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REFLECTIONS IN A FERRITE FILLED WAVEGUIDE

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TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

ABSTRACT

A solution is obtained for the electric field at the air-ferrite interface ($Z = 0$) in a rectangular waveguide filled with ferrite in the semi-infinite half ($Z > 0$) and magnetized in the direction of the electric field. The field is expressed in terms of a Neumann series obtained by iteration of a singular integral equation which satisfies the boundary conditions at the interface. The equivalent circuit for the junction is also presented.

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C. B. Sharpe and D. S. Heim

INTRODUCTION

The mathematical difficulties which are encountered in the solution of boundary value problems involving gyromagnetic media have been pointed out by several authors.[1,2,3] In most of the problems treated to date some form of perturbation theory or the assumption of quasi-stationary fields has been necessary to make the problem tractable. The formulation of these problems is usually straightforward but the imposition of the boundary conditions at the isotropic-to-anisotropic interface usually introduces serious complications in obtaining a direct solution. The problem discussed here in which the anisotropic media is a semi-infinite slab of ferrite filling a rectangular waveguide conforms to the established pattern. This problem appears to present the essential difficulties common to the solution of many ferrite boundary-value problems in a relatively simple form.

Referring to Fig. 1 we consider an infinite rectangular waveguide which is filled with a ferrite medium for $z > 0$ and air for $z < 0$. The ferrite region is magnetized in the y-direction with an internal field H. A TE_{10} wave is incident from the left at the air-ferrite interface ($z = 0$). The problem is to determine the electric and magnetic fields at the interface and the equivalent circuit for the junction. The ferrite medium is assumed to be lossless and characterized by a tensor permeability

$$(\mu) = \begin{pmatrix} \mu & 0 & j\kappa \\ 0 & \mu_0 & 0 \\ -j\kappa & 0 & \mu \end{pmatrix} ,$$

1. P. S. Epstein, Rev. Mod. Phys., vol 28, pp. 3-17; January, 1956.

2. A. A. Th. M. VanTrier, Appl. Sci. Res. sect. B, vol. 3, pp. 305-370; 1953.

3. H. Suhl and L. R. Walker, BSTJ, Part I: vol. 33, pp. 579-659, May, 1954; Part II: vol. 33, pp. 939-986, July, 1954; Part III: vol. 33, pp. 1133-1194, September, 1954.

where

$$\frac{\mu}{\mu_o} = 1 + \frac{\Gamma^2 M_s H}{\Gamma^2 H^2 - \omega^2} ,$$

$$\frac{\kappa}{\mu_o} = \frac{\omega \Gamma M_s}{\Gamma^2 H^2 - \omega^2} .$$

In rationalized mks units, which are used throughout, the gyromagnetic ratio Γ is given by

$$\Gamma = -0.22 \times 10^6 \text{ meters/ampere-second.}$$

M_s is defined as the magnetization at saturation using the convention

$$B = \mu_o (H + M) .$$

All field quantities are taken proportional to exp ($j\omega t$).

THE EQUIVALENT CIRCUIT

It has been shown by several authors that in the case where the electric and magnetic fields are independent of the y variable, the y component of the electric field in the ferrite medium satisfies the scalar wave equation[4]

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} + \omega \mu_{\perp} \epsilon_f E_y = 0 ,$$

where $\mu_{\perp}$ is defined as

$$\mu_{\perp} = \frac{\mu^2 - \kappa^2}{\mu} .$$

Assuming the forward traveling electric wave has the form

$$E_y = E_y(x) e^{-\gamma_n^{(2)} z} ,$$

it follows that the transmitted electric field in medium (2) can be expressed by

[4]. See, for example, Ref. 3, Part II.

$$E_y^{(t)} = \sum_{n=1}^{\infty} T_n \sin \frac{n\pi}{a} x \; e^{-\gamma_n^{(2)} z} \;, \tag{1}$$

where

$$\gamma_n^{(2)} = \sqrt{(n\pi/a)^2 - \omega^2 \mu_\perp \epsilon_f} \;.$$

The transverse magnetic field in the ferrite is given by

$$H_x^{(t)} = \sum_{n=1}^{\infty} T_n \left[nM \cos \frac{n\pi}{a} x - Y_n^{(2)} \sin \frac{n\pi}{a} x \right] e^{-\gamma_n^{(2)} z} \;, \tag{2}$$

where we have defined

$$M = \frac{\pi}{a\omega} \frac{\kappa}{\mu^2 - \kappa^2}$$

$$Y_n^{(2)} = \frac{\gamma_n^{(2)}}{j\omega\mu_\perp} \;.$$

In the air-filled waveguide [medium (1)] the incident and reflected waves are given by

$$E_y^{(i)} = \sin \pi x/a \; e^{-\gamma_1^{(1)} z} \tag{3}$$

$$H_x^{(i)} = - Y_1^{(1)} \sin \pi x/a \; e^{-\gamma_1^{(1)} z} \tag{4}$$

$$E_y^{(r)} = \sum_{n=1}^{\infty} R_n \sin n\pi x/a \; e^{\gamma_n^{(1)} z} \tag{5}$$

$$H_x^{(r)} = \sum_{n=1}^{\infty} Y_n^{(1)} R_n \sin n\pi x/a \; e^{\gamma_n^{(1)} z} \;, \tag{6}$$

where

$$\gamma_n^{(1)} = \sqrt{\left(\frac{n\pi}{a}\right)^2 - \omega^2 \mu_o \epsilon_o} \quad \text{and}$$

$$Y_n^{(1)} = \frac{\gamma_n^{(1)}}{j\omega\mu_o} \;.$$

The boundary conditions to be satisfied at z = 0 are

$$\begin{aligned} E_y^{(i)} + E_y^{(r)} &= E_y^{(t)} \;, \\ H_x^{(i)} + H_x^{(r)} &= H_x^{(t)} \;. \end{aligned} \tag{7}$$

It now becomes apparent why an infinite number of reflected and transmitted modes will be required. In effect, the ferrite interface creates a "discontinuity" due to the presence of the cosine terms in (2). The fact that the sine and cosine terms are not orthogonal to each other in the interval $0 \leq x \leq a$ necessitates the use of all possible modes as expressed in equations (1)-(6). The boundary conditions (7) lead to

$$\sin \pi x/a + \sum_{n=1}^{\infty} R_n \sin n\pi x/a = \sum_{n=1}^{\infty} T_n \sin n\pi x/a \tag{8}$$

and

$$- Y_1^{(1)} \sin \pi x/a + \sum_{n=1}^{\infty} Y_n^{(1)} R_n \sin n\pi x/a = \sum_{n=1}^{\infty} T_n \Big[nM \cos n\pi x/a - Y_n^{(2)} \sin n\pi x/a \Big] . \tag{9}$$

It follows from (8) that

$$\begin{aligned} 1 + R_1 &= T_1 \\ R_n &= T_n \; ; \; n > 1 \; . \end{aligned} \tag{10}$$

Using (10) Eq. (9) can be expressed in the form

$$\Big[T_1 \, (Y_1^{(1)} + Y_1^{(2)}) - 2 \, Y_1^{(1)} \Big] \sin \pi x/a = \sum_{n=1}^{\infty} nM \, T_n \cos n\pi x/a - \sum_{n=2}^{\infty} (Y_n^{(1)} + Y_n^{(2)}) \, T_n \sin n\pi x/a \tag{11}$$

Both Epstein[5] and VanTrier[6] have pointed out that Eq. (11) leads to an infinite system of simultaneous linear equations. Unfortunately, this formulation does not lead to a practical solution. Epstein has used a method of successive approximations to obtain a power series expansion for R_1 in terms of $\kappa/(\mu^2 - \kappa^2)$. However, this leaves much to be desired, particularly since $\kappa/(\mu^2 - \kappa^2)$ can be very large. In the solution which follows, Eq. (11) will be expressed as an integral equation in terms of the electric field E_y at the interface. The solution will yield the electric and magnetic fields directly for almost all values of $\kappa/(\mu^2 - \kappa^2)$.

