ_{00} through t^{10}. Similarly, from the recurrence relation with r = 6 we obtain D_4'' and D_4''' and hence, from (A. 8), D_2^{iv} and D_2^{v}, which then give λ_{00} through t^{14}.

To calculate the expansion for the d_r^{00} correct to $O(t^{10})$ it is only necessary to carry this process one stage further (in the course of which the terms in λ_{00} are determined through t^{18}), and the resulting expansions are

$$d_2^{00} = -\frac{t^2}{3^2}\left\{1 - \frac{2}{3^2 7}t^2 - \frac{13}{3^4 5^2 7}t^4 + \frac{46}{3^6 5.7.11}t^6 - \frac{85648}{3^6 5^4 7^3 11.13}t^8 + O(t^{10})\right\} d_0^{00},$$

$$d_4^{00} = \frac{t^4}{3.5^2 7}\left\{1 - \frac{4}{3^2 11}t^2 - \frac{2498}{3^4 5^2 7.11.13}t^4 + \frac{2608}{3^6 5^2 7^2 13}t^6 + O(t^8)\right\} d_0^{00},$$

$$d_6^{00} = -\frac{t^6}{3^3 5.7^2 11}\left\{1 - \frac{2}{3^2 5}t^2 - \frac{143}{3^5 5^2 17}t^4 + O(t^6)\right\} d_0^{00},$$

$$d_8^{00} = \frac{t^8}{3^6 5^2 7.11.13}\left\{1 - \frac{8}{3^2 19}t^2 + O(t^4)\right\} d_0^{00},$$

$$d_{10}^{00} = -\frac{t^{10}}{3^4 5^2 7.11^2 13.17.19}\left\{1 + O(t^2)\right\} d_0^{00},$$

with $d_r^{00} = O(t^{12})\, d_0^{00}$ for $r \geqslant 12$. Substitution into the normalizing condition (A. 2) with n = 0 then gives

$$d_0^{00} = 1 - \frac{1}{2.3^2} t^2 + \frac{67}{2^3.3^4.5^2} t^4 - \frac{3037}{2^4.3^6.5^2.7^2} t^6 + \frac{100403}{2^7.3^8.5^4.7^2} t^8$$

$$+ \frac{89075591701}{2^8.3^{10}.5^4.7^2.11^2.13.17.19} t^{10} + 0(t^{12}). \qquad (A.9)$$

Before leaving the case n = 0 it is convenient to gather together the other expansions which are required in the course of the analysis. The coefficients $d_{\rho/r}^{0n}$ for r > 0 are defined in Flammer (p. 27), and when n = 0

$$d_{\rho/2}^{00} = -\frac{t^2}{2} \left\{ 1 - \frac{2}{3.5} t^2 + \frac{5}{2.3^3 7} t^4 - \frac{41}{3^5 5^2 7} t^6 + \frac{283}{2.3^7 5^3 11} t^8 + 0(t^{10}) \right\} d_0^{00}$$

$$d_{\rho/4}^{00} = \frac{t^4}{2^2.3.5} \left\{ 1 - \frac{4}{3^3} t^2 + \frac{64}{3^4 5.11} t^4 - \frac{512}{3^6 5.11.13} t^6 + 0(t^8) \right\} d_0^{00},$$

$$d_{\rho/6}^{00} = -\frac{t^6}{2.3^4 5.7} \left\{ 1 - \frac{2}{13} t^2 + \frac{173}{2^2 3^2 5^2 13} t^4 + 0(t^6) \right\} d_0^{00},$$

$$d_{\rho/8}^{00} = \frac{t^8}{2^3 3^3 5.7.11.13} \left\{ 1 - \frac{8}{3.17} t^2 + 0(t^4) \right\} d_0^{00},$$

$$d^{00}_{\rho/10} = - \frac{t^{10}}{2.\,3^4.\,5^3.\,7.\,11.\,13.\,17} \left\{1 + 0(t^2)\right\} d^{00}_0 .$$

The function $c^{00}_0(t)$ is

$$c^{00}_0(t) = d^{00}_0 \left\{1 + \frac{d^{00}_2}{d^{00}_0} + \frac{d^{00}_4}{d^{00}_0} + \frac{d^{00}_6}{d^{00}_0} + \ldots\right\}$$

and has the expansion

$$c^{00}_0(t) = d^{00}_0 \left\{1 - \frac{1}{3^2} t^2 + \frac{11}{3^4.\,5^2} t^4 + \frac{2}{3^6.\,5.\,7^2} t^6 - \frac{571}{3^8.\,5^4.\,7} t^8 \right.$$

$$\left. + \frac{1924952}{3^8.\,5^4.\,7^3.\,11^2.\,13} t^{10} + 0(t^{12})\right\} ; \qquad \text{(A.10)}$$

and finally

$$N_{00}(t) = 2\,(d^{00}_0)^2 \left\{1 + \frac{1}{5}\left(\frac{d^{00}_2}{d^{00}_0}\right)^2 + \frac{1}{9}\left(\frac{d^{00}_4}{d^{00}_0}\right)^2 + \ldots\right\}$$

$$= 2\,(d^{00}_0)^2 \left\{1 + \frac{1}{3^4.\,5} t^4 - \frac{4}{3^6.\,5.\,7} t^6 - \frac{47}{3^8.\,5^4.\,7} t^8 \right.$$

$$\left. + \frac{872}{3^9.\,5^4.\,7.\,11} t^{10} - \frac{1587878}{3^{10}.\,5^6.\,7^4.\,11.\,13} t^{12} + 0(t^{14})\right\} . \qquad \text{(A.11)}$$

From an examination of this last result it would appear that $N_{00}(t)$ can have no zeros for which $|t| < 4$.

