Chapter 5
Oblique$_u$ packings of $\mathbf{B}_u^9$

Construct the cluster $\mathbf{X} = \mathbf{B}_u^9$ with symmetry group $\mathbf{Z}_2$

$$\mathbf{Z}_2 = \left\{ \begin{bmatrix} +1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & +1 \end{bmatrix}, \begin{bmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$$

Note: Special cases have additional symmetries

$$u = 0 \quad \rightarrow \quad \{ \begin{bmatrix} +1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & +1 \end{bmatrix}, \begin{bmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{bmatrix}, \begin{bmatrix} 0 & +1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & +1 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & +1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & +1 \end{bmatrix} \}$$

$\mathbf{B}^1$ has 1 body
1 tetrahedron: $\{\langle 1,1,1\rangle, \langle -1,-1,1\rangle, \langle -1,1,-1\rangle, \langle 1,-1,-1\rangle\}$

$\mathbf{E}_u^{4+}$ has 4 bodies
1 tetrahedron: $\{\langle 1,1,1\rangle, \langle -1,-1,1\rangle, o_u^+, p_u^+\}$
1 tetrahedron: $\{\langle 1,1,1\rangle, \langle -1,-1,1\rangle, p_u^+, q_u^+\}$
1 tetrahedron: $\{\langle 1,1,1\rangle, \langle -1,-1,1\rangle, q_u^+, r_u^+\}$
1 tetrahedron: $\{\langle 1,1,1\rangle, \langle -1,-1,1\rangle, r_u^+, s_u^+\}$

$o_u^+ = \langle -\frac{7}{9}u - \frac{4}{9}\sqrt{6-2u^2}, +\frac{7}{9}u + \frac{4}{9}\sqrt{6-2u^2}, 1 + \frac{8}{9}u - \frac{7}{9}\sqrt{6-2u^2}\rangle$
$p_u^+ = \langle +\frac{1}{3}u - \frac{2}{3}\sqrt{6-2u^2}, -\frac{1}{3}u + \frac{2}{3}\sqrt{6-2u^2}, 1 + \frac{4}{3}u + \frac{1}{3}\sqrt{6-2u^2}\rangle$
$q_u^+ = \langle +u, -u, 1 + \sqrt{6-2u^2}\rangle$
$r_u^+ = \langle +\frac{1}{3}u + \frac{2}{3}\sqrt{6-2u^2}, -\frac{1}{3}u - \frac{2}{3}\sqrt{6-2u^2}, 1 - \frac{4}{3}u + \frac{1}{3}\sqrt{6-2u^2}\rangle$
$s_u^+ = \langle -\frac{7}{9}u + \frac{4}{9}\sqrt{6-2u^2}, +\frac{7}{9}u - \frac{4}{9}\sqrt{6-2u^2}, 1 - \frac{8}{9}u - \frac{7}{9}\sqrt{6-2u^2}\rangle$

The domain of u is $-\frac{1}{9} \le u \le +\frac{1}{9}$, which correspond to face-to-face intersections

$o_u^+ = \langle -1,+1,-1\rangle \quad \rightarrow \quad u = -\frac{1}{9}$
$s_u^+ = \langle +1,-1,-1\rangle \quad \rightarrow \quad u = +\frac{1}{9}$

$$\mathbf{E}_u^{4-} = \begin{bmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{E}_u^{4+}, \mathbf{E}_u^{5-} = \begin{bmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{E}_u^{5+}$$

$\mathbf{E}_u^{5+} = \mathbf{B}^1 \cup \mathbf{E}_u^{4+}$, $\mathbf{E}_u^{5-} = \mathbf{B}^1 \cup \mathbf{E}_u^{4-}$

$\mathbf{B}_u^9 = \mathbf{E}_u^{5+} \cup \mathbf{E}_u^{5-}$

Construct the packing with symmetry group 1

There are 2 cosets $\{+\mathbf{X}\langle 0,0,0\rangle, -\mathbf{X}(c) = -\mathbf{X}(d)\}$

$a = \langle g,h,0\rangle, b = \langle -h,g,0\rangle$
$c = \langle i,j,k\rangle, d = \langle -j,i,-k\rangle$

The lattice basis is $\{2a, 2b, c-d\}$

The layer basis is $\{2a, 2b\}$, with offsets c, d
above layer: $-\mathbf{L} + c$
central layer: $+\mathbf{L}$
below layer: $-\mathbf{L} + d$

above layer: There are 4 neighbors
$-\mathbf{X}+c, -\mathbf{X}+c-2a, -\mathbf{X}+c-2b, -\mathbf{X}+c-2a+2b$

upper central layer: There are 4 neighbors
$+\mathbf{X}+2a-2b, +\mathbf{X}-2a+2b, +\mathbf{X}-2b, +\mathbf{X}+2b$

lower central layer: Similar to upper central layer

below layer: Similar to above layer

Note: Apply the transformation $\begin{bmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Write the intersection equations

above layer: There are 4 neighbors, 3 face-face intersections, 3 equations
$G_0^+ : F[\langle +1,+1,+1\rangle, q_u^+, r_u^+] = F[\langle -1,-1,-1\rangle, -q_u^+, -r_u^+] + c$
$G_a^+ : F[\langle -1,-1,+1\rangle, q_u^+, p_u^+] = F[\langle +1,+1,-1\rangle, -q_u^+, -p_u^+] + c - 2a$
$G_b^+ : F[\langle -1,-1,+1\rangle, q_u^+, r_u^+] = F[\langle +1,+1,-1\rangle, -q_u^+, -r_u^+] + c - 2b$
$G_{ab}^+ : F[\langle +1,+1,+1\rangle, q_u^+, p_u^+] \neq F[\langle -1,-1,-1\rangle, -q_u^+, -p_u^+] + c - 2a + 2b$

upper central layer: There are 4 neighbors, 4 edge-edge intersections, 2 equations
$G_{a-b}^0 \begin{cases} E[\langle +1,+1,+1\rangle, r_u^+] = E[\langle -1,-1,+1\rangle, p_u^+] + 2a - 2b \\ E[\langle -1,-1,+1\rangle, p_u^+] = E[\langle +1,+1,+1\rangle, r_u^+] - 2a + 2b \end{cases}$
$G_b^0 \begin{cases} E[\langle -1,-1,+1\rangle, r_u^+] = E[\langle +1,+1,+1\rangle, p_u^+] - 2b \\ E[\langle +1,+1,+1\rangle, p_u^+] = E[\langle -1,-1,+1\rangle, r_u^+] + 2b \end{cases}$

lower central layer: Similar to upper central layer

below layer: Similar to above layer

Note: Apply the transformation $\begin{bmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Write the free variables

Write the (common denominator) Wronskian W in terms of the free variables

Write the lattice vectors a, b, c, d in terms of the free variables

Write the lattice volume $V = \det(2a, 2b, c-d)$ in terms of the free variables

Write the equations of minimum lattice volume V

Evaluate the free variables which minimize the lattice volume V

Evaluate the vertices of $\mathbf{E}_u^{5+}$

Evaluate the lattice vectors a, b, c, d

Evaluate the lattice volume $V = \det(2a, 2b, c-d) = \det \begin{bmatrix} 2g & 2h & 0 \\ -2h & 2g & 0 \\ i+j & -i+j & 2k \end{bmatrix} = 8k(g^2+h^2)$

Evaluate the packing density $D = 48/V$

Note: Each tetrahedron has volume $\frac{8}{3}$

5.1 $\mathbf{B}_0^9$

$\mathbf{B}_0^9\ Q_3^+\ P_z$

2 cosets

18 tetrahedra

$U = 48$

48.000000000000

$g = -\frac{168}{71}+\frac{106}{71}\sqrt{6}$

1.290787503310

$h = -\frac{88}{71}+\frac{42}{71}\sqrt{6}$

.209557312632

$i = \frac{374}{71}-\frac{358}{213}\sqrt{6}$

1.150622873632

$j = -\frac{306}{71}+\frac{422}{213}\sqrt{6}$

.543120523261

$k = -\frac{262}{71}+\frac{238}{71}\sqrt{6}$

4.520824771583

$V = -\frac{730200320}{357911}+\frac{307139840}{357911}\sqrt{6}$

61.846569901642

$D = \frac{1711407}{4477040}+\frac{419859}{4477040}\sqrt{6}$

.776114181859

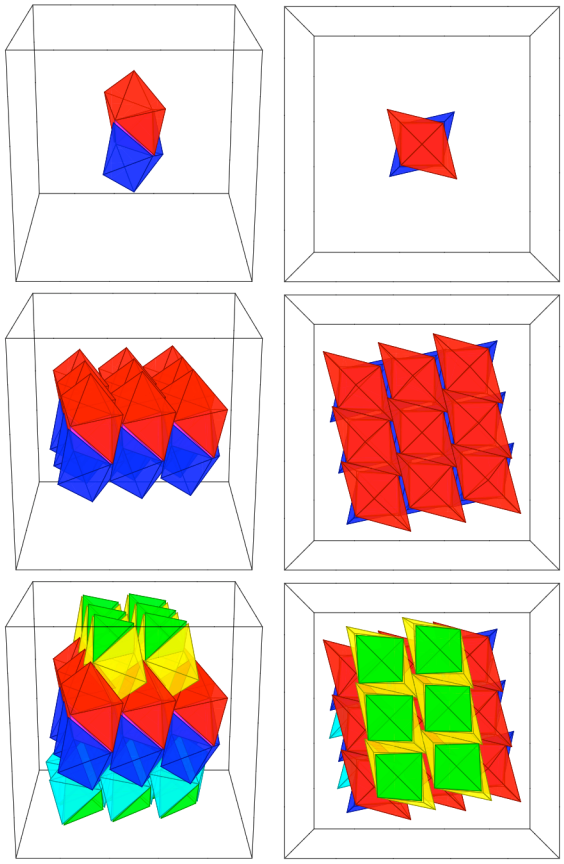
basis {2*a*,2*b*,*c*−*d*}

$a = \langle g,h,0\rangle$

$b = \langle -h,g,0\rangle$

$c = \langle i,j,k\rangle$

$d = \langle -j,i,-k\rangle$



The equations are $\begin{cases} G_0^+\ ,\ G_a^+\ ,\ G_b^+ \\ G_{a-b}^0\ ,\ G_b^0 \\ u = 0 \end{cases}$

There are no free variables

The common denominator is $W = 1$

The lattice vectors are $\begin{cases} a = \langle g,h,0\rangle, b = \langle -h,g,0\rangle \\ c = \langle i,j,k\rangle, d = \langle -j,i,-k\rangle \end{cases}$

$g = \frac{1}{71}(-168+106\sqrt{6})$, $h = \frac{1}{71}(-88+42\sqrt{6})$
$i = \frac{1}{213}(+1122-358\sqrt{6})$, $j = \frac{1}{213}(-918+422\sqrt{6})$, $k = \frac{1}{71}(-262+238\sqrt{6})$

The lattice volume is $V = \frac{1}{357911}(-730200320+307139840\sqrt{6})$

There are no equations of minimum lattice volume

There are no free variables

The vertices of $\mathbf{E}_u^{5+}$ are $\begin{cases} o_u^+ = \langle -\frac{4}{9}\sqrt{6}, +\frac{4}{9}\sqrt{6}, +1-\frac{7}{9}\sqrt{6}\rangle \\ p_u^+ = \langle -\frac{2}{3}\sqrt{6}, +\frac{2}{3}\sqrt{6}, +1+\frac{1}{3}\sqrt{6}\rangle \\ q_u^+ = \langle 0, 0, +1+\sqrt{6}\rangle \\ r_u^+ = \langle +\frac{2}{3}\sqrt{6}, -\frac{2}{3}\sqrt{6}, +1+\frac{1}{3}\sqrt{6}\rangle \\ s_u^+ = \langle +\frac{4}{9}\sqrt{6}, -\frac{4}{9}\sqrt{6}, +1-\frac{7}{9}\sqrt{6}\rangle \end{cases}$

The vertices of $\mathbf{E}_u^{5-}$ are $\begin{cases} o_u^- = \langle -\frac{4}{9}\sqrt{6}, -\frac{4}{9}\sqrt{6}, -1+\frac{7}{9}\sqrt{6}\rangle \\ p_u^- = \langle -\frac{2}{3}\sqrt{6}, -\frac{2}{3}\sqrt{6}, -1-\frac{1}{3}\sqrt{6}\rangle \\ q_u^- = \langle 0, 0, -1-\sqrt{6}\rangle \\ r_u^- = \langle +\frac{2}{3}\sqrt{6}, +\frac{2}{3}\sqrt{6}, -1-\frac{1}{3}\sqrt{6}\rangle \\ s_u^- = \langle +\frac{4}{9}\sqrt{6}, +\frac{4}{9}\sqrt{6}, -1+\frac{7}{9}\sqrt{6}\rangle \end{cases}$

The lattice vectors are $\begin{cases} g = \frac{1}{71}(-168+106\sqrt{6}) \approx 1.290787503310 \\ h = \frac{1}{71}(-88+42\sqrt{6}) \approx 0.209557312632 \\ i = \frac{1}{213}(+1122-358\sqrt{6}) \approx 1.150622873632 \\ j = \frac{1}{213}(-918+422\sqrt{6}) \approx 0.543120523261 \\ k = \frac{1}{71}(-262+238\sqrt{6}) \approx 4.520824771583 \end{cases}$

The lattice volume is $V = \frac{1}{357911}(-730200320+307139840\sqrt{6}) \approx 61.846569901642$

The packing density is $D = \frac{1}{4477040}(1711407+419859\sqrt{6}) \approx .776114181859$

5.2 $\mathbf{B}^9_{-\frac{1}{9}}$

$\mathbf{B}^9_{-\frac{1}{9}}\ Q_3^+\ P_z$

2 cosets

18 tetrahedra

$U = 48$

48.000000000000

$g = \frac{774}{559}$

1.292153589316

$h = \frac{126}{559}$

.210350584307

$i = \frac{18256}{16173}$

1.128794905089

$j = \frac{8240}{16173}$

.509491127187

$k = \frac{72830}{16173}$

4.503184319545

$V = \frac{13270208640}{214921799}$

61.744358653912

$D = \frac{214921799}{276462680}$

.777398956705

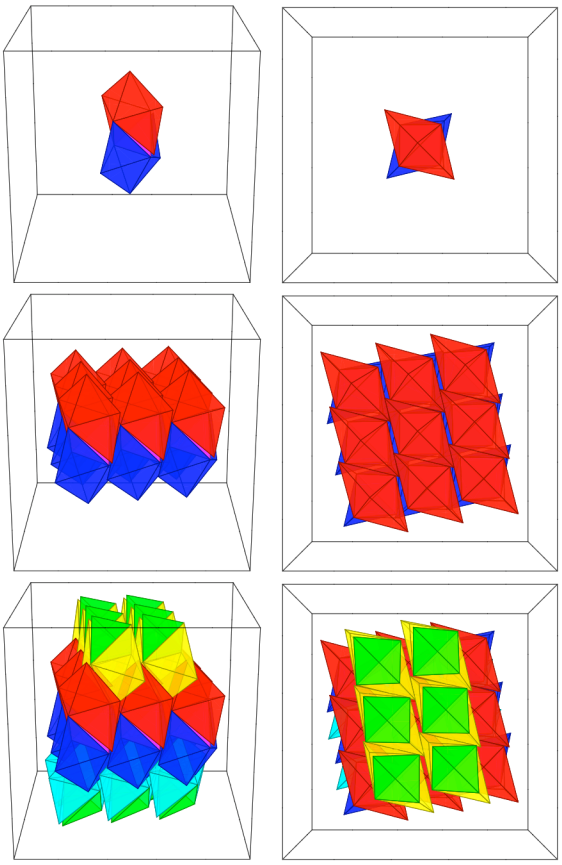
basis $\{2a,2b,c-d\}$

$a = \langle g,h,0\rangle$

$b = \langle -h,g,0\rangle$

$c = \langle i,j,k\rangle$

$d = \langle -j,i,-k\rangle$



The equations are $\begin{cases} G_0^+\ ,\ G_a^+\ ,\ G_b^+ \\ G_{a-b}^0\ ,\ G_b^0 \\ u = -\frac{1}{9} \end{cases}$

There are no free variables

The common denominator is $W = 1$

The lattice vectors are $\begin{cases} a = \langle g,h,0\rangle, b = \langle -h,g,0\rangle \\ c = \langle i,j,k\rangle, d = \langle -j,i,-k\rangle \end{cases}$

$g = \frac{774}{599}, h = \frac{126}{599}, i = \frac{18256}{16173}$
$j = \frac{8240}{16173}, k = \frac{72830}{16173}$

The lattice volume is $V = \frac{13270208640}{214921799}$

There are no equations of minimum lattice volume

There are no free variables

The vertices of E_u^{5+} are $\begin{cases} o_u^+ = \langle -1,+1,-1\rangle \\ p_u^+ = \langle -\frac{5}{3},+\frac{5}{3},+\frac{5}{3}\rangle \\ q_u^+ = \langle -\frac{1}{9},+\frac{1}{9},+\frac{31}{9}\rangle \\ r_u^+ = \langle +\frac{43}{27},-\frac{43}{27},+\frac{53}{27}\rangle \\ s_u^+ = \langle +\frac{95}{81},-\frac{95}{81},-\frac{65}{81}\rangle \end{cases}$

The vertices of E_u^{5-} are $\begin{cases} o_u^- = \langle -1, -1, +1 \rangle \\ p_u^- = \langle -\frac{5}{3}, -\frac{5}{3}, -\frac{5}{3} \rangle \\ q_u^- = \langle -\frac{1}{9}, -\frac{1}{9}, -\frac{31}{9} \rangle \\ r_u^- = \langle +\frac{43}{27}, +\frac{43}{27}, -\frac{53}{27} \rangle \\ s_u^- = \langle +\frac{95}{81}, +\frac{95}{81}, +\frac{65}{81} \rangle \end{cases}$

The lattice vectors are $\begin{cases} g = \frac{774}{599} \approx 1.292153589316 \\ h = \frac{126}{599} \approx 0.210350584307 \\ i = \frac{18256}{16173} \approx 1.128794905089 \\ j = \frac{8240}{16173} \approx 0.509491127187 \\ k = \frac{72830}{16173} \approx 4.503184319545 \end{cases}$

The lattice volume is $V = \frac{13270208640}{214921799} \approx 61.744358653912$

The packing density is $D = \frac{214921799}{276462680} \approx .777398956705$

5.3 $\mathbf{B}^9_{-.084970795847}$

$\mathbf{B}^9_{-.084970795847}\ Q_3^+\ P_z$

2 cosets

18 tetrahedra

U = 48

48.000000000000

g = ?

1.291585634143

h = ?

.210020891576

i = ?

1.133364328111

j = ?

.517959685564

k = ?

4.506610767555

V = ?

61.733437164032

D = ?

.777536489220

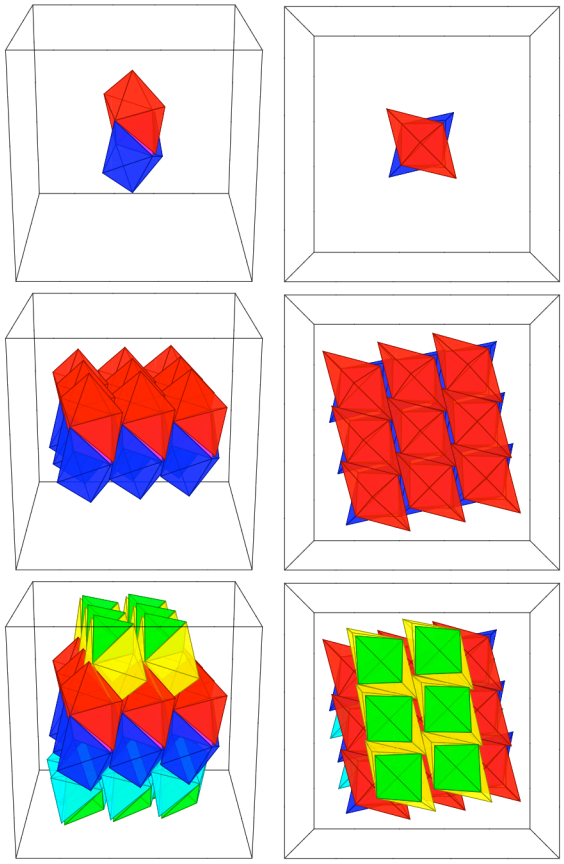
basis {2a,2b,c−d}

$a = \langle g,h,0\rangle$

$b = \langle -h,g,0\rangle$

$c = \langle i,j,k\rangle$

$d = \langle -j,i,-k\rangle$



The equations are $\begin{cases} G_0^+\ ,\ G_a^+\ ,\ G_b^+ \\ G_{a-b}^0\ ,\ G_b^0 \\ \text{minimum lattice volume } V \end{cases}$

The free variable is u

The common denominator is $W = +71 - 42u^2 - u^4$

The lattice vectors are $\begin{cases} a = \langle g,h,0\rangle, b = \langle -h,g,0\rangle \\ c = \langle i,j,k\rangle, d = \langle -j,i,-k\rangle \end{cases}$

$$g = \tfrac{1}{W}\left((-168-8u^2)+(+106+2u^2)\sqrt{6-2u^2}\right)$$
$$h = \tfrac{1}{W}\left((-88+8u^2)+(+42+2u^2)\sqrt{6-2u^2}\right)$$
$$i = \tfrac{1}{3W}\left((-786-972u-196u^2+184u^3-18u^4+4u^5)+(+714+380u-8u^2+12u^3-2u^4)\sqrt{6-2u^2}\right)$$
$$j = \tfrac{1}{3W}\left((+1122+50u-460u^2+20u^3-14u^4+2u^5)+(-358-40u+92u^2-8u^3+2u^4)\sqrt{6-2u^2}\right)$$
$$k = \tfrac{1}{3W}\left((-918+206u+276u^2-20u^3+10u^4-2u^5)+(+422-108u-92u^2+4u^3-2u^4)\sqrt{6-2u^2}\right)$$

The lattice volume is

$$V = \tfrac{1}{3W^3}\left((-2190600960-1670681088u+392051968u^2+531723776u^3+45924864u^4 \right.$$
$$\left. -144384u^5+3287552u^6-1643520u^7+104704u^8-55808u^9+2304u^{10}-512u^{11}\right)$$
$$+(+921419520+680892928u-47891200u^2-100792320u^3+3177984u^4$$
$$\left. -9487360u^5+606720u^6-210944u^7+17152u^8-1536u^9+256u^{10})\sqrt{6-2u^2}\right)$$

The equation of minimum lattice volume is $\partial_u V = 0$

The free variable is $u \approx -0.084970795847$

$$0 = u^{14} - 20u^{13} + 78u^{12} + 540u^{11} + 10505u^{10} - 66800u^9 + 629846u^8 + 380824u^7 + 7789363u^6 + 15599532u^5 + 122520434u^4 + 504389068u^3 + 700251459u^2 + 105710808u - 14354070$$

The vertices of $\mathbf{E}_u^{5+}$ are

$$\begin{cases} o_u^+ = \langle o_{xy}, -o_{xy}, o_z \rangle \approx \langle -1.021262892016, +1.021262892016, -0.978394351685 \rangle \\ p_u^+ = \langle p_{xy}, -p_{xy}, p_z \rangle \approx \langle -1.659350531796, +1.659350531796, +1.702219072127 \rangle \\ q_u^+ = \langle q_{xy}, -q_{xy}, q_z \rangle \approx \langle -0.084970795847, +0.084970795847, +3.446540399770 \rangle \\ r_u^+ = \langle r_{xy}, -r_{xy}, r_z \rangle \approx \langle +1.602703334564, -1.602703334564, +1.928807861053 \rangle \\ s_u^+ = \langle s_{xy}, -s_{xy}, s_z \rangle \approx \langle +1.153439685557, -1.153439685557, -0.827335159068 \rangle \end{cases}$$

The vertices of $\mathbf{E}_u^{5-}$ are

$$\begin{cases} o_u^- = \langle o_{xy}, o_{xy}, -o_z \rangle \approx \langle -1.021262892016, -1.021262892016, +0.978394351685 \rangle \\ p_u^- = \langle p_{xy}, p_{xy}, -p_z \rangle \approx \langle -1.659350531796, -1.659350531796, -1.702219072127 \rangle \\ q_u^- = \langle q_{xy}, q_{xy}, -q_z \rangle \approx \langle -0.084970795847, -0.084970795847, -3.446540399770 \rangle \\ r_u^- = \langle r_{xy}, r_{xy}, -r_z \rangle \approx \langle +1.602703334564, +1.602703334564, -1.928807861053 \rangle \\ s_u^- = \langle s_{xy}, s_{xy}, -s_z \rangle \approx \langle +1.153439685557, +1.153439685557, +0.827335159068 \rangle \end{cases}$$

$$0 = 31381059609o_{xy}^{14} - 460255540932o_{xy}^{13} + 6954197777550o_{xy}^{12} - 148706963937108o_{xy}^{11} + 1851677896431681o_{xy}^{10} - 16189856233016688o_{xy}^9 + 113134355347377942o_{xy}^8 - 581498118651805128o_{xy}^7 + 2278367049218159547o_{xy}^6 - 6628871917191906180o_{xy}^5 + 12648145583692864242o_{xy}^4 - 10984894954581568356o_{xy}^3 + 13075533991563420267o_{xy}^2 + 11182128902372949816o_{xy} - 9505066365941463062$$

$$0 = 31381059609o_z^{14} - 1778260044510o_z^{13} + 43905976597881o_z^{12} - 631714935347100o_z^{11} + 6667269863507469o_z^{10} - 46558548117919890o_z^9 + 823157857616217570o_z^8 + 129991862791499512o_z^7 - 903512493304214152 5o_z^6 + 1381989630479955247 8o_z^5 + 9577377994497204746 7o_z^4 - 5798351683420272875 16o_z^3 + 364288244903842539375o_z^2 + 4157446905314148674514o_z - 396665860776088795 6769$$

$$0 = 6561p_{xy}^{14} - 341172p_{xy}^{13} + 8746542p_{xy}^{12} - 135701892p_{xy}^{11} + 1393037433p_{xy}^{10} - 9627733872p_{xy}^9 + 42018487110p_{xy}^8 - 87977982888p_{xy}^7 - 89980438989p_{xy}^6 + 892483759628p_{xy}^5 - 994396839118p_{xy}^4 - 3414055584596p_{xy}^3 + 7251596096051p_{xy}^2 + 5319604029848p_{xy} - 13643570244134$$

$$0 = 6561p_z^{14} - 65610p_z^{13} + 280665p_z^{12} - 35671428p_z^{11} + 903675285p_z^{10} - 11775260790p_z^9 + 135815940765p_z^8 - 1221447960504p_z^7 + 8647911083715p_z^6 - 48417028375958p_z^5 + 221114048875467p_z^4 - 717928978763396p_z^3 + 1418210587971655p_z^2 - 1369105261682154p_z + 429587642231551$$

$$0 = q_{xy}^{14} - 20q_{xy}^{13} + 78q_{xy}^{12} + 540q_{xy}^{11} + 10505q_{xy}^{10} - 66800q_{xy}^9 + 629846q_{xy}^8 + 380824q_{xy}^7 + 7789363q_{xy}^6 + 15599532q_{xy}^5 + 122520434q_{xy}^4 + 504389068q_{xy}^3 + 700251459q_{xy}^2 + 105710808q_{xy} - 14354070$$

$$0 = q_z^{14} - 82q_z^{13} + 3489q_z^{12} - 94852q_z^{11} + 1768533q_z^{10} - 23198206q_z^9 + 214320005q_z^8 - 1364246520q_z^7 + 5654104547q_z^6 - 13308282158q_z^5 + 10991566947q_z^4 + 6988863676q_z^3 + 23332219911q_z^2 - 10234406594q_z - 228071310281$$

$$0 = 729r_{xy}^{14} - 28188r_{xy}^{13} + 669870r_{xy}^{12} - 11241612r_{xy}^{11} + 133179921r_{xy}^{10} - 1104344592r_{xy}^9 + 6399133062r_{xy}^8 - 25136191992r_{xy}^7 + 61564462747r_{xy}^6 - 70504649820r_{xy}^5 - 34296240398r_{xy}^4 + 134323857156r_{xy}^3 - 49397035013r_{xy}^2 + 398144701704r_{xy} - 667573340582$$

$$0 = 729r_z^{14} - 46170r_z^{13} + 1140561r_z^{12} - 16242660r_z^{11} + 199848429r_z^{10} - 2354384550r_z^9 + 21364031157r_z^8 - 123760770552r_z^7 + 434025830635r_z^6 - 675320201478r_z^5 - 1772657410253r_z^4 + 156190602567 96r_z^3 + 5227593063055r_z^2 - 48626610484794r_z - 8574767926649$$

$$0 = 3486784401s_{xy}^{14} - 159617241468s_{xy}^{13} + 3019383104382s_{xy}^{12} - 33924699918828s_{xy}^{11} + 295677145205433s_{xy}^{10} - 2026013834715408s_{xy}^9 + 8690000198487750s_{xy}^8 - 14069134914372792s_{xy}^7 - 431446734975126 69s_{xy}^6 + 273935898354717252s_{xy}^5 - 505135744311336318s_{xy}^4 - 658583737082645724s_{xy}^3 + 376749220399215029 1s_{xy}^2 + 709196780325225672s_{xy} - 4366079357981753414$$

$$0 = 3486784401s_z^{14} - 73609892910s_z^{13} + 1996722153585s_z^{12} - 49978716235452s_z^{11} + 708708590163045s_z^{10} - 7205790797804610s_z^9 + 556671876577088 85s_z^8 - 277476893544873096s_z^7 + 952746006838374675s_z^6 - 13210643931112160 82s_z^5 - 514099893411571857 3s_z^4 + 167686850434621176 36s_z^3 - 661878328041291465s_z^2 + 101942851025676153474s_z - 90706128722221632089$$

The lattice vectors are $\begin{cases} g \approx 1.291585634143 \\ h \approx 0.210020891576 \\ i \approx 1.133364328111 \\ j \approx 0.517959685564 \\ k \approx 4.506610767555 \end{cases}$

$$0 = 4605216768g^{14} + 1668943872g^{13} - 55135634688g^{12} + 347819434752g^{11} - 1548366821456g^{10} + 3993179160736g^{9} - 6261445428296g^{8} + 6374028678224g^{7} - 4391715655505g^{6} + 2091541626890g^{5} - 691838736726g^{4} + 157264723336g^{3} - 23798940936g^{2} + 2220865728g - 101484096$$

$$0 = 4605216768h^{14} + 14533005312h^{13} + 14842066176h^{12} + 28343723520h^{11} + 3673245040h^{10} + 10551023424h^{9} - 304334776h^{8} + 1657679720h^{7} - 197069813h^{6} + 128810578h^{5} - 25549158h^{4} + 4885448h^{3} - 1429896h^{2} + 100800h - 28224$$

$$0 = 10714113i^{14} - 49315392i^{13} + 632630250i^{12} - 7909953480i^{11} - 154321602066i^{10} + 259629770664i^{9} + 19573383103704i^{8} - 111981486176640i^{7} + 118573959153888i^{6} + 28965778053504i^{5} + 1484193138741376i^{4} - 7668029329679360i^{3} + 14759819805971456i^{2} - 12855062321016832i + 4234665356099584$$

$$0 = 6557037156j^{14} - 107411705640j^{13} + 1035204103656j^{12} - 1334974886316j^{11} - 84554542689723j^{10} + 1602606284735256j^{9} - 2415946639038264j^{8} - 3990170873528400j^{7} - 116076264767900856j^{6} + 725420527659374016j^{5} - 2552951829589390112j^{4} + 795493262488764915 2j^{3} - 1102942607056966912 0j^{2} + 149908564428151664 64j - 5752077384982560768$$

$$0 = 1397493k^{14} - 59250204k^{13} + 743475510k^{12} - 6767663400k^{11} + 147096336501k^{10} - 1085221808460k^{9} - 6993101815260k^{8} + 179974926119872k^{7} - 1297512270479552k^{6} + 5046386677021696k^{5} - 12706270142248704k^{4} + 21085523887730688k^{3} - 23735104070181120k^{2} + 17585512311389184k - 8118200214549504$$

The lattice volume is $V \approx 61.733437164032$

The packing density is $D \approx .777536489220$

Chapter 6
Oblique$_{u,v}$ packings of $\mathbf{B}^9_{u,v}$

Construct the cluster $\mathbf{X} = \mathbf{B}^9_{u,v}$ with symmetry group 1

Note: Special cases have additional symmetries

$$u=+v \quad \rightarrow \quad \begin{bmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad u=0 \quad \rightarrow \quad \begin{bmatrix} 0 & +1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & +1 \end{bmatrix}$$

$$u=-v \quad \rightarrow \quad \begin{bmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad v=0 \quad \rightarrow \quad \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & +1 \end{bmatrix}$$

$\mathbf{B}^1$ has 1 body
1 tetrahedron: $\{\langle 1,1,1\rangle, \langle -1,-1,1\rangle, \langle -1,1,-1\rangle, \langle 1,-1,-1\rangle\}$

$\mathbf{E}^{4+}_u$ has 4 bodies
1 tetrahedron: $\{\langle 1,1,1\rangle, \langle -1,-1,1\rangle, o^+_u, p^+_u\}$
1 tetrahedron: $\{\langle 1,1,1\rangle, \langle -1,-1,1\rangle, p^+_u, q^+_u\}$
1 tetrahedron: $\{\langle 1,1,1\rangle, \langle -1,-1,1\rangle, q^+_u, r^+_u\}$
1 tetrahedron: $\{\langle 1,1,1\rangle, \langle -1,-1,1\rangle, r^+_u, s^+_u\}$

$o^+_u = \langle -\frac{7}{9}u - \frac{4}{9}\sqrt{6-2u^2}, +\frac{7}{9}u + \frac{4}{9}\sqrt{6-2u^2}, 1+\frac{8}{9}u - \frac{7}{9}\sqrt{6-2u^2}\rangle$
$p^+_u = \langle +\frac{1}{3}u - \frac{2}{3}\sqrt{6-2u^2}, -\frac{1}{3}u + \frac{2}{3}\sqrt{6-2u^2}, 1+\frac{4}{3}u + \frac{1}{3}\sqrt{6-2u^2}\rangle$
$q^+_u = \langle +u, -u, 1+\sqrt{6-2u^2}\rangle$
$r^+_u = \langle +\frac{1}{3}u + \frac{2}{3}\sqrt{6-2u^2}, -\frac{1}{3}u - \frac{2}{3}\sqrt{6-2u^2}, 1-\frac{4}{3}u + \frac{1}{3}\sqrt{6-2u^2}\rangle$
$s^+_u = \langle -\frac{7}{9}u + \frac{4}{9}\sqrt{6-2u^2}, +\frac{7}{9}u - \frac{4}{9}\sqrt{6-2u^2}, 1-\frac{8}{9}u - \frac{7}{9}\sqrt{6-2u^2}\rangle$

The domain of u is $-\frac{1}{9} \le u \le +\frac{1}{9}$, which correspond to face-to-face intersections
$o^+_u = \langle -1,+1,-1\rangle \quad \rightarrow \quad u = -\frac{1}{9}$
$s^+_u = \langle +1,-1,-1\rangle \quad \rightarrow \quad u = +\frac{1}{9}$

$\mathbf{E}^{4-}_v$ has 4 bodies
1 tetrahedron: $\{\langle 1,-1,-1\rangle, \langle -1,1,-1\rangle, o^-_v, p^-_v\}$
1 tetrahedron: $\{\langle 1,-1,-1\rangle, \langle -1,1,-1\rangle, p^-_v, q^-_v\}$
1 tetrahedron: $\{\langle 1,-1,-1\rangle, \langle -1,1,-1\rangle, q^-_v, r^-_v\}$
1 tetrahedron: $\{\langle 1,-1,-1\rangle, \langle -1,1,-1\rangle, r^-_v, s^-_v\}$

$o^-_v = \langle -\frac{7}{9}v - \frac{4}{9}\sqrt{6-2v^2}, -\frac{7}{9}v - \frac{4}{9}\sqrt{6-2v^2}, -1-\frac{8}{9}v + \frac{7}{9}\sqrt{6-2v^2}\rangle$
$p^-_v = \langle +\frac{1}{3}v - \frac{2}{3}\sqrt{6-2v^2}, +\frac{1}{3}v - \frac{2}{3}\sqrt{6-2v^2}, -1-\frac{4}{3}v - \frac{1}{3}\sqrt{6-2v^2}\rangle$
$q^-_v = \langle +v, +v, -1-\sqrt{6-2v^2}\rangle$
$r^-_v = \langle +\frac{1}{3}v + \frac{2}{3}\sqrt{6-2v^2}, +\frac{1}{3}v + \frac{2}{3}\sqrt{6-2v^2}, -1+\frac{4}{3}v - \frac{1}{3}\sqrt{6-2v^2}\rangle$
$s^-_v = \langle -\frac{7}{9}v + \frac{4}{9}\sqrt{6-2v^2}, -\frac{7}{9}v + \frac{4}{9}\sqrt{6-2v^2}, -1+\frac{8}{9}v + \frac{7}{9}\sqrt{6-2v^2}\rangle$

The domain of v is $-\frac{1}{9} \le v \le +\frac{1}{9}$, which correspond to face-to-face intersections
$o^-_v = \langle -1,-1,+1\rangle \quad \rightarrow \quad v = -\frac{1}{9}$
$s^-_v = \langle +1,+1,+1\rangle \quad \rightarrow \quad v = +\frac{1}{9}$

$\mathbf{E}^{5+}_u = \mathbf{B}^1 \cup \mathbf{E}^{4+}_u$, $\mathbf{E}^{5-}_v = \mathbf{B}^1 \cup \mathbf{E}^{4-}_v$

$\mathbf{B}^9_{u,v} = \mathbf{E}^{5+}_u \cup \mathbf{E}^{5-}_v$

Construct the packing with symmetry group 1

There are 2 cosets $\{+\mathbf{X}\langle 0,0,0\rangle, -\mathbf{X}(c) = -\mathbf{X}(d)\}$

The lattice basis is $\{2a, 2b, c-d\}$

The layer basis is $\{2a, 2b\}$, with offsets c, d
above layer: $-\mathbf{L}+c$
central layer: $+\mathbf{L}$
below layer: $-\mathbf{L}+d$

above layer: There are 4 neighbors
$-\mathbf{X}+c, -\mathbf{X}+c-2a, -\mathbf{X}+c-2b, -\mathbf{X}+c-2a+2b$

upper central layer: There are 4 neighbors
$+\mathbf{X}+2a-2b, +\mathbf{X}-2a+2b, +\mathbf{X}-2b, +\mathbf{X}+2b$

lower central layer: There are 4 neighbors
$+\mathbf{X}+2a+2b, +\mathbf{X}-2a-2b, +\mathbf{X}+2a, +\mathbf{X}-2a$

below layer: There are 4 neighbors
$-\mathbf{X}+d, -\mathbf{X}+d-2b, -\mathbf{X}+d+2a, -\mathbf{X}+d-2a-2b$

Write the intersection equations

above layer: There are 4 neighbors, 3 face-face intersections, 3 equations
$G_0^+ : F[\langle +1,+1,+1\rangle, q_u^+, r_u^+] = F[\langle -1,-1,-1\rangle, -q_u^+, -r_u^+] + c$
$G_a^+ : F[\langle -1,-1,+1\rangle, q_u^+, p_u^+] = F[\langle +1,+1,-1\rangle, -q_u^+, -p_u^+] + c - 2a$
$G_b^+ : F[\langle -1,-1,+1\rangle, q_u^+, r_u^+] = F[\langle +1,+1,-1\rangle, -q_u^+, -r_u^+] + c - 2b$
$G_{ab}^+ : F[\langle +1,+1,+1\rangle, q_u^+, p_u^+] \neq F[\langle -1,-1,-1\rangle, -q_u^+, -p_u^+] + c - 2a + 2b$

upper central layer: There are 4 neighbors, 4 edge-edge intersections, 2 equations
$G_{a-b}^0 : \begin{cases} E[\langle +1,+1,+1\rangle, r_u^+] = E[\langle -1,-1,+1\rangle, p_u^+] + 2a - 2b \\ E[\langle -1,-1,+1\rangle, p_u^+] = E[\langle +1,+1,+1\rangle, r_u^+] - 2a + 2b \end{cases}$
$G_b^0 : \begin{cases} E[\langle -1,-1,+1\rangle, r_u^+] = E[\langle +1,+1,+1\rangle, p_u^+] - 2b \\ E[\langle +1,+1,+1\rangle, p_u^+] = E[\langle -1,-1,+1\rangle, r_u^+] + 2b \end{cases}$

lower central layer: There are 4 neighbors, 4 edge-edge intersections, 2 equations
$G_{a+b}^0 : \begin{cases} E[\langle -1,+1,-1\rangle, r_v^-] = E[\langle +1,-1,-1\rangle, p_v^-] + 2a + 2b \\ E[\langle +1,-1,-1\rangle, p_v^-] = E[\langle -1,+1,-1\rangle, r_v^-] - 2a - 2b \end{cases}$
$G_a^0 : \begin{cases} E[\langle +1,-1,-1\rangle, r_v^-] = E[\langle -1,+1,-1\rangle, p_v^-] + 2a \\ E[\langle -1,+1,-1\rangle, p_v^-] = E[\langle +1,-1,-1\rangle, r_v^-] - 2a \end{cases}$

below layer: There are 4 neighbors, 3 face-face intersections, 3 equations
$G_0^- : F[\langle -1,+1,-1\rangle, q_v^-, r_v^-] = F[\langle +1,-1,+1\rangle, -q_v^-, -r_v^-] + d$
$G_b^- : F[\langle +1,-1,-1\rangle, q_v^-, p_v^-] = F[\langle -1,+1,+1\rangle, -q_v^-, -p_v^-] + d - 2b$
$G_a^- : F[\langle +1,-1,-1\rangle, q_v^-, r_v^-] = F[\langle -1,+1,+1\rangle, -q_v^-, -r_v^-] + d + 2a$
$G_{ab}^- : F[\langle -1,+1,-1\rangle, q_v^-, p_v^-] \neq F[\langle +1,-1,+1\rangle, -q_v^-, -p_v^-] + d - 2a - 2b$

There are 2 curves which partition the domain into 4 regions
$H_{ab}^- : \begin{cases} E[\langle +1,+1,+1\rangle, s_u^+] = E[p_v^-, o_v^-] + 2a \\ E[p_v^-, o_v^-] = E[\langle +1,+1,+1\rangle, s_u^+] - 2a \end{cases}$
$H_{ab}^+ : \begin{cases} E[p_u^+, o_u^+] = E[\langle +1,-1,-1\rangle, o_v^-] + 2b \\ E[\langle +1,-1,-1\rangle, o_v^-] = E[p_u^+, o_u^+] - 2b \end{cases}$

Write the free variables

Write the (common denominator) Wronskian W in terms of the free variables

Write the lattice vectors a, b, c, d in terms of the free variables

Write the lattice volume $V = \det(2a, 2b, c-d)$ in terms of the free variables

Write the equations of minimum lattice volume V

Evaluate the free variables which minimize the lattice volume V

Evaluate the vertices of $\mathbf{E}_u^{5+}$

Evaluate the vertices of $\mathbf{E}_v^{5-}$

Evaluate the lattice vectors a, b, c, d

Evaluate the lattice volume $V = \det(2a, 2b, c-d)$

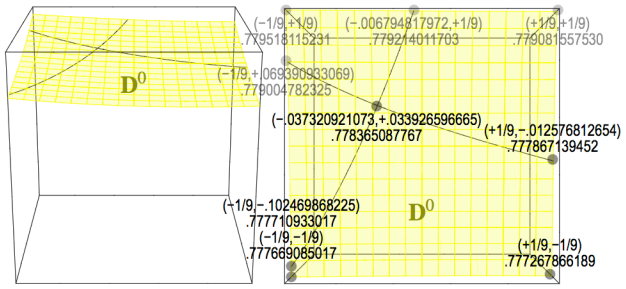
Evaluate the packing density $D = 48/V$

Note: Each tetrahedron has volume $\frac{8}{3}$

virtual density function $\mathbf{D}^0$

5 boundary points

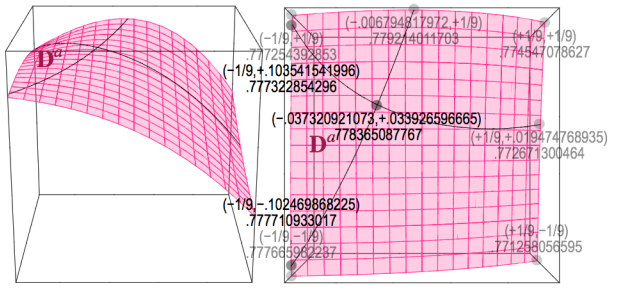
5 boundary curves



virtual density function $\mathbf{D}^a$

3 boundary points

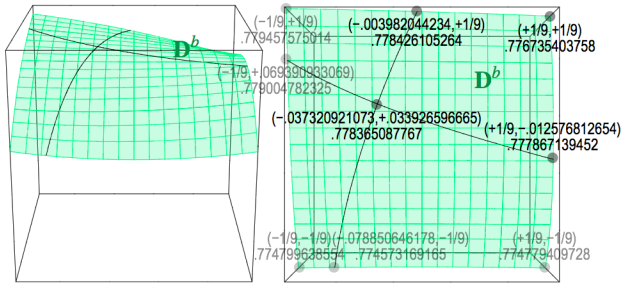
3 boundary curves



virtual density function $\mathbf{D}^b$

4 boundary points

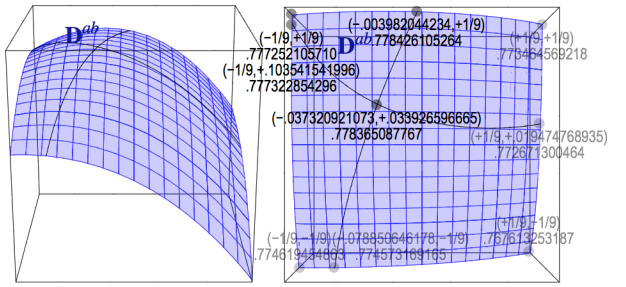
4 boundary curves



virtual density function $\mathbf{D}^{ab}$

4 boundary points

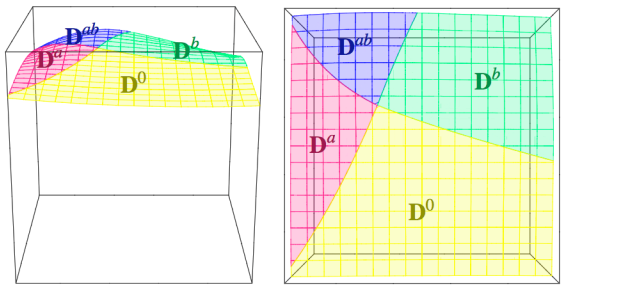
4 boundary curves



composite density function $\mathbf{D}$

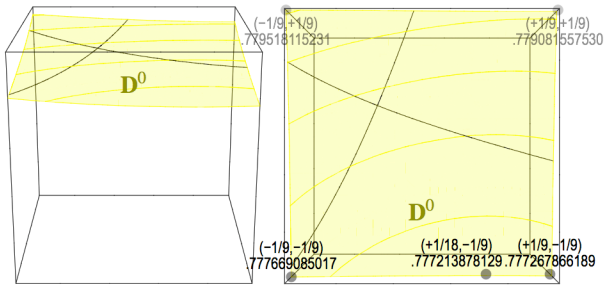
9 boundary points

12 boundary curves



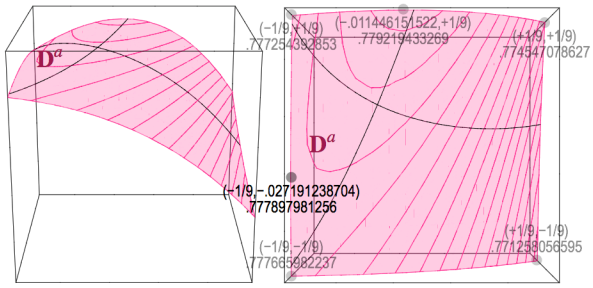
virtual density function $\mathbf{D}^0$

3 critical points



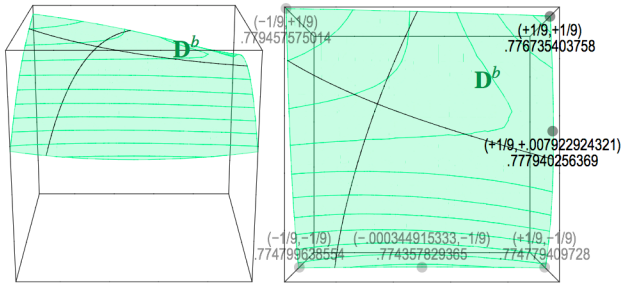
virtual density function $\mathbf{D}^a$

1 critical point



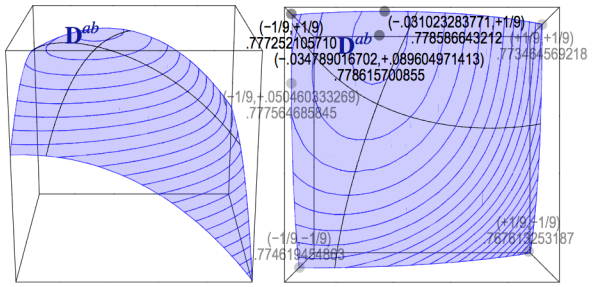
virtual density function $\mathbf{D}^b$

2 critical points



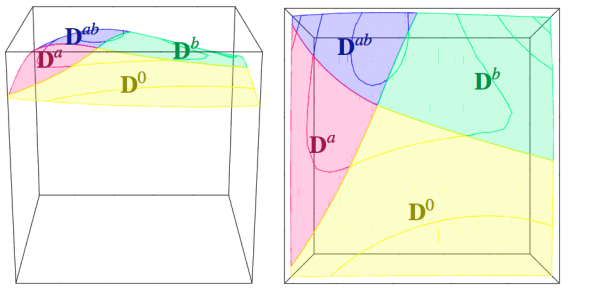
virtual density function $\mathbf{D}^{ab}$

3 critical points



composite density function $\mathbf{D}$

9 critical points



6.1 $\mathbf{B}^9_{-\frac{1}{9},+\frac{1}{9}}$

$\mathbf{B}^9_{-\frac{1}{9},+\frac{1}{9}}\ Q_3^+\ P_z\ \mathbf{D}^{ab}$

2 cosets

18 tetrahedra

$U = 48$

48.000000000000

$a = ?$

+1.287941765701

+0.202195140961

+0.065054781400

$b = ?$

−0.215969909580

+1.294601353287

+0.031649878258

$c = ?$

+1.070183211388

+0.540958063012

+4.574214526410

$d = ?$

−0.441269117432

+1.256162413625

−4.494791862246

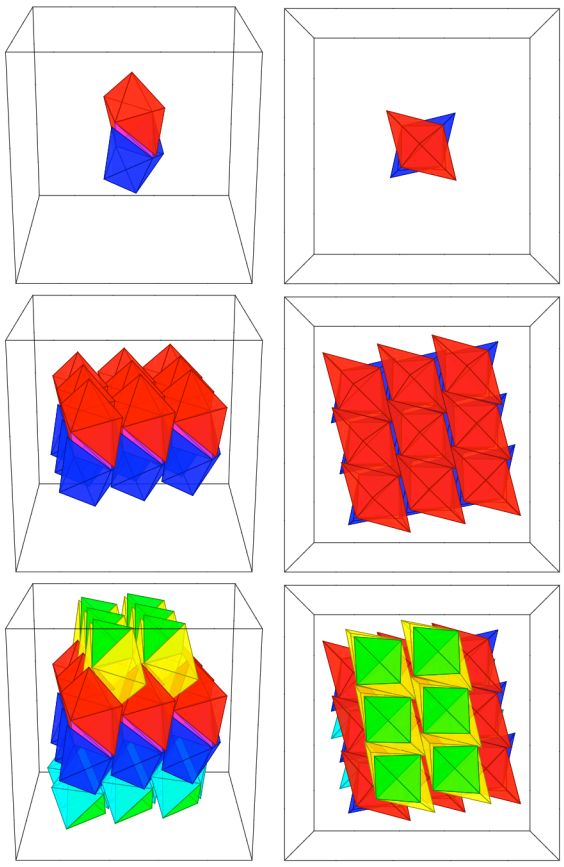
$V = \frac{1129298043089138537578677669 76}{1828644337428063987151456125}$

61.756024393320

$D = \frac{5485933012284191961454368375}{7058112769307115859866735436}$

.777252105710

basis {2*a*,2*b*,*c*−*d*}



The intersection equations are $\begin{cases} G_0^+\ ,\ G_a^+\ ,\ G_b^+ \\ G_{a-b}^0\ ,\ G_b^0 \\ G_{a+b}^0\ ,\ G_a^0 \\ G_0^-\ ,\ G_b^-\ ,\ G_a^- \\ H_a\ ,\ H_b \\ \langle u,v\rangle = \langle -\frac{1}{9},+\frac{1}{9}\rangle \end{cases}$

There are no free variables

The common denominator is $W = 1$

The lattice vectors are $\begin{cases} a = \langle +\frac{1092024102}{847883135}, +\frac{34287570}{169576627}, +\frac{55158852}{847883135}\rangle \\ b = \langle -\frac{183117244}{847883135}, +\frac{1097670654}{847883135}, +\frac{26835398}{847883135}\rangle \\ c = \langle +\frac{4899907600}{4578568929}, +\frac{12384068896}{22892844645}, +\frac{104716782526}{22892844645}\rangle \\ d = \langle -\frac{102039448}{231240855}, +\frac{3195236776}{2543649405}, -\frac{11433174646}{2543649405}\rangle \end{cases}$

The lattice volume is $V = \frac{1129298043089138537578677669 76}{1828644337428063987151456125}$

There are no equations of minimum lattice volume

There are no free variables

The vertices of $\mathbf{E}_u^{5+}$ are $\begin{cases} o_u^+ = \langle -1, +1, -1 \rangle \\ p_u^+ = \langle -\frac{5}{3}, +\frac{5}{3}, +\frac{5}{3} \rangle \\ q_u^+ = \langle -\frac{1}{9}, +\frac{1}{9}, +\frac{31}{9} \rangle \\ r_u^+ = \langle +\frac{43}{27}, -\frac{43}{27}, +\frac{53}{27} \rangle \\ s_u^+ = \langle +\frac{95}{81}, -\frac{95}{81}, -\frac{65}{81} \rangle \end{cases}$

The vertices of $\mathbf{E}_v^{5-}$ are $\begin{cases} o_v^- = \langle -\frac{95}{81}, -\frac{95}{81}, +\frac{65}{81} \rangle \\ p_v^- = \langle -\frac{43}{27}, -\frac{43}{27}, -\frac{53}{27} \rangle \\ q_v^- = \langle +\frac{1}{9}, +\frac{1}{9}, -\frac{31}{9} \rangle \\ r_v^- = \langle +\frac{5}{3}, +\frac{5}{3}, -\frac{5}{3} \rangle \\ s_v^- = \langle +1, +1, +1 \rangle \end{cases}$

The lattice vectors are $\begin{cases} a \approx \langle +1.287941765701, +0.202195140961, +0.065054781400 \rangle \\ b \approx \langle -0.215969909580, +1.294601353287, +0.031649878258 \rangle \\ c \approx \langle +1.070183211388, +0.540958063012, +4.574214526410 \rangle \\ d \approx \langle -0.441269117432, +1.256162413625, -4.494791862246 \rangle \end{cases}$

$a = \langle +\frac{1092024102}{847883135}, +\frac{34287570}{169576627}, +\frac{55158852}{847883135} \rangle$
$b = \langle -\frac{183117244}{847883135}, +\frac{1097670654}{847883135}, +\frac{26835398}{847883135} \rangle$
$c = \langle +\frac{4899907600}{4578568929}, +\frac{12384068896}{22892844645}, +\frac{104716782526}{22892844645} \rangle$
$d = \langle -\frac{102039448}{231240855}, +\frac{3195236776}{2543649405}, -\frac{11433174646}{2543649405} \rangle$

The lattice volume is $V \approx 61.756024393320$

$V = \frac{1129298043089138537578677669 76}{1828644337428063987151456125}$

The packing density is $D \approx .777252105710$

$D = \frac{54859330122841919614543683 75}{70581127693071158598667354 36}$

6.2 $\mathbf{B}^9_{+\frac{1}{9},+\frac{1}{9}}$

$\mathbf{B}^9_{+\frac{1}{9},+\frac{1}{9}}\ Q_3^+\ P_z\ \mathbf{D}^b$

2 cosets

18 tetrahedra

U = 48

48.000000000000

a = ?

+1.285276738077

+0.199628744395

+0.124375806516

b = ?

−0.220237698042

+1.290961331611

+0.103785269574

c = ?

+1.100033554060

+0.561612216353

+4.698786102208

d = ?

−0.288628522805

+1.360240724686

−4.410532566436

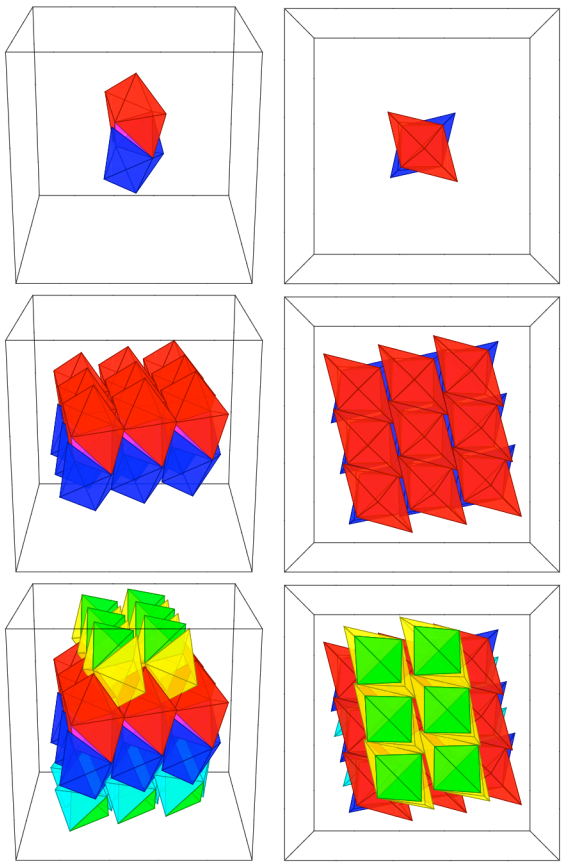
V = ?

61.797105897049

D = ?

.776735403758

basis {2a,2b,c−d}



The intersection equations are
$$\begin{cases} G_0^+\ ,\ G_a^+\ ,\ G_b^+ \\ G_{a-b}^0\ ,\ G_b^0 \\ G_{a+b}^0\ ,\ G_a^0 \\ G_0^-\ ,\ G_b^-\ ,\ G_a^- \\ H_b \\ \langle u,v\rangle = \langle +\frac{1}{9},+\frac{1}{9}\rangle \\ \text{minimum lattice volume } V \end{cases}$$

The free variable is $z = b_z$

The common denominator is $W = 1$

The lattice vectors are

$$a = \langle +\tfrac{452286}{345187} - \tfrac{83106}{345187}z, +\tfrac{252074}{1035561} - \tfrac{2621543}{6213366}z, -\tfrac{304168}{1035561} + \tfrac{25030531}{6213366}z\rangle$$
$$b = \langle -\tfrac{2046}{11903} - \tfrac{5545}{11903}z, +\tfrac{15270}{11903} + \tfrac{928}{11903}z, z\rangle$$
$$c = \langle +\tfrac{14151644}{9320049} - \tfrac{112711873}{27960147}z, +\tfrac{2228716}{9320049} + \tfrac{86877571}{27960147}z, +\tfrac{38698214}{9320049} + \tfrac{147266642}{27960147}z\rangle$$
$$d = \langle -\tfrac{9511412}{9320049} + \tfrac{394355387}{55920294}z, +\tfrac{8996548}{9320049} + \tfrac{212802581}{55920294}z, -\tfrac{43070038}{9320049} + \tfrac{56761187}{27960147}z\rangle$$

The lattice volume is

$$V = +\tfrac{24560542244081634 56}{38293852553802189} - \tfrac{1712544128543841872}{38293852553802189}z + \tfrac{8045283123402433972}{38293852553802189}z^2 + \tfrac{3557791847779632 6980}{1033934018952659103}z^3$$

The equation of minimum lattice volume is $\partial_z V = 0$

The free variable is $z \approx +0.103785269574$

$$z = -\frac{1810188702765547 6437}{8894479619449081745} + \frac{3}{8894479619449081745}\sqrt{4021674876351861077961342989149403 6101}$$

The vertices of $\mathbf{E}_u^{5+}$ are $\begin{cases} o_u^+ = \langle -\frac{95}{81}, +\frac{95}{81}, -\frac{65}{81} \rangle \\ p_u^+ = \langle -\frac{43}{27}, +\frac{43}{27}, +\frac{53}{27} \rangle \\ q_u^+ = \langle +\frac{1}{9}, -\frac{1}{9}, +\frac{31}{9} \rangle \\ r_u^+ = \langle +\frac{5}{3}, -\frac{5}{3}, +\frac{5}{3} \rangle \\ s_u^+ = \langle +1, -1, -1 \rangle \end{cases}$

The vertices of $\mathbf{E}_v^{5-}$ are $\begin{cases} o_v^- = \langle -\frac{95}{81}, -\frac{95}{81}, +\frac{65}{81} \rangle \\ p_v^- = \langle -\frac{43}{27}, -\frac{43}{27}, -\frac{53}{27} \rangle \\ q_v^- = \langle +\frac{1}{9}, +\frac{1}{9}, -\frac{31}{9} \rangle \\ r_v^- = \langle +\frac{5}{3}, +\frac{5}{3}, -\frac{5}{3} \rangle \\ s_v^- = \langle +1, +1, +1 \rangle \end{cases}$

The lattice vectors are $\begin{cases} a \approx \langle +1.285276738077, +0.199628744395, +0.124375806516 \rangle \\ b \approx \langle -0.220237698042, +1.290961331611, +0.103785269574 \rangle \\ c \approx \langle +1.100033554060, +0.561612216353, +4.698786102208 \rangle \\ d \approx \langle -0.288628522805, +1.360240724686, -4.410532566436 \rangle \end{cases}$

$$a_x = +\frac{160122601154808362 16}{8894479619449081745} - \frac{249318}{3070258736398770180311315}\sqrt{4021674876351861077961342989149403 6101}$$
$$a_y = +\frac{5881573133567126 1911}{5336687771669449 0470} - \frac{2621543}{1842155241839262 1081867890}\sqrt{4021674876351861077961342989149403 6101}$$
$$a_z = -\frac{4532155429429650 11287}{5336687771669449 0470} + \frac{25030531}{1842155241839262 1081867890}\sqrt{4021674876351861077961342989149403 6101}$$
$$b_x = +\frac{1380775573669777293}{1778895923889816349} - \frac{3327}{2117419818206048 4002147}\sqrt{4021674876351861077961342989149403 6101}$$
$$b_y = +\frac{9999172698254490138}{8894479619449081745} + \frac{2784}{1058709909103024200 10735}\sqrt{4021674876351861077961342989149403 6101}$$
$$b_z = -\frac{1810188702765547 6437}{8894479619449081745} + \frac{3}{8894479619449081745}\sqrt{4021674876351861077961342989149403 6101}$$
$$c_x = +\frac{2334881401617623868481}{2401509497251252 07115} - \frac{112711873}{8289698588276679486840 5505}\sqrt{4021674876351861077961342989149403 6101}$$
$$c_y = -\frac{1461215870779700450947}{2401509497251252 07115} + \frac{86877571}{8289698588276679486840 5505}\sqrt{4021674876351861077961342989149403 6101}$$
$$c_z = -\frac{1577118768696335382824}{2401509497251252 07115} + \frac{147266642}{8289698588276679486840 5505}\sqrt{4021674876351861077961342989149403 6101}$$
$$d_x = -\frac{7383602728703046648119}{4803018994502504 14230} + \frac{394355387}{1657939717655335897368 11010}\sqrt{4021674876351861077961342989149403 6101}$$
$$d_y = -\frac{3256216295773142653817}{4803018994502504 14230} + \frac{212802581}{1657939717655335897368 11010}\sqrt{4021674876351861077961342989149403 6101}$$
$$d_z = -\frac{2101992369574867283609}{2401509497251252 07115} + \frac{56761187}{8289698588276679486840 5505}\sqrt{4021674876351861077961342989149403 6101}$$

The lattice volume is $V \approx 61.797105897049$

$$V = +\frac{1570583388342464428651640975247235915092856}{2136017727921467214473487787539690615675} - \frac{7830438445858942050434106 2728}{7373255514480275033624598289174631845530062 25}\sqrt{4021674876351861077961342989149403 6101}$$

The packing density is $D \approx .776735403758$

$$D = +\frac{1708009434822430066158659560581369057663 4809}{4213106760850164924911169733826105326294 5142} + \frac{2936414417197103268912789852 3}{5014860977439951310121765334173213169888360 25226}\sqrt{4021674876351861077961342989149403 6101}$$

6.3 $\mathbf{B}^9_{-\frac{1}{9},-\frac{1}{9}}$

$\mathbf{B}^9_{-\frac{1}{9},-\frac{1}{9}}\ Q_3^+\ P_z\ \mathbf{D}^0$

2 cosets

18 tetrahedra

U = 48

48.000000000000

a = ?

+1.293959410666

+0.214323454133

+0.024558730384

b = ?

−0.204778173941

+1.289490557792

−0.038221041613

c = ?

+1.087128071952

+0.583575840582

+4.515455306429

d = ?

