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THE SCATTERING APPROACH TO THE SYNTHESIS OF NONUNIFORM LINES

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ABSTRACT

The analysis of a nonuniform transmission line is formulated as a one-dimensional scattering problem by transforming the transmission line equations into an integral equation. Approximate solutions of the integral equation for the input reflection coefficient of a nonuniform line terminated in an arbitrary impedance are obtained by employing the Neumann and Fredholm series expansions. Two examples in the application of these solutions are given. It is shown that the Fredholm solution leads to Orlov's synthesis formula by means of which the impedance variation of a nonuniform line can be calculated in terms of the input and output reflection coefficients.

THE SCATTERING APPROACH TO THE SYNTHESIS OF NONUNIFORM LINES

1. INTRODUCTION

The analysis of nonuniform transmission lines can be approached in several different ways. However, only one formulation of the solution has proved generally useful in the synthesis problem. For a lossless line of length "ℓ" this formulation is the well-known Fourier relationship (Refs. 1, 2).

$$\Gamma(k) = \int_0^{\ell} p(x)e^{-2jkx}\, dx; \quad p(x) \equiv \frac{1}{2}\frac{d\ell n Z_o(x)}{dx} \qquad (1)$$

between the input reflection coefficient $\Gamma(k)$ and the "reflection distribution function," $p(x)$. The independent variable can always be normalized so that the propagation constant in (1) may be taken as $k = 2\pi/\lambda$, where λ is the wavelength in free space. This formula, which is approximate, is restricted to gradual tapers for which the reflection coefficient along the line defined by

$$\Gamma(x) = \frac{V(x)/I(x) - Z_o(x)}{V(x)/I(x) + Z_o(x)} \qquad (2)$$

satisfies the inequality, $\Gamma(x)^2 \ll 1$ for $0 \leq x \leq \ell$. As a result of this restriction (1) is not a valid approximation when the line is terminated by a load impedance corresponding to a substantial mismatch at $x = \ell$. In many practical applications, such as the broadband matching of crystal mixers or antennas, this case is of interest. The present paper treats the general problem of a nonuniform line terminated by an arbitrary load impedance as a one-dimensional scattering problem. By formulating the problem in terms of an integral equation for the unknown wave function it is shown that the application of classical perturbation techniques leads to a useful generalization of (1) valid for any given load impedance.

2. FORMULATION OF THE SCATTERING PROBLEM

The essential assumption in nonuniform line theory is the existence of a unique voltage V(z) and current I(z) which are related at any point z on the line by the equations

$$\frac{dV(z)}{dz} = -Z(z)I(z) \tag{3a}$$

$$\frac{dI(z)}{dz} = -Y(z)V(z) \tag{3b}$$

where Z and Y are the series impedance and shunt admittance, respectively, per unit length. The application of these equations to a nonuniform structure implies that the electric and magnetic fields supported by the structure at the frequency of interest correspond, within some acceptable approximation, to the principal (TEM) mode. As in the uniform case a characteristic impedance $Z_o(z) = \sqrt{Z(z)/Y(z)}$ and a propagation constant $\gamma(z) = \sqrt{Z(z)Y(z)}$ can be defined, although the physical interpretation of these quantities is now restricted by the fact that they have meaning only in the local sense, that is, at the point z. For the lossless line considered here $\gamma(z) \equiv j\beta(z)$ is imaginary and $Z_o(z)$ is real. It will be convenient to normalize (3a) and (3b) by making the following substitutions:

$$x = \frac{1}{k} \int_{0}^{z} \beta(z)\, dz \tag{4}$$

$$V(x) = u(x)\sqrt{Z_0(x)} \; ; \qquad I(x) = v(x)/\sqrt{Z_0(x)} \; . \tag{5}$$

Then (3a) and (3b) become

$$\frac{du(x)}{dx} + p(x)u(x) + jkv(x) = 0 \tag{6a}$$

$$\frac{dv(x)}{dx} - p(x)v(x) + jku(x) = 0 \; , \tag{6b}$$

where p(x) is defined in (1). Combining (6a) and (6b) one obtains the one-dimensional form of the time-independent Schroedinger wave equation,