When n = 1 the appropriate expansions can be obtained in a similar manner to the above, and because the coefficients now fall off more rapidly with increasing powers of t, a smaller number of terms proves to be sufficient. We have

$$d_3^{01} = -\frac{t^2}{5^2}\left\{1+\frac{2}{3.^2 5^2}t^2 - \frac{229}{3.^2 5.^3 7.^2 11}t^4 - \frac{12542}{3.^2 5.^6 7.^2 11.13}t^6 + 0(t^8)\right\} d_1^{01} \quad ,$$

$$d_5^{01} = \frac{t^4}{3.^2 5.7^2}\left\{1 + \frac{4}{5.^2 13}t^2 - \frac{386}{3.^2 5.^3 7.^2 13}t^4 + 0(t^6)\right\} d_1^{01} \quad ,$$

$$d_7^{01} = -\frac{t^6}{3.^4 5.7.11.13}\left\{1 + \frac{6}{5.^2 17}t^2 + 0(t^4)\right\} d_1^{01},$$

$$d_9^{01} = \frac{t^8}{3.^3 5.^2 7.11.^2 13.17}\left\{1 + 0(t^2)\right\} d_1^{01} \quad ,$$

with $d_r^{01} = 0(t^{10})\, d_1^{01}$ for $r > 9$, giving

$$d_1^{01} = 1 - \frac{3}{2.5^2}t^2 + \frac{543}{2.^3 5.^4 7^2}t^4 - \frac{42827}{2.^4 3.^4 5.^6 7^2}t^6 + \frac{18922109}{2.^7 3.^3 5.^7 7.^4 11^2}t^8 + 0(t^{10}) \tag{A.12}$$

Also,

$$d^{01}_{\rho/1} = \frac{t^2}{2.3}\left\{1 - \frac{2}{3.5}t^2 + \frac{53}{2.3.5^3 7}t^4 - \frac{11}{5^5 7}t^6 + 0(t^8)\right\} d^{01}_1 ,$$

$$d^{01}_{\rho/3} = - \frac{t^4}{2^2 3^2}\left\{1 - \frac{4}{5.7}t^2 + \frac{8}{3^2 5^3}t^4 + 0(t^6)\right\} d^{01}_1 ,$$

$$d^{01}_{\rho/5} = \frac{t^6}{2.3^3 5.7}\left\{1 - \frac{6}{5.11}t^2 + 0(t^4)\right\} d^{01}_1 ,$$

$$d^{01}_{\rho/7} = - \frac{t^8}{2^3 3^4 5.7.11}\left\{1 + 0(t^2)\right\} d^{01}_1 ,$$

$$c^{01}_0(t) = d^{01}_1\left\{1 - \frac{t^2}{5^2} + \frac{3}{5^4 7^2}t^4 + \frac{226}{3^4 5^5 7^2}t^6 + \frac{2187259}{3^4 5^8 7^4 11^2}t^8 + 0(t^{10})\right\} \tag{A.13}$$

$$N_{01}(t) = \frac{2}{3}(d^{01}_1)^2\left\{1 + \frac{3}{5^4 7}t^4 + \frac{4}{3.5^6 7}t^6 - \frac{1147}{3.5^8 7^4}t^8 - \frac{69336}{5^{10}7^4 11.13}t^{10} + 0(t^{12})\right\} , \tag{A.14}$$

and from this last result it is obvious that $N_{01}(t)$ can have no zeros for which $|t| > 5$.

APPENDIX B

THE RAYLEIGH SERIES FOR A DISC

As noted in § 5, the problem of diffraction by a circular disc (and the related problem of diffraction by an aperture) can be solved by methods other than those involving spheroidal coordinates, and this is particularly true at the low frequency end of the spectrum. Thus, for example, using an integral equation approach first proposed by Jones (1956), Bazer and Brown (1959) have derived a sequence of terms in the low frequency expansion for a circular hole and from this the solution for diffraction by a disc can be found by Babinet's principle.