It will be convenient to normalize Eq. (11). Consider the case where only the first mode propagates in the ferrite medium. That is,

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5. Op. cit.

6. Op. cit.

$$(\pi/a)^2 < \omega^2 \mu_1 \epsilon_f < (2\pi/a)^2 \quad .$$

It is assumed in all cases that only the dominant mode propagates in the air-filled section. The transmission line circuit illustrated in Fig. 2 will be equivalent to the waveguide junction if we can identify the voltage and current waves on the line with the fundamental components of the transverse electric and magnetic fields, respectively, in the waveguide. The analogy which we shall employ here makes the constant of proportionality between current and voltage equal to the wave admittance for the dominant mode in the corresponding wave guide. The quantity Re $[V^{(j)} I^{(j)*}]$, $j = i, r, t$, (see Fig. 2) will be proportional to the power flow in the corresponding wave guide. It should be noted that with the coordinate system of Fig. 1 the power will flow in the direction of negative current. In the air-filled section the following definitions will be convenient:

$$E_{y1}^{(i)} \equiv V(z)^{(i)} \sin\frac{\pi x}{a} \quad ; \quad E_{y1}^{(r)} \equiv V(z)^{(r)} \sin\frac{\pi x}{a}$$

$$H_{x1}^{(i)} = -Y_1^{(1)} E_{y1}^{(i)} \equiv I(z)^{(i)} \sin\frac{\pi x}{a} \; ; \; H_{x1}^{(r)} = Y^{(1)} E_{y1}^{(r)} \equiv I(z)^{(r)} \sin\frac{\pi x}{a} \; .$$

The subscript 1 denotes the first mode. However, in the ferrite filled section a fundamental difficulty occurs since $H_{x1}^{(t)}$ is not simply proportional to $E_{y1}^{(t)}$. Nevertheless, a similar correspondence can be made:

$$E_{y1}^{(t)} \equiv V(z)^{(t)} \sin\frac{\pi x}{a}$$

$$I(z)^{(t)} \sin\frac{\pi x}{a} \equiv -Y_1^{(2)} E_{y1}^{(t)} \quad .$$

Power flow in the ferrite medium is still proportional to Re $[V^{(t)} I^{(t)*}]$ since the cosine term in (2) does not contribute to the integral of Poynting's vector over the cross section of the waveguide. Of course, the continuity of H_x at the junction no longer implies the continuity of current flow in the equivalent circuit. This discontinuity in current flow at the junction is accounted for by the current I flowing through the impedance Z. The ideal transformer is taken as 1:1 since the electric field is continuous at the junction. Thus at z = 0:

$$V^{(i)} = 1$$

$$V^{(r)} = R_1$$

$$V^{(t)} = T_1$$

From Fig. 2,

$$V^{(i)} + V^{(r)} = V^{(t)}$$

$$I^{(i)} + I^{(r)} = I + I^{(t)} .$$

It follows that

$$I = T_1 (Y_1^{(1)} + Y_1^{(2)}) - 2 Y_1^{(1)} . \qquad (12)$$

It will be useful to make the change of variable,

$$\phi = \pi x/a .$$

Then, at z = 0,

$$E_y^{(t)}(\phi) = \sum_{n=1}^{\infty} T_n \sin n\phi \; ; \; 0 \le \phi \le \pi , \qquad (13)$$

where

$$T_n = 2/\pi \int_0^{\pi} E_y^{(t)}(\phi) \sin n\phi \, d\phi . \qquad (14)$$

Following Schwinger and Miles,[7] we define a normalized field proportional to $E_y^{(t)}$,

$$E_y^{(t)}(\phi) = I \, \mathcal{E}(\phi) . \qquad (15)$$

Then,

$$V^{(t)} = T_1 = 2/\pi \, I \int_0^{\pi} \mathcal{E}(\phi) \sin n\phi \, d\phi ,$$

and

$$Z = \frac{V^{(t)}}{I} = 2/\pi \int_0^{\pi} \mathcal{E}(\phi) \sin \phi \, d\phi . \qquad (16)$$

If $\mathcal{E}(\phi)$ is expanded in a Fourier series

$$\mathcal{E}(\phi) = \sum_{n=1}^{\infty} (A_n + jB_n) \sin n\phi \; ; \; 0 \le \phi \le \pi ,$$

the impedance Z will be given by

$$Z = A_1 + jB_1 . \qquad (17)$$

It is shown in the Appendix that for a lossless ferrite medium $A_1 = 0$. With the help of Eqs. (12)-(15) and the fact that

7. J. W. Miles, "The Equivalent Circuit for a Plane Discontinuity in a Cylindrical Wave Guide," Proc. IRE, vol. 34, pp. 728-742, October, 1946.

$$\mathcal{E}'(\phi) = 2/\pi \sum_{n=1}^{\infty} n \int_0^{\pi} \mathcal{E}(\phi') \sin n\phi' \cos n\phi \, d\phi' \quad ,$$

Eq. (11) can be written,

$$\sin \phi = M \mathcal{E}'(\phi) - 2/\pi \sum_{n=2}^{\infty} (Y_n^{(1)} + Y_n^{(2)}) \int_0^{\pi} \mathcal{E}(\phi') \sin n\phi' \sin n\phi \, d\phi' \; ; \quad 0 \le \phi \le \pi \; . \tag{18}$$

$\mathcal{E}'(\phi)$ denotes the derivative of $\mathcal{E}(\phi)$ with respect to ϕ.

When all modes are cut-off in the ferrite,

$$\omega^2 \mu_\perp \epsilon_f < (\pi/a)^2 \quad ,$$

and $Y_1^{(2)}$ is imaginary. The equivalent circuit of Fig. 2 will suffice for this case also with the understanding that no power is transmitted away from the junction to the right. We cannot neglect $I^{(t)}$ since it gives rise to an inductive susceptance in parallel with Z. The impedance Z accounts for the discontinuity in the sinusoidal component of H_x as before. The theory which follows will therefore be valid for both the case where only the dominant mode propagates in the ferrite and the case where all modes are cut-off.

