METRIC CURRENTS AND DIFFERENTIABLE STRUCTURES

by

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To Juha Heinonen

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CHAPTER I

Introduction

In [1], Ambrosio and Kirchheim introduced a definition of currents in metric spaces, extending the theory of normal and integral currents developed by Federer and Fleming [11] for Euclidean spaces. The extension of these classes of currents allows the formulation of variational problems in metric spaces, and the validity of the compactness and closure theorems of [11], proven in the metric setting in [1], allows for their solution.

This thesis is primarily an investigation into the theory of metric currents in spaces that admit differentiable structures in the sense of Cheeger [6] and Keith [27]; that is, spaces in which a generalization of Rademacher's differentiation theorem holds. Our main results for this generality (Theorems 1.6 and 1.8) describe a close relationship between the Cheeger differentials $d^\pi f$ and the "metric forms" df in the theory of metric currents (see Definition 2.27 and equation (2.7), respectively).

We perform a more extensive analysis of metric currents in Carnot groups, equipped with the usual Carnot-Carathéodory metrics, characterizing currents of absolutely continuous mass in these spaces (Theorem 1.9).

We conclude with a somewhat tangential discussion of quasiconformal mappings, proving (Theorem 1.12) the equivalence between (one-sided) geometric and analytic definitions of quasiconformal mappings in metric spaces, avoiding the usual assumption of the Loewner condition. While this result does not in and of itself relate to metric currents, the key technical tool used for the proof (Theorem 4.29) is motivated by a similar and simpler fact about currents, as we discuss in Remark 4.32.

1.1 Background

Metric currents

The classical theory of currents goes back to de Rham [8]. A current, in the sense of de Rham, is a member of the dual space to the space of smooth differential forms, in analogy with distributions being dual to smooth functions (in fact, distributions are 0-dimensional currents). A prototypical example of a k-dimensional current in $\mathbb{R}^n$ is the map $\omega \mapsto \int_M \omega$, where $M \subseteq \mathbb{R}^n$ is an embedded Riemannian submanifold of dimension k. With this example in mind, one defines a boundary operator via Stokes' theorem, in a similar manner to how one differentiates distributions using integration by parts. Likewise, the push-forward of a current along a map is defined through duality by pulling back forms.

Federer and Fleming studied various classes of currents with finite and locally finite mass [11]. Continuing with the analogy between distributions and currents, one should think of a current of finite mass as being analogous to a measure, and in fact, this can be made precise if one is willing to consider vector valued measures. The authors of [11] introduced the classes of normal currents (currents with finite mass whose boundaries also have finite mass) and integral currents (normal currents represented by integration along a rectifiable set). They then proved a number of compactness and closure theorems, providing new tools for the formulation and solution of area minimization problems in $\mathbb{R}^n$, including the well-known Plateau problem.

Ambrosio and Kirchheim [1] extended the Federer-Fleming theory to general metric spaces by replacing the space of smooth forms with a space $\mathcal{D}^k(X)$ of Lipschitz k-tuples $(f, g^1, \ldots, g^k)$, written suggestively as $f\, dg^1 \wedge \cdots \wedge dg^k$. They then define a current to be real-valued function on $\mathcal{D}^k(X)$ that is linear in each argument, continuous in an appropriate sense, vanishes where it ought to (namely, on forms $f dg^1 \wedge \cdots \wedge dg^k$ where one of the functions g^i is constant on the support of f), and satisfies a finite mass condition. Having defined metric currents, the authors continue on to show that most of the results of [11] carry over to this more general setting, and that moreover, the classes of classical and metric normal currents are naturally isomorphic in the Euclidean case. In another part of the paper, they show that rectifiable currents can be classified using the metric and weak* differentiation theorems from the paper [2], mentioned below.

Lang [32] has introduced a variation of the Ambrosio-Kirchheim theory tailored specifically to locally compact spaces. In this setting, the finite mass axiom is eliminated. In spite of this, a number of results from [1], including the Leibniz rule and a chain rule, remain true, though the powerful closure and compactness theorems still require assumptions on the masses of currents and their boundaries [32], as is the case for the corresponding results in [1] and [11].

Franchi, Serapioni, and Serra Cassano [12] have recently developed an extension of the Federer-Fleming theory to Heisenberg groups, prototypical examples of the Carnot groups described below. The results in Section 1.2 will address the relationship between the currents of [12] and those of [1].

Differentiable structures

To formulate the most general of our results below, we will need the notion of a differentiable structure, defined by Keith [27], and motivated by Cheeger's [6] differentiation theorem, as well as the generalization of the theorem in [27].

A (strong measured) differentiable structure on a metric measure space X is a measurable covering of X by coordinate charts (Y, π). Here $\pi\colon Y \to \mathbb{E}$ is a Lipschitz map into a Euclidean space $\mathbb{E}$. The defining property of a differentiable structure is the existence, for any Lipschitz function f, of a measurable map $y \mapsto d^\pi f_y \in \mathbb{E}^*$, satisfying

$$f(x) = f(y) + \langle d^\pi f_y, \pi(x) - \pi(y)\rangle + \mathrm{o}(\mathrm{dist}(x, y)) \tag{1.1}$$

at almost every $y \in Y$.

The differentiation theorems of [6] and [27] state, that a nice enough metric measure space X (e.g., in the case of [6], a space with a doubling measure satisfying a generalization of the Poincaré inequality, as defined in [23]) has a covering of measurable coordinate patches X_α, possibly of different dimensions, on each of which one can differentiate Lipschitz functions. This differentiation determines a "measurable cotangent bundle" on the space. In the Euclidean case (or on a Riemannian manifold), it is easy to check that the usual cotangent bundle coincides with Cheeger's [6].

The theorems of [6] and [27] generalize a number of earlier results. The classical version of Rademacher's theorem states that a Lipschitz map between Euclidean spaces is differentiable almost everywhere. Pansu [35] generalized the theorem to maps between Carnot groups, stratified Lie groups equipped with the so-called Carnot-Carathéodory metric. Ambrosio and Kirchheim [2] proved analogs of Rademacher's theorem for maps from Euclidean spaces into general metric spaces, using Banach spaces as an intermediary tool. Cheeger and Kleiner [7] have recently extended the original differentiation theorem from [6] to Banach space-valued maps.

Carnot groups

A Carnot group is a simply connected nilpotent Lie group $\mathbb{G}$ whose Lie algebra $\mathfrak{g}$ admits a stratification

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_n$$

such that $[V_i, V_j] = V_{i+j}$ with the convention that $V_m = 0$ for $m > n$. The subspace $H = V_1$, together with its left translates, forms what is known as the horizontal bundle. It is a result of Chow and Rashevsky (see [16], e.g., for a proof) that any two points in a Carnot group can be joined by a path whose velocity is horizontal at each point. This leads to a natural definition of a metric on Carnot groups, the so-called Carnot-Carathéodory metric, given by the shortest horizontal path between two points (with respect to some invariant Riemannian metric).

The simplest non-Riemannian examples of Carnot groups are the Heisenberg groups $\mathbb{H}^n$. The Lie algebra of $\mathbb{H}^n$ is spanned by vector fields $X_1,\dots,X_n,Y_1,\dots,Y_n$, and Z satisfying $[X_i, Y_j] = \delta^i_j Z$ and $[X_i, Z] = [Y_j, Z] = 0$, and thus admits a stratification

$$\text{Span}(X_1, \dots, X_n, Y_1, \dots, Y_n) \oplus \text{Span}(Z).$$

The geometry of $\mathbb{H}^n$ is highly non-Euclidean. One can show, for example, that its topological dimension is $2n + 1$, whereas its Hausdorff dimension is $2n + 2$ [16].

Jerison [26] proved that a Carnot group satisfies a Poincaré inequality, and so by the result of [6], it admits a differentiable structure. The differentiation theorem of Pansu [35] actually gives an explicit formulation of this structure. In fact, Cheeger's cotangent bundle is given by the dual to the horizontal sub-bundle of the classical tangent bundle (see [41] and [6]).

Definitions of quasiconformality.

One important tool in geometric function theory is the theory of quasiconformal mappings, a generalization of conformal mappings. One immediate motivation for such a generalization of conformality, even in the Euclidean setting, is the observation that, by a theorem of Liouville, the only conformal mappings of a domain in $\mathbb{R}^n$, for $n \geq 3$, are restrictions of Möbius transformations. A quasiconformal mapping in the classical sense is a mapping which infinitesimally sends spheres to ellipsoids with controlled dilatation (though there are important further technical assumptions, in particular, Sobolev regularity.)

We are mostly concerned here with the issue of equivalence between different definitions of quasiconformality, rather than applications. We will not be particularly interested here in the classical theory, which goes back to Ahlfors and Teichmüller in the thirties, but instead refer the interested reader to [21]. For a detailed introduction to the classical theory see, e.g., [40]. We do note, however, that there are a number of equivalent definitions for quasiconformal mappings, which we discuss in brief. The analytic definition, discussed above, states that a homeomorphism $F\colon U \to V$ between domains in $\mathbb{R}^n$ is K-quasiconformal if it is in the Sobolev class $W^{1,n}(U, V)$ and satisfied the condition

$$|dF_x|^n \leq K J_F(x) \tag{1.2}$$

almost everywhere. Note here that the Jacobian $J_F(x)$ is equal, almost everywhere, to the Radon-Nikodym derivative $\frac{dF_{\#}^{-1}\nu}{d\mu}$, where μ and ν are the Lebesgue n-measures restricted to U and V, respectively. The metric definition, easily adapted to the general setting of a metric space, says that F is H-quasiconformal if $H(x) \leq H$ for every $x \in U$, where $H(x)$ is defined by

$$H(x) = \limsup_{r \to 0} \frac{\sup_{|x-y| \leq r} |F(x) - F(y)|}{\inf_{|x-y| \geq r} |F(x) - F(y)|}. \tag{1.3}$$

The geometric definition of quasiconformality states that the conformal modulus of a curve family Γ is quasi-preserved, i.e.,

$$K^{-1} \operatorname{Mod}_n(\Gamma) \leq \operatorname{Mod}_n(F(\Gamma)) \leq K \operatorname{Mod}_n(\Gamma). \tag{1.4}$$

Finally, F is said to be η-quasisymmetric, where $\eta \colon [0, \infty) \to [0, \infty)$ is a homeomorphism, if

$$\frac{|F(x) - F(y)|}{|F(x) - F(z)|} \leq \eta \left(\frac{|x - y|}{|x - z|} \right) \tag{1.5}$$

for all distinct triples $(x, y, z) \in U^3$.

The quantitative equivalence of the three definitions of quasiconformality, along with a localized version of quasisymmetry, has long been established for the classical case (see, e.g., [40] or [22].) However, there has recently been much interest in extending the theory of quasiconformal mappings to non-smooth spaces. Heinonen and Koskela [22] proved the equivalence of the infinitesimal "metric definition" with the stronger quasisymmetry condition in the setting of Carnot groups, and later began a systematic study of quasiconformal and quasisymmetric mappings in spaces admitting a generalization of the Poincaré inequality [23].

To compare the various definitions of quasiconformality in the setting of a general metric space, one first needs an appropriate analog for the classical Sobolev spaces $W^{1,p}(U, V)$. In the last decade, Shanmugalingam's "Newton-Sobolev Spaces" $N^{1,p}(X, Y)$ (defined in [38] for $Y = \mathbb{R}$, and extended to the general case in [24]) have emerged to fill this role. In the latter paper [24], the authors proved that, in the setting of a doubling metric space with "sufficiently many rectifiable curves" (i.e., one admitting a local version of the Poincaré inequality, or the equivalent Loewner condition, both introduced in [23]), the four definitions remain equivalent.

If there is no hypothesis on the existence of rectifiable curves, the equivalence of the definitions breaks down. In the extreme case, where there are no rectifiable curves at all, both the analytic and geometric definitions become trivial (the modulus of a family of unrectifiable curves vanishes by definition, and the Sobolev spaces $N^{1,p}(X,Y)$ reduce to $L^p(X,Y)$ [38]). However, the definitions are still related. Tyson [39] showed that quasisymmetry implies geometric quasiconformality. Heinonen, Koskela, Shanmugalingam and Tyson ([24], Theorem 8.8) proved that quasisymmetric mappings are in $N^{1,p}_{\mathrm{loc}}(X,Y)$, and Balogh, Koskela, and Rogovin [4] extended this result to metrically quasiconformal maps. The preceding results assume nothing about rectifiability, though they do all assume a version of Q-dimensionality known as Q-regularity, which says that the Q-dimensional Hausdorff measure of a ball of radius R is comparable to R^Q.

As we will prove in Theorem 1.12, the analytic and (one-sided) geometric definitions, though weaker than metric quasiconformality and quasisymmetry, remain equivalent to each other, even under very mild hypotheses.

1.2 Results.

We organize our results into three main areas, each of which comprises a chapter in the thesis.

Metric currents and differentiation.

The first results, discussed in Chapter II, concern the compatibility of the theory of metric currents with the differentiable structures of [27]. The most fundamental of these states that metric forms (see Section 2.1) that are equivalent in the sense of differentiable structures are also equivalent in the sense of currents, provided the current is concentrated where the forms are defined.

Theorem 1.6. *Let $X = (X, d, \mu)$ be a metric measure space admitting a strong measured differentiable structure, let (Y, π) be chart, $\pi\colon Y \to \mathbb{E}$,and let*

$$\omega = \sum_{s \in S} \beta_s \, dg_s^1 \wedge \cdots \wedge dg_s^k \in \tilde{\mathcal{E}}_c^k(X),$$

where S is finite. Denote by $Y_\omega \subset Y$ the set of points $y \in Y$ such that all of the functions g_s^i, for $i = 1, \dots, k$, $s \in S$ are differentiable at y, and such that

$$\sum_s \beta(y) \, d_y^\pi g_s^1 \wedge \cdots \wedge d_y^\pi g_s^k = 0.$$

Then for every $T \in \mathbf{M}_k(X)$ concentrated on Y_ω,

$$T(\omega) = 0. \tag{1.7}$$

See Sections 2.1 and 2.2 for the precise definitions.

From Theorem 1.6, we are able to derive a number of results, including Theorem 2.40, which generalizes the chain rule in [1] and [32] (Theorem 2.25 below) to mappings into arbitrary spaces admitting differentiable structures.

We also prove that certain currents are given by integration against a vector measure. This is already known in the classical case (see, e.g., [10] 4.1.5), and in the metric theory for $\mathbb{R}^n$ [1], though we are forced in our generality to restrict our attention to currents with absolutely continuous mass relative to the underlying measure μ of the metric measure space X.

We define a *k-precurrent* T to be a functional on the space of metric forms given by integration against a measurable "k-vector field" $\hat{\lambda}\colon Y \to \bigwedge^k \mathbb{E}$.

$$T(\beta \, dg^1 \wedge \cdots \wedge dg^m) = \int_Y \langle \hat{\lambda}, \beta \, d^\pi g^1 \wedge \cdots \wedge d^\pi g^k \rangle \, d\mu,$$

See Definition 2.42 for the precise definition.

Theorem 1.8. *Let $X = (X, \mathrm{dist}, \mu)$ be a metric measure space admitting a differentiable structure. Then every metric k-current T in X, with $||T|| \ll \mu$, is a k-precurrent.*

The converse of Theorem 1.8 is true in Euclidean space [1], but, as we shall see in the next result, precurrents need not be currents in the general case.

Currents in Carnot groups.

The non-commutativity of Carnot groups will prevent certain precurrents from satisfying the continuity axiom for currents. Our main result for these spaces is the following theorem, which characterizes exactly which precurrents are currents in a given Carnot group.

Theorem 1.9. *Let $\mathbb{G} = (\mathbb{G}, d_{CC}, \mu)$ be a Carnot group, equipped with its Carnot-Carathéodory metric d_{CC} and Haar measure μ, and let $k \geq 2$. Then a k-precurrent T is a current if and only if*

$$T\lfloor_{d\theta} = 0 \tag{1.10}$$

for every vertical 1-form $\theta \in \Omega^1(\mathbb{G})$. Moreover, every current in $T \in \mathbf{M}_k^{loc}(\mathbb{G})$ satisfies equation (1.10).

Here by "vertical 1-form" we mean one that vanishes on the horizontal bundle. For example, the contact form in a Heisenberg group is a vertical form. We will see in Chapter III that as a consequence of Theorem 1.8,for every vertical 1-form θ, we have $T\lfloor_{\theta} = 0$. We will define vertical forms and restrictions to smooth forms precisely in Section 3.3 (the general restriction operator for metric forms is defined earlier in Definition 2.19).

Theorem 1.9 has a number of implications when applied to Heisenberg groups. As observed by Rumin ([37], Section 2), the space $\Omega^k(\mathbb{H}^n)$ of smooth k-forms, for

$k \geq n$, consists entirely of forms $\alpha \wedge \theta + \beta \wedge d\theta$, where θ is the contact form. It follows that there are no nonzero k-currents in $\mathbb{H}^n$, for $k > n$. This generalizes a result of Ambrosio and Kirchheim [2], who showed that nonzero *rectifiable* currents do not exist for $k > n$. For $k \leq n$, the condition of vanishing on the contact form mirrors the defining property of another theory of currents, tailored specifically to Heisenberg groups, developed recently by Franchi, Serapioni, and Serra Cassano [12]. This suggests, perhaps, that metric currents, rectifiable or not, might best be thought of as fundamentally low-dimensional objects.

We discuss the issue of rectifiability in more detail in Section 3.6, where we explore the relationship between our results and the rectifiability theorem of [1], as well as a more general result due to Magnani [34].

Definitions of Quasiconformality

Our last major result is somewhat estranged from our main topic, though the key technical device used (Theorem 4.29) has a close philosophical connection to the theory of currents, as we discuss further in Remark 4.32.

Our main result in this area shows that the analytic definition is equivalent to a 1-sided version of the geometric definition in significantly greater generality than that of previous results.

Define a mapping F between two metric measure spaces (X, μ) and (Y, ν) to be *geometrically lower K-quasiconformal* (with exponent Q) if F satisfies

$$K^{-1} \operatorname{Mod}_Q(\Gamma) \leq \operatorname{Mod}_Q(F(\Gamma)) \tag{1.11}$$

for every curve family Γ in X. Here the modulus (see Definition 4.4) is taken with respect to the underlying measures μ and ν, respectively.

Theorem 1.12. *Let $F\colon X \to Y$ be a homeomorphism between separable, locally finite metric measure spaces $X = (X, \mu)$ and $Y = (Y, \nu)$.*

Then F is geometrically lower K-quasiconformal (with exponent Q) if and only if $F \in N^{1,Q}_{loc}(X,Y)$ and the inequality

$$\rho^{(}_{F}x)^{Q} \leq K \frac{dF^{-1}_{\#}\nu}{d\mu}(x) \tag{1.13}$$

holds for μ-almost every $x \in X$.

Here the Borel function $\frac{dF^{-1}_{\#}\nu}{d\mu}$ denotes the Radon-Nikodym derivative of the measure $F^{-1}_{\#}\nu$ with respect to the measure μ. The function ρ_F is a minimal "p-gradient", a generalization of the gradient of a function on $\mathbb{R}^n$, introduced in [23] under the name "very weak upper gradient". We define p-gradients, as well as the class $N^{1,Q}_{\mathrm{loc}}(X,Y)$, in Section 4.1.

Note that there are no geometric assumptions on the spaces X and Y, other than the selection of a particular metric on each space. The measures need only be locally finite, rather than finite on balls, and number Q need not have anything to do with the Hausdorff dimension, or any other geometric property of the spaces. Most importantly, we assume nothing about the presence or absence of rectifiable curves in either space.

Theorem 1.12 is trivial in the case that X admits no rectifiable curves. Moreover, the theorem is already known to be true by a theorem of Heinonen, Koskela, Shanmugalingam, and Tyson ([24], Theorem 9.8) when there are many rectifiable curves in X (more precisely, when X is locally a Loewner space). Theorem 1.12 extends these results to the "intermediate" cases, in which X admits some, but not too many, rectifiable curves.

The one-sided nature of Theorem 1.12 is unavoidable in the generality with which we are concerned, or even under the assumption of Q-regularity on the two spaces. For example, if $X = \mathbb{R}^n$ with the "snowflaked" distance $\mathrm{dist}(x_1, x_2) = |x_1 - x_2|^{n/Q}$, and $Y = \mathbb{G}$ is a Carnot group with topological dimension n and homogeneous dimension Q, equipped with its Carnot-Carathéodory metric $\mathrm{dist}_{\mathrm{CC}}$, then X and Y are homeomorphic. The space X has no rectifiable curves, whereas Y, a Loewner space [23], has many curve families of positive Q-modulus. It follows that *every* homeomorphism from X to Y is geometrically lower quasiconformal, whereas *no* such homeomorphism is geometrically upper quasiconformal. This contrasts starkly with the case where both spaces satisfy the Loewner property of [23] (and are Q-regular), as in that setting, by Theorem 9.8 of [24], the one-sided analytic and geometric conditions in Theorem 1.12 imply the two-sided definitions.