For a plane wave normally incident on a hard disc the expression for the far field amplitude is

$$f(\eta, \pi) = -\frac{2c}{\pi}\int_0^1 \sinh\,(ct\cos\theta)\, g_1(t)\,dt \tag{B.1}$$

(see Bazer and Brown, 1959) where $g_1(t)$ is given by the integral equation

$$t\,g_1(t) = t\sinh ct + \frac{1}{\pi i}\int_{-1}^{1}\left\{\frac{\sinh\,[c(t-s)]}{t-s} - \frac{\cosh ct\,\sinh cs}{s}\right\} s\,g_1(s)\,ds,$$

and since the kernel is of order c^3 for small c, it is a straight-forward matter to obtain the expansion

$$g_1(t) = ct\left[1 + c^2\,\frac{t^2}{6} + \frac{2ic^3}{\pi}\cdot\frac{1}{9} + c^4\,\frac{t^4}{120} + \frac{2ic^5}{\pi}\left(\frac{4}{225} + \frac{t^2}{90}\right)\right.$$

$$+ c^6\left(\frac{t^6}{5040} - \frac{4}{81\,\pi^2}\right) + \frac{2\,i\,c^7}{\pi}\left(\frac{1}{735} + \frac{t^2}{525} + \frac{t^4}{2520}\right)$$

$$+ c^8\left(\frac{t^s}{362880} - \frac{28}{2025\,\pi^2} - \frac{2}{405\,\pi^2}\,t^2\right) + \frac{2\,i\,c^9}{\pi}\left(\frac{8}{127575} - \frac{4}{729\,\pi^2}\right.$$

$$\left.+ \frac{31}{198450}\,t^2 + \frac{t^4}{14175} + \frac{t^6}{136080}\right) + 0(c^{10})\Bigg]. \qquad \text{(B. 3)}$$

If $f(\eta, \pi)$ is now written in the form

$$f(\eta, \pi) = -\,2\,i \sum_{n=0}^{\infty} \tilde{A}_n\,\mathbf{P}_n\,(\cos\theta) \qquad \text{(B. 4)}$$

the integral relation

$$\int_{-1}^{1} e^{c\,t\,\cos\theta}\;P_n(\cos\theta)\,d(\cos\theta) = 2\,(-i)^n\;j_n(ict) \qquad \text{(B. 5)}$$

can be combined with equation (B. 1) to give

$$\tilde{A}_n = \frac{c}{2\,\pi}\;(2\,n+1)\,(-i)^{n+1}\left\{1 - (-1)^n\right\}\int_0^1 j_n(ict)\;g_1(t)\;dt,$$

implying that

$$\tilde{A}_n = 0$$

for n even, and

$$\tilde{A}_n = \frac{c}{\pi}\;(2\,n+1)\,(-i)^{n+1}\int_0^1 j_n(ict)\;g_1(t)\;dt \qquad \text{(B. 6)}$$

for n odd. Moreover,

$$j_1(ict) = i\,\frac{ct}{3}\left\{1 + \frac{(ct)^2}{10} + \frac{(ct)^4}{280} + \frac{(ct)^6}{15120} + \frac{(ct)^8}{1330560} + 0(c^{10})\right\}$$

$$j_3(ict) = -\,i\,\frac{(ct)^3}{105}\left\{1 + \frac{(ct)^2}{18} + \frac{(ct)^4}{792} + \frac{(ct)^6}{61776} + \frac{(ct)^8}{7413120} + 0(c^{10})\right\}$$

$$j_5(ict) = i\,\frac{(ct)^5}{10395}\left\{1 + \frac{(ct)^2}{26} + \frac{(ct)^4}{1560} + \frac{(ct)^6}{159120} + \frac{(ct)^8}{24186240} + 0(c^{10})\right\}$$

and hence, by substitution of (B. 3) into (B. 6),

$$\tilde{A}_1 = -\,i\,\frac{c^3}{3\pi}\left\{1 + \frac{4}{25}c^2 + \frac{2i}{9\pi}c^3 + \frac{3}{245}c^4 + \frac{2i}{\pi}c^5 \cdot \frac{7}{225}\right.$$

$$+ \frac{1}{81}\left(\frac{8}{175} - \frac{4}{\pi^2}\right)c^6 + \frac{2i}{\pi}c^7 \cdot \frac{403}{91875}$$

$$\left. + \frac{2}{405}\left(\frac{3}{847} - \frac{4}{\pi^2}\right)c^8 + \frac{2i}{\pi}c^9 \cdot \frac{1}{729}\left(\frac{1824}{6125} - \frac{4}{\pi^2}\right) + 0(c^{10})\right\}$$

$$\tilde{A}_3 = -\,i\,\frac{c^5}{75\pi}\left\{1 + \frac{10}{63}c^2 + \frac{2i}{9\pi}c^3 + \frac{28}{2673}c^4 + \frac{2i}{\pi}c^5\,\frac{61}{2025}\right.$$

$$+ \frac{1}{81}\left(\frac{360}{11011} - \frac{4}{\pi^2}\right)c^6 + \frac{2i}{\pi}c^7\,\frac{31583}{7858620}$$

$$\left. + \frac{88}{18225}\left(\frac{225}{104104} - \frac{4}{\pi^2}\right)c^8 + \frac{2i}{\pi}c^9\,\frac{1}{729}\left(\frac{321171}{1226225} - \frac{4}{\pi^2}\right) + 0(c^{10})\right\}$$

$$\tilde{A}_5 = -\,i\,\frac{c^7}{6615\,\pi}\left\{1 + \frac{56}{351}c^2 + \frac{2i}{9\pi}c^3 + \frac{7}{715}c^4 + \frac{2i}{\pi}c^5\,\frac{29}{975}\right.$$

$$+ \frac{1}{81}\left(\frac{396}{14365} - \frac{4}{\pi^2}\right) c^6 + \frac{2i\,c^7}{\pi} \frac{5087}{1289925}$$

$$+ \frac{14}{2925}\left(\frac{1001}{610470} - \frac{4}{\pi^2}\right) c^8 + \frac{2i\,c^9}{\pi} \frac{1}{729}\left(\frac{47938}{193375} - \frac{4}{\pi^2}\right) + 0(c^{10})\Big\}$$