THEORY

In order to solve Eq. (18) for $\mathcal{E}(\phi)$ it is necessary to make a commonly used assumption; namely,

$$\gamma_n^{(1)} = \gamma_n^{(2)} \cong n\pi/a \quad ; \quad n > 1 \; .$$

Then $Y_n^{(1)} + Y_n^{(2)}$ can be approximated by

$$Y_n^{(1)} + Y_n^{(2)} \cong -jKn \quad ; \quad n > 1 \; , \tag{19}$$

where

$$K = \pi/a\omega \left[1/\mu_o + 1/\mu_\perp\right] \; .$$

Eq. (19) is usually a reasonable assumption to make for problems where only the first mode propagates. However, we shall find in the present problem that this assumption is of critical importance for the case where M/K = 1.

With this assumption, Eq. (18) can be written,

$$C \sin \phi = M \mathcal{E}'(\phi) + jK\, 2/\pi \sum_{n=1}^{\infty} n \int_0^{\pi} \mathcal{E}_i(\phi') \sin n\phi' \sin n\phi \, d\phi' \quad , (20)$$

where

$$C = 1 + jK\, 2/\pi \int_0^{\pi} \mathcal{E}(\phi) \sin \phi \, d\phi = 1 - KB_1 \; . \quad (21)$$

Integrating the last term in (20) by parts and employing the identity[8]

$$\sin n\phi = 1/\pi \int_0^{\pi} \frac{\sin \phi \cos n\psi}{\cos \psi - \cos \phi} \, d\psi \; ; \; n = 0, 1, 2, \ldots$$

one obtains

$$C \sin \phi = M \mathcal{E}'(\phi) + jK\, 2/\pi^2 \sin \phi \sum_{n=1}^{\infty} \int_0^{\pi} \mathcal{E}'(\phi') \cos n\phi' d\phi' \int_0^{\pi} \frac{\cos n\psi}{\cos \psi - \cos \phi} \, d\psi \; . \quad (22)$$

Interchanging the order of integration and summation in (22) and using the fact that

$$\mathcal{E}'(\psi) = 2/\pi \sum_{n=1}^{\infty} \int_0^{\pi} \mathcal{E}'(\phi') \cos n\phi' \cos n\psi \, d\phi' \; ,$$

we obtain a singular integral equation of the second kind.

$$C \sin \phi = M \mathcal{E}'(\phi) + jK/\pi \int_0^{\pi} \frac{\mathcal{E}'(\psi) \sin \phi}{\cos \psi - \cos \phi} \, d\psi \; . \quad (23)$$

It is convenient to define

$$\mathcal{E}(\phi) = \mathcal{E}_r(\phi) + j\mathcal{E}_i(\phi) \; .$$

Equation (23) can then be expressed by the system,

$$C \sin \phi = M \mathcal{E}_r'(\phi) - K/\pi \int_0^{\pi} \frac{\mathcal{E}_i'(\psi) \sin \phi}{\cos \psi - \cos \phi} \, d\psi \quad (24)$$

$$0 = M \mathcal{E}_i'(\phi) + K/\pi \int_0^{\pi} \frac{\mathcal{E}_r'(\psi) \sin \phi}{\cos \psi - \cos \phi} \, d\psi \quad (25)$$

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8. W. Magnus and F. Oberhettinger, Formulas and Theorems for the Functions of Mathematical Physics, Chelsea Pub. Co., New York, 1954.

Schmeidler[9] has shown how this system of integral equations can be solved by a process of iteration when $|M/K| < 1$. It is interesting to note that the example discussed by Schmeidler which gives rise to these equations is a problem in elasticity. The real and imaginary components of $\mathcal{E}'(\phi)$ are analogous to the horizontal and vertical components of pressure, respectively, at the base of a dam. The depth of water as a variable has the same significance as the magnitude of the magnetizing field H in the electromagnetic field problem.

Solution for Small M/K

Since a solution is desired for all values of M and K the procedure which is presented here differs somewhat from that given by Schmeidler. We shall have need for the following integral equation and its solution:

$$f(s) = 1/\pi \int_0^\pi \frac{\sin s}{\cos t - \cos s} g(t)\, dt \tag{26}$$

$$g(t) = 1/\pi \int_0^\pi g(t)\, dt - 1/\pi \int_0^\pi \frac{f(s) \sin s}{\cos s - \cos t} ds . \tag{27}$$

Recalling that $\mathcal{E}(\phi) = 0$ for $\phi = 0, \pi$, the application of (26) and (27) to Eqs. (24) and (25) yields

$$\mathcal{E}_i'(\phi) + M/K \frac{1}{\pi} \int_0^\pi \frac{\mathcal{E}_r'(\psi) \sin \psi}{\cos \psi - \cos \phi} d\psi = C/K \frac{1}{\pi} \int_0^\pi \frac{\sin^2 \psi}{\cos \psi - \cos \phi} d\psi$$

$$= -C/K \cos \phi , \tag{28}$$

and

$$\mathcal{E}_r'(\phi) = M/K \frac{1}{\pi} \int_0^\pi \frac{\mathcal{E}_i'(\psi) \sin \psi}{\cos \psi - \cos \phi} d\psi . \tag{29}$$

Solving for $\mathcal{E}_i'(\phi)$,

$$\mathcal{E}_i'(\phi) = -C/K \cos \phi - (M/K)^2 \frac{1}{\pi^2} \int_0^\pi \frac{\sin \tau\, d\tau}{\cos \phi - \cos \tau} \int_0^\pi \frac{\mathcal{E}_i'(\psi) \sin \psi}{\cos \tau - \cos \psi} d\psi. \tag{30}$$

Equation (30) can be identified as an integral equation of the form

9. W. Schmeidler, Integralgleichungen mit Anwendungen in Physik und Technik, Geest and Portig K.-G., Leipzig, 1955.

$$\mathcal{E}_i'(\phi) = f(\phi) + \lambda \int_0^\pi K(\phi,\tau)\, d\tau \int_0^\pi \mathcal{E}_i'(\psi)\, K(\tau,\psi)\, d\psi \tag{31}$$

by making the correspondence,

$$K(\phi,\tau) = 1/\pi \frac{\sin\tau}{\cos\phi - \cos\tau} \tag{32}$$

$$f(\phi) = -\, C/K \cos\phi$$

$$\lambda = -\,(M/K)^2 \quad .$$

Iterating (31) yields,

$$\mathcal{E}'(\phi) = f(\phi) + \lambda \int_0^\pi K(\phi,\tau)\, d\tau \int_0^\pi f(\psi)\, K(\tau,\psi)\, d\psi$$

$$+ \lambda^2 \int_0^\pi K(\phi,\tau)\, d\tau \int_0^\pi K(\tau,\sigma)\, d\sigma \int_0^\pi K(\sigma,\rho)\, d\rho \int_0^\pi \mathcal{E}_i'(\psi)\, K(\rho,\psi) d\psi .$$

Repeating the process leads to a Neumann series for $\mathcal{E}_i'(\phi)$. It can be shown[10] that for the kernel (32), the above series converges for $|\lambda| < 1$. Thus, the first order approximation to $\mathcal{E}'(\phi)$ can be taken as

$$\mathcal{E}_{i_1}'(\phi) = f(\phi)$$

and the second order approximation as

$$\mathcal{E}_{i_2}'(\phi) = f(\phi) + \lambda \int_0^\pi K(\phi,\tau)\, d\tau \int_0^\pi f(\psi)\, K(\tau,\psi)\, d\psi . \tag{33}$$