We should mention that our argument for the implication (1.13) $\Rightarrow$ (1.11) is somewhat standard, though we know of no written proof for the generality here. Theorem 9.8 of [24] refers to an argument involving absolute continuity in measure of F, which is not present in our generality. Our method is more in the spirit of the discussion in Section 4 of [4], particularly Remarks 4.1 and 4.3, though here we eschew the assumption of Q-regularity and the issue of metric quasiconformality in favor of greater generality.

Theorem 1.12 offers an alternative approach to some known results in the theory of quasisymmetric mappings. For example, the implication (1.13)$\Rightarrow$(1.11), in combination with Theorem 8.8 of [24] on the Sobolev regularity of quasisymmetric embeddings, immediately implies a theorem of Tyson ([39], Theorem 1.4) on the geometric quasiconformality of such maps. Conversely, the reverse implication (1.11)$\Rightarrow$(1.13), in combination with Tyson's theorem, implies Theorem 8.8 of [24], in the special case that X is Q-regular.

More generally, [4, Theorem 1.1], combined with Remark 4.1, states that metric quasiconformality implies analytic quasiconformality (still in the setting of Q-regular metric spaces). As suggested in [4, Remark 4.3], this easily (for example, via the implication (1.13)⇒(1.11)) shows that metrically quasiconformal mappings are geometrically quasiconformal. Given this result, one might attempt to use methods along the lines of [39] to show directly that the metric definition implies the geometric definition. Such an approach, in combination with the implication (1.11)⇒(1.13) of Theorem 1.12, would then give alternate proofs, and hopefully new insight, into some of the results of [4].

1.3 Notation

Throughout this paper, $X = (X, \mathrm{dist})$ will denote a metric space.We will frequently make use of the notation $|x_1 - x_2| = \mathrm{dist}(x_{1,x}2)$ when $x_1, x_2 \in X$, and the metric dist is unambiguous. A Euclidean space is a finite dimensional vector space whose metric is given by an inner product. Typically, we will use the notation $\mathbb{E}$ to refer to a Euclidean space of unspecified dimension.

The term "function" will always denote a real valued map, and we will denote the support of a function f by $\mathrm{Spt}(f)$. This is defined to be the smallest closed set outside of which f vanishes.

We recall that the Lipschitz constant of a map $F\colon X \to Y$ is given by

$$L(F) = \sup_{x_1, x_2 \in X, x_1 \neq x_2} \frac{|F(x_1) - F(x_2)|}{|x_1 - x_2|}$$

and that F is said to be *Lipschitz* if $L(F)$ is finite.

When X is locally compact, we will write $\mathrm{Lip}_{\mathrm{c}}(X)$, $\mathrm{Lip}_{\mathrm{loc}}(X)$, and $\mathrm{Lip}_1(X)$ to denote, respectively, the spaces of Lipschitz functions with compact support, locally Lipschitz functions, and functions with Lipschitz constant at most 1.

We will be most interested in the spaces $\mathrm{Lip_c}(X)$ and $\mathrm{Lip_{loc}}(X)$. These spaces are to be equipped with notions of convergence. We will not define topologies on the spaces, since we will be interested only in convergence of sequences, and knowledge of convergent sequences is generally insufficient to describe a vector space topology, if the topology is allowed to be non-metrizable. Instead, we will simply say that a sequence of functions $f_i \in \mathrm{Lip_c}(X)$ *converges* to $f \in \mathrm{Lip_c}(X)$ if the sequence converges to f pointwise, and all of the functions f_i and f have uniformly bounded Lipschitz constant, as well as uniformly compact support. Similarly, $f_i \in \mathrm{Lip_{loc}}(X)$ converges to $f \in \mathrm{Lip_{loc}}(X)$ if it converges to f pointwise, and for any compact $K \subset X$, all of the functions f_i and f have uniformly bounded Lipschitz constants when restricted to K. Though we will not discuss them here, [32] describes locally convex vector space topologies which yield the same notion of convergent sequences.

We will say that a subset $S \subset \mathrm{Lip_c}(X)$ is *dense* if every function in $\mathrm{Lip}_c(X)$ is a limit of a sequence of functions in S. Note that if X is locally compact and separable, then each subset S_K^M, where

$S_K^M = \{f \in \mathrm{Lip_c}(X) : L(f) < M,\ |f(x)| < M$ for all $x \in X$, and f is supported on $K.\}$,*iscompact*(

M¿0*and*K$\subset X$ is compact. It follows that $\mathrm{Lip_c}(X)$ is *separable* in the sense that it has a countable dense subset.

We denote by $\mathcal{B}^\infty(X)$, $\mathcal{B}^\infty_{\mathrm{c}}(X)$ and $\mathcal{B}^\infty_{\mathrm{loc}}(X)$ the spaces of Borel functions that are, respectively, bounded, bounded with compact support, and locally bounded.

If μ is a Borel measure on a space X, and $F\colon X \to Y$ is Borel measurable, then $F_\#\mu$ will denote the pushforward of μ by F; that is, $F_\#\mu$ is the Borel measure on Y given by $F_\#\mu(A) = \mu(F^{-1}(A))$ for every Borel set $A \subseteq Y$.

If X is a metric space, and μ and ν are σ-finite Borel measures on X, then $\frac{d\nu}{d\mu}$ is the Radon-Nikodym derivative of ν with respect to μ. That is, $\frac{d\nu}{d\mu}$ is the Borel function (defined uniquely up to sets of μ-measure 0) such that $\nu_a(A) = \int_A \frac{d\nu}{d\mu}\, d\mu$, whenever $A \subset X$ is a Borel set. Here $\nu = \nu_a + \nu_s$ is the unique decomposition of ν into two Borel measures on X such that $\nu_a \ll \mu$ and ν_s is concentrated on a set of μ-measure 0.

We say that $X = (X, \mathrm{dist}, \mu)$ is a *locally finite metric measure space* if X is separable, and μ is a locally finite Borel regular measure on X, with $\mu(U) > 0$ on each open subset $U \subset X$. Here, unlike much of the literature, we do not assume that $\mu(B)$ is finite on every ball, though in many applications this is the case.

Following [24], if $X = (X, \mathrm{dist}, \mu)$ is a metric measure space, with μ locally finite (i.e., every point has a neighborhood of finite measure), and Y is an arbitrary metric space, with basepoint y_0, we say that a mapping $F\colon X \to Y$ is *locally p-integrable*, written $F \in L^p_{\mathrm{loc}}(X, Y)$, if the function $d_{y_0} \circ F$ is locally p-integrable (that is, $d_{y_0} \circ F \in L^p_{\mathrm{loc}}(X)$), where $d_{y_0}\colon Y \to \mathbb{R}$ is given by $d_{y_0}(y) = |y - y_0|$. If μ is finite, then we say $F \in L^p(X)$, or F *is p-integrable*, provided $d_{y_0} \circ F$ is in $L^p(X)$. As noted in [24], these notions are clearly independent of the choice of base point $y_0 \in Y$. Note also that by the local finiteness of μ, every continuous map $F\colon X \to Y$ is locally p-integrable.

By $\mathcal{H}^k(A)$ we denote the k-dimensional Hausdorff measure of a subset $A \subseteq X$. If V is a vector space, $\bigwedge^k V$ denotes the k^{th} exterior power of V. Finally, we denote by $\Lambda_{k,n}$ the set of k-indices of the form $(i_1, \ldots, i_k)$ satisfying $1 \le i_i < \cdots < i_k \le n$.

CHAPTER II

Currents and Differentiation

In this chapter we analyze the general compatibility between the theories of metric currents and differentiable structures. Sections 2.1 and 2.2 recall the necessary background on metric currents and differentiable structures. Our results are proved in Section 2.3.

2.1 Metric k-currents

Let X be a locally compact metric space. We recall a few definitions from [32]. We will follow [32] throughout this section, except as noted otherwise. One small addition will be our linearization of the spaces of "forms" via tensor products and exterior powers, as described below.

First we define the space $\mathcal{D}^k_{\mathrm{c}}(X)$ of compactly supported simple metric k-forms by

$$\mathcal{D}^k_{\mathrm{c}}(X) = \mathrm{Lip}_{\mathrm{c}}(X) \times \mathrm{Lip}_{\mathrm{loc}}(X)^k.$$

The motivation for calling elements of this space "simple forms" will be explained below. We say that a sequence of k-forms $\omega_i = (f_i, g^1_i, \ldots, g^k_i)$ converges to $\omega = (f, g^1, \ldots, g^k)$ if f_i converges to f and g^j_i converges to g^j for $j = 1, \ldots, k$. Here and throughout, unless otherwise stated, the convergence of a sequence of Lipschitz functions is defined as in Section 1.3.

We also define a number of other spaces, in which we will not concern ourselves with notions of convergence:

$$\tilde{\mathcal{D}}_{\mathrm{c}}^{k}(X) = \mathrm{Lip_c}(X) \otimes \bigwedge^{k} \mathrm{Lip_{loc}}(X).$$

$$\mathcal{D}^{k}(X) = \mathrm{Lip_{loc}}(X)^{k+1}.$$

$$\tilde{\mathcal{D}}^{\mathrm{k}}(X) = \mathrm{Lip_{loc}}(X) \otimes \bigwedge^{k} \mathrm{Lip_{loc}}(X).$$

$$\mathcal{E}^{k}(X) = \mathcal{B}_{\mathrm{loc}}^{\infty}(X) \times \mathrm{Lip_{loc}}(X)^{k}.$$

$$\tilde{\mathcal{E}}^{\mathrm{k}}(X) = \mathcal{B}_{\mathrm{loc}}^{\infty}(X) \otimes \bigwedge^{k} \mathrm{Lip_{loc}}(X).$$

$$\mathcal{E}_{\mathrm{c}}^{k}(X) = \mathcal{B}_{\mathrm{c}}^{\infty}(X) \times \mathrm{Lip_{loc}}(X)^{k}.$$

$$\tilde{\mathcal{E}}_{\mathrm{c}}^{k}(X) = \mathcal{B}_{\mathrm{c}}^{\infty}(X) \otimes \bigwedge^{k} \mathrm{Lip_{loc}}(X).$$

Remark 2.1. The number of different spaces of “forms” may at first appear daunting, but we do not require deep study for most of them. As stated before, we do not topologize any of these additional spaces - any time we speak of convergence of a sequence of forms, we *always* refer to a sequence of simple forms in $\mathcal{D}_{\mathrm{c}}^{k}(X)$.

Our use of tensor and exterior products here is a deviation from both [1] and [32]. The motivation for this is two-fold. Philosophically, in order to complete the analogy between “metric forms” and classical differential forms, we would like for metric forms to constitute a linear space. More practically, in our formulation and proof of Theorem 1.9, we will need to deal with metric forms that are not simple. However, it should be noted that this deviation from the theory is entirely cosmetic, due to our lack of topological considerations on any of the additional spaces. We use them only to more naturally phrase statements that would otherwise require repeated discussion of linear combinations of forms.

Definition 2.2. A **metric k-current** on X is a map $T\colon\ \mathcal{D}^k_{\mathrm{c}}(X) \to \mathbb{R}$ satisfying the following axioms:

1. Linearity: T is linear in each argument.

2. Continuity: $T(\omega_i)$ converges to $T(\omega)$ whenever ω_i converges to ω.

3. Locality: $T((f, g^1, \ldots, g^k)) = 0$ provided that for some i, g^i is constant on $\mathrm{Spt}(f)$.

The space of metric k-currents on X is denoted $\mathcal{D}_k(X)$.

We will frequently drop the adjective "metric" in the future.

Remark 2.3. We should point out that a priori the locality axiom as defined in [32] is only required to hold when g^i is constant on a *neighborhood* of $\mathrm{Spt}(f)$, but it is later proven there that this is equivalent to the above definition. Also, as a consequence of the locality axiom, we may modify any of the functions g^i away from $\mathrm{Spt}(f)$, or in turn modify f away from $\mathrm{Spt}(g^i)$, without changing the value of $T((f, g^1, \ldots, g^k))$ (to see that the second statement is true, note that f vanishes on a neighborhood of $\mathrm{Spt}(g^i)$ if and only if g^i vanishes on a neighborhood of $\mathrm{Spt}(f)$). In particular, if $(f, g^1, \ldots, g^k) \in \mathcal{D}^k(X)$, and one of the functions g^i is $\mathrm{Lip}_{\mathrm{c}}(X)$, we may unambiguously define

$$T((f, g^1, \ldots, g^k)) = T((\sigma f, g^1, \ldots, g^k)),$$

where $\sigma \in \mathrm{Lip}_{\mathrm{c}}(X)$ is any function satisfying $\sigma \equiv 1$ on some neighborhood of $\mathrm{Spt}(g^i)$.

The following theorem provides some intuition for the use of the term "form" above.

Theorem 2.4 ([1] Theorem 3.5, [32] Proposition 2.4)**.** *If $T\colon \mathcal{D}^k_c(X) \to \mathbb{R}$ is a k-current, then T satsifies the alternating property and the Leibniz rule:*

$$T((f, g^1, \dots, g^i, \dots, g^j, \dots, g^k)) = -T((f, g^1, \dots, g^j, \dots, g^i, \dots, g^k)). \tag{2.5}$$

$$T((f, g^1, \dots, g^k)) + T((g^1, f, \dots, g^k)) = T((1, fg^1, g^2, \dots, g^k)). \tag{2.6}$$

Notice that the right hand side of equation (2.6) is well-defined by Remark 2.3.

Although we are using the definition of currents from [32], in light of Theorem 2.4 we will borrow the suggestive notation

$$f\, dg^1 \wedge \cdots \wedge dg^k = (f, g^1, \dots, g^k) \tag{2.7}$$

from [1].

Moreover, if one of the functions g^i is compactly supported, we define

$$dg^1 \wedge \cdots \wedge dg^k = (1, g^1, \dots, g^k). \tag{2.8}$$

This latter notation is justified by Remark 2.3, and the locality property.

With this new notation, equations (2.5) and (2.6) can be rewritten.

$$T(f\, dg^1 \wedge \cdots \wedge dg^i \wedge \cdots \wedge dg^j \wedge \cdots \wedge dg^k) = -T(f\, dg^1 \wedge \cdots \wedge dg^j \wedge \cdots \wedge dg^i \wedge \cdots \wedge dg^k) \tag{2.9}$$

$$T(f\, dg^1 \wedge \cdots \wedge dg^k) + T(g^1\, df \wedge dg^2 \wedge \cdots \wedge dg^k) = T(d(fg^1) \wedge dg^2 \wedge \cdots \wedge dg^k). \tag{2.10}$$

Since T is linear in each variable, and satisfies the alternating property (2.9), there is a unique linear map, which we also denote by $T\colon \tilde{\mathcal{D}}^k_c(X) \to \mathbb{R}$, satisfying $T(f \otimes g^1 \wedge \cdots \wedge g^k) = T(f\, dg^1 \wedge \cdots \wedge dg^k)$. We will therefore use the notation

$$f\, dg^1 \wedge \cdots \wedge dg^k = f \otimes g^1 \wedge \cdots \wedge g^k. \tag{2.11}$$

Since any $(k+1)$-linear functional T satisfies $T(f \otimes g^1 \wedge \cdots \wedge g^k) = T((f, g^1, \ldots, g^k))$, there is no potential for ambiguity between the notations introduced with equations (2.7) and (2.11); the only situations in which we consider metric forms involve pairing the forms with currents, with the one exception being that we discuss convergence of forms in their own right. In this latter context, we only deal with convergence of simple forms in $\mathcal{D}_c^k(X)$, and so in such a setting, we presume the forms are in that space, rather than $\tilde{\mathcal{D}}_c^k(X)$.

With our introduction of the space $\tilde{\mathcal{D}}_c^k(X)$, we are able to rephrase the definition of mass from [32] ([32] Definition 4.1, but see also [1] equation (3.7)). We first make a definition that is somewhat reminiscent of the usual notion of comass for differential forms.

Definition 2.12. Let $\omega \in \tilde{\mathcal{D}}_c^k(X)$. The **comass** of ω, written $||\omega||$, is the number

$$||\omega|| = \inf_{S \text{ finite}} \sum_{s \in S} |f_s|$$

where the functions f_s satisfy

$$\omega = \sum_{s \in S} f_s \, dg_s^1 \wedge \cdots \wedge dg_s^k$$

for some functions $g_s^i \in \mathrm{Lip}_{\mathrm{loc}}(X)$ such that $L(g_s^i|_{\mathrm{Spt}(f)}) = 1$.

We now state the definition of mass from [32] using Definition 2.12.

Definition 2.13. Let $T\colon \mathcal{D}_c^k(X) \to \mathbb{R}$ be any function that is linear in each argument. The **mass** of T is the Borel regular outer measure $||T||$ on X given on open sets U by

$$||T||(U) = \sup_{\omega \in \tilde{\mathcal{D}}_c^k(U), ||\omega|| \leq 1} T(\omega),$$

and on arbitrary sets A by

$$||T||(A) = \inf_{U \supset A,\, U \text{ open}} ||T||(U).$$

It follows from [32, Theorem 4.3] (and the succeeding remarks) that $||T||$ is indeed a Borel regular outer measure. Notice that we do not require any continuity restrictions on T.

We denote by $\mathbf{M}_k(X)$ (resp. $\mathbf{M}_k^{\text{loc}}(X)$) the space of metric k-currents of (resp. locally) finite mass, that is,

$$\mathbf{M}_k(X) = \{T \in \mathcal{D}_k(X) : ||T||(X) < \infty\},$$

and

$$\mathbf{M}_k^{\text{loc}}(X) = \{T \in \mathcal{D}_k(X) : ||T||(A) < \infty \text{ whenever } \overline{A} \subset X \text{ is compact.}\}.$$

It can be shown ([32], Proposition 4.2) that $\mathbf{M}_k(X)$ is a Banach space under the mass norm $||T||(X)$.

We recall from [32] and [1] that for a k-current T of locally finite mass, there is a canonical extension of T to $\mathcal{E}_{\text{c}}^k(X)$, and hence to $\tilde{\mathcal{E}}_{\text{c}}^k(X)$, such that if $f_i \in \text{Lip}_{\text{c}}(X)$, $\beta \in \mathcal{B}_{\text{c}}^\infty(X)$, and $\{f_i\}$ converges to β in $L^1(X, ||T||)$, then for every ordered k-tuple $(g^1, \ldots, g^k) \in \text{Lip}_{\text{loc}}(X)^k$,

$$T(\beta\, dg^1 \wedge \cdots \wedge dg^k) = \lim_{i\to\infty} T(f_i\, dg^1 \wedge \cdots \wedge dg^k). \tag{2.14}$$

The mass measure $||T||$ can be characterized alternatively (see [1], Definition 3.5, and [32], Theorem 4.3) as the minimal Borel measure satisfying

$$T(f\, dg^1 \wedge \cdots \wedge dg^k) \le \prod_{i=1}^k L(g^i) \int_X |f|\, d||T|| \tag{2.15}$$

for every $f\, dg^1 \wedge \cdots \wedge dg^k \in \mathcal{D}_{\text{c}}^k(X)$.

From the definition, the extension of T to $\mathcal{E}_{\text{c}}^k(X)$ also satisfies equation (2.15).

$$T(\beta\, dg^1 \wedge \cdots \wedge dg^k) \le \prod_{i=1}^k L(g^i) \int_X |\beta|\, d||T||. \tag{2.16}$$

In the case $k = 0$, currents of locally finite mass act on functions by integration against a signed Radon measure, absolutely continuous with respect to $||T||$. The following lemma, and proof, were communicated to the author by Urs Lang.

Lemma 2.17. *Let $T \in \mathbf{M}_0^{loc}(X)$. Then there is a function $\lambda \in L^\infty(X, ||T||)$ such that for every $\beta \in \mathcal{E}_c^0(X) = \mathcal{B}_c^\infty(X)$,*

$$T(\beta) = \int_X \beta\lambda \, d||T||.$$

Proof. The mapping T is continuous in the norm of $L^1(X, ||T||)$, by inequality (2.15). Moreover, the compactly supported Lipschitz functions are dense in this norm, and so T extends to a map $\hat{T} \in L^1(X, ||T||)^*$ (This is, in fact, precisely the argument used in [32] to define the extension (2.14) above). Thus the existence and uniqueness of λ follows from the Riesz representation theorem. □

As with the classical definition, the boundary of a current is defined through duality:

Definition 2.18 (Boundary)**.** Let $T\colon \mathcal{D}_c^k(X) \to \mathbb{R}$ be a k-current. The **boundary** of T is the map $\partial T\colon \mathcal{D}_c^{k-1}(X) \to \mathbb{R}$ given by

$$\partial T(f\, dg^1 \wedge \cdots \wedge dg^{k-1}) = T(df \wedge dg^1 \wedge \cdots \wedge dg^{k-1}).$$

One can check [32] that the boundary ∂T is a well-defined $(k-1)$-current[1], that the map $T \mapsto \partial T$ is linear, and that $\partial(\partial T) = 0$.

Typically we do not expect the boundary of a current to have finite mass. If a current and its boundary *do* each have finite (resp. locally finite) mass, the current is said to be a normal (resp. locally normal) current. The space of such currents will be denoted $\mathbf{N}_k(X)$ (resp. $\mathbf{N}_k^{\mathrm{loc}}(X)$).