The corresponding convergence coefficients $|a_r^n|$ defined in the manner of equation (7) are shown in Table XIX and plotted in Figure XVIII. It will be observed that for n = 1, 3 and 5 the curves are almost identical. This is also true for larger values of n, and for large r it would appear that the common asymptotic limit lies somewhere between 2·0 and 2·3. It is therefore concluded that all the $\tilde{A}_n$ for odd n have the same radius of convergence for their low frequency expansions, and accordingly the radius of convergence of the Rayleigh series for the hard disc lies within the range 2·0 to 2·3.

The significance of this result in terms of the amplitude coefficients A_{on} is most easily seen by observing that

$$\tilde{A}_n = \sum_{r=1}^{\infty}{}' d_n^{or} \frac{S_{or}(-ic, -1)}{N_{or}(-ic)} A_{or} \tag{B.7}$$

as a consequence of equations (12) and (B.4). Since S_{or} is free of singularities in the complex c plane, whilst N_{or} has no zeros which can affect the discussion, the radius of convergence of the low frequency expansion for $\tilde{A}_n$ is equal to the smallest radius of convergence for the individual A_{or} (r = 1, 3, 5, ...), and the fact that

TABLE XIX CONVERGENCE COEFFICIENTS FOR A HARD DISC

r	$\lvert a_r^1 \rvert$	$\lvert a_r^3 \rvert$	$\lvert a_r^5 \rvert$
1	∞	∞	∞
2	2·500	2·512	2·504
3	2·418	2·418	2·418
4	3·006	3·126	3·179
5	2·191	2·205	2·211
6	2·466	2·452	2·447
7	2·317	2·346	2·352
8	2·200	2·178	2·184
9	2·802	2·714	2·686