−0.520271340930

+1.097067498905

−4.546256075856

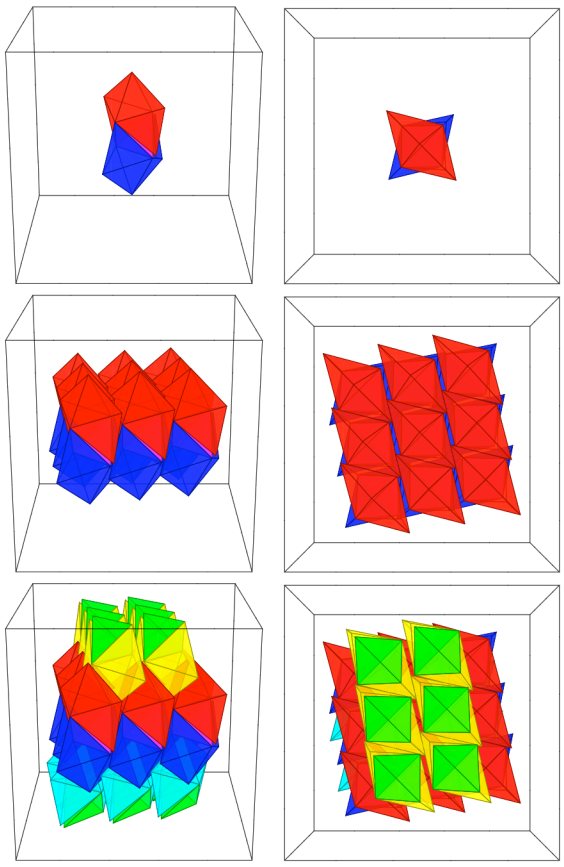
V = ?

61.722911357575

D = ?

.777669085017

basis {2a,2b,c−d}



The intersection equations are
$$\begin{cases} G_0^+\ ,\ G_a^+\ ,\ G_b^+ \\ G_{a-b}^0\ ,\ G_b^0 \\ G_{a+b}^0\ ,\ G_a^0 \\ G_0^-\ ,\ G_b^-\ ,\ G_a^- \\ \langle u,v\rangle = \langle -\frac{1}{9},-\frac{1}{9}\rangle \\ \text{minimum lattice volume } V \end{cases}$$

The free variables are $\langle x,y\rangle = \langle a_x,a_y\rangle$

The common denominator is $W = 1$

The lattice vectors are

$a = \langle x, y, -36+29x-7y\rangle$
$b = \langle +\frac{18}{7}-\frac{18}{7}x+\frac{18}{7}y, -\frac{25}{7}+\frac{527}{126}x-\frac{18}{7}y, -\frac{851}{7}+\frac{13015}{126}x-\frac{396}{7}y\rangle$
$c = \langle -\frac{2323}{63}+\frac{114319}{3402}x-\frac{541}{21}y, +\frac{8641}{63}-\frac{399673}{3402}x+\frac{1513}{21}y, -\frac{2228}{63}+\frac{55753}{1701}x-\frac{248}{21}y\rangle$
$d = \langle -\frac{34450}{189}+\frac{28801}{189}x-\frac{13604}{189}y, -\frac{28654}{189}+\frac{24391}{189}x-\frac{12596}{189}y, -\frac{21898}{189}+\frac{17950}{189}x-\frac{10208}{189}y\rangle$

The lattice volume is

$$V = -\frac{26710576}{147}+\frac{119546864}{189}x-\frac{16795136}{49}y-\frac{8087473388}{11907}x^2+\frac{955999400}{1323}xy-\frac{25133536}{147}y^2 + \frac{24860687200}{107163}x^3-\frac{622374580}{1701}x^2y+\frac{231334256}{1323}xy^2-\frac{3558256}{147}y^3$$

The equations of minimum lattice volume are $\langle \partial_x V, \partial_y V\rangle = \langle 0,0\rangle$

The free variables are $\langle x, y\rangle \approx \langle +1.293959410666, +0.214323454133\rangle$

$$\begin{aligned}0 &= 59845829998195956188337277 00x^4 - 31082933704411639939516317312x^3\\ &\quad +58277059386499851379601043303x^2 - 47055738139494649252640620362x\\ &\quad +13877727923250858970137686256\\ 0 &= 14362999199567029485200946480y^4 - 21371364375247880686608765856y^3\\ &\quad -69214883471751049462947909 1y^2 + 29352312548132557854213791 78y\\ &\quad -41720315921152694630128 4976\end{aligned}$$

The vertices of $\mathbf{E}_u^{5+}$ are $\begin{cases} o_u^+ = \langle -1, +1, -1\rangle \\ p_u^+ = \langle -\frac{5}{3}, +\frac{5}{3}, +\frac{5}{3}\rangle \\ q_u^+ = \langle -\frac{1}{9}, +\frac{1}{9}, +\frac{31}{9}\rangle \\ r_u^+ = \langle +\frac{43}{27}, -\frac{43}{27}, +\frac{53}{27}\rangle \\ s_u^+ = \langle +\frac{95}{81}, -\frac{95}{81}, -\frac{65}{81}\rangle \end{cases}$

The vertices of $\mathbf{E}_v^{5-}$ are $\begin{cases} o_v^- = \langle -1, -1, +1\rangle \\ p_v^- = \langle -\frac{5}{3}, -\frac{5}{3}, -\frac{5}{3}\rangle \\ q_v^- = \langle -\frac{1}{9}, -\frac{1}{9}, -\frac{31}{9}\rangle \\ r_v^- = \langle +\frac{43}{27}, +\frac{43}{27}, -\frac{53}{27}\rangle \\ s_v^- = \langle +\frac{95}{81}, +\frac{95}{81}, +\frac{65}{81}\rangle \end{cases}$

The lattice vectors are $\begin{cases} a \approx \langle +1.293959410666, +0.214323454133, +0.024558730384\rangle \\ b \approx \langle -0.204778173941, +1.289490557792, -0.038221041613\rangle \\ c \approx \langle +1.087128071952, +0.583575840582, +4.515455306429\rangle \\ d \approx \langle -0.520271340930, +1.097067498905, -4.546256075856\rangle \end{cases}$

$$\begin{aligned}0 &= 59845829998195956188337277 00a_x^4 - 31082933704411639939516317312a_x^3\\ &\quad +58277059386499851379601043303a_x^2 - 47055738139494649252640620362a_x\\ &\quad +13877727923250858970137686256\\ 0 &= 14362999199567029485200946480a_y^4 - 21371364375247880686608765856a_y^3\\ &\quad -69214883471751049462947909 1a_y^2 + 29352312548132557854213791 78a_y\\ &\quad -41720315921152694630128 4976\\ 0 &= 71814995997835147426004732400a_z^4 + 27249624768668635442430984 5984a_z^3\\ &\quad -10402258577958834164123663739095a_z^2 - 44091895875241610656848643744516a_z\\ &\quad +10891108476204216389384584873 32\end{aligned}$$

$$\begin{aligned}0 &= 59845829998195956188337277 00b_x^4 - 45260574459943529706143466 72b_x^3\\ &\quad -73510345452489553147671600 21b_x^2 + 45458137813639546986176153 88b_x\\ &\quad +11897526557947512969496500 12\\ 0 &= 71814995997835147426004732400b_y^4 - 25936404494201888014592437 8368b_y^3\\ &\quad +26110555622003832047260542462 1b_y^2 - 14071014120546937271086412 93b_y\\ &\quad -74791818181322575004364072504\\ 0 &= 14362999199567029485200946480b_z^4 + 48792265898123761111625385 74496b_z^3\\ &\quad -49589498623240685009183594154 79b_z^2 + 10882782876141249814135252689487b_z\\ &\quad +42322278184979774344452130 2342\end{aligned}$$

$$\begin{aligned}0 &= 23853395805053442864514613119617 75c_x^4 + 26938896958841490466682535467621262c_x^3\\ &\quad -15085884337211053133292051468948926 7c_x^2 - 11939982746542522357138077216893526 18c_x\\ &\quad +14383778047882828996234244956331316 80\\ 0 &= 95413583220213771458058452478471 00c_y^4 - 43562954553396444127150419565940133 6c_y^3\\ &\quad -19840916658836104184945999576928911 13c_y^2 + 18977731707345734212005260775090182985c_y\\ &\quad -10313770249674633638695852583357454916\\ 0 &= 95413583220213771458058452478471 00c_z^4 - 10690135661723101779434106937704076 8c_z^3\\ &\quad -76065929802229560536625073868372877 9c_z^2 + 69802225880994206965217414843086965 30c_z\\ &\quad -10134024308565269818645068992854236424\end{aligned}$$

$$\begin{aligned}0 &= 35338364155634730169651278695730 0d_x^4 + 15807630008060744465336085151786368d_x^3\\ &\quad -42119044829091575139583701070969652 1d_x^2 - 10969572450819359093286505038544690 86d_x\\ &\quad -45450636248294809384360589325999 1304\\ 0 &= 35338364155634730169651278695730 0d_y^4 + 12481263097770850178598691518379968d_y^3\\ &\quad -28290929625636258641572588587141896 1d_y^2 + 10457210095855122710571193188953093 58d_y\\ &\quad -82372098722684425291234413096132 3368\\ 0 &= 17669182077817365084825639347865d_z^4 + 89295945147651899683150003340301 6d_z^3\\ &\quad +54947236343368682764439984118356 19d_z^2 + 78331847844100805139122981153685 12d_z\\ &\quad -15976497309861332388864754272704 92\end{aligned}$$

The lattice volume is $V \approx 61.722911357575$

$$\begin{aligned}0 = {} & 16318315846238099125459606185646220843044061242298609006 25V^4 \\ & -14723388680741601826578503540955137391865483479779492448359592V^3 \\ & +12565717887814139235124436260650321665307038853708605966699943468V^2 \\ & -22987819424440837652959474885734228848934027223498617428450956608 64V \\ & +97454149617329854065371600805784924038361014646639010388972785369088\end{aligned}$$

The packing density is $D \approx .777669085017$

$$\begin{aligned}0 = {} & 31723355995224561870238151303966446627070642788619469527660411904D^4 \\ & -35918467850688808832749179508959732576459417536716589731954619701D^3 \\ & +94242884158606044263433271954877412489802791402814544750249576 01D^2 \\ & -53004199250669766575682612747438494610715740527206172814094531 2D \\ & +28198049782299435288794199488796669616780137826691996362800 00\end{aligned}$$

6.4 $\mathbf{B}^9_{+\frac{1}{9},-\frac{1}{9}}$

$\mathbf{B}^9_{+\frac{1}{9},-\frac{1}{9}}\ Q_3^+\ P_z\ \mathbf{D}^0$

2 cosets

18 tetrahedra

U = 48

48.000000000000

a = ?

+1.293311994399

+0.224004640816

−0.061984648141

b = ?

−0.197284751868

+1.284397667741

+0.133460898584

c = ?

+1.330303283651

+0.311532230928

+4.449770879700

d = ?

−0.479308445712

+1.204769769330

−4.403272824247

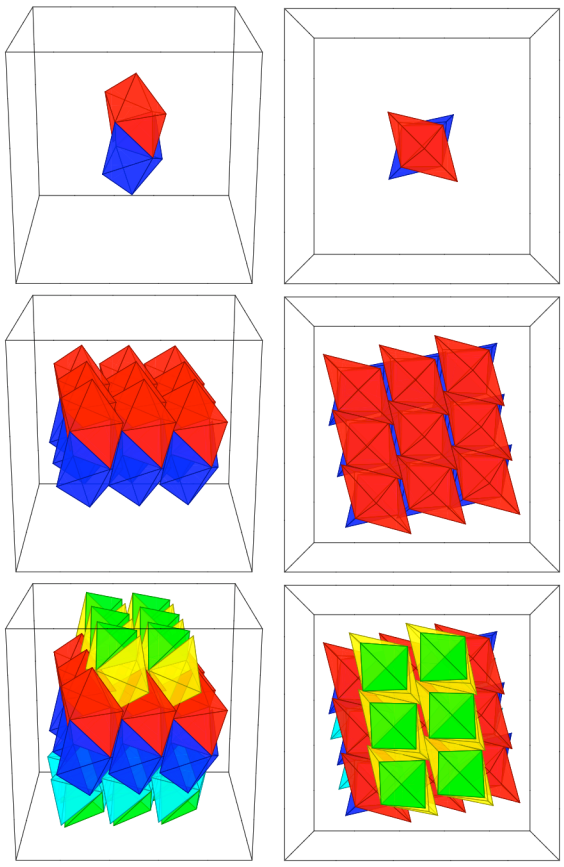
V = ?

61.754772180890

D = ?

.777267866189

basis {2a,2b,c−d}



The intersection equations are
$$\begin{cases} G_0^+\ ,\ G_a^+\ ,\ G_b^+ \\ G_{a-b}^0\ ,\ G_b^0 \\ G_{a+b}^0\ ,\ G_a^0 \\ G_0^-\ ,\ G_b^-\ ,\ G_a^- \\ \langle u,v\rangle = \langle +\frac{1}{9},-\frac{1}{9}\rangle \\ \text{minimum lattice volume } V \end{cases}$$

The free variables are $\langle x,y\rangle = \langle b_x,b_y\rangle$

The common denominator is $W = 1$

The lattice vectors are

$a = \langle -\frac{25}{7}+\frac{18}{7}x+\frac{527}{126}y,-\frac{18}{7}+\frac{18}{7}x+\frac{18}{7}y,-\frac{851}{7}+\frac{396}{7}x+\frac{13015}{126}y\rangle$
$b = \langle x,y,+36-7x-29y\rangle$
$c = \langle +\frac{401}{3}-\frac{491}{9}x-\frac{18049}{162}y,-\frac{485}{3}+\frac{545}{9}x+\frac{21937}{162}y,-\frac{2638}{21}+\frac{3658}{63}x+\frac{62476}{567}y\rangle$
$d = \langle -\frac{8641}{63}+\frac{1513}{21}x+\frac{399673}{3402}y,-\frac{2323}{63}+\frac{541}{21}x+\frac{114319}{3402}y,+\frac{2228}{63}-\frac{248}{21}x-\frac{55753}{1701}y\rangle$

The lattice volume is

$$V = -\frac{8058656}{49}+\frac{14811408}{49}x+\frac{109438984}{189}y-\frac{7037696}{49}x^2-\frac{852937376}{1323}xy-\frac{7451430920}{11907}y^2 +\frac{919984}{49}x^3+\frac{7337032}{49}x^2y+\frac{559684276}{1701}xy^2+\frac{23001192610}{107163}y^3$$

The equations of minimum lattice volume are $\langle \partial_x V,\partial_y V\rangle = \langle 0,0\rangle$

The free variables are $\langle x,y\rangle \approx \langle -0.197284751868, +1.284397667741\rangle$

$$\begin{aligned}0 &= 47106912694709299750887533340x^4 + 59876817472288347418120617 48x^3\\ &\quad -16624680374443076554744356 53x^2 - 10046394433064293140180826 90x\\ &\quad -94653878300338651947494201\\ 0 &= 13085253526308138819690981 5y^4 - 66432197034092286805879996 2y^3\\ &\quad +12110213435031782001762064 83y^2 - 94592864148071668158471217 2y\\ &\quad +26864044914207086285985 2664\end{aligned}$$

The vertices of $\mathbf{E}_u^{5+}$ are $\begin{cases} o_u^+ = \langle -\frac{95}{81}, +\frac{95}{81}, -\frac{65}{81}\rangle \\ p_u^+ = \langle -\frac{43}{27}, +\frac{43}{27}, +\frac{53}{27}\rangle \\ q_u^+ = \langle +\frac{1}{9}, -\frac{1}{9}, +\frac{31}{9}\rangle \\ r_u^+ = \langle +\frac{5}{3}, -\frac{5}{3}, +\frac{5}{3}\rangle \\ s_u^+ = \langle +1, -1, -1\rangle \end{cases}$

The vertices of $\mathbf{E}_v^{5-}$ are $\begin{cases} o_v^- = \langle -1, -1, +1\rangle \\ p_v^- = \langle -\frac{5}{3}, -\frac{5}{3}, -\frac{5}{3}\rangle \\ q_v^- = \langle -\frac{1}{9}, -\frac{1}{9}, -\frac{31}{9}\rangle \\ r_v^- = \langle +\frac{43}{27}, +\frac{43}{27}, -\frac{53}{27}\rangle \\ s_v^- = \langle +\frac{95}{81}, +\frac{95}{81}, +\frac{65}{81}\rangle \end{cases}$

The lattice vectors are $\begin{cases} a \approx \langle +1.293311994399, +0.224004640816, -0.061984648141\rangle \\ b \approx \langle -0.197284751868, +1.284397667741, +0.133460898584\rangle \\ c \approx \langle +1.330303283651, +0.311532230928, +4.449770879700\rangle \\ d \approx \langle -0.479308445712, +1.204769769330, -4.403272824247\rangle \end{cases}$

$$\begin{aligned}0 &= 52341014105232555278763926 0a_x^4 - 19261580689225504773409954 68a_x^3\\ &\quad +19558232379166857675858212 71a_x^2 + 21443352433336753273849884a_x\\ &\quad -59674535146291807497332 4021\\ 0 &= 13085253526308138819690981 5a_y^4 + 65346849488523670860734034a_y^3\\ &\quad -19264246773227217803800937 7a_y^2 - 92271804682360075878908520a_y\\ &\quad +29271769153587582498961632\\ 0 &= 52341014105232555278763926 0a_z^4 + 17682764527102252974430631916a_z^3\\ &\quad -21108522302642452244640166 7129a_z^2 + 53982904184517513138715328 2884a_z\\ &\quad +34276326475686314762336174811\end{aligned}$$

$$\begin{aligned}0 &= 47106912694709299750887533340b_x^4 + 59876817472288347418120617 48b_x^3\\ &\quad -16624680374443076554744356 53b_x^2 - 10046394433064293140180826 90b_x\\ &\quad -94653878300338651947494201\\ 0 &= 13085253526308138819690981 5b_y^4 - 66432197034092286805879996 2b_y^3\\ &\quad +12110213435031782001762064 83b_y^2 - 94592864148071668158471217 2b_y\\ &\quad +26864044914207086285985 2664\\ 0 &= 47106912694709299750887533340b_z^4 - 26701177998492285352077752868b_z^3\\ &\quad -70828780102635909076987846 1645b_z^2 + 35879979317099683768007232064 42b_z\\ &\quad -46617956031870473633296918 4957\end{aligned}$$

$$\begin{aligned}0 &= 77267113547496928916393276659 35c_x^4 - 29653963861176894550677667048762 2c_x^3\\ &\quad -39729571663981758055373154650000 67c_x^2 + 64778126516823529404546847511340 78c_x\\ &\quad -91255752231184923150555578290 6738\\ 0 &= 77267113547496928916393276659 35c_y^4 + 27939792761921942001782687981797 2c_y^3\\ &\quad -97954656595907071338677121859021 88c_y^2 - 19930624676198165915794078116568128c_y\\ &\quad +71511843880969240070905548633447 56\\ 0 &= 77267113547496928916393276659 35c_z^4 + 13038334184385151616729954058031 8c_z^3\\ &\quad -66828557733918547956122858192797 77c_z^2 + 60316176752569534589552761565488848c_z\\ &\quad -15058661565696059497378258210926 9568\end{aligned}$$

$$\begin{aligned}0 &= 25755704515832309638797758886 45d_x^4 + 11274086759680750014111056000158 8d_x^3\\ &\quad -87342919313072820080613066464684 4d_x^2 - 66276227708205239159944488184284 8d_x\\ &\quad -29637384385247411021437688592227 08\\ 0 &= 25755704515832309638797758886 45d_y^4 + 24820730486353368876234214094742d_y^3\\ &\quad -21810087794136947851539842674673 7d_y^2 - 13925409551016691943014528536053 18d_y\\ &\quad +19454283469608838888853015739544962\\ 0 &= 25755704515832309638797758886 45d_z^4 + 25579290305418148685727319516686d_z^3\\ &\quad -25618789662201617576588181233070 3d_z^2 - 20174192589430994680702388370373 64d_z\\ &\quad -27004803401028961531240232723601 52\end{aligned}$$

The lattice volume is $V \approx 61.754772180890$

$$\begin{aligned}0 = {} & 37446658148705020546180045655985081909833154637171950075V^4 \\ & -34841199948115571864260275705456082841044892915411037156350 4V^3 \\ & +35278909773378124846670600837878739632128751890142830576070828 8V^2 \\ & -81677767415652383571181778527571444345054416788475472479150870528V \\ & +37800874179954212300174427938566577583187777848905712312082223595 52\end{aligned}$$

The packing density is $D \approx .777267866189$

$$\begin{aligned}0 = {} & 23071822619600959655868181114847764638176133940982490424854872 8D^4 \\ & -23929033422554409249369661677999446585465161168498673577876231 6D^3 \\ & +49610966868812988065630532428266977607681057345513355497599603D^2 \\ & -23517809964978011008375686101182855917705302717902450080536 52D \\ & +12132717240180426656962334792539166538785942102443711824300\end{aligned}$$

6.5 $\mathbf{B}^9_{+\frac{1}{18},-\frac{1}{9}}$

$\mathbf{B}^9_{+\frac{1}{18},-\frac{1}{9}}\ Q_3^+\ P_z\ \mathbf{D}^0$

2 cosets

18 tetrahedra

U = 48

48.000000000000

a = ?

+1.292439532287

+0.217825760495

−0.044033887144

b = ?

−0.203431163904

+1.288038019831

+0.091024937931

c = ?

+1.269139981949

+0.379065460215

+4.472707183694

d = ?

−0.488706232365

+1.182775403580

−4.431708874667

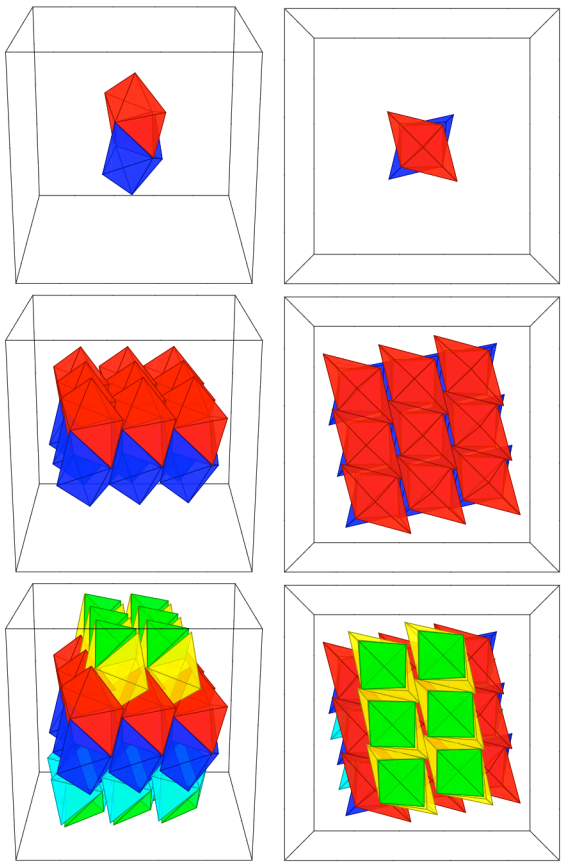
V = ?

61.759061888575

D = ?

.777213878129

basis {2a,2b,c−d}



The intersection equations are
$$\begin{cases} G_0^+\ ,\ G_a^+\ ,\ G_b^+ \\ G_{a-b}^0\ ,\ G_b^0 \\ G_{a+b}^0\ ,\ G_a^0 \\ G_0^-\ ,\ G_b^-\ ,\ G_a^- \\ v = -\frac{1}{9} \\ \text{minimum lattice volume } V \end{cases}$$

Use numerical methods to find $u = +\frac{1}{18}$

The free variables are $\langle x, y\rangle = \langle a_x, a_y\rangle$

The common denominator is $W = 1$

The lattice vectors are

$a = \langle x, y, -36 + 29x - 7y\rangle$

$b_x = (+\frac{3815}{1292} - \frac{97}{1292}\sqrt{1942}) + (+\frac{7187}{2584} - \frac{655}{10336}\sqrt{1942})x + (-\frac{24727}{5814} + \frac{10595}{93024}\sqrt{1942})y$
$b_y = (-\frac{3169}{1292} + \frac{97}{1292}\sqrt{1942}) + (-\frac{31211}{11628} + \frac{6541}{93024}\sqrt{1942})x + (+\frac{12533}{2584} - \frac{1249}{10336}\sqrt{1942})y$
$b_z = (-\frac{59303}{646} + \frac{873}{323}\sqrt{1942}) + (-\frac{3100235}{23256} + \frac{115477}{46512}\sqrt{1942})x + (+\frac{4800685}{23256} - \frac{200077}{46512}\sqrt{1942})y$

$c_x = (-\frac{133981}{1292} + \frac{27229}{7752}\sqrt{1942}) + (-\frac{518883875}{2511648} + \frac{294007}{78489}\sqrt{1942})x + (+\frac{243269485}{837216} - \frac{2592673}{418608}\sqrt{1942})y$
$c_y = (+\frac{10207}{1292} - \frac{14029}{7752}\sqrt{1942}) + (+\frac{294454727}{2511648} - \frac{1597861}{1255824}\sqrt{1942})x + (-\frac{113445169}{837216} + \frac{123707}{52326}\sqrt{1942})y$
$c_z = (-\frac{152413}{3876} + \frac{97}{646}\sqrt{1942}) + (+\frac{29489011}{1255824} + \frac{387775}{2511648}\sqrt{1942})x + (+\frac{861377}{418608} - \frac{210413}{837216}\sqrt{1942})y$

$$d_x = (-\tfrac{5263781}{34884} + \tfrac{11155}{3876}\sqrt{1942}) + (-\tfrac{2307679}{23256} + \tfrac{737357}{279072}\sqrt{1942})x + (+\tfrac{604480}{2907} - \tfrac{1277857}{279072}\sqrt{1942})y$$
$$d_y = (-\tfrac{4194005}{34884} + \tfrac{11155}{3876}\sqrt{1942}) + (-\tfrac{2850319}{23256} + \tfrac{737357}{279072}\sqrt{1942})x + (+\tfrac{619984}{2907} - \tfrac{1277857}{279072}\sqrt{1942})y$$
$$d_z = (-\tfrac{772457}{8721} + \tfrac{2425}{969}\sqrt{1942}) + (-\tfrac{719905}{5814} + \tfrac{160295}{69768}\sqrt{1942})x + (+\tfrac{550576}{2907} - \tfrac{277795}{69768}\sqrt{1942})y$$

The lattice volume is

$$\begin{aligned}V = {} & (-\tfrac{16300935552}{104329} + \tfrac{1146943519}{312987}\sqrt{1942}) + (+\tfrac{70607960479}{11267532} - \tfrac{393145765}{11267532}\sqrt{1942})x \\ & +(+\tfrac{4889433644113}{11267532} - \tfrac{107912272481}{11267532}\sqrt{1942})y + (+\tfrac{88077904802941}{202815576} - \tfrac{8337372523447}{811262304}\sqrt{1942})x^2 \\ & +(-\tfrac{3131482335823}{5633766} + \tfrac{559868830571}{45070128}\sqrt{1942})xy + (-\tfrac{18266640028523}{67605192} + \tfrac{1725088861747}{270420768}\sqrt{1942})y^2 \\ & +(+\tfrac{8263754368901221}{292054442944} - \tfrac{181387688602285}{29205442944}\sqrt{1942})x^3 + (-\tfrac{28311227543365 75}{3245049216} + \tfrac{641290156758 25}{3245049216}\sqrt{1942})x^2y \\ & +(+\tfrac{13783780800791}{21209472} - \tfrac{11691619081}{785536}\sqrt{1942})xy^2 + (+\tfrac{515871602496649}{9735147648} - \tfrac{11350527023747}{9735147648}\sqrt{1942})y^3\end{aligned}$$

The equations of minimum lattice volume are $\langle \partial_x V, \partial_y V \rangle = \langle 0, 0 \rangle$

The free variables are $\langle x, y \rangle \approx \langle +1.292439532287, +0.217825760495 \rangle$

$$\begin{aligned}0 = {} & 587780789826296525664028452220689012693789567616607789198198332240945 04x^4 \\ & +(-22723418284431642326110434542661473089722069167660905623297591720875438 4 \\ & \quad -7347256906197040865653085726348440800357122980336072011770196408621 60\sqrt{1942})x^3 \\ & +(26497677404284350850366594805085701718447226255233497575777533655473306 2 \\ & \quad +29813257978727729014835859386924534690141741086747350427609958837580 50\sqrt{1942})x^2 \\ & +(-13370777958938357649453149588285371445060441460593061855131663519296450 8 \\ & \quad -23347593877749745574967854217171401140401334390552048797647535160694 79\sqrt{1942})x \\ & +(81326817947776736001578817557251745554120816048693274070812011629558924 \\ & \quad -93372630703059221101315048992154117150082503500906511850921011350141 0\sqrt{1942}) \\ 0 = {} & 17633423694788895769920853566620670380813687028498233675945949967228351 2y^4 \\ & +(-31148721206757015125614819957174632289657915540091889136106322774812004 8 \\ & \quad +10706685263559819466920930494073170712200899839461948325626627985464048\sqrt{1942})y^3 \\ & +(44055150865180848222428295759227798690818787312680949926124333967669565 0 \\ & \quad -17788507833301435857061574861344928100156804353636100416827875904901650\sqrt{1942})y^2 \\ & +(-50460813956442992930499531632372496112648091305951943680996096050620712344 \\ & \quad +80511753309951164232968728951821600105359505477121987520169283332811 91\sqrt{1942})y \\ & +(16708545184331584782526191090811618321760439397898145481879916514413660 4 \\ & \quad -27279587908551564294023259903897714523054004094258360340566652055715 66\sqrt{1942})\end{aligned}$$

The vertices of $\mathbf{E}_u^{5+}$ are $\begin{cases} o_u^+ = \langle -\frac{7}{162} - \frac{2}{81}\sqrt{1942}, +\frac{7}{162} + \frac{2}{81}\sqrt{1942}, +\frac{85}{81} - \frac{7}{162}\sqrt{1942} \rangle \\ p_u^+ = \langle +\frac{1}{54} - \frac{1}{27}\sqrt{1942}, -\frac{1}{54} + \frac{1}{27}\sqrt{1942}, +\frac{29}{27} + \frac{1}{54}\sqrt{1942} \rangle \\ q_u^+ = \langle +\frac{1}{18}, -\frac{1}{18}, +1 + \frac{1}{18}\sqrt{1942} \rangle \\ r_u^+ = \langle +\frac{1}{54} + \frac{1}{27}\sqrt{1942}, -\frac{1}{54} - \frac{1}{27}\sqrt{1942}, +\frac{25}{27} + \frac{1}{54}\sqrt{1942} \rangle \\ s_u^+ = \langle -\frac{7}{162} + \frac{2}{81}\sqrt{1942}, +\frac{7}{162} - \frac{2}{81}\sqrt{1942}, +\frac{77}{81} - \frac{7}{162}\sqrt{1942} \rangle \end{cases}$

The vertices of $\mathbf{E}_v^{5-}$ are $\begin{cases} o_v^- = \langle -1, -1, +1 \rangle \\ p_v^- = \langle -\frac{5}{3}, -\frac{5}{3}, -\frac{5}{3} \rangle \\ q_v^- = \langle -\frac{1}{9}, -\frac{1}{9}, -\frac{31}{9} \rangle \\ r_v^- = \langle +\frac{43}{27}, +\frac{43}{27}, -\frac{53}{27} \rangle \\ s_v^- = \langle +\frac{95}{81}, +\frac{95}{81}, +\frac{65}{81} \rangle \end{cases}$

The lattice vectors are $\begin{cases} a \approx \langle +1.292439532287, +0.217825760495, -0.044033887144 \rangle \\ b \approx \langle -0.203431163904, +1.288038019831, +0.091024937931 \rangle \\ c \approx \langle +1.269139981949, +0.379065460215, +4.472707183694 \rangle \\ d \approx \langle -0.488706232365, +1.182775403580, -4.431708874667 \rangle \end{cases}$

The lattice volume is $V \approx 61.759061888575$

The packing density is $D \approx .777213878129$

6.6 $\mathbf{B}^9_{-.031023283771,+\frac{1}{9}}$

$\mathbf{B}^9_{-.031023283771,+\frac{1}{9}}\ Q_3^+\ P_z\ \mathbf{D}^{ab}$

2 cosets

18 tetrahedra

U = 48

48.000000000000

a = ?

+1.284966120611

+0.198933243472

+0.128515206569

b = ?

−0.218075978142

+1.293497355960

+0.059228407199

c = ?

+1.033832639155

+0.593991129873

+4.657761155069

d = ?

−0.324791752415

+1.320795681430

−4.457956768476

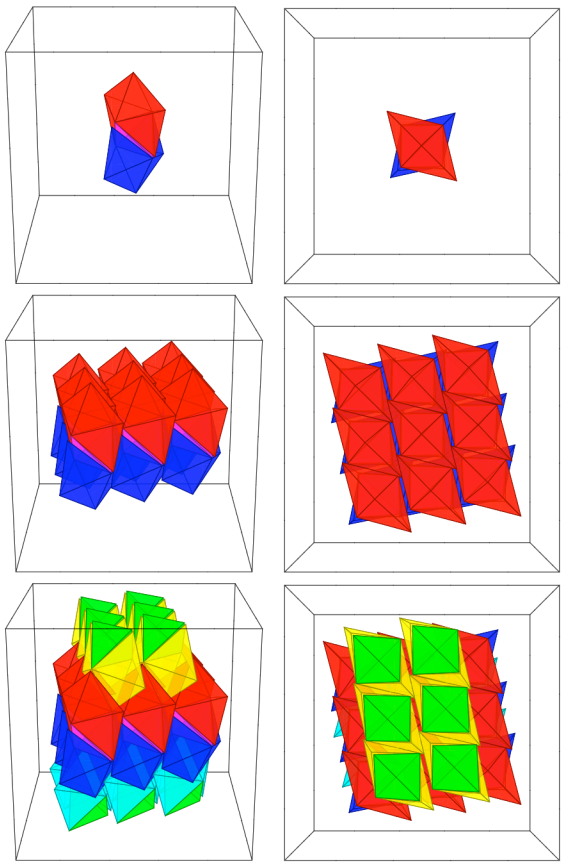
V = ?

61.650171395143

D = ?

.778586643212

basis {2a,2b,c−d}



The intersection equations are
$$\begin{cases} G_0^+\ ,\ G_a^+\ ,\ G_b^+ \\ G_{a-b}^0\ ,\ G_b^0 \\ G_{a+b}^0\ ,\ G_a^0 \\ G_0^-\ ,\ G_b^-\ ,\ G_a^- \\ H_a\ ,\ H_b \\ v = +\frac{1}{9} \\ \text{minimum lattice volume } V \end{cases}$$

The free variable is $u = u$

The common denominator is

$$W = +6560512229749622 - 5493901227933696u - 28683886381349950u^2 + 4007485428433968u^3 + 39821164086708122u^4 + 19256845338831936u^5 - 7111659893194098u^6 - 9553268568534768u^7 - 2763437531079456u^8$$

The lattice vectors are

$$
\begin{aligned}
a_x = \tfrac{1}{W}\big(&(+9152200898769526 - 7480925154914736u - 40437225649109050u^2 + 4719186799513716u^3 \\
&+56325082591955178u^4 + 28447784988778224u^5 - 9177401201531382u^6 - 13310523156410964u^7 \\
&-3913399814263632u^8) \\
&+(-298711362199194 + 307523160924714u + 1098142524353340u^2 - 206483794848000u^3 \\
&-12617222087741 58u^4 - 874722829152042u^5 - 270098132706468u^6 - 26460385780752u^7)\sqrt{6-2u^2}\big) \\
a_y = \tfrac{1}{W}\big(&(+5394498209904982 - 4210615926102072u - 22264543445263498u^2 - 26835122158604u^3 \\
&+28442370112426410u^4 + 1844096475467023 2u^5 - 1771078614113286u^6 - 6239197828270836u^7 \\
&-2008252038049488u^8) \\
&+(-1675583373890682 + 1494629445524886u + 5701650526513620u^2 - 617241398726160u^3 \\
&-5679119003437134u^4 - 3254008989103398u^5 - 683580823068924u^6 + 26460385780752u^7)\sqrt{6-2u^2}\big) \\
a_z = \tfrac{1}{W}\big(&(+8526101676005012 - 10307926195800216u - 15792170021280236u^2 + 5534599685414856u^3 \\
&-768897258222900u^4 - 2652579193927176u^5 - 2272671609370452u^6 - 1586881729229544u^7 \\
&-52920771561504u^8) \\
&+(-3066454113458148 + 1544234451857496u + 8065420479348480u^2 + 1667340259508880u^3 \\
&-3163888969609356u^4 + 2588899121685432u^5 + 3047780087005104u^6 + 952573888107072u^7)\sqrt{6-2u^2}\big)
\end{aligned}
$$

$$
\begin{aligned}
b_x = \tfrac{1}{W}\big(&(+3118132717679195 - 329860690439930u - 12778279862948813u^2 - 6615953484580888u^3 \\
&+10763829074676477u^4 + 10370196736782378u^5 + 2105092813222197u^6 - 1698937426561320u^7 \\
&-883251030893616u^8) \\
&+(-1863724231432893 + 881251050015315u + 6115442976865968u^2 + 1778837865381648u^3 \\
&-4100489934956499u^4 - 3868502738960763u^5 - 1134052682711256u^6 - 68866899781680u^7)\sqrt{6-2u^2}\big) \\
b_y = \tfrac{1}{W}\big(&(+5187870845426675 - 4274714144517350u - 24946435237730597u^2 + 5498173542197048u^3 \\
&+37464717954053 33u^4 + 14563729039380102u^5 - 9036714700420227u^6 - 9278603086781160u^7 \\
&-2577666000160944u^8) \\
&+(+1346289699496023 - 1155576928756635u - 4601073113948808u^2 + 449096910327432u^3 \\
&+4532243453669169u^4 + 2264017116694203u^5 + 378647661817656u^6 - 53103744484560u^7)\sqrt{6-2u^2}\big) \\
b_z = \tfrac{1}{W}\big(&(+6655971305504374 + 3926553120667736u - 20217896484894250u^2 - 2620733834997708 8u^3 \\
&-7432229280983670u^4 + 10488763443341400u^5 + 10172762857690506u^6 + 2619215717679792u^7 \\
&-157631552971 20u^8) \\
&+(-2521078484521350 - 3055337961556572u + 10512933064807128u^2 + 1421532839778 9768u^3 \\
&-1112205093234258u^4 - 7081713798608412u^5 - 4033774118642400u^6 - 847207484633664u^7)\sqrt{6-2u^2}\big)
\end{aligned}
$$

$$
\begin{aligned}
c_x = \tfrac{1}{3W}\big(&(+8647075587412857 + 47964321277817638u - 76686491678606597u^2 - 139162220421970038u^3 \\
&+31574351341829603u^4 + 86009237371234298u^5 + 23256309329324781u^6 - 7486016906747610u^7 \\
&-3620935274591844u^8 + 443714072144112u^9) \\
&+(+4660754407586587 - 22640340904103294u - 9301232836731693u^2 + 52913957785235228u^3 \\
&+50918536039807829u^4 - 4315153805476122u^5 - 28113216203681235u^6 - 1392604436199517 2u^7 \\
&-21130267778484488u^8)\sqrt{6-2u^2}\big) \\
c_y = \tfrac{1}{3W}\big(&(+41027447801819601 - 58282211382917750u - 139948670851280661u^2 + 74839891258252102u^3 \\
&+191526575537326563u^4 + 60169366626757078u^5 - 40849868566511043u^6 - 3964297305650635 8u^7 \\
&-113816144578455 00u^8 - 443714072144112u^9) \\
&+(-11834160245027831 + 15229447973765322u + 33070327163315053u^2 - 13762168568734284u^3 \\
&-41724812285057273u^4 - 16410828648757506u^5 + 11052528469442163u^6 + 967215765552022 8u^7 \\
&+21130267778484488u^8)\sqrt{6-2u^2}\big) \\
c_z = \tfrac{1}{3W}\big(&(+77073476998024836 - 53637434611915830u - 29236011588361530 8u^2 - 12176366094504010u^3 \\
&+317705627699996900u^4 + 21861451531405 1302u^5 - 32604260705799252u^6 - 10102280161191312 6u^7 \\
&-40178827203992136u^8 - 4226053555696896u^9) \\
&+(+6278354357574294 - 20551363848440120u - 16604593144330690u^2 + 40292424007718156u^3 \\
&+38827917034267690u^4 + 21158950444715352u^5 + 1220627572126578u^6 - 3880339591626828u^7 \\
&-443714072144112u^8)\sqrt{6-2u^2}\big)
\end{aligned}
$$

$$
\begin{aligned}
d_x = \tfrac{1}{9W}\big(&(+71020272224617559 - 28230863813865498u + 10560230071078079u^2 - 144789176370561924u^3 \\
&-54428550523108947 1u^4 - 236768207915982054u^5 + 95643648419283513u^6 + 107194624987361076u^7 \\
&+32767056185260320u^8) \\
&+(-35565372605841765 - 25179894725201379u + 109142622832666188u^2 + 141258647481842088u^3 \\
&+23577279719301237u^4 + 17583852990397779u^5 + 13769287740950940u^6 + 4417461956188512u^7)\sqrt{6-2u^2}\big) \\
d_y = \tfrac{1}{9W}\big(&(+103343681127061403 - 32274327826645194u - 293255795992060429u^2 - 120840189665994588u^3 \\
&+51110998709756093u^4 + 59416227522273930u^5 + 11015271355351 41u^6 - 2353801480002046 8u^7 \\
&-76954358089889 28u^8) \\
&+(-9605524650770841 - 39464882023806999u + 38452279661252148u^2 + 12998918332588672 8u^3 \\
&+57717708096883 45u^4 + 43602073051401 99u^5 - 76585447792425 2u^6 - 27268422046145 28u^7)\sqrt{6-2u^2}\big) \\
d_z = \tfrac{1}{9W}\big(&(-280552001408374208 + 313406229984729024u + 129180319619361 4504u^2 - 378620718085772232u^3 \\
&-20762361899480540 48u^4 - 958475486443557120u^5 + 390906937514697624u^6 + 493857648357653448u^7 \\
&+139952991791799648u^8) \\
&+(+7565006081312556 - 55306284669069156u - 6547639795293288u^2 + 140509449403902432u^3 \\
&+90349116561348900u^4 - 44623671407380 44u^5 - 24058216702538568u^6 - 8219465491561632u^7)\sqrt{6-2u^2}\big)
\end{aligned}
$$

The lattice volume is

$$
\begin{aligned}
V = \frac{1}{W^3}((&-589536383759956762660785636153449884199412428544\\
&-9722676469333735942045564712892363033712984597632u\\
&+160161519524916663834545411551929344390822871022 08u^2\\
&+107910166821105821965360951038487112052430601327616u^3\\
&-46006362987509305873386883854861679920109417135872u^4\\
&-56380060592766319991151187967166860482328953863628 8u^5\\
&-24494205438614485163675764809838802372801695187 1744u^6\\
&+15820276327705740121877122371439153424025484738314 24u^7\\
&+17732918484103293375747409508831307336968769951966 72u^8\\
&-21087042266158323004390900764457632576796741608015 36u^9\\
&-45249671578786811045065680682169962579692639097044 48u^{10}\\
&-38091661765994154745832695822715279624644345067520u^{11}\\
&+53587849456408319477811564525275383765497958487662 08u^{12}\\
&+37975931272792005475915527252263088398365930873728 00u^{13}\\
&-16215108895943314173703832575581875409524422738828 80u^{14}\\
&-35969543421671411564521058473016904670114776242749 44u^{15}\\
&-15195789257731457292745992617479170957913318604974 08u^{16}\\
&+62197535801687629049244995922483234074415124098854 4u^{17}\\
&+93316293261427998440768350245628606192105976856729 6u^{18}\\
&+35623709803907607376229099145248755475022182629478 4u^{19}\\
&-29569873395230312994256936393525405421985420625920u^{20}\\
&-83867670674593952623733532223708260333697022607360u^{21}\\
&-36640106049848355165213731414370021765753619611648u^{22}\\
&-77223851464448973263557665513814366975323743738 88u^{23}\\
&-69352288183971687161502610636714656605390241792 0u^{24})\\
+(&73493777600281768740711336387794439422166662659 20\\
&-1401628121358261017419279654561906394135293904582 4u\\
&-81144646850196700186102741862475871506465892056576u^2\\
&+11224376572193613950668485150065991332061131163 4944u^3\\
&+44318422785065611377064979930687629177847502377779 2u^4\\
&-30779232238691382539746604506682214714542777093452 8u^5\\
&-14619435890084878233398511201459711764721316586657 28u^6\\
&+44712367917197725584389990068751084132613929221632u^7\\
&+28563506614472935161814564435403892393288069905798 40u^8\\
&+15969022809588935577915727155483513314945204653090 56u^9\\
&-27186330324204458406008827772791876954283580808066 56u^{10}\\
&-34840867547509638643920973167994597956150997998382 08u^{11}\\
&+53681287213985393256468470417139203323182176991744u^{12}\\
&+25479118928697372018919204952435924520322829126622 72u^{13}\\
&+16657914451305067244269981547938752431045515787066 88u^{14}\\
&-14891690946122828796503742498011087827873524270028 8u^{15}\\
&-74395399403423929302552246457523996168858613222086 4u^{16}\\
&-38893440204176756047998417312385902685500250735200 0u^{17}\\
&-24709345372165537960100064411214189456394953185024u^{18}\\
&+66471735134013480788520839580255741889478428376064u^{19}\\
&+36594668638911155288049808204683051491305498767360u^{20}\\
&+91376765964063504600376590326893919389090195046 40u^{21}\\
&+10616188935665588290795255345491719035331716055 04u^{22}\\
&+30397212588820452597057126840318131367302922240u^{23})\sqrt{6-2u^2})
\end{aligned}
$$

The equation of minimum lattice volume is $\partial_u V = 0$

The free variable is $u \approx -.031023283771$

$$\begin{aligned}0 = {} & 330314310497097063040753686458927341971850121483523518192025 6u^{30} \\ & -5070537930373619650875855156119089369077229351453845335167795 2u^{29} \\ & +3377561310266547125116578099315600222369107613870956500718530 56u^{28} \\ & -1170652046797435342391776237579335268504385078300982329260910592u^{27} \\ & +1644033422699054524907403445204107366903922097495011671082123824u^{26} \\ & +2893916850974628452661189841281767638046887622379599171083218432u^{25} \\ & -1676401028695197450852893450917014798190582927453042317517184244 0u^{24} \\ & +2391496258576574570933059742599640169608718000121438887130371103 2u^{23} \\ & +1816138376973629460181755040606166687048233951730728315066114666 5u^{22} \\ & -1024098315920823920851011832957131768396387381690115363067749085 0u^{21} \\ & +8392641127525163819327027766314004920804696105505100752018769538 7u^{20} \\ & +1389377466465748195345025504487597646409309940160879743542931812 60u^{19} \\ & -2755833077032683444751535597345609408899784665811473693694003758 01u^{18} \\ & -4323141246054002487971240279049925281912320734919725856715446113 8u^{17} \\ & +4064365172393497200737460656989761385648398052727586641054394281 65u^{16} \\ & -1013412082619521523069168063386308968307599610439357830150472099 52u^{15} \\ & -3852843469413645433820888964091800078998995938887476143406437317 66u^{14} \\ & +1414724105534358110607221524101440392306874254873476381272373597 72u^{13} \\ & +2742891381797225842816502887254069885048872820103911390532375207 02u^{12} \\ & -7289541924746975457743863412688321454878953494598550673114760277 6u^{11} \\ & -1492661291871939947618298338598623711600122136068071591600437268 34u^{10} \\ & -5271346633596455452459220335960794858342201288229738559499546516u^{9} \\ & +5715207406352759864793646759119649086674597005205514125305242561 8u^{8} \\ & +2100339068788852633963209533818933331365572151821385721937697767 2u^{7} \\ & -1038851306826570930191020274004344148794351161675887648090516749 1u^{6} \\ & -8295658238250952129515996691421546511494228612441370590486588218u^{5} \\ & +1065702506830380746068474979470537403421664788623614903023270839u^{4} \\ & +1930424086338702957647454264359227074751182787069455120415378348u^{3} \\ & -2266755543805286858130996328513068058207647647598672733782499 09u^{2} \\ & -2873943265949387771042540210353447738328080534467310265289145 22u \\ & -8641330001388376318141614956327254949283622035064476769714271\end{aligned}$$

The vertices of $\mathbf{E}_u^{5+}$ are $\begin{cases} o_u^+ \approx \langle -1.064358243774, +1.064358243774, -0.932429315090 \rangle \\ p_u^+ \approx \langle -1.643072291318, +1.643072291318, +1.775001220003 \rangle \\ q_u^+ \approx \langle -0.031023283771, +0.031023283771, +3.449096795092 \rangle \\ r_u^+ \approx \langle +1.622390102137, -1.622390102137, +1.857729976725 \rangle \\ s_u^+ \approx \langle +1.112616685196, -1.112616685196, -0.877276810608 \rangle \end{cases}$

The vertices of $\mathbf{E}_v^{5-}$ are $\begin{cases} o_v^- = \langle -\frac{95}{81}, -\frac{95}{81}, +\frac{65}{81} \rangle \\ p_v^- = \langle -\frac{43}{27}, -\frac{43}{27}, -\frac{53}{27} \rangle \\ q_v^- = \langle +\frac{1}{9}, +\frac{1}{9}, -\frac{31}{9} \rangle \\ r_v^- = \langle +\frac{5}{3}, +\frac{5}{3}, -\frac{5}{3} \rangle \\ s_v^- = \langle +1, +1, +1 \rangle \end{cases}$

The lattice vectors are $\begin{cases} a \approx \langle +1.284966120611, +0.198933243472, +0.128515206569 \rangle \\ b \approx \langle -0.218075978142, +1.293497355960, +0.059228407199 \rangle \\ c \approx \langle +1.033832639155, +0.593991129873, +4.657761155069 \rangle \\ d \approx \langle -0.324791752415, +1.320795681430, -4.457956768476 \rangle \end{cases}$

The lattice volume is $V \approx 61.650171395143$

The packing density is $D \approx .778586643212$

6.7 $\mathbf{B}^9_{-\frac{1}{9},-.027191238704}$

$\mathbf{B}^9_{-\frac{1}{9},-.027191238704}\ Q_3^+\ P_z\ \mathbf{D}^a$

2 cosets

18 tetrahedra

U = 48

48.000000000000

a = ?

+1.291020198017

+0.208783278532

+0.041340072462

b = ?

−0.210156291688

+1.291889868484

−0.006287855769

c = ?

+1.080809750129

+0.562593540905

+4.541128219558

d = ?

−0.481923104554

+1.164812162346

−4.520656449075

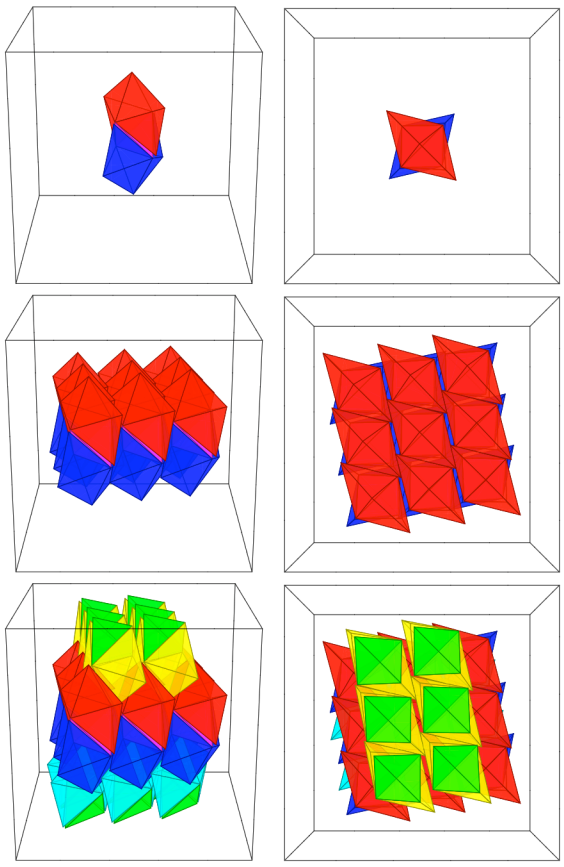
V = ?

61.704749410054

D = ?

.777897981256

basis {2a,2b,c−d}



The intersection equations are
$$\begin{cases} G_0^+\ ,\ G_a^+\ ,\ G_b^+ \\ G_{a-b}^0\ ,\ G_b^0 \\ G_{a+b}^0\ ,\ G_a^0 \\ G_0^-\ ,\ G_b^-\ ,\ G_a^- \\ H_a \\ u = -\frac{1}{9} \\ \text{minimum lattice volume } V \end{cases}$$

The free variables are $\langle v,z\rangle = \langle v,a_z\rangle$

The common denominator is

$$W = (2-11v)(3207514+2995920v-235612v^2-998640v^3-309942v^4)$$

The lattice vectors are

$$\begin{aligned} a_x &= \tfrac{(2-11v)}{W}((1764435+574370z+2061120v+266760zv+60990v^2+49210zv^2-542640v^3-194085v^4) \\ &\quad +(962211-82080z+769500v+497979zv+190959v^2+499320zv^2-6840v^3+154971zv^3)\sqrt{6-2v^2}) \\ a_y &= \tfrac{(2-11v)}{W}((-1763675+869630z-1358880v+411160zv+265050v^2-49210zv^2+597360v^3+194085v^4) \\ &\quad +(962401+278920z+945060v+667459zv+272469v^2+499320zv^2+6840v^3+154971zv^3)\sqrt{6-2v^2}) \\ a_z &= z \end{aligned}$$

$$
\begin{aligned}
b_x = \tfrac{1}{72W}\big(&(-326479282+93137108z+2725907734v-647243394zv+2449490156v^2-352816344zv^2\\
&-298783332v^3-28937248zv^3-919229634v^4+10365156zv^4-277708122v^5+3409362zv^5)\\
&+(88720657+64104067z-896314214v+81845888zv-810557022v^2-517919972zv^2\\
&-220016286v^3-510631488zv^3+2228985v^4-161944695zv^4)\sqrt{6-2v^2}\big)\\
b_y = \tfrac{1}{72W}\big(&(481770842+10669508z-3378970034v+130747326zv-3540720956v^2+56782616zv^2\\
&+172833852v^3+6308512zv^3+1156361034v^4+10365156zv^4+371645262v^5+3409362zv^5)\\
&+(49843047-65815587z+249937522v+4021888Zv+203857158v^2+169322764zv^2\\
&+55622298v^3+137485872zv^3+1081575v^4+39207663zv^4)\sqrt{6-2v^2}\big)\\
b_z = \tfrac{1}{36W}\big(&(-2470876566+480687744z-1493493480v-369515652zv+553303508v^2-411509272zv^2\\
&+689939400v^3-9806944zv^3+116689098v^4+186572808zv^4-1655280v^5+61368516zv^5)\\
&+(1033246481-729961777z+486994320v+344777984zv+118979214v^2+642460176zv^2\\
&+36466320v^3+206334936zv^3+23484285v^4+1704681zv^4)\sqrt{6-2v^2}\big)
\end{aligned}
$$

$$
\begin{aligned}
c_x = \tfrac{1}{1944W}\big(&(-85931007322+8668909196z-11964087662v+15945425802zv+37812505596v^2+37170305992zv^2\\
&+13038524196v^3-2482794576zv^3-8723604234v^4-14231359188zv^4-3335448654v^5-4681054026zv^5)\\
&+(41466587609-23098455997z-23950599074v+14763143376zv-34137223974v^2-1646838740zv^2\\
&-9276602826v^3-16296650496zv^3+890060985v^4-7471616823zv^4)\sqrt{6-2v^2}\big)\\
c_y = \tfrac{1}{1944W}\big(&(-85931007322+8668909196z-11964087662v+15945425802zv+37812505596v^2+37170305992zv^2\\
&+13038524196v^3-2482794576zv^3-8723604234v^4-14231359188zv^4-3335448654v^5-4681054026zv^5)\\
&+(41466587609-23098455997z-23950599074v+14763143376zv-34137223974v^2-1646838740zv^2\\
&-9276602826v^3-16296650496zv^3+890060985v^4-7471616823zv^4)\sqrt{6-2v^2}\big)\\
c_z = \tfrac{1}{972W}\big(&(32569972766+3671562764z-181739392694v-10936010502zv-191552197908v^2-21716493752zv^2\\
&+9442666932v^3+1008297216zv^3+63198003822v^4+9152432748zv^4+20273662602v^5+3010466646zv^5)\\
&+(-1702707619-1869327409z+20905774918v-1126732896zv+19169805474v^2+12576423604zv^2\\
&+5274934782v^3+11921322096zv^3-16543395v^4+3758821605zv^4)\sqrt{6-2v^2}\big)
\end{aligned}
$$

$$
\begin{aligned}
d_x = \tfrac{1}{72W}\big(&(-5373308806+1144472090z+1905325804v-3617505666zv+6982705754v^2-1638326874zv^2\\
&+1762495568v^3+2011306888zv^3-1582462050v^4+969204662zv^4-839240892v^5-276496902zv^5\\
&-73659618v^6-201152358zv^6)\\
&+(2147308452-1409282538z-542267625v+691274235zv-1840444880v^2+1260086332zv^2\\
&-364661676v^3-30676136zv^3+330378228v^4-382017834zv^4+124808661v^5-100576179zv^5)\sqrt{6-2v^2}\big)\\
d_y = \tfrac{1}{72W}\big(&(-4588116838+1130299610z-1673889860v-3615800226zv+2827594970v^2-1246805274zv^2\\
&+1988456240v^3+2164294888zv^3-221631234v^4+969204662zv^4-395019180v^5-276496902zv^5\\
&-73659618v^6-201152358zv^6)\\
&+(2147299332-1580571210z-643025721v+1471974411zv-1372210832v^2+2103086044zv^2\\
&+86599620v^3+214797928zv^3+456334740v^4-382017834zv^4+124808661v^5-100576179zv^5)\sqrt{6-2v^2}\big)\\
d_z = \tfrac{1}{36W}\big(&(-2952252107+311759917z-939711364v+1608657222zv+3991277853v^2-838122019zv^2\\
&+2213108072v^3-1541633056zv^3-551672685v^4+8515283zv^4-638830548v^5+382017834zv^5\\
&-124808661v^6+100576179zv^6)\\
&+(808067922-751006557z+2162991841v+299714227zv+965049136v^2+1272158408zv^2\\
&-317676072v^3+384026152zv^3-308565018v^4-138248451zv^4-36829809v^5-100576179zv^5)\sqrt{6-2v^2}\big)
\end{aligned}
$$

The lattice volume is

$$
\begin{aligned}
V = \tfrac{(2-11v)}{4374W^3}\big(&(-32176094065228062086812213+204104893520841899606871549z+507394893565920013501639328v\\
&-1346904629622319336696857 45z^2+1980928804773618519382662 24zv+432967470341668320123733025v^2\\
&+8231276536085455736802883 7z^3-3710598759744696480553560z^2v+1376124473659999943152980831zv^2\\
&-6552578271266323996508204 88v^3-9410246114835621230837989 6z^3v-9821160894131715408946508 1z^2v^2\\
&+1556064180975753932559429624zv^3-1022585966570256802622520429v^4-2777528079343741614818906 47z^3v^2\\
&+4181957960434663947706658 16z^2v^3-1195561621474528437154170963zv^4-3380102348563851785183813 04v^5\\
&-1008693916131890943828713 20z^3v^3+9613285400982616840406743 71z^2v^4-2940401271486528242908680168zv^5\\
&+1830824015458881441054942 09v^6+2503798516728490521707035 85z^3v^4+2751537408506777693612620 8z^2v^5\\
&-1299006235253414175276163257zv^6+1917507623917988955027316 32v^7+3615192949517145224582664 80z^3v^5\\
&-1027029840178968594153566757z^2v^6+7268611449126193113189725 28zv^7+8929830547839759497217404 1v^8\\
&+1426693677274516185286176 21z^3v^6-6679585916904759486031092 96z^2v^7+9112668762693610954159291 35zv^8\\
&+3647605839388128779112008v^9-9189283920341342910048148 8z^3v^7+1500216380990193384380123 01z^2v^8\\
&+2506089170712105174741301 44zv^9+4104215974547579451952251v^{10}-1148123636822597643944874 09z^3v^8\\
&+3308095207317037082306706 00z^2v^9-7793129982950891845933034 7zv^{10}-9447549875256420464206824v^{11}\\
&-3189365242115921592499173 6z^3v^9+1141529425930150153450284 69z^2v^{10}-6988816848667665191005706 4zv^{11}\\
&-6579756771959158404139575v^{12}+1088459497472942645005365 9z^3v^{10}-2021777960155768701557546 4z^2v^{11}\\
&-1817316492411206916251669 7zv^{12}-1798828633991828831651352v^{13}+9084791477074623470428392z^3v^{11}\\
&-2568801393218521690045436 7z^2v^{12}-1798659987916788518013288zv^{13}-1880462000010578194953 09v^{14}\\
&+1898877992870878343871723z^3v^{12}-7299394790983311941264304z^2v^{13}-2125649777554997243225 1zv^{14}\\
&+3896527931920409575 68z^3v^{13}-7455421392568246947671 91z^2v^{14}-3616153135860424774536 9z^3v^{14}\big)\\
&+(2785673990832224141070450 0-9106711414729929813902251 5z-3275102763002749140439909 74v\\
&+1394204774031914807799086 22z^2-5876891288318384686556485 2zv-1426787326892534512745919 16v^2\\
&-2165831481030298186924037 7z^3+4722762600804016331599577 4z^2v-5701003389931556964680819 58zv^2\\
&+1047731055398252237032730480v^3+3948090373531876517887125 6z^3v-8282321365276850591237599 2z^2v^2
\end{aligned}
$$

$$
\begin{aligned}
&-6493719010901832313008416162v^3+149853527049416656218116307 2v^4+2302339985526168658209417 2z^3v^2\\
&-3072102826922654610400638842z^2v^3+4880194248055682088980868 99zv^4+4578272629238908713916058 58v^5\\
&-7237177506608484066990932802z^3v^3-5127446559411080050949943 06z^2v^4+1132144265160251350684431252zv^5\\
&-5500219804061585194847683441v^6-5664116932742343468964416 1z^3v^4-2768784962895228137600547 90z^2v^5\\
&+4041266972234148189631873082v^6-5610643565551167430587258 72v^7+3588187348740683450289656 0z^3v^5\\
&+1539165155403774889611869282z^2v^6-3625151535638227570785400 80zv^7-1520194905628386616932219 00v^8\\
&+5544768823953770542482807 22z^3v^6+2426383155469544578992098 16z^2v^7-3598563551579528077925614 53zv^8\\
&+5312821849537526667203491 8v^9+1911055956614809833971196 8z^3v^7+7581206681479047604389244 2z^2v^8\\
&-7600902981020073647196937 2zv^9+5178583667169162863843072 4v^{10}-1588470085343356185687795 z^3v^8\\
&-2574772051178081945343223 8z^2v^9+4290295514779401575288466 6zv^{10}+1613404417086616008962419 2v^{11}\\
&-3106329644511567527031144z^3v^9-2265447699871074816070341 6z^2v^{10}+3072500630766507106168876 8zv^{11}\\
&+2309150544417330304779864v^{12}-2133402147210424395877380z^3v^{10}-4413211625305620243340236z^2v^{11}\\
&+7660091458630548334243053zv^{12}+1172520830624700829933 98v^{13}-1380036642571597161433392z^3v^{11}\\
&+2719406709870325711817 22z^2v^{12}+7351133267142478363695900zv^{13}-4918023568621060425365 31z^3v^{12}\\
&+1402326841591722929495 58z^2v^{13}-6886898551175768563996 8z^3v^{13})\sqrt{6-2v^2})
\end{aligned}
$$

The equations of minimum lattice volume are $\langle \partial_v V, \partial_z V \rangle = \langle 0,0 \rangle$

The free variables are $\langle v,z \rangle \approx \langle -0.027191238704, +0.041340072462 \rangle$

$$
\begin{aligned}
0 = {} & 4761577735588714897147214052108157976082937933825933234820049661360521987668104318849165530106818772267 68v^{50}\\
&-6640224702981663487981032709265248680573290890974011616704484573795932980786880441500140540171119349682176v^{49}\\
&+3775031438493617314500880637752458559960590930881781753320808446579320336494571521748849898477775176426893 8v^{48}\\
&-1758527213672518371037236815380260783245530963866727304870570730364304181535777775818237746776494277302276 48v^{47}\\
&+1647845287271436231385386684773097472914927257327833744414676363286092561244189270258007392683839219326583 87v^{46}\\
&-1408497760960925206136111362749825088245275193202758875663626180939102713592346351433302752898762193528983 2644v^{45}\\
&+6823428227944877995937625781503313070517963945434796441138708913778145386721842086598701504769074502793845 1804v^{44}\\
&-1449443960806201501172359404391825343894266764474598885047689844436832911411066109546581501346348348638610 97336v^{43}\\
&-2129137473872303910292496414104776920231784184351977016877738821048414444275193126738543137184757971802533 4736v^{42}\\
&+2180289808638797797647844185698944849302816983825791690929416664254588293892870857541361107646075410640744075 396v^{41}\\
&-4932410684602669290161999136951388098407121198027260758457378403761427790212750438468945203444858503789570 755222v^{40}\\
&-1545738065223145589759943183160625628410061467138340987683231299970493049471417104355530861082371625800425 212296v^{39}\\
&+3153014604098123708716987283935509010052653324557909139114510440566134353158623946014305821513736629252753 0104785v^{38}\\
&-5917324212604332952977659962789179764934689560943235454578940439102103025548002273442299469911362007312923 7277732v^{37}\\
&-1037558804394044792306453421399820933082490799575318436333279358601319981925079612945503192129385306660146345 9142v^{36}\\
&+1916123344058704113478427231168612344272642077947944983020253017578179551113578020428333180276720369809169768 24432v^{35}\\
&-1935742344027913778886152087233584775702301303519414278796138489596425651353786041212413984120976034780022497 04424v^{34}\\
&-2692181684114937274757062599010536565910789983466059555432935247951221911097199161883515990778455616608241981 71372v^{33}\\
&+6717899222281051991744441437122742716136287359208682768476379684581833103434571912936611221446642324292249194 6646370v^{32}\\
&-3932872655937898464519253449525709369277373210285408263741451482450210881872429707690038787186729868652910595 7280v^{31}\\
&-1167677581344337767578882298937617206182712741361388101399860866571227364197233236445248351605621448390441430 81250v^{30}\\
&+9709745167826088913726179510141168642398622814582874298359423034250157707913199774301637140482113626592346918 32888v^{29}\\
&+1094781985513188714743062542617124433676081920409820709360214243378689071724847454187082955645182521459804441 080912v^{28}\\
&-2293383266911683985749443611908960062113550442695983666239982415686139194610181070941749063846472686165767121 148240v^{27}\\
&-3494389237109265736071345031754451818526996465465092326681117508780745473801997757114947926860948315183616060 34032v^{26}\\
&+3973957058749557666903528741655397781960197203844624928340631809636373091120254775261769424248762549047862664 169352v^{25}\\
&-5548552329791428323094032803723256249382500377507549367840826398931896084479265347060619490393795438211343794 4580v^{24}\\
&-6142891335591544048543959940735970919685432312061898970411856107253740693685956375157442708152531029284016075 197168v^{23}\\
&+1519788231583706531801114343932256848117910608036726330098718547289721458490134250668893973363409544572047432 40834v^{22}\\
&+8524781051607017065443007797036553330870175565211502923734528887797769035338270822090753817315882388395020384 5978448v^{21}\\
&+3269229746912186881482101870008759601877709658192604191729601189045712470430566893551237324361736082516881832 099132v^{20}\\
&-9387865046742247046473049019715209450886713891600197452126031149995816911504204262615408517485896149765322770 425984v^{19}\\
&-9250174787266530772523340850902423215512169371741140841730379907649406903033933467628176646939917869824992503 504848v^{18}\\
&+6041003979660104920498840470226548269471600203871881182335396052458165629100459613562695550791350265137919132 093768v^{17}\\
&+1345739746070520246228832936157616470346727358717301453528353737612414992957388242683981841217293232494955539 4196070v^{16}\\
&+1087906867055623006900599275643919247432244015676102098155826512793013273935770524025015794660968903993202764 290976v^{15}\\
&-1099485665088432615984428083614357306633717967845603029180823736168832212361309858886729251820008544119811524 1471201v^{14}\\
&-6216895206090378325349602215965753429116248287368033585303954297340732101626727888050837293722295790745441967 754484v^{13}\\
&+3686914734132714786764201494313080667234466815147382691963077799258909167374150242957061471345459818246389954 289764v^{12}\\
&+4869171336788960234992280364595218288448785931858546330417240004929083617551846043815600261964314776859700282 786440v^{11}\\
&+9107927530784075822529434233702409055492068737700940952984986156941817289000694400339714737477785080084598256 2752v^{10}\\
&-9840070401420014673727522862609718584847947925387466648336551878204136284289176133859397717565990755545665826 48332v^{9}\\
&-6754955368962911130204563087629442028703573141758616026096962649457162111122208549191173373940943914615982029 626v^{8}\\
&-2290386763080472209349498209450845695850752026026470857038968481578603590533797003294954677296453135404029041 87848v^{7}\\
&-6941486970564896752427877471995178190985529163715893932119459899821354369667344035466784949568597033660643 1156483v^{6}\\
&-7225929024424967516387538178272610833651096415400863854274116914087022259823624445751509049189029214677411 476916v^{5}\\
&+7550082419716058912839066087294420287035731411758616026096962649457162111122208549191173373940943914615982029 626v^{4}\\
&+2516038210280769219673727996242660622867833140728804681126220907108230975926181804204642721250386893321451 758160v^{3}\\
&-5402621555207837041679894956922217037979583684352971868976691068843367351683108519944422820063100659260168 6232v^{2}\\
&-7185605427629841167220772889364602580901010858432155743187046359689977224578644337268501862829731045906213 1804v\\
&-1867536149247291217744250858847393591636847344549239964097495734810993246161530776852516621741245692467 917730
\end{aligned}
$$

The vertices of $\mathbf{E}_u^{5+}$ are $\begin{cases} o_u^+ = \langle -1, +1, -1 \rangle \\ p_u^+ = \langle -\frac{5}{3}, +\frac{5}{3}, +\frac{5}{3} \rangle \\ q_u^+ = \langle -\frac{1}{9}, +\frac{1}{9}, +\frac{31}{9} \rangle \\ r_u^+ = \langle +\frac{43}{27}, -\frac{43}{27}, +\frac{53}{27} \rangle \\ s_u^+ = \langle +\frac{95}{81}, -\frac{95}{81}, -\frac{65}{81} \rangle \end{cases}$

The vertices of $\mathbf{E}_v^{5-}$ are $\begin{cases} o_v^- \approx \langle -1.067379205592, -1.067379205592, +0.929093896870 \rangle \\ p_v^- \approx \langle -1.641855666444, -1.641855666444, -1.780140975166 \rangle \\ q_v^- \approx \langle -0.027191238704, -0.027191238704, -3.449187880314 \rangle \\ r_v^- \approx \langle +1.623728173974, +1.623728173974, -1.852650945043 \rangle \\ s_v^- \approx \langle +1.109676688020, +1.109676688020, +0.880753916951 \rangle \end{cases}$

The lattice vectors are $\begin{cases} a \approx \langle +1.291020198017, +0.208783278532, +0.041340072462 \rangle \\ b \approx \langle -0.210156291688, +1.291889868484, -0.006287855769 \rangle \\ c \approx \langle +1.080809750129, +0.562593540905, +4.541128219558 \rangle \\ d \approx \langle -0.481923104554, +1.164812162346, -4.520656449075 \rangle \end{cases}$

The lattice volume is $V \approx 61.704749410054$

The packing density is $D \approx .777897981256$

6.8 $\mathbf{B}^9_{+\frac{1}{9},+.007922924321}$

$\mathbf{B}^9_{+\frac{1}{9},+.007922924321}\ Q_3^+\ P_z\ \mathbf{D}^b$

2 cosets

18 tetrahedra

U = 48

48.000000000000

a = ?

