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ON THE USE OF GENERALIZED IMPEDANCE BOUNDARY CONDITIONS

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Abstract

When higher order boundary conditions are applied to an edged structure such as a wedge or half plane, complications associated with the uniqueness and reciprocity of the solution generally arise. To illustrate this fact, second order boundary conditions are introduced and discussed, and then applied to the diffraction of a plane wave by a half plane. It is shown that reciprocity must be explicitly imposed, but even when this is done and the usual edge conditions applied, the Wiener-Hopf solution still contains an arbitrary constant. The constant is related to the surface values of certain field components at the edge, and the specification of this information, derived from a consideration of the actual structure being modelled, is therefore necessary for a unique solution. The generalization to higher order boundary conditions is also discussed, and for Nth order conditions, the solution contains N-1 unknown constants which can be related to the surface values of N-1 field quantities at the edge.

I. Introduction

Many targets whose radar scattering is of interest involve non-metallic materials, possibly in the form of a dielectric or other coating applied to a metallic substrate, and this makes necessary the development of procedures for simulating the effect of the material. One method which is now attracting attention is the use of approximate boundary conditions applied at a single surface. Such conditions may involve field derivatives of higher order than the first and since they are generalizations of the standard (first order) impedance boundary condition, they have been referred to as generalized impedance boundary conditions (GIBCs) whose order is specified by the highest derivative present.

A version appropriate to a plane surface was originally proposed in [1] to study the surface waves supported by a dielectric coating, and the conditions were subsequently invoked to simulate a perfectly absorbing surface in a finite element analysis [2,3]. The generalization to a curved surface and the accuracy with which the scattering from a metal–backed layer can be simulated are discussed in [4,5], and the diffraction by a wedge subject to these conditions is treated in [6,7]. Analogous results for a half plane obtained using the Maliuzhinets and Wiener-Hopf techniques are given in [8 and 9].

A problem closely related to that of a metal-backed layer is the the modeling of a thin semi-transparent layer using transition conditions applied at a single interface. For a very thin layer composed of a lossy non-magnetic material a resistive sheet provides an adequate simulation, but if the layer is not lossy and/or is thicker, the normal component of the polarization current in the layer is no longer negligible. This component can be simulated using a conductive sheet [10] or, more accurately, by introducing a "modified" conductive sheet [11] distinguished by the presence of a second derivative. The resulting second order conditions are identical to those developed by Weinstein [12], and have been used [13-17] to treat the problem of a plane wave incident on a dielectric half plane.

Unfortunately, difficulties arise when boundary (or transition) conditions of higher order than the first are applied to an edged structure, and because of these, most of the solutions present in the literature are either incorrect or, at best, incomplete through failure to impose constraints adequate to ensure uniqueness. As noted in [8,9], the reciprocity condition concerning interchange of receiver and transmitter is no longer satisfied automatically and must be explicitly enforced, and the solutions in [6,14] violate reciprocity. Moreover, the simple specification of an edge condition is not sufficient [9] for uniqueness, and most of the solutions cited contain one or more arbitrary constants or, even worse, undetermined functions [18].

Since higher order conditions are better able to simulate the material properties of a layer or coating, particularly when the thickness is not very small compared with the wavelength, it is important to address the difficulties that have been found in the case of an edged structure such as a half plane or wedge, and this is no less essential if the solution technique employed is numerical rather than analytical. This is the purpose of the present report. The boundary conditions themselves are discussed in Section 2, and since the difficulties arise in going from a standard (first order) condition to a second order one, it is sufficient to concentrate on a second order GIBC. In Section 3 the solution for a linearly polarized plane wave incident on a half plane subject to the same second order boundary conditions on the two faces is derived using the Wiener-Hopf technique with particular attention to the validity of the mathematical operations. Even when reciprocity is enforced, the standard edge condition still leaves a single constant undetermined, and it is shown how this can be specified using the physical properties of the structure being simulated. The implications of this result are discussed in the last section, where the extension to higher order boundary conditions is described.

2. The Boundary Conditions

The impedance boundary condition generally attributed to Leontovich [19] is a widely-used means for simulating the material properties of a surface in scattering analyses. If $\hat{n}$ is the outward unit vector normal to the surface, the boundary condition can be written as

$$\hat{n} \times \hat{n} \times \overline{E} = -\eta Z \hat{n} \times \overline{H} \qquad (1)$$

where η is the surface impedance normalized to the impedance Z of the surrounding free space medium. In the special case of a planar surface y=constant in a Cartesian coordinate system x,y,z, (1) is equivalent to [20]

$$\frac{\partial E_y}{\partial y} + ik\eta E_y = 0, \qquad \frac{\partial H_y}{\partial y} + \frac{ik}{\eta} H_y = 0, \qquad (2)$$

and these can be obtained from (1) by tangential differentiation. We caution that even for a planar surface in other coordinate systems, it is not permissible to replace y by the normal coordinate.

The boundary condition (1), or where applicable, (2), is well-posed and ensures a unique solution. The resulting boundary value problem is self-adjoint implying a symmetric Green's function, and the reciprocity condition concerning the interchange of the transmitter and receiver is therefore satisfied. In the case of an edged structure such as a wedge or half plane, the standard edge condition is required and, in particular, a current component perpendicular to the edge must be zero at the edge.