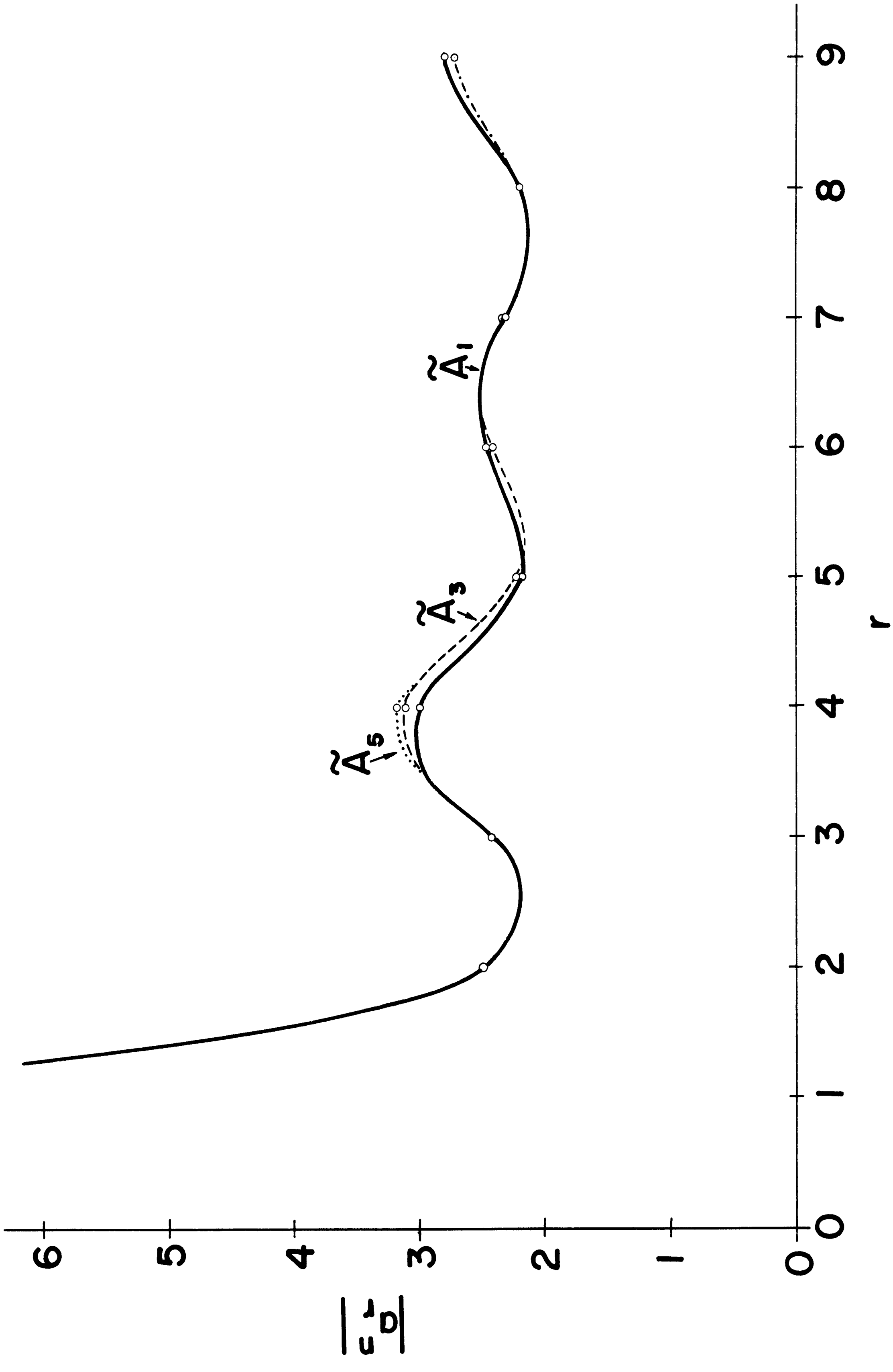

FIGURE XVIII. CONVERGENCE COEFFICIENTS FOR HARD DISC this is true for all (odd) n accounts for the similarity of the various curves shown in Figure XVIII. It follows immediately that one of the A_{or} must have a radius of convergence between 2·0 and 2·3, which is in agreement with the results of the more rigorous analysis of §5.

Turning now to the problem of a soft disc, the far field amplitude for a plane wave which is normally incident is

$$f(\eta, \pi) = -\frac{2c}{\pi} \int_0^1 \cosh(ct \cos\theta)\, g_2(t)\, dt \tag{B.8}$$

where $g_2(t)$ is given by the integral equation

$$g_2(t) = \cosh ct + \frac{1}{\pi i} \int_{-1}^{1} \frac{\sinh[c(t-s)]}{t-s}\, g_2(s)\, ds\,. \tag{B.9}$$

The integrand is here of order c for small c, so that a solution by iteration is again possible, and though it proves desirable to calculate a somewhat larger number of terms than Bazer and Brown (1959) have provided, it is a straight forward matter to show that

$$g_2(t) = 1 - \frac{2ic}{\pi} + c^2\left\{\frac{t^2}{2!} - \frac{4}{\pi^2}\right\} - \frac{2ic^3}{\pi}\left\{\frac{t^2}{3!} - \left(\frac{4}{\pi^2} - \frac{2}{9}\right)\right\}$$

$$+ c^4\left\{\frac{t^4}{4!} - \frac{2t^2}{3\pi^2} + \frac{16}{\pi^4} - \frac{4}{3\pi^2}\right\} - \frac{2ic^5}{\pi}\left\{\frac{t^4}{5!} - t^2\left(\frac{2}{3\pi^2} - \frac{2}{45}\right)\right.$$

$$\left. + \left(\frac{16}{\pi^4} - \frac{16}{9\pi^2} + \frac{2}{75}\right)\right\} + c^6\left\{\frac{t^6}{6!} - \frac{t^4}{30\pi^2} + t^2\left(\frac{8}{3\pi^4} - \frac{34}{135\pi^2}\right)\right.$$

$$-\left(\frac{64}{\pi^6} - \frac{80}{9\pi^4} + \frac{508}{2025\pi^2}\right)\Bigg] - \frac{2ic^7}{\pi}\left\{\frac{t^6}{7!} - t^4\left(\frac{1}{30\pi^2} - \frac{1}{420}\right)\right.$$

$$+ t^2\left(\frac{8}{3\pi^4} - \frac{44}{135\pi^2} + \frac{11}{1575}\right) - \left(\frac{64}{\pi^6} - \frac{32}{3\pi^4} + \frac{4}{9\pi^2} - \frac{4}{2205}\right)\Bigg\}$$

$$+c^8\left\{\frac{t^8}{8!} - \frac{t^6}{1260\pi^2} + t^4\left(\frac{2}{15\pi^4} - \frac{5}{378\pi^2}\right) - t^2\left(\frac{32}{3\pi^6} - \frac{8}{5\pi^4}\right.\right.$$

$$\left.\left. + \frac{746}{14175\pi^2}\right) + \left(\frac{256}{\pi^8} - \frac{448}{9\pi^6} + \frac{1856}{675\pi^4} - \frac{1112}{33075\pi^2}\right)\right\}$$

$$- \frac{2ic^9}{\pi}\left\{\frac{t^8}{9!} - t^6\left(\frac{1}{1260\pi^2} - \frac{1}{17010}\right) + t^4\left(\frac{2}{15\pi^4} - \frac{16}{945\pi^2} + \frac{23}{56700}\right)\right.$$

$$-t^2\left(\frac{32}{3\pi^6} - \frac{256}{135\pi^4} + \frac{50}{567\pi^2} - \frac{11}{19845}\right) + \left(\frac{256}{\pi^8} - \frac{512}{9\pi^6}\right.$$