We shall obtain only the second order approximation, although the extension to higher order approximations is obvious. It will be convenient to set

$$\xi = -\cos\phi , \qquad \eta = -\cos\tau , \qquad \zeta = -\cos\psi .$$

Then Eq. (33) becomes,

$$\mathcal{E}_{i_2}'(\phi) = \frac{C}{K}\zeta - \frac{C}{K}\left(\frac{M}{K}\right)^2 \frac{1}{\pi^2} \int_1^{+1} \frac{d\,\eta}{\xi - \eta} \int_{-1}^{1} \frac{\zeta}{\eta - \zeta}\, d\zeta \quad .$$

With the aid of the definite integral,

10. Schmeidler, op. cit.

$$\int_{-1}^{1} \eta \, \ell n \frac{1 + \eta}{1 - \eta} \frac{d\eta}{\xi - \eta} = \frac{\xi}{2} \left[\left(\ell n \frac{1 + \xi}{1 - \xi} \right)^2 - \pi^2 \right] ,$$

it follows that

$$\mathcal{E}'_{i_2}(\phi) = - \frac{C}{K} \left[1 + 1/2 \left(\frac{M}{K} \right)^2 \right] \cos\phi - \frac{2C}{\pi^2 K} \left(\frac{M}{K} \right)^2 \ell n \frac{1 + \cos\phi}{1 - \cos\phi}$$

$$+ \frac{C}{2\pi^2 K} \left(\frac{M}{K} \right)^2 \cos\phi \left(\ell n \frac{1 + \cos\phi}{1 - \cos\phi} \right)^2 ; \quad \left| \frac{M}{K} \right| < 1 . \tag{34}$$

The solution for $\mathcal{E}'_r(\phi)$ follows in the same manner. Substituting (28) in (29),

$$\mathcal{E}'_r(\phi) + \left(\frac{M}{K} \right)^2 1/\pi^2 \int_0^{\pi} \frac{\sin\tau \, d\tau}{\cos\tau - \cos\phi} \int_0^{\pi} \frac{\mathcal{E}'_r(\psi) \sin\psi}{\cos\psi - \cos\tau} d\psi$$

$$= - \frac{C}{\pi K} \left(\frac{M}{K} \right) \left[2 - \cos\phi \, \ell n \frac{1 + \cos\phi}{1 - \cos\phi} \right] . \tag{35}$$

The iteration of (35) requires the following integrals:[11]

$$\int_{-1}^{1} \ell n \frac{1 + \eta}{1 - \eta} \frac{d\eta}{\xi - \eta} = \frac{1}{2} \left(\ell n \frac{1 + \xi}{1 - \xi} \right)^2 - \frac{\pi^2}{2}$$

$$\int_{-1}^{1} \eta \left(\ell n \frac{1 + \eta}{1 - \eta} \right)^2 \frac{d\eta}{\xi - \eta} = \frac{1}{3} \xi \left(\ell n \frac{1 + \xi}{1 - \xi} \right)^3 - \frac{2\pi^2}{3} \xi \, \ell n \frac{1 + \xi}{1 - \xi} - \frac{2\pi^2}{3} .$$

There results,

$$\mathcal{E}'_{r_2}(\phi) = - \frac{C}{\pi K} \left[\frac{5}{3} \left(\frac{M}{K} \right)^3 + 2 \left(\frac{M}{K} \right) \right] + \frac{C}{\pi K} \left[\left(\frac{M}{K} \right) + \frac{5}{6} \left(\frac{M}{K} \right)^3 \right] \cos\phi \, \ell n \frac{1 + \cos\phi}{1 - \cos\phi}$$

$$+ \frac{C}{\pi^3 K} \left(\frac{M}{K} \right)^3 \left[\left(\ell n \frac{1 + \cos\phi}{1 - \cos\phi} \right)^2 - 1/6 \cos\phi \left(\ell n \frac{1 + \cos\phi}{1 - \cos\phi} \right)^3 \right] ;$$

$$\left| \frac{M}{K} \right| < 1 . \tag{36}$$

11. ibid.

It remains to determine C, which is a function of the unknown reactance B_1. From (16),

$$B_1 = 2/\pi \int_0^\pi \mathcal{E}_i^{\prime}(\phi) \cos\phi \, d\phi \ . \tag{37}$$

Applying (21) and (37) to (36) and using the following integrals:

$$\int_0^\pi \cos\phi \, \ell n \frac{1+\cos\phi}{1-\cos\phi} \, d\phi = 2\pi$$

$$\int_0^\pi \cos^2\phi \left(\ell n \frac{1+\cos\phi}{1-\cos\phi}\right)^2 d\phi = \frac{\pi^3}{2} + 4\pi \ ,$$

we obtain,

$$K B_1 = \frac{1 + \frac{4}{\pi^2}\left(\frac{M}{K}\right)^2}{\frac{4}{\pi^2}\left(\frac{M}{K}\right)^2} \tag{38}$$

and

$$C = -\frac{\pi^2}{4}\left(\frac{K}{M}\right)^2 \ .$$

Therefore, the second order approximations to $\mathcal{E}_i^{\prime}(\phi)$ and $\mathcal{E}_r^{\prime}(\phi)$ when $|M/K| < 1$ are given by

$$\mathcal{E}_{i_2}^{\prime}(\phi) = \frac{\pi^2}{8K}\left[1 + 2\left(\frac{K}{M}\right)^2\right]\cos\phi + \frac{1}{2K}\,\ell n\,\frac{1+\cos\phi}{1-\cos\phi}$$
$$- \frac{1}{8K}\cos\phi\left(\ell n\,\frac{1+\cos\phi}{1-\cos\phi}\right)^2 \tag{39}$$

$$\mathcal{E}_{r_2}^{\prime}(\phi) = \frac{\pi}{2K}\left[\left(\frac{K}{M}\right) + \frac{5}{6}\left(\frac{M}{K}\right)\right] - \frac{\pi}{4K}\left[\left(\frac{K}{M}\right) + \frac{5}{6}\left(\frac{M}{K}\right)\right]\cos\phi\left(\ell n\,\frac{1+\cos\phi}{1-\cos\phi}\right)$$
$$- \frac{1}{4\pi K}\left(\frac{M}{K}\right)\left[\left(\ell n\,\frac{1+\cos\phi}{1-\cos\phi}\right)^2 - \frac{1}{6}\cos\phi\left(\ell n\,\frac{1+\cos\phi}{1-\cos\phi}\right)^3\right] . \tag{40}$$