To improve the accuracy of the simulation and to increase the variety of materials that can be modelled using a boundary condition applied at a single surface, generalized versions of the boundary condition (1) and (2) have been proposed. For a planar surface y = constant, the generic form of these new conditions is [4]

$$\prod_{m=1}^{M}\left(\frac{\partial}{\partial y}+ik\Gamma_m\right)E_y=0, \qquad \prod_{m=1}^{M'}\left(\frac{\partial}{\partial y}+ik\Gamma_m'\right)H_y=0 \tag{3}$$

where Γ_m and Γ_m' are constants chosen to reproduce the desired scattering properties. One way to choose them is to examine the reflection coefficients. For an incident plane wave

$$E_y^i = e^{-ik(x\cos\phi + y\sin\phi)}$$

the reflection coefficient implied by (3) is

$$R(\phi) = -\prod_{m=1}^{M}\frac{\Gamma_m - \sin\phi}{\Gamma_m + \sin\phi},$$

with a similar expression for the reflection coefficient $R'(\phi)$ associated with the component H_y. The required Γ_m and Γ_m' can be found from theoretical or experimental data for the actual reflection coefficients of the surface as functions of $\sin\phi$.

There are four points to be observed. Since a knowledge of E_y or H_y alone is not in general sufficient to determine an electromagnetic field, the constants Γ_m and Γ_m' cannot be chosen independently of one another, and when duality is imposed, a specific relationship is obtained. Thus, for the second order conditions ($M = M' = 2$) we require that

$$\frac{\Gamma_1' + \Gamma_2'}{\Gamma_1'\Gamma_2' + 1} = \frac{\Gamma_1\Gamma_2 + 1}{\Gamma_1 + \Gamma_2}. \tag{4}$$

There is also the restriction imposed by the fact that the surface is passive. For the first order condition the requirement is Re. $\Gamma_1, \Gamma_1' > 0$, but for the higher order conditions, one or more of the Γ_m and Γ_m' can have negative real parts. Indeed, for the second order conditions the restriction derived from a consideration of the reflection coefficient is

$$\frac{\mathrm{Re.}\,\Gamma_1}{|\Gamma_1|^2 + \sin^2\phi} + \frac{\mathrm{Re.}\,\Gamma_2}{|\Gamma_2|^2 + \sin^2\phi} > 1 \tag{5}$$

for $0 \leq \phi \leq \pi/2$, with a similar result for the Γ_m'.

To extend (3) to a surface other than y = constant, e.g. a curved surface, it is necessary to express the boundary conditions in terms of the tangential field components. The procedure is analogous to that involved in going from (2) to (1), implying a tangential integration, and when duality is imposed, the second order conditions become

$$\hat{n} \times \left(\hat{n} \times \left\{ \overline{E} + \frac{1}{ik(\Gamma_1 + \Gamma_2)} \nabla(\hat{n} \cdot \overline{E}) \right\} \right) = - \frac{\Gamma_1 \Gamma_2 + 1}{\Gamma_1 + \Gamma_2} Z \, \hat{n} \times \left\{ \overline{H} + \frac{1}{ik(\Gamma_1' + \Gamma_2')} \nabla(\hat{n} \cdot \overline{H}) \right\} \tag{6}$$

(see [4], where the third order result is also given). The nature of the generalization of the standard impedance boundary condition (1) is evident. Finally, there is the matter of reciprocity. If the boundary condition is of higher order than the first, the boundary value problem is not in general self-adjoint, and the reciprocity condition is not then satisfied automatically. This is certainly the case for an edged structure, and since reciprocity is an essential feature of a physically-meaningful solution, it must be explicitly enforced. Fortunately, the arbitrariness inherent in the solution when only the standard edge condition is imposed allows this to be done.

To determine the additional information necessary to specify a unique solution, it is sufficient to consider the problem of a plane wave incident on a half plane subject to the same second order boundary conditions on the two faces.

3. Second Order Impedance Half Plane

The half plane occupies the portion $x \geq 0$, $-\infty < z < \infty$ of the plane $y=0$ of the Cartesian coordinate system x, y, z, and is illuminated by the H-polarized plane wave

$$\overline{H}^{\,i}(x, y) = \hat{z}\, e^{-ik(x \cos \phi_o + y \sin \phi_o)}. \tag{7}$$

On the half plane the same boundary condition (6) is imposed on the two faces, and since the entire problem is independent of z, the boundary condition reduces to

$$E_x = \pm Z \frac{\Gamma_1 \Gamma_2 + 1}{\Gamma_1 + \Gamma_2} H_z \mp \frac{1}{ik(\Gamma_1 + \Gamma_2)} \frac{\partial E_y}{\partial x} \tag{8}$$

on $y = \pm 0$, $x \geq 0$. This can be expressed as

$$\left\{ \frac{\partial^2}{\partial x^2} + k^2(\Gamma_1 \Gamma_2 + 1) \right\} H_z \mp ik(\Gamma_1 + \Gamma_2) \frac{\partial H_z}{\partial y} = 0 \tag{9}$$

showing that the problem is a scalar one for the component H_z. For simplicity it is assumed that Re. $\Gamma_1, \Gamma_2 > 0$.

An integral representation for H_z is

$$H_z(x, y) = H_z^i(x, y) - \frac{i}{4} \int_0^\infty \left\{ \frac{\partial H_z}{\partial y'} \Big|_-^+ + H_z \Big|_-^+ \frac{\partial}{\partial y} \right\} H_o^{(1)}\left(k\sqrt{(x - x')^2 + y^2} \right) dx'$$

$$= H_z^i(x, y) - \frac{kY}{4} \int_0^\infty \left\{ J_z^*(x') + \frac{iZ}{k} J_x(x') \frac{\partial}{\partial y} \right\} H_o^{(1)}\left(k\sqrt{(x - x')^2 + y^2} \right) dx' \tag{10}$$

where

$$\bar{J}(x) = \hat{y} \times \bar{H} \Big|_-^+ = \hat{x} H_z \Big|_-^+ \tag{11}$$

is the total electric current supported by the half plane, and

$$\bar{J}^*(x) = -\hat{y} \times \bar{E} \Big|_-^+ = \hat{z} E_x \Big|_-^+ = \hat{z} \frac{iZ}{k} \frac{\partial H_z}{\partial y} \Big|_-^+ \tag{12}$$

is the total magnetic current. The solution is required subject to the edge condition $J_x(x) = O\{(kx)^{\varepsilon_1}\}$ and $J_z^*(x) = O\{(kx)^{-1+\varepsilon_2}\}$ for small kx where $\varepsilon_1 > 0$ and $0 < \varepsilon_2 \le 1$. Accordingly, the integral in (10) converges and the Fourier transforms of $J_x(x)$ and $J_z^*(x)$ both exist. Nevertheless, it is not possible to apply the derivatives in (9) to the integrand in (10), and were we to do so, the application of a Fourier transform to the resulting integral could not be justified.

