AN OPTIMAL ALGORITHM FOR FINDING ALL VISIBLE EDGES IN A SIMPLE POLYGON

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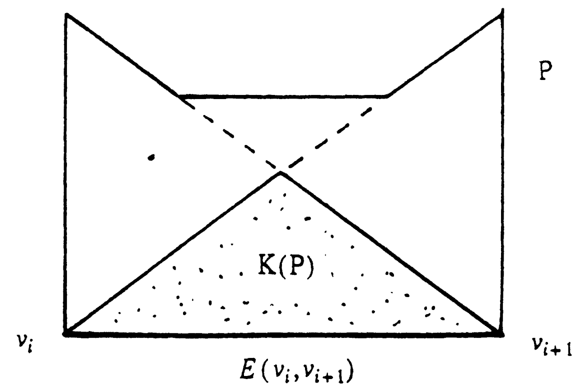
Abstract

Whether the interior of a simple polygon is visible to a given edge can be determined in $O(n)$ time [1], where n is the total number of edges in the polygon. In this paper, we find all such visible edges in $O(n)$ time. It is made possible by showing that there can be at most three visible edges in a simple polygon with an empty kernel.

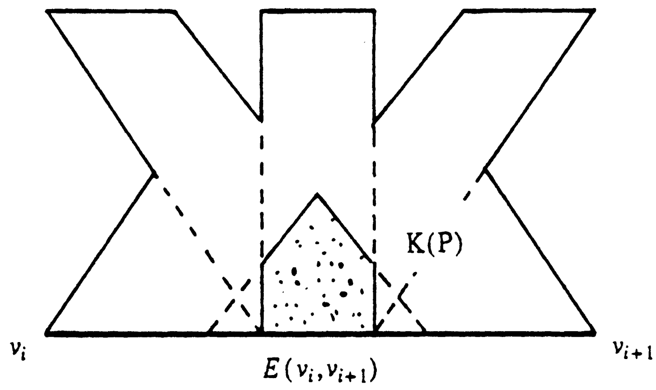
1. **Introduction**

The visibility of a polygon from an edge is considered by Avis and Toussaint[1]. In it,they developed an $O(n)$ algorithm for determining if a simple polygon is visible from a given edge. They also introduce three kinds of edge visibility—weak, strong and complete, that are illustrated in Figure 1.1. In this paper, we pose the question : Given a simple polygon, can we find all the visible edges, if there are any, in $O(n)$ time? If the kernel of the given polygon exists, then finding all the completely visible edges is trivial since they must be contained in the kernel. Likewise, finding all the strongly visible edges in a polygon with a kernel is also straightforward. (It may be noted that if the kernel of a polygon does not share any point with the boundary of the polygon, then these two kinds of visible edges do not exist.) The only interesting variation is to find all the weakly visible edges. Shin and Woo[7,8] described a linear time algorithm for finding all weakly visible edges in a polygon for which a kernel exists. In this paper, we examine the problem of finding all weakly visible edges in a simple polygon in which a kernel does not exist. A straightforward approach would be to apply the Avis and Toussaint's linear time test[1] for edge visibility to each of n edges hence yeielding an $O(n^2)$ algorithm. We give an $O(n)$ algorithm for finding all weakly visible edges, if any, in a simple polygon with an empty kernel. This algorithm together with the Shin and Woo's algorithm[8] make it possible to find all weakly visible edges of a simple polygon in linear time.

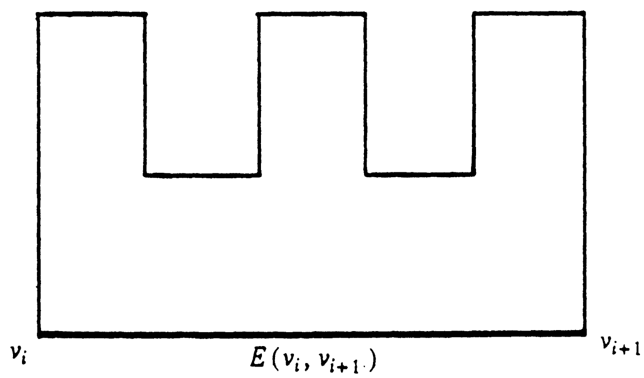
< Insert Figure 1.1 >



(a) P is completely visible from $E(v_i, v_{i+1})$



(b) P is strongly visible from $E(v_i, v_{i+1})$



(c) P is weakly visible from $E(v_i, v_{i+1})$

Figure 1.1 : P is visible from $E(v_i, v_{i+1})$

2. Preliminaries

Let P be a simple polygon with n vertices in the plane. Every vertex v_i, where i is taken modulo n, is represented by its Cartesian coordinates (x_i, y_i). These vertices in sequence are maintained as a circular doubly linked list so that the interior of P lies to the left as the boundary $B(P)$ of P is traversed in the counter-clockwise order. We assume that no three consecutive vertices of P are colinear.

Definition 2.1 : $L(u,w)$ denotes a directed line segment joining two points, u and w with the direction from u to w. An open line segment $\bar{L}(u,w)$ is the line segment $L(u,w)$ excluding its two endpoints, u and w.

Definition 2.2 : An edge $E(v_i, v_{i+1})$ is the segment $L(v_i, v_{i+1})$ joining two adjacent vertices v_i and v_{i+1} on the boundary $B(P)$ of P. An open edge $\bar{E}(v_i, v_{i+1})$ is the edge $E(v_i, v_{i+1})$ excluding its two endpoints, v_i and v_{i+1}.

Definition 2.3 : Let u and w be two distinct points on the boundary $B(P)$ of P. A chain $C_h(u,w)$ is the portion of $B(P)$ from u to w as $B(P)$ is traversed in the counter-clockwise sense. An open chain $\bar{C}_h(u,w)$ is the chain $C_h(u,w)$ excluding its two endpoints u and w.

Definition 2.4 : A ray $RAY(u,w)$ is a halfline starting from u with the direction from u to w.

Definition 2.5 : Two points u and w in P is said to be visible if the line segment joining them is completely contained in P.

Definition 2.6 : The kernel of P, denoted by $K(P)$, is the set of all points in P such that every point u in P is visible from any point w in $K(P)$.

Definition 2.7 : P is said to be weakly visible from $L(u,w)$ in P if, for every point s in P, there exists a point t in $L(u,w)$ (depending on s) such that s and t are visible. P is line-visible if there exists a line segment $L(u,w)$ in P from which P is weakly visible. If such a line segment $L(u,w)$ is contained in an edge of P, then P is said to be edge-visible.

Definition 2.8 : An edge $E(v_i,v_{i+1})$ is said to be a visible edge if P is weakly visible from $E(v_i,v_{i+1})$.

Definition 2.9 : Given a polygon P and a point z in P, the visibility polygon $V(z,P)$ from z is defined to be the set of all points in P, which is visible from z.

Avis and Toussaint gave an elegant characterization of a visible edge. We state a part of their results[1] without proof.

Lemma 2.1 : A polygon P is weakly visible from an edge $E(v_i,v_{i+1})$ if and only if its boundary $B(P)$ is weakly visible from $E(v_i,v_{i+1})$[1].

Shin and Woo [7,8] exhibited an $O(n)$ algorithm for finding all visible edges for a polygon with a non-empty kernel. Their basic idea was to discard all edges not satisfying **Lemma** 2.1. Linear time was made possible by exploiting point-visibility of a star-shaped polygon. In this paper, we find all visible edges in a polygon with an empty kernel. We employ Helly's theorem [6] to discard edges not satisfying **Lemma** 2.1.

Lemma 2.2(Helly's theorem) : Let $\{S_i | i \in I$, where I is an index set$\}$ be a finite collection of convex sets in R^d. If every subcollection consisting of $d+1$ or fewer sets in the collection has a nonempty intersection, then the entire collection has a nonempty intersection[6].

Let $HP(u,w)$ denoted the half plane which lies to the left of $L(u,w)$. It is wellknown that

$$K(P)=\bigcap_{i=0}^{n-1} HP(v_i,v_{i+1}) \qquad (2.1)$$

Suppose that $K(P)=\emptyset$. Since $HP(v_i,v_{i+1})$, $0\le i< n$, is a convex set in R^2, the following result is immediate from **Lemma** 2.2.

<u>Lemma 2.3</u> : If the kernel $K(P)$ of a simple polygon P is empty, then there exist three distinct edges, $E(v_i,v_{i+1})$, $E(v_j,v_{j+1})$, and $E(v_k,v_{k+1})$ such that

$$HP(v_i,v_{i+1})\bigcap HP(v_j,v_{j+1})\bigcap HP(v_k,v_{k+1})=\emptyset.$$

Assuming that $K(P)=\emptyset$, consider three distinct edges, $E(v_i,v_{i+1})$, $E(v_i,v_{i+1})$ and $E(v_k,v_{k+1})$ such that

$$HP(v_i,v_{i+1})\bigcap HP(v_j,v_{j+1})\bigcap HP(v_k,v_{k+1})=\emptyset \qquad (2.2)$$

Suppose that $E(v_a,v_{a+1})$ is a visible edge. Then, every point z in $E(v_r,v_{r+1})$, $r=i,j,k$ is visible from some point (depending on z) in $E(v_a,v_{a+1})$. In particular, each midpoint of $E(v_r,v_{r+1})$, $r=i,j,k$, is visible from $E(v_a,v_{a+1})$. Since visibility is a symmetric relation, the converse is also true, i.e., there exists a point in $E(v_a,v_{a+1})$, which is visible from the midpoint of $E(v_r,v_{r+1})$ for each $r=i,j,k$. In section 3, we show that there exist at most <u>three</u> edges in P such that each of them contains a point (depending on r) which is visible from the midpoint of $E(v_r,v_{r+1})$ for all $r=i,j,k$. In section 4, we present a linear time algorithm for finding all such edges. Although each of these edges contains a point which is visible from the midpoint of $E(v_r,v_{r+1})$, $r=i,j,k$, it may not be a visible edge since they do not necessarily satisfy **Lemma** 2.1. We submit each of these edges to Avis and Toussaint's test [1] for determining its edge visibility. Finally, in section 5, we present a linear time algorithm for finding all visible edges in a simple polygon.

3. Characterization of a Candidate Visible Edge

Let m_r be a point in an open edge $\bar{E}(v_r,v_{r+1})$, i.e.,

$$m_r \in \bar{E}(v_r,v_{r+1}), r=i,j,k \tag{3.1}$$

Suppose that $K(P)=\emptyset$. From **Lemma** 2.3, there exist three edges $E(v_i,v_{i+1})$, $E(v_j,v_{j+1})$ and $E(v_k,v_{k+1})$ such that

$$HP(v_i v_{i+1}) \cap HP(v_j,v_{j+1}) \cap HP(v_k v_{k+1}) = \emptyset \tag{3.2}$$

Since $V(m_r,P) \subset HP(v_r,v_{r+1}), r=i,j,k$,

$$V(m_i,P) \cap V(m_j,P) \cap V(m_k,P) = \emptyset. \tag{3.3}$$

It is obvious that a visible edge shares at least a point with each of three visibility polygons, $V(m_r,P), r=i,j,k$. Conversely, if an edge shares at least a point with each of them, it can possibly be a visible edge.

Definition 3.1 : Let $HP(v_i,v_{i+1}) \cap HP(v_j,v_{j+1}) \cap HP(v_k,v_{k+1}) = \emptyset$. An edge of P is said to be a candidate if it shares at least one point with each of $V(m_i,P)$, $V(m_j,P)$, and $V(m_k,P)$.

We show that there are at most three candidates if $K(P)=\emptyset$. From Equation (3.3), there are two possibilities:

Case 1 : Among three visibility polygons, $V(m_r,P), r=i,j,k$, there exist a pair of them which do not have a common intersection.