Solution for Large M/K

A Neumann series valid for $|M/K| > 1$ can also be obtained from Eqs. (24) and (25). By direct substitution,

$$\mathcal{E}_i'(\phi) = \frac{C}{\pi M}\left(\frac{K}{M}\right) \sin\phi \, \ell n \, \frac{1+\cos\phi}{1-\cos\phi} - \left(\frac{K}{M}\right)^2 \frac{1}{\pi^2} \int_0^\pi \frac{\sin\phi \, d\tau}{\cos\phi - \cos\tau} \int_0^\pi \frac{\mathcal{E}_i'(\psi) \sin\tau}{\cos\tau - \cos\psi} d\psi \tag{41}$$

$$\mathcal{E}_r'(\phi) = \frac{C}{M} \sin\phi - \left(\frac{K}{M}\right)^2 \frac{1}{\pi^2} \int_0^\pi \frac{\sin\phi \, d\tau}{\cos\phi - \cos\tau} \int_0^\pi \frac{\mathcal{E}_r'(\psi) \sin\tau}{\cos\tau - \cos\psi} d\psi \, . \tag{42}$$

Both (41) and (42) reduce to an equation of the same form as (31) if we take,

$$K(\phi,\tau) = \frac{1}{\pi} \frac{\sin\phi}{\cos\phi - \cos\tau}$$

$$\lambda = -\left(\frac{K}{M}\right)^2 .$$

Iteration again yields a Neumann series in terms of λ. It can be shown that the series will converge for $|\lambda| < 1$, that is for $1 < |M/K|$. The proof parallels that given by Schmeidler for the previous case. Second order approximations to $\mathcal{E}_i'(\phi)$ and $\mathcal{E}_r'(\phi)$ are found to be,

$$\mathcal{E}_{i_2}'(\phi) = \frac{C}{\pi K}\left[\left(\frac{K}{M}\right)^2 + \frac{5}{6}\left(\frac{K}{M}\right)^4\right] \sin\phi \, \ell n \, \frac{1+\cos\phi}{1-\cos\phi}$$

$$- \frac{1}{6\pi^3} \frac{C}{K}\left(\frac{K}{M}\right)^4 \sin\phi \left(\ell n \, \frac{1+\cos\phi}{1-\cos\phi}\right)^3 \tag{43}$$

$$\mathcal{E}_{r_2}'(\phi) = \frac{C}{K}\left[\left(\frac{K}{M}\right) + \frac{1}{2}\left(\frac{K}{M}\right)^3\right] \sin\phi - \frac{C}{2\pi^2 K}\left(\frac{K}{M}\right)^3 \sin\phi \left(\ell n \, \frac{1+\cos\phi}{1-\cos\phi}\right)^2 , \tag{44}$$

where we have employed the integral,

$$\int_{-1}^{1} \left(\ell n \, \frac{1+\eta}{1-\eta}\right)^2 \frac{d\eta}{\xi - \eta} = \frac{1}{3}\left(\ell n \, \frac{1+\xi}{1-\xi}\right)^3 - \frac{2\pi^2}{3} \ell n \, \frac{1+\xi}{1-\xi} \, .$$

With the aid of the following definite integrals:

$$\int_0^\pi \sin\phi \cos\phi \, \ell n \, \frac{1+\cos\phi}{1-\cos\phi} \, d\phi = 2$$

$$\int_0^\pi \sin\phi \cos\phi \left(\ell n \, \frac{1+\cos\phi}{1-\cos\phi}\right)^3 d\phi = 2\pi^2 ,$$

the constants B_1 and C are found to be,

$$K\,B_1 = \frac{\left(\frac{K}{M}\right)^2 + \frac{2}{3}\left(\frac{K}{M}\right)^4}{\frac{\pi^2}{4} + \left(\frac{K}{M}\right)^2 + \frac{2}{3}\left(\frac{K}{M}\right)^4} \qquad (45)$$

$$C = \frac{1}{1 + \frac{4}{\pi^2}\left[\left(\frac{K}{M}\right)^2 + \frac{2}{3}\left(\frac{K}{M}\right)^4\right]}$$

DISCUSSION

The question naturally arises as to the range of frequency and magnetizing field over which the foregoing solution is applicable. One might consider the assumption that no higher modes than the first propagate in the ferrite to be rather restrictive in view of the infinite range of values which the effective permeability $\mu_\perp$ can take on in the ideal lossless case. Figure 3 illustrates the regions in the $f - f_o$ plane which correspond to the permitted TE modes of propagation for a typical ferrite ($M_s = .231 \times 10^6$ A/M, $\epsilon_f = 13\ \epsilon_o$) in an x-band waveguide. In deriving these curves we have employed the relation

$$\mu_\perp / \mu_o = \frac{(f_o + f_s)^2 - f^2}{f_o(f_o + f_s) - f^2},$$

where

$$f_o = |\Gamma| \ H/2\pi$$

$$f_s = |\Gamma| \ M_s/2\pi \ .$$

It should be noted that the "no-propagation" region above $\mu_\perp$ resonance occupies a substantial portion of the graph. Much of this region corresponds to negative values of $\mu_\perp$. There is nothing inherent in the method of the preceding section which restricts the solution to the case considered. However, the process of iteration employed becomes exceedingly tedious as higher order modes are permitted to propagate in the ferrite.

For purposes of illustration the derivative of the normalized electric field at the air-ferrite interface is plotted in Figs. 4 and 5 for two different values of M/K. It can be shown that

$$\frac{M}{K} = \frac{f\, f_s}{2f^2 - (f_o + f_s)\ (2f_o + f_s)} \ .$$

Both $(M/K)^2$ values of 1/3 and 5 give solutions for f and f_o which lie in the "no-propagation" region above $\mu_\perp$ resonance in Fig. 3. The existance of both a real and imaginary part to the field at the interface is unique to boundary value problems involving anisotropic media. Although the field strength at each position across the waveguide varies sinusoidally with time, the phase of this variation differs from one point to the next. Thus, the field if plotted sequentially as in Fig. 5 appears to exhibit a periodic "shimmy."

The value of $(M/K) = 1$ seems to be a critical point in the analysis. Not only do the fields display a markedly different appearance for values of M/K on either side of unity but the series solution itself probably does not converge for this critical value. One would suspect the assumption of Eq. (19) to be the source of the difficulty. The value of $(M/K) = 0$ leads to an indeterminate solution for the normalized field $\mathcal{E}_1'(\phi)$ because the normalizing factor I is also zero at this point. This is of little consequence, however, since for $M = 0$ the problem reduces to the case of an isotropic dielectric. It is interesting to note, again in contrast to isotropic problems, that B_1 may be either inductive or capacitive since K can be positive or negative.