+1.292645512658
+0.217536802418
+0.026995711564

b = ?

−0.201824499211
+1.284671215113
+0.157306256212

c = ?

+1.250986321694
+0.370175845742
+4.555185512753

d = ?

−0.362609981734
+1.311039599492
−4.379630039835

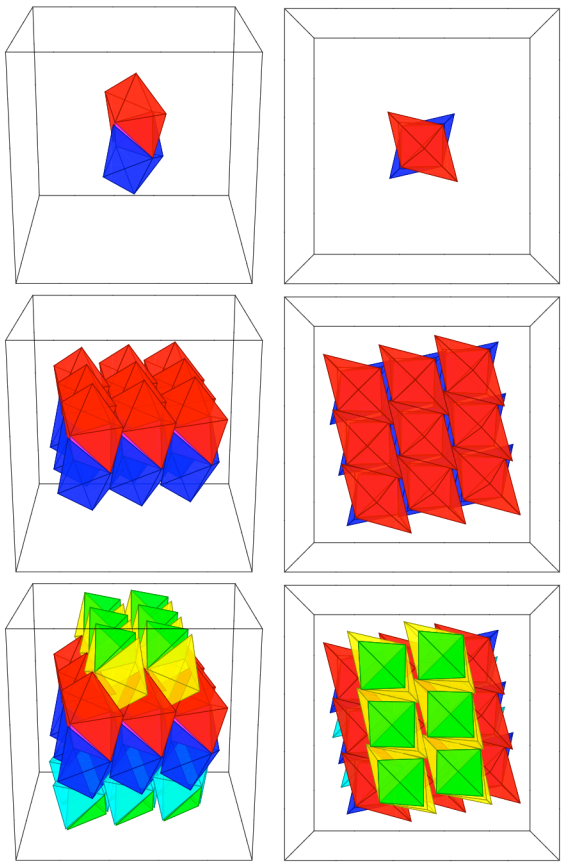
V = ?

61.701396228064

D = ?

.777940256369

basis {$2a$,$2b$,$c-d$}



The intersection equations are
$$\begin{cases} G_0^+\ ,\ G_a^+\ ,\ G_b^+ \\ G_{a-b}^0\ ,\ G_b^0 \\ G_{a+b}^0\ ,\ G_a^0 \\ G_0^-\ ,\ G_b^-\ ,\ G_a^- \\ H_b \\ u = +\frac{1}{9} \\ \text{minimum lattice volume } V \end{cases}$$

The free variables are $\langle v, z\rangle = \langle v, b_z\rangle$

The common denominator is

$$W = (3-649v^2)((-22176-25232v)+18746\sqrt{6-2v^2})$$

The lattice vectors are

$$\begin{aligned} a_x = \tfrac{1}{2W}\,&((+107328-33672z-2210904v+1180302zv+5026328v^2+179776zv^2 \\ &+32186568v^3-9973034zv^3+757944v^4-1757448zv^4) \\ &+(+31860+10803z+781972v-202432zv-19284204v^2+4478663zv^2 \\ &+585684v^3+1050588zv^3)\sqrt{6-2v^2}) \\ a_y = \tfrac{1}{2W}\,&((+274944-22224z-2058840v+1173558zv-31217896v^2-2340488zv^2 \\ &-709944v^3-8514082zv^3-2855688v^4+7691992zv^4) \\ &+(-97524-3171z+796588v-179416zv+8705868v^2+7501705zv^2 \\ &-2576244v^3-3928540zv^3)\sqrt{6-2v^2}) \end{aligned}$$

$$
\begin{aligned}
a_z = \tfrac{1}{W}\big(&(+2518560-566028z-37159344v+20487000zv-36380880v^2-5584956zv^2\\
&+8902800v^3-15439624zv^3+995280v^4+1438976zv^4)\\
&+(-1082052+565449z+14126652v-3563220zv+11031084v^2+357349zv^2\\
&-524436v^3+1483636zv^3)\sqrt{6-2v^2}\big)
\end{aligned}
$$

$$
\begin{aligned}
b_x &= W^{-1}(3-649v^2)\big((+36288-1008z+15600v+7822vz)+(-15996-4273z)\sqrt{6-2v^2}\big)\\
b_y &= W^{-1}(3-649v^2)\big((-36288+1008z-35088v-1018vz)+(+27132+385z)\sqrt{6-2v^2}\big)\\
b_z &= z
\end{aligned}
$$

$$
\begin{aligned}
c_x = \tfrac{1}{18W}\big(&(-40700352+8786568z+620651544v-342625410zv+302176840v^2+240103136zv^2\\
&+10187512v^3+266241798zv^3-25777752v^4-176792zv^4)\\
&+(+18213492-9341253z-236192068v+59643136zv-210246412v^2-30704913zv^2\\
&+777084v^3-37395820zv^3)\sqrt{6-2v^2}\big)\\
c_y = \tfrac{1}{18W}\big(&(+38958528-7806792z-622920408v+343840410zv+74637752v^2-452061344zv^2\\
&+480643400v^3-529086798zv^3+25777752v^4+176792zv^4)\\
&+(-16577076+8455761z+236192068v-59643136zv-143764916v^2+222266349zv^2\\
&-777084v^3+37395820zv^3)\sqrt{6-2v^2}\big)\\
c_z = \tfrac{1}{9W}\big(&(+19602624-5370816z-358713888v+195352020zv+617965280v^2-61502432zv^2\\
&+871596896v^3-58264956zv^3+14730144v^4+101024zv^4)\\
&+(-7376844+5654106z+134966896v-34081792zv-535536476v^2-50869134zv^2\\
&-444048v^3+21369040zv^3)\sqrt{6-2v^2}\big)
\end{aligned}
$$

$$
\begin{aligned}
d_x = \tfrac{1}{2w}\big(&(+3186672-62436z-45836304v+33707844zv-118556744v^2+2821546zv^2\\
&-32828056v^3+2363820zv^3+2618792v^4-1510586zv^4+5872152v^5-10939544zv^5)\\
&+(-1488812+1048661z+15800700v-315318zv+45709588v^2-24274867zv^2\\
&+12099868v^3+3960242zv^3+3613632v^4+8213744zv^4)\sqrt{6-2v^2}\big)\\
d_y = \tfrac{1}{2W}\big(&(+7281600-2328048z-102722880v+47981604zv-112239896v^2+2201550zv^2\\
&+51469432v^3-8087764zv^3+8039240v^4+1978438zv^4+5872152v^5-10939544zv^5)\\
&+(-2974756+1430671z+40590612v-13983534zv+27664220v^2-21431961zv^2\\
&+10744756v^3+11919578zv^3+3613632v^4+8213744zv^4)\sqrt{6-2v^2}\big)\\
d_z = \tfrac{1}{W}\big(&(-153648-24606z+2387352v-1373058zv+5043528v^2+12472524zv^2\\
&-70341448v^3+51710550zv^3-11422312v^4-7939910zv^4-3613632v^5-8213744zv^5)\\
&+(-71580+53430z-796996v+197035zv+27863392v^2-18349118zv^2\\
&+2664508v^3+116963zv^3+2936076v^4-5469772zv^4)\sqrt{6-2v^2}\big)
\end{aligned}
$$

The lattice volume is

$$
\begin{aligned}
V = \tfrac{(3-649v^2)}{W^3}\big(&(-11065973625586621440+8484758971399941120z+296782393907677280256v\\
&-1401535979882485056z^2-233863879161924297216zv-156418443809545431859 2v^2\\
&-45808437377128896z^3+45732726909021945600z^2v+132279837225186012825 6zv^2\\
&-563210374058354074828 8v^3+108549767911181961 6z^3v-27866258899256430316 8z^2v^2\\
&+354083008459069106073 6zv^3-437125611149376344064 0v^4+119078851970087136 0z^3v^2\\
&-302888351254469879040z^2v^3+208322933061613977344 0zv^4-172285513678140182528 0v^5\\
&-118905320931539851104z^3v^3+334887095611104440000z^2v^4-224221579800885852160zv^5\\
&+764409109356353947648v^6+424100355564393326 72z^3v^4-299337240478531427 84z^2v^5\\
&-982415898676080767488zv^6+708174027592697081856v^7+182561616767991523616z^3v^5\\
&-418617929587841653696z^2v^6-171724955808705764352zv^7+423940159161965107 20v^8\\
&+101270700839060554048z^3v^6-389187200089230773 76z^2v^7+168650422815104648704zv^8\\
&-155201091394172313 6v^9-477168220987218729 28z^3v^7+341323453843741402 88z^2v^8\\
&-124832682818922496zv^9-634498028316776448v^{10}-387595442762729358 08z^3v^8\\
&+160280953723964057 60z^2v^9-276898755203288064 0zv^{10}+300411177219873843 2z^3v^9\\
&+844307599866316185 6z^2v^{10}+342506304557299097 6z^3v^{10})\\
&+(+452284055277746688 0-3490074733414129920z-121094028992300156928v\\
&+651520081489902336z^2+946518282483495313 92zv+638919811829788154880v^2\\
&+22995370805322816z^3-165112979030044561 92z^2v-547782152132766286080zv^2\\
&+228715006029147435417 6v^3-400991582465785920z^3v+125930837445611777856z^2v^2\\
&-143929271513461755622 4zv^3+203476493212627999846 4v^4-183129009070134480 0z^3v^2\\
&+844221954854990947 84z^2v^3-912959333797552725248zv^4+608014877460097366016v^5\\
&+359736650748628184 32z^3v^3-103081125552122008192z^2v^4-132284346859144808 96zv^5\\
&+353142139981625668608v^6+218543005890693478 4z^3v^4+459294254076960985 60z^2v^5\\
&+291004429702036001 28zv^6-447851487012399022 08v^7-930617890942071537 28z^3v^5\\
&+118152804422986062144z^2v^6+470974316769718192 64zv^7+103836554791022592v^8\\
&-408448531350928754 24z^3v^6+680739818079880092 16z^2v^7-612354962385717196 8zv^8\\
&+336278274631630848v^9+196030622033304396 8z^3v^7-241079202100328345 6z^2v^8\\
&-283795515483914649 6zv^9+695817877200328960 0z^3v^8-378191661462378086 4z^2v^9\\
&+417739843729473126 4z^3v^9)\sqrt{6-2v^2}\big)
\end{aligned}
$$

The equations of minimum lattice volume are $\langle \partial_v V, \partial_z V \rangle = \langle 0, 0 \rangle$

The free variables are $\langle v, z \rangle \approx \langle +0.007922924321, +0.157306256212 \rangle$

$$
\begin{aligned}
0 = {} & 54731791932600878088632686202284150969193547194417972205169725438918585750538393467052972195184 64 v^{50} \\
& + 87748257017427187032496278769429812486209917408506234946130617446578674824719769167139901747494912 v^{49} \\
& - 18586553777823714153657344387990296910026514276538899706590176630061453712826112441193525680062464 0 v^{48} \\
& - 53106686634930953819788391271082411959007149084866189269708083733131351187120338269473251885553582 08 v^{47} \\
& + 15833383877377495488142992635847124146577705532052630270122481157878352434140428889268897488963423232 v^{46} \\
& + 11408834358789313969056632928219522361327178824871225760371375234082901332346115801043471103404751411 2 v^{45} \\
& - 47756339842407566724426769444104556233753701317251321169182026198060960923356703913267587503801949689 6 v^{44} \\
& - 10525015273982746607693046389921932901969477837217499359606834804099148670640865934690271017870540645 76 v^{43} \\
& + 71054539768434343431516858067993117490294542728294394111841333428334008117785536247817791043538844630 00 v^{42} \\
& + 24725762551736725702488228544044092416022156222946454774943302165124672432222240428598883579503103033 12 v^{41} \\
& - 62055175767540502316203891906990506791428890368134649905256426171950930752340995639578720069344096233 000 v^{40} \\
& + 39343996912295318234596400730331229070940185592791127318585135517809228021639205372681654122764263322 816 v^{39} \\
& + 35042134511677463185375087666629015166019652249365091001190219792028097164152669509285385971166045929 9872 v^{38} \\
& - 48785565329757743391357366245976337990251145446249371975766979155863141966017572647308949905908957 3877992 v^{37} \\
& - 13078063113972437803208162638123769433783005381747671295247958394922163401316782113209356080502186585 67605 v^{36} \\
& + 29914168040085543367190859348128724493966331669634564695535411584605011517055975700587170797879730 60011132 v^{35} \\
& + 29106226684958110956119590780334527729457746225392713649575739290866755400339673373271937471644341425290 83 v^{34} \\
& - 12119461586703837941472973570957862049939487832652026876629645001579291804703730223446962939273877325754048 v^{33} \\
& - 11497290554149802389153183279962081267195890134645828460008324092639914641193284050793836737467822005725 01 v^{32} \\
& + 34718304936885637197197938202464467166271389276580470488873380373614084353525687656272820635223964638076260 v^{31} \\
& - 18627410383768150278599516021830668504413941854189574206982615109620898726776502754409105348537287833552513 v^{30} \\
& - 70209905985331709390255024715589294253214351890594888513980610514600890183374932401934364014795830124773384 v^{29} \\
& + 83266652662033948588231823681110522385507901203272164858312914434221061071255794068742852852028612752066995 v^{28} \\
& + 88904985553395669047446737754105921772350384833822839673417558360049900773459949829760773979195178158250612 v^{27} \\
& - 20878411963531344122804179555998390330305261263648645282834167993740615772133749124440133031764068829541354 5 v^{26} \\
& - 18308794763627390406055652699231439915871957772783547785711370951840968726568828131522987921258092221314192 0 v^{25} \\
& + 34189974951976867231068482041329463973234386848227665369535348492102453821149118550826785316802830628522488 3 v^{24} \\
& - 20402755545775276898736449246276842200940626860369084764202865874488058830301337676272917899450247830527510 0 v^{23} \\
& - 33370962380900347690164657892684270832254634561677720879355861061390187058028846332027434691424947816152664 1 v^{22} \\
& + 49915753134530438703342159537542076785937252293959744125108283846043299599913672866929772243773077829198861 6 v^{21} \\
& + 83502854904364480961689697502788367856901322016655383095895654077086415562445845324719988381806206062040409 v^{20} \\
& - 63925627692997854790731278418008455245001713346743539815915040288630261133447637395776998093008558693243355 6 v^{19} \\
& + 25955808658426638265485107942235958761565106851251900813147749536834116515075108629533177670748268544298196 5 v^{18} \\
& + 51065786434782215382486472650016647644600924045442198897344656419589266000642914384927376164471734035330016 v^{17} \\
& - 42350019688862452339271001289228182720563902800340655358823746552845625001685347491242578933771843605454283 1 v^{16} \\
& - 26654138586675675204243346860349683396136953318982248075224143132477870561866568913345502720148159129545573 2 v^{15} \\
& + 34858240237617222095184700594773089799161342197643931334101652990805715694252993111246217473105246328881014 9 v^{14} \\
& + 93622828986878567615562018150623033628487313997696812048380151247898614831127761197395882725871185300334184 v^{13} \\
& - 18810021513878264926646679918648158498458576716657604843186129836031116686041972109318669800198699796916645 5 v^{12} \\
& - 21526062194007819770779924879719277627464595016545020395321213039122321579703291951049916856173785244409828 v^{11} \\
& + 69712009324846094101032891403525517281283770665750990794903223656407216644211696783645539518643448225597181 v^{10} \\
& + 23372403009575921556641262859133332645888308601707495229573498820770597492692128572375130497431432166094889 v^{9} \\
& - 16652241919349102265808386915628969790674626546904667925149337386055855767536152940125898402403232693091487 v^{8} \\
& + 75345721354730017548876455361181928296093125872677238130245347764572526668501503390228290660930292189868 v^{7} \\
& + 22027855967096724336049390555983265098930297143424514347423304167692681079173270493591374712374143879812 29 v^{6} \\
& - 31925386711904164801558564023997465364783043058878314776457629451585003484454202922190996416076133175296 v^{5} \\
& - 10866485279039380886295245249822739666675293800733338178273877782864771244184314477418059281920656982876 8 v^{4} \\
& + 67404235542870020784079841888020078205347560045843076125788882241214111203855171693130989161923449429 20 v^{3} \\
& + 40341322773481321764996057880094927334927990732208729960665538991303683287764811470775937524448335760 8 v^{2} \\
& - 24024258088496477939019664704344941225118226388688516657722852792272171615229320937849010475876216480 v \\
& + 16209535936578904432310419550550393288975301183379290898362934495047536582318415287346422282775548 0
\end{aligned}
$$

The vertices of $\mathbf{E}_u^{5+}$ are $\begin{cases} o_u^+ = \langle -\frac{95}{81}, +\frac{95}{81}, -\frac{65}{81} \rangle \\ p_u^+ = \langle -\frac{43}{27}, +\frac{43}{27}, +\frac{53}{27} \rangle \\ q_u^+ = \langle +\frac{1}{9}, -\frac{1}{9}, +\frac{31}{9} \rangle \\ r_u^+ = \langle +\frac{5}{3}, -\frac{5}{3}, +\frac{5}{3} \rangle \\ s_u^+ = \langle +1, -1, -1 \rangle \end{cases}$

The vertices of $\mathbf{E}_v^{5-}$ are $\begin{cases} o_v^- \approx \langle -1.094812992601, -1.094812992601, +0.898096157328 \rangle \\ p_v^- \approx \langle -1.630335102419, -1.630335102419, -1.827051937692 \rangle \\ q_v^- \approx \langle +0.007922924321, +0.007922924321, -3.449464115789 \rangle \\ r_v^- \approx \langle +1.635617051967, +1.635617051967, -1.805924139501 \rangle \\ s_v^- \approx \langle +1.082488443657, +1.082488443657, +0.912181356122 \rangle \end{cases}$

The lattice vectors are $\begin{cases} a \approx \langle +1.292645512658, +0.217536802418, +0.026995711564 \rangle \\ b \approx \langle -0.201824499211, +1.284671215113, +0.157306256212 \rangle \\ c \approx \langle +1.250986321694, +0.370175845742, +4.555185512753 \rangle \\ d \approx \langle -0.362609981734, +1.311039599492, -4.379630039835 \rangle \end{cases}$

The lattice volume is $V \approx 61.701396228064$

The packing density is $D \approx .777940256369$

6.9 $\mathbf{B}^9_{-.034789016702,+.089604971413}$

$\mathbf{B}^9_{-.034789016702,+.089604971413}\ Q_3^+\ P_z\ \mathbf{D}^c$

2 cosets

18 tetrahedra

$U = 48$

48.000000000000

$a = ?$

+1.286101228632
+0.200477756509
+0.117304750804

$b = ?$

−0.216654244567
+1.293299854677
+0.064005626691

$c = ?$

+1.049828831839
+0.572626599359
+4.643104387948

$d = ?$

−0.331247602500
+1.312711170924
−4.452573714586

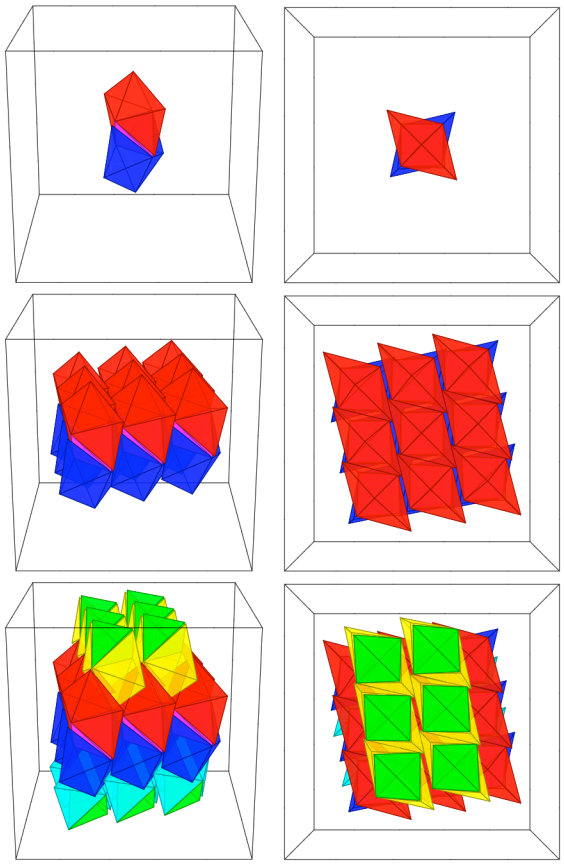
$V = ?$

61.647870634123

$D = ?$

.778615700855

basis {2*a*,2*b*,*c*−*d*}



The intersection equations are
$$\begin{cases} G_0^+\ ,\ G_a^+\ ,\ G_b^+ \\ G_{a-b}^0\ ,\ G_b^0 \\ G_{a+b}^0\ ,\ G_a^0 \\ G_0^-\ ,\ G_b^-\ ,\ G_a^- \\ H_a\ ,\ H_b \\ \text{minimum lattice volume } V \end{cases}$$

The free variables are $\langle u,v\rangle = \langle u,v\rangle$

The common denominator is

$$\begin{aligned} W = {} & (-56064 - 95904u + 52896v - 78720u^2 + 284460uv + 118400v^2 - 86832u^3 - 528u^2v + 62352uv^2 \\ & -78032v^3 - 30240u^4 - 50580u^3v + 115372u^2v^2 - 13620uv^3 - 48160v^4 + 5184u^4v - 12888u^3v^2 - 8728u^2v^3 \\ & -2496uv^4 + 8604u^4v^2 - 6900u^3v^3 + 6364u^2v^4 + 648u^4v^3 - 312u^3v^4 + 1548u^4v^4) \\ & +(-63936u + 11520v - 109512u^2 - 3168uv - 44160v^2 - 30240u^3 + 26172u^2v + 99648uv^2 - 17592v^3 \\ & -2592u^3v - 8604u^2v^2 + 1692uv^3 + 6240v^4 + 8604u^3v^2 + 3156u^2v^3 + 12384uv^4 + 780u^2v^4 - 324u^3v^3 \\ & +1548u^3v^4)\sqrt{6-2u^2} \\ & +(+65280u - 35264v + 71040u^2 + 3168uv + 37832v^2 + 2232u^3 - 24448u^2v - 39012uv^2 + 14560v^3 \\ & -12960u^4 + 828u^3v + 2164u^2v^2 - 4128uv^3 - 3744u^4v - 5916u^3v^2 - 1924u^2v^3 - 1620u^4v^2 - 516u^3v^3 \\ & -468u^4v^3)\sqrt{6-2v^2} \\ & +(+18432 + 10944u - 8256v + 7704u^2 - 55152uv - 10776v^2 + 6480u^3 - 2070u^2v - 4230uv^2 \\ & +10320v^3 - 3744u^3v - 276u^2v^2 - 3744uv^3 + 810u^3v^2 + 1290u^2v^3 - 468u^3v^3)\sqrt{6-2u^2}\sqrt{6-2v^2} \end{aligned}$$

The lattice vectors are

$$
\begin{aligned}
a_x = \tfrac{1}{W}\big(&(-147456+110592u+48768v+65808u^2+310080uv+310800v^2-87840u^3+11838u^2v-176616uv^2\\
&-85472v^3-43200u^4-64944u^3v+151590u^2v^2-163384uv^3-77792v^4-5886u^4v+31992u^3v^2+22206u^2v^3\\
&-35664uv^4+7254u^4v^2+6360u^3v^3+14706u^2v^4+7218u^4v^3+10224u^3v^4+2322u^4v^4)\\
&+(+61440-95424u+6912v-86688u^2-81192uv-114976v^2-16956u^3+31032u^2v+160978uv^2\\
&-3872v^3-24516u^3v+25608u^2v^2+14332uv^3+151590u^2v^2+18888v^4+19404u^3v^2+13584u^2v^3+13176uv^4\\
&-1584u^3v^3+936u^2v^4+702u^3v^4)\sqrt{6-2u^2}\\
&+(+19968-3264u-67968v+37968u^2+64008uv+2648v^2-7776u^3-40098u^2v+27220uv^2\\
&+28808v^3-17820u^4+1560u^3v-46848u^2v^2-4476uv^3+2196u^4v-2940u^3v^2-8820u^2v^3-3708u^4v^2\\
&-1260u^3v^3-702u^4v^3)\sqrt{6-2v^2}\\
&+(+3072+15936u+5632v-17712u^2-59680uv+176v^2-1296u^3+8448u^2v-7345uv^2\\
&+3648v^3+6327u^3v-5124u^2v^2+4968uv^3-2115u^3v^2+1548u^2v^3+1161u^3v^3)\sqrt{6-2u^2}\sqrt{6-2v^2})\\
a_y = \tfrac{1}{W}\big(&(-49152+221184u-24960v+237936u^2+20160uv+186128v^2+36576u^3+60210u^2v-283320uv^2\\
&+23328v^3-21600u^4+144144u^3v-9242u^2v^2-140056uv^3-39392v^4+16254u^4v+75624u^3v^2-58622u^2v^3\\
&-35664uv^4+4518u^4v^2-1416u^3v^3+2226u^2v^4-5922u^4v^3+10224u^3v^4+162u^4v^4)\\
&+(+61440+18624u-9984v+37728u^2-89208uv-84448v^2+22140u^3+24984u^2v+28786uv^2\\
&+13408v^3-11772u^3v-26664u^2v^2+13324uv^3+18888v^4+1116u^3v^2+5808u^2v^3+21816uv^4-2952u^3v^3\\
&+936u^2v^4+1782u^3v^4)\sqrt{6-2u^2}\\
&+(-29184-50880u-17792v-28176u^2+122760uv+17496v^2-25056u^3-43630u^2v+21388uv^2\\
&-15352v^3-8100u^4-20040u^3v+9800u^2v^2-4476uv^3+6444u^4v-996u^3v^2+5532u^2v^3+468u^4v^2\\
&-1260u^3v^3+1782u^4v^3)\sqrt{6-2v^2}\\
&+(-3072-25152u+14848v-45360u^2-29872uv-4144v^2-12960u^3+16656u^2v+32111uv^2\\
&+3648v^3+2313u^3v-3180u^2v^2-4968uv^3+333u^3v^2+1548u^2v^3-81u^3v^3)\sqrt{6-2u^2}\sqrt{6-2v^2})\\
a_z = \tfrac{1}{W}\big(&(-321408+2304u-154608v-345816u^2+163968uv+220656v^2-271296u^3-260760u^2v-205800uv^2\\
&+95696v^3-39096u^4-14688u^3v+270432u^2v^2-38624uv^3-13456v^4-12744u^4v+54072u^3v^2+36552u^2v^3\\
&+8952uv^4+18288u^4v^2+5184u^3v^3+3288u^2v^4+1368u^4v^3+2520u^3v^4-1080u^4v^4)\\
&+(+49152-239952u+103680v-17280u^2-50694uv+37216v^2+10800u^3+177120u^2v+84800uv^2\\
&-20992v^3+11070u^3v-12888u^2v^2-91582uv^3-7296v^4-1368u^3v^2+5184u^2v^3-6570u^3v^3-3096u^2v^4\\
&-1080u^3v^4)\sqrt{6-2u^2}\\
&+(+52608+34944u+67440v+140880u^2-62256uv-992v^2+79488u^3-50778u^2v-135208uv^2\\
&-58592v^3+11664u^4+73584u^3v-39216u^2v^2-35664uv^3+4014u^4v+4536u^3v^2+8466u^2v^3-2448u^4v^2\\
&+10224u^3v^3+1242u^4v^3)\sqrt{6-2v^2}\\
&+(+13056+75792u-47968v+19872u^2+32890uv-36512v^2+4860u^3-3312u^2v+48676uv^2\\
&+18888v^3+2124u^3v+4536u^2v^2+17496uv^3+2088u^3v^2+936u^2v^3+1242u^3v^3)\sqrt{6-2u^2}\sqrt{6-2v^2})\\
b_x = \tfrac{1}{W}\big(&(+49152+123264u-18432v+226320u^2-86592uv+121168v^2+80928u^3+10584u^2v-144654uv^2\\
&+30688v^3-12960u^4+125640u^3v-143014u^2v^2-12496uv^3-26080v^4+21168u^4v+81426u^3v^2-107784u^2v^3\\
&-31778uv^4-10962u^4v^2+14712u^3v^3+11946u^2v^4-10512u^4v^3+5070u^3v^4+1710u^4v^4)\\
&+(+29184+97536u-23232v+68184u^2-74952uv-45808v^2+30456u^3+27252u^2v-57086uv^2\\
&+32928v^3+22788u^3v+3360u^2v^2-51672uv^3+19620v^4-12204u^3v^2+3876u^2v^3-4548uv^4-5868u^3v^3\\
&+2820u^2v^4-1314u^3v^4)\sqrt{6-2u^2}\\
&+(-61440-39168u+16064v-42336u^2+112536uv+42080v^2-22752u^3-28610u^2v-3320uv^2\\
&-19580v^3+1080u^4-24492u^3v+27432u^2v^2+11540uv^3+23544u^4v-7296u^3v^2-10020u^2v^3+6840u^4v^2\\
&+1824u^3v^3+1314u^4v^3)\sqrt{6-2v^2}\\
&+(+3072-43776u+8896v-43056u^2-9952uv-22704v^2-13824u^3+18609u^2v+43968uv^2\\
&-2880v^3-648u^3v-4020u^2v^2+6105uv^3+5436u^3v^2+1131u^2v^3+855u^3v^3)\sqrt{6-2u^2}\sqrt{6-2v^2})\\
b_y = \tfrac{1}{W}\big(&(-147456-123264u+43008v-167184u^2+219072uv+104400v^2-193824u^3+66744u^2v+68994uv^2\\
&-74720v^3-56160u^4-162792u^3v+209574u^2v^2-114544uv^3-32512v^4-21168u^4v+20658u^3v^2+90072u^2v^3\\
&-13730uv^4+17298u^4v^2-2328u^3v^3-1242u^2v^4+10512u^4v^3+7326u^3v^4+162u^4v^4)\\
&+(+19968-137472u+30912v-174168u^2+51912uv-38320v^2-47736u^3-17964u^2v+174610uv^2\\
&-41760v^3-22788u^3v+37416u^2v^2+57336uv^3-5532v^4+13788u^3v^2-6972u^2v^3+20220uv^4+5868u^3v^3\\
&-324u^2v^4+1782u^3v^4)\sqrt{6-2u^2}\\
&+(+61440+56064u-65856v+75168u^2-24648uv-28768v^2+40032u^3-32670u^2v-2216uv^2\\
&+26308v^3-1080u^4+13020u^3v-82296u^2v^2-23212uv^3+1224u^4v+3552u^3v^2-540u^2v^3-6840u^4v^2\\
&-2520u^3v^3+1782u^4v^3)\sqrt{6-2v^2}\\
&+(+3072+48384u-4928v+38736u^2-54160uv+18864v^2+13824u^3+2607u^2v-39984uv^2\\
&+12048v^3+6840u^3v+4956u^2v^2-1383uv^3-5436u^3v^2+2997u^2v^3-81u^3v^3)\sqrt{6-2u^2}\sqrt{6-2v^2})\\
b_z = \tfrac{1}{W}\big(&(-321408-284688u-492288v+121968u^2-33024uv+294360v^2+225072u^3-135576u^2v-233688uv^2\\
&+127424v^3+78192u^4+24480u^3v+102000u^2v^2+41632uv^3-6728v^4+45576u^4v-55800u^3v^2-21432u^2v^3\\
&+11672uv^4-25992u^4v^2+8256u^3v^3+10560u^2v^4-11736u^4v^3+696u^3v^4-3096u^4v^4)\\
&+(+52608+29328u+119424v-50976u^2-58032uv-56720v^2-21600u^3-133848u^2v+292262uv^2\\
&-46464v^3-21168u^3v-61632u^2v^2+80112uv^3-9168v^4+14130u^3v^2-7224u^2v^3-4722uv^4+10512u^3v^3\\
&-4128u^2v^4-774u^3v^4)\sqrt{6-2u^2}\\
&+(+49152+26880u+30480v+16416u^2-58950uv-106880v^2-29952u^3+40416u^2v-133024uv^2\\
&-29296v^3-27648u^4+80514u^3v+67512u^2v^2-22754uv^3-7488u^4v-2496u^3v^2+5352u^2v^3+10872u^4v^2\\
&+6198u^3v^3+936u^4v^3)\sqrt{6-2v^2}\\
&+(+13056+19296u+23792v+5472u^2+48682uv+25440v^2-1080u^3+46116u^2v-51888uv^2\\
&+7044v^3-11160u^3v+2184u^2v^2+7836uv^3-6840u^3v^2+1248u^2v^3+234u^3v^3)\sqrt{6-2u^2}\sqrt{6-2v^2})
\end{aligned}
$$

$$
\begin{aligned}
c_x = \tfrac{1}{3W}\big(&(+491904-998688u-1331568v+1912200u^2-847224uv-84648v^2+1697712u^3+399570u^2v-625528uv^2\\
&+216720v^3-287352u^4+43824u^3v+592808u^2v^2+212144uv^3+87312v^4-272592u^5-57834u^4v-450742u^3v^2\\
&-40414u^2v^3+154828uv^4+72576u^5v+142608u^4v^2-143320u^3v^3-91520u^2v^4-35082u^5v^2-9738u^4v^3-28406u^3v^4\\
&+9072u^5v^3+15504u^4v^4-126u^5v^4)\\
+&(+62016+1337640u+263232v+163608u^2-165918uv-209356v^2-194580u^3-622248u^2v+465998uv^2\\
&-120784v^3-18576u^4+28674u^3v-295840u^2v^2+160490uv^3-32888v^4+64800u^4v+420852u^3v^2-115108u^2v^3\\
&-79820uv^4-44370u^4v^2+21186u^3v^3-25202u^2v^4+8100u^4v^3+34566u^3v^4-5256u^4v^4)\sqrt{6-2u^2}\\
+&(-215424-147360u-81456v-715680u^2-84824uv-228816v^2-480984u^3+315076u^2v+334666uv^2\\
&+79800v^3+145260u^4+150718u^3v+104140u^2v^2-13444uv^3+133488u^5-90822u^4v-295424u^3v^2-15766u^2v^3\\
&+58896u^5v+54720u^4v^2-41764u^3v^3+16686u^5v^2+1416u^4v^3+7362u^5v^3)\sqrt{6-2v^2}\\
+&(+10752-329616u+138472v-176208u^2+128227uv+170200v^2-13716u^3+86464u^2v-155768uv^2\\
&-7960v^3-4536u^4-42771u^3v-47057u^2v^2-9538uv^3+27360u^4v-7524u^3v^2+37220u^2v^3-567u^4v^2\\
&+1146u^3v^3+3420u^4v^3)\sqrt{6-2u^2}\sqrt{6-2v^2}\big)\\
c_y = \tfrac{1}{3W}\big(&(-1130112+635712u+851184v-1510344u^2+2404296uv+1124136v^2-1576176u^3+411918u^2v-937528uv^2\\
&-294096v^3+270072u^4-328512u^3v-22056u^2v^2-768928uv^3-283536v^4+272592u^5+99306u^4v+648430u^3v^2\\
&+40254u^2v^3-175556uv^4-72576u^5v-164352u^4v^2+27400u^3v^3+116064u^2v^4+35082u^5v^2+14922u^4v^3+47942u^3v^4\\
&-9072u^5v^3-17952u^4v^4+126u^5v^4)\\
+&(+308544-1339416u+282816v-594600u^2+109086uv-345772v^2+73620u^3+744936u^2v+276318uv^2\\
&-85840v^3+18576u^4-18306u^3v-31496u^2v^2-115002uv^3+78664v^4-64800u^4v-363108u^3v^2+37060u^2v^3\\
&+158508uv^4+44370u^4v^2-19890u^3v^3-4510u^2v^4-8100u^4v^3-25458u^3v^4+5256u^4v^4)\sqrt{6-2u^2}\\
+&(+137088+186720u-252144v+930720u^2-24788uv+240816v^2+434760u^3-566076u^2v-114074uv^2\\
&-57096v^3-248940u^4+9506u^3v-113676u^2v^2-107644uv^3-133488u^5+155334u^4v+315008u^3v^2+38862u^2v^3\\
&-58896u^5v-67680u^4v^2+43792u^3v^3-16686u^5v^2+6648u^4v^3-7362u^5v^3)\sqrt{6-2v^2}\\
+&(+19968+351120u-198008v+94128u^2-340173uv-110264v^2-12204u^3-69736u^2v+234288uv^2\\
&+87704v^3+4536u^4-30525u^3v+63329u^2v^2+7110uv^3-27360u^4v+4284u^3v^2-25796u^2v^3+567u^4v^2\\
&-10308u^3v^3-3420u^4v^3)\sqrt{6-2u^2}\sqrt{6-2v^2}\big)\\
c_z = \tfrac{1}{3W}\big(&(-2259072-2407104u+185328v-1014312u^2+2802024uv+1390416v^2-1100256u^3-421812u^2v+1592460uv^2\\
&-737040v^3-449640u^4-1717320u^3v+500744u^2v^2-270720uv^3-387072v^4-37152u^5+78084u^4v-107096u^3v^2\\
&+422348u^2v^3+53112uv^4+129600u^5v+835584u^4v^2-199400u^3v^3-171848u^2v^4-88740u^5v^2+44964u^4v^3+27656u^3v^4\\
&+16200u^5v^3+69312u^4v^4-10512u^5v^4)\\
+&(+105984-1525296u+636480v-1682304u^2+704934uv-142512v^2+97272u^3-106896u^2v+1343888uv^2\\
&-517584v^3+272592u^4+63018u^3v+399898u^2v^2+199022uv^3-97632v^4-72576u^4v-101856u^3v^2+94576u^2v^3\\
&+267016uv^4+35082u^4v^2+10386u^3v^3+21038u^2v^4-9072u^4v^3-7440u^3v^4+126u^4v^4)\sqrt{6-2u^2}\\
+&(+747648+835200u-415536v+563280u^2-751104uv-105936v^2+174624u^3-145622u^2v-646578uv^2\\
&+151200v^3-79272u^4+171536u^3v-372608u^2v^2-418872uv^3-9072u^5-34710u^4v-91936u^3v^2-44986u^2v^3\\
&+54720u^5v-21528u^4v^2+61216u^3v^3-1134u^5v^2+8646u^4v^3+6840u^5v^3)\sqrt{6-2v^2}\\
+&(+154368+594672u-262560v+383568u^2-428858uv+16104v^2-158220u^3-8290u^2v-197504uv^2\\
&+185760v^3-133488u^4+100614u^3v+288494u^2v^2+33812uv^3-58896u^4v-56340u^3v^2+59860u^2v^3-16686u^4v^2\\
&-192u^3v^3-7362u^4v^3)\sqrt{6-2u^2}\sqrt{6-2v^2}\big)\\
\\
d_x = \tfrac{1}{W}\big(&(-266112-188112u-697408v+29976u^2-373464uv+90328v^2+217584u^3-247604u^2v-423190uv^2\\
&+422128v^3+95040u^4+241824u^3v-146328u^2v^2-111152uv^3+55000v^4-8496u^4v-40326u^3v^2+32090u^2v^3\\
&-28146uv^4-27632v^5-75888u^4v^2-32808u^3v^3+128272u^2v^4+21888uv^5-5670u^4v^3+13038u^3v^4+2162u^2v^5\\
&+13104u^4v^4+2736u^3v^5+702u^4v^5)\\
+&(+59520+98448u+194336v+179280u^2+26476uv+109472v^2+57456u^3+14766u^2v+32388uv^2\\
&-67384v^3+37296u^3v-68964u^2v^2-34486uv^3-28284v^4+64224u^2v^3-41814u^3v^2-4446uv^4+7248v^5\\
&+216u^3v^3-1080u^2v^4-26352uv^5+3384u^3v^4+906u^2v^5-3294u^3v^5)\sqrt{6-2u^2}\\
+&(-4544-21696u+64856v-57892u^2-164054uv-212008v^2-50160u^3-48546u^2v-213520uv^2\\
&-74428v^3-10080u^4+50802u^3v+135592u^2v^2-64662uv^3+61520v^4-18504u^4v+5340u^3v^2+81296u^2v^3\\
&+30816uv^4+30618u^4v^2+7338u^3v^3-35510u^2v^4+14382u^4v^3+3852u^3v^4-5400u^4v^4)\sqrt{6-2v^2}\\
+&(+19456-12616u+14608v-16872u^2+146883uv+41488v^2-13104u^3+24936u^2v+53896uv^2\\
&-11100v^3-26172u^3v+22221u^2v^2+62793uv^3-31080v^4+4644u^3v^2-12852u^2v^3+25056uv^4+8064u^3v^3\\
&-3885u^2v^4+3132u^3v^4)\sqrt{6-2u^2}\sqrt{6-2v^2}\big)\\
d_y = \tfrac{1}{W}\big(&(-478848-422736u-671648v-414120u^2+116376uv+318872v^2-98640u^3-264124u^2v-120854uv^2\\
&+278320v^3+24192u^4-25392u^3v+209448u^2v^2+46048uv^3-1960v^4+4608u^4v-54534u^3v^2-50838u^2v^3\\
&-20466uv^4-27632v^5-54288u^4v^2-46680u^3v^3+126912u^2v^4+21888uv^5-4158u^4v^3+13998u^3v^4+2162u^2v^5\\
&+13824u^4v^4+2736u^3v^5+702u^4v^5)\\
+&(+33408-176400u+270304v-47952u^2-15256uv+123168v^2-5616u^3+74622u^2v+219292uv^2\\
&-99880v^3+23616u^3v-77508u^2v^2-108214uv^3-47484v^4-13086u^3v^2+67296u^2v^3+32418uv^4+7248v^5\\
&-7668u^3v^3-3480u^2v^4-26352uv^5+7992u^3v^4+906u^2v^5-3294u^3v^5)\sqrt{6-2u^2}\\
+&(+118976+80448u+55720v+174148u^2-216022uv-211480v^2+49008u^3+3690u^2v-257248uv^2\\
&-78908v^3-17568u^4+74946u^3v+57360u^2v^2-73494uv^3+61520v^4-12024u^4v-15060u^3v^2+74112u^2v^3\\
&+30816uv^4+18090u^4v^2+6234u^3v^3-35510u^2v^4+13554u^4v^3+3852u^3v^4-5400u^4v^4)\sqrt{6-2v^2}\\
+&(+29696+95656u-34032v+8376u^2+83381uv+9328v^2-4176u^3-18576u^2v+36688uv^2\\
&+10980v^3-28764u^3v+17133u^2v^2+83385uv^3-31080v^4+2484u^3v^2-10092u^2v^3+25056uv^4+10638u^3v^3\\
&-3885u^2v^4+3132u^3v^4)\sqrt{6-2u^2}\sqrt{6-2v^2}\big)\\
d_z = \tfrac{1}{W}\big(&(+110208-590544u-571968v-190992u^2-1054008uv-697032v^2+503856u^3+85500u^2v+602148uv^2\\
&+768096v^3+253152u^4+258048u^3v-507456u^2v^2+548888uv^3+249656v^4+68472u^4v-93180u^3v^2-276632u^2v^3\\
&+143148uv^4-123040v^5+1152u^4v^2+29880u^3v^3-168136u^2v^4-61632uv^5-65592u^4v^3-12948u^3v^4+71020u^2v^5\\
&-31032u^4v^4-7704u^3v^5+10800u^4v^5)
\end{aligned}
$$

$$
\begin{aligned}
&+(-144000+282384u+101760v+267984u^2+368256uv+243152v^2+35424u^3-206010u^2v-392426uv^2\\
&\quad -174848v^3+69768u^3v-25056u^2v^2+67312uv^3-12360v^4+27666u^3v^2-65568u^2v^3-170946uv^4+62160v^5\\
&\quad +13680u^3v^3+21384u^2v^4-50112uv^5-21798u^3v^4+7770u^2v^5-6264u^3v^5)\sqrt{6-2u^2}\\
&+(+62976-26688u+162384v-148560u^2-288174uv-102144v^2+10512u^3+141360u^2v+17632uv^2\\
&\quad -2600v^3+37152u^4-46086u^3v+88970u^2v^2-16050uv^3-27632v^4-63576u^4v-31392u^3v^2+131440u^2v^3\\
&\quad +21888uv^4+17046u^4v^2+14550u^3v^3+2162u^2v^4+14400u^4v^3+2736u^3v^4+702u^4v^4)\sqrt{6-2v^2}\\
&+(-25344+10080u+30256v+85752u^2+198146uv+39536v^2+7344u^3-45144u^2v-117466uv^2\\
&\quad -58524v^3-26100u^3v+79014u^2v^2+21474uv^3+7248v^4-12168u^3v^2-4860u^2v^3-26352uv^4+6624u^3v^3\\
&\quad +906u^2v^4-3294u^3v^4)\sqrt{6-2u^2}\sqrt{6-2v^2})
\end{aligned}
$$

The lattice volume is

$$
\begin{aligned}
V=\tfrac{1}{3W^3}((&-827512053294956544-651029415060307968u-94834811628946432v\\
&-107286447267407462 40u^2-11908327676546425344uv-222485937861427200v^2\\
&-583264792749146112 0u^3-322981098742952755 2u^2v-1474766032098793881 6uv^2\\
&+49731094225590681 6v^3+4935885324056395776u^4-1125389038186384588 8u^3v\\
&+37974045142173745152u^2v^2+46880534939864924 16uv^3+2400946585744048128v^4\\
&+49627264854815539 2u^5-21444738076127821824u^4v+18081985041154879488u^3v^2\\
&+10770456546921443328u^2v^3-215920425975706091 52uv^4+62770644853840281 6v^5\\
&+10248058547232768 0u^6-15109688128383074304u^5v-42743951239823953 92u^4v^2\\
&-36486575101395025 92u^3v^3-17458852331991619584u^2v^4-92487244315571159 04uv^5\\
&-33165791704729190 4v^6+3395438889176530944u^7+108563364855845683 2u^6v\\
&-17727355235405131776u^5v^2+79415854430354472 96u^4v^3-11563660501205142528u^3v^4\\
&-13561306854154567 68u^2v^5+7044956043545567232uv^6+259048963184984064v^7\\
&+82986998655387648u^8+71942580513191116 8u^7v-52311911903087431 68u^6v^2\\
&-18070442574388844544u^5v^3+24956600314318230528u^4v^4-18709463946285775872u^3v^5\\
&+25845364217882193 92u^2v^6+41125058968118476 80uv^7-10370576885847982 08v^8\\
&-11387122989158891 52u^9+38263378779908997 12u^8v+29930531531390484 48u^7v^2\\
&-16716688403994630144u^6v^3+17780862880577691648u^5v^4+37861076970429843456u^4v^5\\
&-11400598093340545024u^3v^6+26773583407787540 48u^2v^7+18598578763556454 40uv^8\\
&-5939006517818818 56v^9-42985384151600332 8u^{10}+15417855598237286 4u^9v\\
&+11313564596458168 32u^8v^2+13482433779029821 44u^7v^3-66364085819784867 84u^6v^4\\
&+11166686228451910656u^5v^5+60298027115622328 32u^4v^6-27859881338483292 16u^3v^7\\
&+59431317772265062 4u^2v^8-15880320543359795 2uv^9+33241319127338188 8v^{10}\\
&+39543402546855936u^{11}-32821552543120588 8u^{10}v+10215711640638259 2u^9v^2\\
&+20729518342310246 40u^8v^3-57050583717683957 76u^7v^4-54952607849665720 32u^6v^5\\
&-38382858503117373 44u^5v^6-63268999813226096 64u^4v^7+25976660446693580 8u^3v^8\\
&-96386131531804672u^2v^9-80550203067079065 6uv^{10}+94492117697298432v^{11}\\
&+20900877065256960u^{12}-1503942678282240u^{11}v+10354622733685555 2u^{10}v^2\\
&-20318848756047360 0u^9v^3+55006066876957286 4u^8v^4+16155239119843553 28u^7v^5\\
&+26260454109198438 4u^6v^6-24959854887067996 16u^5v^7-13545193916945981 44u^4v^8\\
&-15718918419044966 4u^3v^9+35362103572657766 4u^2v^{10}+27392019974979584uv^{11}\\
&-29120642045706240v^{12}+12256719615590400u^{12}v+17828122821112627 2u^{11}v^2\\
&-16886942644094668 8u^{10}v^3-55969447596063744u^9v^4-13066830862635110 4u^8v^5\\
&+23136963952182528 00u^7v^6+82600666296761651 2u^6v^7-82196225270887116 8u^5v^8\\
&+44673057248657920 0u^4v^9-17145672293007667 2u^3v^{10}-21303969366168371 2u^2v^{11}\\
&+30686914248671232uv^{12}+5218983155780812 8u^{12}v^2-14224660307656704u^{11}v^3\\
&-29636371036804608 0u^{10}v^4+12924839894716108 8u^9v^5-14889798567401472u^8v^6\\
&-19766133470505676 8u^7v^7+69720831267818700 8u^6v^8+33485742860588032u^5v^9\\
&+14807797872134553 6u^4v^{10}+9303506790275481 6u^3v^{11}+41929983081676800u^2v^{12}\\
&+47838725897378 4u^{12}v^3+22932559919075328u^{11}v^4+10901189189081088 0u^{10}v^5\\
&+25993083122032435 2u^9v^6-96517488858267 52u^8v^7-32514955032196915 2u^7v^8\\
&+37249683480811929 6u^6v^9+24291729730451251 2u^5v^{10}+10316409264742604 8u^4v^{11}\\
&-89059790425780224u^3v^{12}-22355344759652352u^{12}v^4-93241272168990 72u^{11}v^5\\
&+65953555802265 60u^{10}v^6-75851974172694528u^9v^7-19495946360706048u^8v^8\\
&+18718565550163968u^7v^9-54576702745801728u^6v^{10}-36912238833342464u^5v^{11}\\
&+39522716406890496u^4v^{12}-3342091023793152u^{12}v^5+63929923877283 84u^{11}v^6\\
&+40434839029951488u^{10}v^7-50584584720577536u^9v^8-69057244122098688u^8v^9\\
&+37140873519676416u^7v^{10}+17734974529686528u^6v^{11}-39916269786470 40u^5v^{12}\\
&-25110239708989 44u^{12}v^6-12483395755149312u^{11}v^7+16366688920866816u^{10}v^8\\
&+68962459937310 72u^9v^9-37494268522113024u^8v^{10}-66872195300413 44u^7v^{11}\\
&-86016681131212 8u^6v^{12}+14521113879828 48u^{12}v^7-25990916774952 96u^{11}v^8\\
&-41040826154864 64u^{10}v^9+21070619210833 92u^9v^{10}-59773080150558 72u^8v^{11}\\
&+50074101565194 24u^7v^{12}+78307121425305 6u^{12}v^8+77908780173312u^{11}v^9\\
&-26801966659368 96u^{10}v^{10}+13578336639129 60u^9v^{11}-30193302679603 20u^8v^{12}\\
&+42093138751488u^{12}v^9+116440713934848u^{11}v^{10}-83903223704371 2u^{10}v^{11}\\
&+10297802559098 88u^9v^{12}-51525678569472u^{12}v^{10}+12919929131520 0u^{11}v^{11}\\
&-43078660854681 6u^{10}v^{12}-24749011869696u^{12}v^{11}+65365205050368u^{11}v^{12}\\
&-16609716652032u^{12}v^{12})
\end{aligned}
$$

$$
\begin{aligned}
&+(-7376035368665088-186124614827507712 0u+361755696973300992v\\
&-858700648452980736u^2-421906402300133376uv+1662370959713107968v^2\\
&-404919565041401856 0u^3-387615912548106240 0u^2v+282328845665579827 2uv^2\\
&-567083951304081408v^3-406741137918374707 2u^4-908201022281927884 8u^3v\\
&+5302671641837125632u^2v^2-257786945342948966 4uv^3-282314654837047296 0v^4\\
&+256450056689958912 0u^5-882344092783671705 6u^4v+131915880824086978 56u^3v^2\\
&+155627230489749504 0u^2v^3-145535636988798566 4uv^4+406765712043245568v^5\\
&+345651536927180390 4u^6-333941047652538777 6u^5v+568058532011093606 4u^4v^2\\
&+141140955066908866 56u^3v^3-789338323740178022 4u^2v^4+266174541567019008 0uv^5\\
&+144155243535443558 4v^6+243639057403256832u^7+174685549624528896 0u^6v\\
&-578376138892925952u^5v^2+109199715690912307 2u^4v^3+336647002838939596 8u^3v^4\\
&-604061158115518568u^2v^5+170962286228091699 2uv^6-268020520199897088v^7\\
&-443618443111071744u^8+2335603073340837888u^7v-250046846676191232 0u^6v^2\\
&-520467262885169510 4u^5v^3+546151448248890624 0u^4v^4+378487139579119104 0u^3v^5\\
&-329556753684253696u^2v^6+759436656777543680uv^7-204447779847667712v^8\\
&+38837256471134208u^9+589073809855463424u^8v+872608390658426888u^7v^2\\
&-728710004722042828 8u^6v^3+708165384708441600u^5v^4-360815483771848396 8u^4v^5\\
&-364867947038507212 8u^3v^6+170051575925255168u^2v^7-339134861204303872uv^8\\
&-39482138859864064v^9+79750606536179712u^{10}+36102419739426816u^9v\\
&+242398124003819528u^8v^2-283118908435721164 8u^7v^3-136119175575507302 4u^6v^4\\
&+844887792068831232 0u^5v^5-552941557590622208 0u^4v^6-256192789788937728 0u^3v^7\\
&+769477483530068992u^2v^8-116075444929478656uv^9-117162627634593792v^{10}\\
&+12438425795493888u^{11}+215273806054686720u^{10}v-344572964048185344u^9v^2\\
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&+441486765382789120 0u^5v^6-143260770481423308 8u^4v^7+657808392623770112u^3v^8\\
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&+1264510434484224u^{11}v^2-7975580720062464u^{10}v^3+165381316624401408u^9v^4\\
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&-335781408652125696u^7v^7-109906593044387328u^6v^8+272757762979049472u^5v^9\\
&-166324550015471616u^4v^{10}-917343805423861 76u^3v^{11}+693296322711552 0u^2v^{12}\\
&+78948394066391 04u^{11}v^4+245947242848256 00u^{10}v^5-999554438195619 84u^9v^6\\
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&+209856457559946 24u^8v^8+807623715972833 28u^7v^9-259964527319436 8u^6v^{10}\\
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&-206804845386086 4u^{11}v^7+280629329639296u^{10}v^8+91185641135585 28u^9v^9\\
&+156297048181862 4u^8v^{10}+57185760764262 4u^7v^{11}+23921117861345 28u^6v^{12}\\
&+213921770176512u^{11}v^8-197675963829964 8u^{10}v^9+63137510086348 8u^9v^{10}\\
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&+259587237169152u^{10}v^{10}+116976965378918 4u^9v^{11}+389642135671296u^8v^{12}\\
&-123178244935 68u^{11}v^{10}-144787290462720u^{10}v^{11}-230587191728640u^9v^{12}\\
&+547884614200 32u^{11}v^{11}+224003004318 72u^{10}v^{12}-138333427568 64u^{11}v^{12})\sqrt{6-2u^2}\\
&+(+539890954090315776-11192235756355584u+461692003866547 20v\\
&+3549620247415750656u^2-136899826727937638 4uv-156142843062858547 2v^2\\
&+3472668107073257472u^3+343828597621771468 8u^2v-459124441879412736 0uv^2\\
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&-433856971808730316 8u^2v^2+615444674505965568uv^3+208638499636150272 0v^4\\
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&-741846352808057088u^2v^6-117897465139151257 6uv^7+304982821538758656v^8\\
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&+102043044615403407 36u^6v^3+125013285999157248u^5v^4-628111944199389542 4u^4v^5\\
&+141603021978731576 32u^3v^6+111895951494525952u^2v^7-631068630795173888uv^8\\
&+123003980444073984v^9+235699340469092352u^{10}+148527792425189376u^9v\\
&-333042383335136256u^8v^2-597754157830216704u^7v^3+405411252304192204 8u^6v^4
\end{aligned}
$$

$$
\begin{aligned}
&-2294843043052426240u^5v^5-3108008685269313024u^4v^6+3326354917870289408u^3v^7\\
&-153026407907391488u^2v^8-26834797615742976uv^9+67785720800083968v^{10}\\
&-32597718159458304u^{11}+262260514424733696u^{10}v+390324329459159040u^9v^2\\
&-1527059873674971648u^8v^3-597728979424028160u^7v^4-160305260079715256u^6v^5\\
&-287430574937442612u^5v^6+654410683187003392u^4v^7+127194031857318604 8u^3v^8\\
&+39948175578667008u^2v^9-11220633792688947 2uv^{10}+5271504299163648v^{11}\\
&-16541617285693440u^{12}+1555267228434432 0u^{11}v+139114089451376 64u^{10}v^2\\
&-124519246063368192u^9v^3+435552591847265 28u^8v^4+111920013829093 6320u^7v^5\\
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&-9983852494848000u^{12}v-740383396931174 40u^{11}v^2+995774376460124 16u^{10}v^3\\
&+993367733898608 64u^9v^4-172740010540608000u^8v^5-188317877962357248u^7v^6\\
&+667243240360867 84u^6v^7-178564928089486336u^5v^8+261174456480055296u^4v^9\\
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&-22653352036551168u^8v^6+514466335759044096u^7v^7+207546794194784256u^6v^8\\
&-22132838569766707 2u^5v^9-60838351775156224u^4v^{10}-159803706518323 20u^3v^{11}\\
&-57279986550374 4u^{12}v^3+73813578242946 56u^{11}v^4+652659550437888 00u^{10}v^5\\
&+16842842347852800u^9v^6-103475387539137024u^8v^7-106989186183214 08u^7v^8\\
&+54999763100989440u^6v^9+353087862278850 56u^5v^{10}+905054498963251 2u^4v^{11}\\
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&+27764093349328896u^9v^7+364314975116037 12u^8v^8+288392666124979 20u^7v^9\\
&+14154914553999 36u^6v^{10}+11712409409249 28u^5v^{11}+43146271300976 64u^{12}v^5\\
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&-20104053260458 4u^{12}v^6-589446650161152u^{11}v^7+148401365737881 6u^{10}v^8\\
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&-10345798154465 28u^9v^{10}-926311679880192u^8v^{11}+406182377088 00u^{12}v^8\\
&+33417998291404 8u^{11}v^9+41269164421017 6u^{10}v^{10}+9890960078131 2u^9v^{11}\\
&-33743869512192u^{12}v^9-369639904896 00u^{11}v^{10}-183374370533376u^{10}v^{11}\\
&+41517649681920u^{12}v^{10}+1300586264064u^{11}v^{11}-5210600306688u^{12}v^{11})\sqrt{6-2v^2}\\
+(&-74474692647321600+865377471468404736u-193074700065177600v\\
&+64184549797527552 0u^2+931580968207122432uv+4128129276287385 6v^2\\
&+12334536934344622 08u^3+790931071046909952u^2v-154084926605583 9744uv^2\\
&+71015029554216960 0v^3+10760601437518233 60u^4+12226596762502103 04u^3v\\
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&+16528775514469 1712u^8v^2+124455992066646816u^7v^3+299477313878925312u^6v^4\\
&-32917633501753 66144u^5v^5+190669178300850 1760u^4v^6+335267488902969856u^3v^7\\
&-23392797032415 4368u^2v^8-763580827684945 92uv^9-6885512529969152v^{10}\\
&-12144915560005 632u^{11}-894793212220538 88u^{10}v+135930285979140096u^9v^2\\
&+23117679119495 1168u^8v^3+530833048434643968u^7v^4+346820268180705792u^6v^5\\
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&-70357410245922 816u^2v^9+243907797111930 88uv^{10}-384822563320627 2v^{11}\\
&-14412537441533 952u^{11}v+401249999811870 72u^{10}v^2+123125411764909 44u^9v^3\\
&-19949173318298 4192u^8v^4-454473014017946112u^7v^5-133100381570076672u^6v^6\\
&+130827576399390 208u^5v^7+665856399283906 56u^4v^8+103963612828362240u^3v^9\\
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&+18136736367567 2064u^7v^6+708907764337274 88u^6v^7-269499563996361216u^5v^8\\
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&-20169099055042 56u^{11}v^3+936378847099699 2u^{10}v^4-749981772718387 2u^9v^5\\
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&-12849305244645 888u^8v^7-305795530506424 32u^7v^8-375652930823531 52u^6v^9\\
&-91169813213767 68u^5v^{10}-932360325341184u^4v^{11}+163388531053209 6u^{11}v^5\\
&-26832005329121 28u^{10}v^6-793249194131712 0u^9v^7-129376215552430 08u^8v^8
\end{aligned}
$$

$$+9993539281236480u^7v^9+2254003233039360u^6v^{10}+2012839402268160u^5v^{11}$$
$$+1392368988750336u^{11}v^6+973507470787584u^{10}v^7+169905759045120u^9v^8$$
$$-50591168775567736u^8v^9-3738733389771264u^7v^{10}-2985567972950016u^6v^{11}$$
$$-148777154818560u^{11}v^7-504207215354880u^{10}v^8+595424618827776u^9v^9$$
$$-356543487452160u^8v^{10}+970027585952256u^7v^{11}-21953199218688u^{11}v^8$$
$$+62363407901184u^{10}v^9-518270051966976u^9v^{10}-458744101337088u^8v^{11}$$
$$-28536720629 76u^{11}v^9-6412150563840u^{10}v^{10}+137484923885568u^9v^{11}$$
$$-29943951063552u^{11}v^{10}-16403224412160u^{10}v^{11}+5211690190848u^{11}v^{11})\sqrt{6-2u^2}\sqrt{6-2v^2})$$

The equations of minimum lattice volume are $\langle \partial_u V, \partial_v V \rangle = \langle 0, 0 \rangle$

The free variables are $\langle u, v \rangle \approx \langle -0.034789016702, +0.089604971413 \rangle$

The vertices of $\mathbf{E}_u^{5+}$ are $\begin{cases} o_u^+ \approx \langle -1.061384364770, +1.061384364770, -0.935697925928 \rangle \\ p_u^+ \approx \langle -1.644260072209, +1.644260072209, +1.769946511051 \rangle \\ q_u^+ \approx \langle -0.034789016702, +0.034789016702, +3.448995599962 \rangle \\ r_u^+ \approx \langle +1.621067394407, -1.621067394407, +1.862717222257 \rangle \\ s_u^+ \approx \langle +1.115500612974, -1.115500612974, -0.873850785124 \rangle \end{cases}$

The vertices of $\mathbf{E}_v^{5-}$ are $\begin{cases} o_v^- \approx \langle -1.156897066820, -1.156897066820, +0.822958681256 \rangle \\ p_v^- \approx \langle -1.600938143111, -1.600938143111, -1.934876528675 \rangle \\ q_v^- \approx \langle +0.089604971413, +0.089604971413, -3.446209700372 \rangle \\ r_v^- \approx \langle +1.660674790719, +1.660674790719, -1.695929938240 \rangle \\ s_v^- \approx \langle +1.017511555733, +1.017511555733, +0.982256408212 \rangle \end{cases}$

The lattice vectors are $\begin{cases} a \approx \langle +1.286101228632, +0.200477756509, +0.117304750804 \rangle \\ b \approx \langle -0.216654244567, +1.293299854677, +0.064005626691 \rangle \\ c \approx \langle +1.049828831839, +0.572626599359, +4.643104387948 \rangle \\ d \approx \langle -0.331247602500, +1.312711170924, -4.452573714586 \rangle \end{cases}$

The lattice volume is $V \approx 61.647870634123$

The packing density is $D \approx .778615700855$

Chapter 7
Optimal oblique$_{u,v}$ packing of $\mathbf{B}^9_{u,v}$

"A Dense Packing of Regular Tetrahedra"

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Abstract

We construct a dense packing of regular tetrahedra, with packing density $D > .7786157$.