Case 2 : The three visibility polygons are pairwise intersecting.

For Case 1, let $V(m_i,P) \cap V(m_j,P) = \emptyset$ without loss of generality. By taking out from P all points in $V(m_i,P)$, the remaining part of P is partitioned into disjoint regions as shown in Figure 3.1.

< Insert Figure 3.1 >

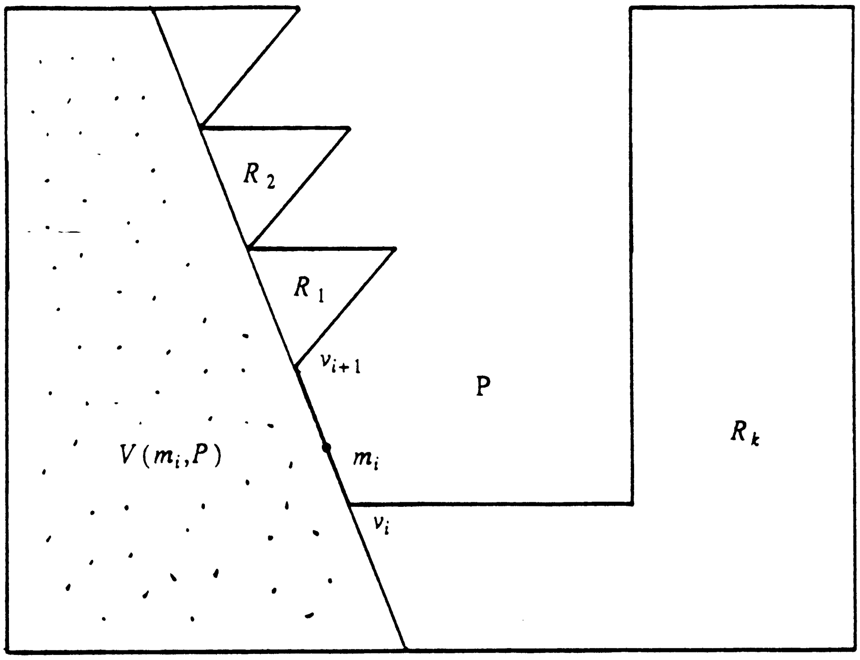


Figure 3.1 : $V(m_i,P)$ Partitions P

Each region $R_f, f=0,1,2,\cdots$ is a simple polygon bounded by a portion of $B(P)$ and a line segment S on the boundary of $V(m_i,P)$. Let m_j be contained in R_f for some f. We show that $V(m_j,P)$ does not share any point with another remaining region.

Lemma 3.1 : Suppose that $V(m_i,P) \cap V(m_j,P)=\emptyset$. Let $R_f, f=0,1,2,\cdots$ be the remaining region after subtracting $V(m_i,P)$ from P. If $m_j \in R_f$ for some f, then $V(m_j,P) \subset R_f$.

[Proof] Suppose that $V(m_j,P)$ shares a point z with another remaining region $R_g, g \neq f$. Then

$$L(m_j,z) \subset V(m_j,P) \subset P \tag{3.4}$$

R_g is bounded by a portion of $B(P)$ and a line segment S on the boundary of $V(m_i,P)$. Since $V(m_i,P) \cap V(m_j,P)= \emptyset$, $S \cap L(m_j,z)=\emptyset$. Since the remaining regions are disjoint, $m_j \notin R_g$. Therefore, $L(m_j,z)$ crosses an edge of P which bounds R_g. This contradicts Equation (3.4), which asserts that $L(m_j,z) \subset P$. Hence, $V(m_j,P)$ does not intersect any remaining region except R_f. □

An implication of **Lemma** 3.1 is that m_j is not visible from any edge which does not intersect R_f. Thus, only the edges sharing one or more points with R_f can possibly be a visible edge. However, a visible edge is required to intersect $V(m_i,P)$. By the way in which R_f is constructed, there are only two such edges, i.e., the edges which are contained (or partly contained) in R_f and intersect $V(m_i,P)$. Therefore, there are at most two candidates in Case 1.

Lemma 3.2 : Let $HP(v_i,v_{i+1}) \cap HP(v_j,v_{j+1}) \cap HP(v_k,v_{k+1})=\emptyset$. If $V(m_r,P), r=i,j,k$ does not pairwise intersect, then there exist at most two candidates.

Now, consider Case 2, where $V(m_r,P), r=i,j,k$ are pairwise intersecting. Let

$L_r =$ the line containing $E(v_r,v_{r+1}), r=i,j,k$

i_{pq} = the intersection point of lines L_p and L_q.

By assumption, $HP(v_i,v_{i+1}) \cap HP(v_j,v_{j+1}) \cap HP(v_k,v_{k+1}) = \emptyset$. Since $V(m_r,P) \subset HP(v_r,v_{r+1})$, $HP(v_r,v_{r+1})$, $r=i,j,k$ are pairwise intersecting. Therefore, the following is true:

(1) i_{ij}, i_{jk}, and i_{ki} are well-defined and distinct

(2) i_{ij}, i_{jk}, and i_{ki} are not colinear

As illustrated in Figure 3.2, no interior point of the triangle (i_{ij},i_{jk},i_{ki}) is contained in $HP(v_r,v_{r+1})$ for any $r=i,j,k$. In the following lemma, we show that the triangle (i_{ij},i_{jk},i_{ki}) is entirely contained in P.

< Insert Figure 3.2 >

<u>Lemma 3.3</u> : Suppose that the following is true :

(1) $HP(v_i,v_{i+1}) \cap HP(v_j,v_{j+1}) \cap HP(v_k,v_{k+1}) = \emptyset$.

(2) $V(m_r,P), r=i,j,k$ are pairwise intersecting.

Then, every point in the triangle, (i_{ij},i_{jk},i_{ki}) is contained in P.

[Proof] Take three points, u_{ij}, u_{jk}, and u_{ki} such that

$$\begin{aligned} u_{ij} &\in V(m_i,P) \cap V(m_j,P) \\ u_{jk} &\in V(m_j,P) \cap V(m_k,p) \\ u_{ki} &\in V(m_k,P) \cap V(m_i,P) \end{aligned} \tag{3.5}$$

Since $HP(v_i,v_{i+1}) \cap HP(v_j,v_{j+1}) \cap HP(v_k,v_{k+1}) = \emptyset$,

$$V(m_i,P) \cap V(m_j,P) \cap V(m_k,P) = \emptyset \tag{3.6}$$

From Equations (3.5) and (3.6), three points, u_{ij}, u_{jk}, and u_{ki} are necessarily distinct. Futhermore,

$$\begin{aligned} u_{ki} \in V(m_i,P) &\text{ and } u_{ij} \in V(m_i,P) \\ u_{ij} \in V(m_j,P) &\text{ and } u_{jk} \in V(m_j,P) \\ u_{jk} \in V(m_k,P) &\text{ and } u_{ki} \in V(m_k,P) \end{aligned} \tag{3.7}$$

Let $P(w,z)$ be a simple path between two points, w and z. From Equation

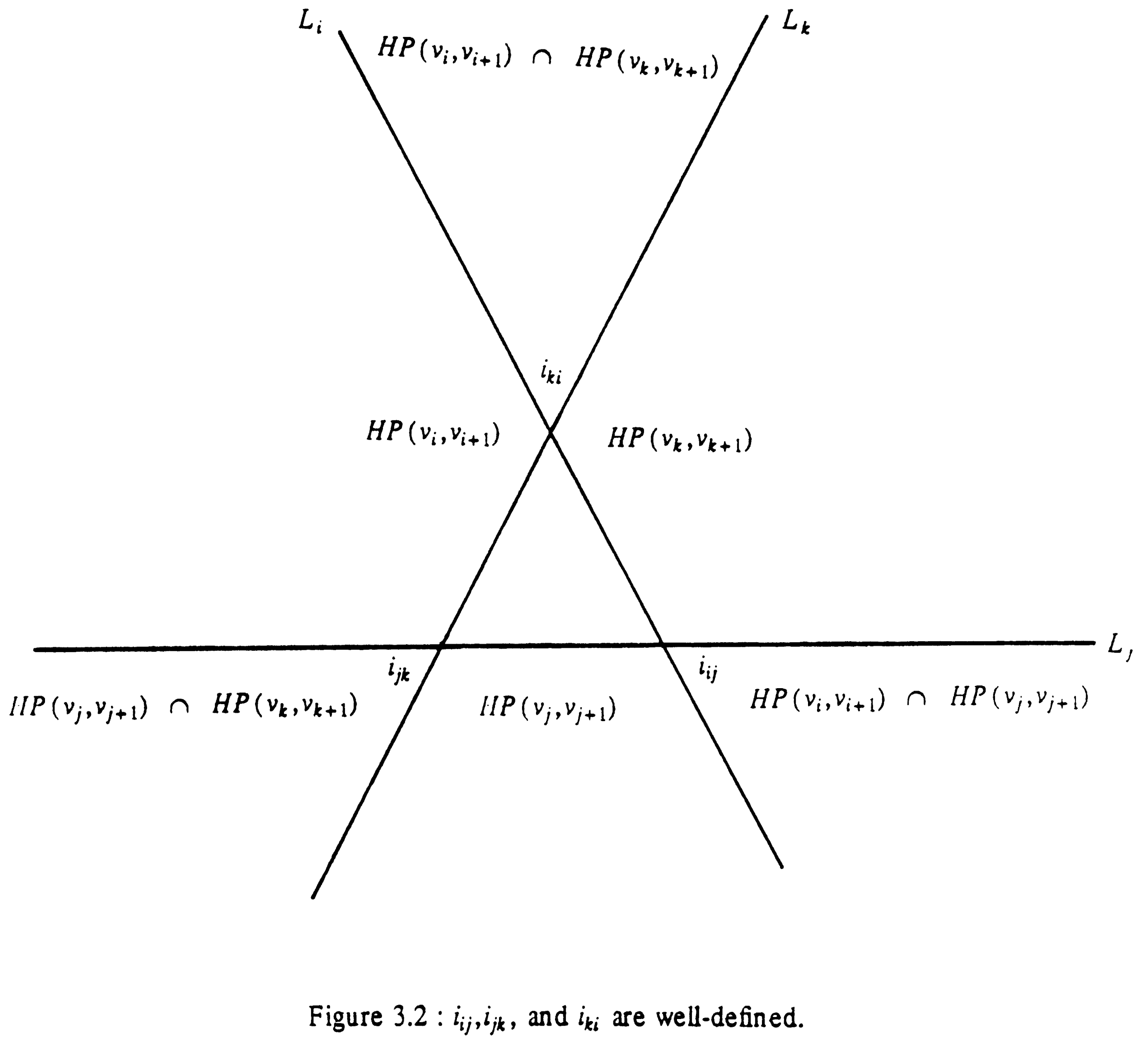


Figure 3.2 : i_{ij}, i_{jk}, and i_{ki} are well-defined.