APPENDIX

A Variational Expression for B_1

The real and imaginary parts of Eq. (20) can be written as,

$$\sin \phi = M \mathcal{E}_r'(\phi) - \frac{2K}{\pi} \sum_{n=2}^{\infty} n \int_0^{\pi} \mathcal{E}_i(\phi') \sin n\phi' \sin n\phi \, d\phi' \qquad \text{(A-1)}$$

$$0 = M \mathcal{E}_i'(\phi) + \frac{2K}{\pi} \sum_{n=2}^{\infty} n \int_0^{\pi} \mathcal{E}_r(\phi') \sin n\phi' \sin n\phi \, d\phi' . \qquad \text{(A-2)}$$

Multiplying (A-1) by $\mathcal{E}_r(\phi)$ and (A-2) by $\mathcal{E}_i(\phi)$ and integrating,

$$\int_0^{\pi} \mathcal{E}_r(\phi) \sin \phi \, d\phi = M \int_0^{\pi} \mathcal{E}_r(\phi) \mathcal{E}_r'(\phi) d\phi - \frac{2K}{\pi} \sum_{n=2}^{\infty} n \int_0^{\pi} \mathcal{E}_i(\phi') \sin n\phi' \sin n\phi \, \mathcal{E}_r(\phi) d\phi \, d\phi' \qquad \text{(A-3)}$$

$$0 = M \int_0^{\pi} \mathcal{E}_i(\phi) \mathcal{E}_i'(\phi) d\phi + \frac{2K}{\pi} \sum_{n=2}^{\infty} n \int_0^{\pi} \mathcal{E}_r(\phi') \sin n\phi \sin n\phi' \, \mathcal{E}_i(\phi) d\phi \, d\phi' . \qquad \text{(A-4)}$$

Since, integrating by parts,

$$\int_0^{\pi} \mathcal{E}_i(\phi) \mathcal{E}_i'(\phi) d\phi = \int_0^{\pi} \mathcal{E}_r(\phi) \mathcal{E}_r'(\phi) d\phi = 0 ,$$

Eqs. (A-3) and (A-4) yield,

$$\int_0^{\pi} \mathcal{E}_r(\phi) \sin \phi \, d\phi = 0 .$$

It follows from (16) and (17) that $A_1 = 0$.

If one multiplies (A-1) by $\mathcal{E}_i(\phi)$ and (A-2) by $\mathcal{E}_r(\phi)$ and integrates,

$$\int_0^{\pi} \mathcal{E}_i(\phi) \sin \phi \, d\phi = M \int_0^{\pi} \mathcal{E}_r'(\phi) \mathcal{E}_i(\phi) d\phi - \frac{2K}{\pi} \sum_{n=2}^{\infty} n \int_0^{\pi} \mathcal{E}_i(\phi') \sin n\phi' \sin n\phi \, \mathcal{E}_i(\phi) d\phi \, d\phi' \qquad \text{(A-5)}$$

$$0 = M\int_0^\pi \mathcal{E}_r(\phi)\mathcal{E}_i'(\phi)d\phi + \frac{2K}{\pi}\sum_{n=2}^{\infty} n\int_0^\pi \mathcal{E}_r(\phi')\sin n\phi' \sin n\phi\, \mathcal{E}_r(\phi)d\phi\, d\phi' . \tag{A-6}$$

Noting that,

$$\int_0^\pi \mathcal{E}_r'(\phi)\,\mathcal{E}_i(\phi)d\phi = -\int_0^\pi \mathcal{E}_r(\phi)\,\mathcal{E}_i'(\phi)d\phi ,$$

Eqs. (A-5) and (A-6) yield

$$\int_0^\pi \mathcal{E}_i(\phi)\sin\phi d\phi = \frac{2K}{\pi}\sum_{n=2}^{\infty} n\left\{\left[\int_0^\pi \mathcal{E}_r(\phi)\sin n\phi\, d\phi\right]^2 - \left[\int_0^\pi \mathcal{E}_i(\phi)\sin n\phi\, d\phi\right]^2\right\}. \tag{A-7}$$

Since we have used the derivative of the field in the foregoing theory it will be convenient to express (A-7) in terms of $\mathcal{E}_i'(\phi)$. From (A-2),

$$\int_0^\pi \mathcal{E}_r(\phi)\sin n\phi d\phi = -1/n\,\frac{M}{K}\int_0^\pi \mathcal{E}_i'(\phi)\sin n\phi d\phi ; \quad n > 1 .$$

Dividing (A-7) by

$$\left(\frac{\pi B_1}{2}\right)^2 = \left[\int_0^\pi \mathcal{E}_i(\phi)\sin\phi d\phi\right]^2$$

$$= \left[\int_0^\pi \mathcal{E}_i'(\phi)\cos\phi d\phi\right]^2 ,$$

it follows that

$$1/B_1 = \frac{K\sum_{n=2}^{\infty} 1/n\left\{\left(\frac{M}{K}\right)^2\left[\int_0^\pi \mathcal{E}_i'(\phi)\sin n\phi\, d\phi\right]^2 - \left[\int_0^\pi \mathcal{E}_i'(\phi)\cos n\phi\, d\phi\right]^2\right\}}{\left[\int_0^\pi \mathcal{E}_i'(\phi)\cos\phi\, d\phi\right]^2} .$$

Alternatively,

$$1/B_1 = \frac{\int_0^\pi\int_0^\pi \mathcal{E}_i'(\phi)\,G(\phi,\phi')\,\mathcal{E}_i'(\phi')d\phi\, d\phi'}{\left[\int_0^\pi \mathcal{E}_i'(\phi)\cos\phi\, d\phi\right]^2} , \tag{A-8}$$

where,

$$G(\phi,\phi') = K \sum_{n=2}^{\infty} 1/n \left[\left(\frac{M}{K}\right)^2 \sin n\phi \sin n\phi' - \cos n\phi \cos n\phi' \right] .$$

It can be shown that (A-8) is stationary with respect to variations in $\mathcal{E}_1'(\phi)$. The correct field will establish the extremum for B_1^{-1}. Therefore, this expression could be employed to improve the approximations previously obtained for B_1. There seems little point in this, however, since higher order terms for the field can be obtained more easily by extending the Neumann series.