$$\frac{d^2u(x)}{dx^2} + k^2u(x) = P(x)u(x) \; , \tag{7}$$

where the role of the potential function is played by

$$P(x) \equiv p^2(x) - \frac{dp(x)}{dx} \; . \tag{8}$$

The application of (7) and its three-dimensional counterpart to the quantum theory of particle scattering has been treated extensively in the literature (Ref. 3). The late W. W. Hansen (Ref. 4) appears to have been first to employ the scattering approach in the analysis of nonuniform lines. The present analysis parallels the formulation of the one-dimensional scattering problem with one major difference. In the transmission line problem under consideration the incident wave is "scattered" by the load which terminates the line as well as by the line itself. Accordingly, in the solution of (7) the boundary condition imposed by the load replaces the boundary condition at $x = \infty$ commonly designated in the problem of scattering by a central force field. If the load impedance is Z_ℓ the continuity of current and voltage at $x = \ell$ requires that

$$v(\ell) = \frac{Z_o(\ell)}{Z_\ell} u(\ell) . \tag{9}$$

It follows from (6a) that the solution to (7) must satisfy

$$\frac{du(x)}{dx}\Big|_{x=\ell} = - u(\ell) \left(p_\ell + jk \frac{Z_{o\ell}}{Z_\ell} \right), \tag{10}$$

where $Z_{o\ell} \equiv Z_o(\ell)$ and $p_\ell \equiv p(\ell)$. The boundary condition at $x = -\infty$ will be satisfied if, as $x \to -\infty$,

$$u(x) \to e^{-jkx} + \Gamma e^{+jkx} \tag{11}$$

Referring to Fig. 1, the incident wave is taken as e^{-jkx} and the reflected or scattered wave as Γe^{jkx}. $\Gamma(k)$ is defined as the reflection coefficient of the line at $x = 0$. The line is assumed to be continuous for $-\infty < x \leq \ell$, although it will be seen later that this restriction can be lifted to permit small step discontinuities. The synthesis problem for nonuniform lines can now be stated as follows: given the reflection coefficient $\Gamma(k)$, $-\infty < k < \infty$, solve (7) for P(x)

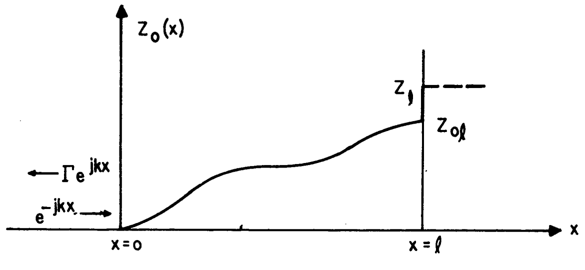

Fig. 1. Typical impedance variation for a nonuniform line.

subject to boundary conditions (10) and (11) with $P(x) = 0$, $x \leq 0$. Even when a solution to this problem exists, it cannot be found in general. Instead of attacking the synthesis problem directly, we will apply the perturbation techniques (Ref. 5) commonly employed in the scattering problem to obtain an approximate relationship between $\Gamma(k)$ and $P(x)$ in the form of a Fourier integral.

In the scattering formulation of the problem the solution to (7) is expressed as the sum of two wave functions:

$$u(x) = u_o(x) + u_s(x) , \tag{12}$$

where the "unperturbed wave $u_o(x)$ is taken as a solution of the homogeneous equation,

$$\frac{d^2u_o(x)}{dx^2} + k^2u_o(x) = 0 . \tag{13}$$

The scattered wave $u_s(x)$, representing the unknown perturbation due to the nonuniformities of the line, consequently satisfies

$$\frac{d^2u_s(x)}{dx^2} + k^2u_s(x) = P(x)u(x) . \tag{14}$$

The unperturbed wave in the present problem in general possesses both an incident and a reflected component owing to the presence of a mismatch at $x = \ell$. The unperturbed wave will therefore be taken as

$$u_o(x) = e^{-jkx} + \Gamma_o e^{jkx} , \tag{15}$$

where the constant Γ_o is chosen so that $u_s(x)$ will satisfy homogeneous boundary conditions at $x = \ell$. It is readily shown that if

$$\Gamma_o = \frac{1 - \left(\frac{p_\ell}{jk} + \frac{Z_{o\ell}}{Z_\ell}\right)}{1 + \left(\frac{p_\ell}{jk} + \frac{Z_{o\ell}}{Z_\ell}\right)} e^{-2jk\ell} , \tag{16}$$