(3.7), it is possible to construct three simple paths, $P(u_{ki},u_{ij})$, $P(u_{ij},u_{jk})$, and $P(u_{jk},u_{ki})$ such that

$$\begin{aligned} P(u_{ki},u_{ij}) &\subset V(m_i,P) \\ P(u_{ij},u_{jk}) &\subset V(m_j,P) \\ P(u_{jk},u_{ki}) &\subset V(m_k,P) \end{aligned} \tag{3.8}$$

Moreover, these three paths can be constructed so that they pairwise intersect only at u_{ij},u_{jk}, and u_{ki}. Since $V(m_r,P)\subset P, r=i,j,k$,

$$P(u_{ki},u_{ij})\subset P,\ P(u_{ij},u_{jk})\subset P, \text{ and } P(u_{jk},u_{ki})\subset P \tag{3.9}$$

In words, each of these simple paths is completely contained in P. Since $V(m_r,P)\subset HP(v_r,v_{r+1}), r=i,j,k$, the following is immediate from (3.8) :

$$\begin{aligned} P(u_{ki},u_{ij}) &\subset HP(v_i,v_{i+1}) \\ P(u_{ij},u_{jk}) &\subset HP(v_j,v_{j+1}) \\ P(u_{jk},u_{ki}) &\subset HP(v_k,v_{k+1}) \end{aligned} \tag{3.10}$$

Equations (3.10) implies that each of these three paths does not share any point with the interior of the triangle (i_{ij},i_{jk},i_{ki}). This is true since $HP(v_r,v_{r+1}), r=i,j,k$ does not share any point with the interior of the triangle. Thus the triangle (i_{ij},i_{jk},i_{ki}) is completely contained in R bounded by the three paths (See Figure 3.3). From Equations (3.9), these three paths lie completely in P. Hence, the triangle (i_{ij},i_{jk},i_{ki}) is completely contained in P. □

< Insert Figure 3.3 >

From **Lemma** 3.3, every interior point of the triangle (i_{ij},i_{jk},i_{ki}) is an interior point of P. Let u be an interior point of the triangle (i_{ij},i_{jk},i_{ki}). Since u is also interior point of P, there exists a line segment $L(w_r,z_r)$ for each $r=i,j,k$ satisfying the following properties (See Figure 3.4) :

(P1) $u\in \tilde{L}(w_r,z_r)\subset P$

(P2) $L(w_r,z_r)\cap B(P)=\{w_r,z_r\}$ and $\tilde{L}(w_r,z_r)\cap B(P)=\emptyset$

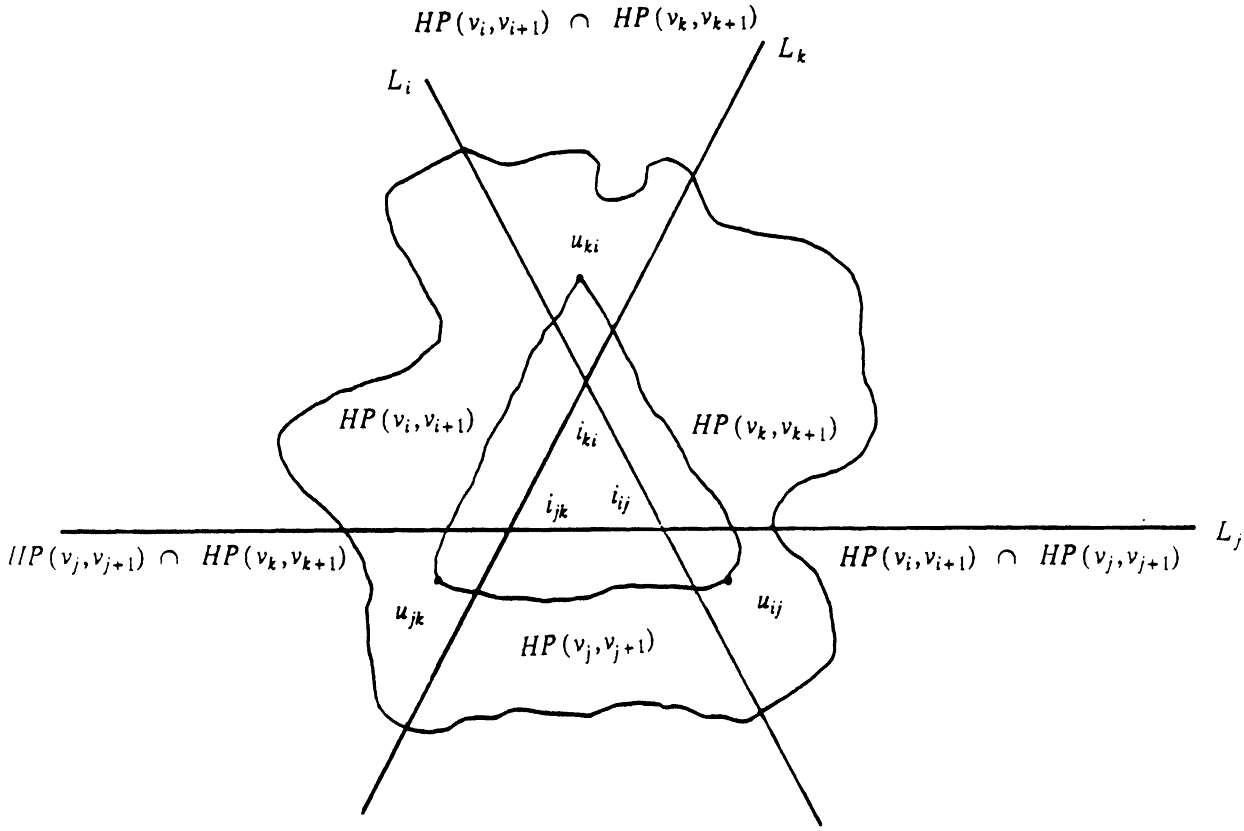


Figure 3.3 : The triangle (i_{ij}, i_{jk}, i_{ki}) is contained in P.

(P3) $L(w_r,z_r)$ is parallel with $E(v_r,v_{r+1})$

(P4) the direction of $L(w_r,z_r)$ is opposite to that of $E(v_r,v_{r+1})$.

< Insert Figure 3.4 >

$HP(v_r,v_{r+1}),r=i,j,k$ is determined by the line L_r containing $E(v_r,v_{r+1})$ which is parallel with $L(w_r,z_r)$. Therefore, $L(w_r,z_r)$ either is contained in $HP(v_r,v_{r+1})$ or does not share any point with $HP(v_r,v_{r+1})$. However, no interior point of the triangle (i_{ij},i_{jk},i_{ki}) is contained in $HP(v_r,v_{r+1})$. In particular, $u \notin HP(v_r,v_{r+1})$ for any $r=i,j,k$. Therefore,

$$L(w_r,z_r) \cap HP(v_r,v_{r+1}) = \varnothing, r=i,j,k. \tag{3.11}$$

From Equation (3.11), the following two lemmas are immediate.

<u>Lemma 3.4</u> : Let $HP(v_i,v_{i+1}) \cap HP(v_j,v_{j+1}) \cap HP(v_k,v_{k+1}) = \varnothing$. If $V(m_r,P),r=i,j,k$, are pairwise intersecting, then

$$HP(v_r,v_{r+1}) \subset HP(z_r,w_r), r=i,j,k.$$

[Proof] Let $V(m_r,P),r=i,j,k$ be pairwise intersecting. By property (P4), $L(w_r,z_r)$ has the opposite direction to $E(v_r,v_{r+1})$. Thus, $L(z_r,w_r)$ and $E(v_r,v_{r+1})$ have the same direction. From property (P3), $L(z_r,w_r)$ and $E(v_r,v_{r+1})$ are parallel. Therefore, either $HP(v_r,v_{r+1}) \subset HP(z_r,w_r)$ or $HP(z_r,w_r) \subset HP(v_r,v_{r+1})$. If $HP(v_r,v_{r+1}) \subset HP(z_r,w_r)$, then $L(z_r,w_r) \subset HP(v_r,v_{r+1})$ (or equivalently $L(w_r,z_r) \subset HP(v_r,v_{r+1})$), which contradicts Equation (3.11). Hence, $HP(v_r,v_{r+1}) \subset HP(z_r,w_r)$. □

<u>Lemma 3.5</u> : Let $HP(v_i,v_{i+1}) \cap HP(v_j,v_{j+1}) \cap HP(v_k,v_{k+1}) = \varnothing$. If $V(m_r,P),r=i,j,k$ are pairwise intersecting, then no point in $L(w_r,z_r)$ is visible from m_r, i.e.,

$$L(w_r,z_r) \cap V(m_r,P) = \varnothing, r=i,j,k.$$

[Proof] From Equation (3.11), $L(w_r,z_r) \cap HP(v_r,v_{r+1}) = \varnothing$. Since $V(m_r,P) \subset HP(v_r,v_{r+1})$, $L(w_r,z_r) \cap V(m_r,P) = \varnothing$. □

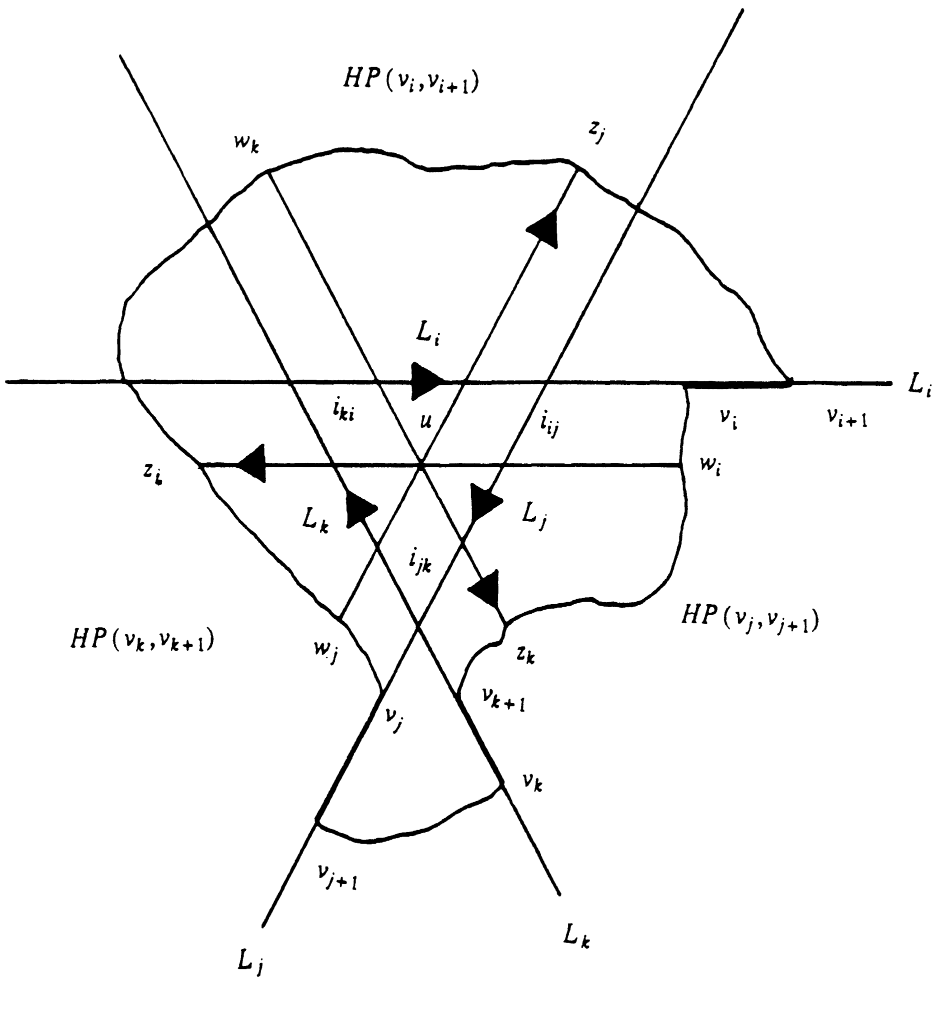


Figure 3.4 : $HP(v_r, v_{r+1}) \cap L(w_r, z_r) = \phi, r = i, j\ k$.