To avoid this difficulty, we consider the integrals with respect to x of the various field quantities. If

$$\mathcal{H}_z(x, y) = \int^x H_z(x', y)\, dx' \tag{13}$$

with similar definitions for the other script quantities, the boundary conditions on $\mathcal{H}_z(x, y)$ are (see (9))

$$\left\{\frac{\partial}{\partial x^2} + k^2(\Gamma_1 \Gamma_2 + 1)\right\}\mathcal{H}_z \mp ik(\Gamma_1 + \Gamma_2)\frac{\partial \mathcal{H}_z}{\partial y} = A^\pm \tag{14}$$

on $y = \pm 0$, $x \ge 0$ where $A^\pm$ are arbitrary constants, and the representation for $\mathcal{H}_z(x, y)$ is (see (10))

$$\mathcal{H}_z(x, y) = \mathcal{H}_z^i(x, y) - \frac{kY}{4}\int_0^\infty \left\{\mathcal{J}_z^*(x') + \frac{iZ}{4}\mathcal{J}_x(x')\frac{\partial}{\partial y}\right\} H_o^{(1)}\left(k\sqrt{(x - x')^2 + y^2}\right) dx' \tag{15}$$

where

$$\mathcal{H}_z^i(x, y) = -\frac{1}{ik \cos \phi_o} e^{-ik(x \cos \phi_o + y \sin \phi_o)} . \tag{16}$$

By addition and subtraction of the boundary conditions (14), we have

$$\left\{\frac{\partial^2}{\partial x^2} + k^2(\Gamma_1 \Gamma_2 + 1)\right\}\left(\mathcal{H}_z^+ + \mathcal{H}_z^-\right) = k^2 Y(\Gamma_1 + \Gamma_2)\mathcal{J}_z^* + A^+ + A^- \tag{17}$$

$$ik(\Gamma_1 + \Gamma_2)\left(\frac{\partial \mathcal{H}_z^+}{\partial y} + \frac{\partial \mathcal{H}_z^-}{\partial y}\right) = \left\{\frac{\partial^2}{\partial x^2} + k^2(\Gamma_1 \Gamma_2 + 1)\right\}\mathcal{J}_x - A^+ + A^- , \tag{18}$$

and when these are applied to the representation (15), we obtain

$$k^2 Y(\Gamma_1 + \Gamma_2)\mathcal{J}_z^*(x) = -A^+ - A^- + \frac{2ik}{\cos \phi_o}(\Gamma_1 \Gamma_2 + \sin^2 \phi_o)\, e^{-ikx \cos \phi_o}$$

$$-\frac{kY}{2}\left\{\frac{\partial^2}{\partial x^2} + k^2(\Gamma_1\Gamma_2 + 1)\right\}\int_0^\infty \mathcal{J}_z^*(x')\, H_o^{(1)}(k\,|x - x'|)\, dx' \tag{19}$$

$$\left\{\frac{\partial^2}{\partial x^2} + k^2(\Gamma_1\Gamma_2 + 1)\right\}\mathcal{J}_x(x) = A^+ - A^- + 2ik \tan \phi_o\, (\Gamma_1 + \Gamma_2)\, e^{-ikx \cos \phi_o}$$

$$-\frac{k}{2}(\Gamma_1 + \Gamma_2)\left(\frac{\partial^2}{\partial x^2} + k^2\right)\int_0^\infty \mathcal{J}_x(x')\, H_o^{(1)}(k\,|x - x'|)\, dx' \tag{20}$$

valid for $x \geq 0$. These are Wiener-Hopf integral equations for $\mathcal{J}_z^*(x)$ and $\mathcal{J}_x(x)$ and can be solved in the usual manner. For simplicity it will be assumed that k has a small positive imaginary part which can be put equal to zero at the conclusion of the analysis.

Consider (20). We first extend the validity of the equation to $-\infty < x < \infty$ by letting $\Phi_1(x)$ be the value of the integral portion of the right hand side when $x < 0$. If the Fourier transform of a function h(x) is defined as

$$\bar{h}\,(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} e^{-i\xi x}\, h(x)\, dx \,,$$

application of a Fourier transform to the extended version of (20) gives

$$\bar{\Phi}_1\,(\xi) + \frac{i}{\xi\sqrt{2\pi}}(A^+ - A^-) - k\sqrt{\frac{2}{\pi}}\;(\Gamma_1 + \Gamma_2)\,\frac{\tan\phi_o}{\xi + k\cos\phi_o} =$$

$$-\,\Gamma_1\,\Gamma_2\,(k^2 - \xi^2)\left(\frac{1}{\Gamma_1} + \frac{k}{\sqrt{k^2 - \xi^2}}\right)\left(\frac{1}{\Gamma_1} + \frac{k}{\sqrt{k^2 - \xi^2}}\right)\mathcal{J}_x(\xi)\,. \qquad (21)$$