1 Introduction

"How can one arrange most densely in space an infinite number of equal solids of given form, e.g., spheres with given radii or regular tetrahedra with given edges (or in prescribed position), that is, how can one so fit them together that the ratio of the filled to the unfilled space may be as great as possible?" (excerpt from David Hilbert's 18th problem [9]).

This is a very old problem. Aristotle [1] believed that you could tile space with regular tetrahedra. Everyone believed him for the next 1800 years, until Johannes Müller (aka Regiomontanus) proved everyone wrong (histories by Dirk Struik [15] and Marjorie Senechal [14]).

This is a very understandable mistake, because tetrahedra almost tile space locally, but not quite. The cluster $\mathbf{E}_5$ (consisting of 5 tetrahedra joined symmetrically about an edge) has total solid angle $5 \cdot 2\cos^{-1}\frac{1}{3} \approx 12.309594173408$ about the edge, and local density $D \approx .979566380077$. The cluster $\mathbf{V}_{20}$ (consisting of 20 tetrahedra joined symmetrically about a vertex) has total solid angle $20 \cdot (-\pi + 3\cos^{-1}\frac{1}{3}) \approx 11.025711968651$ about the vertex, and local density $D \approx .877398280459$.

For the sphere, the analogous problem was very challenging. A long time ago, Johannes Kepler conjectured that the densest packing is the hexagonal close-packing (HCP), with density $D = \pi/\sqrt{18} \approx .740480489693$. Carl Friedrich Gauss proved that the HCP is the densest lattice packing. Only recently, Thomas Hales & Samuel Ferguson [7, 8] proved that the HCP is the densest packing in general.

Helmut Grömer [6] constructed a lattice packing of the single tetrahedron $\mathbf{B}_1$ with density $D = \frac{18}{49} \approx .367346938775$. Douglas Hoylman [10] proved that Grömer's packing was the densest lattice packing.

Andrew Hurley [11] constructed the tetrahelix. I calculated its packing density $D = \sqrt{\frac{50000}{177417}} \approx .531273435694$.

Hermann Minkowski [13] proved that the densest lattice packing of any convex body must satisfy certain constraints. Ulrich Betke & Martin Henk [2, 3] developed an efficient computer algorithm to compute the densest lattice packing of any convex body, and they applied it to the Archimedean solids. I used Betke & Henk's program to calculate the packing density of the convex hull of $\mathbf{V}_{20}$, $D \approx .716796401602$.

John Conway & Salvatore Torquato [5] used Betke & Henk's program to calculate the packing density of $\mathbf{V}_{20}$ inscribed in a regular icosahedron, $D > .7165598$. They slightly improved the packing density by wiggling the tetrahedra, $D > .717455$. They raised the question whether the packing density of regular tetrahedra is, not only less than that of the sphere, but perhaps even the lowest of any convex body.

In this paper [4], we will construct a dense packing of regular tetrahedra with packing density $D \approx .778615700855$, which beats the densest sphere packing!

2 Description by Construction

Feel free to look at the pictures in Sec. 3, as you read the descriptions in Sec. 2.

We construct a 2-parameter family of clusters of regular tetrahedra. Each cluster (9 tetrahedra) is the union of 1 'central' tetrahedron, 4 'upper' tetrahedra attached to an edge of the central tetrahedron, and 4 'lower' tetrahedra attached to the opposite edge of the central tetrahedron. The 4 upper tetrahedra can rotate about its attached edge, by an angle parametrized by u, and the 4 lower tetrahedra can rotate about its attached edge, by an angle parametrized by v. Clusters have 2 orientations: 'positive' and 'negative' (scalar mult. by -1).

For each parameter value $\langle u, v \rangle$, we construct an optimized packing of clusters, which is crystallographic. The clusters pack in layers with alternating orientations. The full symmetry group of the packing acts transitively on all clusters. The direct symmetry group, which is also the translation symmetry group, acts transitively on a coset. There are 2 cosets, which correspond to the 2 orientations = 2 translation classes. So the fundamental domain (under the lattice of translations) contains 2 clusters = 18 tetrahedra.

Since every cluster is equivalent to all other clusters, we restrict our attention to the cluster at the origin and its immediate neighbors. Also, since the cluster (9 tetrahedra) is extremely non-convex, we think of it as the union of (the convex hull of) the 'upper half-cluster' (5 tetrahedra) and (the convex hull of) the 'lower half-cluster' (5 tetrahedra). In order to check that clusters don't overlap in the packing, it's convenient to check that half-clusters don't overlap.

Each cluster has transverse edge-to-edge intersections with neighboring clusters in the same layer, and partial face-to-face intersections with neighboring clusters in adjacent layers. For intersecting edges, the separating plane contains the union of the edges, and for non-intersecting edges, the separating plane is between and parallel to both edges. For intersecting faces, the separating plane contains the union of the faces, and for non-intersecting faces, the separating plane is between and parallel to both faces.

The intersections determine equations in terms of the cluster coordinates and lattice vectors, which we solve in terms of $\langle u, v \rangle$ and optimize over all packings in the family. The maximum occurs at $\langle u, v \rangle \approx \langle -.034789016702, +.089604971413 \rangle$, with packing density $D \approx .778615700855$.

2.1 The cluster $\mathbf{B}^9_{u,v}$ [See Fig. 1]

We construct the cluster $\mathbf{B}^1$, consisting of 1 tetrahedron. It is the convex hull of the 'upper edge' $E[\langle+1,+1,+1\rangle,\langle-1,-1,+1\rangle]$ and 'lower edge' $E[\langle+1,-1,-1\rangle,\langle-1,+1,-1\rangle]$.

We attach 4 additional tetrahedra face-to-face along the 'upper edge', and we call this cluster $\mathbf{E}^{4+}_u$. There are now 5 tetrahedra along the 'upper edge', and they form the 'upper half-cluster' $\mathbf{E}^{5+}_u = \mathbf{B}^1 \cup \mathbf{E}^{4+}_u$.

We attach 4 additional tetrahedra face-to-face along the 'lower edge', and we call this cluster $\mathbf{E}^{4-}_v$. There are now 5 tetrahedra along the 'lower edge', and they form the 'lower half-cluster' $\mathbf{E}^{5-}_v = \mathbf{B}^1 \cup \mathbf{E}^{4-}_v$.

There are 9 total tetrahedra, and they form the 'cluster' $\mathbf{B}^9_{u,v} = \mathbf{E}^{5+}_u \cup \mathbf{E}^{5-}_v$. Clusters have 2 orientations: 'positive' clusters $+\mathbf{B}^9_{u,v}$ and 'negative' clusters $-\mathbf{B}^9_{u,v}$ (point reflection). The set of clusters $\{\mathbf{B}^9_{u,v} : -\frac{1}{9} \le u \le +\frac{1}{9}, -\frac{1}{9} \le v \le +\frac{1}{9}\}$ forms a 2-parameter family of clusters.

2.2 The swivel parameters $\langle u,v\rangle$ [See Fig. 2]

The cluster $\mathbf{E}^{4+}_u$ can rotate about the 'upper edge' through small angles, without overlapping with $\mathbf{B}^1$. The 'upper swivel parameter' $-\frac{1}{9} \le u \le +\frac{1}{9}$ corresponds to the x-coordinate of the highest vertex q^+_u, and the rotation angle is given by $u = \sqrt{3}\sin\theta_u$.

The cluster $\mathbf{E}^{4-}_v$ can rotate about the 'lower edge' through small angles, without overlapping with $\mathbf{B}^1$. The 'lower swivel parameter' $-\frac{1}{9} \le v \le +\frac{1}{9}$ corresponds to the x-coordinate of the lowest vertex q^-_v, and the rotation angle is given by $v = \sqrt{3}\sin\theta_v$.

2.3 The optimized packing $\mathbf{P}_{u,v}$ [See Fig. 3]

The packing is made in layers. The 'central' layer consists of translates of 'positive' clusters, in a 2-dimensional lattice $\mathbf{L}^2_{u,v} = \mathbf{Z}\cdot 2a + \mathbf{Z}\cdot 2b$ with basis $\{2a, 2b\}$. (When $\langle u,v\rangle = \langle 0,0\rangle$, this lattice is a square lattice. For our small values of $\langle u,v\rangle$, this lattice is almost a square lattice.)

The layers alternate between 'positive' layers containing 'positive' clusters, and 'negative' layers containing 'negative' clusters. The layer 'above' the central layer consists of translates of 'negative' clusters, in the offset $c - \mathbf{L}^2_{u,v}$. The layer 'below' the central layer consists of translates of 'negative' clusters, in the offset $d - \mathbf{L}^2_{u,v}$.

The 'positive' clusters live in a 3-dimensional lattice $\mathbf{L}^3_{u,v} = \mathbf{L}^2_{u,v} + \mathbf{Z}(c-d)$ with basis $\{2a, 2b, c-d\}$, and the 'negative' clusters live in the offset $c - \mathbf{L}^3_{u,v} = d - \mathbf{L}^3_{u,v}$. The packing consists of 2 cosets of the lattice $\mathbf{L}^3_{u,v}$, which correspond to the 2 orientations. For each cluster $\mathbf{B}^9_{u,v}$, we find the optimized packing and call it $\mathbf{P}_{u,v}$.

The packing is periodic and crystallographic, and has a large isometry group which acts transitively on clusters. There is a direct isometry between any two clusters of the same orientation, and an indirect isometry between any two clusters of opposite orientations. The direct isometry group is the translation group $\mathbf{L}^3_{u,v} = \mathbf{Z}\cdot 2a + \mathbf{Z}\cdot 2b + \mathbf{Z}(c-d)$. The point group is (generically) trivial.

The set of optimized packings $\{\mathbf{P}_{u,v} : -\frac{1}{9} \le u \le +\frac{1}{9}, -\frac{1}{9} \le v \le +\frac{1}{9}\}$ forms a 2-parameter family of packings.

2.4 The neighbors & separating planes of $\mathbf{B}^9_{u,v}$ [See Fig. 4]

Since $\mathbf{B}^9_{u,v}$ is non-convex, we consider the half-clusters $\mathbf{E}^{5+}_u, \mathbf{E}^{5-}_v$ separately. Since each half-cluster is disjoint from its neighbors, the union is also disjoint from its neighbors. Therefore, we can verify that we have a packing.

Separating planes separate the upper half-cluster $\mathbf{E}^{5+}_u$ from each of its neighbors: the plane $\mathbf{R}\cdot 2a+\mathbf{R}\cdot 2b+q^+_u$ for the upper above half-layer, 4 planes for the 6 neighbors in the lower above half-layer, 4 planes for the 8 neighbors in the upper central half-layer, 6 planes for the 8 neighbors in the lower central half-layer, and the plane $\mathbf{R}\cdot 2a+\mathbf{R}\cdot 2b+d+q^+_u$ for the upper below half-layer. (q^+_u is the highest point of the upper half-cluster.)

Separating planes separate the lower half-cluster $\mathbf{E}^{5-}_v$ from each of its neighbors: the plane $\mathbf{R}\cdot 2a+\mathbf{R}\cdot 2b+c+q^-_v$ for the lower above half-layer, 6 planes for the 8 neighbors in the upper central half-layer, 4 planes for the 8 neighbors in the lower central half-layer, 4 planes for the 6 neighbors in the upper below half-layer, and the plane $\mathbf{R}\cdot 2a+\mathbf{R}\cdot 2b+q^-_v$ for the lower below half-layer. (q^-_v is the lowest point of the lower half-cluster.)

2.5 The intersections & separating planes of $\mathbf{B}^9_{u,v}$ [See Fig. 5]

In the same layer, clusters touch transversely edge-to-edge, intersecting at only 1 point. Each upper half-cluster has 4 such contacts with neighboring upper half-clusters, and each lower half-cluster has 4 such contacts with neighboring lower half-clusters. These intersections give 4 equations. Each upper half-cluster has ≤ 2 such contacts with neighboring lower half-clusters, and each lower half-cluster has ≤ 2 such contacts with neighboring upper half-clusters. These intersections give ≤ 2 equations.

For non-intersecting skew edges $E[a,b] \neq E[c,d]+w$, there are many choices for the separating plane. We can choose the plane which contains $E[a,b]$ and parallel to $E[c,d]$ (ie: the plane through the points $a,b,b+c-d$), the plane which contains $E[c,d]$ and parallel to $E[a,b]$ (ie: the plane through the points $c,d,d+a-b$), or any parallel plane in between.

For the transverse edge-to-edge intersection $E[a,b] = E[c,d]+w$, the separating plane contains the union of the edges $E[a,b]\cup E[c,d]$. We solve the virtual intersection equation $sa+(1-s)b = tc+(1-t)d+w$, between lines with parameters s,t. For our (small) values of $\langle u,v\rangle$, our edges intersect in their interiors (ie: $0\leq s\leq 1$ and $0\leq t\leq 1$).

In adjacent layers, clusters touch partially face-to-face. Each upper half-cluster has 3 such contacts with neighboring half-clusters in the above layer, and each lower half-cluster has 3 such contacts with neighboring half-clusters in the below layer. These intersections give 6 equations.

For non-intersecting parallel faces $F[a,b,c] \neq F[d,e,f]+w$, there are many choices for the separating plane. We can choose the plane containing $F[a,b,c]$, the plane containing $F[d,e,f]$, or any parallel plane in between.

For the partial face-to-face intersection $F[a,b,c] = F[d,e,f]+w$, the separating plane is the plane which contains the union of the faces $F[a,b,c]\cup F[d,e,f]$. There are two ways to write the intersection equation.

We can reduce it to a transverse edge-to-edge intersection of any non-parallel pair of edges from $\{E[b,c],E[c,a],E[a,b]\}$ and $\{E[e,f],E[f,d],E[d,e]\}$). If we choose carefully, we can find two edges which intersect in their interiors.

Alternatively, we can reduce it to an incidence between a triangle $F[a,b,c]$ and a vertex from $\{d,e,f\}$, or vice versa. Then we solve the virtual intersection equation $sa+tb+(1-s-t)c=d+w$, with plane parameters s,t. If we choose carefully, we can find a point which lies inside the triangle (ie: $0 \le s \le 1$ and $0 \le t \le 1$).

3 Pictures & Equations

The figures are really super-figures which contain multiple sub-figures. Figs. 1-5 are floor-plan diagrams which show how the various layers fit together (ie: the top layer is placed at the top, and the bottom layer is placed at the bottom). For the purpose of perspective, we draw the coordinate axes as solid objects. For the purpose of comparison, we provide both top views (perspective vector $\langle 0,0,1\rangle$) & side views (perspective vector $\langle 0,-\cos\frac{\pi}{10},\sin\frac{\pi}{10}\rangle$).

- Fig. 0 shows the orientation of the $\langle x,y,z\rangle$ coordinate axes (for Figs. 1-5).
- Fig. 1 (2 pages) shows the cluster $\mathbf{B}^9_{u,v}$.
- Fig. 2 (2 pages) shows the swivel parameters $\langle u,v\rangle$.
- Fig. 3 (2 pages) shows the packing $\mathbf{P}_{u,v}$.
- Fig. 4 (2 pages) shows the neighbors & separating planes (general overview).
- Fig. 5 (4 pages) shows the intersections & separating planes (fine details).

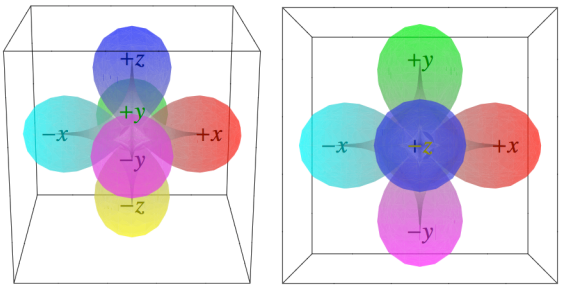


Fig. 0 The $\langle x,y,z\rangle$ coordinate axes (*side view* & *top view*), for Figs. 1–5

$\mathbf{B}^1$: 1 tetrahedron

coordinates: $B[\langle+1,+1,+1\rangle,\langle+1,-1,-1\rangle,\langle-1,+1,-1\rangle,\langle-1,-1,+1\rangle]$

$\mathbf{E}_u^{4+}$: 4 tetrahedra, joined face↔face

attached to the 'upper edge' $E[\langle+1,+1,+1\rangle,\langle-1,-1,+1\rangle]$

coordinates: $B[\langle+1,+1,+1\rangle,\langle-1,-1,+1\rangle,o_u^+,p_u^+]$

$B[\langle+1,+1,+1\rangle,\langle-1,-1,+1\rangle,p_u^+,q_u^+]$

$B[\langle+1,+1,+1\rangle,\langle-1,-1,+1\rangle,q_u^+,r_u^+]$

$B[\langle+1,+1,+1\rangle,\langle-1,-1,+1\rangle,r_u^+,s_u^+]$

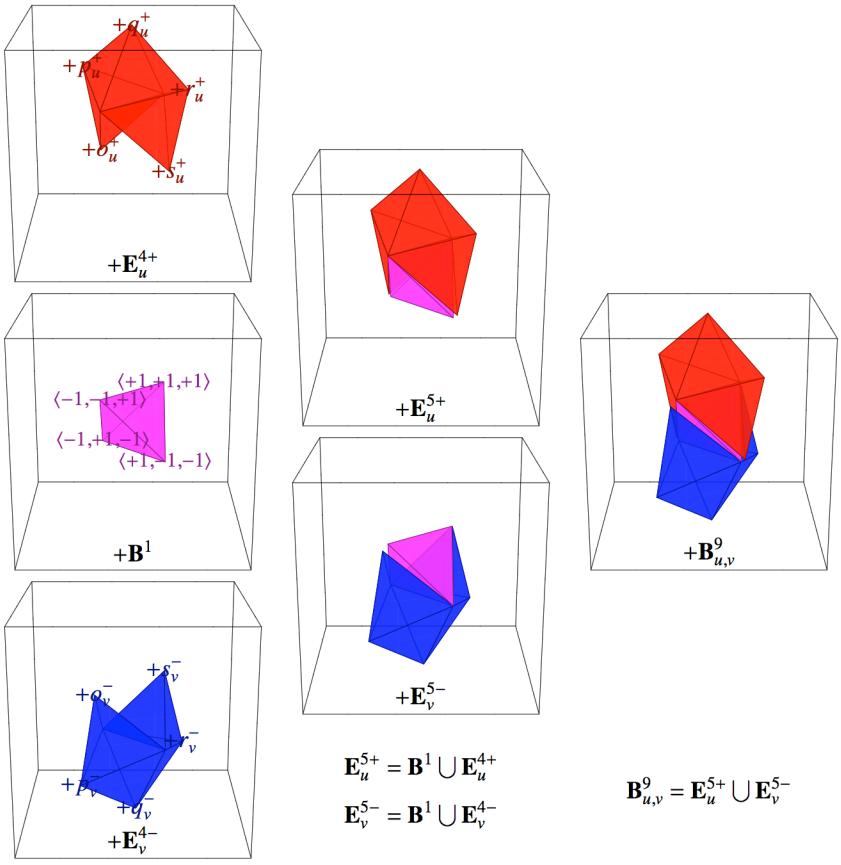
$o_u^+ = \langle -\frac{7}{9}u-\frac{4}{9}\sqrt{6-2\,u^2}\,,+\frac{7}{9}u+\frac{4}{9}\sqrt{6-2\,u^2}\,,+1+\frac{8}{9}u-\frac{7}{9}\sqrt{6-2\,u^2}\,\rangle$

$p_u^+ = \langle +\frac{1}{3}u-\frac{2}{3}\sqrt{6-2\,u^2}\,,-\frac{1}{3}u+\frac{2}{3}\sqrt{6-2\,u^2}\,,+1+\frac{4}{3}u+\frac{1}{3}\sqrt{6-2\,u^2}\,\rangle$

$q_u^+ = \langle +u,-u,+1+\sqrt{6-2\,u^2}\,\rangle$

$r_u^+ = \langle +\frac{1}{3}u+\frac{2}{3}\sqrt{6-2\,u^2}\,,-\frac{1}{3}u-\frac{2}{3}\sqrt{6-2\,u^2}\,,+1-\frac{4}{3}u+\frac{1}{3}\sqrt{6-2\,u^2}\,\rangle$

$s_u^+ = \langle -\frac{7}{9}u+\frac{4}{9}\sqrt{6-2\,u^2}\,,+\frac{7}{9}u-\frac{4}{9}\sqrt{6-2\,u^2}\,,+1-\frac{8}{9}u-\frac{7}{9}\sqrt{6-2\,u^2}\,\rangle$



$\mathbf{E}_u^{5+} = \mathbf{B}^1 \bigcup \mathbf{E}_u^{4+}$

$\mathbf{E}_v^{5-} = \mathbf{B}^1 \bigcup \mathbf{E}_v^{4-}$

$\mathbf{B}_{u,v}^9 = \mathbf{E}_u^{5+} \bigcup \mathbf{E}_v^{5-}$

Fig. $\mathbf{1}_{\text{side}}^+$ The positive cluster $+\mathbf{B}_{u,v}^9$ (*side view*)

Clusters have 2 possible orientations:
'positive' or 'negative' (point reflection = scalar mult. by −1)

$\mathbf{E}_v^{4-}$: 4 tetrahedra, joined face↔face
attached to the 'lower edge' $E[\langle+1,-1,-1\rangle,\langle-1,+1,-1\rangle]$
coordinates: $B[\langle+1,-1,-1\rangle,\langle-1,+1,-1\rangle,o_v^-,p_v^-]$
$B[\langle+1,-1,-1\rangle,\langle-1,+1,-1\rangle,p_v^-,q_v^-]$
$B[\langle+1,-1,-1\rangle,\langle-1,+1,-1\rangle,q_v^-,r_v^-]$
$B[\langle+1,-1,-1\rangle,\langle-1,+1,-1\rangle,r_v^-,s_v^-]$

$$o_v^- = \langle -\tfrac{7}{9}v - \tfrac{4}{9}\sqrt{6-2v^2}\ , -\tfrac{7}{9}v - \tfrac{4}{9}\sqrt{6-2v^2}\ , -1 - \tfrac{8}{9}v + \tfrac{7}{9}\sqrt{6-2v^2}\ \rangle$$
$$p_v^- = \langle +\tfrac{1}{3}v - \tfrac{2}{3}\sqrt{6-2v^2}\ , +\tfrac{1}{3}v - \tfrac{2}{3}\sqrt{6-2v^2}\ , -1 - \tfrac{4}{3}v - \tfrac{1}{3}\sqrt{6-2v^2}\ \rangle$$
$$q_v^- = \langle +v, +v, -1 - \sqrt{6-2v^2}\ \rangle$$
$$r_v^- = \langle +\tfrac{1}{3}v + \tfrac{2}{3}\sqrt{6-2v^2}\ , +\tfrac{1}{3}v + \tfrac{2}{3}\sqrt{6-2v^2}\ , -1 + \tfrac{4}{3}v - \tfrac{1}{3}\sqrt{6-2v^2}\ \rangle$$
$$s_v^- = \langle -\tfrac{7}{9}v + \tfrac{4}{9}\sqrt{6-2v^2}\ , -\tfrac{7}{9}v + \tfrac{4}{9}\sqrt{6-2v^2}\ , -1 + \tfrac{8}{9}v + \tfrac{7}{9}\sqrt{6-2v^2}\ \rangle$$

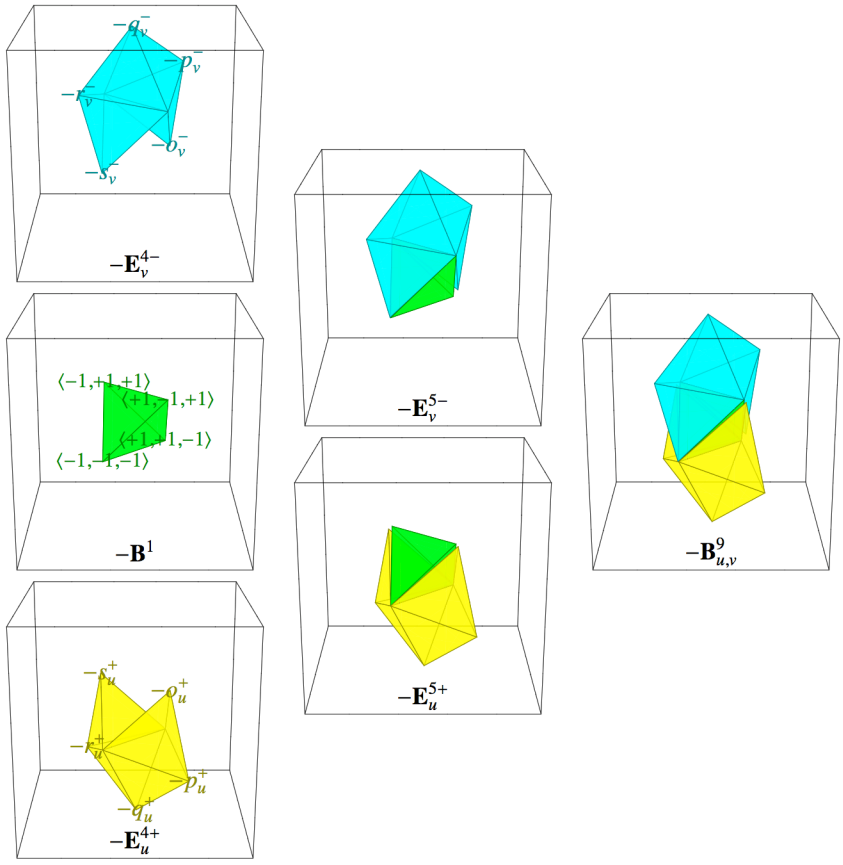


Fig. 1$^-_{\text{side}}$ The negative cluster $-\mathbf{B}_{u,v}^{9}$ (*side view*)

Inside the cluster $\mathbf{E}_u^{5+}$, there is a small gap
around the 'upper edge' $E[\langle+1,+1,+1\rangle,\langle-1,-1,+1\rangle]$
→ the cluster $\mathbf{E}_u^{4+}$ can rotate about $\mathbf{B}^1$, by small angles

u parametrizes the $\langle x,-y\rangle$·coordinates of q_u^+ (indirectly, the angle)
→ the endpoints of u are determined by intersections

$o_u^+ = \langle-1,+1,-1\rangle \to u = -\frac{1}{9}$
$s_u^+ = \langle+1,-1,-1\rangle \to u = +\frac{1}{9}$

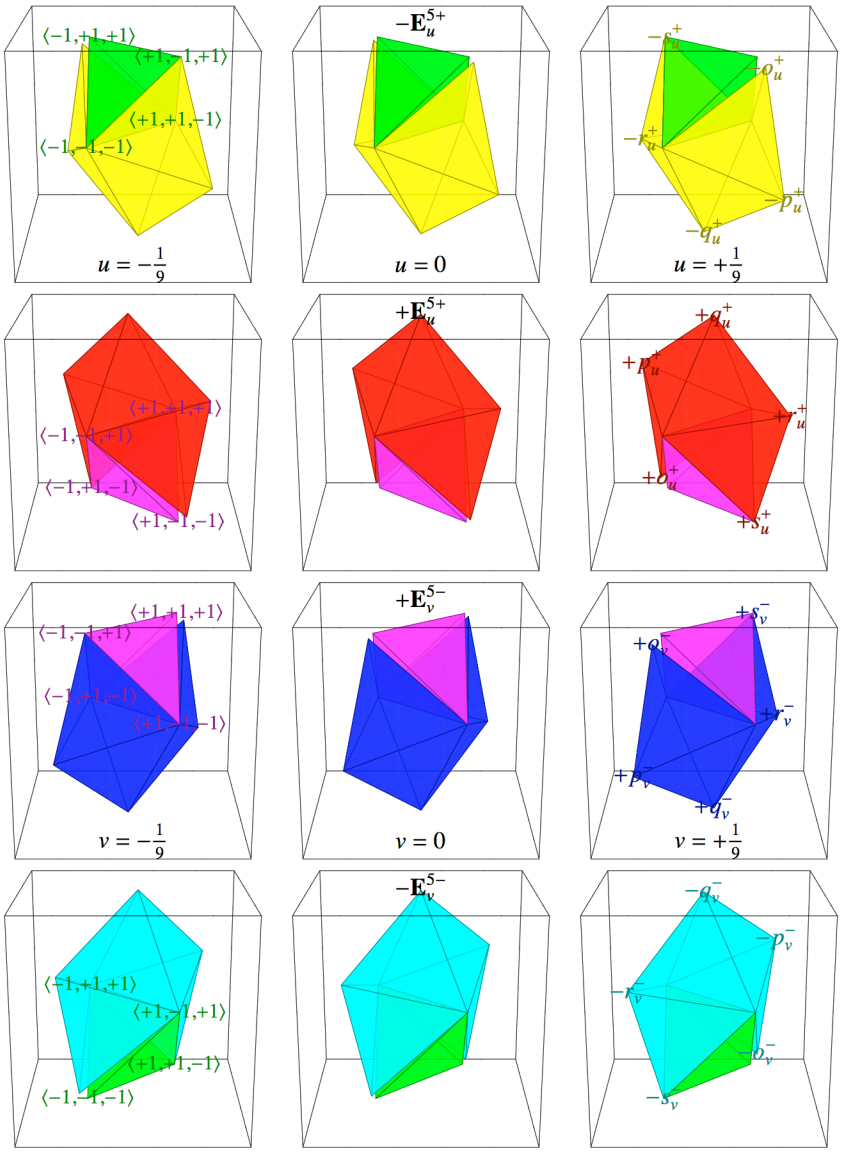


Fig. 2$_{\text{side}}$ The swivel parameters $\langle u,v\rangle$ (*side view*)

Inside the cluster $\mathbf{E}_v^{5-}$, there is a small gap
around the 'lower edge' $E[\langle+1,-1,-1\rangle,\langle-1,+1,-1\rangle]$
→ the cluster $\mathbf{E}_v^{4-}$ can rotate about $\mathbf{B}^1$, by small angles

v parametrizes the $\langle x,+y\rangle$·coordinates of q_v^- (indirectly, the angle)
→ the endpoints of v are determined by intersections

$o_v^- = \langle-1,-1,+1\rangle \rightarrow v = -\frac{1}{9}$
$s_v^- = \langle+1,+1,+1\rangle \rightarrow v = +\frac{1}{9}$

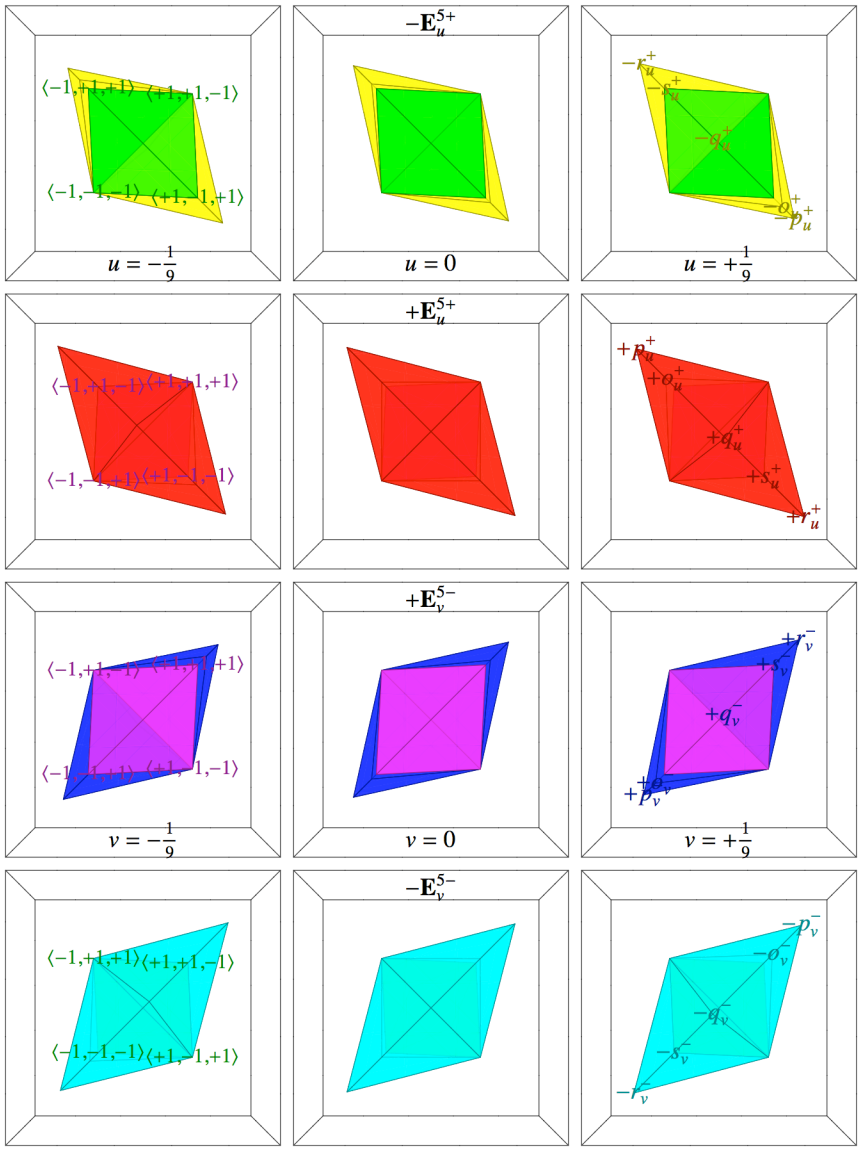


Fig. 2$_{\text{top}}$ The swivel parameters $\langle u,v\rangle$ (*top view*)

Clusters pack very efficiently in layers
the 2D·lattice $\mathbf{L}^2_{u,v}$ has almost·square basis $\{2a,2b\}$

Each layer is identical to any other layer (under some isometry)
→ we need to examine only the layer at the origin, and its neighbors

$z > 0$: 'above layer' at offset c
$z = 0$: 'central layer' at offset 0
$z < 0$: 'below layer' at offset d

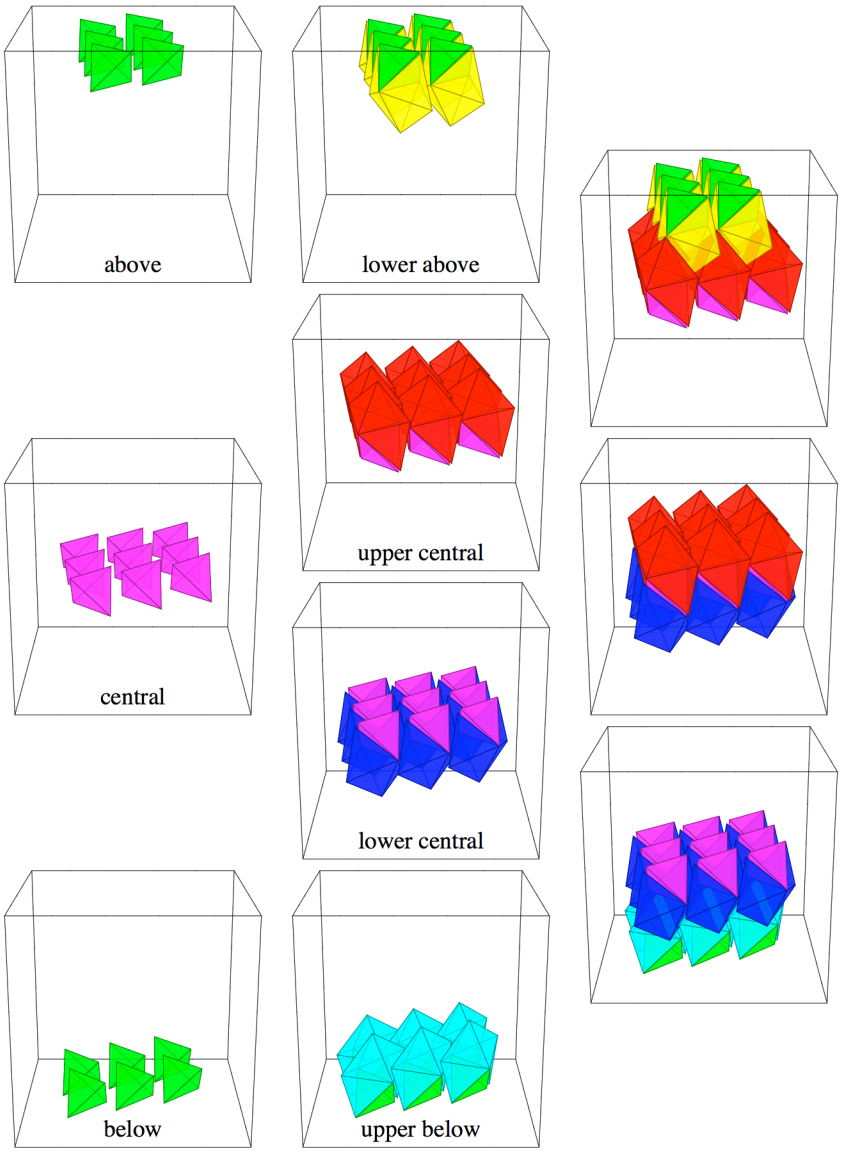


Fig. 3$_{\text{side}}$ The packing $\mathbf{P}_{u,v}$ (*side view*)

→ the 3D·lattice $\mathbf{L}^3_{u,v}$ has basis $\{2a,2b,c-d\}$

Layers have 2 possible orientations:

‘positive’ layers contain ‘positive’ clusters

‘negative’ layers contain ‘negative’ clusters

Adjacent layers have alternating orientations

→ the packing $\mathbf{P}_{u,v}$ has 2 cosets, corresponding to the 2 orientations

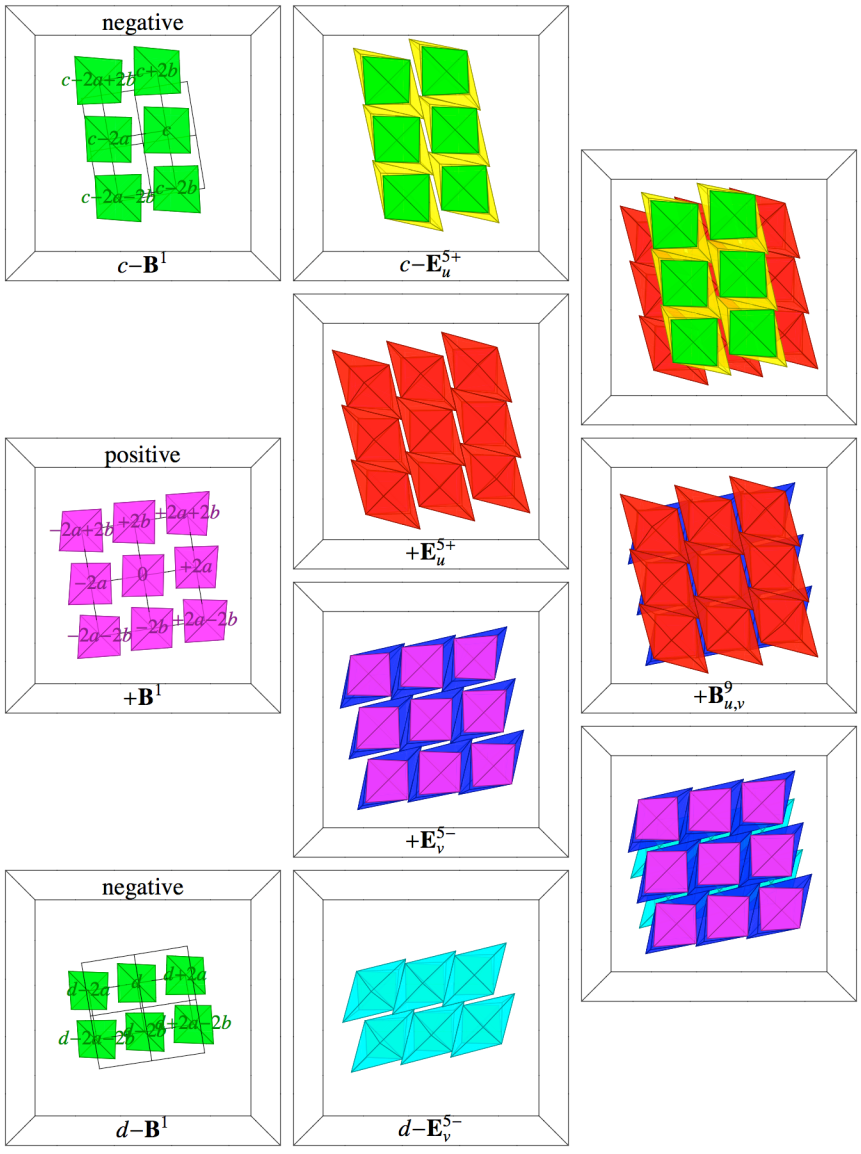


Fig. 3$_{\text{top}}$ The packing $\mathbf{P}_{u,v}$ (*top view*)

Planes separate
the half-cluster $\mathbf{E}_u^{5+}$ from
all of its neighboring
half-clusters $\mathbf{E}_u^{5+}$ & $\mathbf{E}_v^{5-}$
→ there is no overlap

$\mathbf{B}_{u,v}^{9} = \mathbf{E}_u^{5+} \bigcup \mathbf{E}_v^{5-}$
→ planes separate
the cluster $\mathbf{B}_{u,v}^{9}$ from
all of its neighboring
clusters $\mathbf{B}_{u,v}^{9}$

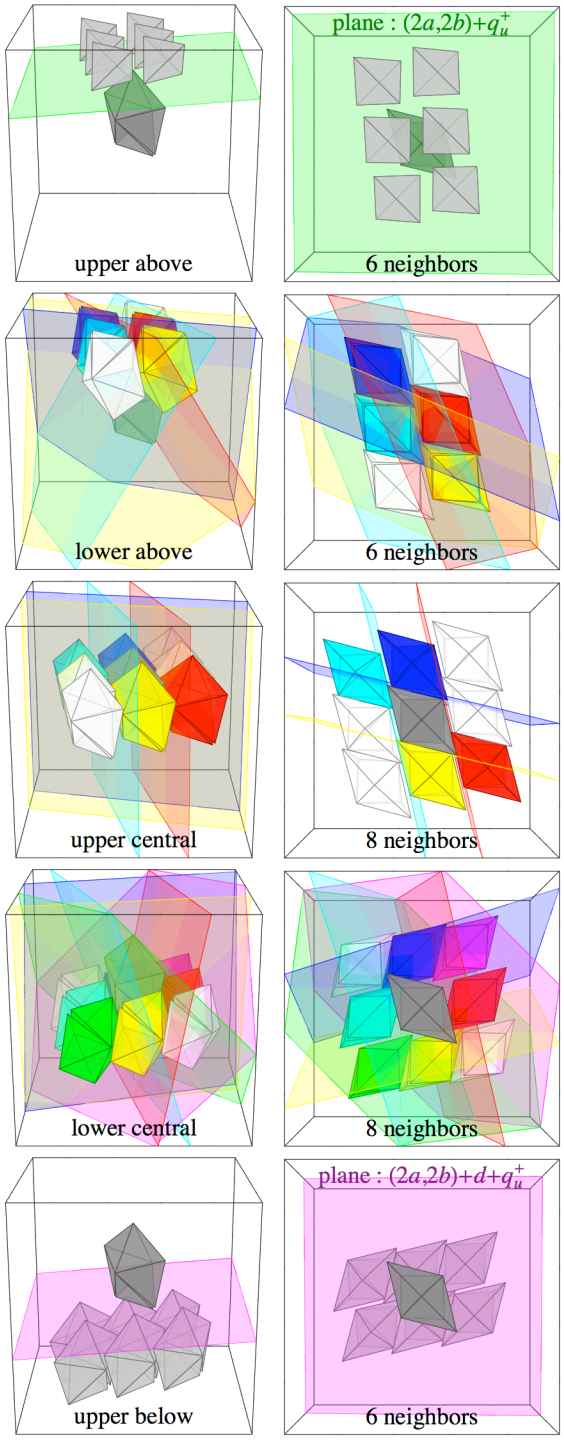


Fig. 4$^+$ The neighbors & separating planes of the upper half-cluster $\mathbf{E}_u^{5+}$

Planes separate
the half-cluster $\mathbf{E}_v^{5-}$ from
all of its neighboring
half-clusters $\mathbf{E}_u^{5+}$ & $\mathbf{E}_v^{5-}$
→ there is no overlap

→ we have a packing!

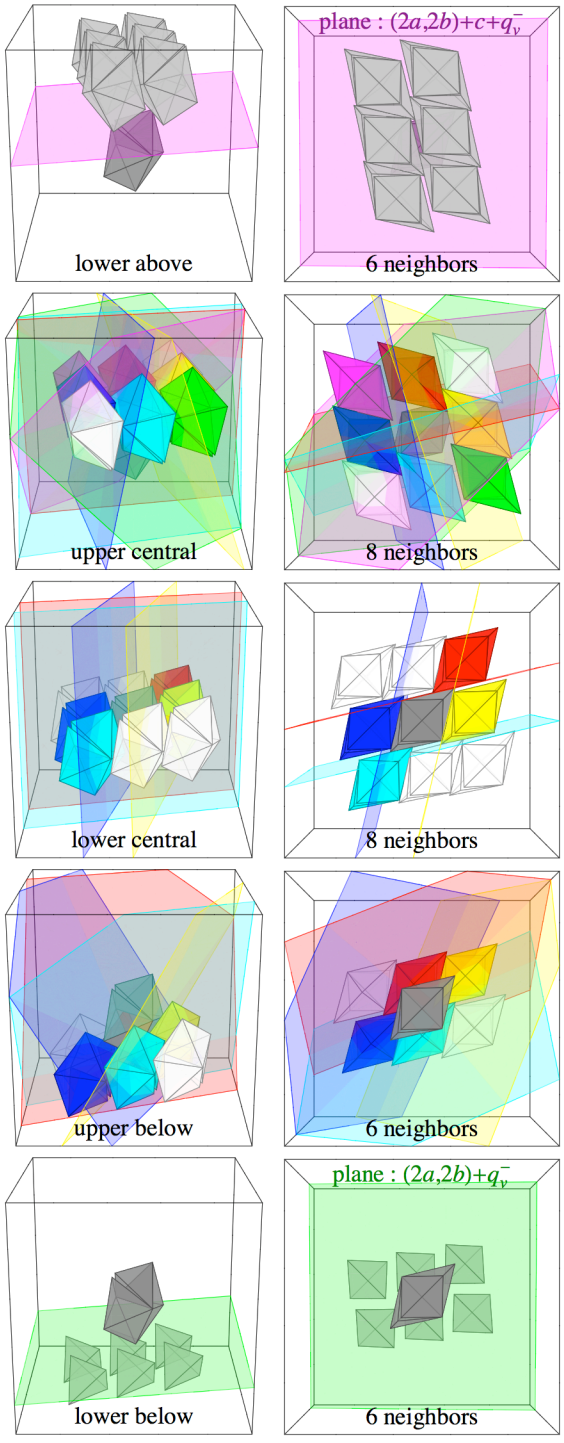
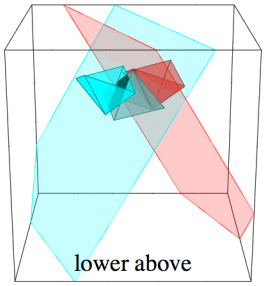
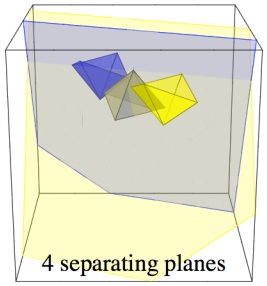


Fig. 4⁻ The neighbors & separating planes of the lower half-cluster $\mathbf{E}_v^{5-}$

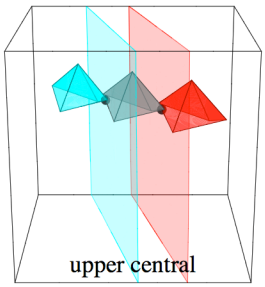


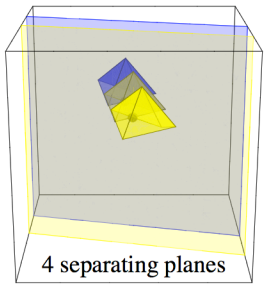


The faces between the ‘upper central’ and the ‘lower above’ layers have opposite orientation (point reflection = scalar mult. by −1)
→ the faces are parallel
→ the intersections are partial face↔face

$\mathbf{K}_0^+$	plane thru $F[\langle+1,+1,+1\rangle,q_u^+,r_u^+],c-F[\langle+1,+1,+1\rangle,q_u^+,r_u^+]$	sep $\mathbf{E}_u^{5+},c-\mathbf{E}_u^{5+}$
$\mathbf{K}_a^+$	plane thru $F[\langle-1,-1,+1\rangle,q_u^+,p_u^+],c-F[\langle-1,-1,+1\rangle,q_u^+,p_u^+]-2a$	sep $\mathbf{E}_u^{5+},c-\mathbf{E}_u^{5+}-2a$
$\mathbf{K}_b^+$	plane thru $F[\langle-1,-1,+1\rangle,q_u^+,r_u^+],c-F[\langle-1,-1,+1\rangle,q_u^+,r_u^+]-2b$	sep $\mathbf{E}_u^{5+},c-\mathbf{E}_u^{5+}-2b$
$\mathbf{K}_{ab}^+$	plane thru $F[\langle+1,+1,+1\rangle,q_u^+,p_u^+] \parallel c-F[\langle+1,+1,+1\rangle,q_u^+,p_u^+]-2a+2b$	sep $\mathbf{E}_u^{5+},c-\mathbf{E}_u^{5+}-2a+2b$

plane containing (the union of) intersecting faces is well-defined
plane parallel to parallel faces has 1 degree of freedom between the faces

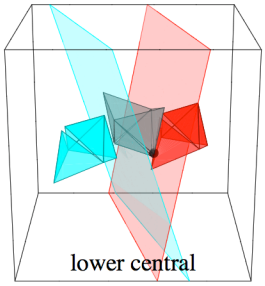


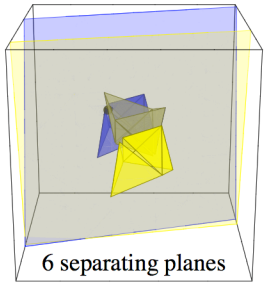


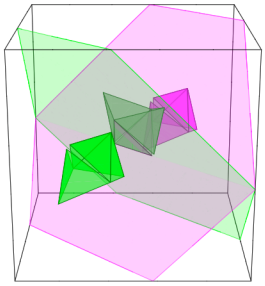
Each cluster is congruent to any other cluster (under some isometry)
→ we need to examine only the half-cluster $\mathbf{E}_u^{5+}$
22 neighbors
14 separating planes
9 intersection points
7 equations

$\mathbf{K}_{+a-b}^0$	plane thru $E[\langle+1,+1,+1\rangle,r_u^+],E[\langle-1,-1,+1\rangle,p_u^+]+2a-2b$	sep $\mathbf{E}_u^{5+},\mathbf{E}_u^{5+}+2a-2b$
$\mathbf{K}_{-a+b}^0$	plane thru $E[\langle-1,-1,+1\rangle,p_u^+],E[\langle+1,+1,+1\rangle,r_u^+]-2a+2b$	sep $\mathbf{E}_u^{5+},\mathbf{E}_u^{5+}-2a+2b$
$\mathbf{K}_{-b}^0$	plane thru $E[\langle-1,-1,+1\rangle,r_u^+],E[\langle+1,+1,+1\rangle,p_u^+]-2b$	sep $\mathbf{E}_u^{5+},\mathbf{E}_u^{5+}-2b$
$\mathbf{K}_{+b}^0$	plane thru $E[\langle+1,+1,+1\rangle,p_u^+],E[\langle-1,-1,+1\rangle,r_u^+]+2b$	sep $\mathbf{E}_u^{5+},\mathbf{E}_u^{5+}+2b$

plane containing (the union of) intersecting edges is well-defined
plane parallel to skew edges has 1 degree of freedom between the edges

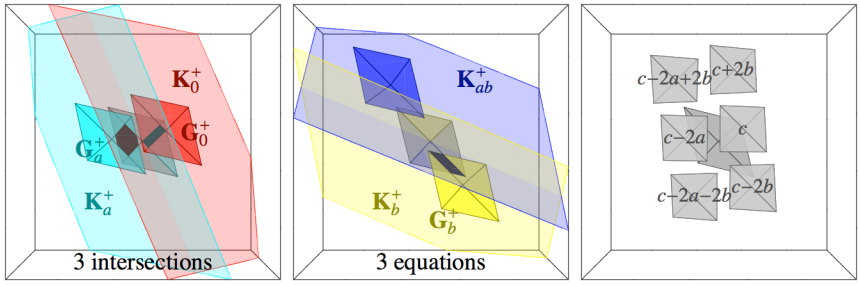




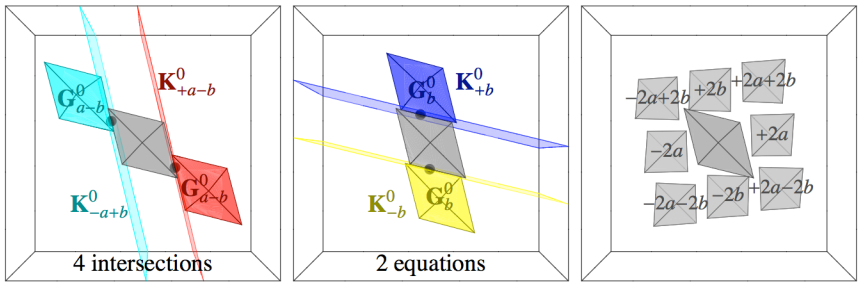


$\mathbf{L}_{ab}^-$	plane thru $E[\langle+1,+1,+1\rangle,s_u^+],E[p_v^-,o_v^-]+2a$	sep $\mathbf{E}_u^{5+},\mathbf{E}_v^{5-}+2a$
$\mathbf{L}_a^-$	plane thru $E[\langle-1,-1,+1\rangle,o_u^+] \parallel E[r_v^-,s_v^-]-2a$	sep $\mathbf{E}_u^{5+},\mathbf{E}_v^{5-}-2a$
$\mathbf{L}_b^+$	plane thru $E[r_u^+,s_u^+] \parallel E[\langle-1,+1,-1\rangle,s_v^-]-2b$	sep $\mathbf{E}_u^{5+},\mathbf{E}_v^{5-}-2b$
$\mathbf{L}_{ab}^+$	plane thru $E[p_u^+,o_u^+],E[\langle+1,-1,-1\rangle,o_v^-]+2b$	sep $\mathbf{E}_u^{5+},\mathbf{E}_v^{5-}+2b$
$\mathbf{L}_{-a-b}^0$	plane thru $F[\langle-1,-1,+1\rangle,\langle-1,+1,-1\rangle,\langle+1,-1,-1\rangle]$	sep $\mathbf{E}_u^{5+},\mathbf{E}_v^{5-}-2a-2b$
$\mathbf{L}_{+a+b}^0$	plane thru $F[\langle+1,+1,+1\rangle,\langle+1,-1,-1\rangle,\langle-1,+1,-1\rangle]$	sep $\mathbf{E}_u^{5+},\mathbf{E}_v^{5-}+2a+2b$

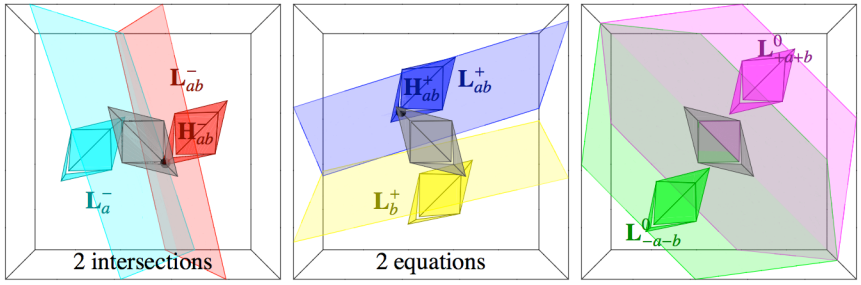
Fig. 5_{side}^+ The intersections & separating planes of $\mathbf{E}_u^{5+}$ (*side view*)



$\mathbf{G}_0^+$ $F[\langle+1,+1,+1\rangle,q_u^+,r_u^+] = c - F[\langle+1,+1,+1\rangle,q_u^+,r_u^+]$
$\mathbf{G}_a^+$ $F[\langle-1,-1,+1\rangle,q_u^+,p_u^+] = c - F[\langle-1,-1,+1\rangle,q_u^+,p_u^+] - 2a$
$\mathbf{G}_b^+$ $F[\langle-1,-1,+1\rangle,q_u^+,r_u^+] = c - F[\langle-1,-1,+1\rangle,q_u^+,r_u^+] - 2b$
$\mathbf{G}_{ab}^+$ $F[\langle+1,+1,+1\rangle,q_u^+,p_u^+] \neq c - F[\langle+1,+1,+1\rangle,q_u^+,p_u^+] - 2a + 2b$
3 intersections (partial face↔face)
1 non-intersection (parallel faces)

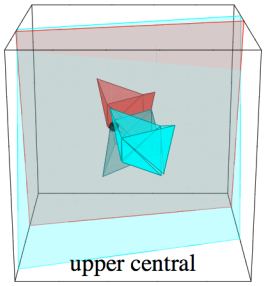


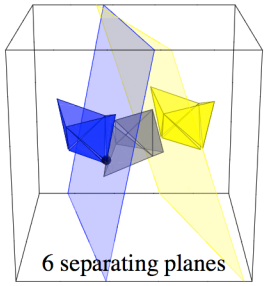
$\mathbf{G}_{a-b}^0$ $E[\langle+1,+1,+1\rangle,r_u^+] = E[\langle-1,-1,+1\rangle,p_u^+] + 2a - 2b$
$\mathbf{G}_{a-b}^0$ $E[\langle-1,-1,+1\rangle,p_u^+] = E[\langle+1,+1,+1\rangle,r_u^+] - 2a + 2b$
$\mathbf{G}_b^0$ $E[\langle-1,-1,+1\rangle,r_u^+] = E[\langle+1,+1,+1\rangle,p_u^+] - 2b$
$\mathbf{G}_b^0$ $E[\langle+1,+1,+1\rangle,p_u^+] = E[\langle-1,-1,+1\rangle,r_u^+] + 2b$
4 intersections (transverse edge↔edge)
0 non-intersections (skew edges)

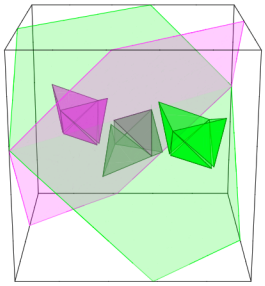


$\mathbf{H}_{ab}^-$ $E[\langle+1,+1,+1\rangle,s_u^+] = E[p_v^-,o_v^-] + 2a$
$\mathbf{H}_a^-$ $E[\langle-1,-1,+1\rangle,o_u^+] \neq E[r_v^-,s_v^-] - 2a$
$\mathbf{H}_b^+$ $E[r_u^+,s_u^+] \neq E[\langle-1,+1,-1\rangle,s_v^-] - 2b$
$\mathbf{H}_{ab}^+$ $E[p_u^+,o_u^+] = E[\langle+1,-1,-1\rangle,o_v^-] + 2b$
2 intersections (transverse edge↔edge)
2 non-intersections (skew edges)

Fig. 5_{top}^+ The intersections & separating planes of $\mathbf{E}_u^{5+}$ (*top view*)







$\mathbf{L}^{+}_{b}$ plane thru $E[\langle-1,+1,-1\rangle,s^{-}_{v}]$ ‖ $E[r^{+}_{u},s^{+}_{u}]+2b$ sep $\mathbf{E}^{5-}_{v},\mathbf{E}^{5+}_{u}+2b$

$\mathbf{L}^{+}_{ab}$ plane thru $E[\langle+1,-1,-1\rangle,o^{-}_{v}],E[p^{+}_{u},o^{+}_{u}]-2b$ sep $\mathbf{E}^{5-}_{v},\mathbf{E}^{5+}_{u}-2b$

$\mathbf{L}^{-}_{a}$ plane thru $E[r^{-}_{v},s^{-}_{v}]$ ‖ $E[\langle-1,-1,+1\rangle,o^{+}_{u}]+2a$ sep $\mathbf{E}^{5-}_{v},\mathbf{E}^{5+}_{u}+2a$

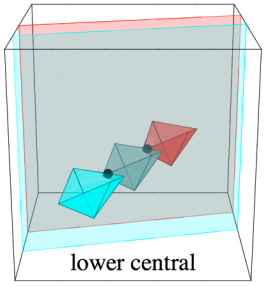
$\mathbf{L}^{-}_{ab}$ plane thru $E[p^{-}_{v},o^{-}_{v}],E[\langle+1,+1,+1\rangle,s^{+}_{u}]-2a$ sep $\mathbf{E}^{5-}_{v},\mathbf{E}^{5+}_{u}-2a$

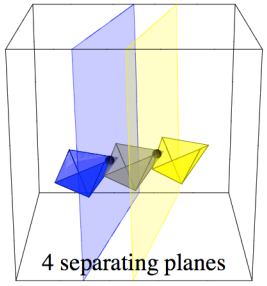
$\mathbf{L}^{0}_{+a-b}$ plane thru $F[\langle+1,-1,-1\rangle,\langle+1,+1,+1\rangle,\langle-1,-1,+1\rangle]$ sep $\mathbf{E}^{5-}_{v},\mathbf{E}^{5+}_{u}+2a-2b$