[1]Here it is important that we are following [32]. In [1], ∂T need not be a current, since currents are axiomatized to have finite mass, which need not be the case for ∂T.

Though one of the highlights of [32] is the elimination of the assumption of finite mass, or even locally finite mass, as a necessary axiom for the theory of currents, all of the currents we consider from now on will have locally finite mass. Indeed, our motivation for following [32] rather than [1] is solely for the reason that the former allows for locally finite mass, rather than just finite mass. For this reason, we introduce the following convention:

Throughout the rest of this paper, except where otherwise noted, the word "current" will denote a metric current of locally finite mass.

Given a k-current and a j-form, with $0 \leq j \leq k$, there is a natural way to produce a $(k-j)$-current.

Definition 2.19. Let $T \in \mathbf{M}_k^{\text{loc}}(X)$, and $\omega = \beta\, dh^1 \wedge \cdots \wedge dh^j \in \mathcal{E}^j(X)$. We define the restriction of the current T by the form ω to be the $(k-j)$-current $T\lfloor_\omega \in \mathbf{M}_{k-j}^{\text{loc}}(X)$, given by

$$T\lfloor_\omega(f\, dg^1 \wedge \cdots \wedge dg^{k-j}) = T(\beta f\, dh^1 \wedge \cdots \wedge dh^j \wedge dg^1 \wedge \cdots \wedge dg^{k-j}). \tag{2.20}$$

If $A \subseteq X$ is a Borel set, we define

$$T\lfloor_A = T\lfloor_{\chi_A}.$$

As noted in [32] and [1], the restriction defined by equation (2.20) is genuinely a current. Note that for a fixed current $T \in \mathbf{M}_k$, the restriction map $T\lfloor\colon \mathcal{E}^j(X) \to \mathbf{M}_{k-j}^{\text{loc}}(X)$ is linear in each argument, and thus induces a linear map $T\lfloor\colon \tilde{\mathcal{E}}^j(X) \to \mathbf{M}_{k-j}^{\text{loc}}(X)$.

It is a fact [1] that $||T\lfloor_A|| = ||T||\lfloor_A$. Using restrictions, we also have notions of concentration and support for currents.

Definition 2.21. We say that T is **concentrated** on a Borel set $A \subseteq X$ if $T\lfloor_A = T$, or equivalently, if $||T||$ is concentrated on A. The **support** of a current T, denoted $\mathrm{Spt}(T)$, is the smallest closed set on which T is concentrated.

Definition 2.21 lets us update equation (2.16):

$$T(\beta\, dg^1 \wedge \cdots \wedge dg^k) \leq \prod_{i=1}^{k} L(g^i|_{\mathrm{Spt}(T)}) \int_X |\beta|\, d||T||. \tag{2.22}$$

We recall the notion of the push-forward of a current.

Definition 2.23. Let $F\colon X \to Y$ be a proper Lipschitz map between metric spaces X and Y. The **push-forward** of a current $T \in \mathbf{M}_k^{\mathrm{loc}}(X)$ along F is the current $F_\# T \in \mathbf{M}_k^{\mathrm{loc}}(Y)$ given by

$$F_\# T(f\, dg^1 \wedge \cdots \wedge dg^k) = T((f \circ F)\, d(g^1 \circ F) \wedge \cdots \wedge d(g^k \circ F)).$$

Remark 2.24. If T is compactly supported, we may drop the assumption that F is proper. Indeed, in this case we define

$$F_\# T(f\, dg^1 \wedge \cdots \wedge dg^k) = T(\sigma \cdot (f \circ F)\, d(g^1 \circ F) \wedge \cdots \wedge d(g^k \circ F)),$$

where σ is any compactly supported Lipschitz function such that $\sigma|_{\mathrm{Spt}(T)} \equiv 1$. It follows immediately from the definition of $\mathrm{Spt}(T)$ that this is well-defined, and coincides, in the case of a proper map, with the Definition 2.23.

We conclude with a version of the chain rule for metric currents given in [1] and [32].

Theorem 2.25 (Ambrosio-Kirchheim)**.** *Let $T \in \mathbf{M}_k(Z)$, and let $g^1, \ldots, g^k$ be smooth functions on an open subset $U \subset \mathbb{R}^n$, with $k \leq n$. Then for every locally Lipschitz function $F = (F^1, \ldots, F^n)\colon Z \to \mathbb{R}^n$, and every $\beta \in \mathcal{B}_c^\infty(Z)$, we have*

(2.26)

$$T(\beta\, d(g^1 \circ F) \wedge \cdots \wedge d(g^k \circ F)) = T\left(\sum_{a \in \Lambda_{k,n}} \beta \det \left(\frac{\partial g^i}{\partial x_{a_j}} \circ F \right) dF^{a_1} \wedge \cdots \wedge dF^{a_k} \right).$$

2.2 Strong measured differentiable structures

We recall the notion of a differentiable structure from [27], inspired by [6].

Definition 2.27. Let $X = (X, d, \mu)$ be a metric measure space. Suppose $Y \subseteq X$, and let $\pi = (\pi^1, \ldots, \pi^n)\colon Y \to \mathbb{E}$ be a Lipschitz map, where $\mathbb{E} \cong \mathbb{R}^n$ is a Euclidean space. We call (Y, π) a **coordinate patch** if the following holds: For any $f \in \mathrm{Lip}(X)$ there is a measurable function $d^\pi f\colon Y \to \mathbb{E}^*$ (written $y \mapsto d^\pi f_y$), defined uniquely up to sets of measure zero, and a set $Y_f \subset Y$, with $\mu(Y \backslash Y_f) = 0$, such that for every $y \in Y_f$,

$$f(x) = f(y) + \langle d^\pi f_y, \pi(x) - \pi(y)\rangle + E^f_y(x), \tag{2.28}$$

where

$$\lim_{x \to y} \frac{E^f_y(x)}{\mathrm{dist}(x, y)} = 0. \tag{2.29}$$

I am greatly indebted to Stefan Wenger for suggesting the following useful fact, which has strengthened the result of Theorem 1.6 while at the same time simplifying its proof (also, compare [2, Section 3]).

Lemma 2.30. *Let X be locally compact and separable, $Y \subset X$, and let $\pi\colon Y \to \mathbb{E}$ be a coordinate patch as in Definition 2.27. Let $f \in \mathrm{Lip}_c(X)$, $\epsilon > 0$, and let ν be a Radon measure concentrated on Y_f. Then there is a compact set $Z = Z(f, \nu, \epsilon) \subseteq Y_f$, with $\nu(Y \backslash Z) < \epsilon$, such as r approaches 0, the Lipschitz constant $L(E^f_z|_{Z \cap B_r(x)})$ of the restricted error function $E^f_z|_{Z \cap B_r(x)}$ converges to 0 uniformly in z, for $z \in Z$. That is, there is a continuous function $\eta\colon [0, \infty) \to [0, \infty)$, with $\eta(0) = 0$, such that for all $z \in Z$,*

$$L(E^f_z|_{Z \cap B_r(x)}) \leq \eta(r). \tag{2.31}$$

Proof. By the local compactness of X, there is some number $\delta > 0$ such that the neighborhood $N = N_\delta(\mathrm{Spt}(f)) = \{x \in X : \mathrm{dist}(x, \mathrm{Spt}(f)) \leq \delta\}$ is compact.

For any $y_0 \in Y_f \backslash N$, the restriction $f|_{B_\delta(y_0)} \equiv 0$. By uniqueness, at almost every such y_0, $d^\pi f_{y_0} = 0$, so there is a subset $\tilde{N} \subset Y_f$, with $\mu(\tilde{N}) = 0$, such that $d^\pi f_{y_0} = 0$. In this case, we also have $E^f_y 0|_{B_\delta(y_0)} \equiv 0$, and in particular $L(E^f_y 0|_{B_\delta(y_0)}) = 0$. By Egorov's Theorem, there is a compact subset $Z_1 \subset (Y_f \cap \tilde{N}$ with $\mu(Y_f \backslash Z_1) < \epsilon/3$.

Consider the functions $E_r \colon Z_1 \to \mathbb{R}$ given by

$$E_r(z) = \sup_{x \in B_r(z), x \neq z} \frac{E^f_z(x)}{\mathrm{dist}(x, z)}.$$

Observe that for a fixed value $x \in X$, the function $z \mapsto E^f_z(x)$ is measurable in z by the measurability of d^π. Therefore for any countable dense subset $S \subset X$, we have

$$E_r(z) = \sup_{x \in S, x \neq z} \begin{cases} \frac{E^f_z(x)}{\mathrm{dist}(x,z)} & \text{if } \mathrm{dist}(x, z) \leq r, \\ 0 & \text{otherwise.} \end{cases}$$

Such a subset S exists by the separability of X.Thus E_r is the supremum of a countable family of measurable functions, and is therefore measurable.

By equation (2.29), the functions E_r converge to 0 pointwise on Z_1. Thus by Egorov's Theorem, there is a subset $Z_2 \subset Z_1$, with $\nu(Z_1 \backslash Z_2) \leq \epsilon/3$, on which the functions E_r converge uniformly. That is, there is a continuous function $\eta_1 \colon [0, \infty) \to [0, \infty)$, with $\eta_1(0) = 0$, such that $E_r(z) \leq \eta_1(r)$ for all $z \in Z_1$. On the other hand, by Lusin's theorem, the measurability of the function $d^\pi f$ guarantees the existence of a subset $Z_3 \subset Z_1$, with $\nu(Z_1 \backslash Z_3) \leq \epsilon$, on which $d^\pi f$ is uniformly continuous, i.e., there is a continuous function $\eta_2 \colon [0, \infty) \to [0, \infty)$, with $\eta(0) = 0$, such that $||d^\pi f_x - d^\pi f_y|| \leq \eta_2(\mathrm{dist}(x, y))$.

Let $Z = Z_2 \cap Z_3$. Then $\mu(Y_f \backslash Z) < \epsilon$, and for every $z \in Z$ and every $x, y \in$

$B_r(z) \cap Z$, $x \neq y$, we have

$$\begin{aligned}
& \left| \frac{E_z^f(x) - E_z^f(y)}{\mathrm{dist}(x,y)} \right| \\
= & \left| \frac{f(x) - f(y) - \langle d^\pi f_z, \pi(x) - \pi(y) \rangle}{\mathrm{dist}(x,y)} \right| \\
\leq & \frac{|f(x) - f(y) - \langle d^\pi f_y, \pi(x) - \pi(y) \rangle| + |\langle d^\pi f_z - d^\pi f_y, \pi(x) - \pi(y) \rangle|}{\mathrm{dist}(x,y)} \\
\leq & E_r(y) + \eta_2(r) \frac{||\pi(x) - \pi(y)||}{\mathrm{dist}(x,y)} \\
\leq & \eta_1(r) + \eta_2(r) L(\pi).
\end{aligned}$$

Letting $\eta(r) = \eta_1(r) + \eta_2(r)L(\pi)$ completes the proof. □

2.3 Currents and differentiation

In this section, we prove Theorems 1.6 and 1.8, as well as some other useful results for relating metric currents and differentiable structures. All of our other results in this section stem from Theorem 1.6, which we now prove.

Proof of Theorem 1.6. Fix $T \in \mathbf{M}_1^{\mathrm{loc}}(X)$ with $||T||$ concentrated on Y_ω. We assume with no loss of generality that $L(g_s^i) \leq 1$ for $i = 1, \ldots, k$ and all $s \in S$, that $|\beta_s(x)| \leq 1$ for every $x \in X$ and $s \in S$, and that $L(\pi) = 1$. It is enough to show that for every $\epsilon > 0$, equation (1.7) holds when T is replaced with $T\lfloor_Z$, where

$$Z = \left(\bigcup_s \mathrm{Spt}(\beta_s) \right) \cap \bigcap_{s,i} Z(g_s^i, ||T||, \epsilon),$$

and where each set $Z(g_s^i, ||T||, \epsilon)$ is chosen as in Lemma 2.30, so that for every $z \in Z$, each restricted error function $E_z^{g_s^i}|_{B_r(z)}$ has Lipschitz constant $L(E_z^{g_s^i}|_{B_r(z)}) < \eta(r)$. Indeed, if this is the case, then by the mass criterion (2.22), we have

$$|T(\omega)| = |T\lfloor_{Y_f \setminus Z}(\omega)| \leq k\#S \cdot ||T||(Y_f \setminus Z) \leq k\#S \cdot \epsilon,$$

from which the result follows upon passing to the limit as ϵ approaches 0.

By the remarks in the previous paragraph, we may assume without loss of generality that $T = T\lfloor_Z$. Further, we will assume that for each i and s, $||d^\pi g_s^i|| \le 1$ on Z, where $||\cdot||$ is the dual norm to the Euclidean norm on $\mathbb{E}$. This is a harmless assumption, as the differentials are measurable and finite almost everywhere, and hence bounded by some number M away from a set of arbitrarily small $||T||$-measure on Z. Rescaling the functions allows us to assume $M = 1$. Notice that under this assumption, for all $z \in Z$, the Lipschitz constant of the function $y \mapsto \langle d^\pi g^i_{s,z}, \pi(y)\rangle$ is at most 1, that is,

$$L(\langle d^\pi g^i_{s,z}, \pi\rangle) \le 1. \tag{2.32}$$

Fix $r > 0$, cover the compact set Z with finitely many disjoint Borel subsets $C_1, \dots, C_m$, each of diameter at most r, and choose a point $c_j \in C_j$ for each j. For each $s \in S$, we have

$$g_s^1 = g_s^1(c_j) + \langle d^\pi g^1_{s,c_j}, \pi - \pi(c_j)\rangle + E^{g_s^1}_{c_j}|_{B_r(c_j)} = C + \langle d^\pi g^1_{s,c_j}, \pi\rangle + E^{g_s^1}_{c_j}|_{B_r(c_j)}$$

for some constant C.

Then by equation (2.28) and the locality axiom, we have

$$T(\omega) = \sum_{j=1}^m \sum_s T\lfloor_{C_j}(\beta_s\, dg_s^1 \wedge \cdots \wedge dg_s^k) = \sum_{j=1}^m \sum_s T\lfloor_{C_j}\Big(\beta_s\, d\left(\langle d^\pi g^1_{s,c_j}, \pi\rangle + E^{g_s^1}_{c_j}|_{B_r(c_j)}\right) \wedge dg_s^2 \wedge \cdots \wedge d$$

Therefore, since for each j, $L(E_z^{g_s^1}|_{B_r(c_j)}) < \eta(r)$, we have

$$\begin{aligned}
&\left|T(\omega) - \sum_{j=1}^{m}\sum_{s} T\lfloor_{C_j}\left(\beta_s\, d(\langle d^\pi g^1_{s,c_j},\pi\rangle)\wedge dg_s^2\wedge\cdots\wedge dg_s^k\right)\right| \\
&\le \left|\sum_{j=1}^{m}\sum_{s} T\lfloor_{C_j}\left(\beta_s\, d(E_{c_j}^{g_s^1}|_{B_r(c_j)})\wedge dg_s^2\wedge\cdots\wedge dg_s^k\right)\right| \\
&\le \eta(r)\cdot \#S\cdot \sum_{j=1}^{m} ||T||(C_j) \\
&= \eta(r)\cdot \#S\cdot ||T||(Z).
\end{aligned}$$

Arguing similarly for $i = 2,\ldots,k$, and additionally using inequality (2.32), we have

(2.33)
$$\left|T(\omega) - \sum_{j=1}^{m}\sum_{s} T\lfloor_{C_j}\left(\beta_s\, d(\langle d^\pi g^1_{s,c_j},\pi\rangle)\wedge\cdots\wedge d(\langle d^\pi g^k_{s,c_j},\pi\rangle)\right)\right| \le k\eta(r)\cdot\#S\cdot||T||(Z).$$

We next claim that for each $j = 1,\ldots,m$,

$$\sum_{s\in S} T\lfloor_{C_j}\left(\beta_s\, d(\langle d^\pi g^1_{s,c_j},\pi\rangle)\wedge\cdots\wedge d(\langle d^\pi g^k_{s,c_j},\pi\rangle)\right) = 0. \tag{2.34}$$

Indeed, equation (2.34) may be rewritten

$$\sum_{s\in S} F((d^\pi g^1_{s,c_j},\ldots,\langle d^\pi g^k_{s,c_j})) = 0, \tag{2.35}$$

where $F\colon (\mathbb{E}^*)^k \to \mathbb{R}$ is given by

$$F(\theta_1,\ldots,\theta_k) = T\lfloor_{C_j}(\beta_s\, d(\langle\theta_1,\pi\rangle)\wedge\cdots\wedge d(\langle\theta_k,\pi\rangle))\,.$$

By the linearity and alternating properties of currents, F is linear and alternating, and therefore induces a linear map $\tilde{F}\colon \bigwedge^k \mathbb{E}^* \to \mathbb{R}$ such that $\tilde{F}(\theta_1\wedge\cdots\wedge\theta_k) = F(\theta_1,\cdots,\theta_k)$ for all $(\theta_1,\cdots,\theta_k)\in(\mathbb{E}^*)^k$. Therefore, we have

$$\begin{aligned}
\sum_{s\in S} F((d^\pi g^1_{s,c_j},\ldots,\langle d^\pi g^k_{s,c_j})) &= \sum_{s\in S}\tilde{F}(d^\pi g^1_{s,c_j}\wedge\cdots\wedge d^\pi g^k_{s,c_j}) \\
&= \tilde{F}\left(\sum_{s\in S} d^\pi g^1_{s,c_j}\wedge\cdots\wedge d^\pi g^k_{s,c_j}\right) \\
&= 0,
\end{aligned}$$

and so the claim is proved.

Combining equation (2.34) with inequality (2.33), we see that

$$|T(\omega)| \leq k\eta(r) \cdot \#S \cdot ||T||(Y).$$

Passing to the limit as r approaches 0 completes the proof. □

Theorem 1.6 gives us an immediate bound on the dimension of most currents.

Corollary 2.36. *Suppose the chart $\pi\colon Y \to \mathbb{E}$ has dimension n, i.e.,* $\dim(\mathbb{E}) = n$*. Then there is a subset $Y_0 \subset Y$, with $\mu(Y \backslash Y_0) = 0$, such that every nonzero current concentrated on Y_0 has dimension at most n.*

Proof. Let $\mathcal{G}$ be a countable subset of $\mathrm{Lip}_{\mathrm{loc}}(X)$, dense in $\mathrm{Lip}_{\mathrm{loc}}(X)$. Recall from Section **??** that such a subset exists. Since $\mathcal{G}$ is countable, the set $Y_0 = \bigcap_{g \in \mathcal{G}} Y_g$ has full measure in Y. On the other hand, for $k > n$, $\bigwedge^k \mathbb{E}^* = 0$, so Proposition 2.3 implies that every k-current T concentrated on Y_0 must satisfy

$$T(f\, dg^1 \wedge \cdots \wedge dg^k) = 0$$

whenever each $g^i \in \mathcal{G}$. The density of $\mathcal{G}$ in $\mathrm{Lip}_{\mathrm{loc}}(X)$ then implies that $T = 0$. □

Though Theorem 1.6 itself is entirely coordinate free, there are a number of consequences when coordinate functions are chosen for the differentiable structure. Let $e_1, \ldots, e_n$ be a basis for $\mathbb{E}$, an let $x^1, \ldots, x^n \in \mathbb{E}^*$ be the corresponding dual basis. Also, let $\pi^i = x^i \circ \pi$. For every $g \in \mathrm{Lip}_{\mathrm{loc}}(X)$ and each $y \in Y_g$, let $\frac{\partial g}{\partial \pi^i}(y) = \langle e_i, d^\pi g_y \rangle$, so that

$$d^\pi g_y = \sum_{i=1}^{n} \frac{\partial g}{\partial \pi^i}(y) d^\pi \pi^i_y. \tag{2.37}$$

Corollary 2.38. *Let (Y, π) be a coordinate chart on X, let $\beta\, dg^1 \wedge \cdots \wedge dg^k \in \tilde{\mathcal{E}}^k_c(X)$, and let $Y_{\mathcal{G}} = \cap_{i=1}^k Y_{g_i}$. Then for any current $T \in \mathbf{M}_k(X)$ such that $T\lfloor_\beta$ is concentrated* *on* $Y_{\mathcal{G}}$,

$$T(\beta\, dg^1 \wedge \cdots \wedge dg^k) = T\left(\sum_{a\in\Lambda_{k,n}} \beta \det\left(\frac{\partial g^i}{\partial \pi^j}\right) d\pi^{a_1} \wedge \cdots \wedge d\pi^{a_k}\right). \tag{2.39}$$

Proof. The corresponding differential forms for both sides are equal when defined, i.e.,

$$\beta\, d^\pi g^1 \wedge \cdots \wedge d^\pi g^k - \beta \sum_{a\in\Lambda_{k,n}} \left(\det\left(\frac{\partial g^i}{\partial \pi^j}\right) d^\pi \pi^{a_1} \wedge \cdots \wedge d^\pi \pi^{a_k}\right) = 0$$

almost everywhere. Applying Theorem 1.6 then completes the proof. □

Corollary 2.38 generalizes to an extension of the chain rule in Theorem 2.25 to spaces with differentiable structures.