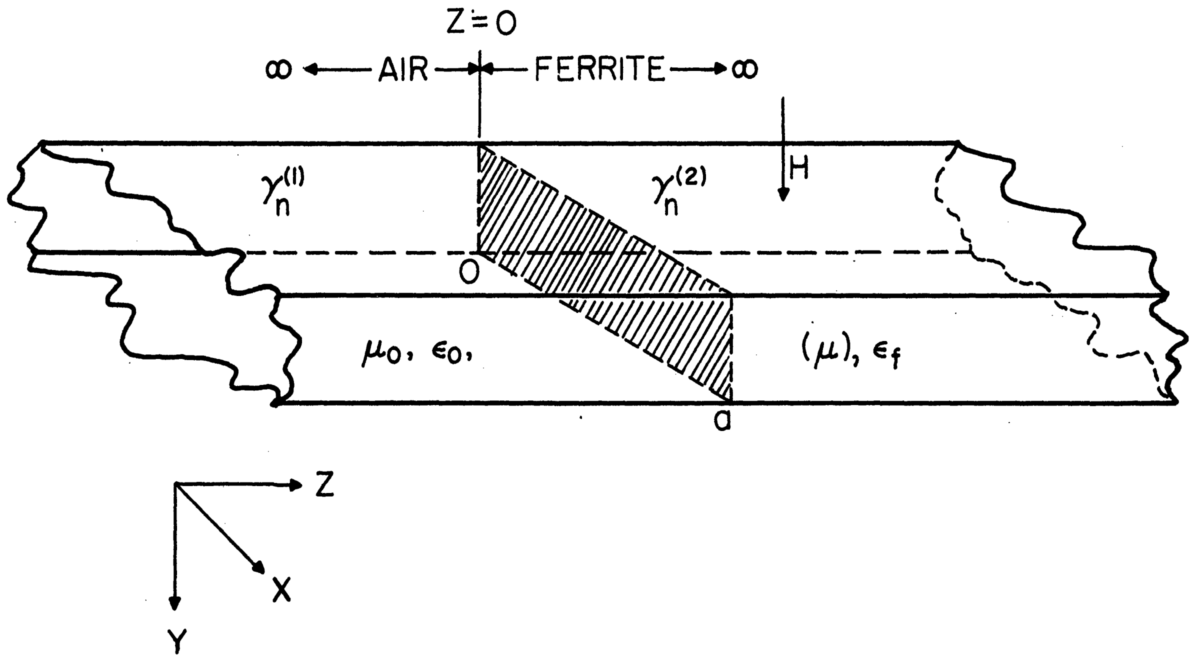


FIG. 1
FERRITE - FILLED WAVEGUIDE

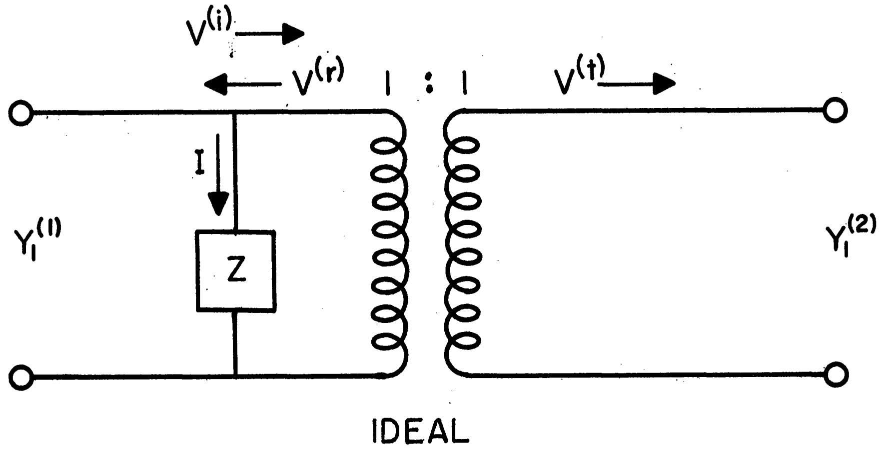


FIG. 2.

EQUIVALENT CIRCUIT FOR THE AIR-FERRITE JUNCTION

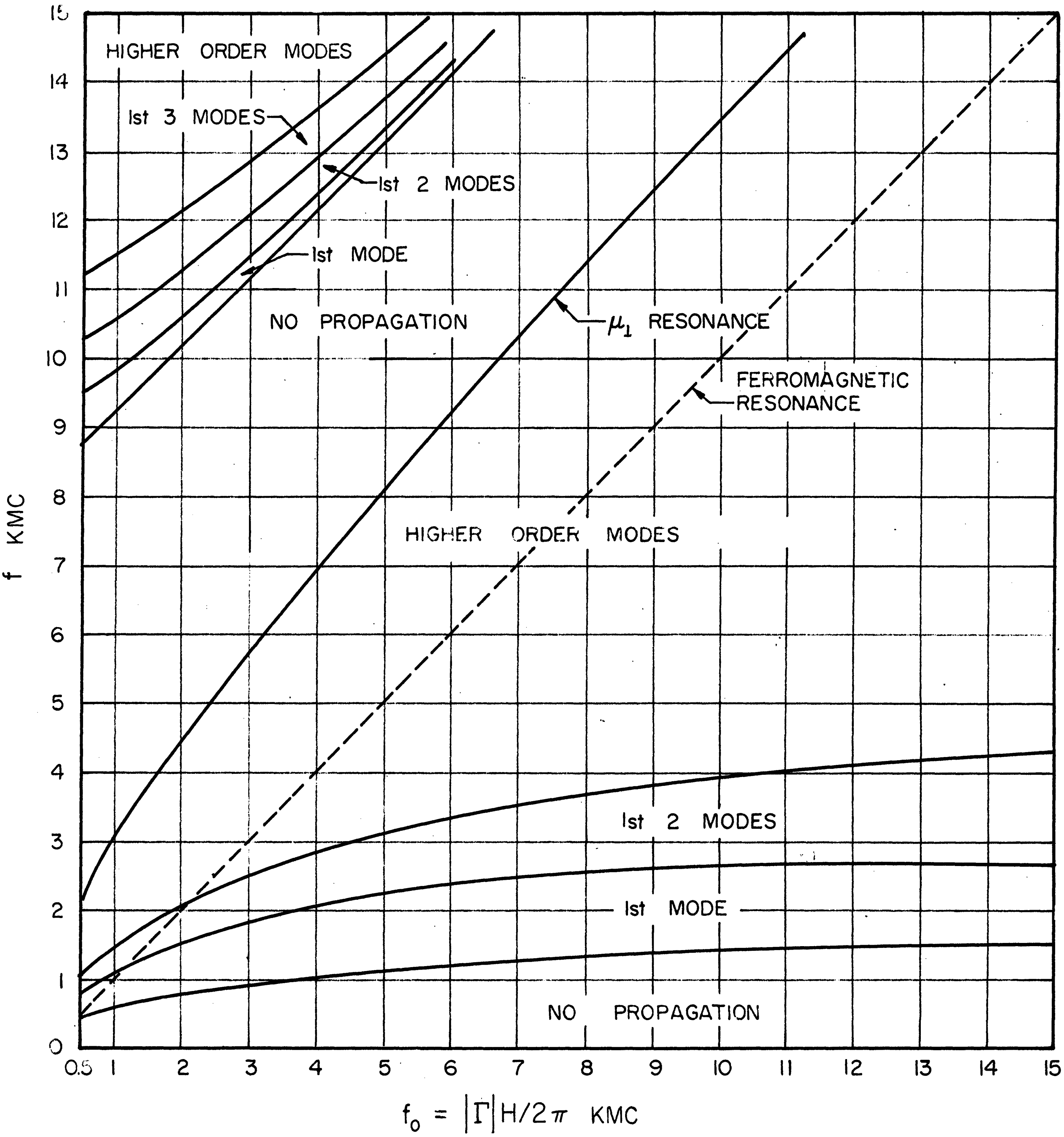
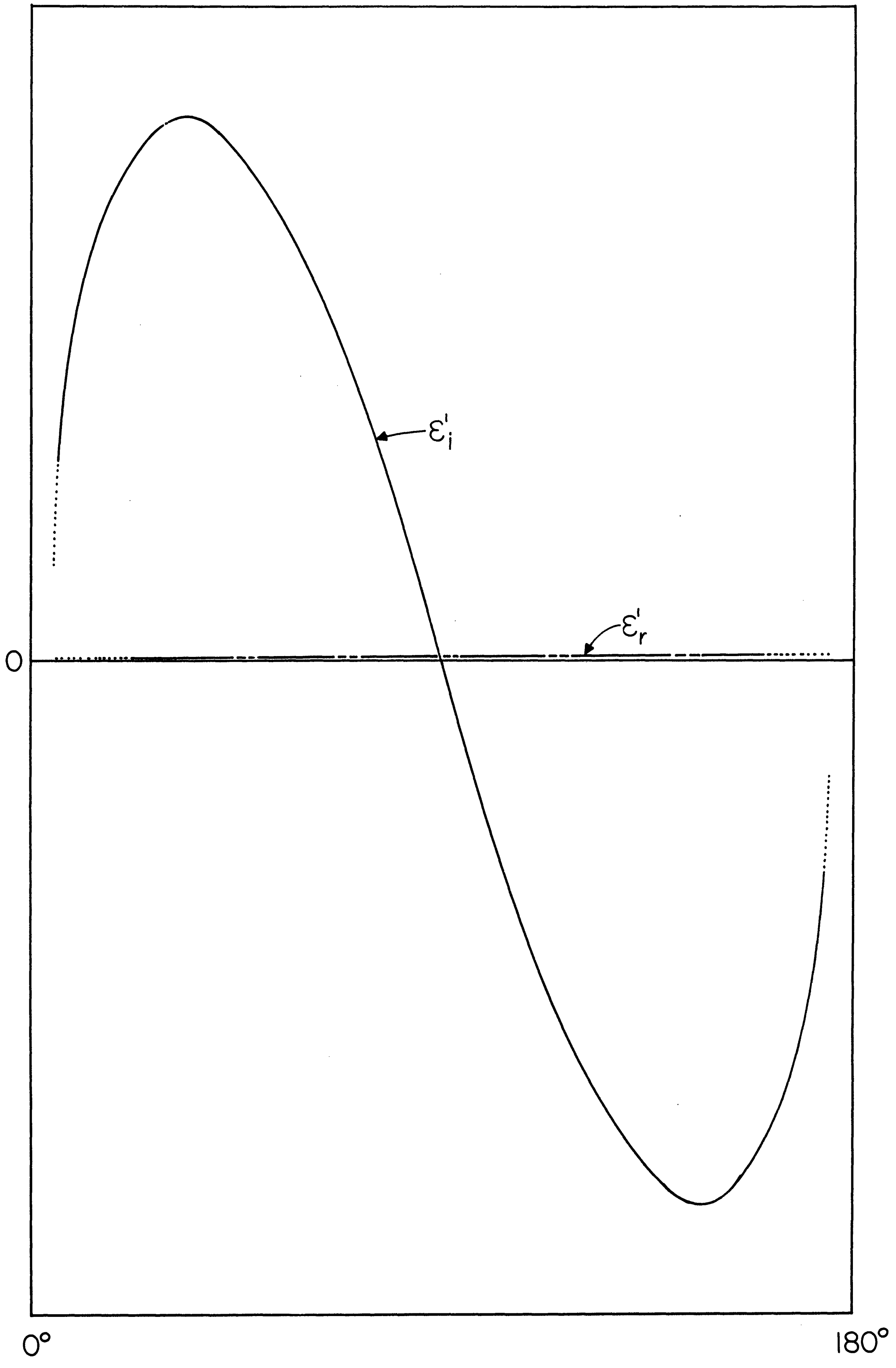