$$\left.\left. + \frac{7936}{2025\pi^4} - \frac{2864}{35721\pi^2} + \frac{2}{25515}\right)\right\} + c^{10}\left\{\frac{t^{10}}{10!} - \frac{t^8}{90720\pi^2}\right.$$

$$+ t^6\left(\frac{1}{315\pi^4} - \frac{11}{34020\pi^2}\right) - t^4\left(\frac{8}{15\pi^6} - \frac{26}{315\pi^4} + \frac{407}{141750\pi^2}\right)$$

$$+ t^2\left(\frac{128}{3\pi^8} - \frac{1184}{135\pi^6} + \frac{22448}{42525\pi^4} - \frac{3844}{496125\pi^2}\right)$$

$$\left. - \left(\frac{1024}{\pi^{10}} - \frac{256}{\pi^8} + \frac{14272}{675\pi^6} - \frac{78224}{127575\pi^4} + \frac{2852}{626875\pi^2}\right)\right\} + 0(c^{11}) \qquad \text{(B.10)}$$

If the far field amplitude is written in the form (B. 4), the coefficients $\tilde{A}_n$ for the soft disc are given by the equation

$$\tilde{A}_n = \frac{c}{2\pi}(2n+1)(-i)^{n+1}\left\{1+(-1)^n\right\}\int_0^1 j_n(ict)\, g_2(t)\, dt$$

and hence

$$\tilde{A}_n = \frac{c}{\pi}(2n+1)(-i)^{n+1}\int_0^1 j_n(ict)\, g_2(t)\, dt \qquad \text{(B. 11)}$$

for n even, and

$$\tilde{A}_n = 0$$

for n odd. Since

$$j_o(ict) = 1 + \frac{(ct)^2}{6} + \frac{(ct)^4}{120} + \frac{(ct)^6}{5040} + \frac{(ct)^8}{362880} + \frac{(ct)^{10}}{39916800} + 0(c^{12})$$

$$j_2(ict) = -\frac{(ct)^2}{15}\left\{1 + \frac{(ct)^2}{14} + \frac{(ct)^4}{504} + \frac{(ct)^6}{33264} + \frac{(ct)^8}{3459456} + \frac{(ct)^{10}}{518918400} + 0(c^{12})\right\}$$

$$j_4(ict) = \frac{(ct)^4}{945}\left\{1 + \frac{(ct)^2}{22} + \frac{(ct)^4}{1144} + \frac{(ct)^6}{102960} + \frac{(ct)^8}{14002560} + \frac{(ct)^{10}}{2660486400} + 0(c^{12})\right\},$$

we have, by substituting (B. 10) into (B. 11)

$$\tilde{A}_o = -\frac{ic}{\pi}\left\{1 - \frac{2ic}{\pi} - \left(\frac{4}{\pi^2} - \frac{2}{9}\right)c^2 + \frac{2ic^3}{\pi}\left(\frac{4}{\pi^2} - \frac{1}{3}\right)\right.$$

$$+\left(\frac{16}{\pi^4} - \frac{16}{9\pi^2} + \frac{2}{75}\right)c^4 - \frac{2ic^5}{\pi}\left(\frac{16}{\pi^4} - \frac{20}{9\pi^2} + \frac{127}{2025}\right)$$

$$-\left(\frac{64}{\pi^6} - \frac{32}{3\pi^4} + \frac{4}{9\pi^2} - \frac{4}{2205}\right)c^6 + \frac{2ic^7}{\pi}\left(\frac{64}{\pi^6}\right.$$

$$\left. - \frac{112}{9\pi^4} + \frac{464}{675\pi^2} - \frac{278}{33075}\right) + \left(\frac{256}{\pi^8} - \frac{512}{9\pi^6}\right.$$

$$\left. + \frac{7936}{2025\pi^4} - \frac{2864}{35721\pi^2} + \frac{2}{25515}\right)c^8$$

$$-\frac{2ic^9}{\pi}\left(\frac{256}{\pi^8} - \frac{64}{\pi^6} + \frac{3568}{675\pi^4} - \frac{19556}{127575\pi^4} + \frac{713}{826875}\right)$$

$$-\left(\frac{1024}{\pi^{10}} - \frac{2560}{9\pi^8} + \frac{18496}{675\pi^6} - \frac{36928}{35721\pi^4} + \frac{254684}{22325625\pi^2}\right.$$

$$\left.\left. - \frac{4}{1715175}\quad c^{10}\right) + 0(c^{11})\right\}$$

$$\tilde{A}_2 = -\frac{ic^3}{9\pi}\left\{1 - \frac{2ic}{\pi} - \left(\frac{4}{\pi^2} - \frac{12}{35}\right)c^2 + \frac{2ic^3}{\pi}\left(\frac{4}{\pi^2} - \frac{23}{63}\right)\right.$$

$$+\left(\frac{16}{\pi^4} - \frac{40}{21\pi^2} + \frac{5}{147}\right)c^4 - \frac{2ic^5}{\pi}\left(\frac{16}{\pi^4} - \frac{148}{63\pi^2} + \frac{38}{525}\right)$$