Corollary 2.40. *Let* $F\colon Z \to Y$ *be a Lipschitz map, where* (Y,π) *is a coordinate chart. Let* $\beta \in \mathcal{B}^\infty_{\mathrm{c}}(Z)$, *and let* $(g^1,\ldots,g^k) \in \mathrm{Lip}_{\mathrm{loc}}(Y)^k$. *Let* $Y_{\mathcal{G}} = \cap_{i=1}^k Y_{g_i}$. *Then for any current* $T \in \mathbf{M}_k(Z)$ *such that* $F_\#(T\lfloor_\beta)$ *is concentrated on* $Y_{\mathcal{G}}$,

(2.41)

$$T(\beta\, dg^1{\circ}F\wedge\cdots\wedge dg^k{\circ}F) = T\left(\sum_{a\in\Lambda_{k,n}} \beta \det\left(\frac{\partial g^i}{\partial \pi^j}\circ F\right) d(\pi^{a_1}\circ F) \wedge \cdots \wedge d(\pi^{a_k}\circ F)\right).$$

Note that since β has compact support, so does $T\lfloor_\beta$, so that $F_\#(T\lfloor_\beta)$ is well defined, by Remark 2.24.

Proof. Apply Corollary 2.38 to the current $F_\#(T\lfloor_\beta)$ and the form $dg^1\wedge\cdots\wedge dg^k$. □

We are about ready to prove Theorem 1.8, but first we must define precurrents precisely.

Definition 2.42. A linear map $T\colon \mathcal{E}^k_{\mathrm{c}}(X) \to \mathbb{R}$ is a k**-precurrent on** Y if

$$T(\beta\, dg^1 \wedge \cdots \wedge dg^m) = \int_Y \langle \hat\lambda, \beta\, d^\pi g^1 \wedge \cdots \wedge d^\pi g^k\rangle\, d\mu,$$

where $\hat{\lambda}\colon Y \to \bigwedge^k \mathbb{E}$ is locally p-integrable. Such a map is called a **(measurable) k-vector field**. If $T\lfloor_Y$ is a k-precurrent on Y for every coordinate patch Y, we simply say that T is a **k-precurrent**.

Note that the linearity and locality axioms from Definition 2.2 are easily seen to be satisfied, but the continuity axiom need not be, as we will see in the next chapter.

Proof of Theorem 1.8. We must show that $T\lfloor_Y$ is a precurrent for the chart Y, whenever Y is a coordinate chart. We fix such a chart Y.

The correspondence between 0-currents and measures given in Lemma 2.17 says that there are functions $\lambda^a \in L^\infty(Y, ||T||)$ such that

$$T(\beta\, d\pi^{a_1} \wedge \cdots \wedge d\pi^{a_k}) = T\lfloor_{d\pi^{a_1}\wedge\cdots\wedge d\pi^{a_k}}(\beta) = \int_Y \beta\lambda^a\, d\mu \tag{2.43}$$

for any $\beta \in \mathcal{B}^\infty_{\mathrm{c}}(X)$. Since $||T|| \ll \mu$, the functions λ^a are locally μ-integrable. Let $\hat{\lambda}\colon Y \to \bigwedge^k \mathbb{E}$ be given by

$$\hat{\lambda}^\alpha = \sum_{a\in\Lambda_{k,n_\alpha}} \lambda^{\alpha,a} e_{a_1} \wedge \cdots \wedge e_{a_k}.$$

Since $Y_{\mathcal{G}}$ has full μ-measure, and by assumption, T is concentrated on Y with $||T|| \ll \mu$, we see that T is concentrated on $Y_{\mathcal{G}}$. Thus we may invoke Corollary 2.38. Applying equation (2.43) to the right hand side of (2.39), we see that

$$\begin{aligned} T(f\, dg^1 \wedge \cdots \wedge dg^k) &= \sum_{a\in\Lambda_{k,n}} T\left(f \det\left(\frac{\partial g^i}{\partial \pi^{a_j}}\right) d\pi^{a_1} \wedge \cdots \wedge d\pi^{a_k}\right) \\ &= \sum_{a\in\Lambda_{k,n}} \int_Y f\lambda^a \det\left(\frac{\partial g^i}{\partial \pi^{a_j}}\right) d\mu = \int_Y \langle \hat{\lambda}, f\, d^\pi g^1 \wedge \cdots \wedge d^\pi g^k\rangle\, d\mu. \end{aligned}$$

$\square$

CHAPTER III

Currents in Carnot groups

In this chapter we study metric currents in Carnot groups, equipped with their Carnot-Carathéodory metrics. In Section 3.1, we recall the basic facts we will need from the theory of Carnot groups. In Sections 3.2 and 3.3 we do some preliminary analysis of currents in general metric groups, and of precurrents in Carnot groups, respectively. Section 3.4 provides a characterization of invariant currents in Carnot groups, which we use in Section 3.5 to prove Theorem 1.9. Lastly, we relate our results to previous results on rectifiability in Carnot groups in Section 3.6.

3.1 Carnot groups.

We recall some definitions and facts about stratified Lie groups, also known as Carnot groups, equipped with their Carnot-Carathéodory metrics. All of this material is surveyed in [19] and in Section 11 of [17]. A much more in-depth study of Carnot-Carathéodory spaces can be found in [16], and of Carnot groups specifically, in [35].

Definition 3.1. A **Carnot group** is a connected, simply connected Lie group $\mathbb{G} = (\mathbb{G}, \cdot)$, with unit element e and left Haar measure μ, whose Lie algebra $\mathfrak{g} = T_e\mathbb{G}$, with bracket $[\cdot,\cdot]$, admits a stratification, i.e., a direct sum decomposition

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_m$$

such that $[V_1, V_j] = V_{j+1}$ for $j < m$, and $[\mathfrak{g}, V_m] = 0$.

We call $\mathbb{G}$ a Carnot group of step m.

For $p \in \mathbb{G}$, let τ_p denote the left-translation map $q \mapsto p \cdot q$. Throughout this chapter, we will take the point of view that k-vector fields and k-forms, respectively, are maps from $\mathbb{G}$ into $\bigwedge^k \mathfrak{g}$ and $\bigwedge^k \mathfrak{g}^*$, where g^* is the dual vector space of linear maps from $\mathfrak{g}$ to $\mathbb{R}$. Notice that this agrees with the usual notion by way of the canonical identification between T_p and $\mathfrak{g} = T_e$ given by the translation map τ_{p*}.

We assume $\mathfrak{g}$ is equipped with an inner product $\langle\cdot,\cdot\rangle$, so that $\mathbb{G}$ has a left-invariant Riemannian structure. We denote by $\mathrm{dist}_{\mathrm{R}}(\cdot,\cdot)$ the metric induced by this structure.

We refer to $H = V_1$ as the *horizontal subspace*. The vector bundle $\mathcal{H} = \bigcup_{p\in\mathbb{G}} \tau_{p*}H$ is called the *horizontal bundle*. A piecewise smooth path $\gamma\colon I \to \mathbb{G}$ is said to be **horizontal** if $\frac{d\gamma}{dt} \in \mathcal{H}$ for all but finitely many $t \in I$.

Definition 3.2. The Carnot-Carathéodory distance between two points $p, q \in \mathbb{G}$ is

$$\mathrm{dist}_{\mathrm{CC}}(p,q) = \inf\{l(\gamma) : \gamma \text{ is a horizontal path joining } p \text{ and } q.\}$$

The following theorem is a special case of a deep result due to Chow and Rashevsky. See, e.g., [16] for a proof.

Theorem 3.3 (Chow-Rashevsky)**.** *Let* $\mathbb{G}$ *be a Carnot group. Then* dist_{CC} *is a metric on* $\mathbb{G}$.

If $v \in \mathfrak{g}$, we denote by X^v the unique left invariant vector field on $\mathbb{G}$ satisfying $X^v_e = v$.

Finally, if $f\colon \mathbb{G} \to \mathbb{R}$ is differentiable (in the usual sense, as opposed to the Pansu-differentiability described below) at $p \in \mathbb{G}$, we write $d_r f_p\colon \mathfrak{g} \to \mathbb{R}$ for the differential of f, as the symbol *df* has already been expropriated for metric currents. The "r" is to emphasize that this differential is the one that should exist almost everywhere (by Rademacher's theorem) for functions that are Lipschitz in the *Riemannian* metric on $\mathbb{G}$. A theorem of Pansu (Theorem 3.8 below) provides an analogous differential, d_c, for Lipschitz functions in the Carnot-Carathéodory metric.

A Carnot group's Lie algebra $\mathfrak{g}$ is equipped with a one-parameter family of linear dilations $\delta_r\colon \mathfrak{g} \to \mathfrak{g}$ given by $\delta_r(v_j) = r^j v_j$ for $v_j \in V_j$. The maps δ_r are Lie algebra homomorphisms, and so induce Lie group homomorphisms $\Delta_r\colon \mathbb{G} \to \mathbb{G}$ via the exponential map, such that the $\Delta_{r*}(e) = \delta_r$. Notice that since Δ_r is a homomorphism, we have $\Delta_r \circ \tau_p = \tau_{\Delta(p)} \circ \Delta_r$ for every $p \in \mathbb{G}$. It follows that for every $u \in H$, $p \in \mathbb{G}$, and $r > 0$, we have

$$\Delta_{r*} X_p^u = \Delta_{r*} \tau_{p*} u = \tau_{\Delta_r(p)*} \Delta_{r*} u = r \tau_{\Delta_r(p)*} u = r X^u_{\Delta_r(p)}. \tag{3.4}$$

Thus the dilation Δ_r rescales the metric $\mathrm{dist}_{\mathrm{CC}}$ by a factor of r, as the name implies.

The number $Q = \sum_{i=1}^m i \dim(V_i)$ is called the *homogeneous dimension* of $\mathbb{G}$. As motivation, we note that the dilations Δ_r have Jacobian r^Q. A Carnot $\mathbb{G}$ with homogeneous dimension Q has Hausdorff dimension Q as well, and is in fact Ahlfors Q-regular [19]. Since the metric $\mathrm{dist}_{\mathrm{CC}}$ is invariant under left translations, and the Hausdorff Q-measure $\mathcal{H}^Q$ is positive and finite on balls, we adopt the convention that the Haar measure $\mu = \mathcal{H}^Q$. Note that this implies

$$\Delta_{r\#} \mu = r^{-Q} \mu \tag{3.5}$$

for each $r > 0$. For a noncommutative Carnot group (i.e., one of step $m > 1$), Q

always exceeds the topological dimension, and so such groups give us a rich supply of fractal spaces to study.

Lastly, we note that Carnot groups, being nilpotent, are unimodular [36].

Example 3.6. The n^{th} Heisenberg group $\mathbb{H}^n$ is a $(2n+1)$-dimensional Lie group whose Lie algebra is spanned by vector fields X_i, Y_i and Z, for $i = 1, \ldots, n$, satisfying the relations

$$[X_i, Y_i] = Z$$

with all other generators commuting. The group $\mathbb{H}^n$ is a step-2 Carnot group with stratification $\operatorname{Span}(X_1, Y_1, \ldots, X_n, Y_n) \oplus \operatorname{Span}(Z)$. The homogeneous dimension Q is $2n+2$, one more than the topological dimension.

Density of smooth functions

The following lemma will allow us to employ the smooth structure of a Carnot group $\mathbb{G}$ in our analysis of $\mathbf{M}_k(\mathbb{G})$. The proof is a standard convolution argument, essentially the same the proof for the case $\mathbb{G} = \mathbb{R}^n$ given in [32]. Similar convolution arguments have previously been used in the setting of homogeneous groups; see [13] for example.

Lemma 3.7. *The space $C_c^\infty(\mathbb{G})$ of smooth functions on $\mathbb{G}$ with compact support is dense in $\mathrm{Lip_c}(\mathbb{G})$. Similarly, $C^\infty(\mathbb{G})$ is a dense subset of $\mathrm{Lip_{loc}}(\mathbb{G})$.*

Proof. Let $f \in \mathrm{Lip_c}(\mathbb{G})$, with Lipschitz constant L. Let $\phi\colon \mathbb{G} \to [0, \infty)$ be a smooth function supported on $B_1(e)$ such that $\int_{\mathbb{G}} \phi \, d\mu = 1$. For every $\epsilon > 0$, define $\phi_\epsilon(p) = \epsilon^{-Q} \phi \circ \Delta_\epsilon$. Note that ϕ_ϵ is supported on $B_\epsilon(e)$, and that $\int_{\mathbb{G}} \phi_\epsilon = 1$. We then define smooth functions $f_\epsilon\colon \mathbb{G} \to \mathbb{R}$ by

$$f_\epsilon(p) = \int_{\mathbb{G}} f(q^{-1}p)\phi_\epsilon(q) d\mu(q) = \int_{\mathbb{G}} f(z)\phi_\epsilon(pz) d\mu(z).$$

Then at every $p \in \mathbb{G}$, and for every $\epsilon > 0$,

$$|f_\epsilon(p) - f(p)| \leq \int_{\mathbb{G}} |f(z)\phi_\epsilon(pz) - f(p)| d\mu(z).$$

By continuity of ϕ, the right hand side converges to 0 with ϵ, so that f_ϵ converges pointwise to f. Moreover, for any $p_1, p_2 \in \mathbb{G}$, we have

$$\begin{aligned} |f_\epsilon(p) - f_\epsilon(q)| &= \left| \int_{\mathbb{G}} \left(f(q^{-1}p_1) - f(q^{-1}p_1) \right) \phi_\epsilon(q) d\mu(q) \right| \\ &\leq \int_{\mathbb{G}} \left| f(q^{-1}p_1) - f(q^{-1}p_1) \right| \phi_\epsilon(q) d\mu(q) \\ &\leq \int_{\mathbb{G}} \left| L \operatorname{dist}_{\mathrm{CC}}(p_1, p_2) \right| \phi_\epsilon(q) d\mu(q) \\ &= L \operatorname{dist}_{\mathrm{CC}}(p_1, p_2). \end{aligned}$$

Thus the functions f_ϵ have uniformly bounded Lipschitz constant. Moreover, for $\epsilon < 1$, they are supported on the relatively compact neighborhood $\mathcal{N}_1(\operatorname{Spt}(f)) = \{p \in \mathbb{G} : \operatorname{dist}_{\mathrm{CC}}(p, \operatorname{Spt}(f)) < 1\}$. Therefore f_ϵ converges in $\operatorname{Lip}_{\mathrm{c}}(\mathbb{G})$ to f.

The same argument shows the density of $C^\infty(\mathbb{G})$ in $\operatorname{Lip}_{\mathrm{loc}}(\mathbb{G})$. The only different part of the argument is to show that the functions are locally uniformly Lipschitz. To see this, note that for any compact set $K \subset \mathbb{G}$, if $f|_{\mathcal{N}_1(K)}$ is L-Lipschitz, then $f_\epsilon|_K$ is L-Lipschitz for $\epsilon < 1$, and so the the Lipschitz constants of $f_\epsilon|_K$ are uniformly bounded for each K. □

Differentiable structure

According to a result of Jerison [26], a Carnot group admits a Poincaré inequality, and thus by Cheeger's differentiation theorem, also admits a differentiable structure. In fact, the structure can be described by differentiating in the horizontal directions, as stated precisely in the differentiation Theorems 3.8 and 3.13 below, due to Pansu and Cheeger-Weaver, respectively.

Before we state the theorem, a number of remarks are in order. First, the Lie subalgebra $\mathfrak{v} = 0 \oplus V_2 \oplus \cdots \oplus V_m$ is an ideal, and so the corresponding Lie subgroup $\mathbb{V} \subset \mathbb{G}$ is normal [18]. Moreover, we can identify H with $\mathfrak{g}/\mathfrak{v}$ by way of the quotient map $\pi_* \colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{v}$ (here $\pi \colon \mathbb{G} \to \mathbb{G}/\mathbb{V}$ is the quotient map). By way of this identification, we equip $\mathfrak{g}/\mathfrak{v}$ with the inner product from H, and notice that with respect to this inner product, the map π_* is 1- Lipschitz. It follows that the map π is Lipschitz with respect to the Carnot Carathéodory metric on $\mathbb{G}$ and the Riemannian metric on $\mathbb{G}/\mathbb{V}$ (which is just a Euclidean metric). In the future, we will denote by $\mathbb{H}$ the group $\mathbb{G}/\mathbb{V}$, equipped with the aforementioned metric. We will also denote by $\mathfrak{h} = \mathfrak{g}/\mathfrak{v}$ the Lie algebra of the group $\mathbb{H}$.

The following generalization of Rademacher's differentiation theorem was proved by Pansu [35].

Theorem 3.8 (Pansu)**.** *Let $f \colon \mathbb{G}_1 \to \mathbb{G}_2$ be a Lipschitz mapping between two Carnot groups. For every $p \in \mathbb{G}$ and $t > 0$, define $f_p^t \colon \mathbb{G}_1 \to \mathbb{G}_2$ to be the rescaling*

$$f_p^t(q) = \Delta_t^{-1}(f(p)^{-1} \cdot f(p \cdot \Delta_t(q))). \tag{3.9}$$

Then at almost every $p \in \mathbb{G}_1$, there is a Lie group homomorphism $D_c f_p \colon \mathbb{G}_1 \to \mathbb{G}_2$, commuting with each dilation Δ_r, given by

$$D_c f_p(q) = \lim_{t \to \infty} f_p^t(q) \tag{3.10}$$

We call $D_c f_p$ the *Pansu differential* at p. When it is defined, we say f is *Pansu differentiable* at p. Notice that each of the maps f_p^t are Lipschitz with the same Lipschitz constant as f, so that if it exists, $D_c f_p$ is Lipschitz. We also define $d_c f_p$ to be the induced Lie algebra homomorphism $d_c f_p = (D_c f_p)_*$.

We are interested in the case where $\mathbb{G}_2 = \mathbb{R}$. In this case, since $\mathbb{R}$ is Abelian, the map $D_c f_p$ vanishes on $\mathbb{V}$, since the latter group is the commutator of $\mathbb{G}$, as follows from the stratification of $\mathfrak{g}$. Therefore there is an induced homomorphism $D^h_c f_p\colon \mathbb{H} \to \mathbb{R}$ such that $D^h_c f_p \circ \pi = D_c f_p$. Note then that $d^h_c f_p \circ \pi_* = d_c f_p\colon \mathfrak{g} \to \mathbb{R}$. Also, since $d_c f_p$ is an element of $\mathfrak{g}^*$, we will write $d_c f_p(u) = \langle d_c f_p, u\rangle$.

The stratification of $\mathbb{G}$ indicates that $\exp(tu) = \exp(\delta_t u) = \Delta_t(\exp(u))$ for every $u \in H$. It follows that at every point $p \in \mathbb{G}$ of Pansu differentiability, and for every $u \in H$, the partial derivatives $X^u_p(f)$ exist, and we have

$$
\begin{aligned}
X^u_p(f) &= \frac{d}{dt}|_{t=0} f(p\cdot \exp(tu)) \\
&= \lim_{t\to 0} \frac{f(p\cdot\exp(tu)) - f(p)}{t} \\
&= \lim_{t\to 0} \frac{f(p\cdot\Delta_t \exp(u)) - f(p)}{t} \\
&= D_c f_p(\exp(u)) \\
&= \langle u, d_c f_p\rangle.
\end{aligned}
$$

Moreover, if $f\colon \mathbb{G} \to \mathbb{R}$ is differentiable (in the usual sense), then for any $q \in \mathbb{G}$, the map $t \mapsto g(p\cdot\Delta_t(q))$ is differentiable at $t = 0$, from which it follows that f is Pansu differentiable at p. From the equation (3.11), then, we have that

$$
\langle u, d_c f_p\rangle = X^u_p(f) = \langle u, d_r f_p\rangle. \tag{3.11}
$$

Note that the Pansu differential is compatible with dilations in the following sense. If $f\colon \mathbb{G} \to \mathbb{R}$ is a Lipschitz function, and $r > 0$, then

$$
\begin{aligned}
D_c(f\circ\Delta_r)_p(q) &= \lim_{t\to 0}(f\circ\Delta_r)^t_p(q) \\
&= \lim_{t\to 0} \frac{f(\Delta_r(p\Delta_t(q))) - f(\Delta_r(p))}{t} \\
&= \lim_{t\to 0} r\cdot \frac{f(\Delta_r(p)\Delta_{rt}(q)) - f(\Delta_r(p))}{rt} \\
&= rD_c f_{\Delta_r(p)}(q).
\end{aligned}
$$

Differentiating, we obtain

$$
d_c(f\circ\Delta_r)_p = r d_c f_{\Delta_r(p)}. \tag{3.12}
$$

As outlined in [6] (Remark 4.66) and [41] (Theorems 39 and 43), differentiation in the horizontal directions provides a concrete description of the differential structure of a Carnot group (and, more generally, of a sub-Riemannian manifold). For completeness, we give here the proof for Carnot groups, using Pansu's theorem. See also [5] for a proof in the special case of the Heisenberg group; there, a version of Theorem 3.13 is proved first, and then employed in the proof of Theorem 3.8.