Let

$$K_i\,(\xi)\,K_i\,(-\xi) = \left(\frac{1}{\Gamma_i} + \frac{k}{\sqrt{k^2 - \xi^2}}\right)^{-1} \qquad (i = 1, 2) \qquad (22)$$

where $K_i(\xi)$ is analytic and free of zeros in an upper half plane. If $\Gamma_i = 1/\eta$, $K_i(\xi)$ is identical to the function $K_+(\xi)$ given in [21], and when (22) is inserted into (21), the terms can be separated according to their half planes of analyticity to give

$$\bar{\Phi}_1(\xi)\,\frac{K_1(\xi)\,K_2(\xi)}{k + \xi} + \frac{i}{\xi\sqrt{2\pi}}(A^+ - A^-)\left\{\frac{K_1(\xi)\,K_2(\xi)}{k + \xi} - \frac{K_1(0)\,K_2(0)}{k}\right\}$$

$$-k\sqrt{\frac{2}{\pi}}\;(\Gamma_1 + \Gamma_2)\,\frac{\tan\phi_o}{\xi + k\cos\phi_o}\left\{\frac{K_1(\xi)\,K_2(\xi)}{k + \xi} - \frac{K_1(-k\cos\phi_o)\,K_2(-k\cos\phi_o)}{k(1 - \cos\phi_o)}\right\}$$

$$= -\,\frac{i}{k\xi\sqrt{2\pi}}(A^+ - A^-)\,K_1(0)\,K_2(0) + \sqrt{\frac{2}{\pi}}\;(\Gamma_1 + \Gamma_2)\,\frac{\tan\phi_o}{\xi + k\cos\phi_o}$$

$$\bullet\,\frac{K_1(-k\cos\phi_o)\,K_2(-k\cos\phi_o)}{1 - \cos\phi_o} - \Gamma_1\,\Gamma_2\,\frac{k - \xi}{K_1(\xi)\,K_2(\xi)}\;\mathcal{J}_x(\xi)\,.$$

Since the half planes overlap producing a common strip of analyticity $-\mathrm{Im}.k < \mathrm{Im}.\xi <$ min.

$\{0, -\mathrm{Im}.k \cos \phi_o\}$, application of Liouville's theorem shows that each side of the equation is at most a constant, and from the order of the functions as $|\xi| \to \infty$, the constant must be zero. Hence

$$\mathcal{J}_x(\xi) = \sqrt{\frac{2}{\pi}}\, \frac{\Gamma_1 + \Gamma_2}{\Gamma_1 \Gamma_2}\, \frac{\tan \phi_o}{\xi + k \cos \phi_o}\, \frac{K_1(-\xi)\, K_2(-k \cos \phi_o)\, K_2(-\xi)\, K_2(-k \cos \phi_o)}{\xi(1 - \cos \phi_o)(k - \xi)}$$

$$\bullet \left\{\xi - B(\xi + k \cos \phi_o)\right\}$$

where

$$B = \frac{i}{2k}\, \frac{1 - \cos \phi_o}{(\Gamma_1 + \Gamma_2) \tan \phi_o}\, \frac{K_1(0)\, K_2(0)}{K_1(-k \cos \phi_o)\, K_2(-k \cos \phi_o)}\, (A^+ - A^-)\, .$$

But from the edge condition $\mathcal{J}_x(\xi) = O(|\xi|^{-2 - \varepsilon_1})$ for large $|\xi|$ with $\varepsilon_1 > 0$, and this is only possible if B=1, implying

$$A^+ - A^- \;=\; -\,2ik(\Gamma_1 + \Gamma_2)\, X\, \frac{\tan \phi_o}{1 - \cos \phi_o} \tag{23}$$

with

$$X = \frac{K_1(-k \cos \phi_o)\, K_2(-k \cos \phi_o)}{K_1(0)\, K_2(0)}\, , \tag{24}$$

and the final expression for $\mathcal{J}_x(\xi)$ is

$$\mathcal{J}_x(\xi) = -k\sqrt{\frac{2}{\pi}}\, \frac{\Gamma_1 + \Gamma_2}{\Gamma_1 \Gamma_2}\, \frac{\sin \phi}{\xi + k \cos \phi_o}\, \frac{K_1(-\xi)\, K_1(-k \cos \phi_o)\, K_2(\xi)\, K_2(-k \cos \phi_o)}{\xi(1 - \cos \phi_o)(k - \xi)}\, . \tag{25}$$

This is $O(|\xi|^{-3})$ for large $|\xi|$, and as will be evident later, it is in accordance with the reciprocity condition.

The solution of the integral equation (19) for the magnetic current can be obtained in a similar manner. On applying a Fourier transform to the equation extended to the whole range $-\infty < x < \infty$, we obtain

$$\bar{\Phi}_2(\xi)\,\frac{K_1(\xi)\,K_2(\xi)}{\sqrt{k+\xi}} - \frac{i}{\xi\sqrt{2\pi}}\,(A^+ + A^-)\left\{\frac{K_1(\xi)\,K_2(\xi)}{\sqrt{k+\xi}} - \frac{K_1(0)\,K_2(0)}{\sqrt{k}}\right\}$$

$$- k\sqrt{\frac{2}{\pi}}\;\frac{\Gamma_1\,\Gamma_2 + \sin^2\phi_o}{\cos\phi_o\,(\xi + k\cos\phi_o)}\left\{\frac{K_1(\xi)\,K_2(\xi)}{\sqrt{k+\xi}} - \frac{K_1(-k\cos\phi_o)\,K_2(-k\cos\phi_o)}{\sqrt{k(1-\cos\phi_o)}}\right\}$$

$$= \frac{i}{\xi\sqrt{2\pi k}}\,(A^+ + A^-)\,K_1(0)\,K_2(0) + \sqrt{\frac{2k}{\pi}}\;\frac{\Gamma_1\,\Gamma_2 + \sin^2\phi_o}{\cos\phi_o(\xi + k\cos\phi_o)}\;\frac{K_1(-k\cos\phi_o)\,K_2(-k\cos\phi_o)}{\sqrt{(1-\cos\phi_o)(k-\xi)}}$$

$$- kY\,\Gamma_1\,\Gamma_2\,\frac{\sqrt{k-\xi}}{K_1(-\xi)\,K_2(-\xi)}\,\mathcal{J}_z^*(\xi)\,.$$