then the scattered wave $u_s(x)$ also satisfies (10). Equation 14 can be expressed as an integral equation by introducing the one-dimensional Green's function defined as a solution of

$$\frac{d^2G(x,\xi)}{dx^2} + k^2G(x,\xi) = - \delta(x - \xi) , \tag{17}$$

which satisfies (10) at $x = \ell$ and the asymptotic condition, $G(x,\xi) \rightarrow A(\xi)\, e^{jkx}$ as $x \rightarrow -\infty$. The desired Green's function can be shown to be

$$G(x,\xi) = \begin{cases} \dfrac{e^{jk(x - \xi)}}{2jk} \left(1 + \Gamma_o e^{2jk\xi}\right), & x < \xi \\[2ex] \dfrac{e^{jk(\xi - x)}}{2jk} \left(1 + \Gamma_o e^{2jkx}\right), & x > \xi . \end{cases} \tag{18}$$

It follows from the theory of Green's functions (Ref. 6) that

$$u_s(x) = - \int_{-\infty}^{\ell} G(x,\xi)\, P(\xi)\, u(\xi)\, d\xi . \tag{19}$$

When (19) is substituted in (12) one obtains a Fredholm integral equation of the second kind in terms of $u(x)$:

$$u(x) = u_o(x) - \int_{-\infty}^{\ell} G(x,\xi)\, P(\xi)\, u(\xi)\, d\xi . \tag{20}$$

Equation 20 is a convenient starting point for the application of perturbation methods. Although the approximate solution of (20) for u(x), given P(x), will yield the complete voltage pattern on the line, it will not be necessary to carry through the solution to this extent in order to obtain the desired reflection coefficient. It will be shown in the next section that an approximate value of $\Gamma(k)$ can be obtained by inspection.

3. APPROXIMATE SOLUTION OF THE INTEGRAL EQUATION

There are two basic methods in the theory of integral equations (Ref. 7) for solving equations of the type given by (20). The Neumann series solution is obtained by a simple procedure of successive approximations. For example, the first iteration of (20) yields:[1]

$$u_1(x) = u_o(x) - \int_{-\infty}^{\ell} G(x,\xi)\, P(\xi)\, u_o(\xi)\, d\xi \, . \tag{21}$$

In general,

$$u_{k+1}(x) = u_o(x) - \int_{-\infty}^{\ell} G(x,\xi)\, P(\xi)\, u_k(\xi)\, d\xi \, . \tag{22}$$

As would be expected the series obtained by repeated iterations converges most rapidly when the contribution of the last term in (21) to $u_1(x)$ is relatively small. Substituting (15) and (18) in (21) gives

$$u_1(x) = e^{-jkx} - \left(e^{-jkx} + \Gamma_o e^{jkx}\right) \frac{1}{2jk} \int_{-\infty}^{x} e^{jk\xi}\, P(\xi)\, u_o(\xi)\, d\xi$$

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[1] When this procedure is applied to the scattering problem, (21) is referred to as Born's approximation.

$$+ e^{jkx} \left[\Gamma_o - \frac{1}{2jk} \int_x^{\ell} \left(e^{-jk\xi} + \Gamma_o e^{jk\xi} \right) P(\xi)\, u_o(\xi)\, d\xi \right] . \qquad (23)$$

The first-order approximation to the reflection coefficient is obtained by examining the behavior of (23) as $x \to \infty$. In the limit the coefficient of e^{jkx} is

$$\Gamma(k) = \Gamma_o - \frac{1}{2jk} \int_{-\infty}^{\ell} \left(e^{-jk\xi} + \Gamma_o e^{jk\xi} \right) P(\xi)\, u_o(\xi)\, d\xi . \qquad (24)$$

Substituting (15) in (24) and noting that $P(x) = 0$, $x \leq 0$,

$$\Gamma(k) = \Gamma_o - \frac{1}{2jk} \int_0^{\ell} \left(e^{-jk\xi} + \Gamma_o e^{jk\xi} \right)^2 P(\xi)\, d\xi . \qquad (25)$$