Theorem 3.13 (Cheeger, Weaver)**.** *Let $\mathbb{G}$ be a Carnot group with H, $\mathfrak{v}$, $\mathfrak{h}$, $\mathbb{V}$, and $\mathbb{H}$ as defined above. Then $\mathbb{G}$ admits a differentiable structure with a single coordinate chart, namely the quotient map $\pi\colon \mathbb{G} \to \mathbb{H}$ defined above. For every $f \in \mathrm{Lip}(\mathbb{G})$, the differential $d^\pi f\colon \mathbb{G} \to \mathfrak{h}^*$ is given by $d^\pi f_p = d_c^h f_p$, whenever the latter is defined. If p is a point of (Pansu) differentiability, then for every $u \in H$, $X_p^u(f)$ exists and satisfies*

$$\langle d^\pi f_p, \pi_* u\rangle = X_p^u(f). \tag{3.14}$$

Proof. At any point p of Pansu differentiability, the functions f_p^t are uniformly Lipschitz with the same Lipschitz constant as f, and so they converge uniformly on compact sets to $D_c f_p$, and in particular, on the ball $B_1(e)$. Note also that since $\mathbb{R}$ is Abelian, we have

$$f_p^t(q) = \frac{f(p\cdot\Delta_t(q)) - f(p)}{t}.$$

Thus, for any $\epsilon > 0$, there is an $\eta = \eta(\epsilon)$ such that

$$\begin{aligned}
&\left|f(q) - f(p) - \langle d_c^h f_p, \pi(q) - \pi(p)\rangle\right| = \left|f(q) - f(p) - D_c^h f_p(\pi(q) - \pi(p))\right| \\
&= \left|f(q) - f(p) - D_c f_p(p^{-1}q)\right| = \left|f(pp^{-1}q) - f(p) - D_c f_p(p^{-1}q)\right| \\
&= \left|t\cdot\left(\frac{f(p\Delta_t(\Delta_t^{-1}(p^{-1}q))) - f(p)}{t} - D_c f_p(\Delta_t^{-1}(p^{-1}q))\right)\right| \\
&= \left|t\cdot\left(f_p^t(\Delta_t^{-1}(p^{-1}q)) - D_c f_p(p^{-1}q)\right)\right| \le \epsilon t
\end{aligned}$$

whenever $t \leq \eta$ and $\Delta_t^{-1}(p^{-1}q) \in B_1(e)$. In particular, the inequality holds for $t = \text{dist}_{\text{CC}}(p, q)$, yielding equation (2.28). Finally, equation (3.14) is a restatement of equation (3.11). □

3.2 Metric groups

We begin our study of currents in Carnot groups with a more general setting. Suppos $X = (\Gamma, \text{dist}, \mu)$, where $\Gamma = (\Gamma, 1_\Gamma)$ is a locally compact group with identity 1_Γ, left-invariant metric dist and left Haar measure μ. In such a group, we will abuse notation and identify an element $\gamma \in \Gamma$ with the associated left translation map $\alpha \mapsto \gamma\alpha$.

Our main result in this section is that on a metric group, the set of k-currents of absolutely continuous mass is weakly dense:

Lemma 3.15. *Let $T \in \mathbf{M}_k^{loc}(\Gamma)$ be a current of locally finite mass in a metric group Γ. Then there are currents $T_\epsilon \in \mathbf{M}_k^{loc}(\Gamma)$ whose masses $||T_\epsilon||$ are absolutely continuous with respect to μ, and such that T_ϵ converges to weakly to T as ϵ converges to 0, i.e.,*

(3.16) $$\lim_{\epsilon \to 0} T_\epsilon(\omega) = T(\omega)$$

for each $\omega \in \mathcal{D}_c^k(\Gamma)$.

Proof. For $\omega \in \mathcal{D}_{\text{c}}^k(\Gamma)$, we define

$$T_\epsilon(\omega) = \fint_{B(1_\Gamma, \epsilon)} (\gamma_\# T)(\omega)\, d\mu(\gamma).$$

We must first check that for each $\epsilon > 0$, T_ϵ is a current. Fix ϵ, and suppose the forms $\omega_i = f_i\, dg_i^1 \wedge \cdots \wedge dg_i^k$ converge to $\omega = f\, dg^1 \wedge \cdots \wedge dg^k$.

Since the functions f_i converge uniformly to f, and the translation maps are isometries, all of the functions $f_i \circ \gamma$ are uniformly bounded in absolute value, say

(3.17) $$|f_i \circ \gamma| \leq M.$$

Similarly, the functions g_i^j have locally uniformly bounded Lipschitz constants, and so there is some $N > 0$ such that $L(g_i^j|_{N_{2\epsilon}(K)}) < N$ for all i and j, where $K = \bigcup \mathrm{Spt}(f_i)$, and $N_{2\epsilon}(K) = \{\gamma \in \Gamma : \mathrm{dist}(\gamma, K) < 2\epsilon\}$. It follows, again because the translation mappings are isometries, that for each $\gamma \in B_\epsilon(1_\Gamma)$, and each i and j,

$$L(g_i^j \circ \gamma|_{N_\epsilon(K)}) < N. \tag{3.18}$$

Inequalities (3.17) and (3.18), as well as the mass criterion (2.15), imply that for all i and j, and for $\gamma \in B_\epsilon(1_\Gamma)$,

$$\gamma_\# T(\omega_i) \leq MN^k ||T||(N_\epsilon(K)).$$

Moreover, by the continuity axiom, for each γ, $T(\omega_i)$ converges to $T(\omega)$. Thus $T_\epsilon(\omega_i)$ converges to $T_\epsilon(\omega)$ by the Lebesgue Dominated Convergence Theorem.

To prove that $||T_\epsilon|| \ll \mu$, it is enough to show that whenever $\mu(A) = 0$, $T_\epsilon \lfloor_A = 0$, since this implies $||T_\epsilon||(A) = ||T_\epsilon \lfloor_A|| = 0$. To establish that $T_\epsilon \lfloor_A = 0$, we argue as follows: If $A \subset \Gamma$ with $\mu(A) = 0$, we use Fubini's theorem and Definition 2.13 to conclude that

$$\begin{aligned}
& |T_\epsilon \lfloor_A (f\, dg^1 \wedge \cdots \wedge dg^k)| \\
= & \left| \fint_{B(1_\Gamma, \epsilon)} T\left((\chi_A f) \circ \gamma\, d(g^1 \circ \gamma) \wedge \cdots \wedge d(g^k \circ \gamma)\right)\, d\mu(\gamma) \right| \\
\leq & \fint_{B(1_\Gamma, \epsilon)} \left| T\left((\chi_A f) \circ \gamma\, d(g^1 \circ \gamma) \wedge \cdots \wedge d(g^k \circ \gamma)\right) \right|\, d\mu(\gamma) \\
\leq & \sup_{|f| \leq 1} \left(\fint_{B(1_\Gamma, \epsilon)} \left(\int_\Gamma |(\chi_A f)(\gamma y)|\, d||T||(y) \right) d\mu(\gamma) \right) \\
= & \sup_{|f| \leq 1} \left(\int_\Gamma \left(\fint_{B(1_\Gamma, \epsilon)} |(\chi_A f)(\gamma y)|\, d\mu(\gamma) \right) d||T||(y) \right) \\
= & \ 0.
\end{aligned}$$

Note that the second to last line vanishes because right translations map null sets to null sets. This follows from the fact that left and right Haar measure are in the same measure class, and thus have the same null sets.

It now remains only to check that (3.16) holds for every $\omega \in \mathcal{D}^k_{\mathrm{c}}(\Gamma)$.

We argue by contradiction. Suppose T_ϵ does not converge weakly to T. Then for some $\omega = f\,dg^1 \wedge \cdots \wedge dg^k \in \mathcal{D}^k_{\mathrm{c}}(\Gamma)$, $\delta > 0$, and some sequence $\{\epsilon_i\}$ with $\epsilon_i \to 0$ as $i \to \infty$, we have

$$|(T_{\epsilon_i} - T)(\omega)| \geq \delta.$$

For each i, we therefore have some $\gamma_i \in B(1_\Gamma, \epsilon_i)$ such that

$$\begin{aligned} \delta &\leq |(\gamma_{i\#}T - T)(\omega)| \\ &\leq |T(f \circ \gamma_i\, d(g^1 \circ \gamma_i) \wedge \cdots \wedge d(g^k \circ \gamma_i)) - T(f\,dg^1 \wedge \cdots \wedge dg^k)|. \end{aligned}$$

On the other hand, $\gamma_i \to 1_\Gamma$, from which it follows that $f \circ \gamma_i$ converges to f in $\mathrm{Lip}_{\mathrm{c}}(\Gamma)$, and $g^j \circ \gamma_i$ converges to g^j in $\mathrm{Lip}_{\mathrm{loc}}(\Gamma)$ for each j. This contradicts the continuity of T. $\square$

Lemma 3.15, in combination with Corollary 2.36 and the alternating property, immediately yields the following result.

Corollary 3.19. *Let Γ be a metric group with a differentiable structure of dimension n. Then Γ admits no nonzero k-currents for $k > n$.*

3.3 Precurrents in Carnot groups.

From Theorem 3.13, we know that precurrents in Carnot groups have the form

$$T(f\,dg^1 \wedge \cdots \wedge dg^k) = \int_{\mathbb{G}} \langle \hat{\lambda}(p), f\, d^h_c g^1_p \wedge \cdots \wedge d^h_c g^k_p \rangle\, d\mu(p), \tag{3.20}$$

where $\hat{\lambda}\colon \mathbb{G} \to \bigwedge^k \mathfrak{h}$ is locally integrable. Since $\bigwedge^k \pi_*|_{\bigwedge^k H}\colon \bigwedge^k H \to \bigwedge^k \mathfrak{h}$ is an isomorphism, it follows that there is a locally integrable k-vector field $\tilde{\lambda}\colon \mathbb{G} \to \bigwedge^k H$

such that

$$
\begin{aligned}
T(f\,dg^1 \wedge \cdots \wedge dg^k) &= \int_{\mathbb{G}} \langle \hat{\lambda}_p, f(p)\, d^h_c g^1_p \wedge \cdots \wedge d^h_c g^k_p \rangle \, d\mu(p) \\
&= \int_{\mathbb{G}} \langle \bigwedge^k \pi_*(\tilde{\lambda}_p), f(p)\, d^h_c g^1_p \wedge \cdots \wedge d^h_c g^k_p \rangle \, d\mu(p) \\
&= \int_{\mathbb{G}} \langle \tilde{\lambda}_p, f(p)\, d_c g^1_p \wedge \cdots \wedge d_c g^k_p \rangle \, d\mu(p) \\
&= \int_{\mathbb{G}} \langle \tilde{\lambda}_p, f(p)\, d_c g^1_p \wedge \cdots \wedge d_c g^k_p \rangle \, d\mu(p).
\end{aligned} \tag{3.21}
$$

We denote the precurrent in the above equation by $T_{\tilde{\lambda}}$. For the rest of this chapter, all k-vector fields under consideration will be locally integrable, and so we generally omit this modifier and simply refer to such an object as a k-vector field.

Let $u_1, \dots, u_n$ be an orthonormal basis for H, dual to the basis $d_r\pi_1|_H, \dots, d_r\pi_n|_H$ of H^*. Then the simple k-vectors $\tilde{u}_a = u_{a_1} \wedge \cdots \wedge u_{a_k}$ form a basis for $\bigwedge^k H$, and so every k-vector field $\tilde{\lambda}$ has the form

$$
\tilde{\lambda} = \sum_{a \in \Lambda_{k,n}} \lambda^a \tilde{u}_a, \tag{3.22}
$$

for locally integrable functions λ^a on $\mathbb{G}$.

An initial observation is that, as with the currents described in this paper, precurrents have locally finite mass.

Lemma 3.23. *Let T be a k-precurrent in $\mathbb{G}$. Then T has locally finite mass.*

Proof. Let $T = T_{\tilde{\lambda}}$, let $\omega \in \mathcal{D}^k_{\mathrm{c}}(U)$ with $||\omega|| \le 1$, and let $U \subset X$, with $\overline{U}$ compact. We may then write $\omega = \sum_{s\in S} f_s\, dg^1_s \wedge \cdots \wedge dg^k_s$, with $\sum_{s \in S} |f_s| \le 2$, and each $g^i_s \in \mathrm{Lip}_1(U)$. Since $||u_j|| = 1$, equation (3.11) implies that $|\langle u_j, (d_c g^i_s)_p \rangle| = |X^u_p(g^i_s)| \le 1$ for every i and j, so that

$$
|\langle \tilde{u}_a, f d_c g^1_s \wedge \cdots \wedge d_c g^k_s \rangle| \le \binom{n}{k} |f|.
$$

We therefore compute

$$
\begin{aligned}
|T_{\tilde\lambda}(\omega)| &\le \sum_{s\in S} |T(f_s\, dg_s^1 \wedge \cdots \wedge dg_s^k)| \\
&\le \sum_{s\in S} \sum_{a\in\Lambda(k,n)} |T_{\lambda^a \tilde u_a}(f_s\, dg_s^1 \wedge \cdots \wedge dg_s^k) \\
&\le \sum_{s\in S} \sum_{a\in\Lambda(k,n)} \int_U |\lambda^a| \cdot |\langle \tilde u_a, f d_c g_s^1 \wedge \cdots \wedge d_c g_s^k\rangle|\, d\mu \qquad (3.24)\\
&\le \sum_{s\in S} \sum_{a\in\Lambda(k,n)} \int_U |\lambda^a| \cdot \binom{n}{k} |f|\, d\mu \\
&\le \binom{n}{k} \sum_{a\in\Lambda(k,n)} \int_U |\lambda^a|\, d\mu,
\end{aligned}
$$

where the last line is finite as a result of the local integrability of the functions λ^a. □

As is the case with currents, the finite mass condition extends the domain of a precurrent to $\mathcal{E}_c^k(\mathbb{G})$, and to $\tilde{\mathcal{E}}_c^k(\mathbb{G})$.

We define restrictions of precurrents exactly as we did in Section 2.1 for currents. That is, if T is a k-precurrent, and $\omega \in \mathcal{E}^j(\mathbb{G})$, we define the restriction $T\lfloor_\omega$ by equation (2.20).

Recall that given a k-vector $\hat u \in \bigwedge^k \mathfrak{g}$ and a j-covector $\hat w \in \bigwedge^j \mathfrak{g}^*$, there is a unique $k-j$ vector $\hat u\lfloor_{\hat w} \in \bigwedge^{k-j}\mathfrak{g}$ such that for all $\hat z \in \bigwedge^{k-j}\mathfrak{g}^*$, $\langle \hat u\lfloor_{\hat w}, \hat z\rangle = \langle \hat u, \hat w \wedge \hat z\rangle$. Thus

$$
\begin{aligned}
& T_{\tilde\lambda}\lfloor_{\beta\, dh^1\wedge\cdots\wedge dh^j}(f\, dg^1 \wedge \cdots \wedge dg^{k-j}) \\
=\ & \int_{\mathbb{G}} \langle \tilde\lambda, \beta f\, d_c h^1 \wedge \cdots \wedge d_c h^j \wedge d_c g^1 \wedge \cdots \wedge d_c g^k\rangle\, d\mu \\
=\ & \int_{\mathbb{G}} \langle \tilde\lambda\lfloor_{\beta\, d_c h^1\wedge\cdots\wedge d_c h^j}, f\, d_c g^1 \wedge \cdots \wedge d_c g^k\rangle\, d\mu \\
=\ & T_{\tilde\lambda}\lfloor_{\beta\, d_c h^1\wedge\cdots\wedge d_c h^j}(f\, dg^1 \wedge \cdots \wedge dg^{k-j}),
\end{aligned}
$$

and so a restriction of a precurrent is again a precurrent.

As a result of the expansion in equation (3.22), every precurrent T has the form

$$T = T_{\tilde{\lambda}} = \sum_{a\in\Lambda_{k,n}} T_{\lambda^a \tilde{u}_a} = \sum_{a\in\Lambda_{k,n}} T_{\tilde{u}_a} \lfloor_{\lambda_a}. \tag{3.25}$$

Remark 3.26. Note that $T = 0$ if and only if $\lambda^a = 0$ almost everywhere for each $a \in \Lambda_{k,n}$. Indeed, if $\lambda^b \neq 0$ on a set of positive measure, then

$$T_{\tilde{\lambda}} \lfloor_{d\pi_{b_1}\wedge\cdots\wedge d\pi_{b_k}} = \sum_{a\in\Lambda_{k,n}} T_{\tilde{u}_a \lfloor_{\lambda^a d_c\pi_{b_1}\wedge\cdots\wedge d_c\pi_{b_k}}} = T_{\tilde{u}_b \lfloor_{\lambda^b d_c\pi_{b_1}\wedge\cdots\wedge d_c\pi_{b_k}}} = T_{\lambda^b} \neq 0,$$

where λ^b is viewed as a 0-vector field. To prove the last line, assume without loss of generality that $\lambda^b > \epsilon > 0$ on a compact set of positive measure $S \subset \mathbb{G}$, so that we have

$$T_{\lambda^b}(\chi_S) = \int_S \lambda^b > \epsilon\mu(S) > 0.$$

It follows from the previous paragraph that the vector field $\tilde{\lambda}$ in the expansion (3.25) is uniquely determined up to null sets, so that $T_{\tilde{\lambda_1}} = T_{\tilde{\lambda_2}}$ if and only if $\tilde{\lambda_1} = \tilde{\lambda_2}$ almost everywhere.

Smooth forms and smooth restrictions

We have already defined the restriction a current or precurrent by a metric form, or extended form. We now discuss the case where a form is smooth.

Definition 3.27. The elements of the subspaces $\mathcal{S}^k(\mathbb{G}) = C^\infty(\mathbb{G})^{k+1} \subset \mathcal{D}^k(\mathbb{G})$ and $\tilde{\mathcal{S}}^k(\mathbb{G}) = C^\infty(\mathbb{G}) \otimes \bigwedge^k C^\infty(\mathbb{G}) \subset \tilde{\mathcal{D}}^{\mathrm{k}}(\mathbb{G})$ are called **simple smooth forms** and **smooth forms**, respectively.

If $\omega = f\,dg^1 \wedge \cdots \wedge dg^k \in \mathcal{S}^k(\mathbb{G})$, we denote by $\hat{\omega}$ the differential form $f\,d_r g^1 \wedge \cdots \wedge dg^k \in \Omega^k(\mathbb{G})$. Because every differential k-form $\theta \in \Omega^k(\mathbb{G})$ can be written

$$\theta = \sum_{a\in\Lambda(k,\dim(\mathbb{G}))} f_a\, d_r x_{a_1} \wedge \cdots \wedge d_r x_{a_k},$$

the map $\tilde{\mathcal{S}}^k(\mathbb{G}) \to \Omega^k(\mathbb{G})$ given by $\omega \to \hat{\omega}$ is surjective. Here we have implicitly invoked the fact that, as $\mathbb{G}$ is nilpotent, connected, and simply connected, the exponential map $\exp\colon \mathfrak{g} \to \mathbb{G}$ is a diffeomorphism, and $\mathfrak{g}$ in turn is diffeomorphic to $\mathbb{R}^{\dim(\mathbb{G})}$. For an arbitrary manifold, of course, we could prove surjectivity by way of a partition of unity argument.

Note that if $\omega = f\, dg^1 \wedge \cdots \wedge dg^k \in \mathcal{S}^k(X)$, then equation (3.11) implies that for every precurrent $T_{\tilde{\lambda}}$, we have

$$T_{\tilde{\lambda}}\lfloor_{\omega} = T_{\tilde{\lambda}\lfloor_{f\, d_c g^1 \wedge \cdots \wedge d_c g^k}} = T_{\tilde{\lambda}\lfloor_{f\, d_r g^1 \wedge \cdots \wedge d_r g^k}} = T_{\tilde{\lambda}\lfloor_{\hat{\omega}}}.$$

In particular, if T is a precurrent and $\hat{\omega}_1 = \hat{\omega}_2$, then $T\lfloor_{\omega_1} = T\lfloor_{\omega_2}$. With this in mind we define the restriction of a precurrent by a smooth differential form.