From Liouville's theorem and the order of the functions involved, each side of the equation is zero and hence

$$\mathcal{J}_z^*(\xi) = Z\sqrt{\frac{2}{\pi k}}\;\frac{1}{\Gamma_1\,\Gamma_2}\;\frac{1}{\xi + k\cos\phi_o}\;\frac{K_1(-\xi)\,K_1(-k\cos\phi_o)\,K_2(-\xi)\,K_2(-k\cos\phi_o)}{\xi\sqrt{(1-\cos\phi_o)(k-\xi)}}$$

$$\cdot\left\{\left(\frac{\Gamma_1\,\Gamma_2 + 1}{\cos\phi_o} - \cos\phi_o\right)\xi + B'(\xi + k\cos\phi_o)\right\} \tag{27}$$

with

$$B' = \frac{i}{2kX}\,\sqrt{1-\cos\phi_o}\;\;(A^+ + A^-)\,. \tag{28}$$

We observe that $\mathcal{J}_z^*(\xi) = O(|\xi|^{-3/2})$ for large $|\xi|$ in accordance with the edge condition, and to satisfy reciprocity we write

$$B' = b - \frac{\Gamma_1 \Gamma_2 + 1}{\cos \phi_o}$$

where b is independent of ϕ_o. In terms of b

$$A^+ + A^- = 2ik\,(\Gamma_1 \Gamma_2 + 1 - b\cos\phi_o)\,\frac{X}{\cos\phi_o \sqrt{1 - \cos\phi_o}} \tag{29}$$

and

$$\mathfrak{J}_z^*(\xi) = -Z\sqrt{\frac{2}{\pi k}}\,\frac{1}{\Gamma_1 \Gamma_2}\,\frac{1}{\xi + k\cos\phi_o}\,\frac{K_1(-\xi)\,K_1(-k\cos\phi_o)\,K_2(-\xi)\,K_2(-k\cos\phi_o)}{\xi\sqrt{(1-\cos\phi_o)(k-\xi)}}$$

$$\bullet \left\{(\Gamma_1 \Gamma_2 + 1)k + \xi\cos\phi_o - b(\xi + k\cos\phi_o)\right\} \tag{30}$$

whose order is independent of the choice of b. Thus, the standard edge condition does not serve to specify b.

From (15), on using the Fourier integral representation of the Hankel function

$$H_z(x, y) = H_z^i(x, y) - \frac{1}{4}\sqrt{\frac{2}{\pi}} \int_C \left\{kY\mathfrak{J}_z^*(\xi) - \frac{y}{|y|}\sqrt{k^2 - \xi^2}\;\mathfrak{J}_x(\xi)\right\}$$

$$\bullet \exp\left(i\xi x + i\,|y|\sqrt{k^2 - \xi^2}\right) \frac{d\xi}{\sqrt{k^2 - \xi^2}}$$

where C is a path extending from $x = -\infty$ to $x = \infty$ in the strip of analyticity, and when the expressions for $\mathfrak{J}_z(\xi)$ and $\mathfrak{J}_z^*(\xi)$ are inserted, we obtain

$$\mathcal{H}_z(x, y) = \mathcal{H}_z^i(x, y) + \frac{1}{2\pi} \int_C \frac{K_1(-\xi)\,K_1(-k\cos\phi_o)\,K_2(-\xi)\,K_2(-k\cos\phi_o)}{\Gamma_1 \Gamma_2\,(\xi + k\cos\phi_o)\,\xi\sqrt{(1-\cos\phi_o)(k-\xi)/k}}$$

$$\bullet \left\{ (\Gamma_1 \Gamma_2 + 1)k + \xi \cos \phi_o - b(\xi + k \cos \phi_o) - \frac{y}{|y|} (\Gamma_1 + \Gamma_2) \sqrt{k(1 + \cos \phi_o)(k+\xi)} \right\}$$

$$\bullet \exp\left(i\xi x + i|y| \sqrt{k^2 - \xi^2} \right) \frac{d\xi}{\sqrt{k^2 - \xi^2}} . \tag{31}$$

It can be verified that the boundary conditions (14) are satisfied when b is related to the constants $A^{\pm}$ as shown in (29). When the differentiations in (14) are carried out and y put equal to ± 0, (22) can be used to show that the integrand is analytic in the half plane above the contour apart from the poles at $\xi = 0$ and $\xi = -k \cos \phi_o$. The residue at the former reproduces the constants $A^{\pm}$, the residue at the latter annuls the incident field contribution, and for $x > 0$ the contour can be closed in the upper half plane. Finally, since the integrand in (31) can be differentiated with respect to x, we have

$$H_z(x, y) = H_z^i(x, y) - \frac{1}{2\pi} \int_C \frac{K_1(-\xi)\, K_1(-k \cos \phi_o)\, K_2(-\xi)\, K_2(-k \cos \phi_o)}{\Gamma_1 \Gamma_2 (\xi + k \cos \phi_o) \sqrt{(1 - \cos \phi_o)(k - \xi)/k}} \left\{ (\Gamma_1 \Gamma_2 + 1)k \right.$$

$$\left. + \xi \cos \phi_o - b(\xi + k \cos \phi_o) - \frac{y}{|y|} (\Gamma_1 + \Gamma_2) \left\{ \sqrt{k(1+\cos \phi_o)(k+\xi)} \right\} \right\}$$

$$\bullet \exp\left(i\xi x + i|y| \sqrt{x^2 - \xi^2} \right) \frac{d\xi}{\sqrt{k^2 - \xi^2}} . \tag{32}$$

This represents a solution of the diffraction problem and we observe that $H_z^s = O\{(kx)^{1/2}\}$, $E_x^s, E_y^s = O\{(kx)^{-1/2}\}$ for small kx. The reciprocity condition is also satisfied as evident from the symmetry in α and ϕ_o of the non-exponential portion of the integrand when the variable of integration is changed to α with $\xi = k \cos \alpha$.