It is not inconsistent with the perturbation assumption to take $P(x) \cong -p'(x)$. From (7) and (8) a sufficient condition for the validity of this approximation is

$$|p(x)|_{max} \ll k \qquad (26)$$

With the additional assumption that $p_\ell = 0$, integration by parts yields

$$\Gamma(k) = \Gamma_0 + \int_0^{\ell} \left(e^{-2jk\xi} - \Gamma_0^2\, e^{2jk\xi} \right) p(\xi)\, d\xi . \qquad (27)$$

Even when $p_\ell \neq 0$, (27) can be considered a useful approximation since, by virtue of (26), the constant term resulting from the integration by parts will be negligible. It can be seen that (27) agrees with (1) when $\Gamma_0 = 0$. The second term in the integrand accounts, to the first order, for the interaction of the wave reflected from the load with the nonuniform line.

It will be instructive to apply (27) to the simple case of a step line terminated in a load having the reflection coefficient

$$\Gamma_\ell = \frac{Z_\ell - Z_{02}}{Z_\ell + Z_{02}} \tag{28}$$

as illustrated in Fig. 2. The applicability of (27) to this problem is questionable because the term $p^2(x)$ which has been neglected in its derivation has no meaning in this case at x = 0. Nevertheless, we will proceed without answering this question now. Integrating (27) by parts one obtains

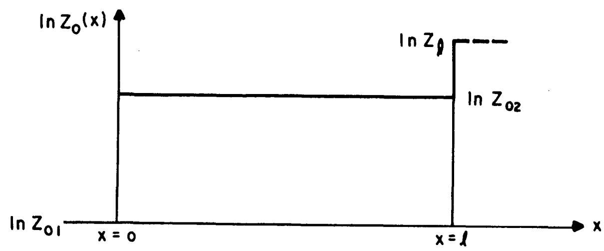

Fig. 2. Impedance variation for the step line.

$$\Gamma = \Gamma_o + \frac{1}{2}\left[\left(e^{-2jk\xi} - \Gamma_o^2\, e^{2jk\xi}\right) \ell n Z_o(\xi)\right]_{\xi = 0}^{\xi = 0^+}$$

$$= \frac{1}{2}\, \ell n\, \frac{Z_{02}}{Z_{01}} + \Gamma_o - \frac{1}{2}\, \Gamma_o^2\, \ell n\, \frac{Z_{02}}{Z_{01}}\,, \tag{29}$$

where $\Gamma_o = e^{-2jk\ell}\, \Gamma_\ell$. The exact solution to this problem is

$$\Gamma = \frac{\left(\frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}}\right) + \Gamma_o}{1 + \Gamma_o\left(\frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}}\right)}\,. \tag{30}$$

Assuming the step is small,

$$\frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} \approx \frac{1}{2} \ln \frac{Z_{02}}{Z_{01}} \ll 1 \; . \tag{31}$$

Expanding (30) in powers of Γ_0 and neglecting higher-order terms in

$$\left(\frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}}\right) ,$$

there results,

$$\Gamma \approx \left(\frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}}\right) + \Gamma_0 - \Gamma_0^2\left(\frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}}\right) + \Gamma_0^3\left(\frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}}\right)^2 - \cdots \tag{32}$$

which agrees with (29) to the second order within the restriction imposed by (31). Thus it is seen that (27) is applicable in the case of a step discontinuity when the step is small. This is not suprising since this type of problem presents no difficulty when the Fourier relation (1) is derived from the differential equation satisfied by the reflection coefficient (2). The restriction that the step be small is equivalent to the condition $\Gamma^2(x) \ll 1$.

There is an inherent limitation in the Neumann series solution to the scattering problem. This can be seen by considering the above example. When $\Gamma_0 \neq 0$ in (30), $\Gamma(k)$ will have countable infinity of poles in the finite k-plane. The larger the discontinuity at the step or the reflection at the load, the closer these poles will be to the real k-axis. The Neumann series will tend to converge slowly or may not converge at all in the vicinity of these poles. This situation is likely to prevail in general since we can think of the arbitrary nonuniform line as consisting of a succession of infinitesimal steps, the input reflection coefficient at one step providing the load reflection coefficient for the next. This suggests that we should seek an approximation involving a series expansion of the denominator as well as the numerator, in this way introducing poles in $\Gamma(k)$. The Fredholm series is such a solution.