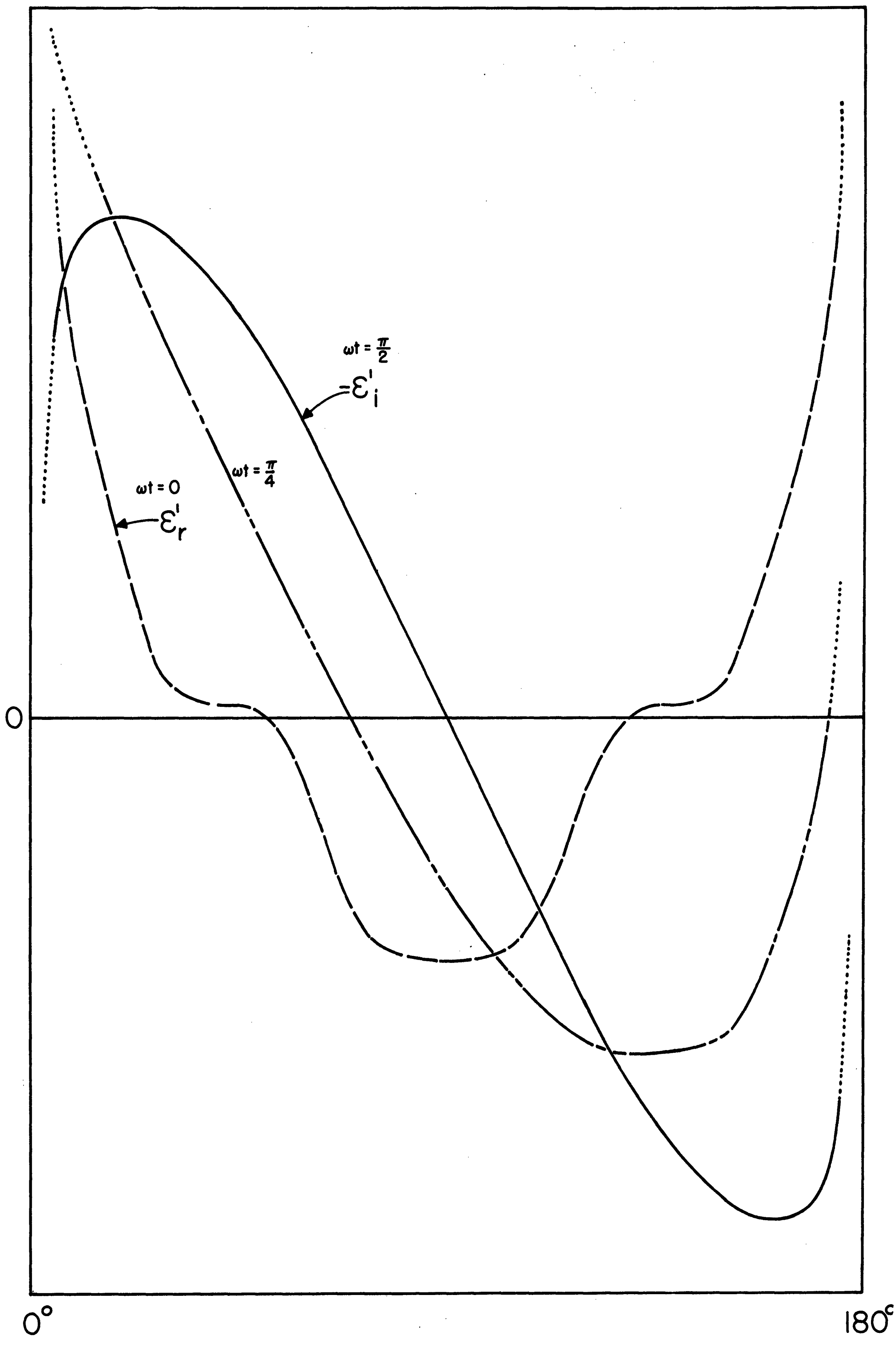
FIG. 3.

TE MODE PROPAGATION IN A FERRITE-FILLED WAVEGUIDE



$\varepsilon'(\phi)$ AT FERRITE-AIR INTERFACE FOR $(M/K)^2 = 5$

FIG. 4.



$\varepsilon'(\phi)$ AT FERRITE-AIR INTERFACE FOR $(M/K)^2 = 1/3$

FIG. 5.

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