$\mathbf{L}^{0}_{-a+b}$ plane thru $F[\langle-1,+1,-1\rangle,\langle-1,-1,+1\rangle,\langle+1,+1,+1\rangle]$ sep $\mathbf{E}^{5-}_{v},\mathbf{E}^{5+}_{u}-2a+2b$

Each cluster is congruent to any other cluster (under some isometry) → we need to examine only the half-cluster $\mathbf{E}^{5-}_{v}$

22 neighbors
14 separating planes
9 intersection points
7 equations





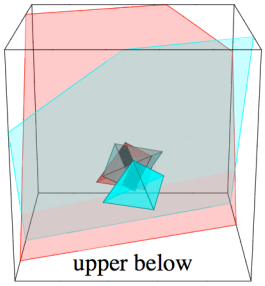
$\mathbf{K}^{0}_{+a+b}$ plane thru $E[\langle-1,+1,-1\rangle,r^{-}_{v}],E[\langle+1,-1,-1\rangle,p^{-}_{v}]+2a+2b$ sep $\mathbf{E}^{5-}_{v},\mathbf{E}^{5-}_{v}+2a+2b$

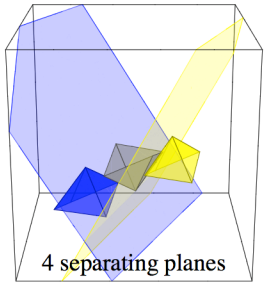
$\mathbf{K}^{0}_{-a-b}$ plane thru $E[\langle+1,-1,-1\rangle,p^{-}_{v}],E[\langle-1,+1,-1\rangle,r^{-}_{v}]-2a-2b$ sep $\mathbf{E}^{5-}_{v},\mathbf{E}^{5-}_{v}-2a-2b$

$\mathbf{K}^{0}_{+a}$ plane thru $E[\langle+1,-1,-1\rangle,r^{-}_{v}],E[\langle-1,+1,-1\rangle,p^{-}_{v}]+2a$ sep $\mathbf{E}^{5-}_{v},\mathbf{E}^{5-}_{v}+2a$

$\mathbf{K}^{0}_{-a}$ plane thru $E[\langle-1,+1,-1\rangle,p^{-}_{v}],E[\langle+1,-1,-1\rangle,r^{-}_{v}]-2a$ sep $\mathbf{E}^{5-}_{v},\mathbf{E}^{5-}_{v}-2a$

plane containing (the union of) intersecting edges is well-defined
plane parallel to skew edges has 1 degree of freedom between the edges





The faces between the ‘lower central’ and the ‘upper below’ layers have opposite orientation (point reflection = scalar mult. by −1) → the faces are parallel → the intersections are partial face↔face

$\mathbf{K}^{-}_{0}$ plane thru $F[\langle-1,+1,-1\rangle,q^{-}_{v},r^{-}_{v}],d-F[\langle-1,+1,-1\rangle,q^{-}_{v},r^{-}_{v}]$ sep $\mathbf{E}^{5-}_{v},d-\mathbf{E}^{5-}_{v}$

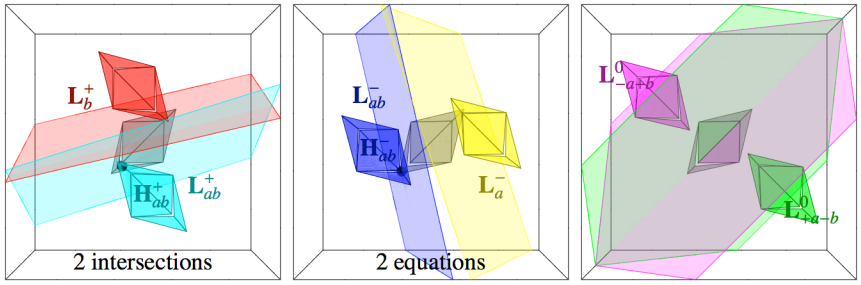
$\mathbf{K}^{-}_{b}$ plane thru $F[\langle+1,-1,-1\rangle,q^{-}_{v},p^{-}_{v}],d-F[\langle+1,-1,-1\rangle,q^{-}_{v},p^{-}_{v}]-2b$ sep $\mathbf{E}^{5-}_{v},d-\mathbf{E}^{5-}_{v}-2b$

$\mathbf{K}^{-}_{a}$ plane thru $F[\langle+1,-1,-1\rangle,q^{-}_{v},r^{-}_{v}],d-F[\langle+1,-1,-1\rangle,q^{-}_{v},r^{-}_{v}]+2a$ sep $\mathbf{E}^{5-}_{v},d-\mathbf{E}^{5-}_{v}+2a$

$\mathbf{K}^{-}_{ab}$ plane thru $F[\langle-1,+1,-1\rangle,q^{-}_{v},p^{-}_{v}]$ ‖ $d-F[\langle-1,+1,-1\rangle,q^{-}_{v},p^{-}_{v}]-2a-2b$ sep $\mathbf{E}^{5-}_{v},d-\mathbf{E}^{5-}_{v}-2a-2b$

plane containing (the union of) intersecting faces is well-defined
plane parallel to parallel faces has 1 degree of freedom between the faces

Fig. 5^{-}_{side} The intersections & separating planes of $\mathbf{E}^{5-}_{v}$ (*side view*)



$\mathbf{H}_b^+$ $E[\langle -1,+1,-1\rangle, s_v^-] \neq E[r_u^+, s_u^+]+2b$

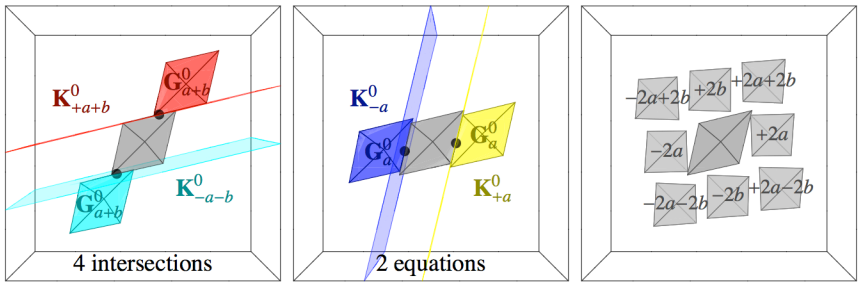
$\mathbf{H}_{ab}^+$ $E[\langle +1,-1,-1\rangle, o_v^-] = E[p_u^+, o_u^+]-2b$

$\mathbf{H}_a^-$ $E[r_v^-, s_v^-] \neq E[\langle -1,-1,+1\rangle, o_u^+]+2a$

$\mathbf{H}_{ab}^-$ $E[p_v^-, o_v^-] = E[\langle +1,+1,+1\rangle, s_u^+]-2a$

2 intersections (transverse edge↔edge)

2 non-intersections (skew edges)



$\mathbf{G}_{a+b}^0$ $E[\langle -1,+1,-1\rangle, r_v^-] = E[\langle +1,-1,-1\rangle, p_v^-]+2a+2b$

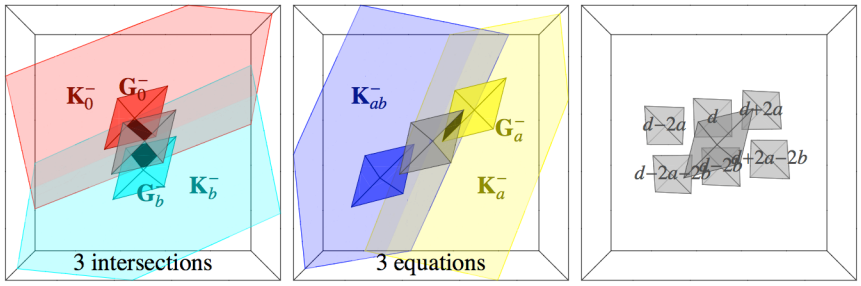
$\mathbf{G}_{a+b}^0$ $E[\langle +1,-1,-1\rangle, p_v^-] = E[\langle -1,+1,-1\rangle, r_v^-]-2a-2b$

$\mathbf{G}_a^0$ $E[\langle +1,-1,-1\rangle, r_v^-] = E[\langle -1,+1,-1\rangle, p_v^-]+2a$

$\mathbf{G}_a^0$ $E[\langle -1,+1,-1\rangle, p_v^-] = E[\langle +1,-1,-1\rangle, r_v^-]-2a$

4 intersections (transverse edge↔edge)

0 non-intersections (skew edges)



$\mathbf{G}_0^-$ $F[\langle -1,+1,-1\rangle, q_v^-, r_v^-] = d-F[\langle -1,+1,-1\rangle, q_v^-, r_v^-]$

$\mathbf{G}_b^-$ $F[\langle +1,-1,-1\rangle, q_v^-, p_v^-] = d-F[\langle +1,-1,-1\rangle, q_v^-, p_v^-]-2b$

$\mathbf{G}_a^-$ $F[\langle +1,-1,-1\rangle, q_v^-, r_v^-] = d-F[\langle +1,-1,-1\rangle, q_v^-, r_v^-]+2a$

$\mathbf{G}_{ab}^-$ $F[\langle -1,+1,-1\rangle, q_v^-, p_v^-] \neq d-F[\langle -1,+1,-1\rangle, q_v^-, p_v^-]-2a-2b$

3 intersections (partial face↔face)

1 non-intersection (parallel faces)

Fig. $\mathbf{5}_{\text{top}}^-$ The intersections & separating planes of $\mathbf{E}_v^{5-}$ (*top view*)

4 Proof by Computation

4.1 Numerical results

We use the following procedure to compute the maximal optimized packing:

1. Write abstract expressions for the virtual cluster vertices $\{o_u^+, p_u^+, q_u^+, r_u^+, s_u^+\}$, $\{o_v^-, p_v^-, q_v^-, r_v^-, s_v^-\}$, in terms of the swivel parameters $\langle u, v\rangle$. [see Fig. 1]

2. Write abstract equations for the virtual intersections between the cluster and its neighbors, in terms of the virtual cluster vertices (ultimately in terms of the swivel parameters), the lattice vectors $\{a, b, c, d\}$ and extra intersection parameters (edge-to-edge or point-in-plane). [see Fig. 5]

$$\mathbf{G} = \{\mathbf{G}_0^+, \mathbf{G}_a^+, \mathbf{G}_b^+, \mathbf{G}_0^-, \mathbf{G}_a^-, \mathbf{G}_b^-, \mathbf{G}_a^0, \mathbf{G}_b^0, \mathbf{G}_{a+b}^0, \mathbf{G}_{a-b}^0\}.$$
$$\mathbf{H} = \{\mathbf{H}_a, \mathbf{H}_b\}$$

3. Solve for the abstract lattice vectors and extra intersection parameters, in terms of the swivel parameters. Clear the common denominator.

4. Write abstract expressions for the lattice volume V & packing density D, in terms of the lattice vectors (ultimately in terms of the swivel parameters). (The fundamental domain contains 2 clusters = 18 tetrahedra. Each tetrahedron has edge length $\sqrt{8}$ and volume $\frac{8}{3}$.)

$$V = \det[2a, 2b, c - d]$$
$$D = 48/V$$

5. Find the relative minimum of the lattice volume, with respect to the swivel parameters. (We could find only a numerical approximation, but not the minimal polynomials, because the degree is too big for our computer program.)

$$\langle u, v\rangle \approx \langle -0.034789016702, +0.089604971413\rangle$$

6^+. Evaluate the vertices of the upper half-cluster.

$$o_u^+ \approx \langle -1.061384364770, +1.061384364770, -0.935697925928\rangle$$
$$p_u^+ \approx \langle -1.644260072209, +1.644260072209, +1.769946511051\rangle$$
$$q_u^+ \approx \langle -0.034789016702, +0.034789016702, +3.448995599962\rangle$$
$$r_u^+ \approx \langle +1.621067394407, -1.621067394407, +1.862717222257\rangle$$
$$s_u^+ \approx \langle +1.115500612974, -1.115500612974, -0.873850785124\rangle$$

6^-. Evaluate the vertices of the lower half-cluster.

$$o_v^- \approx \langle -1.156897066820, -1.156897066820, +0.822958681256\rangle$$
$$p_v^- \approx \langle -1.600938143111, -1.600938143111, -1.934876528675\rangle$$
$$q_v^- \approx \langle +0.089604971413, +0.089604971413, -3.446209700372\rangle$$
$$r_v^- \approx \langle +1.660674790719, +1.660674790719, -1.695929938240\rangle$$
$$s_v^- \approx \langle +1.017511555733, +1.017511555733, +0.982256408212\rangle$$

7. Evaluate the extra intersection parameters and intersection points, in order to verify that the intersections are valid (i.e.: parameter values between 0 and 1).

8. Evaluate the lattice vectors.

$$a \approx \langle +1.286101228632, +0.200477756509, +0.117304750804 \rangle$$
$$b \approx \langle -0.216654244567, +1.293299854677, +0.064005626691 \rangle$$
$$c \approx \langle +1.049828831839, +0.572626599359, +4.643104387948 \rangle$$
$$d \approx \langle -0.331247602500, +1.312711170924, -4.452573714586 \rangle$$

9. Evaluate the lattice volume & packing density.

$$V \approx 61.647870634123$$
$$D \approx 0.778615700855$$

4.2 The actual density function D [see Fig. 7]

For each cluster $\mathbf{B}^9_{u,v}$, we compute the optimized packing $\mathbf{P}_{u,v}$ and packing density $\mathbf{D}_{u,v}$. The set of all packing densities $\{\mathbf{D}_{u,v} : -\frac{1}{9} \le u \le +\frac{1}{9}, -\frac{1}{9} \le v \le +\frac{1}{9}\}$ forms the 'actual' density function $\mathbf{D}$. The range of the graphs is $+.7767 \le w \le +.7787$ for Fig. 7 and $+.7675 \le w \le +.78$ for Fig. 8.

All packings in the family satisfy the set of equations $\mathbf{G}$, which gives all packings the same geometric structure and lattice type. (For small values of the swivel parameters $\langle u, v \rangle$, the intersections from the equations $\mathbf{G}$ remain valid.)

However, when we try to find the maximal optimized packing using only the equations $\mathbf{G}$, there are 2 places where the clusters overlap. In this case, we need to impose the 2 additional equations $\mathbf{H}$ in order to prevent overlap.

For a general cluster $\mathbf{B}^9_{u,v}$, only a subset of the equations $\mathbf{H}$ is necessary. So there are 4 possible optimized packings using the equations $\mathbf{G}$, $\mathbf{G} \cup \{\mathbf{H}_a\}$, $\mathbf{G} \cup \{\mathbf{H}_b\}$, $\mathbf{G} \cup \{\mathbf{H}_a, \mathbf{H}_b\}$, which gives us 4 possible packing densities $\mathbf{D}^0_{u,v}$, $\mathbf{D}^a_{u,v}$, $\mathbf{D}^b_{u,v}$, $\mathbf{D}^{ab}_{u,v}$.

A priori, we don't know which set of equations to use for a given cluster, or which clusters satisfy a given set of equations. So we naïvely assume that all 4 possibilities are valid for all clusters, which gives us 4 'virtual' density functions $\mathbf{D}^0$, $\mathbf{D}^a$, $\mathbf{D}^b$, $\mathbf{D}^{ab}$.

4.3 The virtual density functions $\mathbf{D}^0, \mathbf{D}^a, \mathbf{D}^b, \mathbf{D}^{ab}$ [see Fig. 8]

For each swivel parameter value $\langle u, v \rangle$, we find the 'valid' virtual density function (the set of equations which gives the densest non-overlapping packing for the given cluster). Conversely, for each virtual density function, we find the 'valid' region of swivel parameter values (the set of clusters having densest non-overlapping packings which satsify the given equations).

The 4 virtual density functions $\mathbf{D}^0, \mathbf{D}^a, \mathbf{D}^b, \mathbf{D}^{ab}$ are smooth, and their surfaces are pairwise tangent along their intersection curves $\mathbf{D}^0 \cap \mathbf{D}^a, \mathbf{D}^0 \cap \mathbf{D}^b, \mathbf{D}^a \cap \mathbf{D}^{ab}, \mathbf{D}^b \cap \mathbf{D}^{ab}$. All 4 surfaces are mutually tangent at their common intersection point $\mathbf{D}^0 \cap \mathbf{D}^a \cap \mathbf{D}^b \cap \mathbf{D}^{ab}$, which is also the common intersection point of the 4 intersection curves.

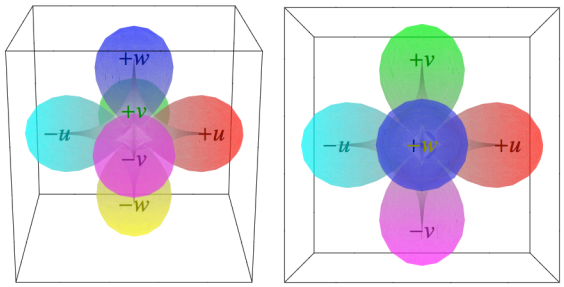


Fig. 6 The $\langle u,v,w\rangle$ coordinate axes (*side view* & *top view*), for Figs. 7–8

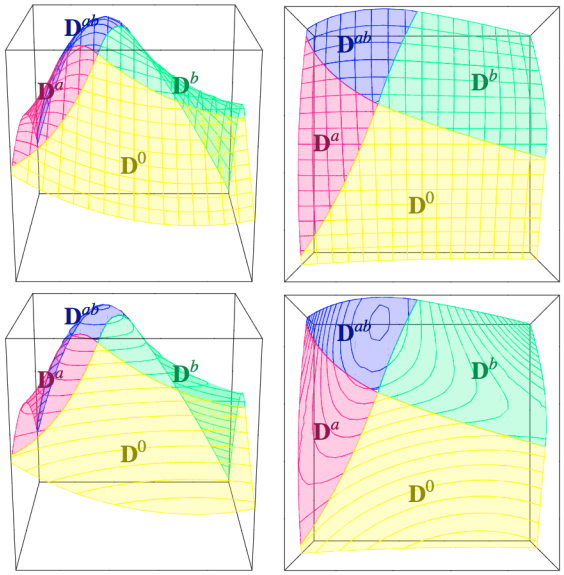


Fig. 7 The composite density function **D** with $\langle u,v\rangle$-traces & w-contours

The actual density function **D** consists of the union of the 4 valid regions of the virtual density functions, which are bounded by the intersection curves. The 4 valid regions are pairwise tangent at the intersection curves, and mutually tangent at their common intersection point. Thus, the actual density function is also smooth.

The common intersection point occurs at

$$\langle u,v,w\rangle \approx \langle -0.037320921073, +0.033926596665, +0.778365087767\rangle.$$

The relative maximum of the actual density function **D** occurs at

$$\langle u,v,w\rangle \approx \langle -0.034789016702, +0.089604971413, +0.778615700855\rangle.$$

- Figure 6 shows the orientation of the $\langle u,v,w\rangle$ coordinate axes (for Figs. 7-8).
- Figure 7 shows the actual density function **D**.
- Figure 8 (2 pages) shows the virtual density functions $\mathbf{D}^0, \mathbf{D}^a, \mathbf{D}^b, \mathbf{D}^{ab}$.

4.4 The highly symmetric cluster $\mathbf{B}^9_{0,0}$ and packing $\mathbf{P}_{\text{sym}}$

The highly symmetric cluster $\mathbf{B}^9_{0,0}$ has symmetry group $\mathbf{I}_{x,y,z}$.

$$\mathbf{I}_{x,y,z} = \{I, P, P^2, P^3, Q, QP, QP^2, QP^3\},\ P = \begin{bmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},\ Q = \begin{bmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The highly symmetric packing $\mathbf{P}_{\text{sym}}$ also has symmetry group $\mathbf{I}_{x,y,z}$. The lattice has a square basis, with lattice vectors of the form

$$\begin{aligned} a &= \langle +2i, +2j, 0 \rangle \\ b &= \langle -2j, +2i, 0 \rangle \\ c &= \langle +i, +j, +k \rangle \\ d &= \langle -j, +i, -k \rangle, \end{aligned}$$

$$\begin{aligned} i &= \tfrac{1}{71}(-168 + 106\sqrt{6}) \approx 1.290787503310 \\ j &= \tfrac{1}{71}(-88 + 42\sqrt{6}) \approx 0.209557312632 \\ k &= \tfrac{1}{71}(-262 + 238\sqrt{6}) \approx 4.520824771583 \end{aligned}$$

and lattice volume & packing density

$$\begin{aligned} V &= \tfrac{1}{357911}(-730200320 + 307139840\sqrt{6}) \approx 61.846569901642 \\ D &= \tfrac{1}{4477040}(1711407 + 719859\sqrt{6}) \approx .776114181859. \end{aligned}$$

The highly symmetric packing $\mathbf{P}_{\text{sym}}$ is *not* a member of the family $\{\mathbf{P}_{u,v}\}$, because adjacent layers have only 2 face-to-face intersections.

The optimized packing $\mathbf{P}_{0,0}$ has trivial symmetry group. As with all members of the family, adjacent layers have 3 face-to-face intersections. In order to get this additional intersection, we need to shift adjacent layers by a small distance (relative to the lattice), which destroys symmetry.

4.5 Isometries of the cluster $\mathbf{B}^9_{u,v}$ and packing $\mathbf{P}_{u,v}$ [see Fig. 9]

We can apply the isometries $\mathbf{I}_{x,y,z}$ in several ways:

1. Apply the isometries to the optimized packing $\mathbf{P}_{u,v}$ (which also applies the isometries to the clusters and intersection equations). The 8 clusters $\mathbf{I}_{x,y,z} \cdot \mathbf{B}^9_{u,v}$ are congruent to each other, the 8 optimized packings $\mathbf{I}_{x,y,z} \cdot \mathbf{P}_{u,v}$ are congruent to each other, and the 8 optimized packing densities are equal to each other.

Each optimized packing corresponds to a point of an actual density function. The 8 different (but analogous) points have the same w-coordinate (packing density), but each point lives in a different (but congruent) surface (actual density functions $\mathbf{I}_{u,v,w} \cdot \mathbf{D}$). (The isometries $\mathbf{I}_{u,v,w}$ are similar to the isometries $\mathbf{I}_{x,y,z}$, except that the w-coordinate (packing density) is always positive.)

$$\mathbf{I}_{u,v,w} = \{I, R, R^2, R^3, S, SR, SR^2, SR^3\},\ R = \begin{bmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & +1 \end{bmatrix},\ S = \begin{bmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{bmatrix}$$

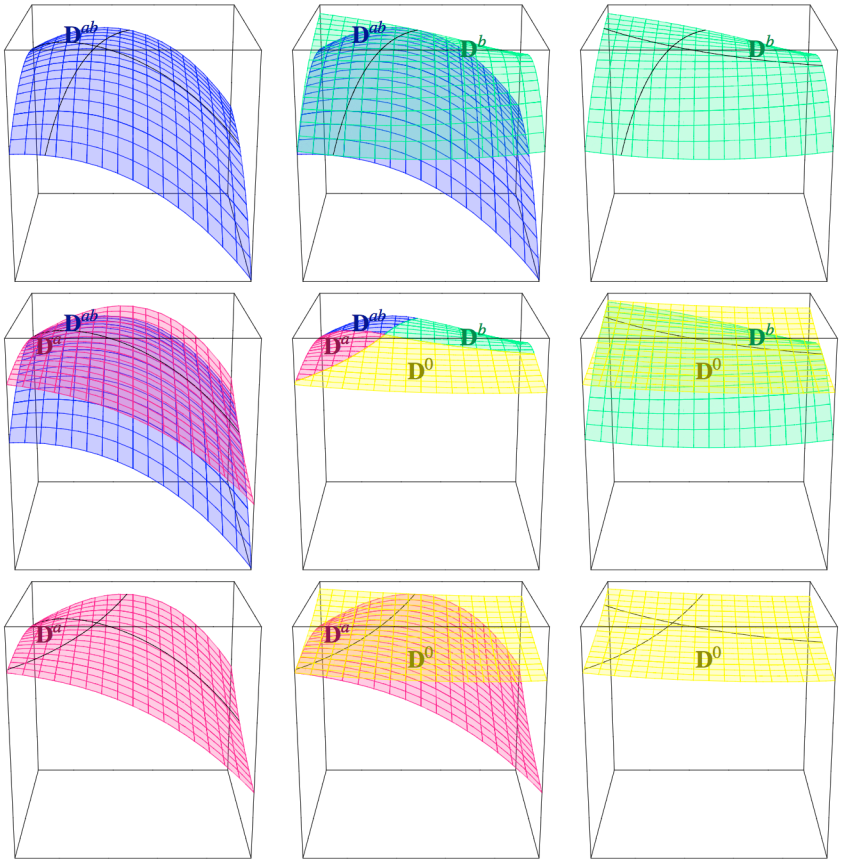


Fig 8$^{u,v}_{\text{side}}$ The virtual density functions $\mathbf{D}^0$,$\mathbf{D}^a$,$\mathbf{D}^b$,$\mathbf{D}^{ab}$ with $\langle u,v\rangle$·traces (*side view*)

2. Apply the isometries to the cluster $\mathbf{B}^9_{u,v}$, but fix the intersection equations $\mathbf{G}\cup\mathbf{H}$. The 8 clusters $\mathbf{I}_{x,y,z}\cdot\mathbf{B}^9_{u,v}$ are identical to

$$\{\mathbf{B}^9_{+u,+v},\mathbf{B}^9_{-v,+u},\mathbf{B}^9_{-u,-v},\mathbf{B}^9_{+v,-u},\mathbf{B}^9_{+v,+u},\mathbf{B}^9_{+u,-v},\mathbf{B}^9_{-v,-u},\mathbf{B}^9_{-u,+v}\}$$

and congruent to each other. The 8 optimized packings

$$\{\mathbf{P}_{+u,+v},\mathbf{P}_{-v,+u},\mathbf{P}_{-u,-v},\mathbf{P}_{+v,-u},\mathbf{P}_{+v,+u},\mathbf{P}_{+u,-v},\mathbf{P}_{-v,-u},\mathbf{P}_{-u,+v}\}$$

are *not* congruent to each other, and the 8 optimized packing densities

$$\{\mathbf{D}_{+u,+v},\mathbf{D}_{-v,+u},\mathbf{D}_{-u,-v},\mathbf{D}_{+v,-u},\mathbf{D}_{+v,+u},\mathbf{D}_{+u,-v},\mathbf{D}_{-v,-u},\mathbf{D}_{-u,+v}\}$$

are *not* equal to each other.

Each optimized packing corresponds to a point of an actual density function. All 8 points live in the same surface (actual density function $\mathbf{D}$), but each point has a different w-coordinate (packing density).

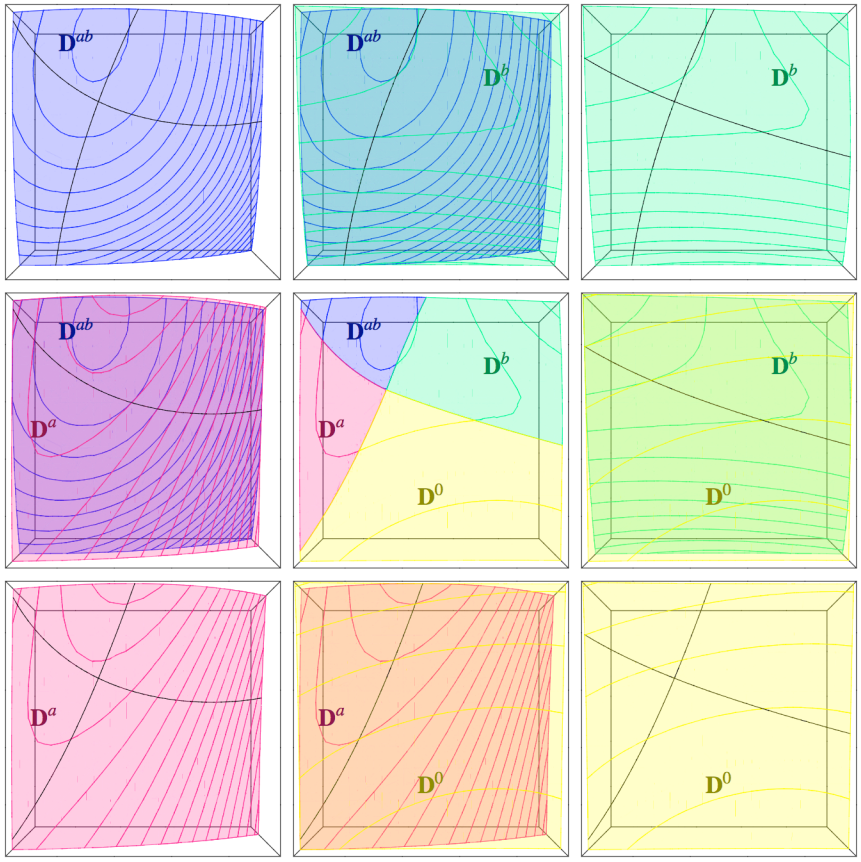


Fig 8$^{w}_{\text{top}}$ The virtual density functions $\mathbf{D}^0$,$\mathbf{D}^a$,$\mathbf{D}^b$,$\mathbf{D}^{ab}$ with w-contours (*top view*)

3. Apply the isometries to the intersection equations $\mathbf{G}\cup\mathbf{H}$, but fix the cluster $\mathbf{B}^9_{u,v}$. This is equivalent to fixing the intersection equations and applying the inverse isometries to the cluster. Since $\mathbf{I}_{x,y,z}$ is a group, the set of packings & densities is identical.

You could also apply the isometries individually to each vector quantity in the equations. For the virtual cluster vertices $\{o_u^+, p_u^+, q_u^+, r_u^+, s_u^+\}$, $\{o_v^-, p_v^-, q_v^-, r_v^-, s_v^-\}$, use the highly symmetric cluster $\mathbf{B}_{0,0}$ as a paradigm. For the abstract lattice vectors $\{a,b,c,d\}$, use the highly symmetric packing $\mathbf{P}_{\text{sym}}$ as a paradigm.

In any case, we don't get any additional packings by considering additional isometries of the cluster, intersection equations or packings. Thus, it suffices to optimize over a single generic cluster $\mathbf{B}^9_{u,v}$, a single set of equations $\mathbf{G}\cup\mathbf{H}$, and a single actual density function $\mathbf{D}$.

The 8 congruences of the cluster $\mathbf{B}^9_{u,v}$ are symmetric (about $\langle u,v\rangle = \langle 0,0\rangle$) as a whole, so the 8 sets of intersection equations $\mathbf{G}\cup\mathbf{H}$ are symmetric as a whole, and the 8 congruences of the density function $\mathbf{D}$ are symmetric as a whole. However, any single set of intersection equations is individually asymmetric, so any single density function is individually asymmetric.

• Figure 9 (2 pages) shows the isometries of the cluster $\mathbf{B}^9_{u,v}$ and packing $\mathbf{P}_{u,v}$.

$$\mathbf{B}^9_{+u,+v} = I\cdot\mathbf{B}^9_{u,v} \qquad \mathbf{B}^9_{+v,-u} = P^3\cdot\mathbf{B}^9_{u,v} \qquad \mathbf{B}^9_{+u,-v} = QP\cdot\mathbf{B}^9_{u,v} \qquad \mathbf{B}^9_{-v,-u} = QP^2\cdot\mathbf{B}^9_{u,v}$$
$$\mathbf{P}_{+u,+v} = I\cdot\mathbf{P}_{u,v} \qquad \mathbf{P}_{+v,-u} \neq P^3\cdot\mathbf{P}_{u,v} \qquad \mathbf{P}_{+u,-v} \neq QP\cdot\mathbf{P}_{u,v} \qquad \mathbf{P}_{-v,-u} \neq QP^2\cdot\mathbf{P}_{u,v}$$

$$I_{x,y,z} = \begin{pmatrix} +1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & +1 \end{pmatrix} \quad P^3_{x,y,z} = \begin{pmatrix} 0 & +1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad QP_{x,y,z} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & +1 \end{pmatrix} \quad QP^2_{x,y,z} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
$$I_{u,v,w} = \begin{pmatrix} +1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & +1 \end{pmatrix} \quad R^3_{u,v,w} = \begin{pmatrix} 0 & +1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & +1 \end{pmatrix} \quad SR_{u,v,w} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & +1 \end{pmatrix} \quad SR^2_{u,v,w} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & +1 \end{pmatrix}$$

$$\mathbf{B}^9_{-v,+u} = P\cdot\mathbf{B}^9_{u,v} \qquad \mathbf{B}^9_{-u,-v} = P^2\cdot\mathbf{B}^9_{u,v} \qquad \mathbf{B}^9_{+v,+u} = Q\cdot\mathbf{B}^9_{u,v} \qquad \mathbf{B}^9_{-u,+v} = QP^3\cdot\mathbf{B}^9_{u,v}$$
$$\mathbf{P}_{-v,+u} \neq P\cdot\mathbf{P}_{u,v} \qquad \mathbf{P}_{-u,-v} \neq P^2\cdot\mathbf{P}_{u,v} \qquad \mathbf{P}_{+v,+u} \neq Q\cdot\mathbf{P}_{u,v} \qquad \mathbf{P}_{-u,+v} \neq QP^3\cdot\mathbf{P}_{u,v}$$

$$P_{x,y,z} = \begin{pmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad P^2_{x,y,z} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{pmatrix} \quad Q_{x,y,z} = \begin{pmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad QP^3_{x,y,z} = \begin{pmatrix} 0 & +1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & +1 \end{pmatrix}$$
$$R_{u,v,w} = \begin{pmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & +1 \end{pmatrix} \quad R^2_{u,v,w} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{pmatrix} \quad S_{u,v,w} = \begin{pmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{pmatrix} \quad SR^3_{u,v,w} = \begin{pmatrix} 0 & +1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & +1 \end{pmatrix}$$

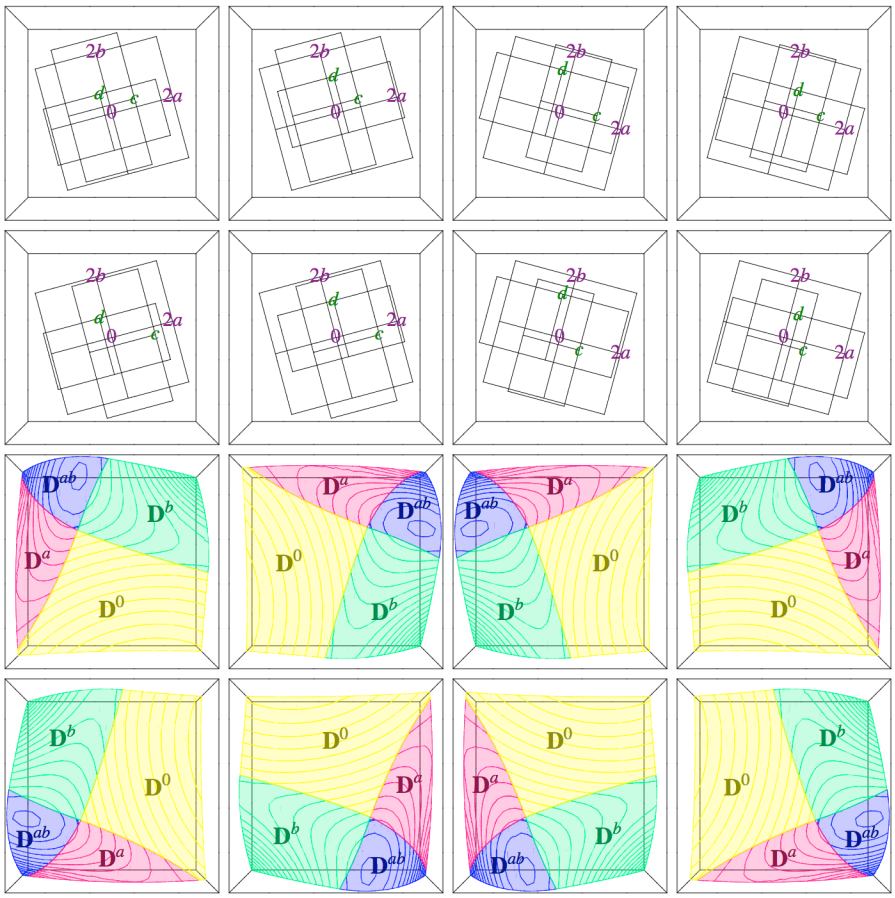


Fig. 9$^{\mathbf{B,P,D}}$ The 8 congruences of the cluster $\mathbf{B}^9_{u,v}$, packing $\mathbf{P}_{u,v}$ & density function $\mathbf{D}$ under the isometries $\mathbf{I}_{x,y,z}$ & $\mathbf{I}_{u,v,w}$

Opposite face/edge pairs: face↔face non−equation $\mathbf{G}^{+}_{*}$ ≈ edge↔edge equation $\mathbf{H}^{+}_{*}$
(non−intersecting) 'far' faces at top of $\mathbf{E}^{5+}_{u}$ ≈ (intersecting) 'near' edges at bottom of $\mathbf{E}^{5+}_{u}$

$\mathbf{G}^{+}_{0}$ $F[\langle+1,+1,+1\rangle,q^{+}_{u},r^{+}_{u}] \neq c - F[\langle+1,+1,+1\rangle,q^{+}_{u},r^{+}_{u}]$ $\quad$ $\mathbf{H}^{+}_{0}$ $E[r^{+}_{u},s^{+}_{u}] = E[\langle-1,+1,-1\rangle,o^{-}_{v}] + 2a$

$\mathbf{G}^{+}_{a}$ $F[\langle-1,-1,+1\rangle,q^{+}_{u},p^{+}_{u}] \neq c - F[\langle-1,-1,+1\rangle,q^{+}_{u},p^{+}_{u}] - 2a$ $\quad$ $\mathbf{H}^{+}_{a}$ $E[p^{+}_{u},o^{+}_{u}] = E[\langle+1,-1,-1\rangle,s^{-}_{v}] - 2a$

$\mathbf{G}^{+}_{b}$ $F[\langle-1,-1,+1\rangle,q^{+}_{u},r^{+}_{u}] \neq c - F[\langle-1,-1,+1\rangle,q^{+}_{u},r^{+}_{u}] - 2b$ $\quad$ $\mathbf{H}^{+}_{b}$ $E[r^{+}_{u},s^{+}_{u}] = E[\langle-1,+1,-1\rangle,s^{-}_{v}] - 2b$

$\mathbf{G}^{+}_{ab}$ $F[\langle+1,+1,+1\rangle,q^{+}_{u},p^{+}_{u}] \neq c - F[\langle+1,+1,+1\rangle,q^{+}_{u},p^{+}_{u}] - 2a + 2b$ $\quad$ $\mathbf{H}^{+}_{ab}$ $E[p^{+}_{u},o^{+}_{u}] = E[\langle+1,-1,-1\rangle,o^{-}_{v}] + 2b$

Opposite face/edge pairs: face↔face non−equation $\mathbf{G}^{-}_{*}$ ≈ edge↔edge equation $\mathbf{H}^{-}_{*}$
(non−intersecting) 'far' faces at bottom of $\mathbf{E}^{5-}_{v}$ ≈ (intersecting) 'near' edges at top of $\mathbf{E}^{5-}_{v}$

$\mathbf{G}^{-}_{0}$ $F[\langle-1,+1,-1\rangle,q^{-}_{v},r^{-}_{v}] \neq d - F[\langle-1,+1,-1\rangle,q^{-}_{v},r^{-}_{v}]$ $\quad$ $\mathbf{H}^{-}_{0}$ $E[r^{-}_{v},s^{-}_{v}] = E[\langle-1,-1,+1\rangle,s^{+}_{u}] + 2b$

$\mathbf{G}^{-}_{b}$ $F[\langle+1,-1,-1\rangle,q^{-}_{v},p^{-}_{v}] \neq d - F[\langle+1,-1,-1\rangle,q^{-}_{v},p^{-}_{v}] - 2b$ $\quad$ $\mathbf{H}^{-}_{b}$ $E[p^{-}_{v},o^{-}_{v}] = E[\langle+1,+1,+1\rangle,o^{+}_{u}] - 2b$

$\mathbf{G}^{-}_{a}$ $F[\langle+1,-1,-1\rangle,q^{-}_{v},r^{-}_{v}] \neq d - F[\langle+1,-1,-1\rangle,q^{-}_{v},r^{-}_{v}] + 2a$ $\quad$ $\mathbf{H}^{-}_{a}$ $E[r^{-}_{v},s^{-}_{v}] = E[\langle-1,-1,+1\rangle,o^{+}_{u}] + 2a$

$\mathbf{G}^{-}_{ab}$ $F[\langle-1,+1,-1\rangle,q^{-}_{v},p^{-}_{v}] \neq d - F[\langle-1,+1,-1\rangle,q^{-}_{v},p^{-}_{v}] - 2a - 2b$ $\quad$ $\mathbf{H}^{-}_{ab}$ $E[p^{-}_{v},o^{-}_{v}] = E[\langle+1,+1,+1\rangle,s^{+}_{u}] - 2a$

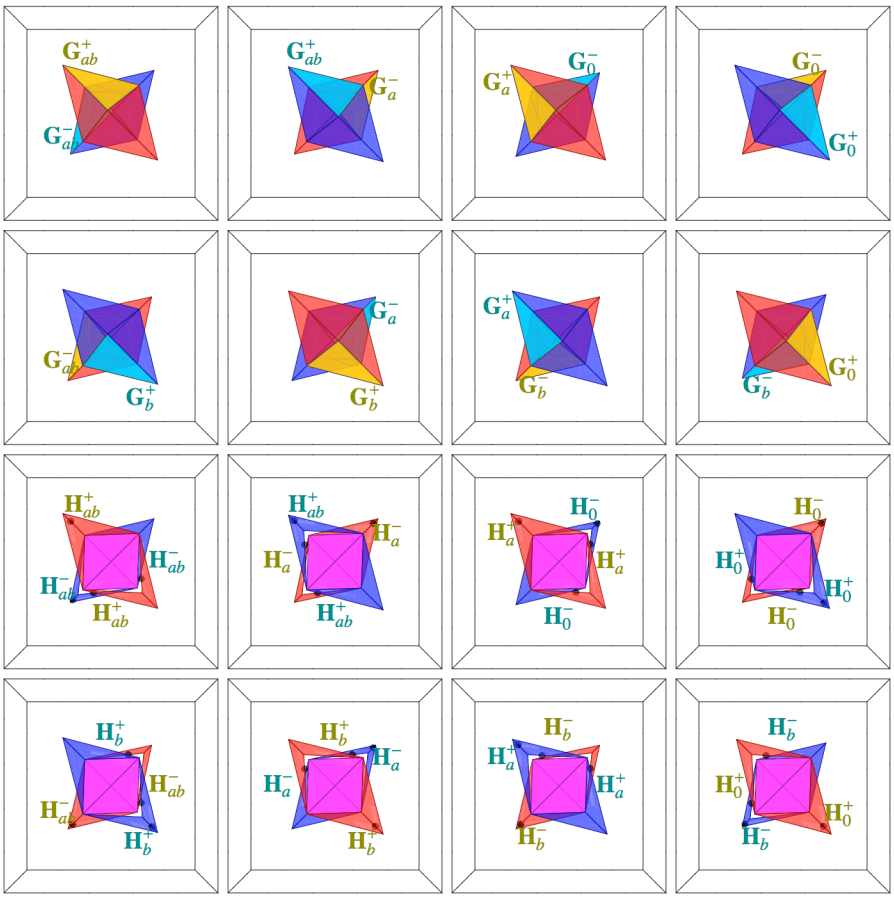


Fig. 9$^{\mathbf{G,H}}$ The 8 sets of intersection equations $\mathbf{G} \bigcup \mathbf{H}$ under the isometries $\mathbf{I}_{x,y,z}$

4.6 Remarks: Dense Packing

This construction gives a dense packing, for several reasons:

1. The half-cluster $\mathbf{E}^5$ (5 tetrahedra joined face-to-face along a common edge) is very dense locally. The local density (fraction of solid angle around the edge) is $D \approx .979566380077$. The relative density of the half-cluster inside its convex hull is $D = \frac{405}{416} \approx .973557692308$.
2. Opposite edges of a regular tetrahedron are skew perpendicular. Inside the cluster $\mathbf{B}^9_{u,v}$, the two half-clusters are oriented perpendicularly. So the clusters fit together very nicely in layers with an almost-square basis.
3. Between adjacent layers, the half-clusters and their neighbors are point reflections (scalar mult. by -1). Therefore, the faces are parallel, and the intersections are partial face-to-face.
4. For small values of $\langle u, v\rangle \approx \langle 0, 0\rangle$, the clusters have an extra almost-symmetry. Because the layers have almost-square basis, point reflection (scalar mult. by -1) is almost-equivalent to rotation by $\frac{\pi}{2}$ radians about the z-axis. Thus, the layers fit together very well to form a dense packing.

5 Conclusion

For the $\mathbf{B}^9$ cluster, if we rotate the 8 non-central tetrahedra independently about the central tetrahedron $\mathbf{B}^1$, we can generalize to an 8-parameter family $\mathbf{B}^9_{a,b,c,d,e,f,g,h} = \mathbf{B}^1 \cup \mathbf{E}^{4+}_{a,b,c,d} \cup \mathbf{E}^{4-}_{e,f,g,h}$.

Intuitively, the maximal packing occurs when there are no gaps among the 4 upper tetrahedra, and no gaps among the 4 lower tetrahedra (ie: the special case of our 2-parameter family $\mathbf{B}^9_{u,v} = \mathbf{B}^9_{u,u,u,u,v,v,v,v}$, where the non-central tetrahedra form $\mathbf{E}^{4+}_u = \mathbf{E}^{4+}_{u,u,u,u}$ and $\mathbf{E}^{4-}_v = \mathbf{E}^{4-}_{v,v,v,v}$).

From a geometric perspective, it's easy to see why. If you start with a gapless cluster, and try to open a gap somewhere, you will only decrease the packing density. The distance between neighbors in the same layer will increase, and the perpendicular distance between layers will not decrease. Therefore, generalizing will not help us.

I found packings for other clusters of tetrahedra $\mathbf{B}_{17}$, $\mathbf{V}_{20}$, $\mathbf{E}_{15}$, $\mathbf{B}_{13}$, $\mathbf{B}_9$ with packing density $D \geq \frac{7}{10}$, but the winner is $\mathbf{B}_9$.

However, it's possible that other arrangements of tetrahedra yet undiscovered could give a denser packing...

Chapter 8
Rect$_u$ packings of $\mathbf{E}_u^{15}$

Construct the cluster $\mathbf{X} = \mathbf{E}_u^{15}$ with the symmetries of a regular pentagon

$\mathbf{E}^5$ has 5 bodies
5 tetrahedra: $\mathbf{Z}_5\{\langle 0,0,+\sqrt{2}\rangle, \langle 0,0,-\sqrt{2}\rangle, \langle 2,+\sqrt{2},0\rangle, \langle 2,-\sqrt{2},0\rangle\}$

$\mathbf{E}_u^{5+}$ has 5 bodies
5 tetrahedra: $\mathbf{Z}_5\{\langle 0,0,+\sqrt{2}\rangle, \langle r_u,0,+\tilde{s}_u\rangle, \langle p_u,+\sqrt{2},+\tilde{q}_u\rangle, \langle p_u,-\sqrt{2},+\tilde{q}_u\rangle\}$

$\mathbf{E}_u^{5-}$ has 5 bodies
5 tetrahedra: $\mathbf{Z}_5\{\langle 0,0,-\sqrt{2}\rangle, \langle r_u,0,-\tilde{s}_u\rangle, \langle p_u,+\sqrt{2},-\tilde{q}_u\rangle, \langle p_u,-\sqrt{2},-\tilde{q}_u\rangle\}$

u parametrizes the z-coordinate $\tilde{q}_u$
The domain of u is $0 \le u \le +.287703459948$

$\mathbf{E}_u^{15} = \mathbf{E}^5 \cup \mathbf{E}_u^{5+} \cup \mathbf{E}_u^{5-}$

$\mathbf{Z}_5 = \{R_{-\frac{4}{5}\pi}, R_{-\frac{2}{5}\pi}, R_0, R_{+\frac{2}{5}\pi}, R_{+\frac{4}{5}\pi}\}$, $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Construct the packing with symmetry group $\mathbf{Z}_2^3 = \left\{ \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \right\}$

There are 8 cosets
$\{+\mathbf{X}_A\langle 0,0,0\rangle, +\mathbf{X}_C\langle 0,0,2c\rangle, +\mathbf{X}_B\langle a,b,0\rangle, +\mathbf{X}_D\langle a,b,2c\rangle,$
$-\mathbf{X}_E\langle 0,0,c\rangle, -\mathbf{X}_G\langle 0,0,-c\rangle, -\mathbf{X}_F\langle a,b,c\rangle, -\mathbf{X}_H\langle a,b,-c\rangle\}$

The lattice basis is $\{\langle 0,2b,2c\rangle, \langle 2a,0,2c\rangle, \langle 2a,2b,0\rangle\}$

Write the intersection equations

Evaluate the vertices of $\mathbf{E}_u^{5+}$

Evaluate the lattice vectors a, b, c

Evaluate the lattice volume $V = \det \begin{bmatrix} 0 & 2b & 2c \\ 2a & 0 & 2c \\ 2a & 2b & 0 \end{bmatrix} = 16abc$

Evaluate the packing density $D = 320/V$

Note: Each tetrahedron has volume $\frac{8}{3}$

8.1 $\mathbf{E}_0^{15}$

$\mathbf{E}_0^{15}$

8 cosets

120 tetrahedra

$U = 320$

320.000000000000

$a = \frac{12}{5}$

2.400000000000

$b = \frac{12}{25}\sqrt{325 - 110\sqrt{5}}$

4.267211405385

$c = 2\sqrt{2}$

2.828427124746

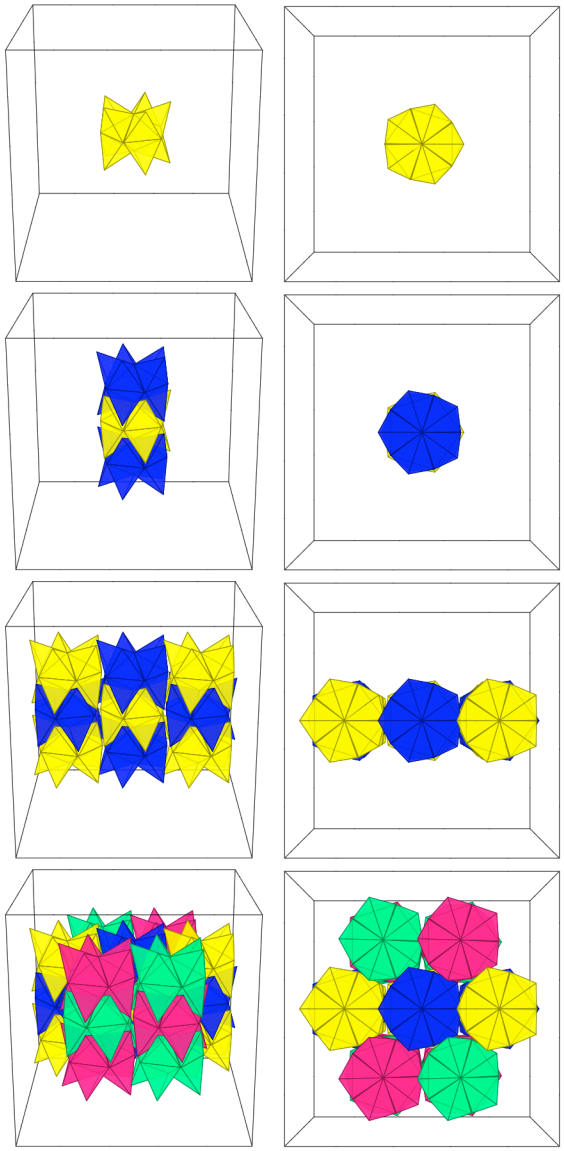
$V = \frac{4608}{125}\sqrt{650 - 220\sqrt{5}}$

463.468665063030

$D = \frac{125}{2736}\sqrt{130 + 44\sqrt{5}}$

.690445814619

basis $\{\langle 0,2b,2c\rangle,\langle 2a,0,2c\rangle,\langle 2a,2b,0\rangle\}$



The intersections are

$V_A\langle 2,\sqrt{2},0\rangle = V_A\langle p,\sqrt{2},\tilde{q}\rangle$
$V_A\langle 2,-\sqrt{2},0\rangle = V_A\langle p,-\sqrt{2},\tilde{q}\rangle$
$V_A\langle 0,0,\sqrt{2}\rangle = V_E\langle 0,0,-\sqrt{2}\rangle + \langle 0,0,c\rangle$
$V_A\langle r,0,-\tilde{s}\rangle = F_E[\langle -p,\sqrt{2},\tilde{q}\rangle,\langle -p,-\sqrt{2},\tilde{q}\rangle,\langle -r,0,\tilde{s}\rangle] + \langle 2a,0,-c\rangle$
$E_A[\langle p,\sqrt{2},-\tilde{q}\rangle,\langle r,0,-\tilde{s}\rangle] = E_E[\langle -p,\sqrt{2},\tilde{q}\rangle,\langle -r,0,\tilde{s}\rangle] + \langle 2a,0,-c\rangle$
$V_A\langle 0,0,-\sqrt{2}\rangle = V_G\langle 0,0,\sqrt{2}\rangle + \langle 0,0,-c\rangle$
$V_A\langle r,0,\tilde{s}\rangle = F_G[\langle -p,\sqrt{2},-\tilde{q}\rangle,\langle -p,-\sqrt{2},-\tilde{q}\rangle,\langle -r,0,-\tilde{s}\rangle] + \langle 2a,0,c\rangle$
$E_A[\langle p,\sqrt{2},\tilde{q}\rangle,\langle r,0,\tilde{s}\rangle] = E_G[\langle -p,\sqrt{2},-\tilde{q}\rangle,\langle -r,0,-\tilde{s}\rangle] + \langle 2a,0,c\rangle$
$R_{+\frac{2}{5}\pi}V_A\langle r,0,\tilde{s}\rangle = R_{+\frac{2}{5}\pi}F_F[\langle -p,-\sqrt{2},-\tilde{q}\rangle,\langle -p,\sqrt{2},-\tilde{q}\rangle,\langle -r,0,-\tilde{s}\rangle] + \langle a,b,c\rangle$
$R_{+\frac{2}{5}\pi}E_A[\langle p,\sqrt{2},\tilde{q}\rangle,\langle r,0,\tilde{s}\rangle] = R_{+\frac{2}{5}\pi}E_F[\langle -p,\sqrt{2},-\tilde{q}\rangle,\langle -r,0,-\tilde{s}\rangle] + \langle a,b,c\rangle$
$R_{+\frac{2}{5}\pi}V_A\langle r,0,-\tilde{s}\rangle = R_{+\frac{2}{5}\pi}F_H[\langle -p,-\sqrt{2},\tilde{q}\rangle,\langle -p,\sqrt{2},\tilde{q}\rangle,\langle -r,0,\tilde{s}\rangle] + \langle a,b,-c\rangle$
$R_{+\frac{2}{5}\pi}E_A[\langle p,\sqrt{2},-\tilde{q}\rangle,\langle r,0,-\tilde{s}\rangle] = R_{+\frac{2}{5}\pi}E_H[\langle -p,\sqrt{2},\tilde{q}\rangle,\langle -r,0,\tilde{s}\rangle] + \langle a,b,-c\rangle$
$\tilde{q} = \sqrt{2} - q$
$\tilde{s} = \sqrt{2} + s$
$p^2 + q^2 = 6$
$r^2 + s^2 = 8$
$pr - qs = 4$

The vertices of $\mathbf{E}_u^{5+}$ are $\begin{cases} p = 2 \approx 2.000000000000 \\ q = \sqrt{2} \approx 1.414213562373 \\ r = \frac{8}{3} \approx 2.666666666667 \\ s = \frac{2}{3}\sqrt{6} \approx 1.632993161855 \end{cases}$

The lattice vectors are $\begin{cases} a = \frac{12}{5} \approx 2.400000000000 \\ b = \frac{12}{25}\sqrt{325 - 110\sqrt{5}} \approx 4.267211405385 \\ c = 2\sqrt{2} \approx 2.828427124746 \end{cases}$

The lattice volume is $V = \frac{4608}{125}\sqrt{650 - 220\sqrt{5}} \approx 463.468665063030$

The packing density is $D = \frac{125}{2736}\sqrt{130 + 44\sqrt{5}} \approx .690445814619$

8.2 $\mathbf{E}^{15}_{+.287703459948}$

$\mathbf{E}^{15}_{+.287703459948}$

8 cosets

120 tetrahedra

$U = 320$

320.000000000000

$a = ?$

2.383005692101

$b = ?$

4.236995445178

$c = 2\sqrt{2}$

2.828427124746

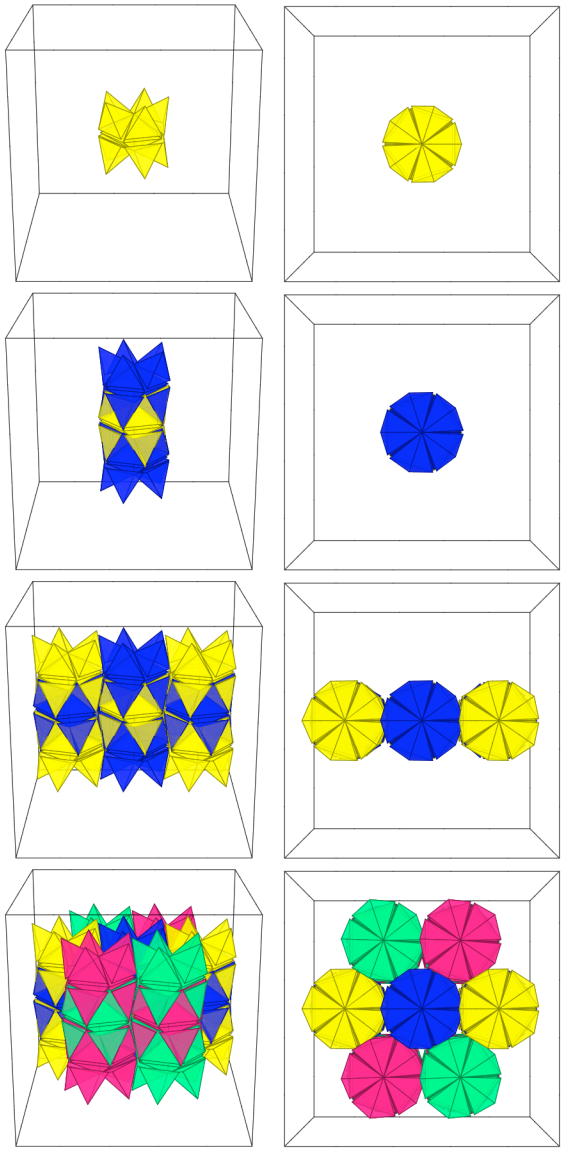
$V = ?$

456.928295726853

$D = ?$

.700328701445

basis $\{\langle 0,2b,2c\rangle,\langle 2a,0,2c\rangle,\langle 2a,2b,0\rangle\}$



The intersections are

$V_A\langle 0,0,\sqrt{2}\rangle = V_E\langle 0,0,-\sqrt{2}\rangle + \langle 0,0,c\rangle$
$E_A[\langle p,-\sqrt{2},\tilde{q}\rangle,\langle r,0,\tilde{s}\rangle] = R_{+\frac{4}{5}\pi}E_E[\langle -p,-\sqrt{2},-\tilde{q}\rangle,\langle -r,0,-\tilde{s}\rangle] + \langle 0,0,c\rangle$
$E_A[\langle p,\sqrt{2},\tilde{q}\rangle,\langle r,0,\tilde{s}\rangle] = R_{-\frac{4}{5}\pi}E_E[\langle -p,\sqrt{2},-\tilde{q}\rangle,\langle -r,0,-\tilde{s}\rangle] + \langle 0,0,c\rangle$
$V_A\langle 0,0,-\sqrt{2}\rangle = V_G\langle 0,0,\sqrt{2}\rangle + \langle 0,0,-c\rangle$
$E_A[\langle p,-\sqrt{2},-\tilde{q}\rangle,\langle r,0,-\tilde{s}\rangle] = R_{+\frac{4}{5}\pi}E_G[\langle -p,-\sqrt{2},\tilde{q}\rangle,\langle -r,0,\tilde{s}\rangle + \langle 0,0,-c\rangle$
$E_A[\langle p,\sqrt{2},-\tilde{q}\rangle,\langle r,0,-\tilde{s}\rangle] = R_{-\frac{4}{5}\pi}E_G[\langle -p,\sqrt{2},\tilde{q}\rangle,\langle -r,0,\tilde{s}\rangle] + \langle 0,0,-c\rangle$
$R_{-\frac{2}{5}\pi}V_A\langle p,-\sqrt{2},\tilde{q}\rangle = R_{+\frac{2}{5}\pi}E_B[\langle p,\sqrt{2},\tilde{q}\rangle,\langle p,-\sqrt{2},\tilde{q}\rangle] + \langle -a,-b,0\rangle$
$R_{-\frac{4}{5}\pi}V_A\langle p,\sqrt{2},\tilde{q}\rangle = R_{+\frac{2}{5}\pi}E_B[\langle p,\sqrt{2},\tilde{q}\rangle,\langle p,-\sqrt{2},\tilde{q}\rangle] + \langle -a,-b,0\rangle$
$R_{+\frac{2}{5}\pi}V_A\langle p,\sqrt{2},\tilde{q}\rangle = R_{-\frac{2}{5}\pi}E_D[\langle p,\sqrt{2},\tilde{q}\rangle,\langle p,-\sqrt{2},\tilde{q}\rangle] + \langle -a,b,0\rangle$
$R_{+\frac{4}{5}\pi}V_A\langle p,-\sqrt{2},\tilde{q}\rangle = R_{-\frac{2}{5}\pi}E_D[\langle p,\sqrt{2},\tilde{q}\rangle,\langle p,-\sqrt{2},\tilde{q}\rangle] + \langle -a,b,0\rangle$
$R_{+\frac{4}{5}\pi}V_A\langle p,\sqrt{2},\tilde{q}\rangle = E_C[\langle p,\sqrt{2},\tilde{q}\rangle,\langle p,-\sqrt{2},\tilde{q}\rangle] + \langle -2a,0,0\rangle$
$R_{-\frac{4}{5}\pi}V_A\langle p,-\sqrt{2},\tilde{q}\rangle = E_C[\langle p,\sqrt{2},\tilde{q}\rangle,\langle p,-\sqrt{2},\tilde{q}\rangle] + \langle -2a,0,0\rangle$
$\tilde{q} = \sqrt{2} - q$
$\tilde{s} = \sqrt{2} + s$
$p^2 + q^2 = 6$
$r^2 + s^2 = 8$
$pr - qs = 4$

The vertices of $\mathbf{E}_u^{5+}$ are $\begin{cases} p \approx 2.175080455784 \\ q \approx 1.126510102425 \\ r \approx 2.512137547189 \\ s \approx 1.299678784932 \end{cases}$

$p = \frac{2}{5}\sqrt{50-20\sqrt{5}} + \frac{1}{5}\sqrt{-50+40\sqrt{5}}$
$q = -\frac{2}{5}\sqrt{25-10\sqrt{5}} + \frac{2}{5}\sqrt{-25+20\sqrt{5}}$
$r = \frac{2}{5}\sqrt{-50+40\sqrt{5}}$
$s = \frac{4}{5}\sqrt{25-10\sqrt{5}}$

The lattice vectors are $\begin{cases} a \approx 2.383005692101 \\ b \approx 4.236995445178 \\ c \approx 2.828427124746 \end{cases}$

$a = \frac{3}{4}\sqrt{5-\sqrt{5}} + \frac{1}{4}\sqrt{5+7\sqrt{5}}$
$b = -\frac{21}{8}\sqrt{2} + \frac{15}{8}\sqrt{10} + \frac{1}{4}\sqrt{-89+69\sqrt{5}}$
$c = 2\sqrt{2}$