$$-\left(\frac{64}{\pi^6} - \frac{704}{63\pi^4} + \frac{7048}{14175\pi^2} - \frac{8}{4455}\right)c^6 + \frac{2ic^7}{\pi}\left(\frac{64}{\pi^6}\right.$$

$$\left. - \frac{272}{21\pi^4} + \frac{132}{175\pi^2} - \frac{18377}{1819125}\right) + \left(\frac{256}{\pi^8} - \frac{3712}{63\pi^6} + \frac{20048}{4725\pi^4}\right.$$

$$\left. - \frac{33472}{363825\pi^2} + \frac{2}{33033}\right)c^8 - \frac{2ic^9}{\pi}\left(\frac{256}{\pi^8} - \frac{4160}{63\pi^6}\right.$$

$$\left. + \frac{16064}{2835\pi^4} - \frac{48392}{280665\pi^2} + \frac{60638}{59594535}\right)$$

$$-\left(\frac{1024}{\pi^{10}} - \frac{2048}{7\pi^8} + \frac{137728}{4725\pi^6} - \frac{11255488}{9823275\pi^4} + \frac{295857592}{22347950625\pi^2}\right.$$

$$\left. - \frac{8}{5589675}\right)c^{10} + 0(c^{11})\Bigg\}$$

$$\tilde{A}_4 = -\frac{ic^5}{525\pi}\Bigg\{1 - \frac{2ic}{\pi} - \left(\frac{4}{\pi^2} - \frac{30}{77}\right)c^2 + \frac{2ic^3}{\pi}\left(\frac{4}{\pi^2} - \frac{37}{99}\right)$$

$$+\left(\frac{16}{\pi^4} - \frac{64}{33\pi^2} + \frac{140}{3861}\right)c^4 - \frac{2ic^5}{\pi}\left(\frac{16}{\pi^4} - \frac{236}{99\pi^2} + \frac{5627}{75075}\right)$$

$$-\left(\frac{64}{\pi^6} - \frac{1120}{99\pi^4} + \frac{94228}{184275\pi^2} - \frac{8}{4719}\right) + \frac{2ic^7}{\pi}\left(\frac{64}{\pi^6}\right.$$

$$\left. - \frac{144}{11\pi^4} + \frac{20872}{27027\pi^2} - \frac{256}{24255}\right) + \left(\frac{256}{\pi^8} - \frac{5888}{99\pi^6}\right.$$

$$+\frac{2925856}{675675\,\pi^4}-\frac{449536}{4729725\,\pi^2}+\frac{1}{20111}\Bigg)c^8$$

$$-\frac{2i}{\pi}c^9\left(\frac{256}{\pi^8}-\frac{6592}{99\,\pi^6}+\frac{11693936}{2027025\,\pi^4}-\frac{906316}{5108103\,\pi^2}\right.$$

$$\left.+\frac{35155667}{33432534135}\right)-\left(\frac{1024}{\pi^{10}}-\frac{9728}{33\,\pi^8}+\frac{2223808}{75075\,\pi^6}\right.$$

$$\left.-\frac{150100864}{127702575\,\pi^4}+\frac{742415224}{54273594375\,\pi^2}-\frac{2}{1962225}\right)c^{10}$$

$$+\,0(c^{11})\Bigg\}\;.$$

The corresponding convergence coefficients $|a_r^n|$ defined in the manner of equation (7) are shown in Table XX and plotted in Figure XIX. For n = 0, 2 and 4 the curves are far less regular than the analogous ones for a hard disc, and even when terms in c^{10} are included the curves still remain apart. It is therefore difficult to estimate the radius of convergence with any certainty, but bearing in mind that for sufficiently large r the curves must all approach a common limit, the more regular curve for $\tilde{A}_4$ can be used to indicate the limit. It would appear from this that the radius of convergence of the Rayleigh series lies somewhere between 3.0 and 3.4.

TABLE XX CONVERGENCE COEFFICIENTS FOR A SOFT DISC

r	$\left\lvert a_r^o \right\rvert$	$\left\lvert a_r^2 \right\rvert$	$\left\lvert a_r^4 \right\rvert$
1	1·571	1·571	1·571
2	2·336	4·003	7·863
3	2·795	3·393	3·679
4	3·106	3·710	3·973
5	3·870	4·081	3·681
6	3·903	3·634	3·499
7	4·280	4·138	3·836
8	4·458	4·223	4·258
9	4·033	3·631	3·521
10	4·230	3·563	3·459