4. DERIVATION OF ORLOV'S RESULT

The Fredholm series solution of

$$u(x) \quad = \quad u_o(x) + \lambda \int_{-\infty}^{\ell} K(x,\xi)\, u(\xi)\, d\xi \tag{33}$$

is given by

$$u(x) \quad = \quad u_o(x) + \lambda \int_{-\infty}^{\ell} \frac{D(x,\xi;\lambda)}{D(\lambda)}\, u_o(\xi)\, d\xi \ , \tag{34}$$

where

$$D(\lambda) = 1 - \lambda \int_{-\infty}^{\ell} K(\xi_1,\xi_1)d\xi_1 + \frac{\lambda^2}{2!} \int_{-\infty}^{\ell} \int_{-\infty}^{\ell} \begin{vmatrix} K(\xi_1,\xi_1) & K(\xi_1,\xi_2) \\ K(\xi_2,\xi_1) & K(\xi_2,\xi_2) \end{vmatrix} d\xi_1 d\xi_2 - \cdots \tag{35}$$

and

$$D(x,\xi;\lambda) \quad = \quad K(x,\xi) - \lambda \int_{-\infty}^{\ell} \begin{vmatrix} K(x,\ \xi) & K(x,\ \xi_1) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) \end{vmatrix} d\xi_1 +$$

$$\frac{\lambda^2}{2!} \int_{-\infty}^{\ell} \int_{-\infty}^{\ell} \begin{vmatrix} K(x,\ \xi) & K(x,\ \xi_1) & K(x,\ \xi_2) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) & K(\xi_1,\xi_2) \\ K(\xi_2,\xi) & K(\xi_2,\xi_1) & K(\xi_2,\xi_2) \end{vmatrix} d\xi_1 d\xi_2 - \cdots \tag{36}$$

Comparing (33) with (20) we will have the desired expansion if we set $K(x,\xi) = G(x,\xi)\ P(\xi)$ and $\lambda = -1$. It is assumed, of course, that $\lambda = -1$ is not an eigenvalue of (33). Although the complexity of this expansion rapidly gets out of hand, the first-order approximation obtained by cutting off (35) with the second term and (36) with the first term is a simple expression:

$$u(x) = u_o(x) - \frac{\int_{-\infty}^{\ell} G(x,\xi)\ P(\xi)\ u_o(\xi)\ d\xi}{1 + \int_{-\infty}^{\ell} G(\xi,\xi)\ P(\xi)\ d\xi} \ . \tag{37}$$

Proceeding as in the previous case the reflection coefficient obtained from (37) is

$$\Gamma(k) = \frac{\Gamma_o - \frac{1}{2jk}\int_0^{\ell}\left(\Gamma_o + e^{-2jk\xi}\right) P(\xi)\ d\xi}{1 + \frac{1}{2jk}\int_0^{\ell}\left(1 + \Gamma_o\ e^{2jk\xi}\right) P(\xi)\ d\xi} \ . \tag{38}$$

If we again take $P(x) \cong -p'(x)$, integration by parts yields

$$\Gamma(k) = \frac{\Gamma_o + \int_0^{\ell} e^{-2jk\xi}\ p(\xi)\ d\xi}{1 + \Gamma_o \int_0^{\ell} e^{2jk\xi}\ p(\xi) d\xi} \ , \tag{39}$$

where the constant term has been neglected by virtue of (26), as in the derivation of (27). It is evident that within the restriction imposed by (31), (39) gives the exact solution for the problem of Fig. 2. The superiority of the Fredholm solution over the Neumann solution is probably a valid conclusion in general for the reasons discussed above. If condition (26) is also imposed on (16) to eliminate the p_ℓ/jk term, then (39) becomes identical to a result obtained by Orlov (Ref. 8). Orlov's derivation was based on an entirely different approach involving a limit process applied to a step approximation to the nonuniform line. It should be noted that Orlov did not explicitly make the assumption (26) although he required $|p(x)|_{max}\,\ell \ll 1$ to justify taking only the first terms of his numerator and denominator series. The somewhat more general result (38) does not appear obtainable by his method.