The lattice volume is $V \approx 456.928295726853$

The packing density is $D \approx .700328701445$

8.3 $\mathbf{E}^{15}_{+.209816513850}$

$\mathbf{E}^{15}_{+.209816513850}$

8 cosets

120 tetrahedra

$U = 320$

320.000000000000

$a = ?$

2.344888169244

$b = ?$

4.169222308397

$c = 2\sqrt{2}$

2.828427124746

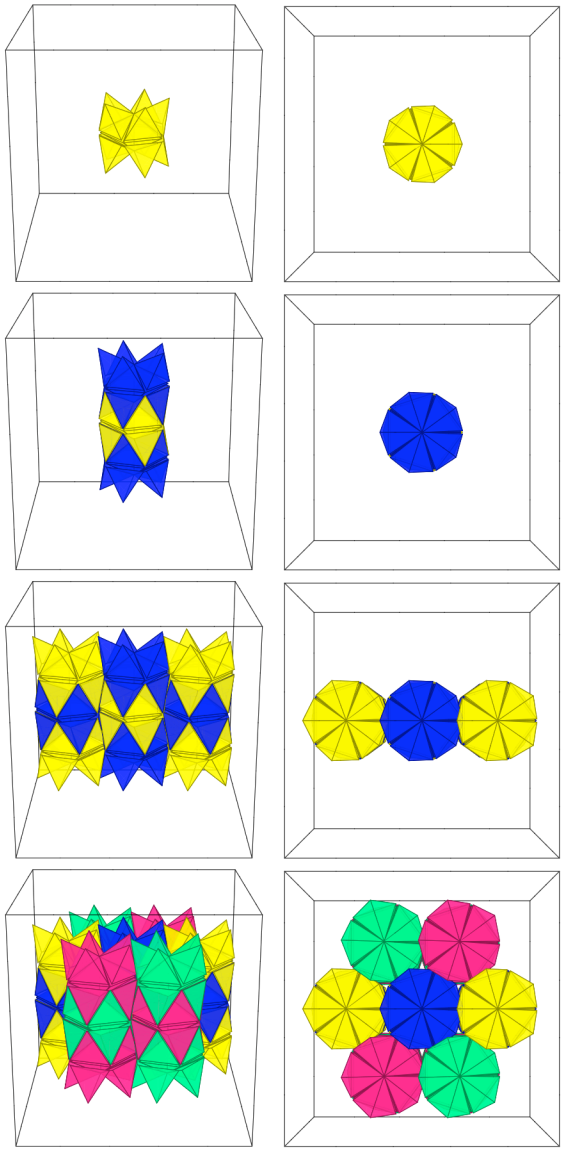
$V = ?$

442.427551867245

$D = ?$

.723282260902

basis $\{\langle 0,2b,2c\rangle,\langle 2a,0,2c\rangle,\langle 2a,2b,0\rangle\}$



The intersections are

$$V_A\langle 0,0,\sqrt{2}\rangle = V_E\langle 0,0,-\sqrt{2}\rangle + \langle 0,0,c\rangle$$
$$E_A[\langle p,\sqrt{2},-\tilde{q}\rangle,\langle p,-\sqrt{2},-\tilde{q}\rangle] = E_E[\langle -p,\sqrt{2},\tilde{q}\rangle,\langle -r,0,\tilde{s}\rangle] + \langle 2a,0,-c\rangle$$
$$E_A[\langle p,\sqrt{2},-\tilde{q}\rangle,\langle r,0,-\tilde{s}\rangle] = E_E[\langle -p,\sqrt{2},\tilde{q}\rangle,\langle -r,0,\tilde{s}\rangle] + \langle 2a,0,-c\rangle$$
$$V_A\langle 0,0,-\sqrt{2}\rangle = V_G\langle 0,0,\sqrt{2}\rangle + \langle 0,0,-c\rangle$$
$$E_A[\langle p,\sqrt{2},\tilde{q}\rangle,\langle p,-\sqrt{2},\tilde{q}\rangle] = E_G[\langle -p,\sqrt{2},-\tilde{q}\rangle,\langle -r,0,-\tilde{s}\rangle] + \langle 2a,0,c\rangle$$
$$E_A[\langle p,\sqrt{2},\tilde{q}\rangle,\langle r,0,\tilde{s}\rangle] = E_G[\langle -p,\sqrt{2},-\tilde{q}\rangle,\langle -r,0,-\tilde{s}\rangle] + \langle 2a,0,c\rangle$$
$$R_{+\frac{2}{5}\pi}E_A[\langle p,\sqrt{2},\tilde{q}\rangle,\langle p,-\sqrt{2},\tilde{q}\rangle] = R_{+\frac{2}{5}\pi}E_F[\langle -p,\sqrt{2},-\tilde{q}\rangle,\langle -r,0,-\tilde{s}\rangle] + \langle a,b,c\rangle$$
$$R_{+\frac{2}{5}\pi}E_A[\langle p,\sqrt{2},\tilde{q}\rangle,\langle r,0,\tilde{s}\rangle] = R_{+\frac{2}{5}\pi}E_F[\langle -p,\sqrt{2},-\tilde{q}\rangle,\langle -r,0,-\tilde{s}\rangle] + \langle a,b,c\rangle$$
$$R_{+\frac{2}{5}\pi}E_A[\langle p,\sqrt{2},-\tilde{q}\rangle,\langle p,-\sqrt{2},-\tilde{q}\rangle] = R_{+\frac{2}{5}\pi}E_H[\langle -p,\sqrt{2},\tilde{q}\rangle,\langle -r,0,\tilde{s}\rangle] + \langle a,b,-c\rangle$$
$$R_{+\frac{2}{5}\pi}E_A[\langle p,\sqrt{2},-\tilde{q}\rangle,\langle r,0,-\tilde{s}\rangle] = R_{+\frac{2}{5}\pi}E_H[\langle -p,\sqrt{2},\tilde{q}\rangle,\langle -r,0,\tilde{s}\rangle] + \langle a,b,-c\rangle$$
$$R_{-\frac{2}{5}\pi}V_A\langle p,-\sqrt{2},\tilde{q}\rangle = R_{+\frac{2}{5}\pi}E_B[\langle p,\sqrt{2},\tilde{q}\rangle,\langle p,-\sqrt{2},\tilde{q}\rangle] + \langle -a,-b,0\rangle$$
$$R_{-\frac{4}{5}\pi}V_A\langle p,\sqrt{2},\tilde{q}\rangle = R_{+\frac{2}{5}\pi}E_B[\langle p,\sqrt{2},\tilde{q}\rangle,\langle p,-\sqrt{2},\tilde{q}\rangle] + \langle -a,-b,0\rangle$$
$$R_{+\frac{2}{5}\pi}V_A\langle p,\sqrt{2},\tilde{q}\rangle = R_{-\frac{2}{5}\pi}E_D[\langle p,\sqrt{2},\tilde{q}\rangle,\langle p,-\sqrt{2},\tilde{q}\rangle] + \langle -a,b,0\rangle$$
$$R_{+\frac{4}{5}\pi}V_A\langle p,-\sqrt{2},\tilde{q}\rangle = R_{-\frac{2}{5}\pi}E_D[\langle p,\sqrt{2},\tilde{q}\rangle,\langle p,-\sqrt{2},\tilde{q}\rangle] + \langle -a,b,0\rangle$$
$$R_{+\frac{4}{5}\pi}V_A\langle p,\sqrt{2},\tilde{q}\rangle = E_C[\langle p,\sqrt{2},\tilde{q}\rangle,\langle p,-\sqrt{2},\tilde{q}\rangle] + \langle -2a,0,0\rangle$$
$$R_{-\frac{4}{5}\pi}V_A\langle p,-\sqrt{2},\tilde{q}\rangle = E_C[\langle p,\sqrt{2},\tilde{q}\rangle,\langle p,-\sqrt{2},\tilde{q}\rangle] + \langle -2a,0,0\rangle$$
$$\tilde{q} = \sqrt{2} - q$$
$$\tilde{s} = \sqrt{2} + s$$
$$p^2 + q^2 = 6$$
$$r^2 + s^2 = 8$$
$$pr - qs = 4$$

The vertices of $\mathbf{E}_u^{5+}$ are $\begin{cases} p \approx 2.132938758968 \\ q \approx 1.204397048524 \\ r \approx 2.557475599648 \\ s \approx 1.208022581413 \end{cases}$

$$0 = 15p^4 + 6\sqrt{50-20\sqrt{5}}\,p^3 - (300-52\sqrt{5})p^2 - 4\sqrt{27890-12244\sqrt{5}}\,p + (2284-760\sqrt{5})$$
$$0 = 15q^4 + (180-76\sqrt{5})q^2 - 32\sqrt{50-22\sqrt{5}}\,q + (704-320\sqrt{5})$$

The lattice vectors are $\begin{cases} a \approx 2.344888169244 \\ b \approx 4.169222308397 \\ c \approx 2.828427124746 \end{cases}$

$$0 = 6a^4 - 3\sqrt{5-\sqrt{5}}\,a^3 - (40+9\sqrt{5})a^2 + 4\sqrt{25+5\sqrt{5}}\,a + (90+30\sqrt{5})$$
$$0 = 12b^4 + \sqrt{2}(21-15\sqrt{5})b^3 - (644-118\sqrt{5})b^2 + \sqrt{2}(-1052+556\sqrt{5})b + (13524-4644\sqrt{5})$$
$$c = 2\sqrt{2}$$

The lattice volume is $V \approx 442.427551867245$

$$0 = 9V^4 - 48\sqrt{107405-4279\sqrt{5}}\,V^3 + (9966080-302080\sqrt{5})V^2$$
$$-12451840\sqrt{47165-2551\sqrt{5}}\,V + (272923361280-6417285120\sqrt{5})$$

The packing density is $D \approx .723282260902$

$$0 = 18766224D^4 - 274360\sqrt{46825-341\sqrt{5}}\,D^3 + (70115225-478325\sqrt{5})D^2$$
$$-5700\sqrt{38623945+275131\sqrt{5}}\,D + (6507000+153000\sqrt{5})$$

Chapter 9
Trigonal packings

Construct a cluster $\mathbf{X}$ with trigonal symmetry $\mathbf{Z}_3$

$$\mathbf{Z}_3 = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\} \qquad \text{rotation by } \{-\tfrac{2}{3}\pi, 0, +\tfrac{2}{3}\pi\} \text{ about } \langle 1,1,1\rangle$$

$\mathbf{F}_2 =$ (2 bodies)
1 tetrahedron: $\{\langle +\frac{4}{3}, +\frac{4}{3}, +\frac{4}{3}\rangle, \langle +\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}\rangle, \langle -\frac{2}{3}, +\frac{4}{3}, -\frac{2}{3}\rangle, \langle -\frac{2}{3}, -\frac{2}{3}, +\frac{4}{3}\rangle\}$
1 tetrahedron: $\{\langle -\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\rangle, \langle +\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}\rangle, \langle -\frac{2}{3}, +\frac{4}{3}, -\frac{2}{3}\rangle, \langle -\frac{2}{3}, -\frac{2}{3}, +\frac{4}{3}\rangle\}$

$\mathbf{F}_8 = \mathbf{F}_2 \cup$ (6 bodies)
3 tetrahedra: $\mathbf{Z}_3\{\langle +\frac{4}{3}, +\frac{4}{3}, +\frac{4}{3}\rangle, \langle -\frac{2}{3}, +\frac{4}{3}, -\frac{2}{3}\rangle, \langle -\frac{2}{3}, -\frac{2}{3}, +\frac{4}{3}\rangle, \langle -\frac{4}{3}, +2, +2\rangle\}$
3 tetrahedra: $\mathbf{Z}_3\{\langle -\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\rangle, \langle -\frac{2}{3}, +\frac{4}{3}, -\frac{2}{3}\rangle, \langle -\frac{2}{3}, -\frac{2}{3}, +\frac{4}{3}\rangle, \langle -\frac{28}{9}, +\frac{2}{9}, +\frac{2}{9}\rangle\}$

$\mathbf{F}_{11} = \mathbf{F}_8 \cup$ (3 bodies)
3 tetrahedra: $\mathbf{Z}_3\{\langle -\frac{2}{3}, +\frac{4}{3}, -\frac{2}{3}\rangle, \langle -\frac{2}{3}, -\frac{2}{3}, +\frac{4}{3}\rangle,$
$\langle -\frac{2}{3} - \sqrt{6}, +\frac{1}{3}, +\frac{1}{3}\rangle, \langle -\frac{2}{3} - \frac{1}{3}\sqrt{6}, +\frac{1}{3} + \frac{2}{3}\sqrt{6}, +\frac{1}{3} + \frac{2}{3}\sqrt{6}\rangle\}$

$\mathbf{V}_{20} =$ (20 bodies)
8 tetrahedra: $\pm\mathbf{Z}_2^2\{\langle 0,0,0\rangle, \langle q_0, q_1, q_2\rangle, \langle q_2, q_0, q_1\rangle, \langle q_1, q_2, q_0\rangle\}$
12 tetrahedra: $\mathbf{A}_4\{\langle 0,0,0\rangle, \langle 0, s_1, s_2\rangle, \langle r_1, r_2, r_0\rangle, \langle -r_1, r_2, r_0\rangle\}$

$P = \frac{1}{2}(1+\sqrt{5})$, $Q = \frac{1}{2}(-1+\sqrt{5})$
$\langle q_0, q_1, q_2\rangle = \frac{1}{3}\langle 4 - P^2\sqrt{2}, 4 + Q^2\sqrt{2}, 4 + \sqrt{10}\rangle \approx \langle .099180275452, 1.513393837825, 2.387425886723\rangle$
$\langle r_0, r_1, r_2\rangle = \frac{1}{3}\langle 4Q - P\sqrt{2}, \sqrt{2}, 4P + Q\sqrt{2}\rangle \approx \langle .061296781243, 1.414213562373, 2.448722667966\rangle$
$\langle 0, s_1, s_2\rangle = \frac{1}{3}\langle 0, 4P - 2Q\sqrt{2}, 4Q + 2P\sqrt{2}\rangle \approx \langle .000000000000, 1.574690619068, 2.349542392514\rangle$

$\mathbf{B}_1 =$ (1 body)
1 tetrahedron: $\{\langle +1,+1,+1\rangle, \langle +1,-1,-1\rangle, \langle -1,+1,-1\rangle, \langle -1,-1,+1\rangle\}$

$\mathbf{B}_4 = \mathbf{B}_1 \cup$ (3 bodies)
3 tetrahedra: $\mathbf{Z}_3\{\langle +1,+1,+1\rangle, \langle -1,+1,-1\rangle, \langle -1,-1,+1\rangle, \langle -\frac{5}{3}, +\frac{5}{3}, +\frac{5}{3}\rangle\}$

$\mathbf{B}_5 = \mathbf{B}_1 \cup$ (4 bodies)
4 tetrahedra: $\mathbf{Z}_2^2\{\langle +1,-1,-1\rangle, \langle -1,+1,-1\rangle, \langle -1,-1,+1\rangle, \langle -\frac{5}{3}, -\frac{5}{3}, -\frac{5}{3}\rangle\}$

$\mathbf{B}_{10} = \mathbf{B}_4 \cup$ (6 bodies)
3 tetrahedra: $\mathbf{Z}_2^2\{\langle +1,+1,+1\rangle, \langle +1,-1,-1\rangle, \langle +\frac{5}{3}, +\frac{5}{3}, -\frac{5}{3}\rangle, \langle +\frac{31}{9}, +\frac{1}{9}, -\frac{1}{9}\rangle\}$
3 tetrahedra: $\mathbf{Z}_2^2\{\langle +1,+1,+1\rangle, \langle +1,-1,-1\rangle, \langle +\frac{5}{3}, -\frac{5}{3}, +\frac{5}{3}\rangle, \langle +\frac{31}{9}, -\frac{1}{9}, +\frac{1}{9}\rangle\}$

$\mathbf{B}_{10}^* = \mathbf{B}_4 \cup$ (6 bodies)
3 tetrahedra: $\mathbf{Z}_2^2\{\langle +1,+1,+1\rangle, \langle +1,-1,-1\rangle, \langle +1+\sqrt{6}, 0, 0\rangle, \langle +1+\frac{1}{3}\sqrt{6}, +\frac{2}{3}\sqrt{6}, -\frac{2}{3}\sqrt{6}\rangle\}$
3 tetrahedra: $\mathbf{Z}_2^2\{\langle +1,+1,+1\rangle, \langle +1,-1,-1\rangle, \langle +1+\sqrt{6}, 0, 0\rangle, \langle +1+\frac{1}{3}\sqrt{6}, -\frac{2}{3}\sqrt{6}, +\frac{2}{3}\sqrt{6}\rangle\}$

$\mathbf{B}_{11} = \mathbf{B}_5 \cup$ (6 bodies)
3 tetrahedra: $\mathbf{Z}_2^2\{\langle +1,+1,+1\rangle, \langle +1,-1,-1\rangle, \langle +\frac{5}{3}, +\frac{5}{3}, -\frac{5}{3}\rangle, \langle +\frac{31}{9}, +\frac{1}{9}, -\frac{1}{9}\rangle\}$
3 tetrahedra: $\mathbf{Z}_2^2\{\langle +1,+1,+1\rangle, \langle +1,-1,-1\rangle, \langle +\frac{5}{3}, -\frac{5}{3}, +\frac{5}{3}\rangle, \langle +\frac{31}{9}, -\frac{1}{9}, +\frac{1}{9}\rangle\}$

$\mathbf{B}_{11}^* = \mathbf{B}_5 \cup$ (6 bodies)
3 tetrahedra: $\mathbf{Z}_2^2\{\langle +1,+1,+1\rangle, \langle +1,-1,-1\rangle, \langle +1+\sqrt{6}, 0, 0\rangle, \langle +1+\frac{1}{3}\sqrt{6}, +\frac{2}{3}\sqrt{6}, -\frac{2}{3}\sqrt{6}\rangle\}$
3 tetrahedra: $\mathbf{Z}_2^2\{\langle +1,+1,+1\rangle, \langle +1,-1,-1\rangle, \langle +1+\sqrt{6}, 0, 0\rangle, \langle +1+\frac{1}{3}\sqrt{6}, -\frac{2}{3}\sqrt{6}, +\frac{2}{3}\sqrt{6}\rangle\}$

$\mathbf{B}_{11} = \mathbf{B}_5 \cup (6 \text{ bodies})$
3 tetrahedra: $\mathbf{Z}_2^2\{\langle -1,+1,-1\rangle, \langle -1,-1,+1\rangle, \langle -\frac{5}{3},-\frac{5}{3},-\frac{5}{3}\rangle, \langle -\frac{31}{9},-\frac{1}{9},-\frac{1}{9}\rangle\}$
3 tetrahedra: $\mathbf{Z}_2^2\{\langle -1,+1,-1\rangle, \langle -1,-1,+1\rangle, \langle -\frac{5}{3},+\frac{5}{3},+\frac{5}{3}\rangle, \langle -\frac{31}{9},+\frac{1}{9},+\frac{1}{9}\rangle\}$

$\mathbf{B}_{11}^* = \mathbf{B}_5 \cup (6 \text{ bodies})$
3 tetrahedra: $\mathbf{Z}_2^2\{\langle -1,+1,-1\rangle, \langle -1,-1,+1\rangle, \langle -1-\sqrt{6},0,0\rangle, \langle -1-\frac{1}{3}\sqrt{6},-\frac{2}{3}\sqrt{6},-\frac{2}{3}\sqrt{6}\rangle\}$
3 tetrahedra: $\mathbf{Z}_2^2\{\langle -1,+1,-1\rangle, \langle -1,-1,+1\rangle, \langle -1-\sqrt{6},0,0\rangle, \langle -1-\frac{1}{3}\sqrt{6},+\frac{2}{3}\sqrt{6},+\frac{2}{3}\sqrt{6}\rangle\}$

$\mathbf{B}_{17} = \mathbf{B}_5 \cup (12 \text{ bodies})$
12 tetrahedra: $\mathbf{A}_4\{\langle -1,+1,-1\rangle, \langle -1,-1,+1\rangle, \langle -\frac{5}{3},-\frac{5}{3},-\frac{5}{3}\rangle, \langle -\frac{31}{9},-\frac{1}{9},-\frac{1}{9}\rangle\}$

$\mathbf{B}_{17}^* = \mathbf{B}_5 \cup (12 \text{ bodies})$
12 tetrahedra: $\mathbf{A}_4\{\langle -1,+1,-1\rangle, \langle -1,-1,+1\rangle, \langle -1-\sqrt{6},0,0\rangle, \langle -1-\frac{1}{3}\sqrt{6},-\frac{2}{3}\sqrt{6},-\frac{2}{3}\sqrt{6}\rangle\}$

Construct a packing with trigonal symmetry $\mathbf{Z}_3$

$T_2^=$ (pure 2-direction) face-centered trigonal
There is 1 coset $\{+\mathbf{X}_A\langle 0,0,0\rangle, *, *, *\}$

$T_1^=$ (pure 1-direction) simple trigonal
There are 2 cosets $\{+\mathbf{X}_A\langle 0,0,0\rangle, *, +\mathbf{X}_C\langle b+c, c-a, -a+b\rangle, *\}$

$T_1^{\neq}$ (mixed 1-direction) simple trigonal
There are 2 cosets $\{+\mathbf{X}_A\langle 0,0,0\rangle, *, -\mathbf{X}_C\langle b+c, c-a, -a+b\rangle, *\}$

$T_3^=$ (pure 3-direction) body-centered trigonal
There are 4 cosets $\{+\mathbf{X}_A\langle 0,0,0\rangle, +\mathbf{X}_B\langle -a+b+c, -a+b+c, -a+b+c\rangle,$
$+\mathbf{X}_C\langle b+c, c-a, -a+b\rangle, +\mathbf{X}_D\langle a-b-c, a-b-c, a-b-c\rangle\}$

$T_3^{\neq}$ (mixed 3-direction) body-centered trigonal
There are 4 cosets $\{+\mathbf{X}_A\langle 0,0,0\rangle, -\mathbf{X}_B\langle -a+b+c, -a+b+c, -a+b+c\rangle,$
$+\mathbf{X}_C\langle b+c, c-a, -a+b\rangle, -\mathbf{X}_D\langle a-b-c, a-b-c, a-b-c\rangle\}$

T_3^+ (positive 3-direction) diamond-crystal trigonal
There are 2 cosets $\{+\mathbf{X}_A\langle 0,0,0\rangle, -\mathbf{X}_B\langle -a+b+c, -a+b+c, -a+b+c\rangle, *, *\}$

T_3^- (negative 3-direction) diamond-crystal trigonal
There are 2 cosets $\{+\mathbf{X}_A\langle 0,0,0\rangle, *, *, -\mathbf{X}_D\langle a-b-c, a-b-c, a-b-c\rangle\}$

Each packing could have several variations

The lattice basis is $\{\langle -2a+b+c, 2b+c-a, 2c-a+b\rangle,$
$\langle 2c-a+b, -2a+b+c, 2b+c-a\rangle, \langle 2b+c-a, 2c-a+b, -2a+b+c\rangle\}$

Write the intersection equations

Evaluate the lattice vectors a, b, c

Evaluate the lattice volume $V = \det \begin{bmatrix} -2a+b+c & 2b+c-a & 2c-a+b \\ 2c-a+b & -2a+b+c & 2b+c-a \\ 2b+c-a & 2c-a+b & -2a+b+c \end{bmatrix} = 4(-a^3+b^3+c^3+3abc)$

Evaluate the packing density D

Note: Each tetrahedron has volume $\frac{8}{3}$

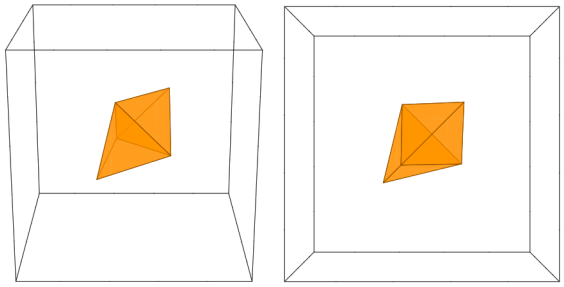
$\mathbf{F}_2$

$U = \frac{16}{3}$

n cosets

2 tetrahedra

5.333333333333



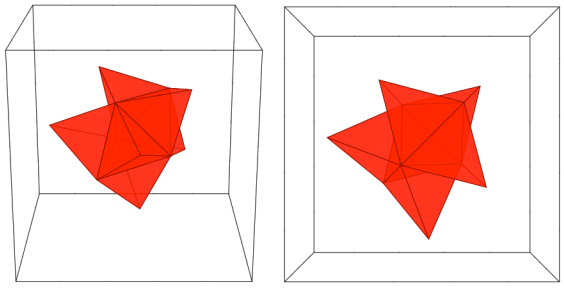
$\mathbf{F}_8$

$U = \frac{64}{3}$

n cosets

8 tetrahedra

21.333333333333



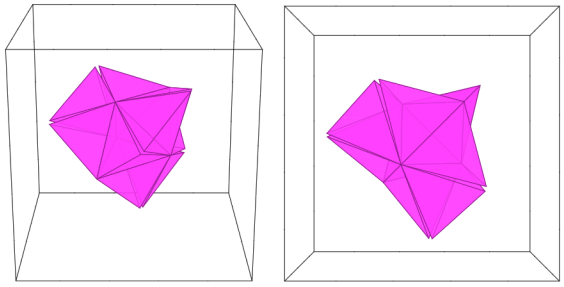
$\mathbf{F}_{11}$

$U = \frac{88}{3}$

n cosets

11 tetrahedra

29.333333333333



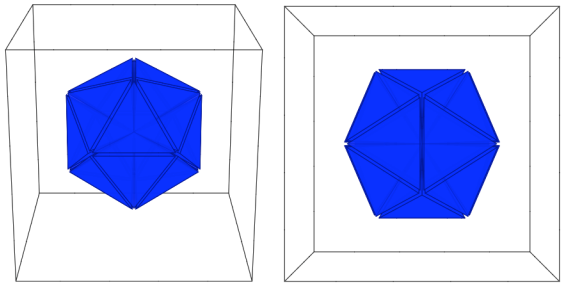
$\mathbf{V}_{20}$

$U = \frac{160}{3}$

n cosets

20 tetrahedra

53.333333333333



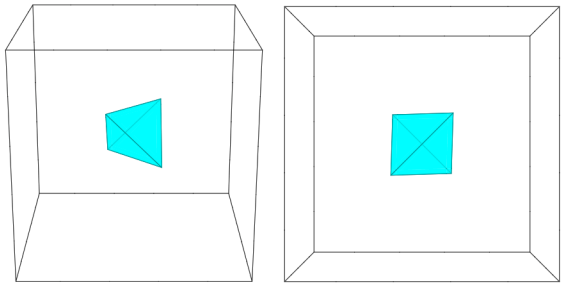
$\mathbf{B}_1$

$U = \frac{8}{3}$

n cosets

1 tetrahedron

2.666666666667



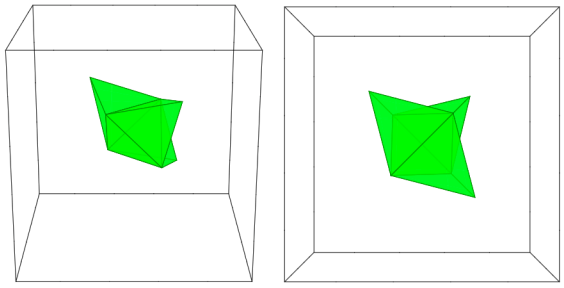
$\mathbf{B}_4$

$U = \frac{32}{3}$

n cosets

4 tetrahedra

10.666666666667



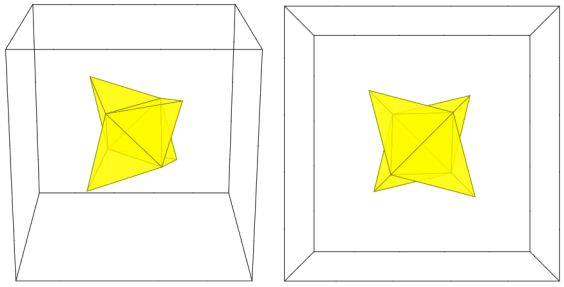
$\mathbf{B}_5$

$U = \frac{40}{3}$

n cosets

5 tetrahedra

13.333333333333



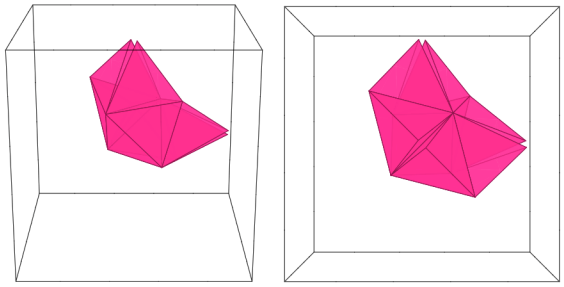
$\mathbf{B}_{10}$

$U = \frac{80}{3}$

n cosets

10 tetrahedra

26.666666666667



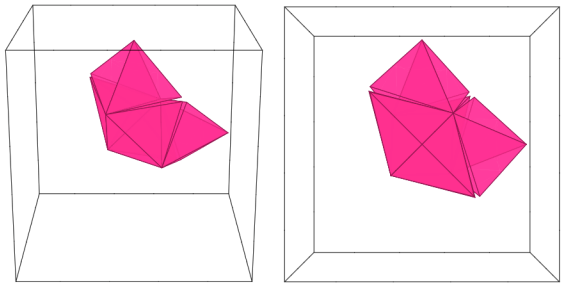
$\mathbf{B}^*_{10}$

$U = \frac{80}{3}$

n cosets

10 tetrahedra

26.666666666667



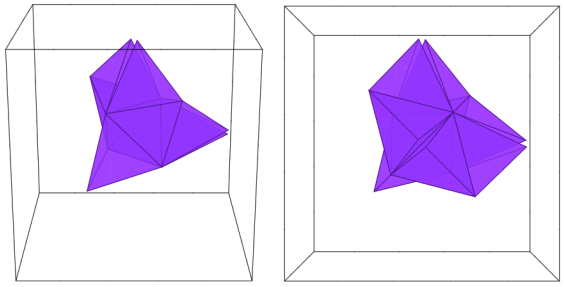
$\mathbf{B}_{11}$

$U = \frac{88}{3}$

n cosets

11 tetrahedra

29.333333333333



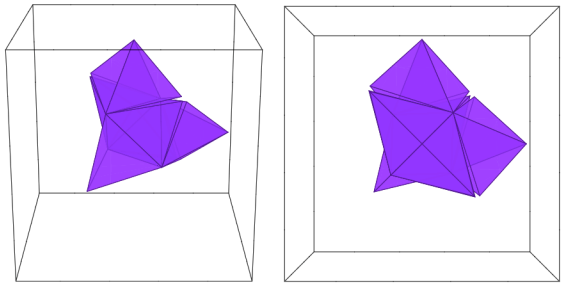
$\mathbf{B}^*_{11}$

$U = \frac{88}{3}$

n cosets

11 tetrahedra

29.333333333333



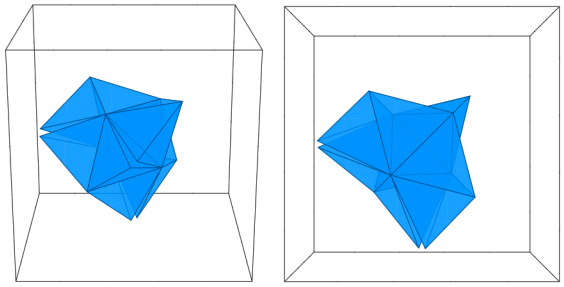
$\mathbf{F}_{11}^{*}$

$U = \frac{88}{3}$

n cosets

11 tetrahedra

29.333333333333



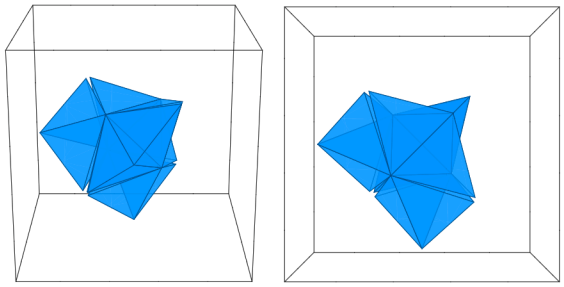
$\mathbf{F}_{11}^{**}$

$U = \frac{88}{3}$

n cosets

11 tetrahedra

29.333333333333



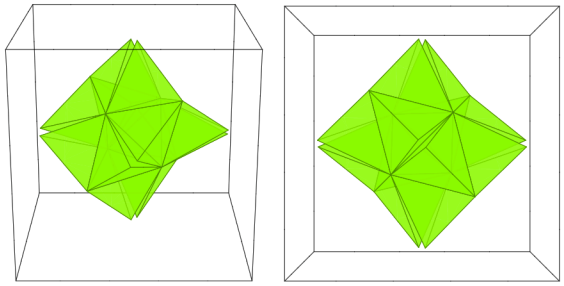
$\mathbf{B}_{17}$

$U = \frac{136}{3}$

n cosets

17 tetrahedra

45.333333333333



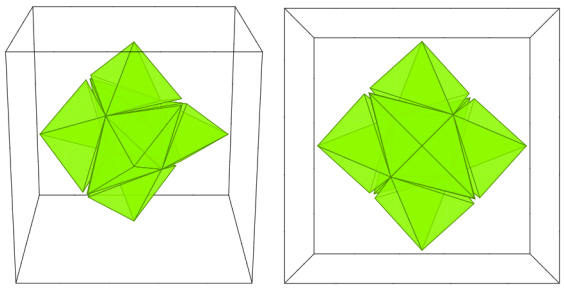
$\mathbf{B}_{17}^{*}$

$U = \frac{136}{3}$

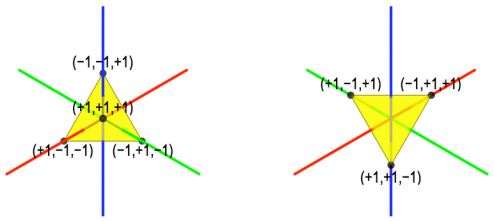
n cosets

17 tetrahedra

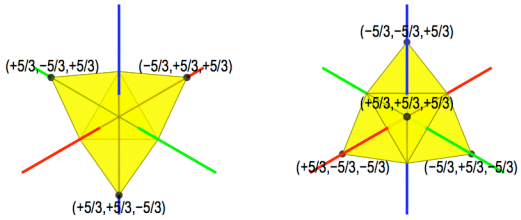
45.333333333333



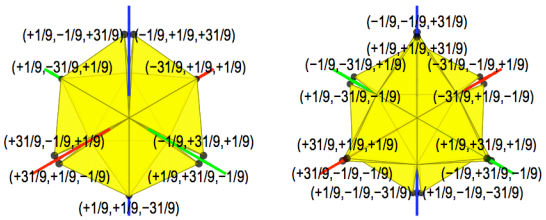
$+\mathbf{B}_1$ $-\mathbf{B}_1$



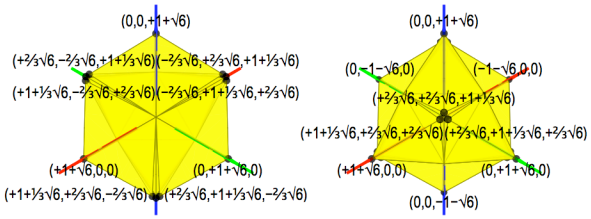
$+\mathbf{B}_5$ $-\mathbf{B}_5$



$+\mathbf{B}_{17}$ $-\mathbf{B}_{17}$

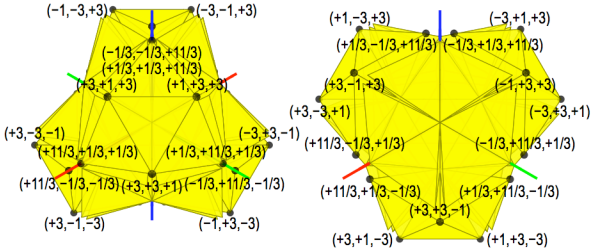


$+\mathbf{B}^*_{17}$ $-\mathbf{B}^*_{17}$



$+\mathbf{B}_{41}$ $-\mathbf{B}_{41}$

$+\mathbf{B}_{57}$ $-\mathbf{B}_{57}$



$+\mathbf{icos}$ $-\mathbf{icos}$

0

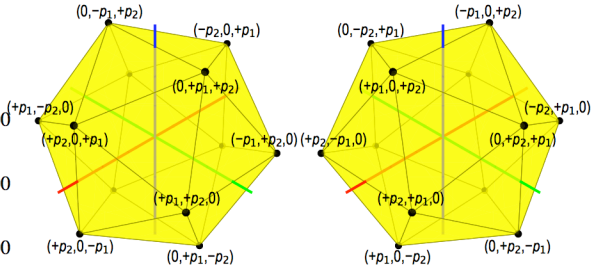
0.000000000000

$p_1 = +6-2\sqrt{5}$

1.527864045000

$p_2 = -2+2\sqrt{5}$

2.472135955000



$+\mathbf{V}_{20}$ $-\mathbf{V}_{20}$

$q_0 = \frac{4}{3}-\frac{1}{2}\sqrt{2}-\frac{1}{6}\sqrt{10}$

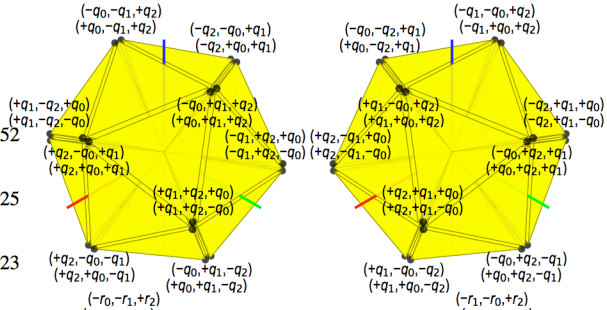
0.099180275452

$q_1 = \frac{4}{3}+\frac{1}{2}\sqrt{2}-\frac{1}{6}\sqrt{10}$

1.513393837825

$q_2 = \frac{4}{3}+\frac{1}{3}\sqrt{10}$

2.387425886723



$r_0 = -\frac{2}{3}+\frac{2}{3}\sqrt{5}-\frac{1}{6}\sqrt{2}-\frac{1}{6}\sqrt{10}$

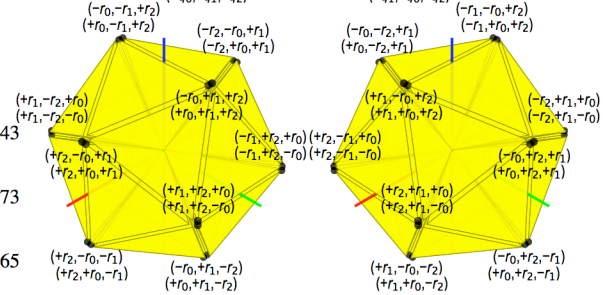
0.061296781243

$r_1 = \sqrt{2}$

1.414213562373

$r_2 = +\frac{2}{3}+\frac{2}{3}\sqrt{5}-\frac{1}{6}\sqrt{2}+\frac{1}{6}\sqrt{10}$

2.448722667965



0

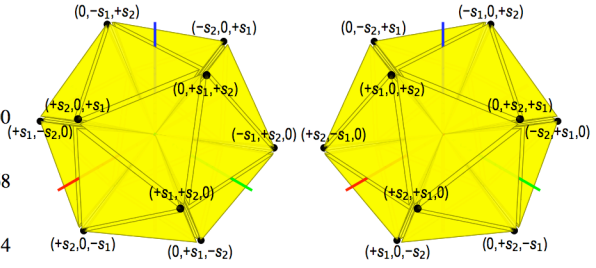
0.000000000000

$s_1 = +\frac{2}{3}+\frac{2}{3}\sqrt{5}+\frac{1}{3}\sqrt{2}-\frac{1}{3}\sqrt{10}$

1.574690619068

$s_2 = -\frac{2}{3}+\frac{2}{3}\sqrt{5}+\frac{1}{3}\sqrt{2}+\frac{1}{3}\sqrt{10}$

2.349542392514



Chapter 10
Hexagonal packing of the tetrahelix

10.1 Vertices

Start with a given tetrahedron $\{\mathbf{a}_{-\frac{3}{2}}, \mathbf{a}_{-\frac{1}{2}}, \mathbf{a}_{+\frac{1}{2}}, \mathbf{a}_{+\frac{3}{2}}\}$

Any 4 sequential points $\{\mathbf{a}_{t-\frac{3}{2}}, \mathbf{a}_{t-\frac{1}{2}}, \mathbf{a}_{t+\frac{1}{2}}, \mathbf{a}_{t+\frac{3}{2}}\}_{t\in\mathbf{Z}}$ define a tetrahedron

For any two adjacent tetrahedra $\{\mathbf{a}_{t-2}, \mathbf{a}_{t-1}, \mathbf{a}_t, \mathbf{a}_{t+1}\}, \{\mathbf{a}_{t-1}, \mathbf{a}_t, \mathbf{a}_{t+1}, \mathbf{a}_{t+2}\}_{t\in\mathbf{Z}\pm\frac{1}{2}}$
the two non-shared vertices are reflected through the center of the shared face

The difference equation is $\frac{1}{2}(\mathbf{a}_{t+2} + \mathbf{a}_{t-2}) = \frac{1}{3}(\mathbf{a}_{t+1} + \mathbf{a}_t + \mathbf{a}_{t-1})$

Let $\zeta = -\frac{2}{3} + \frac{1}{3}i\sqrt{5}$

Let $\alpha = \pi - \cos^{-1}(\frac{2}{3})$ $\quad \begin{cases} \frac{1}{2}\alpha = \cos^{-1}(\frac{1}{6}\sqrt{6}) \\ \frac{3}{2}\alpha = \pi + \cos^{-1}(\frac{7}{18}\sqrt{6}) \end{cases}$

The characteristic is $(T^4+1) - \frac{2}{3}(T^3+T^2+T) = (T-\zeta)(T-\bar{\zeta})(T-1)^2$

$\ln(\zeta) = \ln|\zeta| + i\arg(\zeta) = \ln(1) + i\alpha = i\alpha$
$\zeta^t\mathbf{o} + \bar{\zeta}^t\bar{\mathbf{o}} = \exp(t\ln(\zeta))\mathbf{o} + \exp(t\ln(\bar{\zeta}))\bar{\mathbf{o}} = \exp(i\alpha t)\mathbf{o} + \exp(-i\alpha t)\bar{\mathbf{o}}$
$= \cos(\alpha t)(\mathbf{o}+\bar{\mathbf{o}}) + i\sin(\alpha t)(\mathbf{o}-\bar{\mathbf{o}})$

$\mathbf{a}_t = \zeta^t\mathbf{o} + \bar{\zeta}^t\bar{\mathbf{o}} + (t\mathbf{r}+\mathbf{s})1^t = \cos(\alpha t)\mathbf{p} + \sin(\alpha t)\mathbf{q} + t\mathbf{r} + \mathbf{s}$

$$\begin{cases} \mathbf{p} = \frac{3}{20}\sqrt{6}(-\mathbf{a}_{-\frac{3}{2}} + \mathbf{a}_{-\frac{1}{2}} + \mathbf{a}_{+\frac{1}{2}} - \mathbf{a}_{+\frac{3}{2}}) \\ \mathbf{q} = \frac{3}{100}\sqrt{30}(+\mathbf{a}_{-\frac{3}{2}} - 3\mathbf{a}_{-\frac{1}{2}} + 3\mathbf{a}_{+\frac{1}{2}} - \mathbf{a}_{+\frac{3}{2}}) \\ \mathbf{r} = \frac{1}{10}(-3\mathbf{a}_{-\frac{3}{2}} - \mathbf{a}_{-\frac{1}{2}} + \mathbf{a}_{+\frac{1}{2}} + 3\mathbf{a}_{+\frac{3}{2}}) \\ \mathbf{s} = \frac{1}{20}(+3\mathbf{a}_{-\frac{3}{2}} + 7\mathbf{a}_{-\frac{1}{2}} + 7\mathbf{a}_{+\frac{1}{2}} + 3\mathbf{a}_{+\frac{3}{2}}) \end{cases}$$

$$\text{Let } \begin{cases} \mathbf{a}_{-\frac{3}{2}} = \langle -1, +1, -1\rangle \\ \mathbf{a}_{-\frac{1}{2}} = \langle +1, -1, -1\rangle \\ \mathbf{a}_{+\frac{1}{2}} = \langle +1, +1, +1\rangle \\ \mathbf{a}_{+\frac{3}{2}} = \langle -1, -1, +1\rangle \end{cases} \quad \rightarrow \quad \begin{cases} \mathbf{p} = \frac{3}{5}\sqrt{6}\langle 1,0,0\rangle \\ \mathbf{q} = \frac{3}{5}\sqrt{6}\langle 0, \frac{2}{5}\sqrt{5}, \frac{1}{5}\sqrt{5}\rangle \\ \mathbf{r} = \frac{2}{5}\sqrt{5}\langle 0, -\frac{1}{5}\sqrt{5}, \frac{2}{5}\sqrt{5}\rangle \\ \mathbf{s} = \frac{2}{5}\langle 1,0,0\rangle \end{cases}$$

Apply the transformation $\langle x,y,z\rangle \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{5}\sqrt{5} & \frac{1}{5}\sqrt{5} \\ 0 & -\frac{1}{5}\sqrt{5} & \frac{2}{5}\sqrt{5} \end{bmatrix} \langle x,y,z\rangle - \frac{2}{5}\langle 1,0,0\rangle$

$$\text{Let } \begin{cases} \mathbf{a}_{-\frac{3}{2}} = \langle -\frac{7}{5}, +\frac{1}{5}\sqrt{5}, -\frac{3}{5}\sqrt{5}\rangle \\ \mathbf{a}_{-\frac{1}{2}} = \langle +\frac{3}{5}, -\frac{3}{5}\sqrt{5}, -\frac{1}{5}\sqrt{5}\rangle \\ \mathbf{a}_{+\frac{1}{2}} = \langle +\frac{3}{5}, +\frac{3}{5}\sqrt{5}, +\frac{1}{5}\sqrt{5}\rangle \\ \mathbf{a}_{+\frac{3}{2}} = \langle -\frac{7}{5}, -\frac{1}{5}\sqrt{5}, +\frac{3}{5}\sqrt{5}\rangle \end{cases} \quad \rightarrow \quad \begin{cases} \mathbf{p} = \frac{3}{5}\sqrt{6}\langle 1,0,0\rangle \\ \mathbf{q} = \frac{3}{5}\sqrt{6}\langle 0,1,0\rangle \\ \mathbf{r} = \frac{2}{5}\sqrt{5}\langle 0,0,1\rangle \\ \mathbf{s} = \langle 0,0,0\rangle \end{cases}$$

10.2 Spirals

$$\text{There are 3 types of edges } \begin{cases} \{\mathbf{a}_{t-\frac{1}{2}}, \mathbf{a}_{t+\frac{1}{2}}\}_{t\in\mathbf{Z}} \\ \{\mathbf{a}_{t-1}, \mathbf{a}_{t+1}\}_{t\in\mathbf{Z}\pm\frac{1}{2}} \\ \{\mathbf{a}_{t-\frac{3}{2}}, \mathbf{a}_{t+\frac{3}{2}}\}_{t\in\mathbf{Z}} \end{cases} \rightarrow \text{3 types of spirals } \begin{cases} \mathbf{a}_t^s \\ \mathbf{b}_t^s \\ \mathbf{c}_t^s \end{cases}$$

There are 6 cosets of vertices $\begin{cases} \{\mathbf{a}_{1t}\}_{t\in\mathbf{Z}\pm\frac{1}{2}} \\ \{\mathbf{a}_{2t-\frac{1}{2}}\},\{\mathbf{a}_{2t+\frac{1}{2}}\}_{t\in\mathbf{Z}} \\ \{\mathbf{a}_{3t-1}\},\{\mathbf{a}_{3t}\},\{\mathbf{a}_{3t+1}\}_{t\in\mathbf{Z}\pm\frac{1}{2}} \end{cases}$ $\rightarrow$ 6 spirals $\begin{cases} \mathbf{a}_t^0 \\ \mathbf{b}_t^{-\frac{1}{2}},\mathbf{b}_t^{+\frac{1}{2}} \\ \mathbf{c}_t^{-1},\mathbf{c}_t^{0},\mathbf{c}_t^{+1} \end{cases}$

$$\begin{cases} 1\alpha = \pi - \cos^{-1}(\frac{2}{3}) \\ 2\alpha = \pi + \cos^{-1}(\frac{1}{9}) \\ 3\alpha = 2\pi + \cos^{-1}(\frac{22}{27}) \end{cases} \rightarrow \begin{cases} \alpha = 1\alpha \bmod 2\pi = \alpha = \pi - \cos^{-1}(\frac{2}{3}) \\ \beta = 2\alpha \bmod 2\pi = 2\alpha - 2\pi = -\pi + \cos^{-1}(\frac{1}{9}) \\ \gamma = 3\alpha \bmod 2\pi = 3\alpha - 2\pi = \cos^{-1}(\frac{22}{27}) \end{cases}$$

$$\begin{cases} \mathbf{a}_t^s = \cos(\alpha t + \alpha s)\mathbf{p} + \sin(\alpha t + \alpha s)\mathbf{q} + (1t+s)\mathbf{r} + \mathbf{s} \\ \mathbf{b}_t^s = \cos(\beta t + \alpha s)\mathbf{p} + \sin(\beta t + \alpha s)\mathbf{q} + (2t+s)\mathbf{r} + \mathbf{s} \\ \mathbf{c}_t^s = \cos(\gamma t + \alpha s)\mathbf{p} + \sin(\gamma t + \alpha s)\mathbf{q} + (3t+s)\mathbf{r} + \mathbf{s} \end{cases}$$

Let $\begin{cases} T = t + 1s \\ T = t + \frac{1}{2}s \\ T = t + \frac{1}{3}s \end{cases} \rightarrow \begin{cases} \alpha t + \alpha s = \alpha T + (\alpha - 1\alpha)s = \alpha T \\ \beta t + \alpha s = \beta T + (\alpha - \frac{1}{2}\beta)s = \beta T + \pi s \\ \gamma t + \alpha s = \gamma T + (\alpha - \frac{1}{3}\gamma)s = \gamma T + \frac{2}{3}\pi s \end{cases}$

$$\begin{cases} \mathbf{a}_T^s = \cos(\alpha T)\mathbf{p} + \sin(\alpha T)\mathbf{q} + 1T\mathbf{r} + \mathbf{s} \\ \mathbf{b}_T^s = \cos(\beta T + \pi s)\mathbf{p} + \sin(\beta T + \pi s)\mathbf{q} + 2T\mathbf{r} + \mathbf{s} \\ \mathbf{c}_T^s = \cos(\gamma T + \frac{2}{3}\pi s)\mathbf{p} + \sin(\gamma T + \frac{2}{3}\pi s)\mathbf{q} + 3T\mathbf{r} + \mathbf{s} \end{cases}$$

The spirals are centered (translationally) along the axis $\mathbf{r}$
The spirals are equally spaced (rotationally) about the axis $\mathbf{r}$

10.3 Best approximation

The radius of the spirals is $|\mathbf{p}| = |\mathbf{q}| = \frac{3}{5}\sqrt{6} \approx 1.469693845670$

The midpoint of an edge $\{\mathbf{a}_i, \mathbf{a}_j\}$ is $\frac{1}{2}(\mathbf{a}_i + \mathbf{a}_j)$

If we project normal to the axis $\mathbf{r}$, the image of the midpoint is $\frac{1}{2}(i+j)\mathbf{r} + \mathbf{s}$

The apothem of $\mathbf{a}_t^s$ is $\frac{3}{5} \approx 0.600000000000$

$$\begin{cases} \frac{1}{2}(\mathbf{a}_{-\frac{3}{2}} + \mathbf{a}_{-\frac{1}{2}}) = \langle -\frac{2}{5}, -\frac{1}{5}\sqrt{5}, -\frac{2}{5}\sqrt{5} \rangle \\ -1\mathbf{r} + \mathbf{s} = \langle 0, 0, -\frac{2}{5}\sqrt{5} \rangle \end{cases}$$
$$\begin{cases} \frac{1}{2}(\mathbf{a}_{-\frac{1}{2}} + \mathbf{a}_{+\frac{1}{2}}) = \langle +\frac{3}{5}, 0, 0 \rangle \\ 0\mathbf{r} + \mathbf{s} = \langle 0, 0, 0 \rangle \end{cases}$$
$$\begin{cases} \frac{1}{2}(\mathbf{a}_{+\frac{1}{2}} + \mathbf{a}_{+\frac{3}{2}}) = \langle -\frac{2}{5}, +\frac{1}{5}\sqrt{5}, +\frac{2}{5}\sqrt{5} \rangle \\ +1\mathbf{r} + \mathbf{s} = \langle 0, 0, +\frac{2}{5}\sqrt{5} \rangle \end{cases}$$

The apothem of $\mathbf{b}_t^s$ is $\frac{2}{5}\sqrt{6} \approx 0.979795897113$

$$\begin{cases} \frac{1}{2}(\mathbf{a}_{-\frac{3}{2}} + \mathbf{a}_{+\frac{1}{2}}) = \langle -\frac{2}{5}, +\frac{2}{5}\sqrt{5}, -\frac{1}{5}\sqrt{5} \rangle \\ -\frac{1}{2}\mathbf{r} + \mathbf{s} = \langle 0, 0, -\frac{1}{5}\sqrt{5} \rangle \end{cases}$$
$$\begin{cases} \frac{1}{2}(\mathbf{a}_{-\frac{1}{2}} + \mathbf{a}_{+\frac{3}{2}}) = \langle -\frac{2}{5}, -\frac{2}{5}\sqrt{5}, +\frac{1}{5}\sqrt{5} \rangle \\ +\frac{1}{2}\mathbf{r} + \mathbf{s} = \langle 0, 0, +\frac{1}{5}\sqrt{5} \rangle \end{cases}$$

The apothem of $\mathbf{c}_t^s$ is $\frac{7}{5} \approx 1.400000000000$

$$\begin{cases} \frac{1}{2}(\mathbf{a}_{-\frac{3}{2}} + \mathbf{a}_{+\frac{3}{2}}) = \langle -\frac{7}{5}, 0, 0 \rangle \\ 0\mathbf{r} + \mathbf{s} = \langle 0, 0, 0 \rangle \end{cases}$$

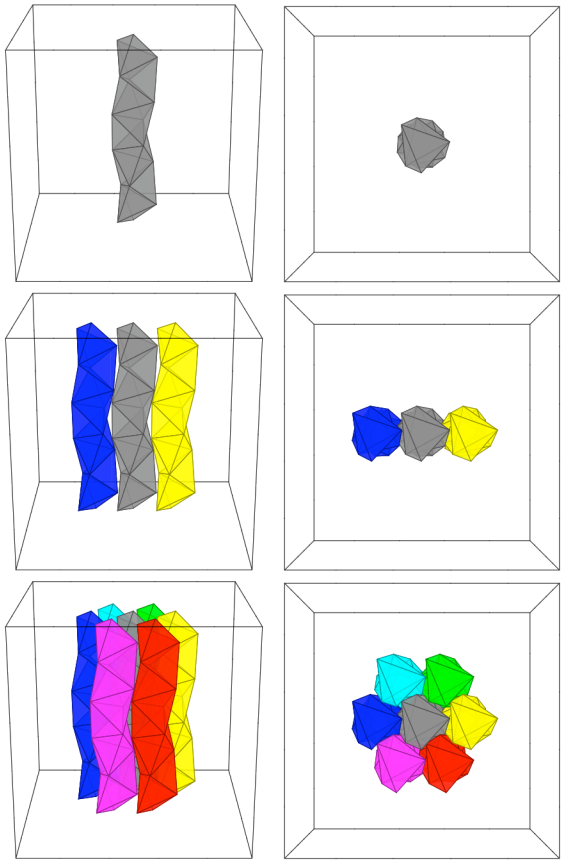
The spirals are contained in a cylinder of radius $|\mathbf{p}| = |\mathbf{q}| = \frac{3}{5}\sqrt{6}$
If we project parallel to the axis $\mathbf{r}$, the image of the cylinder is a circle
Since the angles α, β, γ are irrational,

the images of the vertices form a dense subset of the circle

The spirals $\mathbf{c}_t^s$ give the best approximation for the edges of the tetrahedra

6 neighbors

symmetry group $\mathbf{Z}_6$



10.4 Intersection equations

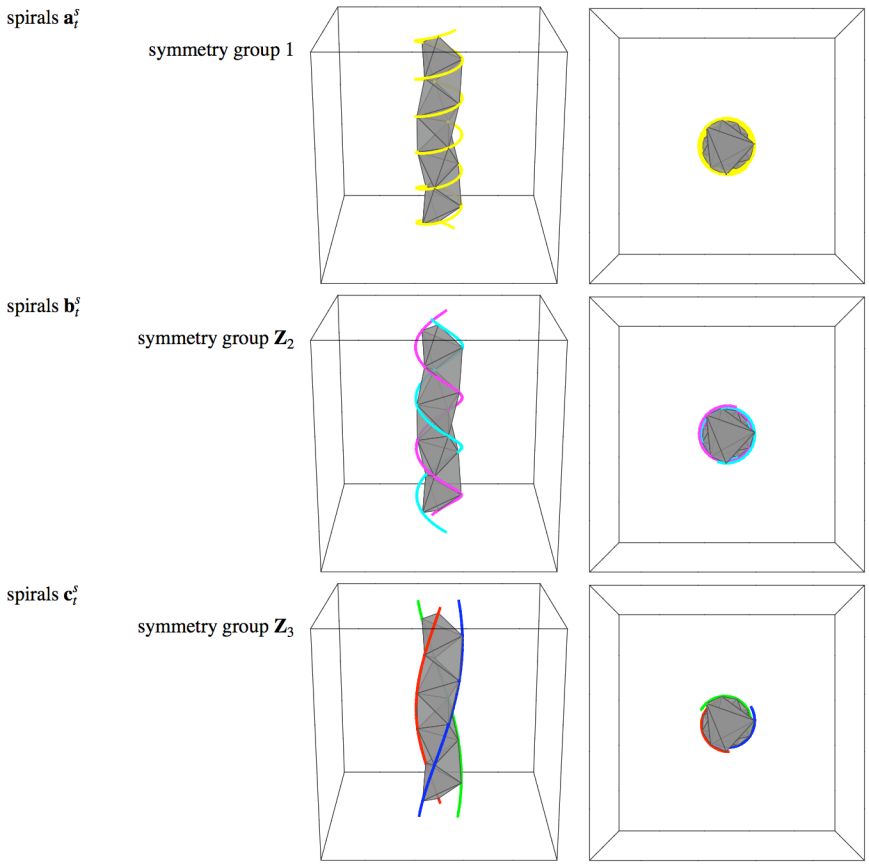
Consider 2 parallel helices

$$\begin{cases} \mathbf{c}_t^s = \cos(\gamma t + \frac{2}{3}\pi s)\mathbf{p} + \sin(\gamma t + \frac{2}{3}\pi s)\mathbf{q} + 3t\mathbf{r} + \mathbf{s} \\ \mathbf{c}_t^S = (u + \cos(\gamma t + \frac{2}{3}\pi S))\mathbf{p} + (v + \sin(\gamma t + \frac{2}{3}\pi S))\mathbf{q} + 3t\mathbf{r} + \mathbf{s} \end{cases}$$

Since $\{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$ is an orthogonal basis of $\mathbf{R}^3$, we equate coefficients

$$\begin{cases} u = \cos(\gamma t + \frac{2}{3}\pi s) - \cos(\gamma t + \frac{2}{3}\pi S) \\ v = \sin(\gamma t + \frac{2}{3}\pi s) - \sin(\gamma t + \frac{2}{3}\pi S) \end{cases}$$

$$\begin{aligned} u^2 + v^2 &= 2 - 2\cos(\gamma t + \tfrac{2}{3}\pi s)\cos(\gamma t + \tfrac{2}{3}\pi S) - 2\sin(\gamma t + \tfrac{2}{3}\pi s)\sin(\gamma t + \tfrac{2}{3}\pi S) \\ &= 2 - 2\cos((\gamma t + \tfrac{2}{3}\pi s) - (\gamma t + \tfrac{2}{3}\pi S)) \\ &= 2 - 2\cos(\tfrac{2}{3}\pi(s - S)) \end{aligned}$$



There are 2 cases $\begin{cases} s-S=0 & (\text{mod } 3) \\ s-S=\pm 1 & (\text{mod } 3) \end{cases}$

case $s-S=0 \pmod 3$

$u^2+v^2=0$

$u=0,\ v=0$

$\mathbf{c}_t^s=\mathbf{c}_t^S \quad \rightarrow \quad$ contradiction

case $s-S=\pm 1 \pmod 3$

$u^2+v^2=3$

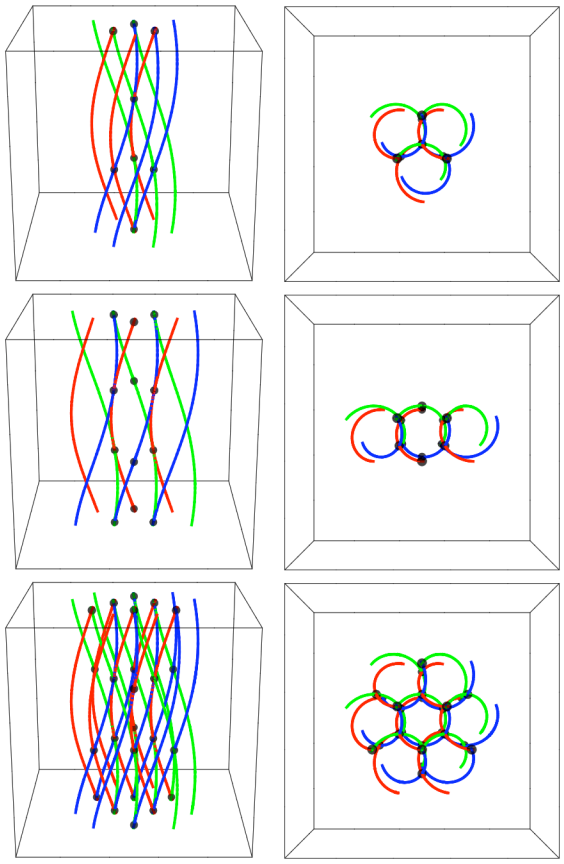
$u=\sqrt{3}\cos(\theta),\ v=\sqrt{3}\sin(\theta)$

$$\begin{cases} u=(\cos(\gamma t)\cos(\frac{2}{3}\pi s)-\sin(\gamma t)\sin(\frac{2}{3}\pi s))-(\cos(\gamma t)\cos(\frac{2}{3}\pi S)-\sin(\gamma t)\sin(\frac{2}{3}\pi S)) \\ v=(\sin(\gamma t)\cos(\frac{2}{3}\pi s)+\cos(\gamma t)\sin(\frac{2}{3}\pi s))-(\sin(\gamma t)\cos(\frac{2}{3}\pi S)+\cos(\gamma t)\sin(\frac{2}{3}\pi S)) \end{cases}$$

$$\begin{cases} \sqrt{3}\cos(\theta)=(\cos(\frac{2}{3}\pi s)-\cos(\frac{2}{3}\pi S))\cos(\gamma t)-(\sin(\frac{2}{3}\pi s)-\sin(\frac{2}{3}\pi S))\sin(\gamma t) \\ \sqrt{3}\sin(\theta)=(\cos(\frac{2}{3}\pi s)-\cos(\frac{2}{3}\pi S))\sin(\gamma t)+(\sin(\frac{2}{3}\pi s)-\sin(\frac{2}{3}\pi S))\cos(\gamma t) \end{cases}$$

intersections

symmetry group $\mathbf{Z}_3$



10.5 Nearest neighbors

There are 6 cases $\quad\rightarrow\quad$ 6 nearest neighbors
The 6 neighbors are equally spaced, with the symmetries of a regular hexagon

case $s=-1, S=0$

$$\begin{cases} \sqrt{3}\cos(\theta) = -\frac{3}{2}\cos(\gamma t) + \frac{1}{2}\sqrt{3}\sin(\gamma t) = \sqrt{3}\cos(\gamma t - \frac{5}{6}\pi) \\ \sqrt{3}\sin(\theta) = -\frac{3}{2}\cos(\gamma t) - \frac{1}{2}\sqrt{3}\sin(\gamma t) = \sqrt{3}\sin(\gamma t - \frac{5}{6}\pi) \end{cases}$$

$$\theta = \gamma t - \tfrac{5}{6}\pi$$

case $s=-1, S=+1$

$$\begin{cases} \sqrt{3}\cos(\theta) = 0\cos(\gamma t) + \sqrt{3}\sin(\gamma t) = \sqrt{3}\cos(\gamma t - \frac{1}{2}\pi) \\ \sqrt{3}\sin(\theta) = 0\cos(\gamma t) - \sqrt{3}\sin(\gamma t) = \sqrt{3}\sin(\gamma t - \frac{1}{2}\pi) \end{cases}$$

$$\theta = \gamma t - \tfrac{1}{2}\pi$$

case $s=0, S=+1$

$$\begin{cases} \sqrt{3}\cos(\theta) = +\frac{3}{2}\cos(\gamma t) + \frac{1}{2}\sqrt{3}\sin(\gamma t) = \sqrt{3}\cos(\gamma t - \frac{1}{6}\pi) \\ \sqrt{3}\sin(\theta) = +\frac{3}{2}\cos(\gamma t) - \frac{1}{2}\sqrt{3}\sin(\gamma t) = \sqrt{3}\sin(\gamma t - \frac{1}{6}\pi) \end{cases}$$