Definition 3.28. Let T be a k-precurrent, and $\theta \in \Omega^j(\mathbb{G})$. We define the **restriction of T by θ** to be the $(k-j)$ precurrent

$$T\lfloor_{\theta} = T\lfloor_{\omega},$$

where $\omega \in \tilde{\mathcal{S}}^j(\mathbb{G})$ is any form such that $\hat{\omega} = \theta$. We also define, for $\omega_1 \in \tilde{\mathcal{D}}^{k_1}_{\mathrm{c}}(\mathbb{G})$, $\omega_2 \in \tilde{\mathcal{D}}^{\mathrm{k}_2}(\mathbb{G})$ and $\theta \in \Omega^{k_3}(\mathbb{G})$, with $k_1 + k_2 + k_3 = k$,

$$T(\omega_1 \wedge \theta \wedge \omega_2) = (-1)^{k_1 k_3} T\lfloor_{\theta}(\omega_1 \wedge \omega_2).$$

Finally, we note that the extension to smooth forms applies to currents as well as precurrents. To see this, suppose $\omega_1, \omega_2 \in \tilde{\mathcal{S}}^k(\mathbb{G})$, with $\hat{\omega_1} = \hat{\omega_2} = \theta \in \Omega^k(\mathbb{G})$. Then for any $T \in \mathbf{M}_k(\mathbb{G})$, with $l \leq k$, with absolutely continuous mass $||T||$, T is a precurrent, and so we have

$$T\lfloor_{\omega_1} = T\lfloor_{\omega_2}. \tag{3.29}$$

By Lemma 3.15, equation (3.29) extends to all currents in $\mathbf{M}_k(\mathbb{G})$, making the restriction to θ well-defined.

Normal currents

If T is a k-precurrent, we can define the boundary ∂T as in Definition 2.18. It is not necessarily the case that ∂T is also a precurrent — the proof of Proposition 3.43 will provide a counterexample to this as well. However, boundary continuity is closely related with the question of which precurrents are currents, as the following proposition indicates.

Proposition 3.30. *Let T be a k-precurrent such that ∂T is a $(k-1)$-precurrent. Then $T \in \mathbf{N}_k^{loc}(\mathbb{G})$.*

Proof. Multi-linearity follows from the linearity of the Pansu-differentiation operator; if f and g are Pansu differentiable at p, then so is $f+g$, and $d_c(f+g)_p = d_c f_p + d_c g_p$. Similarly, locality follows from locality of the Pansu differential; if f is constant on an open set $U \subset \mathbb{G}$, then $d_c g_p = 0$ for $p \in U$. We have already shown precurrents have locally finite mass in Lemma 3.23 above.

All that remains is to check that T is continuous. We wish to show that for every sequence of forms $f_i\, dg_i^1 \wedge \cdots \wedge dg_i^k$ converging to $f\, dg^1 \wedge \cdots \wedge dg^k \in \mathcal{D}_c^k(\mathbb{G})$, we have

$$\lim_{i\to\infty} T(f_i\, dg_i^1 \wedge \cdots \wedge dg_i^k) = T(f\, dg^1 \wedge \cdots \wedge dg^k). \tag{3.31}$$

We first decompose the limit in (3.31).

$$\begin{aligned} &\lim_{i\to\infty} T(f_i\, dg_i^1 \wedge \cdots \wedge dg_i^k) \\ = \;&\lim_{i\to\infty} T(f_i - f\, dg_i^1 \wedge \cdots \wedge dg_i^k) + \lim_{i\to\infty} T(f\, d(g_i^1 - g^1) \wedge dg_i^2 \wedge \cdots \wedge dg_i^k) \\ &+ \lim_{i\to\infty} T(f\, dg^1 \wedge dg_i^2 \cdots \wedge dg_i^k). \end{aligned} \tag{3.32}$$

By the locally finite mass condition mentioned above for T, the first term on the second line of equation (3.32) is 0. By way of the Leibniz rule, we next compute

$$\begin{aligned} & \lim_{i\to\infty} T(f\, d(g_i^1 - g^1) \wedge dg_i^2 \wedge \cdots \wedge dg_i^k) \\ = \;& \lim_{i\to\infty} \partial T(f(g_i^1 - g^1)\, dg_i^2 \wedge \cdots \wedge dg_i^k) - \lim_{i\to\infty} T((g_i^1 - g^1)\, df \wedge dg_i^2 \wedge \cdots \wedge dg_i^k) \\ = \;& 0. \end{aligned}$$

Here both terms in the second line vanish on account of the locally finite mass condition, since T and ∂T are both precurrents.

Equation (3.32) then reduces to

$$\lim_{i\to\infty} T(f_i\, dg_i^1 \wedge \cdots \wedge dg_i^k) = T(f\, dg^1 \wedge dg_i^2 \wedge \cdots \wedge dg_i^k).$$

Moreover, by the alternating property, from equation (3.33) we deduce that for $1 \leq j \leq k$,

$$\lim_{i\to\infty} T(f_i\, dg_i^1 \wedge \cdots \wedge dg_i^k) = T(f\, dg_i^1 \wedge \cdots dg_i^{j-1} \wedge dg^j \wedge dg_i^{j+1} \wedge \cdots \wedge dg_i^k). \tag{3.33}$$

Applying equation (3.33) for $j = 1, \ldots, \infty$ yields equation (3.31). □

1-currents

Though we will see shortly that a precurrent need not be a current, 1-precurrents are always currents.

Lemma 3.34. *Every* 1*-precurrent in* $\mathbb{G}$ *is a current.*

Lemma 3.34 follows from a simple observation, which will itself be of use momentarily, in the proof of Proposition 3.43.

Lemma 3.35. *Let* $\mathbb{G}$ *be a Carnot group. Then an invariant* 1*-precurrent has vanishing boundary.*

Proof. We must show that for any $f \in \mathrm{Lip_c}(X)$, and any $u \in \mathfrak{h}$,

$$T_u(df) = \int_{\mathbb{G}} X^u(f) = 0.$$

Without loss of generality, we will assume $||u|| = 1$. Since $\mathbb{G}$ is unimodular, we recall from the theory of topological groups (see, e.g, [29], Theorem 6.18) that for any unimodular subgroup $S \subset \mathbb{G}$, with Haar measure μ_S, there is a left-invariant measure $\mu_{\mathbb{G}/S}$ on the quotient $\mathbb{G}/S$ such that for any $g \in C_c(\mathbb{G})$, we have

$$\int_{\mathbb{G}} g\, d\mu = \int_{\mathbb{G}/S} \left(\int_S g(ps)\, d\mu_S(s) \right) d\mu_{\mathbb{G}/S}(pS). \tag{3.36}$$

We apply equation (3.36) with $g = X^u(f)$, $S = \exp(\mathrm{Span}\{u\})$, and $d\mu_S = ds$, where ds is the arc-length measure. Notice that since $||u|| = 1$, the map $t \to \exp(tu)$ is an isometry. We thus have

$$\int_{t\in\mathbb{R}} X^u_{(p\cdot\exp(tu))}(f)\, ds(t) = \int_{\mathbb{R}} X^u_{(p\cdot\exp(tu))}(f)\, dt = \int_{\mathbb{R}} X^u_{\exp(tu)}(f\circ\tau_p)\, dt = \int_{\mathbb{R}} f\circ\tau_p\, dt = 0,$$

since $f \circ \tau_p$ has compact support. Thus by equation (3.36) we have

$$\int_{\mathbb{G}} g\, d\mu = 0.$$

□

Proof of Lemma 3.34. It is enough to show that invariant 1-precurrents are actually currents. Indeed, once we have proved this, we see that each of the precurrents T_{u_i} are 1-currents. But restricting a 1-current by a function or form, as in Definition 2.19, gives us another current. Thus every precurrent of the form $T = \sum_{i=1}^n T_{u_i} \lfloor_{\lambda^i}$ is a current. By equation (3.25), every 1-precurrent has this form.

The proof now follows from Lemma 3.35 and Proposition 3.30, since an invariant 1-precurrent T has boundary $\partial T = 0$, and is thus a current. □

Though we will see that precurrents need not be currents, the following corollary to Lemma 3.34 shows that precurrents are separately continuous in each variable.

Corollary 3.37. *Let* $\omega_i = f dg^1 \wedge \cdots \wedge dg^{j-1} \wedge dg^j_i \wedge dg^{j+1} \wedge \cdots \wedge dg^k \in \mathcal{D}^k_c(\mathbb{G})$*, and* $\omega = f\, dg^1 \wedge \cdots \wedge dg^k \in \mathcal{D}^k_c(\mathbb{G})$*, and suppose* g^j_i *converges to* g^j *in the topology of* $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{G})$*. Then for every* k*-precurrent* T*,*

$$\lim_{i\to\infty} T(\omega_i) = T(\omega). \tag{3.38}$$

Equation (3.38) *holds as well for the case* $j = 0$*, if* f_i *converges to* f *in the topology of* $\mathrm{Lip_c}(X)$*, where* $f_i = g^0_i$*, and* $f = g^0$*. If* T *is a* $(k+1)$*-precurrent, then equation* (3.38) *holds when* T *is replaced by* ∂T

Proof. If T is a k-precurrent, then the restriction of T to a metric$(j-1)$-form is a 1-precurrent, and hence a current by Lemma 3.34. Thus

$$\begin{aligned}
\lim_{i\to\infty} T(\omega_i) &= \lim_{i\to\infty} (-1)^{k-j} T\lfloor_{dg^1\wedge\cdots\wedge dg^{j-1}\wedge dg^{j+1}\wedge\cdots\wedge dg^k}(f\, dg^j_i) \\
&= (-1)^{k-j} T\lfloor_{dg^1\wedge\cdots\wedge dg^{j-1}\wedge dg^{j+1}\wedge\cdots\wedge dg^k}(f\, dg^j) \\
&= T(\omega).
\end{aligned}$$

The continuity in the variable f follows from the same argument. Moreover, continuity in the variable f already follows from the locally finite mass condition used in the proof of Proposition 3.30. The argument for ∂T is identical. □

As a consequence of Corollary 3.37 and Lemma 3.7, two precurrents are equal if they are equal when evaluated on smooth forms, and similarly for boundaries of precurrents.

Corollary 3.39. *Suppose* T_1 *and* T_2 *are* k*-precurrents, and that*

$$T_1(\omega) = T_2(\omega) \tag{3.40}$$

for any smooth form $\omega \in \mathcal{S}_c^k(\mathbb{G})$*. Then* $T_1 = T_2$*. Similarly, if* $\partial T_1(\omega') = \partial T_2(\omega')$ *for every* $\omega' \in \mathcal{S}_c^{k-1}(\mathbb{G})$*, then* $\partial T_1 = \partial T_2$*.*

Proof. Suppose there is a number j, $0 \leq j \leq k+1$, such that equation (3.40) holds whenever $\omega = f\, dg^1 \wedge \cdots \wedge dg^k$, where $g^m \in C^\infty(\mathbb{G})$ whenever $j \leq m \leq k$ (letting $g^0 = f$). We claim then that the same is true for $j+1$. Indeed, by Lemma 3.7, there is a sequence of smooth functions g_i^j converging to g^j in $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{G})$. Then by Corollary 3.37, we have

$$\partial T(f \wedge dg^1 \wedge \cdots \wedge dg^k) = \lim_{i \to \infty} T(df \wedge dg^1 \wedge \cdots \wedge dg_i^j \cdots \wedge dg^k) = 0.$$

The result now follows by induction on j, as the case $j = 0$ is true by hypothesis, and the case $j = k+1$ is a restatement of the corollary. The last statement is proved with the exact same argument. □

3.4 Invariant currents.

In this section we prove Proposition 3.43, which characterizes translation invariant currents. From the definition, a precurrent $T = T_{\tilde{\lambda}}$ is invariant if and only if $\tilde{\lambda} \circ \tau_p = \tilde{\lambda}$ almost everywhere, which in turn occurs if and only if $\tilde{\lambda}$ is constant almost everywhere.

To formulate Proposition 3.43, we will need the notion of a "vertical form". We will call a smooth 1-form $\theta \in \Omega^1(\mathbb{G})$ *vertical* if $T\lfloor_\theta = 0$ for every k-precurrent T. Equivalently, θ is vertical if and only if θ annihilates every horizontal vector field, i.e., $\langle \theta, X^u \rangle \equiv 0$ for every $u \in H$.

Example 3.41. In the n^{th} Heisenberg group $\mathbb{H}^n$, the basis $X_1, \ldots, X_n, Y_1, \ldots, Y_n$, Z has a dual basis consisting of forms $dx_1, \ldots, dx_n$, $dy_1, \ldots, dy_n$, θ. The form θ is a vertical form, as it vanishes when paired with every horizontal vector field. θ is sometimes called the *contact form*, as $(\mathbb{H}^n, \theta)$ is a *contact manifold*, meaning that $\theta \wedge (d\theta)^{\wedge n}$ is a volume form on $\mathbb{H}^n$.

It can be shown [37] that θ and $d\theta$ generate $\Omega^k(\mathbb{H}^n)$ for $k > n$; that is, every $\omega \in \Omega^k(\mathbb{H}^n)$ has the form $\omega = \alpha \wedge \theta + \beta \wedge d\theta$.

The following lemma describes the push-forward of an invariant precurrent along the dilation maps Δ_r.

Lemma 3.42. *Let $T = T_{\tilde{\lambda}}$ be an invariant k-precurrent. Then $\Delta_{r\#}T = r^{k-Q}T$.*

Proof. We compute, via equations (3.5) and (3.12),

$$\begin{aligned}
\Delta_{r\#}T(f\, dg^1 \wedge \cdots \wedge dg^k) &= \int_{\mathbb{G}} \langle \tilde{\lambda}, f(\Delta_r(p))\, d_c(g^1 \circ \Delta_r)_p \wedge \cdots \wedge d_c(g^k \circ \Delta_r)_p \rangle\, d\mu(p) \\
&= r^k \int_{\mathbb{G}} \langle \tilde{\lambda}, f(\Delta_r(p))\, d_c g^1_{\Delta_r(p)} \wedge \cdots \wedge d_c g^k_{\Delta_r(p)} \rangle\, d\mu(p) \\
&= r^k \int_{\mathbb{G}} \langle \tilde{\lambda}, f(p)\, d_c g^1_p \wedge \cdots \wedge d_c g^k_p \rangle\, d\Delta_{r\#}\mu(p) \\
&= r^{k-Q} \int_{\mathbb{G}} \langle \tilde{\lambda}, f(p)\, d_c g^1_p \wedge \cdots \wedge d_c g^k_p \rangle\, d\mu(p) \\
&= r^{k-Q} T(f\, dg^1 \wedge \cdots \wedge dg^k).
\end{aligned}$$

□

Proposition 3.43. *Let T be an invariant k-precurrent in a Carnot group $\mathbb{G}$. The following statements are equivalent:*

1. *T is a current.*

2. *$\partial T = 0$.*

3. *$T \lfloor_{d\theta} = 0$ for every vertical 1-form $\theta \in \Omega^k(\mathbb{G})$.*

4. *$T \lfloor_{d\theta} = 0$ for every invariant vertical 1-form $\theta \in \Omega^k(\mathbb{G})$.*

Proof. We will prove $1 \Leftrightarrow 2$, and $2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1$.

$1 \Rightarrow 2$: Suppose T is an invariant k-current. We wish to show that $\partial T = 0$, that is, for $g^1\, dg^2 \wedge \cdots \wedge dg^k \in \mathcal{D}^{k-1}_{\mathrm{c}}(X)$, $T(dg^1 \wedge \cdots \wedge dg^k) = 0$.

We may assume without loss of generality that $T(dg^1 \wedge \cdots \wedge dg^k) \geq 0$. We also assume that each function g_i has compact support, so that each g^i is supported in some ball $B_R(0)$ centered at the origin.

For every $\epsilon > 0$, we define the rescaled functions g^i_ϵ by

$$g^i_\epsilon(p) = \epsilon g^i \circ \delta_{\frac{1}{\epsilon}},$$

and note that g^i_ϵ is supported on $B_{R/\epsilon}$ and has the same Lipschitz constant as g^i. Also, we let $N \subset B_{R/2}(0)$ be a maximal $4\epsilon R$-separated subset of the ball $B_{R/2}(0)$. By the Q-regularity of $\mathbb{G}$, $\#N \geq C\epsilon^{-Q}$. We define the functions $\hat{g}^i_\epsilon$ by

$$\hat{g}^i_\epsilon = \sum_{p \in N} g^i_\epsilon \circ \tau_p.$$

Again, we note that $\hat{g}^i_\epsilon$ has the same Lipschitz constant as g^i. Moreover, for $p, q \in N$, $p \neq q$, $\mathrm{Spt}(g_\epsilon \circ \tau_p) \cap \mathrm{Spt}(g_\epsilon \circ \tau_p) = \emptyset$, and so by the invariance of T under left translations, we have

$$T(dg^1_\epsilon \circ \tau_{p_1} \wedge \cdots \wedge dg^k_\epsilon \circ \tau_{p_k}) = \begin{cases} T(dg^1_\epsilon \wedge \cdots \wedge dg^k_\epsilon) & \text{if } p_1 = \cdots = p_k, \\ 0 & \text{otherwise.} \end{cases}$$

But now, with the help of Lemma 3.42, we compute

$$\begin{aligned}
T(d\hat{g}^1_\epsilon \wedge \cdots \wedge d\hat{g}^k_\epsilon) &= \sum_{p \in N} T(dg^1_\epsilon \wedge \cdots \wedge dg^k_\epsilon) \\
&= \#N \cdot T(dg^1_\epsilon \wedge \cdots \wedge dg^k_\epsilon) \\
&= \#N \cdot \epsilon^k T(dg^1 \circ \delta_{\frac{1}{\epsilon}} \wedge \cdots \wedge dg^k \circ \delta_{\frac{1}{\epsilon}}) \\
&= \#N \cdot \epsilon^k \delta_{\frac{1}{\epsilon}\#} T(dg^1 \wedge \cdots \wedge dg^k) \\
&= \#N \cdot \epsilon^Q T(dg^1 \wedge \cdots \wedge dg^k) \\
&\geq CT(dg^1 \wedge \cdots \wedge dg^k).
\end{aligned}$$

As ϵ approaches 0, the functions $\hat{g}^i_\epsilon$ converge to 0 uniformly and with bounded Lipschitz constant, so the last expression must approach 0 by continuity of T. Since the last expression is independent of ϵ, $T(dg^1 \wedge \cdots \wedge dg^k) = 0$.

$2 \Rightarrow 1$: This follows immediately from Proposition 3.30.

$2 \Rightarrow 3$: If $\partial T = 0$, then for any $f\, dg^1 \wedge \cdots \wedge dg^{k-2} \in \mathcal{S}^{k-2}_{\mathrm{c}}(\mathbb{G})$ and any $\theta \in \mathcal{D}^k_{\mathrm{c}}(\mathbb{G})$, we have

$$T\lfloor_{d\theta}(f\, dg^1 \wedge \cdots \wedge dg^{k-2}) = T(f\, d\theta \wedge dg^1 \wedge \cdots \wedge dg^{k-2})$$
$$= T(d(f\theta) \wedge dg^1 \wedge \cdots \wedge dg^{k-2}) - T(df \wedge \theta \wedge dg^1 \wedge \cdots \wedge dg^{k-2})$$
$$= \partial T(f\theta \wedge dg^1 \wedge \cdots \wedge dg^{k-2}) + T\lfloor_{\theta}(df \wedge dg^1 \wedge \cdots \wedge dg^{k-2}) = 0.$$

Here the second term vanishes because θ is vertical.

$3 \Rightarrow 4$ is clear.

$4 \Rightarrow 2$: Let $T = T_{\tilde{\lambda}} = \sum_{a \in \Lambda(k,n)} T_{\lambda^a \tilde{u}_a}$, where here, since T is invariant, each λ^a is constant. For $\omega = f\, dg^1 \wedge \cdots \wedge dg^k \in \mathcal{S}^k_{\mathrm{c}}(\mathbb{G})$, and each $a \in \Lambda(k,n)$, we compute

$$\partial T_{\tilde{u}_a}(\omega) = T_{\tilde{u}_a}(d\omega) = \int_{\mathbb{G}} \langle \tilde{u}_a, d\omega \rangle$$
$$= \int_{\mathbb{G}} \sum_{i=1}^{k} (-1)^{i-1} X^{u_{a_i}} \left(\langle u_{a_1} \wedge \cdots \wedge \widehat{u_{a_i}} \wedge \cdots \wedge u_{a_k}, \omega \rangle \right) d\mu$$
$$+ \int_{\mathbb{G}} \sum_{i<j} (-1)^{i+j} \langle [u_{a_i}, u_{a_j}] \wedge u_{a_1} \cdots \wedge \widehat{u_{a_i}} \wedge \cdots \wedge \widehat{u_{a_j}} \wedge \cdots \wedge u_{a_k}, \omega \rangle d\mu$$
$$= \sum_{i=1}^{k} (-1)^{i-1} \partial T_{u_{a_i}} \left(\langle u_{a_1} \wedge \cdots \wedge \widehat{u_{a_i}} \wedge \cdots \wedge u_{a_k}, \omega \rangle \right)$$
$$+ \sum_{i<j} (-1)^{i+j} \int_{\mathbb{G}} \langle [u_{a_i}, u_{a_j}] \wedge u_{a_1} \cdots \wedge \widehat{u_{a_i}} \wedge \cdots \wedge \widehat{u_{a_j}} \wedge \cdots \wedge u_{a_k}, \omega \rangle d\mu.$$

See [31], e.g., for the expansion in the second and third lines; here, as in [31], the symbol "$\widehat{\ }$" above a vector means that vector should be omitted.)