4. Specification of the constant

In view of the arbitrary constant b, the solution (32) is not unique, and we now show how the edge condition must be supplemented to ensure uniqueness.

Using Maxwell's equations, the boundary conditions (14) can be written as

$$E_y = ik(\Gamma_1\Gamma_2 + 1)\mathcal{H}_z \mp ik(\Gamma_1 + \Gamma_2)\mathcal{E}_x - \frac{iZ}{k}A^{\pm} \qquad (33)$$

for $y = \pm 0$, $x \geq 0$, and (33) is also evident from (8). As $x \to 0$

$$\mathcal{H}_z \to \mathcal{H}_z^i = -(ik\cos\phi_o)^{-1}, \qquad \mathcal{E}_x \to \mathcal{E}_x^i = -\frac{Z}{ik}\tan\phi_o$$

and hence

$$E_y^{\pm} = -Z\left\{\frac{\Gamma_1\Gamma_2 + 1}{\cos\phi_o} \mp (\Gamma_1 + \Gamma_2)\tan\phi_o + \frac{i}{k}A^{\pm}\right\}$$

where

$$E_y^{\pm} = \lim_{x \to +0} E_y(x, \pm 0) \qquad (34)$$

are the surface values of E_y at the edge. Inserting the expressions for $A^{\pm}$ derived from (23) and (29), it follows that

$$\frac{1}{2}(E_y^+ - E_y^-) = Z(\Gamma_1 + \Gamma_2)\sqrt{\frac{1 + \cos\phi_o}{1 - \cos\phi_o}}\;\frac{1 - \cos\phi_o - X}{\cos\phi_o}, \qquad (35)$$

$$\frac{1}{2}(E_y^+ + E_y^-) = -\frac{Z}{\sqrt{1 - \cos\phi_o}}\left\{(\Gamma_1\Gamma_2 + 1)\frac{\sqrt{1 - \cos\phi_o} - X}{\cos\phi_o} + bX\right\}, \qquad (36)$$

and whereas the first of these is independent of b and therefore specified by the boundary condition, the second is a function of b and can be adjusted.

Both are finite for all ϕ_o, $0 \le \phi_o \le 2\pi$. From the expression for $K_+(\xi)$ in (21) of [21] with $\eta = 1/\Gamma_i$ it is found that as $\cos\phi_o \to 1$, $X \to C_1(1 - \cos\phi_o)$ where C_1 is a constant, implying

$$\frac{1}{2}(E_y^+ - E_y^-) \to 0, \qquad \frac{1}{2}(E_y^+ + E_y^-) \to -Z(\Gamma_1\Gamma_2 + 1).$$

Thus, for incidence along the plane, the surface values at the edge are the same and equal to $-Z(\Gamma_1\Gamma_2 + 1)$. They are also equal at edge-on incidence($\phi_o = \pi$), but near normal incidence for which $|\cos\phi_o| = \varepsilon << 1$,

$$X = 1 - (C_2 + 1)\varepsilon + O(\varepsilon^2)$$

with

$$C_2 = \frac{1}{2\pi}\left\{\frac{1}{\sqrt{\Gamma_1^2 - 1}}\log\frac{\Gamma_1 - \sqrt{\Gamma_1^2 - 1}}{\Gamma_1 + \sqrt{\Gamma_1^2 - 1}} + \frac{1}{\sqrt{\Gamma_2^2 - 1}}\log\frac{\Gamma_2 - \sqrt{\Gamma_2^2 - 1}}{\Gamma_2 + \sqrt{\Gamma_2^2 - 1}}\right\},$$

showing that as $\cos\phi_o \to 0$

$$\frac{1}{2}(E_y^+ - E_y^-) \to Z(\Gamma_1 + \Gamma_2)C_2, \qquad \frac{1}{2}(E_y^+ + E_y^-) \to -Z\left\{(\Gamma_1\Gamma_2 + 1)(C_2 + \frac{1}{2}) + b\right\}.$$

From (30) it is now evident that a unique solution to the diffraction problem is assured by specifying the (finite) value of $\frac{1}{2}(E_y^+ + E_y^-)$ at any one angle of incidence other than grazing or edge-on. This information should be furnished in addition to the standard edge condition, and must be derived from a knowledge of the particular structure which the boundary condition models.

The procedure is applicable to higher order boundary conditions as well, and to illustrate this fact, consider the third order one. The relevant boundary condition is given in

[4] and when specialized to be the case of an H-polarized plane wave incident in a plane perpendicular to the edge, the condition becomes

$$E_x = \pm Z \frac{a_2 + a_o}{a_3 + a_1} H_z \mp \frac{a_2}{ik(a_3 + a_1)} \frac{\partial E_y}{\partial x} - \frac{1}{k^2(a_3 + a_1)} \frac{\partial^2 E_x}{\partial x^2} \tag{37}$$

on $y = \pm 0$, $x \geq 0$, where

$$a_0 = \Gamma_1 \Gamma_2 \Gamma_3, \qquad a_1 = \Gamma_1 \Gamma_2 + \Gamma_2 \Gamma_3 + \Gamma_3 \Gamma_1$$

$$a_2 = \Gamma_1 + \Gamma_2 + \Gamma_3, \qquad a_3 = 1.$$

In scalar form the condition is

$$\left\{a_3 \frac{\partial^2}{\partial x^2} + k^2(a_3 + a_1)\right\} \frac{\partial H_z}{\partial y} \pm \left\{a_2 \frac{\partial^2}{\partial x^2} + k^2(a_2 + a_0)\right\} H_z = 0, \tag{38}$$

but to ensure the validity of the solution technique, it is necessary to employ the double integral with respect to x of all field qualities, e.g.