A pair of points, w_r and z_r for each $r=i,j,k$ partition $B(P)$ into two chains, $C_h(w_r,z_r)$ and $C_h(z_r,w_r)$. From properties (P1) and (P2) of $L(w_r,z_r)$, $L(w_r,z_r)$ partitions P into two simple polygons, and P_r^1 and P_r^2 as follows :

P_r^1 = the simple polygon bounded by $C_h(w_r,z_r)$ and $L(z_r,w_r)$

P_r^2 = the simple polygon bounded by $C_h(z_r,w_r)$ and $L(w_r,z_r)$.

Accordingly,

$$P_r^1 \cup P_r^2 = P \text{ and } P_r^1 \cap P_r^2 = L(w_r,z_r), r=i,j,k. \tag{3.12}$$

By properties (P1) abd (P3) of $L(w_i,z_i)$, $L(w_i,z_i)$ containing u is parallel with $E(v_i,v_{i+1})$, where u is an interior point of the triangle (i_{ij},i_{jk},i_{ki}). Thus, $L(w_i,z_i)$ and $E(v_i,v_{i+1})$ are not colinear. Since $L(i_{ki},i_{ij})$ is contained in the line L_i containing $E(v_i,v_{i+1})$, $L(w_i,z_i)$ and $L(i_{ki},i_{ij})$ are parallel but not colinear. Futhermore, $L(w_i,z_i)$ bisects both $L(i_{ij},i_{jk})$ and $L(i_{jk},i_{ki})$. Therefore, $L(i_{ki},i_{ij})$ and i_{jk} cannot be simultaneously contained in P_i^1 (or P_i^2). Since $L(i_{ki},i_{ij}) \subset L_i \subset HP(v_i,v_{i+1})$, $L(i_{ki},i_{ij}) \subset HP(z_i,w_i)$. This implies that $L(i_{ki},i_{ij})$ lies to the right of $L(w_i,z_i)$. Since the interior of P_i^1 lies to the right of $L(w_i,z_i)$,

$$L(i_{ki},i_{ij}) \subset P_i^1 \text{ and } i_{jk} \in P_i^2 \tag{3.13}$$

Similarly, as illustrated in Figure 3.4,

$$L(i_{ij},i_{jk}) \subset P_j^1 \text{ and } i_{ki} \in P_j^2 \tag{3.14}$$

$$L(i_{jk},i_{ki}) \subset P_k^1 \text{ and } i_{ij} \in P_k^2$$

Thus, the following lemma can be obtained.

<u>Lemma 3.6</u> : Let $HP(v_i,v_{i+1}) \cap HP(v_j,v_{j+1}) \cap HP(v_k,v_{k+1}) = \emptyset$. If $V(m_r,P), r=i,j,k$ are pairwise intersecting, then the following is true :

$$L(i_{ki},i_{ij}) \subset P_i^1 \text{ and } i_{jk} \in P_i^2$$
$$L(i_{ij},i_{jk}) \subset P_j^1 \text{ and } i_{ki} \in P_j^2$$
$$L(i_{jk},i_{ki}) \subset P_k^1 \text{ and } i_{ij} \in P_k^2$$

< Insert Figure 3.5 >

Since $m_r \in E(v_r,v_{r+1})$, every point in $E(v_r,v_{r+1})$ is visible from m_r. From **Lemma** 3.5, neither w_r nor w_r is visible from m_r. Therefore,

$$w_r \notin E(v_r,v_{r+1}) \text{ and } z_r \notin E(v_r,{}_{r+1}). \tag{3.15}$$

This implies that either $E(v_r,v_{r+1}) \subset \tilde{C}_h(w_r,z_r)$ or $E(v_r,v_{r+1}) \subset \tilde{C}_h(z_r,w_r)$. We show that $E(v_r,v_{r+1}) \subset \tilde{C}_h(w_r,z_r)$.

<u>Lemma 3.7</u> : Let $HP(v_i,v_{i+1}) \cap HP(v_j,v_{j+1}) \cap HP(v_k,v_{k+1}) = \emptyset$. If $V(m_r,P), r=i,j,k$ are pairwise intersecting, then

$$E(v_r,v_{r+1}) \subset \tilde{C}_h(w_r,z_r), r=i,j,k.$$

[Proof] Suppose that $E(v_i,v_{i+1}) \subset \tilde{C}_h(z_i,w_i)$. Then,

$$m_i \in \bar{E}(v_i,v_{i+1}) \subset P_i^2. \tag{3.16}$$

We first show that, given $E(v_i,v_{i+1}) \subset \tilde{C}_h(z_i,w_i)$,

$$m_r \in \tilde{C}_h(z_i,w_i), r=i,j,k. \tag{3.17}$$

Assume that $m_j \in C_h(w_i,z_i)$. Then $m_j \in P_i^1$. Since $m_i \in P_i^2$ and $L(w_i,z_i) \cap V(m_i,P) = \emptyset$,

$$V(m_i,P) \subset P_i^2. \tag{3.18}$$

Thus,

$$V(m_i,P) \cap V(m_j,P) \subset P_i^2 \tag{3.19}$$

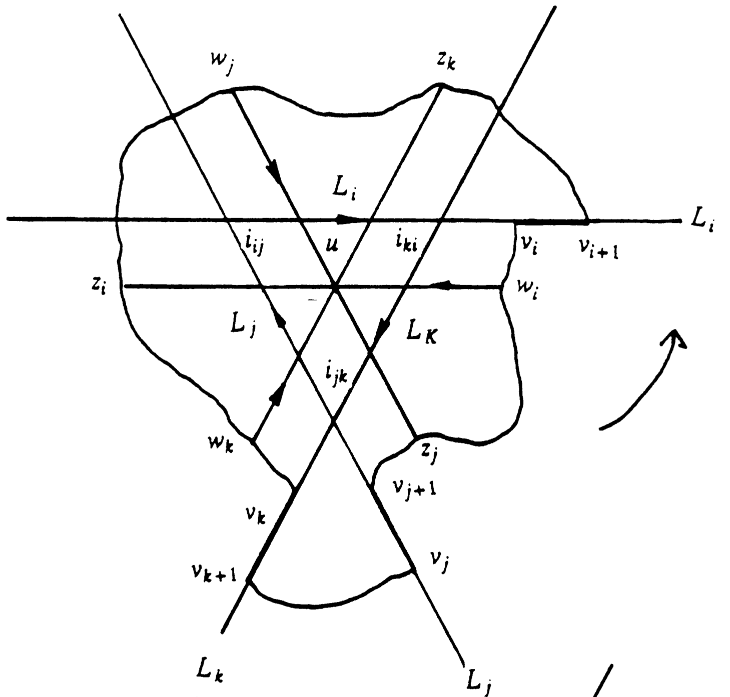
Let a set G be defined as follows :

$$G = \{g \mid g \in V(m_j,P) \text{ and } g \in P_i^2\}.$$

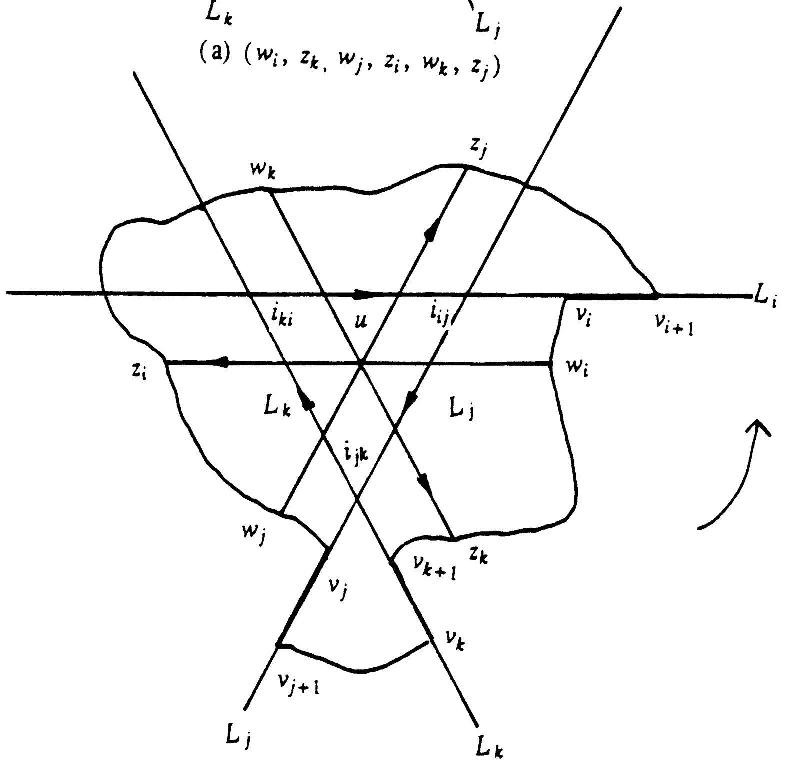
Then, from Equation (3.18),

$$V(m_i,P) \cap V(m_j,P) = V(m_i,P) \cap G. \tag{3.20}$$

By **Lemma** 3.4,



(a) $(w_i,\ z_k,\ w_j,\ z_i,\ w_k,\ z_j)$



(b) $(w_i,\ z_j,\ w_k,\ z_i,\ w_j,\ z_k)$

Figure 3.5 : w_r and z_r, $r = i,\ j,\ k$ are alternating.

$$V(m_i,P) \subset HP(v_i,v_{i+1}) \subset HP(z_i,w_i). \tag{3.21}$$

Therefore, every point in $V(m_i,P)$ lies strictly to the left of $L(z_i,w_i)$ or equivalently lies to the right of $L(w_i,z_i)$. Now, consider G. The interior of P_i^2 lies to the left of $L(w_i,z_i)$. Since $m_j \in C_h(w_i,z_i) \subset P_i^1$, every point g is visible from m_j only through $L(w_i,z_i)$, i.e.,

$$L(m_j,g) \cap L(w_i,z_i) \neq \emptyset \text{ for } all\ g \in G. \tag{3.22}$$

Moreover, $m_j \in \bar{E}(v_j,v_{j+1})$ so that m_j is not an endpoint of $E(v_j,v_{j+1})$. Thus, Equation (3.22) implies that every point g in G lies on or to the left of $L(w_i,z_i)$. Therefore,

$$V(m_i,P) \cap G = \emptyset, \tag{3.23}$$

From Equations (3.20) and (3.23),

$$V(m_i,P) \cap V(m_j,P) = \emptyset,$$

which is a contraction. Thus, $m_j \notin C_h(w_i,z_i)$ or $m_j \in \bar{C}_h(z_i,w_i)$. Similarly, $m_k \in \bar{C}_h(z_i,w_i)$. Hence, given $E(v_i,v_{i+1}) \subset \bar{C}_h(z_i,w_i)$, Equation (3.17) holds true.

Now, we show that $E(v_i,v_{i+1})$ cannot be contained in $\bar{C}_h(z_i,w_i)$ using **Lemma** 3.3 and thus arrive at a contradiction. Consider the simple polygon P_i^2. Since $m_r \in \bar{C}_h(z_i,w_i) \subset P_i^2$, there exists an edge $E(a_r,b_r)$ of P_i^2 for each $r=i,j,k$ such that

(1) $E(a_r,b_r) \subseteq E(v_r,v_{r+1})$ and $m_r \in \bar{E}(a_r,b_r)$

(2) $E(a_r,b_r)$ has the same direction as $E(v_r,v_{r+1})$

By property (1),

$$E(a_r,b_r) \subset L_r, r=i,j,k \tag{3.24}$$

From Equation (3.24) and property (2),

$$HP(a_r,b_r) = HP(v_r,v_{r+1}), r=i,j,k \tag{3.25}$$

Therefore, the triangle determined by L_r containing $E(a_r,b_r),r=i,j,k$ is the triangle (i_{ij},i_{jk},i_{ki}). By **Lemma** 3.6,

$$L(i_{ki},i_{ij})\cap P_i^2=\emptyset, \tag{3.26}$$

which means that the triangle (i_{ij},i_{jk},i_{ki}) is not completely contained in P_i^2. This contradicts **Lemma** 3.3 if P_i^2 satisfies the assumptions for this lemma.