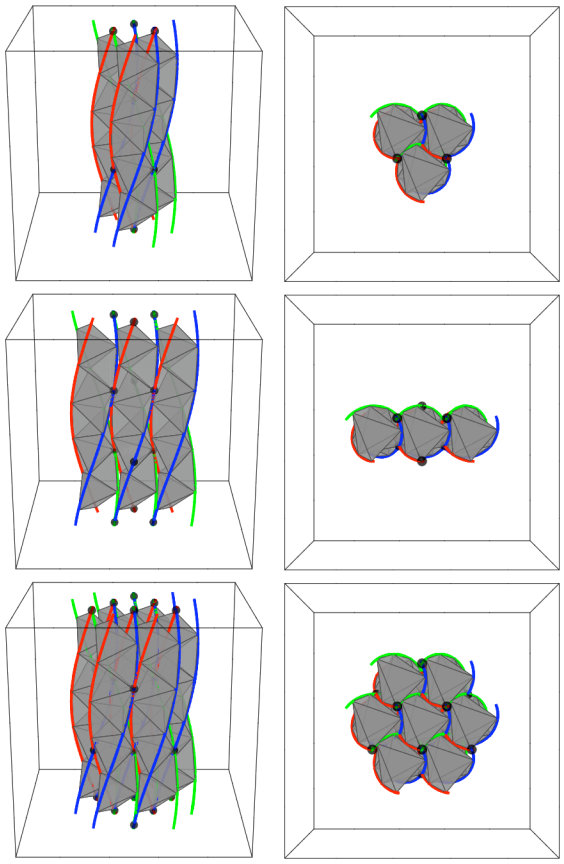
$$\theta = \gamma t - \tfrac{1}{6}\pi$$

intersections

symmetry group $\mathbf{Z}_3$

6 neighbors

symmetry group $\mathbf{Z}_6$



case $s=0$, $S=-1$

$$\begin{cases} \sqrt{3}\cos(\theta) = +\frac{3}{2}\cos(\gamma t) - \frac{1}{2}\sqrt{3}\sin(\gamma t) = \sqrt{3}\cos(\gamma t + \frac{1}{6}\pi) \\ \sqrt{3}\sin(\theta) = +\frac{3}{2}\cos(\gamma t) + \frac{1}{2}\sqrt{3}\sin(\gamma t) = \sqrt{3}\sin(\gamma t + \frac{1}{6}\pi) \end{cases}$$

$$\theta = \gamma t + \tfrac{1}{6}\pi$$

case $s=+1$, $S=-1$

$$\begin{cases} \sqrt{3}\cos(\theta) = 0\cos(\gamma t) - \sqrt{3}\sin(\gamma t) = \sqrt{3}\cos(\gamma t + \frac{1}{2}\pi) \\ \sqrt{3}\sin(\theta) = 0\cos(\gamma t) + \sqrt{3}\sin(\gamma t) = \sqrt{3}\sin(\gamma t + \frac{1}{2}\pi) \end{cases}$$

$$\theta = \gamma t + \tfrac{1}{2}\pi$$

case $s=+1$, $S=0$

$$\begin{cases} \sqrt{3}\cos(\theta) = -\frac{3}{2}\cos(\gamma t) - \frac{1}{2}\sqrt{3}\sin(\gamma t) = \sqrt{3}\cos(\gamma t + \frac{5}{6}\pi) \\ \sqrt{3}\sin(\theta) = -\frac{3}{2}\cos(\gamma t) + \frac{1}{2}\sqrt{3}\sin(\gamma t) = \sqrt{3}\sin(\gamma t + \frac{5}{6}\pi) \end{cases}$$

$$\theta = \gamma t + \tfrac{5}{6}\pi$$

For symmetry, let $t = \{\cdots, -\pi/\gamma, -\frac{2}{3}\pi/\gamma, -\frac{1}{3}\pi/\gamma, 0, +\frac{1}{3}\pi/\gamma, +\frac{2}{3}\pi/\gamma, +\pi/\gamma, \cdots\}$

$\rightarrow \quad \theta = \{\cdots, -\frac{5}{6}\pi, -\frac{1}{2}\pi, -\frac{1}{6}\pi, +\frac{1}{6}\pi, +\frac{1}{2}\pi, +\frac{5}{6}\pi, \cdots\}$

For symmetry, let $t = \{\cdots, -\frac{5}{6}\pi/\gamma, -\frac{1}{2}\pi/\gamma, -\frac{1}{6}\pi/\gamma, +\frac{1}{6}\pi/\gamma, +\frac{1}{2}\pi/\gamma, +\frac{5}{6}\pi/\gamma, \cdots\}$

$\rightarrow \quad \theta = \{\cdots, -\pi, -\frac{2}{3}\pi, -\frac{1}{3}\pi, 0, +\frac{1}{3}\pi, +\frac{2}{3}\pi, +\pi, \cdots\}$

10.6 Intersection points

There are 2 cosets of angles $\begin{cases} t = \{\cdots, -\frac{5}{6}\pi/\gamma, -\frac{1}{6}\pi/\gamma, +\frac{1}{2}\pi/\gamma, \cdots\} \\ t = \{\cdots, -\frac{1}{2}\pi/\gamma, +\frac{1}{6}\pi/\gamma, +\frac{5}{6}\pi/\gamma, \cdots\} \end{cases}$

These correspond to 2 cosets of intersection points
'positive' triangle:

case $t = -\frac{5}{6}\pi/\gamma$

$$\begin{cases} s=+1 & \mathbf{c}_t^s = +\frac{1}{2}\sqrt{3}\mathbf{p} - \frac{1}{2}\mathbf{q} - \frac{1}{2}(\pi/\gamma)\mathbf{r} + \mathbf{s} \\ s=-1 & \mathbf{c}_t^s = \quad 0\mathbf{p} + 1\mathbf{q} - \frac{1}{2}(\pi/\gamma)\mathbf{r} + \mathbf{s} \\ s=0 & \mathbf{c}_t^s = -\frac{1}{2}\sqrt{3}\mathbf{p} - \frac{1}{2}\mathbf{q} - \frac{1}{2}(\pi/\gamma)\mathbf{r} + \mathbf{s} \end{cases}$$

case $t = -\frac{1}{6}\pi/\gamma$

$$\begin{cases} s=0 & \mathbf{c}_t^s = +\frac{1}{2}\sqrt{3}\mathbf{p} - \frac{1}{2}\mathbf{q} - \frac{1}{2}(\pi/\gamma)\mathbf{r} + \mathbf{s} \\ s=+1 & \mathbf{c}_t^s = \quad 0\mathbf{p} + 1\mathbf{q} - \frac{1}{2}(\pi/\gamma)\mathbf{r} + \mathbf{s} \\ s=-1 & \mathbf{c}_t^s = -\frac{1}{2}\sqrt{3}\mathbf{p} - \frac{1}{2}\mathbf{q} - \frac{1}{2}(\pi/\gamma)\mathbf{r} + \mathbf{s} \end{cases}$$

case $t = +\frac{1}{2}\pi/\gamma$

$$\begin{cases} s=-1 & \mathbf{c}_t^s = +\frac{1}{2}\sqrt{3}\mathbf{p} - \frac{1}{2}\mathbf{q} + \frac{3}{2}(\pi/\gamma)\mathbf{r} + \mathbf{s} \\ s=0 & \mathbf{c}_t^s = \quad 0\mathbf{p} + 1\mathbf{q} + \frac{3}{2}(\pi/\gamma)\mathbf{r} + \mathbf{s} \\ s=+1 & \mathbf{c}_t^s = -\frac{1}{2}\sqrt{3}\mathbf{p} - \frac{1}{2}\mathbf{q} + \frac{3}{2}(\pi/\gamma)\mathbf{r} + \mathbf{s} \end{cases}$$

'negative' triangle:

case $t = -\frac{1}{2}\pi/\gamma$

$$\begin{cases} s=-1 & \mathbf{c}_t^s = -\frac{1}{2}\sqrt{3}\mathbf{p} + \frac{1}{2}\mathbf{q} - \frac{3}{2}(\pi/\gamma)\mathbf{r} + \mathbf{s} \\ s=0 & \mathbf{c}_t^s = \quad 0\mathbf{p} - 1\mathbf{q} - \frac{3}{2}(\pi/\gamma)\mathbf{r} + \mathbf{s} \\ s=+1 & \mathbf{c}_t^s = +\frac{1}{2}\sqrt{3}\mathbf{p} + \frac{1}{2}\mathbf{q} - \frac{3}{2}(\pi/\gamma)\mathbf{r} + \mathbf{s} \end{cases}$$

case $t = +\frac{1}{6}\pi/\gamma$

$$\begin{cases} s=+1 & \mathbf{c}_t^s = -\frac{1}{2}\sqrt{3}\mathbf{p} + \frac{1}{2}\mathbf{q} + \frac{1}{2}(\pi/\gamma)\mathbf{r} + \mathbf{s} \\ s=-1 & \mathbf{c}_t^s = \quad 0\mathbf{p} - 1\mathbf{q} + \frac{1}{2}(\pi/\gamma)\mathbf{r} + \mathbf{s} \\ s=0 & \mathbf{c}_t^s = +\frac{1}{2}\sqrt{3}\mathbf{p} + \frac{1}{2}\mathbf{q} + \frac{1}{2}(\pi/\gamma)\mathbf{r} + \mathbf{s} \end{cases}$$

case $t = +\frac{5}{6}\pi/\gamma$

$$\begin{cases} s=0 & \mathbf{c}_t^s = -\frac{1}{2}\sqrt{3}\mathbf{p} + \frac{1}{2}\mathbf{q} + \frac{1}{2}(\pi/\gamma)\mathbf{r} + \mathbf{s} \\ s=+1 & \mathbf{c}_t^s = \quad 0\mathbf{p} - 1\mathbf{q} + \frac{1}{2}(\pi/\gamma)\mathbf{r} + \mathbf{s} \\ s=-1 & \mathbf{c}_t^s = +\frac{1}{2}\sqrt{3}\mathbf{p} + \frac{1}{2}\mathbf{q} + \frac{1}{2}(\pi/\gamma)\mathbf{r} + \mathbf{s} \end{cases}$$

The intersection points are equally spaced, with the symmetries of a regular triangle
They occur periodically, with a period $2\pi/\gamma$ along the axis $\mathbf{r}$

10.7 Hexagonal packing

The tetrahelix packing is a (planar) hexagonal lattice
The (planar) lattice basis is $\{\mathbf{p}, \mathbf{q}\}$

Each tetrahelix has 6 nearest neighbors, at offsets $\sqrt{3}\cos(\theta)\mathbf{p} + \sqrt{3}\sin(\theta)\mathbf{q}$
$\theta = \{\cdots, -\pi, -\frac{2}{3}\pi, -\frac{1}{3}\pi, 0, +\frac{1}{3}\pi, +\frac{2}{3}\pi, +\pi, \cdots\}$

The spirals $\mathbf{c}_t^s$ are equally spaced, with the symmetries of a regular triangle
The packing is consistent

The intersection points are equally spaced, with the symmetries of a regular hexagon
They occur in a circle of radius $|\mathbf{p}| = |\mathbf{q}| = \frac{3}{5}\sqrt{6}$
The inscribed hexagon has area $A = \frac{3}{2}\sqrt{3}|\mathbf{p}|^2 = \frac{3}{2}\sqrt{3}|\mathbf{q}|^2 = \frac{81}{25}\sqrt{3}$

The vertices are $\mathbf{a}_t = \cos(\alpha t)\mathbf{p} + \sin(\alpha t)\mathbf{q} + t\mathbf{r} + \mathbf{s}$
Along the axis $\mathbf{r}$, the spacing between tetrahedra is $h = |\mathbf{r}| = \frac{2}{5}\sqrt{5}$

The hexagonal prism has volume $V = hA = \frac{162}{125}\sqrt{15} \approx 5.019386416685$

The packing density is $D = \frac{8}{3}/V = \sqrt{\frac{50000}{177147}} \approx .531273435694$

Note: Each tetrahedron has volume $\frac{8}{3}$

Chapter 11
Cubic family spacings

Periodic packings with 'cubic' symmetry can be described by a single family
with 4 cluster parameters and 1 spacing parameter

Choose a cluster $\mathbf{X}_A$ with coordinates $\langle 0,0,0\rangle$

Nearest neighbors must occur at directions with 'cubic' symmetry
There are 4 possibilities for the spacing

Each cluster $\mathbf{X}_A$ has 12 neighbors $\mathbf{X}_A$ in the directions $\mathbf{Z}_3\langle\pm 2,\pm 2,0\rangle$ with spacing a_A
The clusters $\mathbf{X}_A$ have coordinates $\mathbf{Z}_3 a_A\langle\pm 2,\pm 2,0\rangle$
$\rightarrow \mathbf{Z}_3 a_A\langle\pm 2,\pm 2,0\rangle \cup \langle 0,0,0\rangle$

Each cluster $\mathbf{X}_A$ has 6 neighbors $\mathbf{X}_C$ in the directions $\mathbf{Z}_3\langle\pm 2,0,0\rangle$ with spacing a_C
The clusters $\mathbf{X}_C$ have coordinates $\mathbf{Z}_3 a_C\langle\pm 2,0,0\rangle$

Each cluster $\mathbf{X}_C$ has 6 neighbors $\mathbf{X}_A$ in the directions $\mathbf{Z}_3\langle\pm 2,0,0\rangle$ with spacing a_C
The clusters $\mathbf{X}_A$ have coordinates $\mathbf{Z}_3 a_C\langle\pm 2,0,0\rangle + \mathbf{Z}_3 a_C\langle\pm 2,0,0\rangle$
$= \mathbf{Z}_3 a_C\langle\pm 2,\pm 2,0\rangle \cup \langle 0,0,0\rangle$

Each cluster $\mathbf{X}_A$ has 4 neighbors $\mathbf{X}_B$ in the directions $\mathbf{Z}_2^2\langle a,a,+a\rangle$ with spacing a_B
The clusters $\mathbf{X}_B$ have coordinates $\mathbf{Z}_2^2 a_B\langle 1,1,1\rangle$

Each cluster $\mathbf{X}_B$ has 4 neighbors $\mathbf{X}_A$ in the directions $\mathbf{Z}_2^2\langle a,a,-a\rangle$ with spacing a_B
The clusters $\mathbf{X}_A$ have coordinates $\mathbf{Z}_2^2 a_B\langle +1,+1,+1\rangle + \mathbf{Z}_2^2 a_B\langle -1,-1,-1\rangle$
$= \mathbf{Z}_3 a_B\langle\pm 2,\pm 2,0\rangle \cup \langle 0,0,0\rangle$

Each cluster $\mathbf{X}_A$ has 4 neighbors $\mathbf{X}_D$ in the directions $\mathbf{Z}_2^2\langle a,a,-a\rangle$ with spacing a_D
The clusters $\mathbf{X}_D$ have coordinates $\mathbf{Z}_2^2 a_D\langle -1,-1,-1\rangle$

Each cluster $\mathbf{X}_D$ has 4 neighbors $\mathbf{X}_A$ in the directions $\mathbf{Z}_2^2\langle a,a,+a\rangle$ with spacing a_D
The clusters $\mathbf{X}_A$ have coordinates $\mathbf{Z}_2^2 a_D\langle -1,-1,-1\rangle + \mathbf{Z}_2^2 a_D\langle +1,+1,+1\rangle$
$= \mathbf{Z}_3 a_D\langle\pm 2,\pm 2,0\rangle \cup \langle 0,0,0\rangle$

The coordinates of the clusters $\mathbf{X}_A$ must be the same in all 4 cases
The spacing is a $(= a_A = a_B = a_C = a_D)$

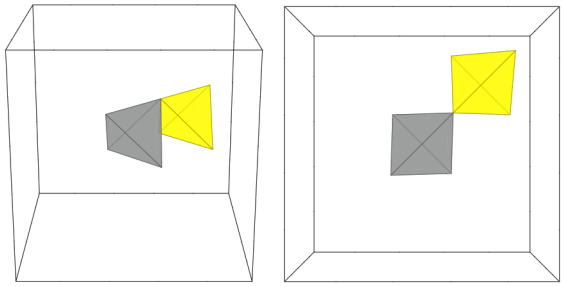
The clusters form a lattice, with 4 cosets $\{\mathbf{X}_A, \mathbf{X}_B, \mathbf{X}_C, \mathbf{X}_D\}$

$\{+\mathbf{B}_1\}$

$a = 1$

$\langle 2a, 2a, 0 \rangle$

1.000000000000

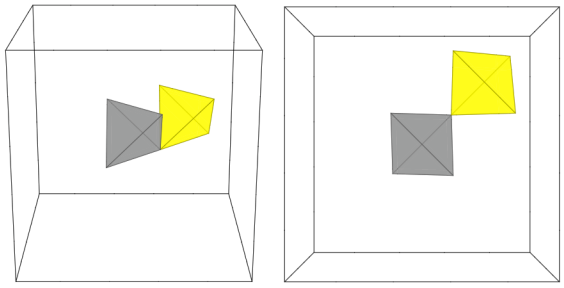


$\{-\mathbf{B}_1\}$

$a = 1$

$\langle 2a, 2a, 0 \rangle$

1.000000000000

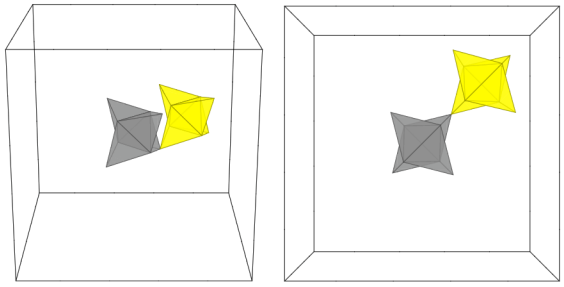


$\{+\mathbf{B}_5\}$

$a = \frac{5}{3}$

$\langle 2a, 2a, 0 \rangle$

1.666666666667

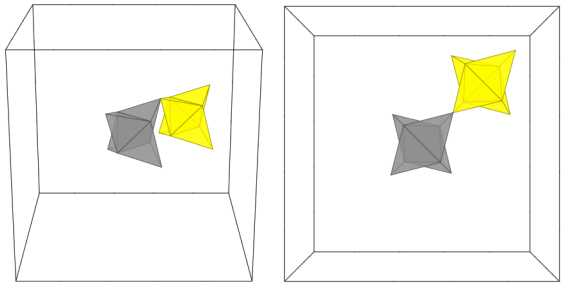


$\{-\mathbf{B}_5\}$

$a = \frac{5}{3}$

$\langle 2a, 2a, 0 \rangle$

1.666666666667

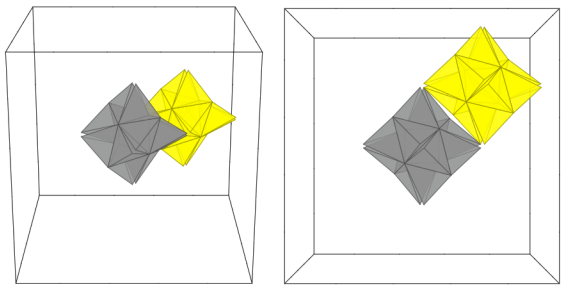


$\{+\mathbf{B}_{17}\}$

$a = \frac{16}{9}$

$\langle 2a,2a,0\rangle$

1.777777777778

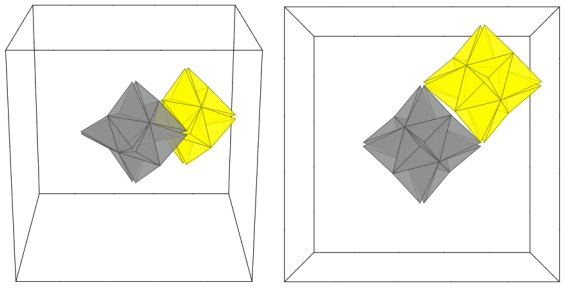


$\{-\mathbf{B}_{17}\}$

$a = \frac{16}{9}$

$\langle 2a,2a,0\rangle$

1.777777777778

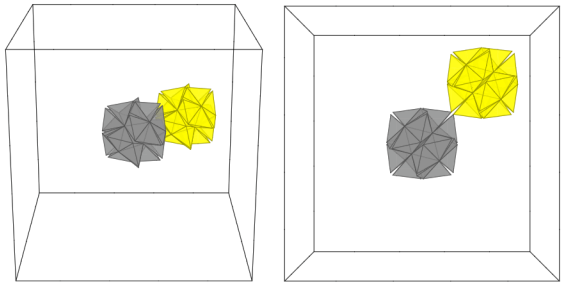


$\{+\mathbf{B}_{41}\}$

$a = \frac{80}{27}$

$\langle 2a,2a,0\rangle$

2.962962962963

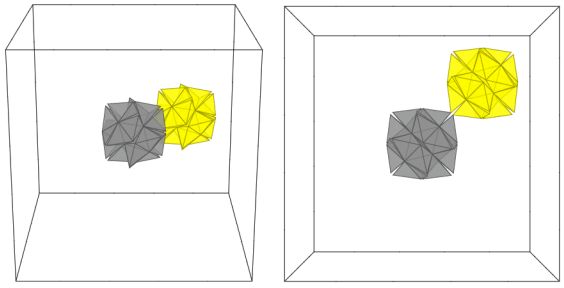


$\{-\mathbf{B}_{41}\}$

$a = \frac{80}{27}$

$\langle 2a,2a,0\rangle$

2.962962962963

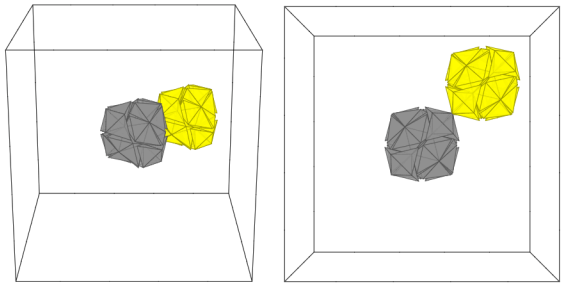


$\{+\mathbf{B}_{57}\}$

$a = 3$

$\langle 2a,2a,0\rangle$

3.000000000000

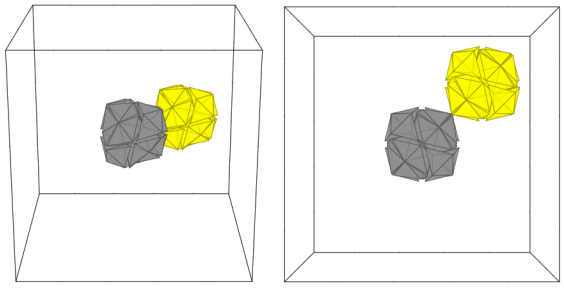


$\{-\mathbf{B}_{57}\}$

$a = 3$

$\langle 2a,2a,0\rangle$

3.000000000000

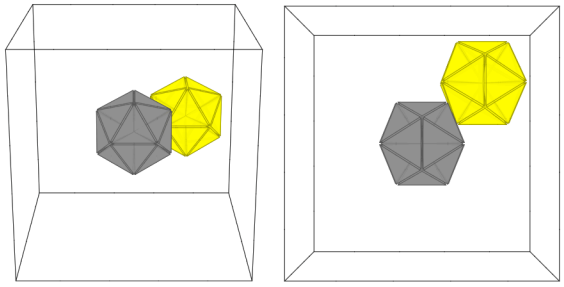


$\{+\mathbf{V}_{20}\}$

$a = \frac{4}{5}\sqrt{5}$

$\langle 2a,2a,0\rangle$

1.788854382000

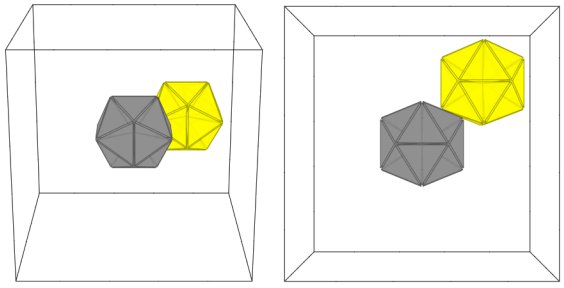


$\{-\mathbf{V}_{20}\}$

$a = \frac{4}{5}\sqrt{5}$

$\langle 2a,2a,0\rangle$

1.788854382000

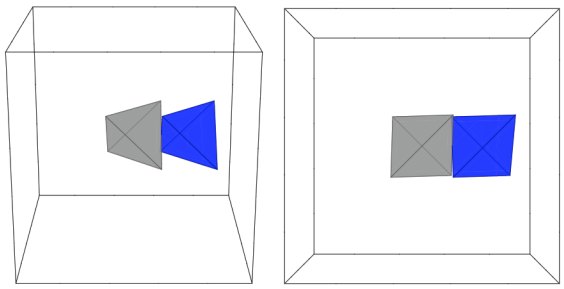


$\{+\mathbf{B}_1,+\mathbf{B}_1\}$

$a = 1$

$\langle 2a,0,0\rangle$

1.000000000000

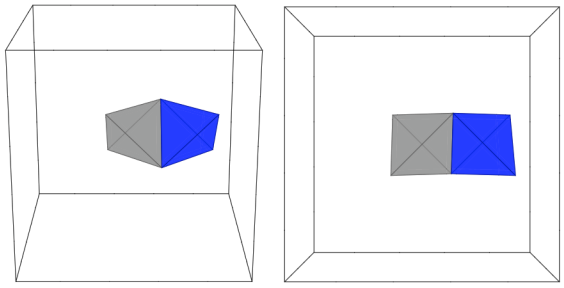


$\{+\mathbf{B}_1,-\mathbf{B}_1\}$

$a = 1$

$\langle 2a,0,0\rangle$

1.000000000000

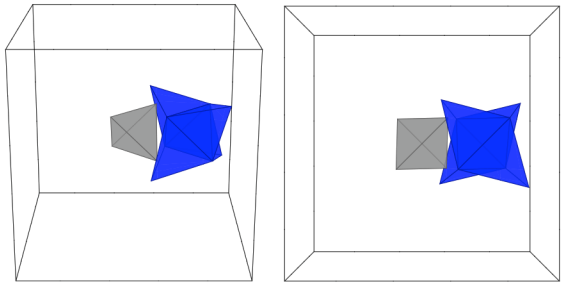


$\{+\mathbf{B}_1,+\mathbf{B}_5\}$

$a = \frac{6}{5}$

$\langle 2a,0,0\rangle$

1.200000000000

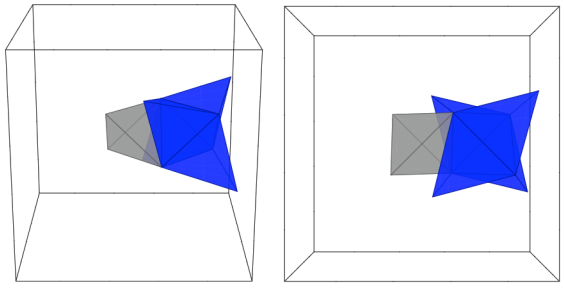


$\{+\mathbf{B}_1,-\mathbf{B}_5\}$

$a = 1$

$\langle 2a,0,0\rangle$

1.000000000000

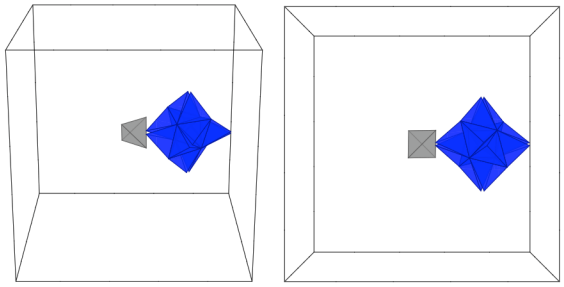


$\{+\mathbf{B}_1,+\mathbf{B}_{17}\}$

$a = \frac{20}{9}$

$\langle 2a,0,0\rangle$

2.222222222222

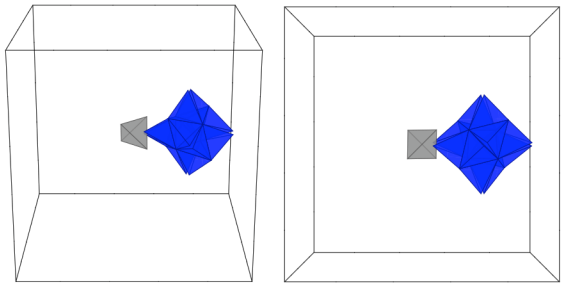


$\{+\mathbf{B}_1,-\mathbf{B}_{17}\}$

$a = \frac{19}{9}$

$\langle 2a,0,0\rangle$

2.111111111111

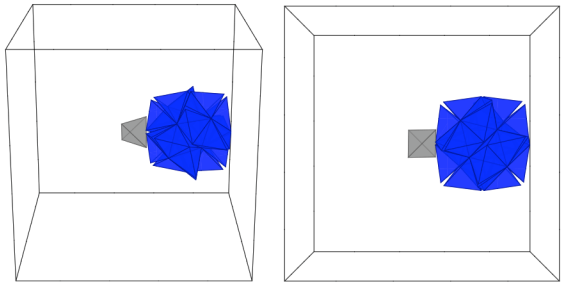


$\{+\mathbf{B}_1,+\mathbf{B}_{41}\}$

$a = \frac{20}{9}$

$\langle 2a,0,0\rangle$

2.222222222222

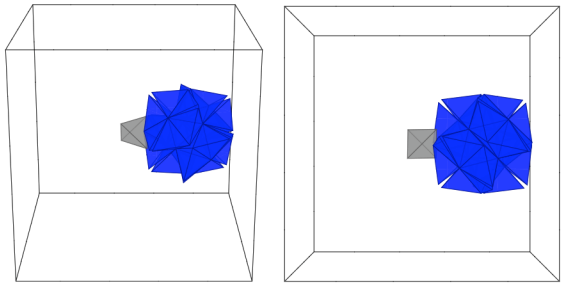


$\{+\mathbf{B}_1,-\mathbf{B}_{41}\}$

$a = \frac{19}{9}$

$\langle 2a,0,0\rangle$

2.111111111111

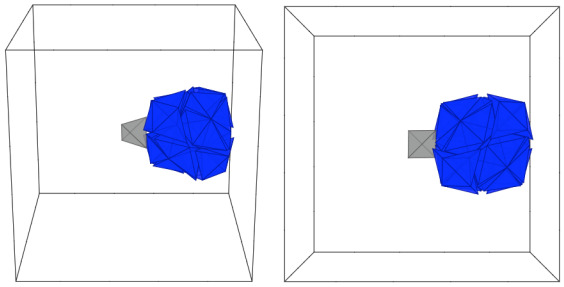


$\{+\mathbf{B}_1,+\mathbf{B}_{57}\}$

$a = \frac{20}{9}$

$\langle 2a,0,0\rangle$

2.222222222222

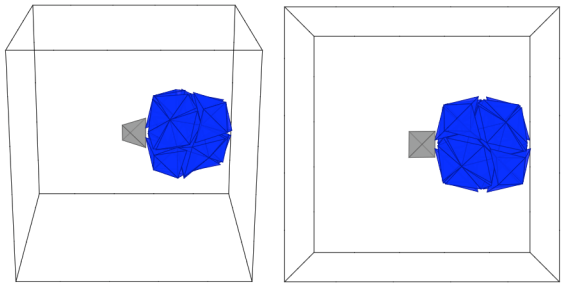


$\{+\mathbf{B}_1,-\mathbf{B}_{57}\}$

$a = \frac{7}{3}$

$\langle 2a,0,0\rangle$

2.333333333333

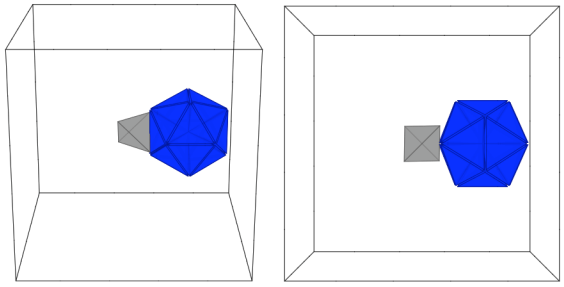


$\{+\mathbf{B}_1,+\mathbf{V}_{20}\}$

$a = \frac{5}{6}+\frac{1}{3}\sqrt{5}-\frac{1}{12}\sqrt{2}+\frac{1}{12}\sqrt{10}$

$\langle 2a,0,0\rangle$

1.724361333983

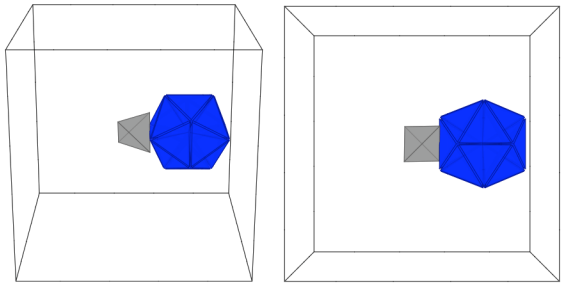


$\{+\mathbf{B}_1,-\mathbf{V}_{20}\}$

$a = \frac{5}{6}+\frac{1}{3}\sqrt{5}-\frac{1}{12}\sqrt{2}+\frac{1}{12}\sqrt{10}$

$\langle 2a,0,0\rangle$

1.724361333983

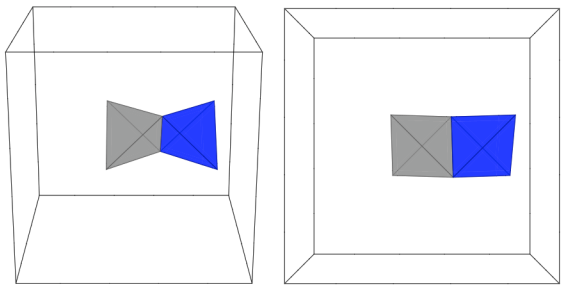


$\{-\mathbf{B}_1,+\mathbf{B}_1\}$

$a = 1$

$\langle 2a,0,0\rangle$

1.000000000000

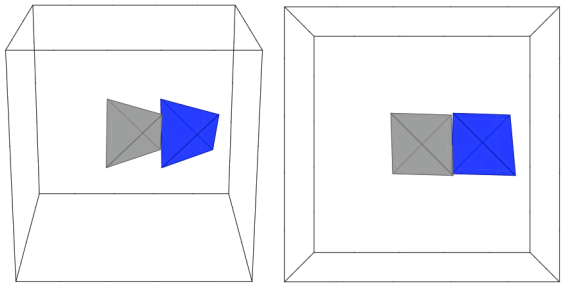


$\{-\mathbf{B}_1,-\mathbf{B}_1\}$

$a = 1$

$\langle 2a,0,0\rangle$

1.000000000000

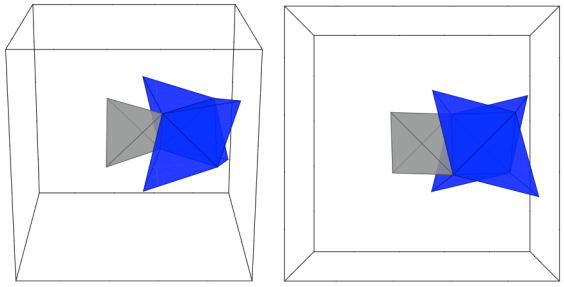


$\{-\mathbf{B}_1,+\mathbf{B}_5\}$

$a = 1$

$\langle 2a,0,0\rangle$

1.000000000000

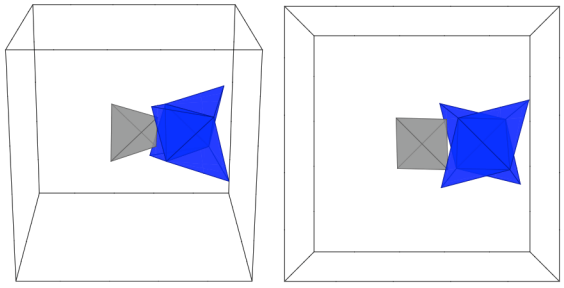


$\{-\mathbf{B}_1,-\mathbf{B}_5\}$

$a = \frac{6}{5}$

$\langle 2a,0,0\rangle$

1.200000000000

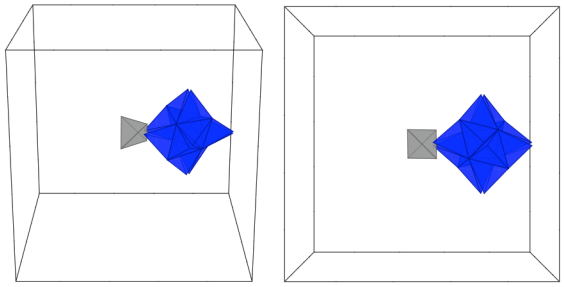


$\{-\mathbf{B}_1,+\mathbf{B}_{17}\}$

$a = \frac{19}{9}$

$\langle 2a,0,0\rangle$

2.111111111111

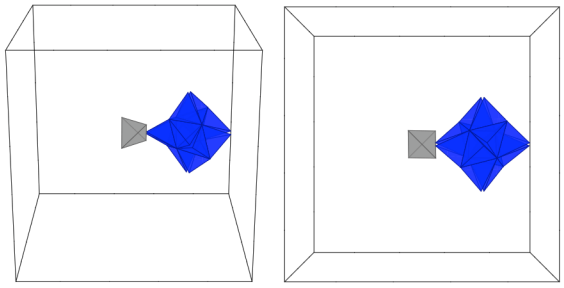


$\{-\mathbf{B}_1,-\mathbf{B}_{17}\}$

$a = \frac{20}{9}$

$\langle 2a,0,0\rangle$

2.222222222222

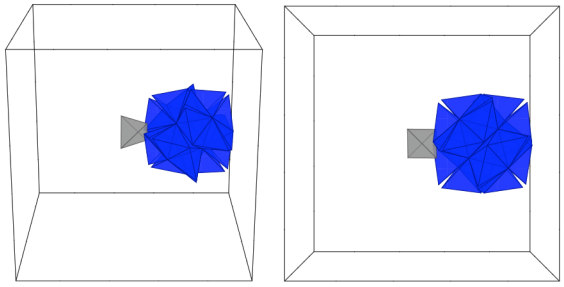


$\{-\mathbf{B}_1,+\mathbf{B}_{41}\}$

$a = \frac{19}{9}$

$\langle 2a,0,0\rangle$

2.111111111111

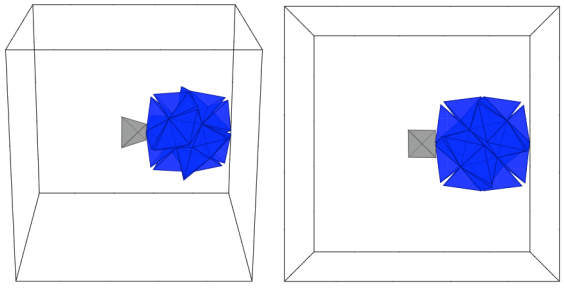


$\{-\mathbf{B}_1,-\mathbf{B}_{41}\}$

$a = \frac{20}{9}$

$\langle 2a,0,0\rangle$

2.222222222222

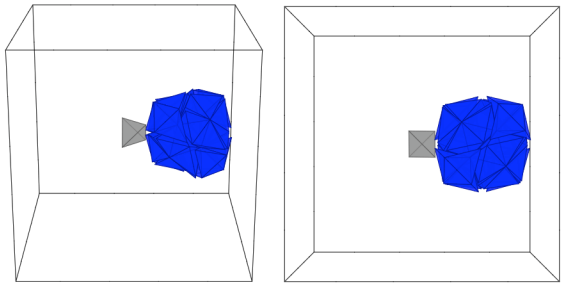


$\{-\mathbf{B}_1,+\mathbf{B}_{57}\}$

$a = \frac{7}{3}$

$\langle 2a,0,0\rangle$

2.333333333333

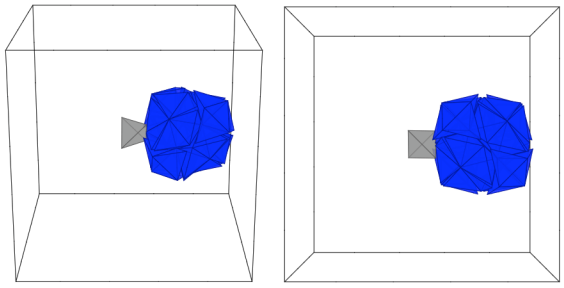


$\{-\mathbf{B}_1,-\mathbf{B}_{57}\}$

$a = \frac{20}{9}$

$\langle 2a,0,0\rangle$

2.222222222222

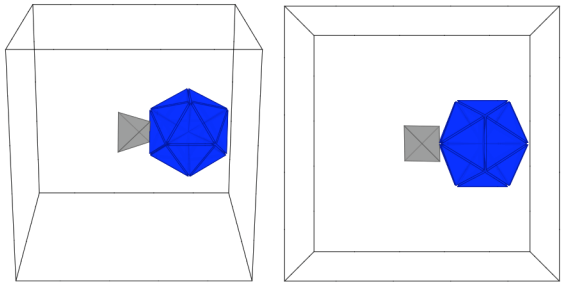


$\{-\mathbf{B}_1,+\mathbf{V}_{20}\}$

$a = \frac{5}{6}+\frac{1}{3}\sqrt{5}-\frac{1}{12}\sqrt{2}+\frac{1}{12}\sqrt{10}$

$\langle 2a,0,0\rangle$

1.724361333983

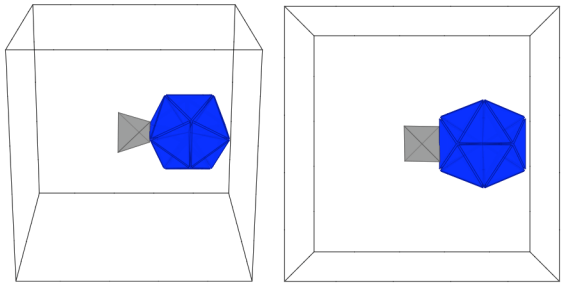


$\{-\mathbf{B}_1,-\mathbf{V}_{20}\}$

$a = \frac{5}{6}+\frac{1}{3}\sqrt{5}-\frac{1}{12}\sqrt{2}+\frac{1}{12}\sqrt{10}$

$\langle 2a,0,0\rangle$

1.724361333983

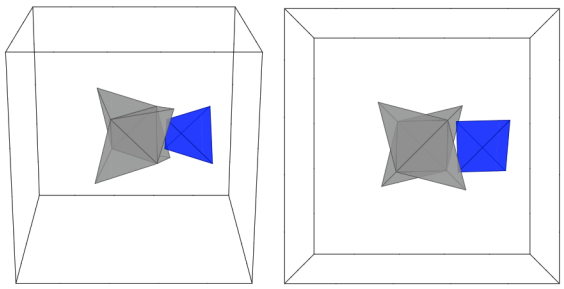


$\{+\mathbf{B}_5,+\mathbf{B}_1\}$

$a = \frac{6}{5}$

$\langle 2a,0,0\rangle$

1.200000000000

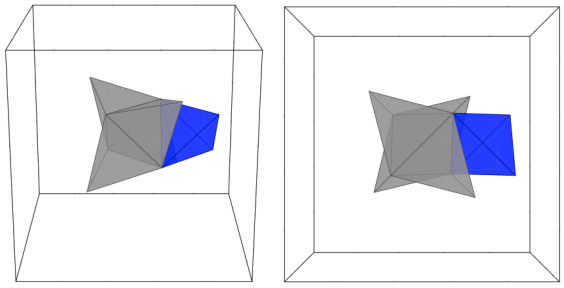


$\{+\mathbf{B}_5,-\mathbf{B}_1\}$

$a = 1$

$\langle 2a,0,0\rangle$

1.000000000000

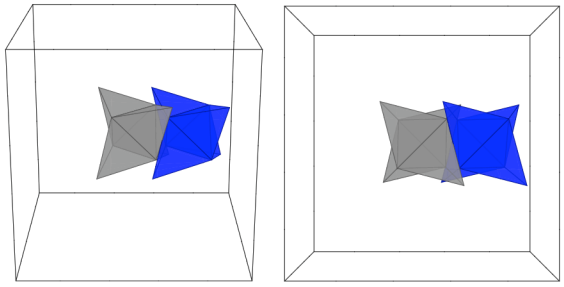


$\{+\mathbf{B}_5,+\mathbf{B}_5\}$

$a = \frac{5}{4}$

$\langle 2a,0,0\rangle$

1.250000000000

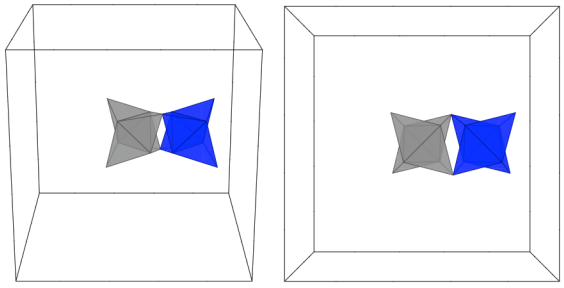


$\{+\mathbf{B}_5,-\mathbf{B}_5\}$

$a = \frac{5}{3}$

$\langle 2a,0,0\rangle$

1.666666666667

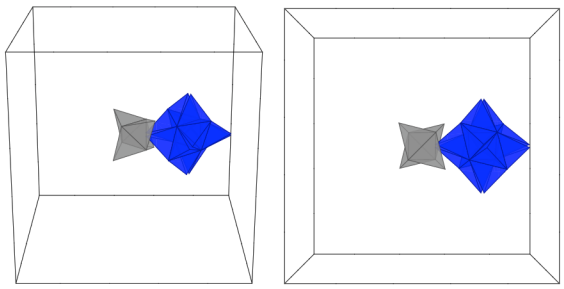


$\{+\mathbf{B}_5,+\mathbf{B}_{17}\}$

$a = \frac{20}{9}$

$\langle 2a,0,0\rangle$

2.222222222222

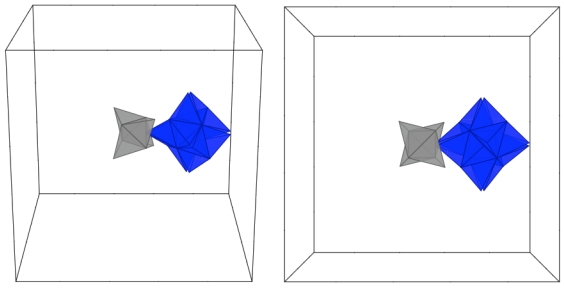


$\{+\mathbf{B}_5,-\mathbf{B}_{17}\}$

$a = \frac{101}{45}$

$\langle 2a,0,0\rangle$

2.244444444444

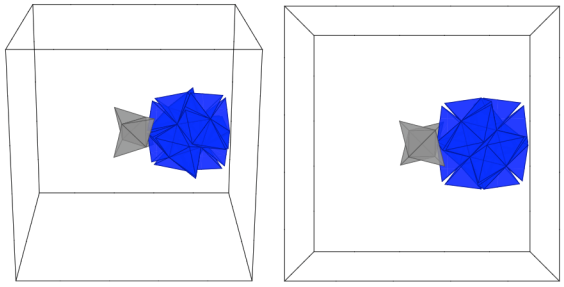


$\{+\mathbf{B}_5,+\mathbf{B}_{41}\}$

$a = \frac{640}{279}$

$\langle 2a,0,0\rangle$

2.293906810036

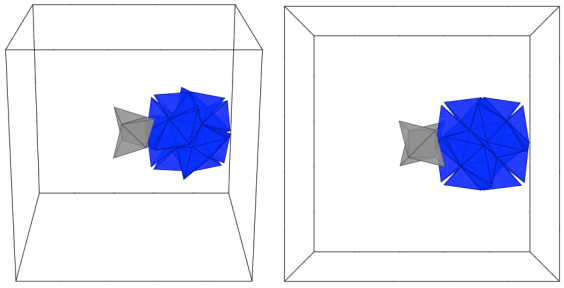


$\{+\mathbf{B}_5,-\mathbf{B}_{41}\}$

$a = \frac{88}{39}$

$\langle 2a,0,0\rangle$

2.256410256410

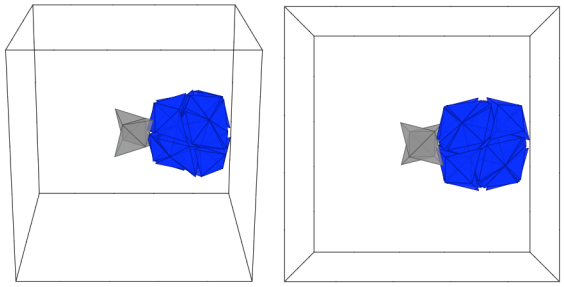


$\{+\mathbf{B}_5,+\mathbf{B}_{57}\}$

$a = \frac{12}{5}$

$\langle 2a,0,0\rangle$

2.400000000000

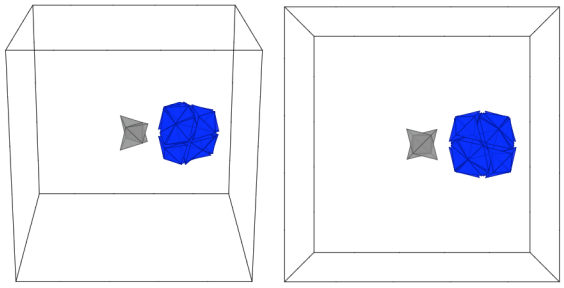


$\{+\mathbf{B}_5,-\mathbf{B}_{57}\}$

$a = \frac{7}{3}$

$\langle 2a,0,0\rangle$

2.333333333333

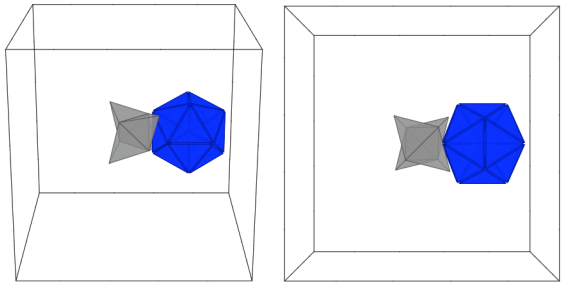


$\{+\mathbf{B}_5,+\mathbf{V}_{20}\}$

$a = \frac{7}{8}+\frac{5}{12}\sqrt{5}-\frac{5}{48}\sqrt{2}+\frac{1}{16}\sqrt{10}$

$\langle 2a,0,0\rangle$

1.857023431638

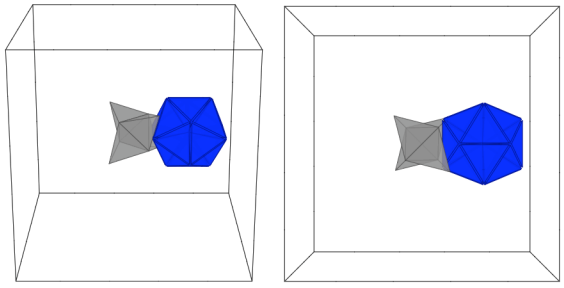


$\{+\mathbf{B}_5,-\mathbf{V}_{20}\}$

$a = \frac{7}{8}+\frac{5}{12}\sqrt{5}-\frac{5}{48}\sqrt{2}+\frac{1}{16}\sqrt{10}$

$\langle 2a,0,0\rangle$

1.857023431638

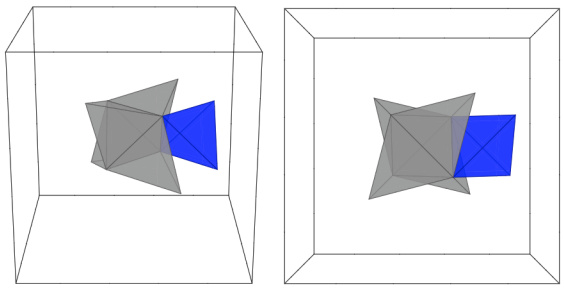


$\{-\mathbf{B}_5,+\mathbf{B}_1\}$

$a = 1$

$\langle 2a,0,0\rangle$

1.000000000000

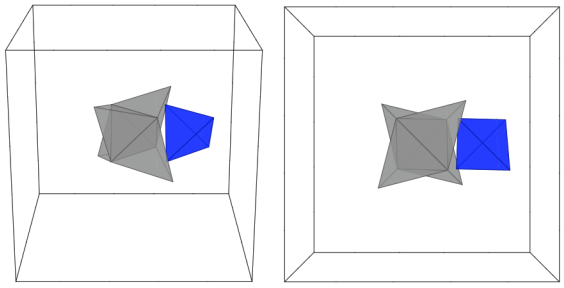


$\{-\mathbf{B}_5,-\mathbf{B}_1\}$

$a = \frac{6}{5}$

$\langle 2a,0,0\rangle$

1.200000000000

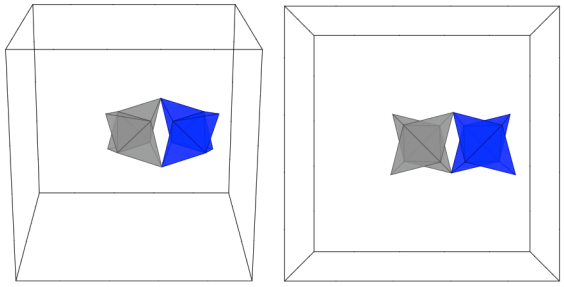


$\{-\mathbf{B}_5,+\mathbf{B}_5\}$

$a = \frac{5}{3}$

$\langle 2a,0,0\rangle$

1.666666666667

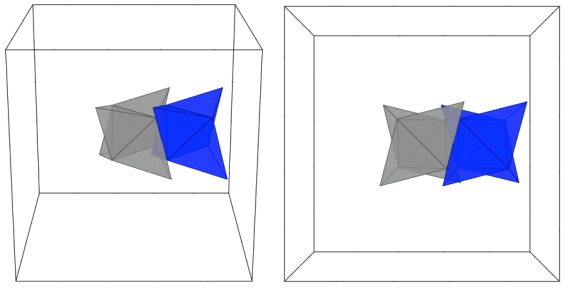


$\{-\mathbf{B}_5,-\mathbf{B}_5\}$

$a = \frac{5}{4}$

$\langle 2a,0,0\rangle$

1.250000000000

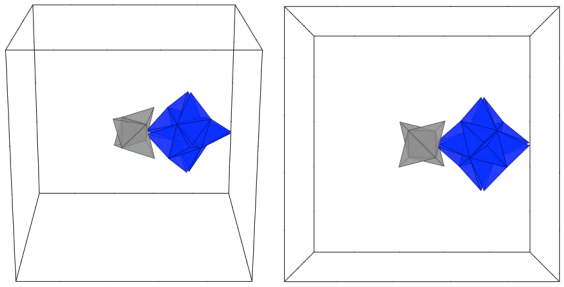


$\{-\mathbf{B}_5,+\mathbf{B}_{17}\}$

$a = \frac{101}{45}$

$\langle 2a,0,0\rangle$

2.244444444444

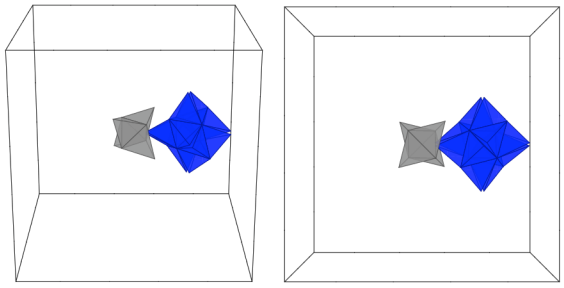


$\{-\mathbf{B}_5,-\mathbf{B}_{17}\}$

$a = \frac{20}{9}$

$\langle 2a,0,0\rangle$

2.222222222222

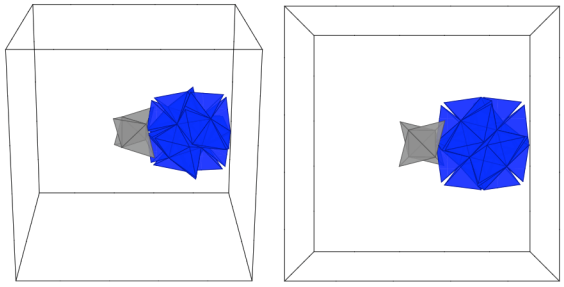


$\{-\mathbf{B}_5,+\mathbf{B}_{41}\}$

$a = \frac{88}{39}$

$\langle 2a,0,0\rangle$

2.256410256410

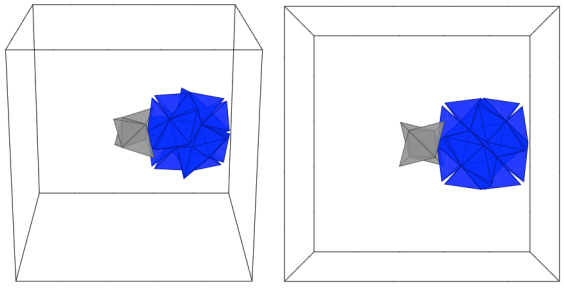


$\{-\mathbf{B}_5,-\mathbf{B}_{41}\}$

$a = \frac{640}{279}$

$\langle 2a,0,0\rangle$

2.293906810036

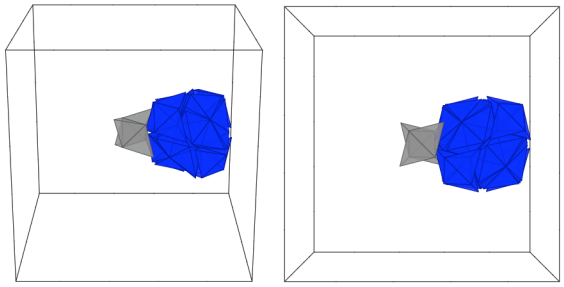


$\{-\mathbf{B}_5,+\mathbf{B}_{57}\}$

$a = \frac{7}{3}$

$\langle 2a,0,0\rangle$

2.333333333333

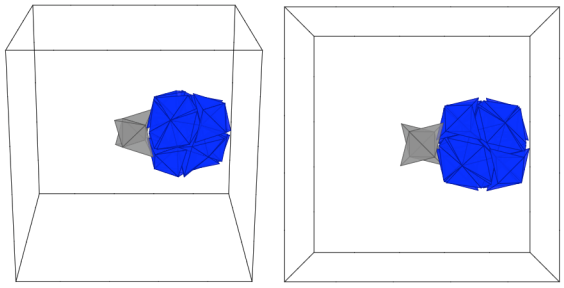


$\{-\mathbf{B}_5,-\mathbf{B}_{57}\}$

$a = \frac{12}{5}$

$\langle 2a,0,0\rangle$

2.400000000000

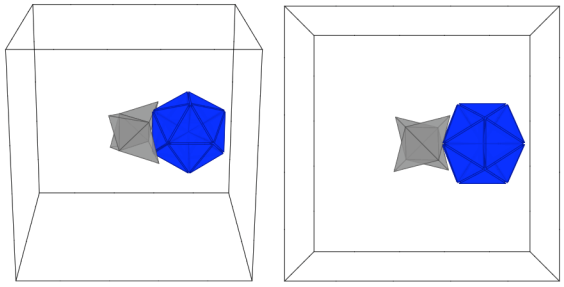


$\{-\mathbf{B}_5,+\mathbf{V}_{20}\}$

$a = \frac{7}{8}+\frac{5}{12}\sqrt{5}-\frac{5}{48}\sqrt{2}+\frac{1}{16}\sqrt{10}$

$\langle 2a,0,0\rangle$

1.857023431638

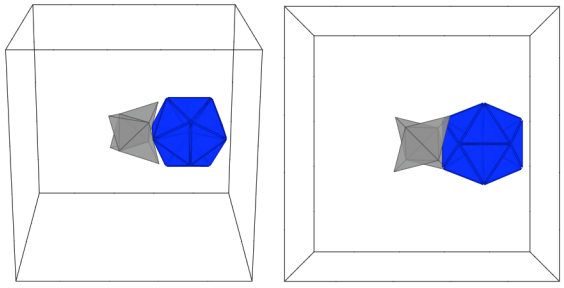


$\{-\mathbf{B}_5,-\mathbf{V}_{20}\}$

$a = \frac{7}{8}+\frac{5}{12}\sqrt{5}-\frac{5}{48}\sqrt{2}+\frac{1}{16}\sqrt{10}$