By Lemma 3.35, $\partial T_{u_{a_i}} = 0$ for all i, so the first sum in the last line vanishes. Since $[u_{a_i}, u_{a_j}] \in V_2$ for all i and j, expanding the second sum in the last line shows that the boundary $\partial T_{\tilde{u}_a}$ satisfies

$$\partial T_{\tilde{u}_a}(\omega) = \sum_{b\in\Lambda_{n,k-2}} \int_{\mathbb{G}} \langle v_b \wedge u_{b_1} \wedge \cdots \wedge u_{b_{k-2}}, \omega\rangle, \tag{3.44}$$

where each $v_b = v_b(a) \in V_2$. Again, this holds for every $\omega \in \mathcal{S}^k_{\mathrm{c}}(\mathbb{G})$. Of course, the vectors v_b do not depend on the choice of ω.

Since $T = \sum_{b\in\Lambda(k,n)} \lambda^b T_{\tilde{u}_b}$, and the boundary operator is linear, it follows that $\partial T(\omega)$ can be written in the form of equation (3.44).

$$\partial T(\omega) = \sum_{b\in\Lambda_{n,k-2}} \int_{\mathbb{G}} \langle v_b \wedge u_{b_1} \wedge \cdots \wedge u_{b_{k-2}}, \omega\rangle. \tag{3.45}$$

In other words, ∂T, when applied to a smooth form, is given by integrating the form against an invariant $(k-1)$-vector field in $V_2 \wedge \left(\bigwedge^{k-2} H\right)$.

Now suppose $T\lfloor_{d\theta} = 0$ for every smooth invariant 1-form $\theta \in \Omega^1(\mathbb{G})$. Then for every such θ, every $a \in \Lambda_{n,k-2}$, and every $f \in C^\infty_{\mathrm{c}}(\mathbb{G})$, we have

$$\begin{aligned}
0 &= T\lfloor_{d\theta}\lfloor_{d\pi^{a_1}\wedge\cdots\wedge d\pi^{a_{k-2}}}(f) \\
&= (\partial T\lfloor_{\theta} + \partial(T\lfloor_{\theta}))\lfloor_{d\pi^{a_1}\wedge\cdots\wedge d\pi^{a_{k-2}}}(f) \\
&= \partial T(f\theta \wedge d\pi^{a_1} \wedge \cdots \wedge d\pi^{a_{k-2}}) \\
&= \sum_{b\in\Lambda_{n,k-2}} \int_{\mathbb{G}} f\langle v_b \wedge u_{b_1} \wedge \cdots \wedge u_{b_{k-2}}, \theta \wedge d_r\pi^{a_1} \wedge \cdots \wedge d_r\pi^{a_{k-2}}\rangle \, d\mu \\
&= \sum_{b\in\Lambda_{n,k-2}} \int_{\mathbb{G}} f\delta^b_a \langle v_b, \theta\rangle \\
&= \int_{\mathbb{G}} f\langle v_a, \theta\rangle.
\end{aligned}$$

Since this holds for all f, and in particular, any nonzero, nowhere negative $f \in C^\infty_{\mathrm{c}}(\mathbb{G})$, we have $\langle v_a, \theta\rangle = 0$. If $v_a \neq 0$, then $v_a \notin H$, so there is an invariant vertical 1-form θ such that $\langle v_a, \theta\rangle \neq 0$, a contradiction. Thus $v_a = 0$. This holds for all $a \in \Lambda(k, n)$, so by equation (3.45), we have $\partial T(\omega) = 0$ for all $\omega \in \mathcal{S}^{k-1}(\mathbb{G})$. Therefore, by Corollary 3.39, $\partial T = 0$. □

3.5 General currents in Carnot groups.

In this section we prove Theorem 1.9. We need to relate arbitrary precurrents to invariant ones, and so we introduce a kind of tangent approximation. Let $T = T_{\tilde{\lambda}}$ be a k-precurrent. At a given point $p \in \mathbb{G}$, we define the pointwise current T^p by the equation

$$T^p = T_{\tilde{\lambda}_p}.$$

Note that T^p is well-defined up to sets of measure 0.

Lemma 3.46. *A k-precurrent T is a current if and only T^p is a current for almost every $p \in \mathbb{G}$.*

Proof. Let $T = T_{\tilde{\lambda}}$, and suppose first that for almost every $p \in \mathbb{G}$, T^p is a current. Now suppose that p is a Lebesgue point of each function λ^a for $a \in \Lambda(k,n)$. Note that since each λ^a is locally integrable, almost every $p \in \mathbb{G}$ satisfies this condition. For every $\epsilon > 0$, there is a number $R = R(\epsilon, p) > 0$ such that for $0 \leq r \leq R$, we have

$$\sum_{a \in \Lambda(k,n)} \int_{B_r(p)} |\lambda^a - \lambda^a_p| \, d\mu \leq \epsilon \mu(B_r(p)).$$

Thus by equation 3.24,

$$||T - T^p||(B_r(p)) \leq \binom{n}{k} \sum_{a \in \Lambda(k,n)} \int_{B_r(p)} |\lambda^a - \lambda^a_p| \, d\mu \leq \binom{n}{k} \epsilon \mu(B_r(p)). \tag{3.47}$$

Since equation (3.47) holds for almost every $p \in \mathbb{G}$, and every $r < R(\epsilon, p)$, by the Vitali Covering Theorem, there is a countable pairwise disjoint collection of balls $B_i = B_{r_i}(p_i)$ such that $\mathbb{G} \backslash \bigcup_{i=1}^{\infty} B_i = 0$, and such that $r_i \leq \min(\epsilon, R(\epsilon, p_i))$.

Let $T_\epsilon = \sum_{i=1}^{\infty} T^{p_i} \lfloor_{B_{r_i}(p_i)}$. We claim this sum converges locally in mass. Indeed, given a relatively compact subset $U \subset \mathbb{G}$, let $U_\epsilon = \{q \in \mathbb{G} : \mathrm{dist}(U, q) < \epsilon\}$. Then

$$\begin{aligned} \sum_{i=1}^{\infty} ||T^{p_i}\lfloor_{B_{r_i}(p_i)}||(U) &\leq \sum_{p_i \in U_\epsilon} ||T^{p_i}\lfloor_{B_{r_i}(p_i)}||(U) \\ &\leq \sum_{p_i \in U_\epsilon} ||T^{p_i}||(B_{r_i}(p_i)) \\ &\leq \sum_{p_i \in U_\epsilon} (||T||(B_{r_i}(p_i)) + \epsilon\mu(B_{r_i}(p_i))) \\ &\leq ||T||(U_\epsilon) + \epsilon\mu(U_\epsilon), \end{aligned}$$

and so the sum converges.

Moreover, we have

$$\begin{aligned} ||T_\epsilon - T||(U) &\leq \sum_{p_i \in U_\epsilon} ||T - T^{p_i}||(B_{r_i}(p_i)) \\ &\leq \epsilon\mu(U_\epsilon). \end{aligned}$$

Thus $T_\epsilon\lfloor_U$ converges to $T\lfloor_U$ in mass as ϵ approaches 0. Since each $T_\epsilon\lfloor_U$ is a current, $T\lfloor_U$ is also a current, by the completeness of the space of currents in the mass norm. Being a current is a local property (indeed, the convergence axiom is satisfied for T if and only if it is satisfied for $T\lfloor_U$ for every relatively compact open set $U \subset \mathbb{G}$), and so T is a current.

Conversely, suppose that T is a current, and let p be a Lebesgue point as above. Let Δ_t^p denote the dilation centered at point p, that is $\Delta_t^p = \tau_p \circ \Delta_t \circ \tau_{p^{-1}}$. From Lemma 3.42, and the invarance of T^p under translations, we have $\Delta^p_{r\#}T = r^{k-Q}T$. Thus by equation (3.47),

$$\begin{aligned} ||(r/R)^{k-Q}\Delta^p_{r/R\#}T - T^p||(B_R(p)) &= (r/R)^{k-Q}||\Delta^p_{(r/R)^{-1}\#}(T - T^p)||(B_R(p)) \\ &= (r/R)^{k-Q}(r/R)^{-k}||T - T^p||(B_r(p)) \\ &\leq (r/R)^{-Q}\epsilon\mu(B_r(p)) \\ &= C\epsilon R^Q. \end{aligned}$$

Thus the currents $(r/R)^{k-Q}\Delta^p_{r/R\#}T$ converge locally in mass to T^p, and therefore T^p is a current. $\square$

We are now ready to prove Theorem 1.9.

Proof of Theorem 1.9. Let $T = T_{\tilde{\lambda}}$ be a k-precurrent. If T is a current, then by Lemma 3.46, T^p is an invariant current for almost every $p \in \mathbb{G}$. Let T_ϵ be as in the proof of Lemma 3.46. Then for any $\omega \in \mathcal{D}^{k-2}_{\mathrm{c}}(\mathbb{G})$, by Proposition 3.43 we have

$$T_\epsilon \lfloor_{d\theta}(\omega) = \sum_{i=1}^{\infty} T^{p_i} \lfloor_{B_{r_i}(p_i)} \lfloor_{d\theta}(\omega) = \sum_{i=1}^{\infty} T^{p_i} \lfloor_{d\theta} \lfloor_{B_{r_i}(p_i)}(\omega) = 0.$$

Since T_ϵ converges locally in mass, and hence weakly, to T, we have $T\lfloor_{d\theta}(\omega) = 0$.

Conversely, suppose T is a k-precurrent, with $T\lfloor_{d\theta}= 0$ for every vertical form θ. Notice that if θ is vertical, then so is $\Delta^{p*}_t\theta$, since translations and dilations both respect the horizontal bundle. Thus we have

$$(\Delta^p_{t\#}T)\lfloor_{d\theta} = \Delta^p_{t\#}(T\lfloor_{\Delta^{p*}_t d\theta}) = \Delta^p_{t\#}(T\lfloor_{d\Delta^{p*}_t\theta}) = 0.$$

Since, as shown in the proof of Lemma 3.46, the currents $(r/R)^{k-Q}\Delta_{r/R\#}T$ converge locally in mass to T^p, it follows that $T^p\lfloor_{d\theta}= 0$. Thus by Proposition 3.43, T^p is a current, and by Lemma 3.46, so is T.

For the last statement of the Theorem, suppose that $T \in \mathbf{M}^{\mathrm{loc}}_k(\mathbb{G})$, and $\theta \in \Omega^1(\mathbb{G})$, is vertical. By Lemma 3.15, T can be approximated weakly by currents T_ϵ of absolutely continuous mass. Since each T_ϵ is both a current and a precurrent (by Theorem 1.8), $T_\epsilon\lfloor_\theta= 0$, and by the first part of the theorem, $T_\epsilon\lfloor_{d\theta}= 0$. Thus we have

$$T\lfloor_{d\theta}(\omega) = T(d\theta \wedge \omega) = \lim_{\epsilon\to 0} T_\epsilon(d\theta \wedge \omega) = \lim_{\epsilon\to 0} T_\epsilon\lfloor_{d\theta}(\omega) = 0,$$

and the same computation shows that $T\lfloor_\theta(\omega) = 0$. $\square$

Corollary 3.48. *The n^{th} Heisenberg group $\mathbb{H}^n$ admits no nonzero currents of dimension greater than n.*

Proof. The contact form θ is a vertical form in $\mathbb{H}^n$, and so for every $T \in \mathbf{M}_k^{\mathrm{loc}}(\mathbb{G})$, $T\lfloor_\theta = 0$ and $T\lfloor_{d\theta} = 0$, by Theorem 1.9. For $k > n$, as discussed in Example 3.41, every differential k-form $\omega \in \Omega^k(\mathbb{G})$ can be written $\omega = \alpha \wedge \theta + \beta \wedge d\theta$, so we have $T(\omega) = 0$. Then by Corollary 3.39, $T = 0$. □

Remark 3.49. Let $\mathbb{G}_{CC} = (\mathbb{G}, \mathrm{dist_{CC}})$, $\mathbb{G}_R = (\mathbb{G}, \mathrm{dist_R})$. Since $\mathrm{dist_{CC}} > \mathrm{dist_R}$, the identity map $I \colon \mathbb{G}_{CC} \to \mathbb{G}_R$ is 1-Lipschitz. As a consequence of Theorem 1.9, the push-forward operator induces a map $I_\# \colon \mathbf{M}_k^{\mathrm{loc}}(\mathbb{G}_{CC}) \to \mathbf{M}_k^{\mathrm{loc}}(\mathbb{G}_R)^H$, where

$$\mathbf{M}_k^{\mathrm{loc}}(\mathbb{G}_R)^H = \{T \in \mathbf{M}_k^{\mathrm{loc}}(\mathbb{G}_R) : T\lfloor_\theta = 0 \text{ and } T\lfloor_{d\theta=0}\},$$

so that $||I_\# T|| \leq ||T||$ for any $T \in \mathbf{M}_k^{\mathrm{loc}}(\mathbb{G})$. Moreover, if $I_\# T = 0$, then $T(\omega) = 0$ for every $\omega \in \Omega^k\,\mathrm{c}(\mathbb{G})$, and so by Lemma 3.39, $T = 0$. Thus $I_\#$ is an embedding.

Remark 3.50. The embedding $I_\#$ discussed in Remark 3.49 can be interpreted in the context of other theories of currents as well.

Let $\mathbf{F}_k(\mathbb{G})$ denote the space of currents in the sense of de Rham, Federer and Fleming. That is $T \in \mathbf{F}_k(\mathbb{G})$ if $T \colon \Omega^k(\mathbb{G}) \to \mathbb{R}$ is a linear functional on differential k-forms that is continuous in the topology of uniform convergence in all partial derivatives. While [11] and [8] deal with currents in Euclidean spaces, a Carnot group $\mathbb{G}$ is diffeomorphic (and, given an invariant Riemannian metric, locally bi-Lipschitz equivalent) to $\mathbb{R}^{\dim(\mathbb{G})}$, and so their theory applies just as well to $\mathbb{G}$.

In [1] and [32], an embedding $F \colon \mathbf{M}_k^{\mathrm{loc}}(\mathbb{G}_R) \to \mathbf{F}_k(\mathbb{G})$ is exhibited, which, in our language, has the property that $F(T)(\omega) = T(\omega)$ for any $\omega \in \Omega^k(\mathbb{G})$. There, as well, only Euclidean currents are considered, but as it is a Riemannian manifold, $\mathbb{G}_R$ is locally bi-Lipschitz equivalent to $\mathbb{R}^{\dim(\mathbb{G})}$, and so the push-forward operator along such an equivalence induces an isomorphism between $\mathbf{M}_k^{\mathrm{loc}}(\mathbb{G}_R)$ and $\mathbf{M}_k^{\mathrm{loc}}(\mathbb{R}^{\dim(\mathbb{G})})$.

By Remark 3.49, then, $F \circ I_\#\colon \mathbf{M}_k^{\mathrm{loc}}(\mathbb{G}_{CC}) \to \mathbf{F}_k(\mathbb{G})^H$ is an embedding. Here $\mathbf{F}_k(\mathbb{G})^H$ denotes the space

$$\mathbf{F}_k(\mathbb{G})^H = \{T \in \mathbf{F}_k(\mathbb{G}_R) : T\lfloor_\theta = 0 \text{ and } T\lfloor_{d\theta=0}\}.$$

In [12], Franchi, Serapioni, and Serra Cassano defined currents in the Heisenberg groups $\mathbb{H}^n$ based on Rumin's complex of differential forms [37]. They show that in low dimensions ($k \leq n$), their space $\mathcal{D}_{\mathbb{H},k}$ of k-dimensional "Heisenberg currents" is naturally isomorphic to $\mathbf{F}_k(\mathbb{H}^n)^H$, and thus we have an embedding of $\mathbf{M}_k^{\mathrm{loc}}(\mathbb{H}^n)$ into $\mathcal{D}_{\mathbb{H},k}$.

3.6 Rectifiability

We interpret our results in the context of rectifiable sets in metric spaces.

Definition 3.51. A metric space X is called k**-rectifiable** if it is the union of countably many Lipschitz images of subsets of $\mathbb{R}^k$ and an $\mathcal{H}^k$-null set. That is,

$$X = \left(\bigcup_i F_i(A_i)\right) \cup N$$

where each $A_i \subseteq \mathbb{R}^k$, $F_i\colon A_i \to X$ is Lipschitz, and $\mathcal{H}^k(N) = 0$. If every k-rectifiable subset S of a space X is trivial (i.e. $\mathcal{H}^k(S) = 0$), X is said to be **purely** k**-unrectifiable**.

Ambrosio and Kirchheim studied rectifiable sets in metric spaces in [2], continuing earlier work by Kirchheim in [28]. With the help of an area formula and a metric differentiation theorem developed in [28], they proved that one can take the maps F_i in Definition 3.51 to be bi-Lipschitz. This immediately implies that a nontrivial k-

rectifiable set must admit nonzero metric k-currents, as one can simply push forward a Euclidean current from one of the sets A_i.

We now examine some consequences of our results in terms of rectifiability. First, Corollary 2.36 has immediate implications for the dimension of a rectifiable subset of a space admitting a differentiable structure.

Corollary 3.52. *Let $X = (X, d, \mu)$ be a proper, doubling, metric measure space admitting a differentiable structure of dimension n. Then there is subset $N \subset X$, with $\mu(N) = 0$, such that $X \backslash N$ is purely k-unrectifiable for any $k > n$.*

The area formula and metric differential are also used in [2] to prove the following theorem. Though stated there for $n = 1$, the proof given in [1] extends to the general case.

Theorem 3.53 (Ambrosio-Kirchheim)**.** *The Heisenberg group $\mathbb{H}^n = (\mathbb{H}^n, \mathrm{dist}_{CC})$ is purely k-unrectifiable for $k > n$.*

In light of the fact that one can use bi-Lipschitz maps in the definition of rectifiability, it is clear that Theorem 3.53 can also be viewed as a consequence of Corollary 3.48. On the one hand, this argument for unrectifiability is not much different from the one in [2], in that it uses the same ingredients, namely, Pansu's differentiation theorem and the area formula. On the other hand, the method of proof by way of currents uses the area formula solely for the purpose of using bi-Lipschitz maps in Definition 3.51. Moreover, this argument relies on differentiation of maps from $\mathbb{H}^1$ into Euclidean spaces, rather than vice-versa. Thus no analysis of the metric differential of any map *into* $\mathbb{H}^1$ is required. Instead, one computes the Cheeger differential of a map from $\mathbb{H}^1$ into a Euclidean space.

In the case of general Carnot groups, the rectifiability question was answered by

Magnani in [34]:

Theorem 3.54 (Magnani)**.** *A Carnot group $\mathbb{G}$ is purely k-unrectifiable if and only if every horizontal Abelian subalgebra of its Lie algebra $\mathfrak{g}$ has rank less than k.*

We can interpret this result in the context of currents as well. Indeed, suppose we pick linearly independent horizontal vectors $u_1, \ldots, u_k \in H$, and let $\tilde{u} = u_1 \wedge \cdots \wedge u_k$.

By the proof of Proposition 3.43, the boundary $\partial T_{\tilde{u}}$ of the simple k-current $T_{\tilde{u}}$ vanishes if and only if $[u_i, u_j] = 0$ for all i and j. This is true if and only if the Lie subalgebra generated by the vectors $u_1, \ldots, u_k$ is Abelian (or, equivalently, is horizontal). Combining this with Theorem 3.43, we obtain the following reformulation of Magnani's Theorem.

Theorem 3.55. *A Carnot group $\mathbb{G}$ has a nontrivial k-rectifiable subset if and only if it has a nonzero, invariant, "simple" k-current $T_{\tilde{u}} = T_{u_1 \wedge \cdots \wedge u_k}$.*

We are unaware if Theorem 3.55 can be deduced independently of Theorem 3.54. In particular, we do not know whether either implication is true in a general metric group with a differentiable structure.

Remark 3.56. It is not true that the absence of k-rectifiable sets in $\mathbb{G}$ implies the nonexistence of arbitrary (i.e., non-simple) k-currents. To construct an explicit counterexample, let $\mathfrak{g}$ have the stratification

$$\mathfrak{g} = \mathrm{Span}(u_1, u_2, u_3, u_4) \oplus \mathrm{Span}(v_1, v_2, v_3, v_4, v_5)$$

satisfying the relations $[u_1, u_2] = [u_3, u_4] = v_1$, $[u_1, u_3] = v_2$, $[u_1, u_4] = v_3$, $[u_2, u_3] = v_4$, and $[u_2, u_4] = v_5$. It is easily verified that any two linearly independent horizontal vectors do not commute, and so by Theorems 3.55 and 3.54, respectively, $\mathbb{G}$ admits no nonzero simple 2-currents, nor any nontrivial 2-rectifiable sets. On the other hand, $T_{u_1 \wedge u_2 - u_3 \wedge u_4}$ is a 2-current, and is in fact a cycle. Thus there are purely k-unrectifiable spaces which still admit normal k-currents, for $k \geq 2$. In this sense, the theory of metric currents is at least somewhat more general than the theory of rectifiable sets. This contrasts strongly with the Euclidean case, where any normal metric current can be identified with a normal current in the sense of Federer and Fleming [1], and where the latter can be approximated in Whitney's flat norm [44] (and hence weakly) by polyhedral chains, which are of course rectifiable.