From Equation (3.25), $HP(a_i,b_i)\cap HP(a_j,b_j)\cap HP(a_k,b_k)=\emptyset$. Thus, P_i^2 satisfies assumption (1) for **Lemma** 3.3. We will be done if we show that P_i^2 also satisfies assumption (2), i.e., $V(m_r,P_i^2),r=i,j,k$ are pairwise intersecting. Since $V(m_i,P)\subset P_i^2$,

$$V(m_i,P_i^2)\cap V(m_j,P_i^2)=V(m_i,P)\cap V(m_j,P)\neq\emptyset \tag{3.27}$$
$$V(m_i,P_i^2)\cap V(m_k,P_i^2)=V(m_i,P)\cap V(m_k,P)\neq\emptyset$$

We need to show that

$$V(m_j,P_i^2)\cap V(m_k,P_i^2)\neq\emptyset. \tag{3.28}$$

From **Lemmma** 3.6, $i_{jk}\in P_i^2$ and $L(i_{ki},i_{ij})\subset P_i^1$. Since $L(w_i,z_i)$ bisects $L(i_{jk},i_{ki})$, i_{jk} lies strictly to the left of $L(w_i,z_i)$. Therefore, every point in $HP(v_j,v_{j+1})\cap HP(v_k,v_{k+1})$ lies strictly to the left of $L(w_i,z_i)$. Thus, every point in $V(m_j,P)\cap V(m_k,P)$ also lies strictly to the left of $L(w_i,z_i)$. Suppose that a point h in $V(m_j,P)\cap V(m_k,P)$ is contained in P_i^1. Since both m_j and m_k are contained in $\bar{C}_h(z_i,w_i)$, h lies to the right of $L(w_i,z_i)$, which is not possible. Therefore,

$$V(m_j,P)\cap V(m_k,P)\subset P_i^2 \tag{3.29}$$

From Equation (3.29), inequality (3.28) follows. Hence, P_i^2 also satisfies assumption (2) for **Lemma** 3.3. However, the triangle (i_{ij},i_{jk},i_{jk}), which is determined by $L_r,r=i,j,k$, containing $E(a_r,b_r)$, is not contained in P_i^2. This contradicts **Lemma** 3.3. Thus, $E(v_i,v_{i+1})\subset\bar{C}_h(w_i,z_i)$. By a similar reasoning, $E(v_j,v_{j+1})\subset\bar{C}_h(w_j,z_j)$ and $E(v_k,v_{k+1})\subset\bar{C}_h(w_k,z_k)$. □

Based on **Lemma** 3.7, we develop a fundamental relationship among w_r and z_r $r=i,j,k$.

Lemma 3.8 : Let $HP(v_i,v_{i+1}) \cap HP(v_j,v_{j+1}) \cap HP(v_k,v_{k+1})=\emptyset$. Suppose that $V(m_r,P),r=i,j,k$ are pairwise intersecting. Then, w_r and $z_r,r=i,j,k$, appear in one of the following order as $B(P)$ is scanned from w_i in the counter-clockwise sense:

(1) $(w_i,z_k,w_j,z_i,w_k,z_j)$

(2) $(w_i,z_j,w_k,z_i,w_j,z_k)$

[Proof] $L(w_r,z_r),r=i,j,k$ are pairwise not parallel and share a common point u from properties (P1) and (P4) of $L(w_r,z_r)$ so that

$$L(w_i,z_i) \cap L(w_j,z_j)=L(w_j,z_j) \cap L(w_k,z_k)=L(w_k,z_k) \cap L(w_i,z_i)=u \qquad (3.30)$$

Therefore, either w_i or z_j(but not both) is to the right of $L(w_i,z_i)$. We consider the following two cases separately :

Case (a) : w_j lies to the right of $L(w_i,z_i)$

Case (b) : z_j lies to the right of $L(w_i,z_i)$

Case (a) : Let $CN(s,t)$ be a cone obtained by rotating the ray $RAY(u,s)$ about u in the counter-clockwise sense until $RAY(u,t)$ is encountered. Since w_j lies to the right of $L(w_i,z_i)$, $RAY(u,w_j)$ partitions $HP(z_i,w_i)$ into two cones, $CN(w_i,w_j)$ and $CN(w_j,z_i)$. Accordingly, by Equation (3,30) and properties (p2) and (P4), $C_h(w_i,z_i)$ is also partitioned into $C_h(w_i,w_j)$ and $C_h(w_j,z_i)$. From **Lemma** 3.4, $HP(v_r,v_{r+1}) \subset HP(z_r,w_r),r=i,j,k$. Therefore,

$$HP(v_i,v_{i+1}) \cap HP(v_j,v_{j+1}) \subset HP(z_i,w_i) \cap HP(z_j,w_j). \qquad (3.31)$$

$HP(v_r,v_{r+1})$ lies strictly to the right of $L(w_r,z_r),r=i,j,k$. Since $u \in \bar{L}(w_r,z_r)$, $HP(v_r,v_{r+1})$ lies strictly to the left of $L(u,w_r)$ and strictly to the right of $L(u,z_r)$. By definition, every point in $CN(w_j,z_j)$ lies to the left of $L(u,w_j)$ and to the right of $L(u,z_i)$. Since every point in $HP(z_i,w_i) \cap HP(z_j,w_j)$ lies to the left of $L(u,w_j)$ and to the right of $L(u,z_i)$,

$$CN(w_j,z_i) = HP(z_i,w_i) \cap HP(z_j,w_j) \tag{3.32}$$

From Equations (3.31) and (3.32),

$$HP(v_i,v_{i+1}) \cap HP(v_j,v_{j+1}) \subset CN(w_j,z_i). \tag{3.33}$$

Since $\bar{L}(w_i,z_i) \cap \bar{L}(w_k,z_k) = u$, either w_k or z_k (but not both) lies to the right of $L(w_i,z_i)$. Suppose that w_k lies to the right of $L(w_i,z_i)$. Then,

$$RAY(u,w_k) \subset HP(z_i,w_i) \tag{3.34}$$

Since $RAY(u,w_j)$ partions $HP(z_i,w_i)$, either $RAY(u,w_k) \subset CN(w_i,w_j)$ or $RAY(u,w_k) \subset CN(w_j,z_i)$. If $RAY(u,w_k) \subset CN(w_i,w_j)$, then

$$CN(w_j,z_i) \subset HP(z_k,w_k), \tag{3.35}$$

since $L(w_r,z_r), w=i,j,k$ are pairwise not parallel and share a point u. From Equations (3.33) and (3.35),

$$HP(v_i,v_{i+1}) \cap HP(v_j,v_{j+1}) \subset HP(z_k,w_k) \tag{3.36}$$

This together with the fact that $HP(v_k,v_{k+1}) \subset HP(z_k,w_k)$ implies that $HP(v_i,v_{i+1}) \cap HP(v_j,v_{j+1}) \cap HP(v_k,v_{k+1}) \neq \emptyset$, which is a contradiction. Thus, $RAY(u,w_k) \not\subset CN(w_i,w_j)$. Now, suppose that $RAY(u,w_k) \subset CN(w_j,z_i)$. $L_r, r=i,j,k$ determining $HP(v_r,v_{r+1})$ is parallel with $L(w_r,z_r)$. Since $HP(v_i,v_{i+1}) \cap HP(v_j,v_{j+1}) \subset CN(w_j,z_i)$, $RAY(u,w_k)$ intersects $HP(v_i,v_{i+1}) \cap HP(v_j,v_{j+1})$. However,

$$RAY(u,w_k) \subset L_k \subset HP(v_k,v_{k+1}) \tag{3.37}$$

Therefore, $HP(v_i,v_{i+1}) \cap HP(v_j,v_{j+1}) \cap HP(v_k,v_{k+1}) \neq \emptyset$, which is also a contradiction. Thus, $RAY(u,w_k) \not\subset CN(w_j,z_i)$, either. Consequently, w_k does not lie to the right of $L(w_i,z_i)$, and z_k lies to the right of $L(w_i,z_i)$, i.e.,

$$RAY(u,z_k) \subset HP(z_i,w_i) \tag{3.38}$$

Thus, either $RAY(u,z_k) \subset CH(w_i,w_j)$ or $RAY(u,z_k) \subset CN(w_j,z_i)$.

Suppose that $RAY(u,z_k) \subset CN(w_j,z_i)$. Using a similar argument as that employed for deriving Equations (3.33) and (3.35),

$$\begin{gathered} HP(v_j,v_{j+1}) \cap HP(v_k,v_{k+1}) \subset CN(w_j,z_k) \\ CN(w_j,z_k) \subset HP(z_i,w_i) \end{gathered} \tag{3.39}$$

Equations (3.39) implies that $HP(v_i,v_{i+1}) \cap HP(v_k,v_{j+1}) \cap HP(v_k,v_{k+1}) \neq \emptyset$, which is a contradiction. Therefore,

$$RAY(u,z_k) \subset CN(w_i,z_j) \tag{3.40}$$

Thus, $z_k \in \tilde{C}_h(w_i,w_j)$. Since $w_j \in \tilde{C}_h(w_i,z_i)$,

$$C_h(w_i,z_k) \subset C_h(w_i,w_j) \subset C_h(w_i,z_i) \tag{3.41}$$

Accordingly, as illustrated in Figure 3.5,

$$C_h(z_i,w_k) \subset C_h(z_i,z_j) \subset C_h(z_i,w_i) \tag{3.42}$$

From Equations (3.41) and (3.42), the result follows.

Case (b) : By a similar argument,

$$\begin{gathered} C_h(w_i,z_j) \subset C_h(w_i,w_k) \subset C_h(w_i,z_i) \\ C_h(z_i,w_j) \subset C_h(z_i,z_k) \subset C_h(z_i,w_i) \end{gathered} \tag{3.43}$$

Hence, the result holds. □

Now, we are ready to present the comparison lemma of **Lemma** 3.2 for Case 2.

<u>Lemma 3.9</u> : Let $HP(v_i,v_{i+1}) \cap HP(v_j,v_{j+1}) \cap HP(v_k,v_{k+1}) \neq \emptyset$. If $V(m_r,P), r=i,j,k$ are pairwise intersecting, then there exist at most three candidates.

[Proof] From **Lemma** 3.5, $V(m_r,P) \cap L(w_r,z_r) = \emptyset, r=i,j,k$. By **Lemma** 3.7,

$$m_r \in E(v_r, v_{r+1}) \subset \tilde{C}_h(w_r, z_r) \subset P_r^1. \qquad (3.44)$$

Therefore, no point in P_r^2 is visible from $m_r, r=i,j,k$. In particular, no point in $C_h(z_r, w_r)$ is visible from m_r. Thus, if an edge of P is completely contained in $C_h(z_r, w_r)$ for some $r=i,j,k$, it can not be a candidate by **Definition** 3.1.

Points w_i and z_i partition $B(P)$ into two chains, $C_h(w_i, z_i)$ and $C_h(z_i, w_i)$. None of the edges, which are completely contained in $C_h(z_i, w_i)$, can be a candidate. However, an edge, which is partly contained in $C_h(z_i, w_i)$, can possibly be a candidate. There are at most two edges which are partly contained in $C_h(z_i, w_i)$, i.e.,

(1) the edge containing w_i if w_i is not a vertex.

(2) the edge containing z_i if z_i is not a vertex.

These edges, if they exist, are also partly contained in $C_h(w_i, z_i)$.