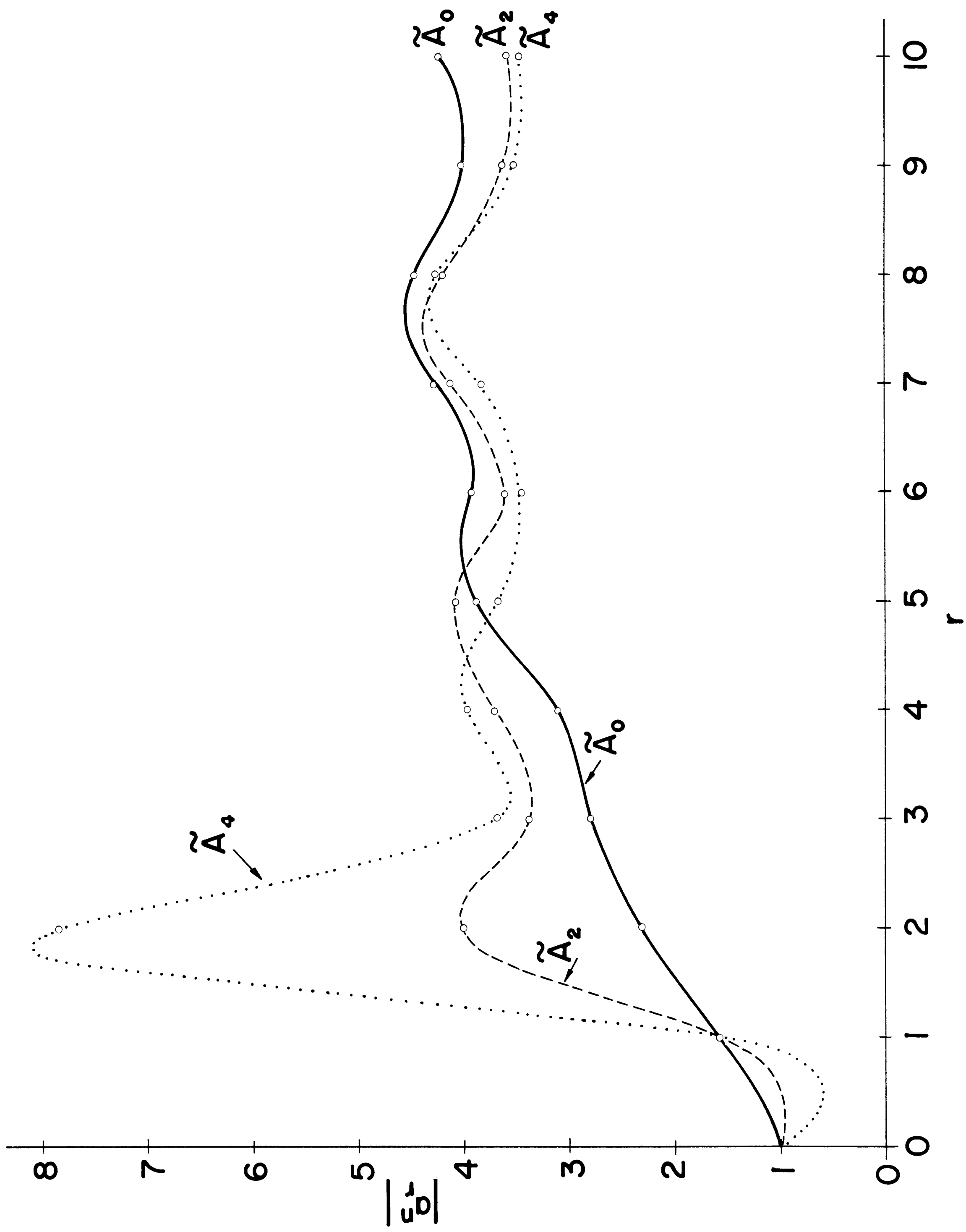

FIGURE XIX. CONVERGENCE COEFFICIENTS FOR SOFT DISC

The relationship between the coefficients $\tilde{A}_n$ and A_{on} is given by

$$\tilde{A}_n = \sum_{r=0}^{\infty}{}' d_n^{or} \frac{S_{or}(-ic,-1)}{N_{or}(-ic)} A_{or} \qquad \text{(B. 12)}$$

(cf equation B. 7) and by the same argument as before it follows that the limiting value of the $|a_r^n|$ for large r represents the smallest radius of convergence of the expansions for the individual A_{or}. It will be observed that this is in good agreement with the radius of convergence of A_{oo} found by the more rigorous analysis of § 5, and this confirms that the singularity of A_{oo} is the one which dictates the radius of convergence of the Rayleigh series for a soft disc.

Unfortunately, for the particular problem of the soft disc difficulties are experienced with both the available methods for estimating the convergence, and it comes as no surprise to find that these have a common origin. In seeking the solution of equation (47) it was found that a large number of terms must be included in the expansion for d_o^{oo} in order to determine the root, and even when terms as high as c^{12} are taken into account it is still not possible to find the solution with an error of less than about 1 %. This is due to the fact that for values of $|c|$ of the same order of magnitude as the root the leading terms in d_o^{oo} do not decrease rapidly and, indeed, the convergence coefficients for d_o^{oo} behave in a similar manner to those shown in Figure XIX. Such a behaviour is reflected in all the d_r^{oo} and hence, through equation (B. 12), in all the $\tilde{A}_n$ for n = 0, 2, 4. . . , which then leads to difficulties in applying the intuitive method. This is in marked contrast to the case of the hard disc.