$\langle 2a,0,0\rangle$

1.857023431638

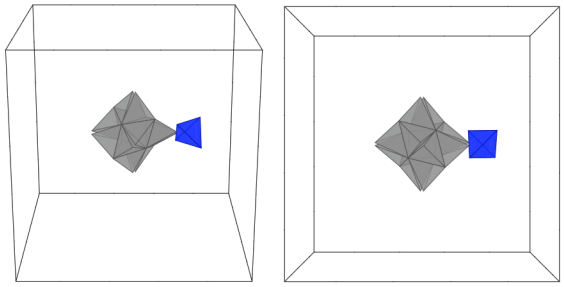


$\{+\mathbf{B}_{17},+\mathbf{B}_1\}$

$a = \frac{20}{9}$

$\langle 2a,0,0\rangle$

2.222222222222

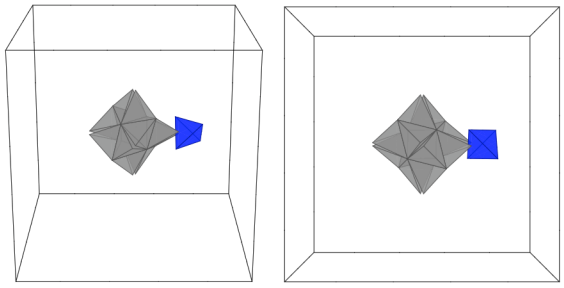


$\{+\mathbf{B}_{17},-\mathbf{B}_1\}$

$a = \frac{19}{9}$

$\langle 2a,0,0\rangle$

2.111111111111

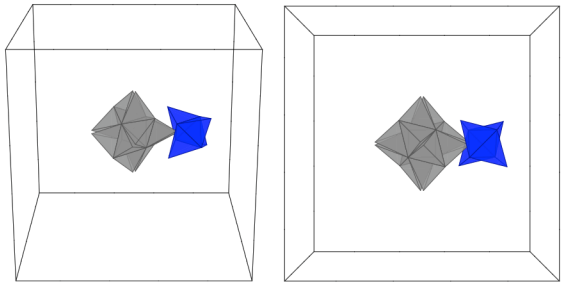


$\{+\mathbf{B}_{17},+\mathbf{B}_5\}$

$a = \frac{20}{9}$

$\langle 2a,0,0\rangle$

2.222222222222

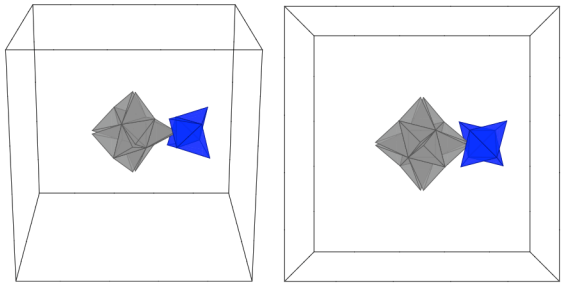


$\{+\mathbf{B}_{17},-\mathbf{B}_5\}$

$a = \frac{101}{45}$

$\langle 2a,0,0\rangle$

2.244444444444

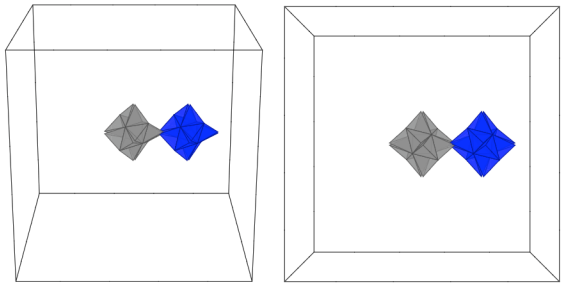


$\{+\mathbf{B}_{17},+\mathbf{B}_{17}\}$

$a = \frac{101}{45}$

$\langle 2a,0,0\rangle$

2.244444444444

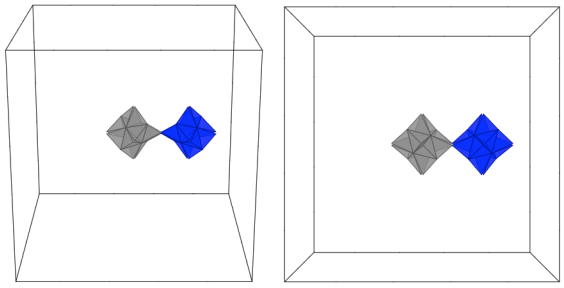


$\{+\mathbf{B}_{17},-\mathbf{B}_{17}\}$

$a = \frac{31}{9}$

$\langle 2a,0,0\rangle$

3.444444444444

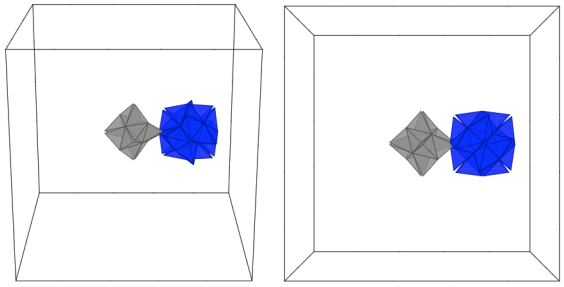


$\{+\mathbf{B}_{17},+\mathbf{B}_{41}\}$

$a = \frac{100}{31}$

$\langle 2a,0,0\rangle$

3.225806451613

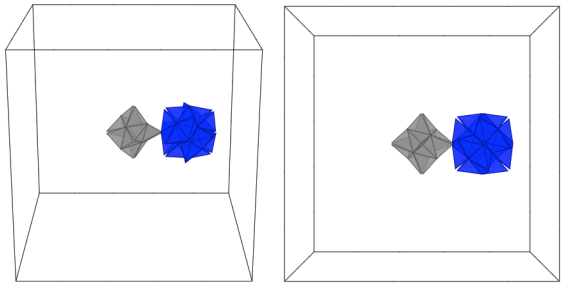


$\{+\mathbf{B}_{17},-\mathbf{B}_{41}\}$

$a = \frac{31}{9}$

$\langle 2a,0,0\rangle$

3.444444444444

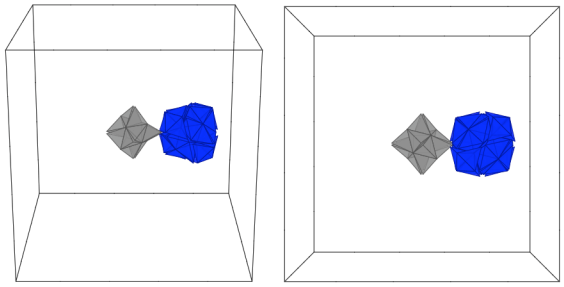


$\{+\mathbf{B}_{17},+\mathbf{B}_{57}\}$

$a = \frac{24}{7}$

$\langle 2a,0,0\rangle$

3.428571428571

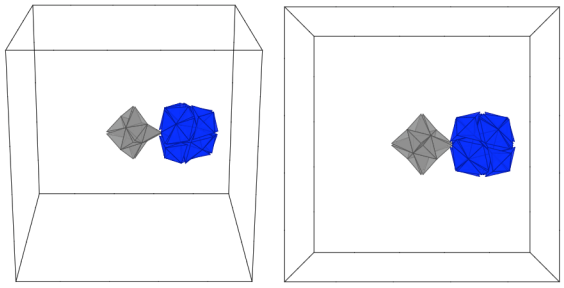


$\{+\mathbf{B}_{17},-\mathbf{B}_{57}\}$

$a = \frac{31}{9}$

$\langle 2a,0,0\rangle$

3.444444444444

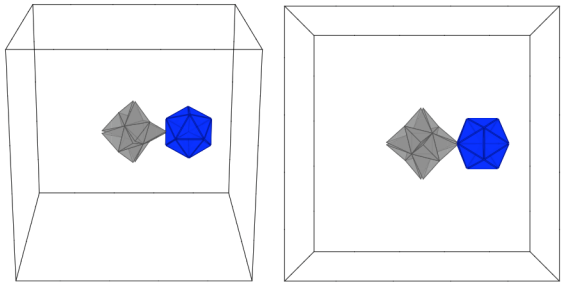


$\{+\mathbf{B}_{17},+\mathbf{V}_{20}\}$

$a = \frac{23}{36}+\frac{37}{36}\sqrt{5}$

$\langle 2a,0,0\rangle$

2.937069865764

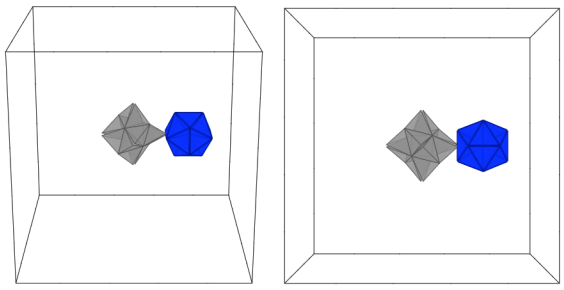


$\{+\mathbf{B}_{17},-\mathbf{V}_{20}\}$

$a = \frac{23}{36}+\frac{37}{36}\sqrt{5}$

$\langle 2a,0,0\rangle$

2.937069865764

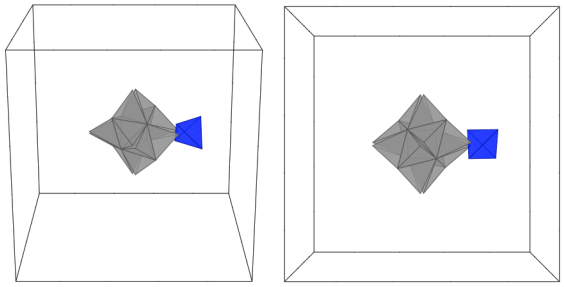


$\{-\mathbf{B}_{17},+\mathbf{B}_1\}$

$a = \frac{19}{9}$

$\langle 2a,0,0\rangle$

2.111111111111

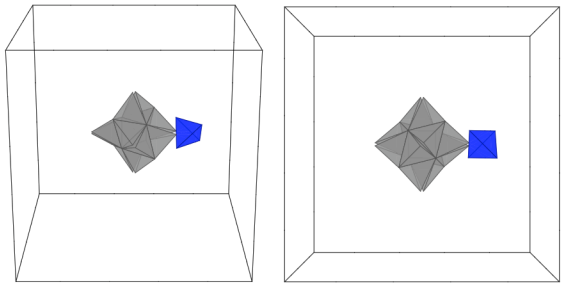


$\{-\mathbf{B}_{17},-\mathbf{B}_1\}$

$a = \frac{20}{9}$

$\langle 2a,0,0\rangle$

2.222222222222

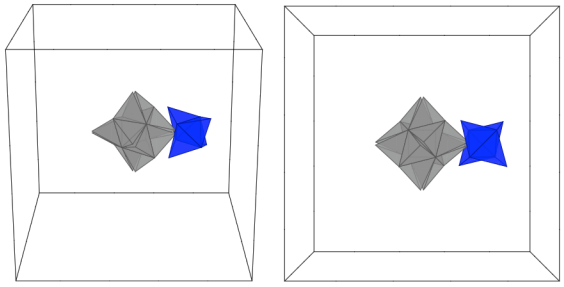


$\{-\mathbf{B}_{17},+\mathbf{B}_5\}$

$a = \frac{101}{45}$

$\langle 2a,0,0\rangle$

2.244444444444

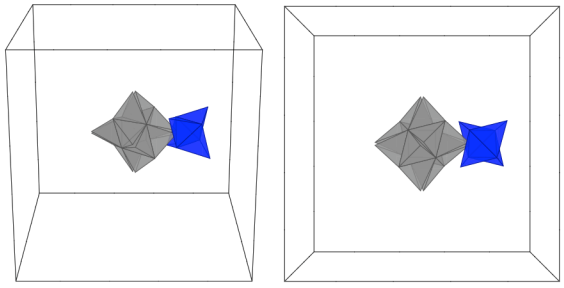


$\{-\mathbf{B}_{17},-\mathbf{B}_5\}$

$a = \frac{20}{9}$

$\langle 2a,0,0\rangle$

2.222222222222

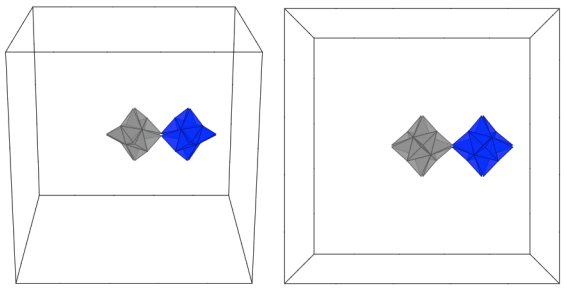


$\{-\mathbf{B}_{17},+\mathbf{B}_{17}\}$

$a = \frac{31}{9}$

$\langle 2a,0,0\rangle$

3.444444444444

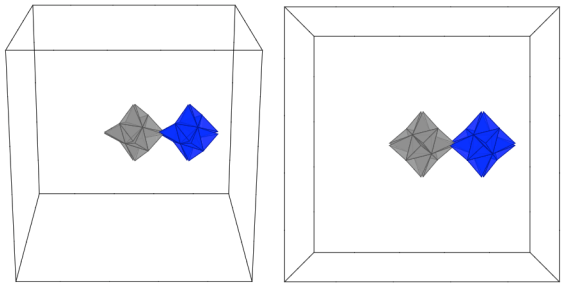


$\{-\mathbf{B}_{17},-\mathbf{B}_{17}\}$

$a = \frac{101}{45}$

$\langle 2a,0,0\rangle$

2.244444444444

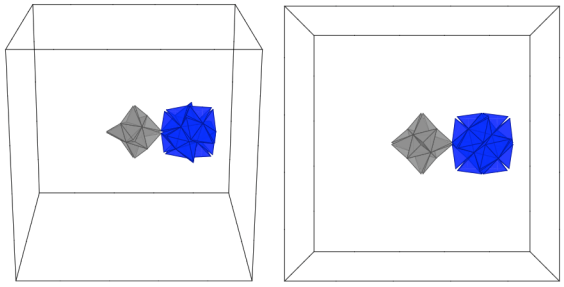


$\{-\mathbf{B}_{17},+\mathbf{B}_{41}\}$

$a = \frac{31}{9}$

$\langle 2a,0,0\rangle$

3.444444444444

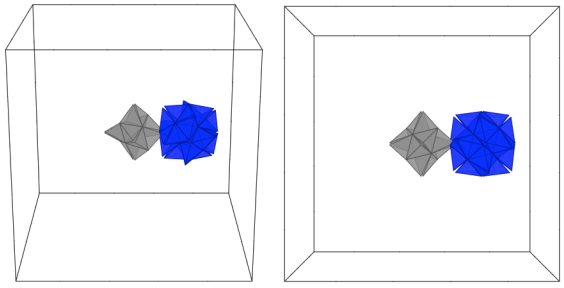


$\{-\mathbf{B}_{17},-\mathbf{B}_{41}\}$

$a = \frac{100}{31}$

$\langle 2a,0,0\rangle$

3.225806451613

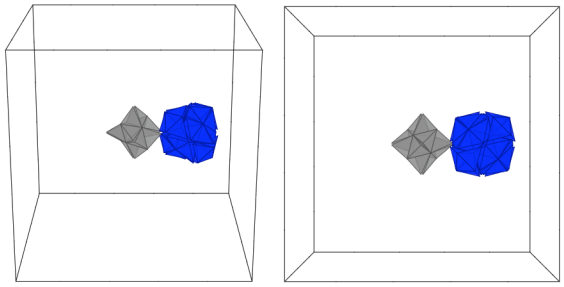


$\{-\mathbf{B}_{17},+\mathbf{B}_{57}\}$

$a = \frac{31}{9}$

$\langle 2a,0,0\rangle$

3.444444444444

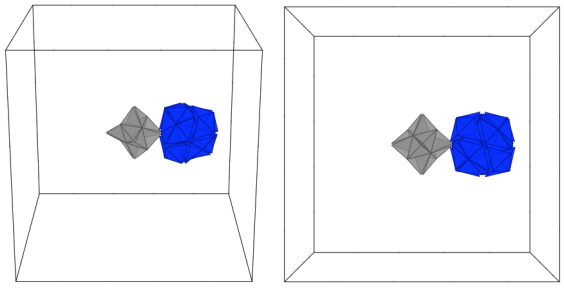


$\{-\mathbf{B}_{17},-\mathbf{B}_{57}\}$

$a = \frac{24}{7}$

$\langle 2a,0,0\rangle$

3.428571428571

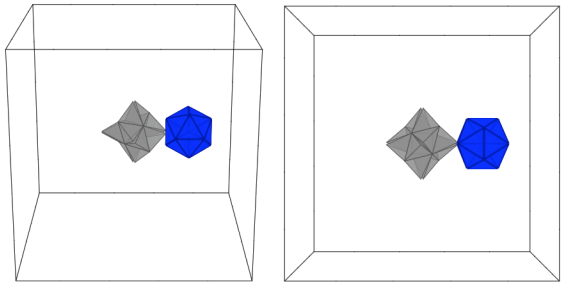


$\{-\mathbf{B}_{17},+\mathbf{V}_{20}\}$

$a = \frac{23}{36}+\frac{37}{36}\sqrt{5}$

$\langle 2a,0,0\rangle$

2.937069865764

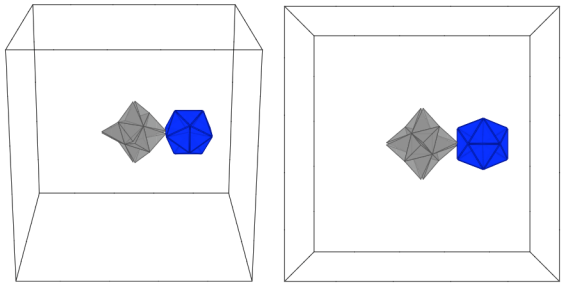


$\{-\mathbf{B}_{17},-\mathbf{V}_{20}\}$

$a = \frac{23}{36}+\frac{37}{36}\sqrt{5}$

$\langle 2a,0,0\rangle$

2.937069865764

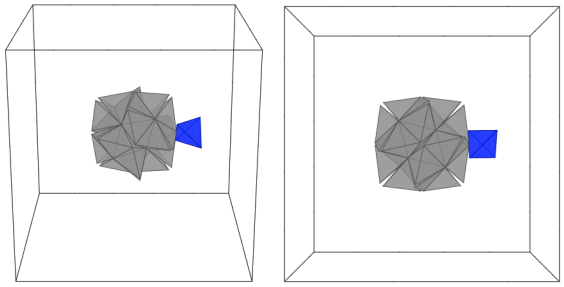


$\{+\mathbf{B}_{41},+\mathbf{B}_1\}$

$a = \frac{20}{9}$

$\langle 2a,0,0\rangle$

2.222222222222

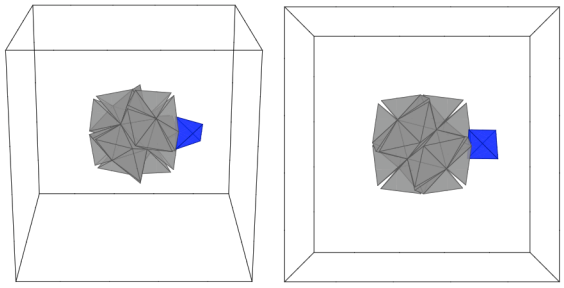


$\{+\mathbf{B}_{41},-\mathbf{B}_1\}$

$a = \frac{19}{9}$

$\langle 2a,0,0\rangle$

2.111111111111

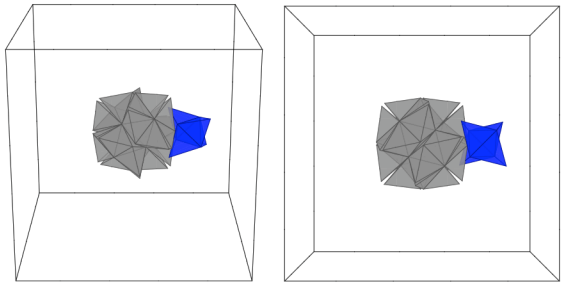


$\{+\mathbf{B}_{41},+\mathbf{B}_5\}$

$a = \frac{640}{279}$

$\langle 2a,0,0\rangle$

2.293906810036

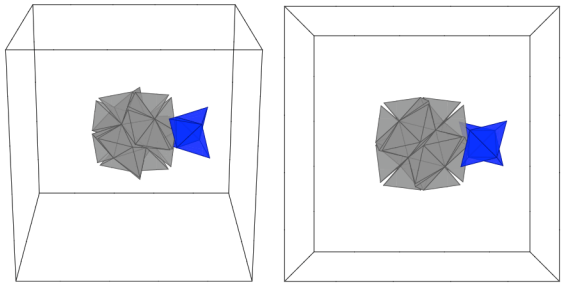


$\{+\mathbf{B}_{41},-\mathbf{B}_5\}$

$a = \frac{88}{39}$

$\langle 2a,0,0\rangle$

2.256410256410

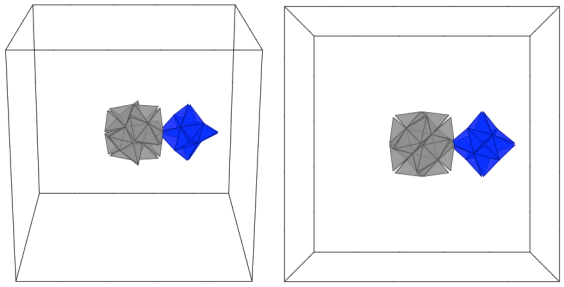


$\{+\mathbf{B}_{41},+\mathbf{B}_{17}\}$

$a = \frac{100}{31}$

$\langle 2a,0,0\rangle$

3.225806451613

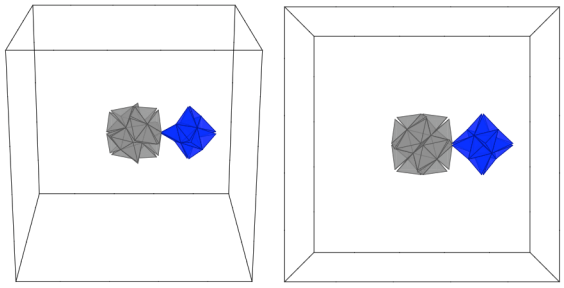


$\{+\mathbf{B}_{41},-\mathbf{B}_{17}\}$

$a = \frac{31}{9}$

$\langle 2a,0,0\rangle$

3.444444444444

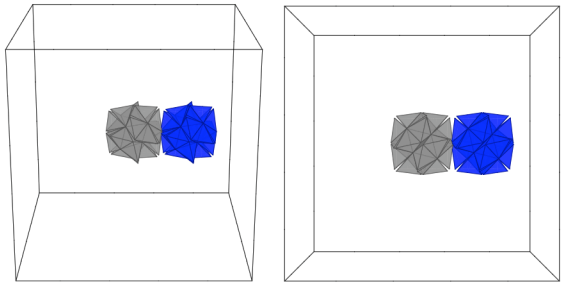


$\{+\mathbf{B}_{41},+\mathbf{B}_{41}\}$

$a = \frac{27}{8}$

$\langle 2a,0,0\rangle$

3.375000000000

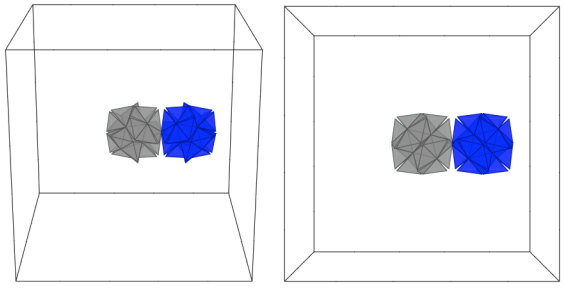


$\{+\mathbf{B}_{41},-\mathbf{B}_{41}\}$

$a = \frac{31}{9}$

$\langle 2a,0,0\rangle$

3.444444444444

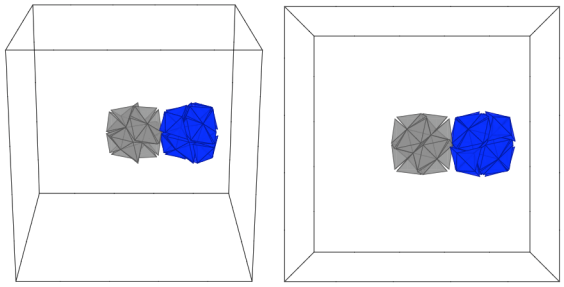


$\{+\mathbf{B}_{41},+\mathbf{B}_{57}\}$

$a = \frac{24}{7}$

$\langle 2a,0,0\rangle$

3.428571428571

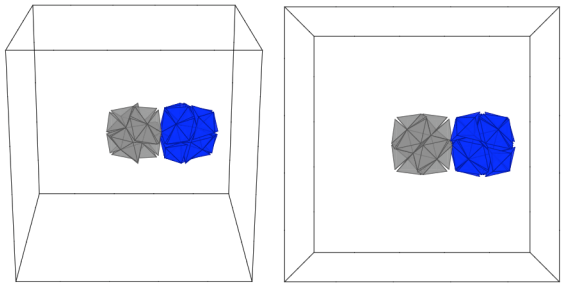


$\{+\mathbf{B}_{41},-\mathbf{B}_{57}\}$

$a = \frac{31}{9}$

$\langle 2a,0,0\rangle$

3.444444444444

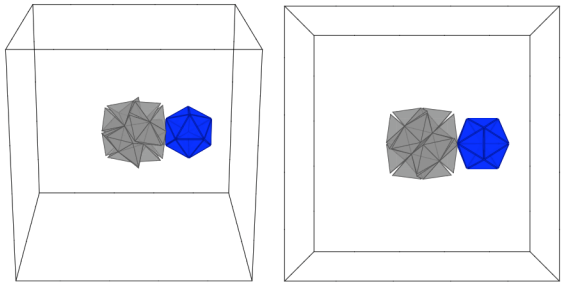


$\{+\mathbf{B}_{41},+\mathbf{V}_{20}\}$

$a = \frac{23}{36}+\frac{37}{36}\sqrt{5}$

$\langle 2a,0,0\rangle$

2.937069865764

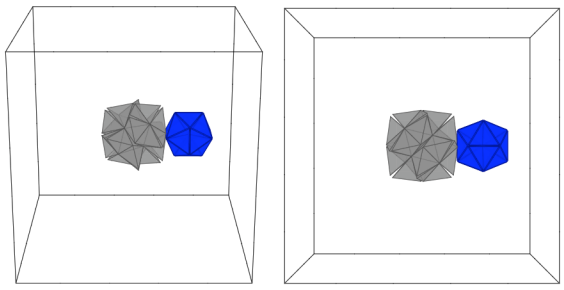


$\{+\mathbf{B}_{41},-\mathbf{V}_{20}\}$

$a = \frac{23}{36}+\frac{37}{36}\sqrt{5}$

$\langle 2a,0,0\rangle$

2.937069865764

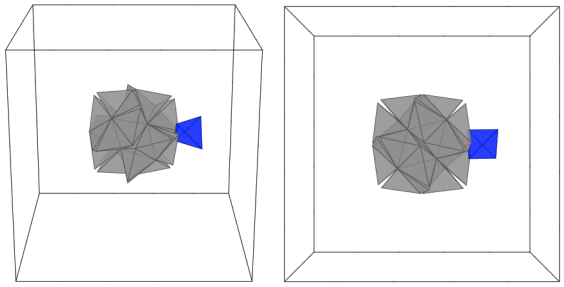


$\{-\mathbf{B}_{41},+\mathbf{B}_1\}$

$a = \frac{19}{9}$

$\langle 2a,0,0\rangle$

2.111111111111

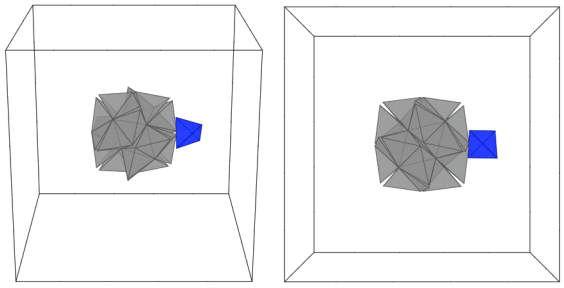


$\{-\mathbf{B}_{41},-\mathbf{B}_1\}$

$a = \frac{20}{9}$

$\langle 2a,0,0\rangle$

2.222222222222

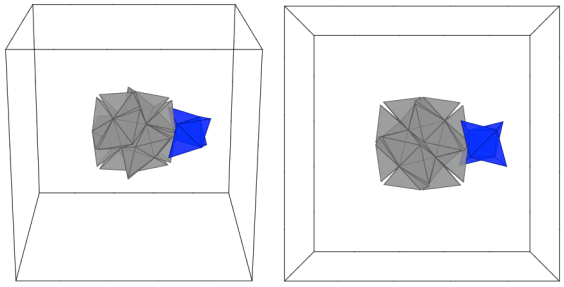


$\{-\mathbf{B}_{41},+\mathbf{B}_5\}$

$a = \frac{88}{39}$

$\langle 2a,0,0\rangle$

2.256410256410

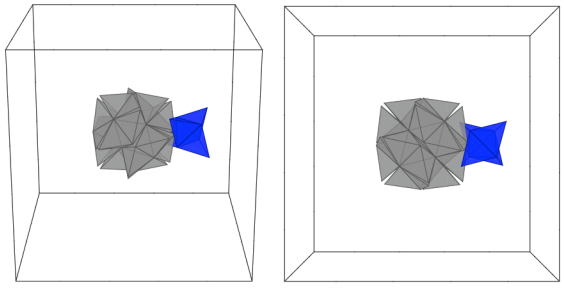


$\{-\mathbf{B}_{41},-\mathbf{B}_5\}$

$a = \frac{640}{279}$

$\langle 2a,0,0\rangle$

2.293906810036

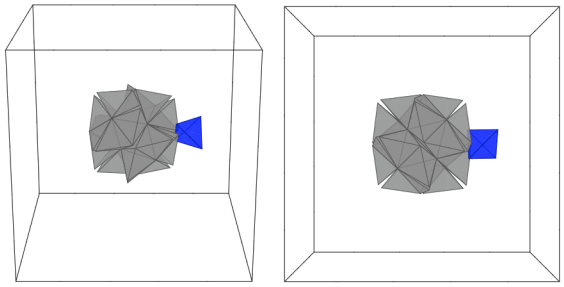


$\{-\mathbf{B}_{41},+\mathbf{B}_1\}$

$a = \frac{19}{9}$

$\langle 2a,0,0\rangle$

2.111111111111

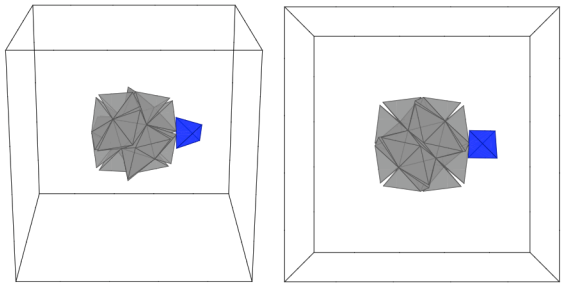


$\{-\mathbf{B}_{41},-\mathbf{B}_1\}$

$a = \frac{20}{9}$

$\langle 2a,0,0\rangle$

2.222222222222

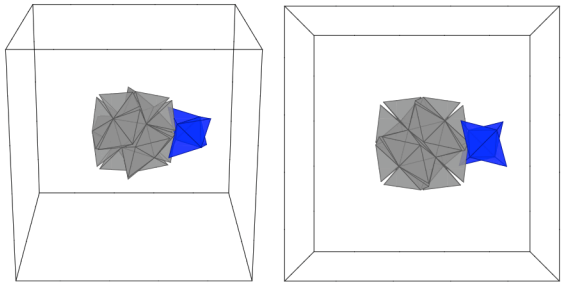


$\{-\mathbf{B}_{41},+\mathbf{B}_5\}$

$a = \frac{88}{39}$

$\langle 2a,0,0\rangle$

2.256410256410

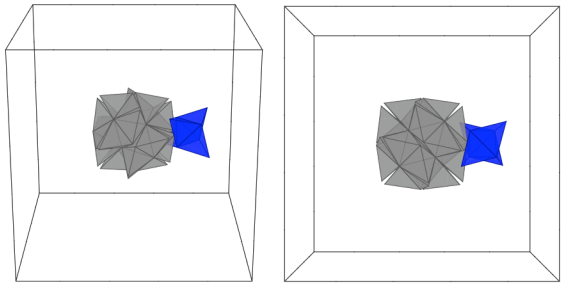


$\{-\mathbf{B}_{41},-\mathbf{B}_5\}$

$a = \frac{640}{279}$

$\langle 2a,0,0\rangle$

2.293906810036

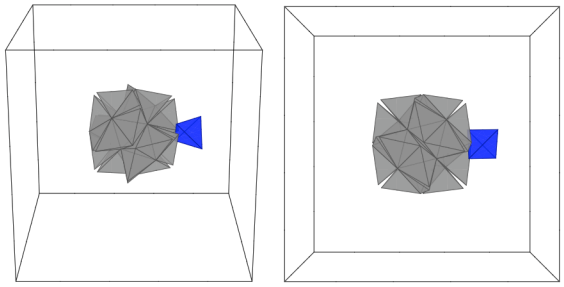


$\{-\mathbf{B}_{41},+\mathbf{B}_1\}$

$a = \frac{19}{9}$

$\langle 2a,0,0\rangle$

2.111111111111

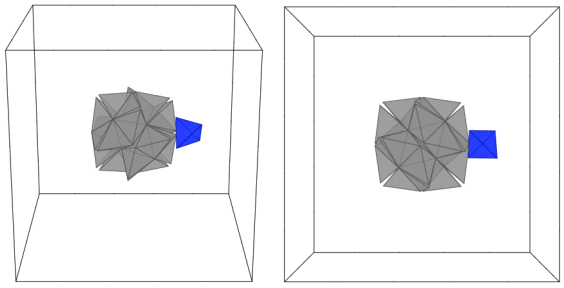


$\{-\mathbf{B}_{41},-\mathbf{B}_1\}$

$a = \frac{20}{9}$

$\langle 2a,0,0\rangle$

2.222222222222

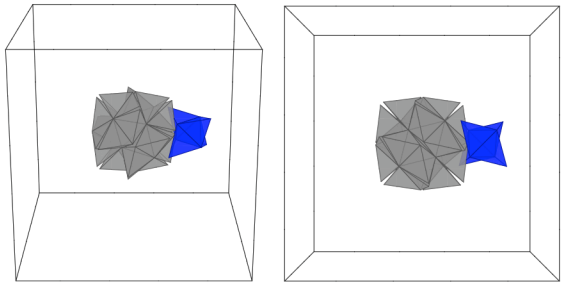


$\{-\mathbf{B}_{41},+\mathbf{B}_5\}$

$a = \frac{88}{39}$

$\langle 2a,0,0\rangle$

2.256410256410

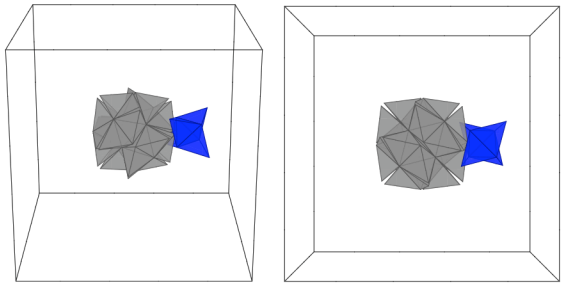


$\{-\mathbf{B}_{41},-\mathbf{B}_5\}$

$a = \frac{640}{279}$

$\langle 2a,0,0\rangle$

2.293906810036

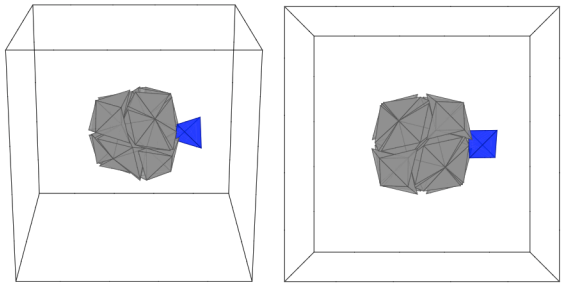


$\{+\mathbf{B}_{57},+\mathbf{B}_1\}$

$a = \frac{20}{9}$

$\langle 2a,0,0\rangle$

2.222222222222

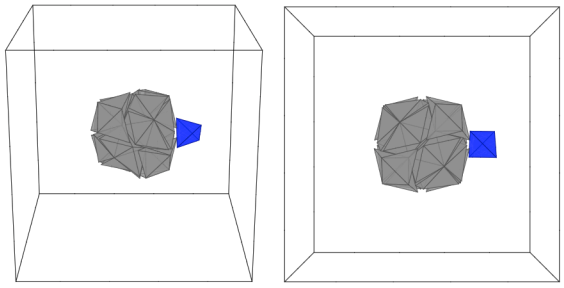


$\{+\mathbf{B}_{57},-\mathbf{B}_1\}$

$a = \frac{7}{3}$

$\langle 2a,0,0\rangle$

2.333333333333

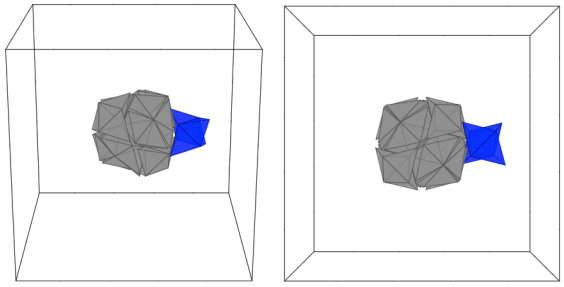


$\{+\mathbf{B}_{57},+\mathbf{B}_5\}$

$a = \frac{12}{5}$

$\langle 2a,0,0\rangle$

2.400000000000

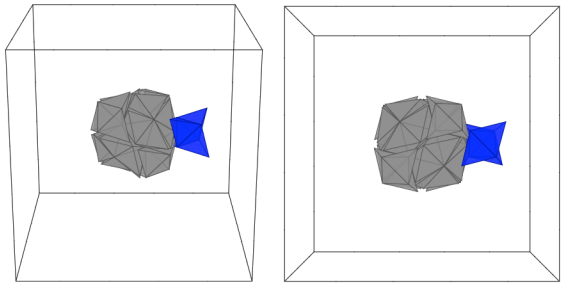


$\{+\mathbf{B}_{57},-\mathbf{B}_5\}$

$a = \frac{7}{3}$

$\langle 2a,0,0\rangle$

2.333333333333

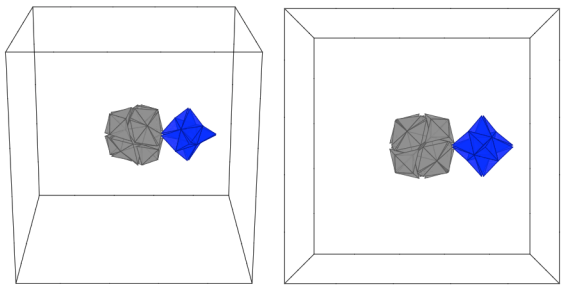


$\{+\mathbf{B}_{57},+\mathbf{B}_{17}\}$

$a = \frac{24}{7}$

$\langle 2a,0,0\rangle$

3.428571428571

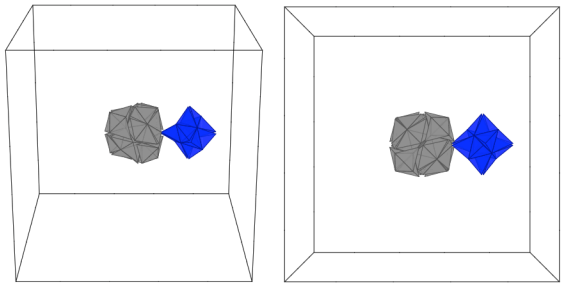


$\{+\mathbf{B}_{57},-\mathbf{B}_{17}\}$

$a = \frac{31}{9}$

$\langle 2a,0,0\rangle$

3.444444444444

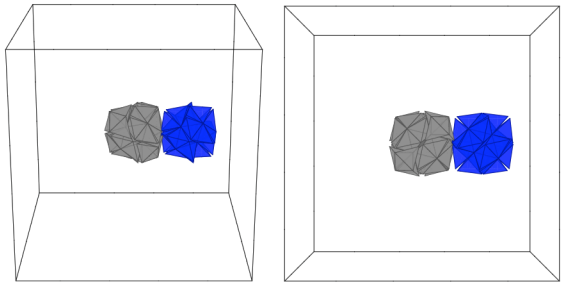


$\{+\mathbf{B}_{57},+\mathbf{B}_{41}\}$

$a = \frac{24}{7}$

$\langle 2a,0,0\rangle$

3.428571428571

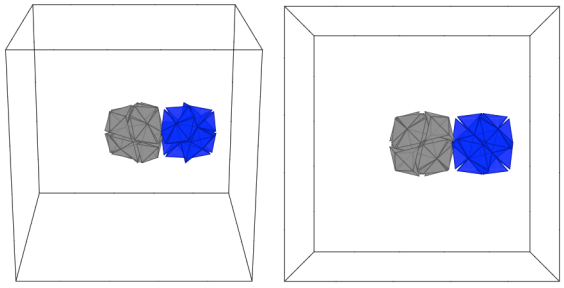


$\{+\mathbf{B}_{57},-\mathbf{B}_{41}\}$

$a = \frac{31}{9}$

$\langle 2a,0,0\rangle$

3.444444444444

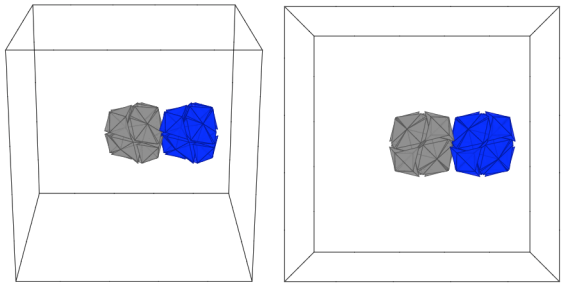


$\{+\mathbf{B}_{57},+\mathbf{B}_{57}\}$

$a = \frac{24}{7}$

$\langle 2a,0,0\rangle$

3.428571428571

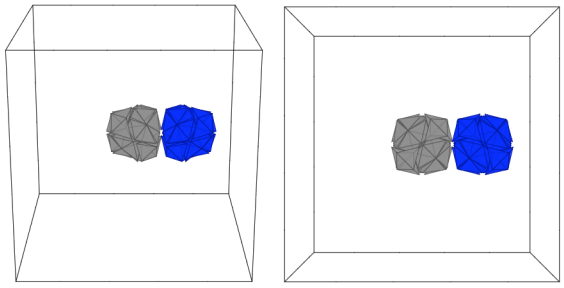


$\{+\mathbf{B}_{57},-\mathbf{B}_{57}\}$

$a = \frac{11}{3}$

$\langle 2a,0,0\rangle$

3.666666666667

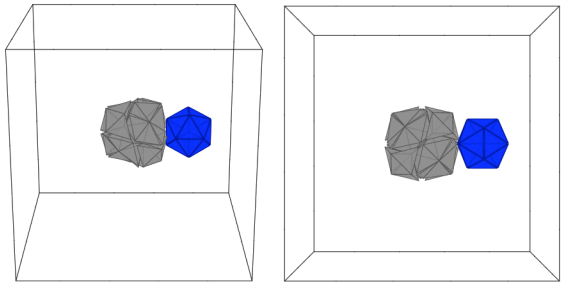


$\{+\mathbf{B}_{57},+\mathbf{V}_{20}\}$

$a = \frac{7}{12}+\frac{13}{12}\sqrt{5}$

$\langle 2a,0,0\rangle$

3.005740308958

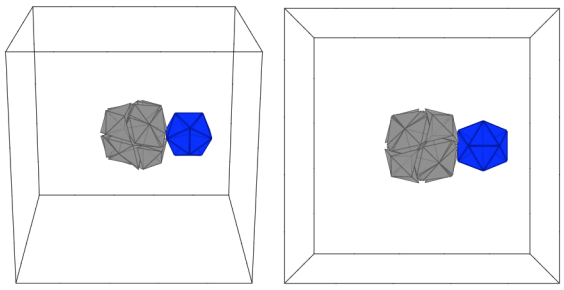


$\{+\mathbf{B}_{57},-\mathbf{V}_{20}\}$

$a = \frac{7}{12}+\frac{13}{12}\sqrt{5}$

$\langle 2a,0,0\rangle$

3.005740308958

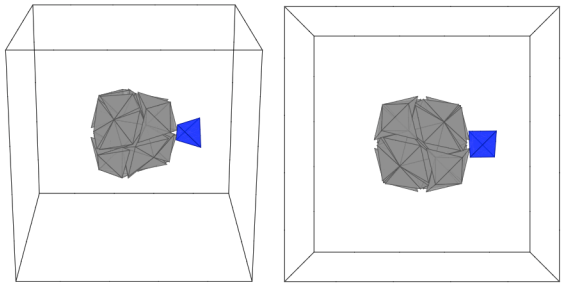


$\{-\mathbf{B}_{57},+\mathbf{B}_1\}$

$a = \frac{7}{3}$

$\langle 2a,0,0\rangle$

2.333333333333

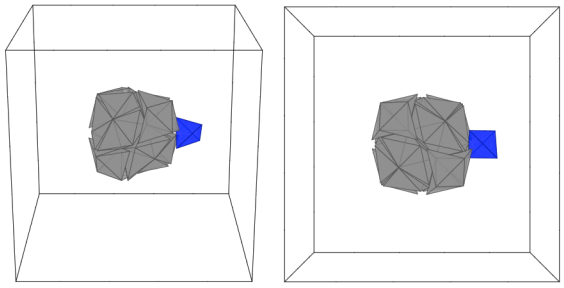


$\{-\mathbf{B}_{57},-\mathbf{B}_1\}$

$a = \frac{20}{9}$

$\langle 2a,0,0\rangle$

2.222222222222

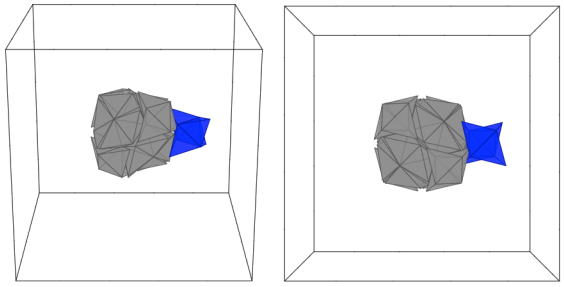


$\{-\mathbf{B}_{57},+\mathbf{B}_5\}$

$a = \frac{7}{3}$

$\langle 2a,0,0\rangle$

2.333333333333

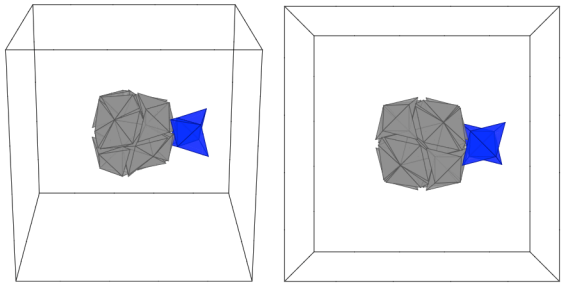


$\{-\mathbf{B}_{57},-\mathbf{B}_5\}$

$a = \frac{12}{5}$

$\langle 2a,0,0\rangle$

2.400000000000

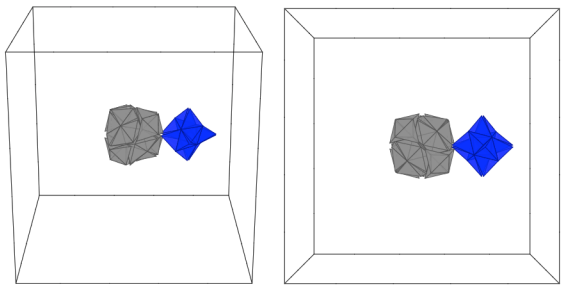


$\{-\mathbf{B}_{57},+\mathbf{B}_{17}\}$

$a = \frac{31}{9}$

$\langle 2a,0,0\rangle$

3.444444444444

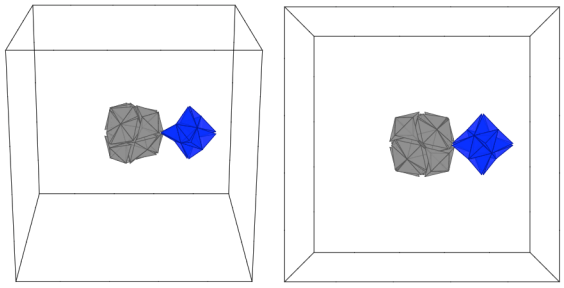


$\{-\mathbf{B}_{57},-\mathbf{B}_{17}\}$

$a = \frac{24}{7}$

$\langle 2a,0,0\rangle$

3.428571428571

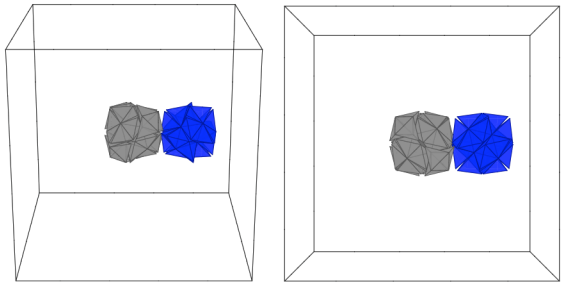


$\{-\mathbf{B}_{57},+\mathbf{B}_{41}\}$

$a = \frac{31}{9}$

$\langle 2a,0,0\rangle$

3.444444444444

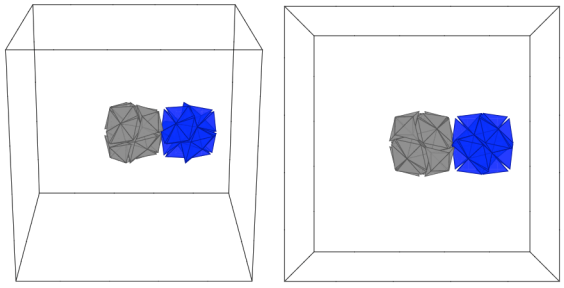


$\{-\mathbf{B}_{57},-\mathbf{B}_{41}\}$

$a = \frac{24}{7}$

$\langle 2a,0,0\rangle$

3.428571428571

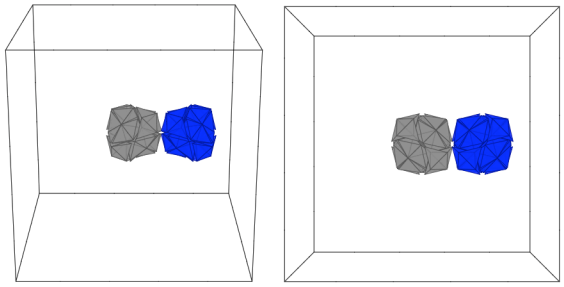


$\{-\mathbf{B}_{57},+\mathbf{B}_{57}\}$

$a = \frac{11}{3}$

$\langle 2a,0,0\rangle$

3.666666666667

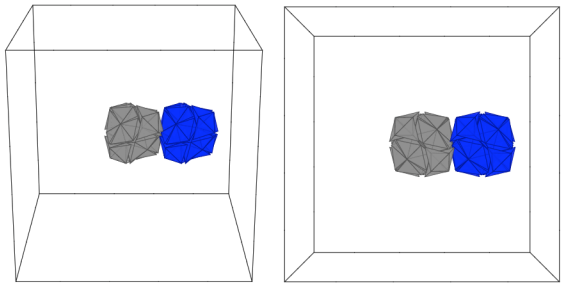


$\{-\mathbf{B}_{57},-\mathbf{B}_{57}\}$

$a = \frac{24}{7}$

$\langle 2a,0,0\rangle$

3.428571428571

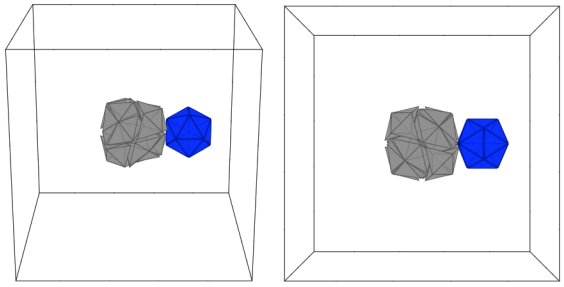


$\{-\mathbf{B}_{57},+\mathbf{V}_{20}\}$

$a = \frac{7}{12}+\frac{13}{12}\sqrt{5}$

$\langle 2a,0,0\rangle$

3.005740308958

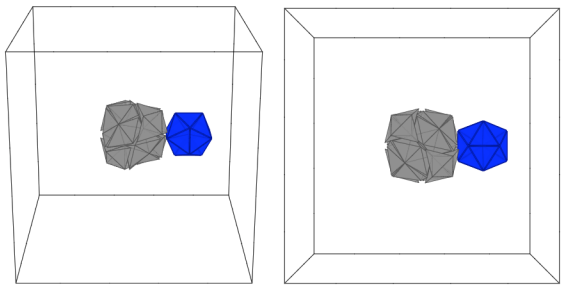


$\{-\mathbf{B}_{57},-\mathbf{V}_{20}\}$

$a = \frac{7}{12}+\frac{13}{12}\sqrt{5}$

$\langle 2a,0,0\rangle$

3.005740308958

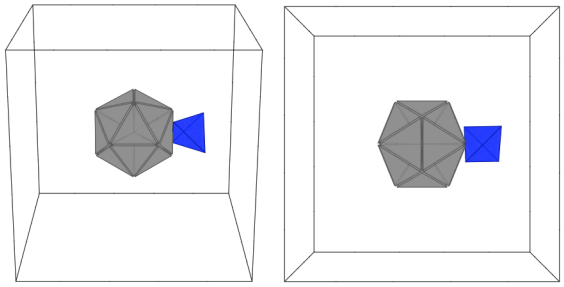


$\{+\mathbf{V}_{20},+\mathbf{B}_1\}$

$\langle 2a,0,0\rangle$

$a = \frac{5}{6}+\frac{1}{3}\sqrt{5}-\frac{1}{12}\sqrt{2}+\frac{1}{12}\sqrt{10}$

1.724361333983

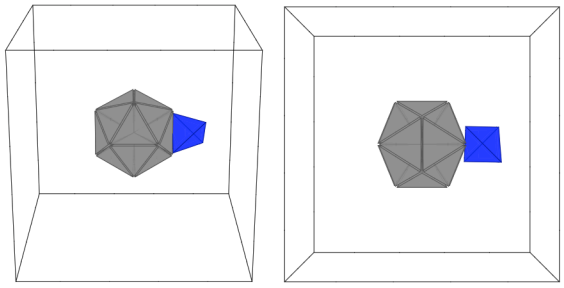


$\{+\mathbf{V}_{20},-\mathbf{B}_1\}$

$\langle 2a,0,0\rangle$

$a = \frac{5}{6}+\frac{1}{3}\sqrt{5}-\frac{1}{12}\sqrt{2}+\frac{1}{12}\sqrt{10}$

1.724361333983

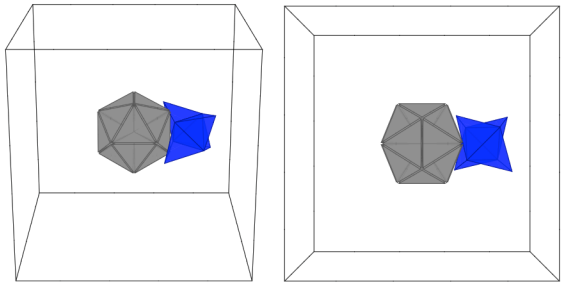


$\{+\mathbf{V}_{20},+\mathbf{B}_5\}$

$\langle 2a,0,0\rangle$

$a = \frac{7}{8}+\frac{5}{12}\sqrt{5}-\frac{5}{48}\sqrt{2}+\frac{1}{16}\sqrt{10}$

1.857023431638

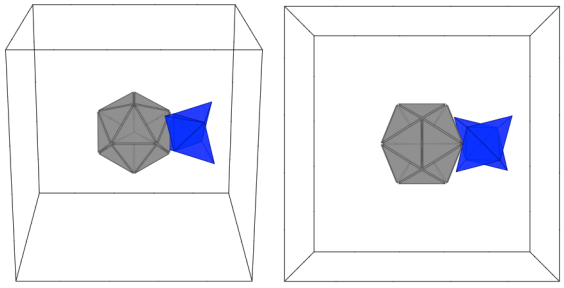


$\{+\mathbf{V}_{20},-\mathbf{B}_5\}$

$\langle 2a,0,0\rangle$

$a = \frac{7}{8}+\frac{5}{12}\sqrt{5}-\frac{5}{48}\sqrt{2}+\frac{1}{16}\sqrt{10}$

1.857023431638

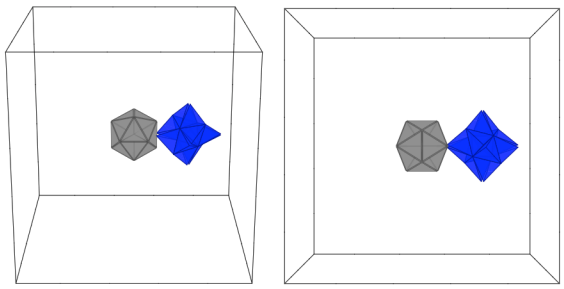


$\{+\mathbf{V}_{20},+\mathbf{B}_{17}\}$

$a = \frac{23}{36}+\frac{37}{36}\sqrt{5}$

$\langle 2a,0,0\rangle$

2.937069865764

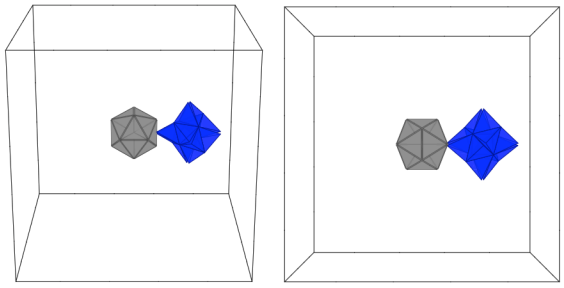


$\{+\mathbf{V}_{20},-\mathbf{B}_{17}\}$

$a = \frac{23}{36}+\frac{37}{36}\sqrt{5}$

$\langle 2a,0,0\rangle$

2.937069865764

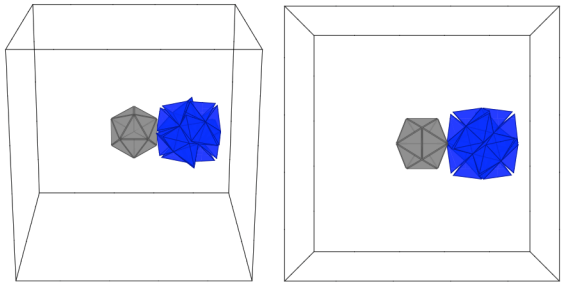


$\{+\mathbf{V}_{20},+\mathbf{B}_{41}\}$

$a = \frac{23}{36}+\frac{37}{36}\sqrt{5}$

$\langle 2a,0,0\rangle$

2.937069865764

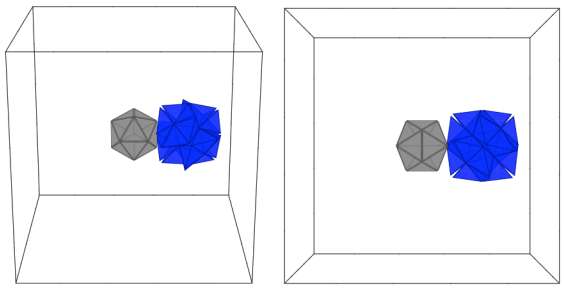


$\{+\mathbf{V}_{20},-\mathbf{B}_{41}\}$

$a = \frac{23}{36}+\frac{37}{36}\sqrt{5}$

$\langle 2a,0,0\rangle$

2.937069865764

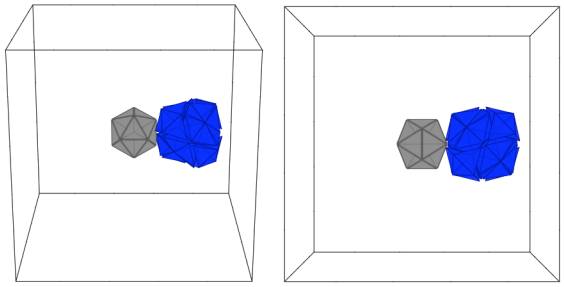


$\{+\mathbf{V}_{20},+\mathbf{B}_{57}\}$

$a = \frac{7}{12}+\frac{13}{12}\sqrt{5}$

$\langle 2a,0,0\rangle$

3.005740308958

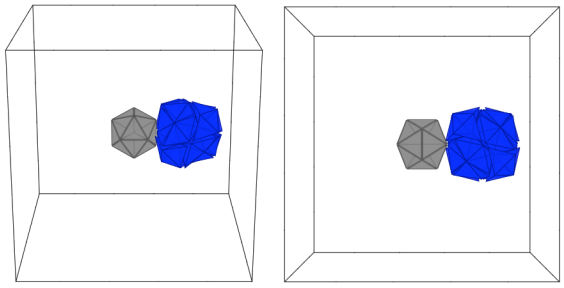


$\{+\mathbf{V}_{20},-\mathbf{B}_{57}\}$

$a = \frac{7}{12}+\frac{13}{12}\sqrt{5}$

$\langle 2a,0,0\rangle$

3.005740308958

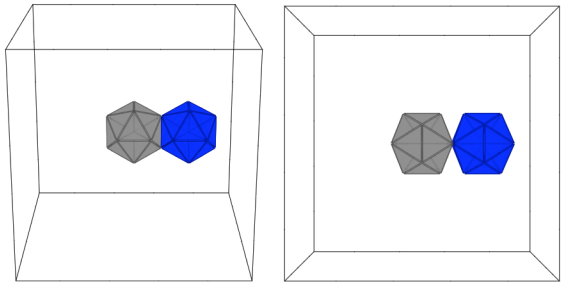


$\{+\mathbf{V}_{20},+\mathbf{V}_{20}\}$

$a = \frac{2}{3}+\frac{2}{3}\sqrt{5}\ -\frac{1}{6}\sqrt{2}\ +\frac{1}{6}\sqrt{10}$

$\langle 2a,0,0\rangle$

2.448722667966

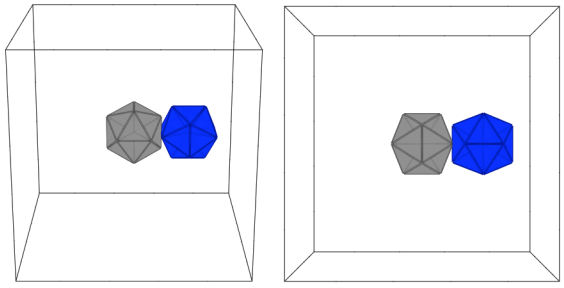


$\{+\mathbf{V}_{20},-\mathbf{V}_{20}\}$

$a = \frac{2}{3}+\frac{2}{3}\sqrt{5}\ -\frac{1}{6}\sqrt{2}\ +\frac{1}{6}\sqrt{10}$

$\langle 2a,0,0\rangle$

2.448722667966

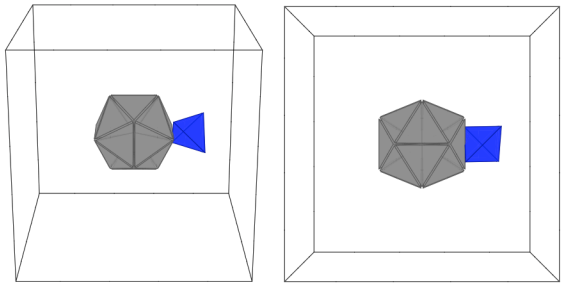


$\{-\mathbf{V}_{20},+\mathbf{B}_1\}$

$\langle 2a,0,0\rangle$

$a = \frac{5}{6}+\frac{1}{3}\sqrt{5}-\frac{1}{12}\sqrt{2}+\frac{1}{12}\sqrt{10}$

1.724361333983

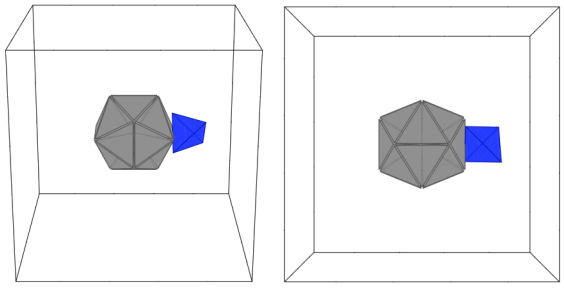


$\{-\mathbf{V}_{20},-\mathbf{B}_1\}$

$\langle 2a,0,0\rangle$

$a = \frac{5}{6}+\frac{1}{3}\sqrt{5}-\frac{1}{12}\sqrt{2}+\frac{1}{12}\sqrt{10}$

1.724361333983

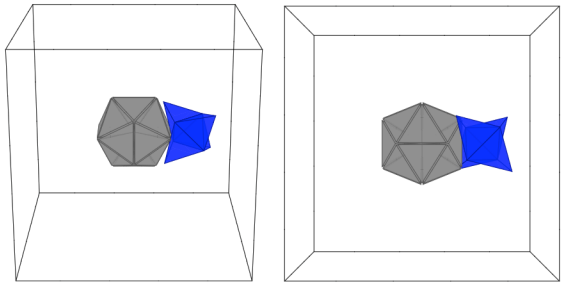


$\{-\mathbf{V}_{20},+\mathbf{B}_5\}$

$\langle 2a,0,0\rangle$

$a = \frac{7}{8}+\frac{5}{12}\sqrt{5}-\frac{5}{48}\sqrt{2}+\frac{1}{16}\sqrt{10}$

1.857023431638

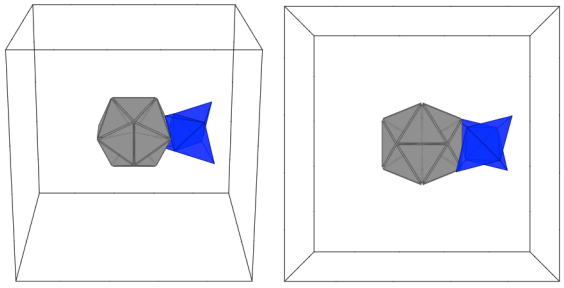


$\{-\mathbf{V}_{20},-\mathbf{B}_5\}$

$\langle 2a,0,0\rangle$

$a = \frac{7}{8}+\frac{5}{12}\sqrt{5}-\frac{5}{48}\sqrt{2}+\frac{1}{16}\sqrt{10}$

1.857023431638

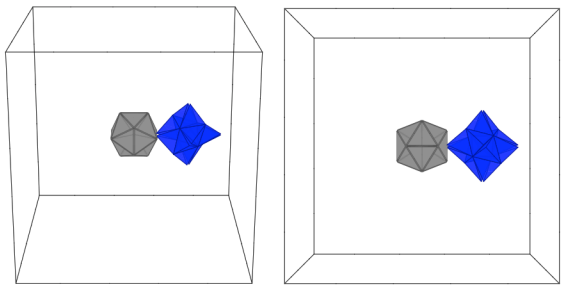


$\{-\mathbf{V}_{20},+\mathbf{B}_{17}\}$

$a = \frac{23}{36}+\frac{37}{36}\sqrt{5}$

$\langle 2a,0,0\rangle$

2.937069865764

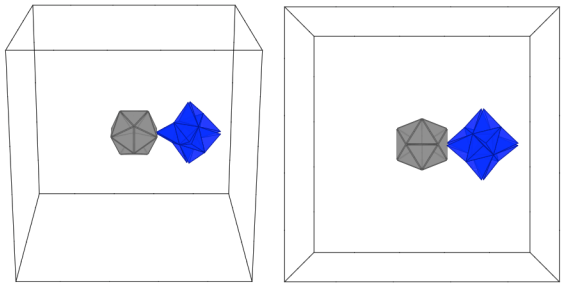


$\{-\mathbf{V}_{20},-\mathbf{B}_{17}\}$

$a = \frac{23}{36}+\frac{37}{36}\sqrt{5}$

$\langle 2a,0,0\rangle$

2.937069865764

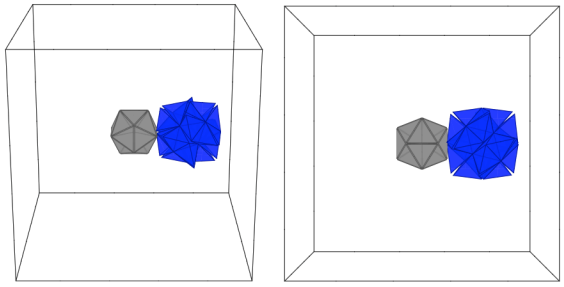


$\{-\mathbf{V}_{20},+\mathbf{B}_{41}\}$

$a = \frac{23}{36}+\frac{37}{36}\sqrt{5}$

$\langle 2a,0,0\rangle$

2.937069865764

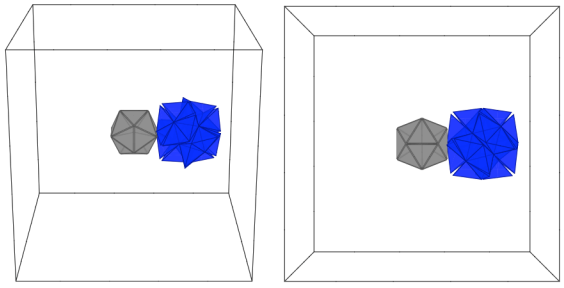


$\{-\mathbf{V}_{20},-\mathbf{B}_{41}\}$

$a = \frac{23}{36}+\frac{37}{36}\sqrt{5}$

$\langle 2a,0,0\rangle$

2.937069865764

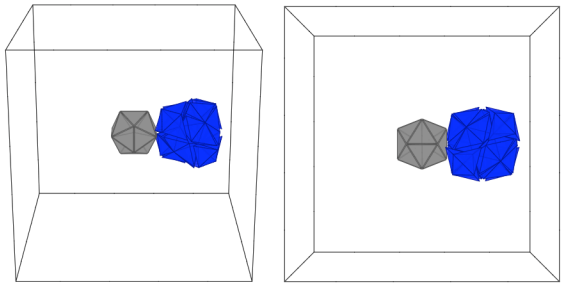


$\{-\mathbf{V}_{20},+\mathbf{B}_{57}\}$

$a = \frac{7}{12}+\frac{13}{12}\sqrt{5}$

$\langle 2a,0,0\rangle$

3.005740308958

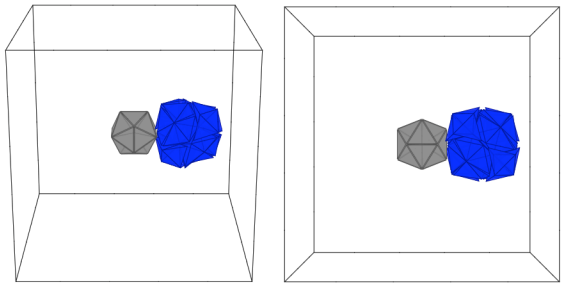


$\{-\mathbf{V}_{20},-\mathbf{B}_{57}\}$

$a = \frac{7}{12}+\frac{13}{12}\sqrt{5}$

$\langle 2a,0,0\rangle$

3.005740308958

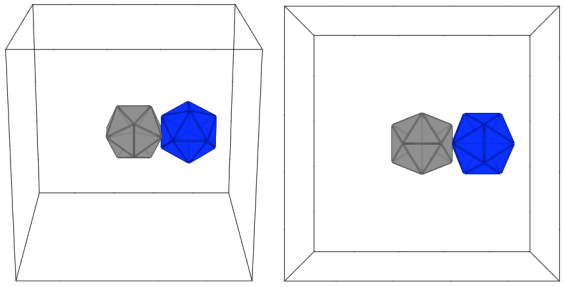


$\{-\mathbf{V}_{20},+\mathbf{V}_{20}\}$

$a = \frac{2}{3}+\frac{2}{3}\sqrt{5}\ -\frac{1}{6}\sqrt{2}\ +\frac{1}{6}\sqrt{10}$

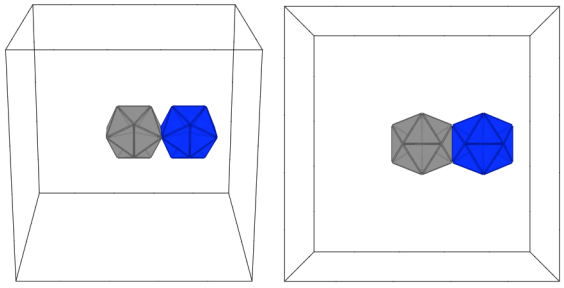
$\langle 2a,0,0\rangle$

2.448722667966



$\{-\mathbf{V}_{20},-\mathbf{V}_{20}\}$

$a = \frac{2}{3}+\frac{2}{3}\sqrt{5}\ -\frac{1}{6}\sqrt{2}\ +\frac{1}{6}\sqrt{10}$

$\langle 2a,0,0\rangle$

2.448722667966