$$\mathcal{H}_z^{(2)}(x, y) = \left(\int^x dx' \right)^2 H_z(x', y), \tag{39}$$

with a similar definition of other script quantities, The boundary condition for $\mathcal{H}_z^{(2)}$ is then

$$\left\{a_3 \frac{\partial^2}{\partial x^2} + k^2(a_3 + a_1)\right\} \frac{\partial \mathcal{H}_z^{(2)}}{\partial y} \pm \left\{a_2 \frac{\partial^2}{\partial x^2} + k^2(a_2 + a_0)\right\} \mathcal{H}_z^{(2)} = A_0^{\pm} + x A_1^{\pm} \tag{40}$$

on $y = \pm 0$, $x \geq 0$, where $A_0^{\pm}$ and $A_1^{\pm}$ are four arbitrary constants.

As a result of the Wiener-Hopf solution, the expressions for the integrated currents each contain a constant related to $(A_0^+ \pm A_0^-) + (A_1^+ \pm A_0^1)$ with the upper sign for the electric currents and the lower sign for the magnetic, and two equations involving $A_0^{\pm}$ and

$A_1^{\pm}$ are specified as well. To obtain a unique solution of the problem it is therefore necessary to provide two additional pieces of information.

The boundary condition (40) can be written as

$$E_x = \pm a_2 ZH_z - k^2(1 + a_1)\mathcal{E}_x^{(2)} + k^2Z(a_2 + a_0)\mathcal{H}_z^{(2)} - \frac{iZ}{k}(A_0^{\pm} + xA_1^{\pm})$$

where we have used the fact that $a_3 = 1$, and because all of the field quantities on the right hand side approach their incident values as $x \to +0$, it follows that

$$E_x^{\pm} = Z\left\{\pm a_2 + (1 + a_1)\frac{\sin\phi_o}{\cos^2\phi_o} - \frac{a_0 + a_2}{\cos^2\phi_o} - \frac{i}{k}A_0^{\pm}\right\}. \tag{41}$$

where

$$E_x^{\pm} = \lim_{x \to +0} E_x(x, \pm 0).$$

Similarly,

$$\frac{\partial E_x^{\pm}}{\partial x} \mp ika_2 E_y^{\pm} = -ikZ\left\{(1 + a_1)\tan\phi_o - \frac{a_0 + a_2}{\cos\phi_o} + \frac{1}{k^2}A_1^{\pm}\right\} \tag{42}$$

and we can specify, for example, E_x^+ and E_x^- or

$$\frac{1}{2}(E_x^+ + E_x^-) \quad \text{and} \quad \frac{1}{2}\left(\frac{\partial E_x^+}{\partial x} + \frac{\partial E_x^-}{\partial x}\right) - \frac{ika_2}{2}\left(E_y^+ - E_y^-\right).$$

Since the left hand side of (42) is simply $-\left(a_3\frac{\partial E_y^{\pm}}{\partial y} \pm ika_2 E_y^{\pm}\right)$ which closely resembles the first two terms of the boundary condition expressed as a function of E_y, the extension of the procedure to boundary conditions of still higher order is evident.

5. Concluding Remarks

The problem of the diffraction of a plane wave by a half plane satisfying second order boundary conditions has been examined with particular attention to the reciprocity and uniqueness of the solution obtained. Whereas self-adjointness automatically ensures a reciprocal solution for boundary conditions of order zero and one, conditions of higher order (greater than or equal to the order of the wave equation) do not in general lead to a self-adjoint problem. The conditions must then allow for reciprocity to be imposed explicitly.

Unfortunately, this still does not ensure uniqueness as evidenced by the arbitrary constant appearing in the solution for the second order problem. In the general case of an Nth order boundary condition, the solution contains N-1 arbitrary constants, either equally divided between the expressions for the magnetic and elective currents (if N is odd), or with the magnetic current having one more than the electric (if N is even). These do not affect the edge behavior of the spatial fields, and to specify them requires N-1 items of supplementary information over and above the standard edge conditions. As we have shown using the second and third order boundary conditions as examples, the information consists of the values of certain field components on the top and bottom surfaces of the half plane at the edge, and this must be derived from a consideration of the physical structure being modelled by the boundary condition.

In addition to these main issues, there are two others worth mentioning. The rigorous analysis of a GIBC problem using the Wiener-Hopf technique must be carried out in items of integrated (with respect to x) field quantities to ensure the existence of the Fourier transforms involved [22]. For the second order problem considered here, a single integration is required with the boundary conditions expressed in terms of the tangential field components, but to use the boundary conditions involving the normal field components, a further integration is necessary. For the Nth order condition, the corresponding numbers are N-1 and N respectively. Although it is possible to arrive at the correct solution without integration, it is not only rigor that is sacrificed; it is then difficult if not impossible to connect the constants which appear in the solution to the surface values of the field components at the edge.