In order to identify other possible candidates, consider $C_h(w_i, z_i)$. From **Lemma** 3.8, there are two possibilities:

(a) $C_h(w_i, z_k) \subset C_h(w_i, w_j) \subset C_h(w_i, z_i)$

(b) $C_h(w_i, z_j) \subset C_h(w_i, w_k) \subset C_h(w_i, z_i)$

Without loss of generality, let

$$C_h(w_i, z_k) \subset C_h(w_i, w_j) \subset C_h(w_i, z_i). \qquad (3.45)$$

(Notice that (a) can be obtained from (b) by swapping (w_j, z_k) and (w_k, z_j).) Then,

$$C_h(w_i, z_i) = C_h(w_i, z_k) \cap C_h(z_k, z_i). \qquad (3.46)$$

Consider $C_h(z_k, z_i)$. Since $C_h(z_k, z_i) \subset C_h(z_k, w_k)$, no point in $C_h(z_k, z_i)$ is visible from m_k. Therefore, if an edge is completely contained in $C_h(z_k, z_i)$, it cannot be a candidate. There are at most two edges which are partly contained in $C_h(z_k, z_i)$, i.e.,

(3) the edge containing z_k if z_k is not a vertex.

(4) the edge containing z_i if z_i is not a vertex.

However, the edges (2) and (4) are the same. Finally, consider $C_h(w_i,z_k)$. Since $C_h(w_i,z_k) \subset C_h(z_j,w_j)$, no point in $C_h(w_i,z_k)$ is visible from m_j. There are at most two edges which are partly contained in $C_h(w_i,z_k)$, i.e.,

(5) the edge containing w_i if w_i is not a vertex

(6) the edge containing z_k if z_k is not a vertex.

However, the edges (5) and (6), if any, are the same as the edges (1) and (3), respectively. Hence, there are at most three candidates in Case 2. □

From **Lemma** 3.2 and 3.9, the following result is immediate.

Theorem 3.1 : If $K(P)=\varnothing$, then there are at most three candidates, and thus at most three visible edges in P.

4. Choosing all Candidates

In this section, assuming that $K(P)=\varnothing$, we show algorithmically that all candidates can be found in $O(n)$ time. Suppose that $K(P)=\varnothing$. From **Lemma** 2.3, there exist three edges, $E(v_r,v_{r+1}), r=i,j,k$, such that

$$HP(v_i,v_{i+1}) \cap HP(v_j,v_{j+1}) \cap HP(v_k,v_{k+1}) = \varnothing \qquad (4.1)$$

We first show that $E(v_r,v_{r+1})$, $r=i,j,k$ can be found in $O(n)$ time. Let

$$K_t(P) = \bigcap_{m=0}^{t} HP(v_m,v_{m+1}), t=0,1,2, \cdots, n-1. \qquad (4.2)$$

Since $K(P)$ is the intersection of n half planes $HP(v_m,v_{m+1})$, $0 \le m < n$, $K_{n-1}(P)=K(P)$. Lee and Preparata[4] presented a linear time algorithm for finding $K(P)$. Their basic idea is to iteratively construct $K_t(P)$, i.e.,

$$K_t(P) = \begin{cases} HP(v_0,v_1), & \text{if } t=0 \\ K_{t-1}(P) \cap HP(v_t,v_{t+1}), & \text{otherwise} \end{cases} \qquad (4.3)$$

The algorithm is terminated if it encounters one of the following conditions :

(1) $K_{t-1}(P) \neq \emptyset$ and $K_t(P) = \emptyset$ for some $0 < t \leq n-1$

(2) $K_{n-1}(P)$ is constructed, i.e., $t = n-1$.

Since $K(P) = \emptyset$, the algorithm is always terminated with condition (1), i.e., $K_{t-1}(P) \neq \emptyset$ and $K_t(P) \neq \emptyset$ for some $0 < t \leq n-1$. Moreover, $2 < t \leq n-1$ since two adjacent edges share a common vertex. Thus, the following result is immediate from reference[4].

<u>Lemma 4.1</u> : Suppose that $K(P) = \emptyset$. Then, there exist $K_{t-1}(P)$ and $E(v_t, v_{t+1})$ for some $2 < t \leq n-1$ such that $K_{t-1}(P) \neq \emptyset$ and $K_t(P) = \emptyset$. Such an edge $E(v_t, v_{t+1})$ can be obtained in $O(n)$ time[4].

Assuming that $K_{t-1}(P) \neq \emptyset$ and $K_t(P) = \emptyset, 0 < t \leq n-1$, let

$$H_{t-1} = \{HP(v_m, v_{m+1}) \mid 0 \leq m < t\}. \tag{4.4}$$

Since $K_{t-1}(P) \neq \emptyset$, every subset of t or fewer halfplanes in H_{t-1} has a non-empty intersection. Consider

$$H_t = H_{t-1} \cup \{HP(v_t, v_{t+1})\}. \tag{4.5}$$

By assumption,

$$K_t(P) = K_{t-1}(P) \cap HP(v_t, v_{t+1}) = \emptyset$$

From **Lemma** 2.2(Helly's theorem), there exist two halfplanes, $HP(v_i, v_{i+1})$ and $HP(v_j, v_{j+1})$ in H_{t-1} such that

$$HP(v_i, v_{i+1}) \cap HP(v_j, v_{j+1}) \cap HP(v_t, v_{t+1}) = \emptyset. \tag{4.6}$$

There are two possibilities :

Case I : $HP(v_m, v_{m+1}) \cap HP(v_t, v_{t+1}) = \emptyset$ for some $0 \leq m < t$

Case II: $HP(v_m, v_{m+1}) \cap HP(v_t, v_{t+1}) \neq \emptyset$ for any $0 \leq m < t$

In case I, take any halfplane $HP(v_j, v_{j+1})$ in H_{t-1}, which is not $HP(v_m, v_{m+1})$. Since $HP(v_m, v_{m+1}) \cap HP(v_t, v_{t+1}) = \emptyset$, $HP(v_m, v_{m+1}) \cap HP(v_j, v_{j+1}) \cap HP(v_t, v_{t+1}) = \emptyset$. By setting $i=m$,

$$HP(v_i,v_{i+1}) \cap HP(v_j,v_{j+1}) \cap HP(v_t,v_{t+1})=\emptyset.$$

Now, consider Case II. In this case, $K_{t-1}(P)=\emptyset$, $K_t(P) = \emptyset$, and there does not exist any halfplane $HP(v_m,v_{m+1})$ in H_{t-1} such that $HP(v_m,v_{m+1}) \cap HP(v_t,v_{t+1}) = \emptyset$. Therefore, the halfplanes in H_t are pairwise intersecting. In particular, $HP(v_r,v_{r+1})$, $r=i,j,t$ are pairwise intersecting (notice that indices i,j, and t are the same as used in Equation (4.6)). Recalling that L_r is the line containing $E(v_r,v_{r+1}),L_r,r=i,j,t$ are pairwise intersecting, and lines $L_r,r=i,j,t$ do not have a common intersection. Thus, three intersection points i_{ij},i_{jt}, and i_{it} of these lines are distinct and well-defined. Futhermore, they are not colinear. Obviously, $i_{ij} \notin HP(v_t,v_{t+1})$, $i_{jt} \notin HP(v_i,v_{i+1})$, and $i_{it} \notin HP(v_j,v_{j+1})$. Otherwise, $HP(v_r,v_{r+1}),r=i,j,t$ would have a non-empty intersection. Since i_{it} is the intersection point of L_t and L_i, $i_{it} \in L_t$. Similarly, $i_{jt} \in L_t$. By definition, $E(v_t,v_{t+1}) \subset L_t$. Therefore, i_{it},i_{jt}, and $E(v_t,v_{t+1})$ are contained in the same line L_t.

Let D_t be a unit vector such that

$$D_t=\frac{(v_{t+1}-v_t)}{||v_{t+1}-v_t||}. \tag{4.7}$$

Since $i_{it} \in L_t, i_{it}$ partitions L_t into two halflines, i.e.,

$$\{x \mid x=i_{ti}+\varepsilon.D_t,\varepsilon\geq 0\} \text{ and} \qquad (4.8)$$
$$\{x \mid x=i_{ti}-\varepsilon.D_t,\varepsilon\geq 0\} \quad .$$

L_t and L_i share only one point i_{it}. Therefore, one of two halflines in Equations (4.8) are completely contained in $HP(v_i,v_{i+1})$ and the other shares only one point i_{it} with $HP(v_i,v_{i+1})$. Assuming that i_{pt} is well-defined, i_{pt} is said to be marked <u>white</u> if $i_{pt}+ \varepsilon_p.D_t$ is in $HP(v_p,v_{p+1})$ for all $\varepsilon_p \geq 0$. Otherwise, i_{pt} is said to be marked <u>black</u>.

Let three index sets U,S, and Q be defined as follows :

$U=$ $\{u \mid HP(v_u,v_{u+1}) \in H_{t-1}\}$

$S=$ $\{s \mid HP(v_s,v_{s+1}) \in H_{t-1}$ and i_{gt} is well-defined$\}$

$Q = \{q \mid HP(v_q, v_{q+1}) \in H_{t-1}$ and i_{gt} is not well-defined$\}$.

Clearly, $U = \{0,1,2, \cdots, t-1\} = S \cup Q$ and $S \cap Q = \emptyset$. Since every point i_{st} for $s \in S$ is marked white or black, S can be partitioned into two subsets, W and B as follows :

$W = \{w \mid w \in S$ and i_{wt} is marked white $\}$

$B = \{b \mid b \in S$ and i_{bt} is marked black $\}$

$S = W \cup B$ and $W \cap B = \emptyset$. Consider two points, i_{it} and i_{jt}. Since i_{it} and i_{jt} are well-defined, both i and j are in S. Suppose that both i and j are in the same subset, say W. Then

$$\{i_{it} + \varepsilon_i \bullet D_t, \varepsilon_i \geq 0\} \subset HP(v_i, v_{i+1}) \tag{4.9}$$
$$\{i_{jt} + \varepsilon_j \bullet D_t, \varepsilon_j \geq 0\} \subset HP(v_j, v_{j+1}).$$

Therefore, either $i_{it} \in HP(v_j, v_{j+1})$ or $i_{jt} \in HP(v_i, v_{i+1})$, whichever implies that $HP(v_i, v_{i+1}) \cap HP(v_j, v_{j+1}) \cap HP(v_t, v_{t+1}) \neq \emptyset$. This contradicts Equation (4.6). Hence, i and j cannot be contained in the same subset, W or B.

Take any point t_* in L_t. Then, every point in L_t can be expressed as $t_* + \varepsilon . D_t$ for some $\varepsilon \in R^1$. Let $i_{dt} = t_* + \varepsilon_d \bullet D_t$ and $i_{ft} = t_* + \varepsilon_f \bullet D_t$ for some ε_d and ε_f in R^1. i_{dt} is said to be to the <u>right</u> of i_{ft} if $\varepsilon_d > \varepsilon_f$. If $\varepsilon_d < \varepsilon_f$, then i_{dt} is said to be the <u>left</u> of i_{ft}. Otherwise, i_{dt} is said to be <u>on</u> i_{ft}. This makes sense if the direction given by D_t is pointing to the right along L_t. Let

$$\varepsilon_g = \max_{w \in W} |\varepsilon_w| \tag{4.10}$$
$$\varepsilon_h = \max_{b \in B} |\varepsilon_b|$$

Then, i_{gt} is <u>a</u> rightmost point among all i_{wt} for $w \in W$, and i_{ht} is <u>a</u> leftmost point among all i_{bt} for $w \in B$. (The article "a" is used rather than "the" since i_{gt} as well as i_{ht} may be on another point in the same subset). The following lemma shows that $HP(v_g, v_{g+1}) \cap HP(v_h, v_{h+1}) \cap HP(v_t, v_{t+1}) = \emptyset$.