As an example in the application of the Fredholm solution, consider the exponential line of Fig. 3, where $p(x) = K = (1/2\ell)\ell n(Z_{02}/Z_{01})$, $0 \le x \le \ell$. Neglecting terms of order K^2, (38) yields

$$\Gamma(k) = \frac{\dfrac{K}{K + 2jk} + \Gamma_o}{\dfrac{-K + 2jk}{K + 2jk} - \dfrac{K}{K + 2jk}\,\Gamma_o}\,, \quad (40)$$

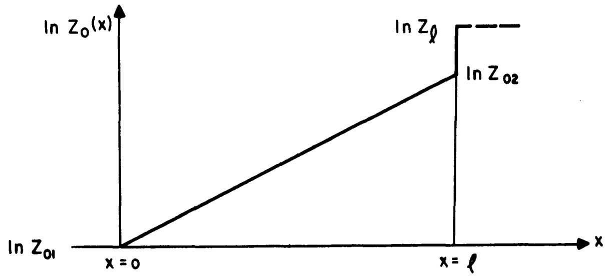

Fig. 3. Impedance variation for the exponential line.

where from (16),

$$\Gamma_o = -\left[\frac{K + jk\left(\dfrac{Z_{02}}{Z_\ell} - 1\right)}{K + jk\left(\dfrac{Z_{02}}{Z_\ell} + 1\right)}\right] e^{-2jk\ell}. \quad (41)$$

The exact solution for this problem is

$$\Gamma(k) = \frac{\left[\dfrac{K + j\left(k - \sqrt{k^2 - K^2}\right)}{K + j\left(k + \sqrt{k^2 - K^2}\right)}\right] + \Gamma_o'}{\left[\dfrac{-K + j\left(k + \sqrt{k^2 - K^2}\right)}{K + j\left(k + \sqrt{k^2 - K^2}\right)}\right] + \left[\dfrac{-K + j\left(k - \sqrt{k^2 - K^2}\right)}{K + j\left(k + \sqrt{k^2 - K^2}\right)}\right]\Gamma_o'}\,, \quad (42)$$

where

$$\Gamma_o' = - \left[\frac{K + j\left(k \frac{Z_{02}}{Z_\ell} - \sqrt{k^2-K^2}\right)}{K + j\left(k \frac{Z_{02}}{Z_\ell} + \sqrt{k^2-K^2}\right)} \right] e^{-2j\ell \sqrt{k^2 - K^2}} . \qquad (43)$$

Equation 40 agrees term-for-term with the exact solution (42) to the extent that $\sqrt{k^2-K^2} \cong k$.

Orlov's (Ref. 9) synthesis formula is readily obtained by taking the complex conjugate of (39) and eliminating the integral involving the negative exponential. Then, noting that $p(x) = 0,\ x \leq 0$ and defining $p(x) = 0,\ x > \ell$, one obtains,

$$\int_{-\infty}^{\infty} p(x)\, e^{-2jkx} dx = \frac{\Gamma\left(1 - |\Gamma_o|^2\right) - \Gamma_o\left(1 - |\Gamma|^2\right)}{1 - |\Gamma|^2\, |\Gamma_o|^2} . \qquad (44)$$

Application of the Fourier integral theorem yields the synthesis formula,

$$p(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left[\frac{\Gamma(k)\left(1 - |\Gamma_o(k)|^2\right) - \Gamma_o(k)\left(1 - |\Gamma(k)|^2\right)}{1 - |\Gamma(k)|^2\, |\Gamma_o(k)|^2} \right] e^{2jkx} dk \qquad (45)$$

(Note that Orlov's equations (16-18) are in error by a factor of 1/2) Given $\Gamma(k)$ and $\Gamma_o(k)$ as a function of real k (45) allows us to calculate $p(x)$ from which $Z_o(x)$ can be obtained by direct integration. However, as Orlov points out, $\Gamma(k)$ and Γ_o (k) can not be chosen arbitrarily since the bracketed term in (45) must satisfy conditions which will guarantee the realizability of $p(x)$ as the distribution function of a sectionally continuous line of finite length. For this reason it is usually necessary to expand the bracketed term in a series of functions which do satisfy the realizability criteria. The reader is referred to a recent report by Cath (Ref. 10) for a discussion of this aspect of the synthesis problem.

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