In carrying out the second order solution it was assumed that Re. $\Gamma_1, \Gamma_2 > 0$, but for the Nth order solution with N>1, one or more of the Γ_m may have negative real parts and in general will. If Re. $\Gamma_i < 0$ the corresponding surface wave pole becomes explicit, and the Wiener-Hopf split must be modified. In particular, if

$$\frac{1}{\Gamma_i} + \frac{k}{\sqrt{k^2 - \xi^2}} = \left\{ \tilde{K}\left(\xi, \frac{1}{\Gamma_i}\right) \tilde{K}\left(-\xi, \frac{1}{\Gamma_i}\right)\right\}^{-1}$$

then

$$\tilde{K}\left(\xi, \frac{1}{\Gamma_i}\right) = \begin{cases} K_i\left(\xi, \frac{1}{\Gamma_i}\right) & (\text{Re. } \Gamma_i > 0) \\ \left\{\frac{i}{\Gamma_i}\left(\xi - k\sqrt{1 - \Gamma_i^2}\right) K_i\left(\xi, \frac{-1}{\Gamma_i}\right)\right\}^{-1} & (\text{Re. } \Gamma_i < 0) \end{cases}$$

where $K_i(\xi, \frac{1}{\Gamma_i}) = K_i(\xi)$ is given in (22), and the branch of $\sqrt{1 - \Gamma_i^2}$ is such that Im. $k\sqrt{1 - \Gamma_i^2} < 0$. Provided Im. $\Gamma_i \neq 0$ implying $\sqrt{1 - \Gamma_i^2} \neq 0$, the non-physical pole at $\xi = k\sqrt{1 - \Gamma_i^2}$ can be excluded from the strip of analyticity. The requirement for this is

$$\left|\{\text{Re. } k\}\left\{\text{Im. } \sqrt{1 - \Gamma_i^2}\right\}\right| > |\text{Im. } k|\left\{1 + \text{Re. } \sqrt{1 - \Gamma_i^2}\right\}$$

which can be achieved by making Im.k sufficiently small.

References

[1] S.N. Karp and F.C. Karal, Jr., "Generalized Impedance Boundary Conditions with Applications to Surface Wave Structures," in Electromagnetic Wave Theory, Part 1, ed. J. Brown, pp. 479-483, Pergamon: New York, 1965.

[2] B. Engquist and A. Majda, "Absorbing Boundary Conditions for the Numerical Simulation of Waves," Math. Comput., vol. 31, pp. 629-651, 1977.

[3] L.N. Trefethen and L. Halpern, "Well-Posedness of One-Way Equations and Absorbing Boundary Conditions," Math. Comp., vol. 47, pp. 421-435, 1986.

[4] T.B.A. Senior and J.L. Volakis, "Derivation and Application of a Class of Generalized Boundary Conditions," to be published in IEEE Trans. Antennas Propagat., vol. AP-37, 1989.

[5] J.L. Volakis, "Numerical Implementation of Generalized Impedance Boundary Conditions," presented at URSI International Electromagnetic Wave Theory Symposium, Stockholm, Sweden, 14-17 August 1989.

[6] J.-M.L. Bernard, "Diffraction by a Metallic Wedge Covered with a Dielectric Material," Wave Motion, vol. 9, pp. 543-561, 1987.

[7] T.B.A. Senior, "Diffraction by a Right-Angled Second Order Impedance Wedge," Electromagnetics, vol. 9, pp. 313-330, 1989.

[8] J.L. Volakis and T.B.A. Senior, "Application of a Class of Generalized Boundary Conditions to Scattering by a Metal-Backed Dielectric Half-Plane," Proc. IEEE, vol. 77, pp. 796-805, 1989.

[9] T.B.A. Senior, "Diffraction by a Generalized Impedance Half Plane," presented at URSI International Electromagnetic Wave Theory Symposium, Stockholm, Sweden, 14-17 August 1989.

[10] T.B.A. Senior, "Combined Resistive and Conductive Sheets," IEEE Trans. Antennas Propagat., vol. AP-33, pp. 577-579, 1985.

[11] T.B.A. Senior and J.L. Volakis, "Sheet Simulation of a Thin Dielectric Layer," Radio Sci., vol. 22, pp. 1261-1272, 1987.

[12] A.L. Weinstein, The Theory of Diffraction and the Factorization Method, Golem Press: Boulder, CO., 1969.

[13] F.G. Leppington, "Traveling Waves in a Dielectric Slab with an Abrupt Change in Thickness," Proc. Roy. Soc. (London), vol. A386, pp. 443-460, 1983.

[14] A. Chakrabarti, "Diffraction by a Dielectric Half-Plane," IEEE Trans. Antennas Propagat., vol. AP-34, pp. 830-833, 1986.

[15] J.L. Volakis and T.B.A. Senior, "Diffraction by a Thin Dielectric Half-Plane," IEEE Trans. Antennas Propagat., vol. AP-35, pp. 1483-1487, 1987.

[16] J.L. Volakis, "High Frequency Scattering by a Material Half-Plane and Strip," Radio Sci., vol. 23, pp. 450-462, 1988.

[17] T.B.A. Senior, "Skew Incidence on a Dielectric Half-Plane," Electromagnetics, vol. 9, pp. 187-200, 1989.

[18] R. Rojas, "Diffraction of EM waves by a Dielectric/Ferrite Half-Plane and Related Configurations," IEEE Trans. Antennas Propagat., vol. AP-37, pp. 751-763, 1989.

[19] M.A. Leontovich, Investigations on Radiowave Propagation, Part II, Moscow: Academy of Sciences, 1948.

[20] T.B.A. Senior, "Some Problems Involving Imperfect Half Planes," in Electromagnetic Scattering, ed. P.L.E. Uslenghi, pp. 185-219, Academic Press: New York, 1978.

[21] T.B.A. Senior, "Diffraction by a Semi-Infinite Metallic Sheet," Proc. Roy. Soc. (London), vol. A213, pp. 436-458, 1952.

[22] T.B.A. Senior, "A Critique of Certain Half Plane Diffraction Analyses," Electromagnetics, vol. 7, pp. 81-90, 1987.