<u>Lemma 4.1</u>: In Case II, $HP(v_g, v_{g+1}) \cap HP(v_h, v_{h+1}) \cap HP(v_t, v_{t+1}) = \emptyset$.

[Proof] Since the halfplanes in H_t are pairwise intersecting,

$$HP(v_g, v_{g+1}) \cap HP(v_h, v_{h+1}) \neq \emptyset. \quad (4.11)$$

By definition of i_{gt}, i_{gt} is on or to the right if i_{wt} for all $w \in W$. Since i_{wt} for $w \in W$ is marked white,

$$i_{gt} \in HP(v_w, v_{w+1}) \text{ for all } w \in W. \quad (4.12)$$

Similarly,

$$i_{ht} \in HP(v_b, v_{b+1}) \text{ for all } b \in W. \quad (4.13)$$

Suppose that i_{gt} is not to the right of i_{ht}. Then, i_{gt} is also contained in $HP(v_b, v_{b+1})$ for all $b \in B$, i.e.,

$$i_{gt} \in \bigcap_{s \in S} HP(v_s, v_{s+1}). \quad (4.14)$$

If $Q = \emptyset$, then $U = S \cup Q = S$. Therefore,

$$i_{gt} \in \bigcap_{s \in S} HP(v_s, v_{s+1}) = \bigcap_{u \in U} HP(v_u, v_{u+1}).$$

This implies that $K_t(P) \neq \emptyset$, which is a contradiction. Thus, Q cannot be empty. Let

$$I_0 = \bigcap_{s \in S} HP(v_s, v_{s+1}) \quad (4.15)$$

$$I_1 = \bigcap_{q \in Q} HP(v_q, v_{q+1})$$

Clearly,

$$K_{t-1}(P) = I_0 \cap I_1 \neq \emptyset. \quad (4.16)$$

From Equations (4.14) and (4.15),

$$i_{gt} \in I_0. \quad (4.17)$$

Since $i_{gt} \in L_t \subset HP(v_t, v_{t+1})$

$$i_{gt} \in I_0 \cap HP(v_t, v_{t+1}). \quad (4.18)$$

Now, consider I_1. Since i_{qt} is not well-defined for all $q \in Q$, L_q and L_t are either parallel or colinear. Suppose that

$$I_1 \cap HP(v_t, v_{t+1}) = \emptyset.$$

Since L_q and L_t are either parallel or colinear for all $q \in Q$,

$$HP(v_q, v_{q+1}) \cap HP(v_t, v_{t+1}) = \emptyset \text{ for } \textit{some } q \in Q.$$

This contradicts that $HP(v_m, v_{m+1}) HP(v_t, v_{t+1}) \neq \emptyset$ for all $0 \leq m < t$.

Thus,

$$I_1 \cap HP(v_t, v_{t+1}) \neq \emptyset, \quad (4.19)$$

which implies that either

$$I_1 \subseteq HP(v_t, v_{t+1}) \textit{ or } L_t \subseteq I_1. \quad (4.20)$$

Suppose that $I_1 \subseteq HP(v_t, v_{t+1})$. Since $K_{t-1}(P) = I_0 \cap I_1$,

$$K_{t-1}(P) \subseteq I_1 \subseteq HP(v_t, v_{t+1}).$$

Therefore, $K_t(P) = K_{t-1}(P) \cap HP(v_t, v_{t+1}) \neq \emptyset$, which is a contradiction. Now, suppose that $L_t \subseteq I_1$. Since $i_{gt} \in L_t$,

$$i_{gt} \in I_1. \quad (4.21)$$

From Equations (4.18) and (4.21),

$$i_{gt} \in I_0 \cap I_1 \cap HP(v_t, v_{t+1}) = K_t(P).$$

This means that $K_t(P) \neq \emptyset$, which is also a contradiction. Hence, i_{gt} is to the right of i_{ht}.

By definition, i_{gt} and i_{ht} are marked white and black, respectively. This together with the fact that i_{gt} is to the right of i_{ht} guarantee that

$$HP(v_g, v_{g+1}) \cap HP(v_h, v_{h+1}) \cap L_t = \emptyset. \quad (4.22)$$

Therefore, either $[HP(v_g, v_{g+1}) \cap HP(v_h, v_{h+1})] \subset HP(v_t, v_{t+1})$, or $[HP(v_g, v_{g+1}) \cap HP(v_h, v_{h+1})] \cap HP(v_t, v_{t+1}) = \emptyset$. Clearly,

$$K_{t-1}(P) \subset [HP(v_g,v_{g+1}) \cap HP(v_h,v_{h+1})].$$

If $[HP(v_g,v_{g+1}) \cap HP(v_h,v_{h+1})] \subset HP(v_t,v_{t+1})$, then $K_{t-1}(P) \subset HP(v_t,v_{t+1})$, which would mean that $K_t(P) \neq \emptyset$. Hence,

$$HP(v_g,v_{g+1}) \cap HP(v_h,v_{h+1}) \cap HP(v_t,v_{t+1}) = \emptyset. \quad \square$$

Given t with $K_{t-1}(P) \neq \emptyset$ and $K_t(P) = \emptyset$, **Lemma** 4.2 suggests an algorithm for finding two edges, $E(v_i,v_{i+1})$ and $E(v_j,v_{j+1})$ such that $HP(v_i,v_{i+1}) \cap HP(v_j,v_{j+1}) \cap HP(v_{t,}v_{t+1}) = \emptyset$.

Algorithm 4.1 : (Finding $E(v_i,v_{i+1})$ and $E(v_j,v_{j+1})$)

```
            Procedure   FIND-IJ(t,i,j)
              begin
Step 0          GO ← true
                i ← 0
                while (GO and i< t) do
                  begin
                    if HP(v_i,v_{I+1}) ∩ HP(v_t,v_{t+1})=∅
                      then GO ← false
                      else i ← i+ 1
                  end
Step 1          if GO =  false then
                  begin
                    j ← any positive integer x such that 0≤ x< t and x≠ i.
                  end
Step 2          else begin
                  m ← 0, W ← ∅, B ← ∅
Step 2a           while ( m <  t ) do
                    begin
                      if the line L_m containing E(v_m,v_{m+1})
                        is neither parallel nor colinear with L_t
                        containing E(v_t,v_{t+1}) then begin
                          compute and mark i_mt
                          if i_mt is white
                            then W ← W ∪ {m}
                            else B ← B ∪ {m}
                        end
                      m ← m+ 1
                    end
Step 2b           i_gt← the rightmost point among all i_wt for w ∈ W.
                  i_ht← the leftmost point among all i_bt for b ∈ B.
                  i←g, i←h
                end
              end
```

Lemma 4.3 : Given t with $K_{t-1}(P) \neq \emptyset$ and $K_t(P)=\emptyset$, Algorithm 4.1 can find, in $O(n)$ time, two edges, $E(v_i,v_{i+1})$ and $E(v_j,v_{j+1})$ such that $HP(v_i,v_{i+1}) \cap HP(v_j,v_{j+1}) \cap HP(v_t,v_{t+1}) = \emptyset$.

[Proof] In Step 0, the algorithm checks if there exists any halfplane $HP(v_i,v_{i+1})$ such that $HP(v_i,v_{i+1}) \cap HP(v_t,v_{t+1})=\emptyset$. If there exists such a halfplane $HP(v_i,v_{i+1})$, the algorithm sets the logical variable GO false. Otherwise, it sets GO true. Obviously, if GO = false, then Case I must occur. Therefore, an edge $E(v_j,v_{j+1}) 0 \leq j \neq i < t$ is chosen by Step 1. Since $HP(v_i,v_{i+1}) \cap HP(v_t,v_{t+1}) = \emptyset$, $HP(v_i,v_{i+1}) \cap HP(v_j,v_{j+1}) \cap HP(v_t,v_{t+1})=\emptyset$ for any j such that $0 \leq j \neq i < t$. If GO = true, Case II must occur. Therefore, Step 2 chooses $E(v_i,v_{i+1})$ and $E(v_j,v_{j+1})$ such that $HP(v_i,v_{i+1}) \cap HP(v_j,v_{j+1}) \cap HP(v_t,v_{t+1})=\emptyset$, which is guaranteed by **Lemma** 4.2. Hence, the correstness of the algorithm is immediate.

The algorithm traverses $C_h(v_0,v_t)$ at most twice; once in Step 0 and once in Step 2a. Therefore, each edge in $C_h(v_0,v_t)$ is visited at most twice. At each edge, a constant number of operations is needed. Thus, Step 0 and Step 2a can be done in $O(t)$ time. Clearly, Step 1 and Step 2b can be done in $O(1)$ and $O(t)$, respectively. Since $t \leq n$, the time complexity of Algorithm 4.1 is $O(n)$. □

If $K(P)=\emptyset$, Algorithm 4.1 together with the kernel finding algorithm[4] can choose, in $O(n)$ time, three edges, $E(v_i,v_{i+1})$, $E(v_j,v_{j+1})$ and $E(v_k,v_{k+1})$ such that $HP(v_i,v_{i+1}) \cap HP(v_j,v_{j+1}) \cap HP(v_k,v_{k+1})=\emptyset$. This is true since $HP(v_t,v_{t+1})$ can be equated to $HP(v_k,v_{k+1})$.

From **Theorem** 3.1, there are at most three candidates in a simple polygon with an empty kernel. By **Definition** 3.1, each of them intersects $V(m_i,P)$, $V(m_j,P)$, and $V(m_k,P)$ such that $HP(v_i,v_{i+1}) \cap HP(v_j,v_{j+1}) \cap HP(v_k,v_{k+1})=\emptyset$, where $m_r=i,j,k$ is a point in $\bar{E}(v_r,v_{r+1})$. A point is said to be marked red if $z \in V(m_i,P)$. If $z \in V(m_j,P)$ then z is said to be marked green. Z is said to be marked yellow if

$z \in V(m_k,P)$. Since $HP(v_i,v_{i+1}) \cap HP(v_j,v_{j+1}) \cap HP(v_k,v_{k+1})=\emptyset$, $V(m_i,P) \cap V(m_k,P) \cap V(m_k,P)=\emptyset$. Therefore, if $K(P)=\emptyset$, a point z in P can not be marked with these three colors, simutaneously. An edge of P is said to be marked "C", if a point in the edge is marked "C", where "C" is red, green, or yellow. An edge is said to be full-colored if it is marked with three colors simultaneously. It is clear that a candidate must be full-colored. Therefore, finding all candidates is equivalent to finding all full-colored edges.

Without loss of generality, let m_r be the mid-point of $E(v_r,v_{r+1})$, i.e.,

$$m_r=\frac{1}{2}\bullet(v_{r+1}+v_r),r=i,j,k. \qquad (4.23)$$

Our strategy for determining all candidates is first marking the edges of P using the three colors assigned to $V(m_r,P),r=i,j,k$ and then choosing all full-colored edges. Therefore, an efficient algorithm for constructing the visibility polygon $V(m_r,P)$ from m_r is needed. $V(m_r,P),r=i,j,k$ can be constructed in $O(n)$ time[2,3,5].

Lemma 4.4 : $V(m_r,P),r=i,j,k$ can be constructed in $O(n)$ time[2,3,5].

In order to efficiently mark the edges of P with three colors, the structure $V(m_r,P),r=i,j,k$ is exploited. Let $B(V(m_r,P)) = (L(s_0,s_1), L(s_1,s_2), \cdots, L(s_{c-2},s_{c-1}), L(s_{c-1},s_0))$, where $B(V(m_r,P))$ is the boundary of $V(m_r,P),r=i,j,k$. $(s_0,s_1,\cdots,s_{c-1})$ is the sequence of all corner points on $B(V(m_r,P))$ such that s_a appears before s_b for all $b>a$ as $B(P)$ is scanned from s_0 in the counter-clockwise order. Therefore, $(s_0,s_1,\cdots,s_{c-1})$ partitions $B(P)$, i.e.,

$$B(P)=\bigcup_{a=0}^{c-1} C_h(s_a,s_{a+1}). \qquad (4.24)$$

As illustrated in Figure 4.1, either $C_h(s_a,s_{a+1})=L(s_a,s_{a+1})$ or $\bar{C}_h(s_a,s_{a+1}) \cap L(s_a,s_{a+1})=\emptyset$, where $C_h(s_a,s_{a+1})$ is the chain $\bar{C}_h(s_a,s_{a+1})$ excluding its two endpoints, s_a and s_{a+1}. If $C_h(s_a,s_{a+1})=L(s_a,s_{a+1})$, then $L(s_a,s_{a+1}) \subseteq E(v_b,v_{b+1})$ for some $0\le b< n$. Otherwise, $\bar{C}_h(s_a,s_{a+1}) \cap V(m_r,P)=\emptyset$ [2,3,5]. Since $L(s_a,s_{a+1}) \subset V(m_r,P)$, an edge of P can be marked with the color assigned to $V(m_r,P)$ if and only if it contains s_a for some $0\le a< c$. Without loss of generality, let $s_0=V_{r+1}$. The following algorithm is for marking the edges of P with the color assigned to $V(m_r,P), r=i,j,k$.

< Insert Figure 4.1>

Algorithm 4.2 : (Coloring the edges of P)

```
                procedure EDGE-CLR(r,C)
                  begin
Step 0               m_r = (1/2)• (v_r + v_{r+1})
                     compute V(m_r,P)
                     b ← r, q ← 0
Step 1               while (q< c) do
                        begin
Step 1a                    while (S_q ∉ E(v_b,v_{b+1})) do
                              begin
                                 b ← b+ 1
                              end
Step 1b                    mark E(v_b,v_{b+1}) with C
                           if S_q ∈ E(v_{b+1},v_{b+2}), then
                              mark E(v_{b+1},v_{b+2}) with C
                           q ← q+ 1
                        end
                  end
```

Lemma 4.5 : Let $E_r=\{E(v_b,v_{b+1}) \mid 0\le b< n$ and $E(v_b,v_{b+1}) \cap V(m_r,P)\ne\emptyset$ for some $r=i,j,k$ }. **Algorithm** 4.2 marks, in $O(n)$ time, all the edges in E_r with the color assigned to $V(m_r,P)$.

[Proof] Let $B(V(m_r,P))=(L(s_0,s_1),L(s_1,s_2),\cdots,L(s_{c-2},s_{c-1}),L(s_{c-1},s_0))$. $(s_0,s_1,\cdots,s_{c-1})$ is in the order appearing on $B(P)$ as $B(P)$ is scanned from s_0 in the countèr-clockwise sense. Therefore, $(s_0,s_1,\cdots,s_{c-1})$ partitions $B(P)$ into c chains, $C_h(s_0,s_1)$, $C_h(s_1,s_2)$, $\cdots$, $C_h(s_{c-2},s_{c-1})$, and $C_h(s_{c-1},s_0)$. **Algorithm** 4.2 traverses $B(P)$ from s_0(or equivalently v_{r+1}) in the counter-clockwise order. Step 0 is mainly for computing $V(m_r,P)$. We inductively show that Step 1 correctly

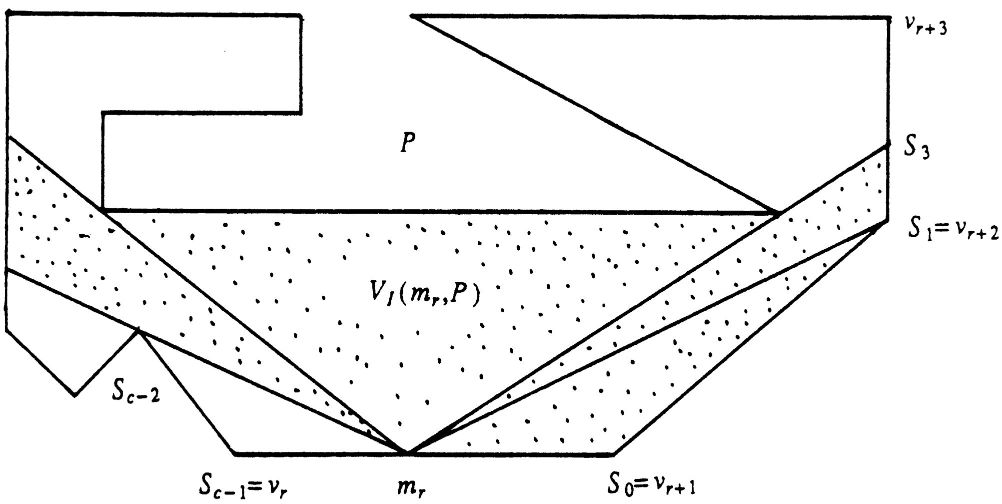


Figure 4.1 : A shape of $V_I(m_r,P)$

marks the edges in E_r with the given color.

$g=0$: Initially, Step 0 sets $q=0$ and $b=r$ since $s_0=v_{r+1}$. Therefore, both $E(v_r,v_{r+1})$ and $E(v_{r+1},v_{r+2})$ are marked C by Step 1. Since s_0 is chosen to be a vertex v_{r+1}, there does not exist any other edge $E(v_a,v_{a+1})$ such that $s_0 \in E(v_a,v_{a+1})$. Thus, Step 1 works correctly when $a=0$.

Assuming that Step 1 marks the edges correctly for $s_f, 0 \le f < c$, consider $q=f+1$. From the previous excution of Step 1, $s_f \in E(v_b,v_{b+1})$. There are two possibilities :

(1) $s_{f+1} \in E(v_b,v_{b+1})$.

(2) $s_{f+1} \notin E(v_b,v_{b+1})$.

If $s_{f+1} \in E(v_b,v_{b+1})$, then Step 1a is skipped and Step 1b marks $E(v_b,v_{b+1})$ with the given color. Futhermore, $E(v_{b+1},v_{b+2})$ is also marked if $s_{f+1} \in E(v_{b+1},v_{b+2})$. If $s_{f+1} \notin E(v_b,v_{b+1})$, then Step 1a scans $C_h(s_f,s_{f+1})$ from s_f until an edge containing s_{f+1} is encountered. After Step 1a, the new $E(v_b,v_{b+1})$ contains s_{f+1} so that Step 1b can be applied. In either case, Step 1 marks every edge $E(v_a,v_{a+1})$ such that $s_{f+1} \in E(v_a,v_{a+1})$. This together with the induction hypothesis give the correctness of Step 1. Hence, the correctness of the algorithm follows.

$V(m_r,P)$ can be computed in $O(n)$ time by **Lemma** 4.4. Therefore, Step 0 can be done in $O(n)$ time. Step 1 also requires $O(n)$ time since $s_q, 0 \le q < c$ is contained at most two edges. Hence, the time complexity of the algorithm is $O(n)$.

□

Now, we are ready to present a linear time algorithm for finding all candidates of a simple polygon P with an empty $K(P)$.

Algorithm 4.3 : (Finding all candidates in P with an Empty $K(P)$)

```
            procedure  FIND-CD (t, CAND)
              begin
Step 0           call FIND-IJ(t,i,j),k←t
Step 1           call EDGE-CLR(i,"red")
                 call EDGE-CLR(j,"green")
```

```
                call EDGE-CLR(k,"yellow")
Step 2          f←0, CAND←∅
                while ( f< n ) do
                    begin
                        if E(v_f, v_{f+1}) is full-closed, then
                            CAND←CAND ∪ {E(v_f, v_{f+1})}
                        f←f+ 1
                    end
            end
```

Theorem 4.1 : Suppose that the kernel $K(P)$ of a simple polygon P is empty. Given t with $K_{t-1}(P) \neq \varnothing$ and $K_t(P)=\varnothing$, **Algorithm** 4.3 finds all candidates, if any, in $O(n)$ time.

[Proof] In Step 0, the algorithm chooses three edges, $E(v_i,v_{i+1})$, $E(v_j,v_{j+1})$, and $E(v_k,v_{k+1})$ such that $HP(v_i,v_{i+1}) \cap HP(v_j,v_{j+1}) \cap HP(v_k,v_{k+1}) = \varnothing$. This can be done in $O(n)$ time by **Lemma** 4.3. Step 1 marks the edges of P with the three colors assigned to $V(m_r,P), r=i,j,k$, which can also be done in $O(n)$ time by **Lemma** 4.5. Now, Step 2 scans $B(P)$ once and picks up all full-colored edges. Clearly, Step 2 takes $O(n)$ time. Since an edge is a candidate if and only if it is full-colored, the result follows. □

5. Determining All Visible Edge

A simple polygon P may or may not have a non-empty kernel $K(P)$. If $K(P)$ is not empty, we employ Shin and Woo's algorithm [8] to find all visible edges.

Lemma 5.1 : Suppose that a simple polygon P has a non-empty kernel $K(P)$. Then, all visible edges of P can be determined in $O(n)$ time [8].

Suppose that $K(P)=\varnothing$. From **Theorem** 3.1, there exist at most three candidates in P. By **Theorem** 4.1, all candidates, if any, can be determined in $O(n)$ time. A visible edge is always a candidate. However, a candidate is not necessarily a visible edge. This is true since **Definition** 3.1 does not guarantee that every point in P is visible from a candidate. In order to check if a candidate is indeed a visible edge, we use Avis and Toussaint's result[1].

Lemma 5.2 : Given an edge $E(v_a,v_{a+1})$ of a simple polygon P, it takes $O(n)$ time for determining whether or not $E(v_a,v_{a+1})$ is a visible edge.

Since there are at most three candidates if $K(P)=\emptyset$, given all candidates, all visible edges can be found in $O(n)$ time using **Lemma** 5.2.

Now, a linear time algorithm for finding all visible edges in a simple polygon P is in order.

Algorithm 5.1 : (Finding all Visible Edges in P)

```
            Procedure V-EDGE
               begin
Step 0            find K(P)
Step 1            if K(P) ≠ ∅, then determine all visible edges using
                    Shin and Woo's algorithm[8]
Step 2            else begin
                       compute t such that K_{t-1}(P)≠ ∅ and K_t(P)=∅
                       call FIND-CD(t, CAND)
                       choose all visible edges in CAND using
                         Avis and Toussaint's algorithm [1]
                  end
               end
```

Theorem 5.1 : **Algorithm** 5.1 can determine all visible edges in a simple polygon in $O(n)$ time.

[Proof] Step 0 is for finding $K(P)$ which can be done in $O(n)$ time [4]. Step 1 determies all visible edges if $K(P)\neq\emptyset$. Otherwise, Step 2 finds all visible edges. Step 1 needs $O(n)$ time by **Lemma** 5.1. From **Theorem** 3.1, 4.1, and **Lemma** 5.2, Step 2 can also be done in $O(n)$ time. Hence, the result follows : □

6. Concluding Remark

A trivial lower bound for determining all visible edges is $\Omega(n)$. Therefore, **Algorithm** 5.1 is optimal within a mutiplicative constant factor. As a direct consequent of **Theorem** 3.1, there are at most three visible edges in a simple polygon P if $K(P)=\emptyset$. **Theorem** 3.1 may be extended for a simple polygon with disjoint holes.

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