CHAPTER IV

Modulus, upper gradients, and quasiconformality

In this chapter we explore the relationship between curve modulus, upper gradients, and quasiconformal maps. In Section 4.1 we recall the necessary definitions and facts on curves, modulus, and upper gradients. In Section 4.2, we analyze upper gradients and minimal upper gradients in detail, culminating in Theorem 4.29, which provides a characterization of minimal upper gradients entirely in terms of curve modulus. Finally, in Section 4.3, we use the results of Section 4.2 to prove Theorem 1.12 on the equivalence of (lower) geometric quasiconformality with analytic quasiconformality.

4.1 Curves, modulus, and upper gradients

Throughout the chapter, except where otherwise specified, F will denote a measurable map $F\colon X \to Y$ from a separable metric measure space $X = (X, \mu)$ to a metric space Y. A *curve* in X is a continuous map $\gamma\colon I \to X$, where $I = [a, b] \subset \mathbb{R}$ is a closed interval. We say that γ is a curve joining points x_1 and x_2 in X if $\gamma(a) = x_1$ and $\gamma(b) = x_2$. A *subcurve* of γ is the restriction $\gamma|_{[c,d]}$ of γ to a subinterval $[c, d] \subseteq [a, b]$. By $F(\gamma)$ we denote the curve $F \circ \gamma$ in Y, provided $F \circ \gamma$ is continuous. If Γ is a family of curves in X, then $F(\Gamma)$ is the corresponding family $\{F(\gamma) : \gamma \in \Gamma \text{ and } F \circ \gamma \text{ is continuous}\}$.

Absolute continuity and arc-length.

Following [9], we define the variation function $v_\gamma\colon [a,b] \to [0,\infty]$ by

$$v_\gamma(s) = \sup_{a\le a_1\le b_1\le\cdots\le a_n\le b_n\le s} \sum_{i=1}^{n} |\gamma(b_i) - \gamma(a_i)|.$$

Note that for $a \le q \le r \le b$,

$$v_\gamma(r) - v_\gamma(q) = \sup_{q\le a_1\le b_1\le\cdots\le a_n\le b_n\le r} \sum_{i=1}^{n} |\gamma(b_i) - \gamma(a_i)|. \tag{4.1}$$

The *length* $l(\gamma)$ of γ is defined by $l(\gamma) = v_\gamma(b)$. If γ has finite length, we say that γ is *rectifiable.* Note that in this case, v_γ is a continuous, increasing function $v_\gamma\colon [a,b] \to [0, l(\gamma)]$. The unique curve $\hat\gamma\colon [0,l(\gamma)] \to X$ such that $\gamma = \hat\gamma \circ v_\gamma$ is called the *arc-length parametrization* of γ. The integral $\int_\gamma \omega\, ds$ of a Borel function $\omega\colon X \to \mathbb{R}$ along γ is defined by

$$\int_\gamma \omega\, ds = \int_0^{l(\gamma)} \omega(\hat\gamma(t))\, dt.$$

We use the following definitions for absolutely continuity of mappings. A curve $\gamma\colon I \to X$ is **absolutely continuous** if for every $\epsilon > 0$, there is a $\delta > 0$ such that for every disjoint collection of intervals $[a_1, b_a] \dots [a_n, b_n]$ in I such that $\sum_{i=1}^n |b_i - a_i| < \delta$, we have $\sum_{i=1}^n |\gamma(b_i) - \gamma(a_i)| < \epsilon$. If $F\colon X \to Y$ is a map, and γ a rectifiable curve in X, we say that F is (resp., absolutely) continuous along γ if $F \circ \hat\gamma$ is (resp., absolutely) continuous.

A detailed introduction to arc-length parametrizations can be found in [40], Chapter 1. For now, we need a few elementary facts.

It follows immediately from the definition that a curve γ is absolutely continuous if and only if the variation function v_γ is absolutely continuous as well. Note therefore that if F is absolutely continuous on γ and γ is also absolutely continuous itself, then

$F\circ\gamma = F\circ\hat{\gamma}\circ v_\gamma$, a composition of two absolutely continuous maps, is also absolutely continuous.

If γ is absolutely continuous, we can evaluate line integrals without employing the arc-length parametrization (see [40], Theorem 4.1).

$$\int_\gamma \omega\, ds = \int_a^b \omega(\gamma(t)) \cdot v_\gamma'(t)\, dt. \tag{4.2}$$

The arc-length parametrization respects mappings as well. That is, if γ is rectifiable, and so is $F\circ\gamma$, (in particular, if F is absolutely continuous along γ), then $\widehat{F\circ\gamma} = \widehat{F\circ\hat{\gamma}}$ (See [40], Theorem 2.6). In particular, this implies that

$$\int_{F\circ\gamma} \omega\, ds = \int_{F\circ\hat{\gamma}} \omega\, ds \tag{4.3}$$

whenever ω is integrable along one (and hence both) of these parametrizations.

For much more on absolutely continuous maps into metric spaces, see [9].

Curve modulus.

Next, we recall the notion of curve modulus.

Definition 4.4. Let Γ be a family of curves in X. A Borel function $\rho\colon X\to[0,\infty]$ is **admissible** for Γ if for every rectifiable curve $\gamma\in\Gamma$,

$$\int_\gamma \rho\, ds \geq 1. \tag{4.5}$$

The p**-modulus** of Γ is

$$\mathrm{Mod}_p(\Gamma) = \inf\left\{\int_X \rho^p\, d\mu : \rho \text{ is admissible for } \Gamma.\right\}. \tag{4.6}$$

The p-modulus satisfies a number of important properties, which can be found in [14].

Lemma 4.7 ([14], Theorem 1)**.** Mod_p *is an outer measure on the set* $\Gamma_c(X)$ *of curves in* X*.*

If $\mathrm{Mod}_p(\Gamma) = 0$, we say $\mathcal{N}$ is *p-exceptional*, or simply *exceptional* if p is understood. In view of Lemma 4.7, we will speak of a property holding for *p-almost every* curve if the property fails only on an exceptional family.

Another property allows us to replace families of curves with certain families of subcurves:

Lemma 4.8 ([14], Theorem 1 (c)). *Suppose Γ_1 and Γ_2 are curve families such that Γ_1 minorizes Γ_2; that is, every curve in Γ_1 has a subcurve in Γ_2. Then $\mathrm{Mod}_p(\Gamma_1) \leq \mathrm{Mod}_p(\Gamma_2)$.*

We will also need the following result from Fuglede.

Theorem 4.9 ([14], Theorem 3 (f)). *Let $\{\rho_i\}$ be a sequence of locally p-integrable Borel functions on X converging to ρ locally in the p-norm. Then there is a subsequence $\{\rho_{i_k}\}$ such that on p-almost every rectifiable curve γ,*

$$\int_\gamma \rho_{i_k}\, ds \to \int_\gamma \rho\, ds.$$

Note that Fuglede's theorem was stated in [14] for the case where the functions ρ_i are actually p-integrable, not merely locally. But the usual diagonalization argument immediately generalizes the theorem to locally integrable functions, provided, as per our standing assumptions, that the metric measure space X is σ-finite. Also, though the theorem is stated there for $X = \mathbb{R}^n$, the proof of the general case is identical.

If $A \subset X$, and γ is rectifiable, we say that γ *spends positive time in A* if $\mathrm{m}_1(\hat{\gamma}^{-1}(A)) > 0$. Another result [14] tells us most curves will not spend positive time in a μ-null set.

Lemma 4.10 ([14] Theorem 3 (d)). *Let $E \subset X$ with $\mu(E) = 0$. Then the set*

$$\{\gamma \in \Gamma_c(X) : \gamma \textit{ is rectifiable and spends positive time in } E.\}$$

is exceptional.

Upper gradients and Newton-Sobolev maps.

In order to define the Newton-Sobolev spaces $N^{1,p}(X,Y)$ we will need the following generalization of gradients to metric spaces, introduced for the case $Y=\mathbb{R}$ in [23] under the name "very weak gradients" and then defined for the general case in [24] (Our abbreviation "p-gradient" below is new, we believe).

Definition 4.11. Let $F\colon X \to Y$ be a Borel function. A Borel function $\rho\colon X \to [0,\infty]$ is called an **upper gradient** for F if

$$\int_\gamma \rho\, ds \geq |F(x_2) - F(x_1)| \tag{4.12}$$

for every rectifiable curve γ joining points x_1 and x_2. If the inequality (4.12) holds merely for p-*almost* every curve, then ρ is said to be a p**-weak upper gradient** (briefly, a p**-gradient**).

By Lemma 2.4 of [30], F has a p-integrable p-gradient if and only if it has a p-integrable gradient.

We define the Newton-Sobolev spaces $N^{1,p}(X,Y)$, and $N^{1,p}_{\mathrm{loc}}(X,Y)$ as introduced in [24].

Definition 4.13. Let $F\colon X \to Y$ be a Borel map, where $X = (X, \mathrm{dist}, \mu)$ is a metric measure space, and Y is an arbitrary metric space. Then we say that F is in the local Newton-Sobolev class $N^{1,p}_{\mathrm{loc}}(X,Y)$ if F is locally p-integrable and has a locally p-integrable p-gradient. If μ is finite, we also say F is in the Newton-Sobolev class $N^{1,p}(X,Y)$ if F is p-integrable and has a p-integrable p-gradient.

Notice that definition 4.13 is slightly different than that in [24], which first requires embedding Y into a Banach space. However, note that by the discussion after [24]

Definition 3.9, a map is a Sobolev map in the sense of Definition 4.13 if and only if the induced map into $l^\infty(Y)$ is a Sobolev mapping in the sense of [24]. We have chosen to avoid the language of Banach spaces, since all of our methods are intrinsic.

The following proposition of [38] provides an analog to the classical ACL ("absolute continuity on lines") property of Sobolev mappings.

Proposition 4.14 ([38], Proposition 3.1)**.** *If $F\colon X \to Y$ has a locally p-integrable p-gradient, then F is absolutely continuous along p-almost every curve.*

Remark 4.15. Although Proposition 3.1 of [38] is stated for the case $Y = \mathbb{R}$, the proof carries over word for word to the general case.

4.2 Analysis of gradients

In this section we collect some facts about p-modulus and upper gradients, and use them in Proposition 4.29 to characterize the p-norm of p-gradients in terms of modulus of certain curve families.

First, we note that the condition of being an upper gradient is local in the following sense.

Lemma 4.16. *A Borel function ρ is an upper gradient (resp., p-gradient) for F if and only if for every $x \in X$, there is a neighborhood U of x such that $\rho|_U$ is an upper gradient (resp., p-gradient) for $F|_U$.*

Proof. The first implication is trivial, for both upper gradients and p-gradients. To prove the second, suppose first that ρ is locally an upper gradient for F. Let γ be a rectifiable curve in X joining x_1 and x_2. Choose a finite cover $\{U_i\}$ of the image $\gamma([a,b])$ such that $\rho|_{U_i}$ is an upper gradient for $F|_{U_i}$, and such that $\gamma([t_{i-1},t_i]) \subset U_i$

for each i, where $a = t_0 < \cdots < t_n = b$ is a partition of $[a, b]$. Then

$$\int_\gamma \rho\, ds = \sum_{i=0}^{n-1} \int_{\gamma|_{[t_i, t_{i+1}]}} \rho\, ds \geq \sum_{i=0}^{n-1} |F(\gamma(t_{i+1})) - F(\gamma(t_i))|$$
$$\geq |F(\gamma(t_n)) - F(\gamma(t_0))| = |F(x_2) - F(x_1)|.$$

Now suppose ρ is locally a p-gradient of F. Since X is separable, we may choose a countable cover of X by open sets U_i, such that $\rho|_{U_i}$ is a p-gradient for $F|_{U_i}$. Let Γ be the family of curves along which (4.12) fails, and let $\Gamma_i \subset \Gamma$ be the subfamily family of curves in Γ supported on U_i. By assumption, $\mathrm{Mod}_p(\Gamma_i) = 0$ for each i. Moreover, by the compactness argument in the previous paragraph, every curve in Γ has a subcurve in Γ_i for some i. Then by Lemma 4.8,

$$\mathrm{Mod}_p(\Gamma) \leq \mathrm{Mod}_p\left(\bigcup_{i=i}^{\infty} \Gamma_i\right) \leq \sum_{i=1}^{\infty} \mathrm{Mod}_p(\Gamma_i) = 0, \tag{4.17}$$

so inequality (4.12) holds for almost every curve in X. □

Next, we note that the p-gradient condition is equivalent to an a priori stronger condition. The following lemma is a fairly standard application of Lemma 4.8, and tends to appear implicitly in other arguments (see, e.g., the proof of [38], Proposition 3.1).

Lemma 4.18. *A Borel function $\rho\colon X \to \mathbb{R}$ is an upper gradient (resp. a p-gradient) for F if and only if for every (resp. p-almost every) curve γ in X, condition* (4.12) *holds for every subcurve of γ.*

Proof. The statement for genuine upper gradients is trivial. To prove it for p-gradients, let Γ be the family of curves where (4.12) fails, and let Γ_0 be the family of curves γ such that (4.12) fails for some subcurve γ' of γ. By definition, Γ_0 minorizes Γ. Therefore, by Lemma 4.8, $\mathrm{Mod}_p(\Gamma_0) \leq \mathrm{Mod}_p(\Gamma)$. On the other hand, $\Gamma \subset \Gamma_0$, so by monotonicity, $\mathrm{Mod}_p(\Gamma) \leq \mathrm{Mod}_p(\Gamma_0)$. Thus $\mathrm{Mod}_p(\Gamma) = \mathrm{Mod}_p(\Gamma_0)$, from which fact the Lemma immediately follows. □

We next provide a characterization of the upper gradient condition (4.12) for a curve γ in terms of the variation function $v_{F\circ\hat{\gamma}}$.

Lemma 4.19. *Let $\gamma : [a,b] \to X$ be absolutely continuous, ρ a Borel function such that $\int_a^b \rho \circ \gamma ds < \infty$, and $F\colon X \to Y$ a map such that F is absolutely continuous along γ. Then inequality* (4.12) *holds for every subcurve of γ if and only if*

$$\rho \circ \gamma \cdot v_\gamma' \geq v_{F\circ\gamma}' \tag{4.20}$$

almost everywhere on $[a,b]$.

Remark 4.21. When written in terms of the arc-length parametrization $\hat{\gamma}$, Inequality (4.20) becomes

$$\rho \circ \hat{\gamma} \geq v_{F\circ\hat{\gamma}}', \tag{4.22}$$

since $v_{\hat{\gamma}}$ is the identity function.

Proof. If (4.20) holds almost everywhere, then by the absolute continuity of $v_{F\circ\hat{\gamma}}$, along every subcurve $\gamma' = \gamma|_{[q,r]}$ of γ joining x_1 and x_2, we have

$$\begin{aligned} \int_{\gamma'} \rho\, ds = \int_q^r \rho \circ \gamma \cdot v_\gamma'\, dt \geq \int_q^r v_{F\circ\gamma}'\, dt = v_{F\circ\gamma}(r) - v_{F\circ\gamma}(q) \\ \geq |F(\gamma(r)) - F(\gamma(q))| = |F(x_2) - F(x_1)|. \end{aligned}$$

Conversely, if inequality (4.12) holds for every subcurve of γ, then combining (4.12) with equation (4.1) yields the inequality

$$\begin{aligned} \int_q^r v_{F\circ\gamma}'\, dt = v_{F\circ\gamma}(r) - v_{F\circ\gamma}(q) = \sup_{q\leq a_1\leq b_1\leq\cdots\leq a_n\leq b_n\leq r} \sum_{i=1}^n |F(\gamma(b_i)) - F(\gamma(a_i))| \\ \leq \sup_{q\leq a_1\leq b_1\leq\cdots\leq a_n\leq b_n\leq r} \sum_{i=1}^n \int_{a_i}^{b_i} \rho\circ\gamma \cdot v_\gamma'\, dt \leq \int_q^r \rho\circ\gamma\cdot v_\gamma'\, dt. \end{aligned}$$

But since this holds for every interval $[q,r] \subset [0, v_\gamma(b)]$, and $\int_\gamma \rho\, ds$ is finite, the inequality (4.20) holds almost everywhere. □

We can now conclude that the family of p-integrable p-gradients of F forms a lattice.

Lemma 4.23. *Let ρ_1 and ρ_2 be p-integrable. Then $\min(\rho_1, \rho_2)$ is a p-gradient for F if and only if ρ_1 and ρ_2 are p-gradients for F.*

Proof. Let Γ_1 (resp. Γ_2, Γ) be the family of curves on which either F fails to be absolutely continuous, or inequality (4.20) fails for $\rho = \rho_1$ (resp. $\rho = \rho_2$, $\rho = \min(\rho_1, \rho_2)$). Then $\Gamma_1 \cup \Gamma_2 = \Gamma$. Indeed, on a given curve γ, (4.19) fails for $\rho = \min(\rho_1, \rho_2)$ if and only if it fails for either $\rho = \rho_1$ or $\rho = \rho_2$.

On the other hand, by Lemma 4.19, Γ_1 (resp. Γ_2, Γ) can also be characterized as the family of curves γ on which either F fails to be absolutely continuous, or inequality (4.12) fails for $\rho = \rho_1$ (resp. $\rho = \rho_2$, $\rho = \min(\rho_1, \rho_2)$) on some subcurve γ' of γ.

Combining Proposition 4.14 and Lemma 4.18, we see that ρ_1 (resp. ρ_2, $\min(\rho_1, \rho_2)$) is a p-gradient for F if and only if $\mathrm{Mod}_p(\Gamma_1) = 0$ (resp. $\mathrm{Mod}_p(\Gamma_2) = 0$, $\mathrm{Mod}_p(\Gamma) = 0$). But from the subadditivity of modulus, $\mathrm{Mod}_p(\Gamma) = \mathrm{Mod}_p(\Gamma_1 \cup \Gamma_2) = 0$ if and only if $\mathrm{Mod}_p(\Gamma_1) = 0$ and $\mathrm{Mod}_p(\Gamma_2) = 0$. □

Definition 4.24. A p-weak upper gradient ρ of F is **minimal** if and only if for every p-gradient ρ' of u, $\rho' \geq \rho$ μ-almost everywhere.

Proposition 4.23 has the following Corollary, which also follows in the case $p > 1$ from a standard argument using Mazur's Lemma and Fuglede's Theorem 4.9 (see, e.g., [24], or [38]).

Corollary 4.25. *Let $p \geq 1$. If F has a locally p-integrable p-gradient ρ, then there is a unique (up to sets of measure 0) minimal p-gradient ρ_0 for F.*

Remark 4.26. Uniqueness up to a set of measure 0 is the most we can hope for in Corollary 4.25. Indeed, if ρ_1 is a p-gradient for u, then every function ρ_2 that agrees with ρ almost everywhere is also a p-gradient for u. To see this, suppose $\rho = \rho'$ on $A \subset X$, with $\mu(X \backslash A) = 0$. By Corollary 4.10, almost every curve is concentrated on A. But on such a curve, inequality (4.12) is satisfied for $\rho = \rho_1$ if and only if it is satisfied by $\rho = \rho_2$.

Proof. We first note that if ρ' is a p-gradient, then there is a p-integrable p-gradient ρ'' with $\rho'' \leq \rho'$ everywhere (namely, the function $\rho'' = \min(\rho, \rho')$). Thus it suffices to find a (unique) minimal p-integrable p-gradient. We let $\{\rho_i\}$ be a sequence of integrable p-gradients that infimizes the p-norm among all such p-gradients. Replacing ρ_i with $\min(\rho_1, \cdots, \rho_i)$, we may assume that the sequence is descending, pointwise. Then by the Lebesgue Dominated Convergence Theorem, ρ_i converges to ρ_0 in the p-norm, where $\rho_0(x) = \lim_{i\to\infty} \rho_i(x)$. By Fuglede's Theorem, $\int_\gamma \rho_i$ converges to $\int_\gamma \rho_0$ for almost every curve γ. Since inequality (4.12) holds for almost every curve for each function ρ_i, countable subadditivity of the p-modulus implies it holds for ρ_0 for almost every curve as well. Thus ρ_0 is a p-gradient, which has minimal p-norm by construction. But now it is clear that ρ_0 is minimal in the sense of Definition 4.24, for if ρ' is a p-gradient with $\rho' < \rho_0$ on a set of positive measure, then $\min(\rho', \rho_0)$ is a p-gradient with strictly smaller p-norm than that of ρ_0, a contradiction. □

In light of Corollary 4.25, we make the following definition.

Definition 4.27. Suppose F has a p-integrable p-gradient. Then the minimal upper gradient will be denoted ρ_F.

By Corollary 4.25, ρ_F is well defined up to sets of measure 0. Moreover, the operation of restricting to an open set commutes with that of taking a minimal upper gradient.

Proposition 4.28. *If X is separable, then F has a locally p-integrable p-gradient if and only if every $x \in X$ has a neighborhood U such that $F|_U$ has a p-integrable p-gradient. In this case, $\rho_{F|_U} = (\rho_F)|_U$ almost everywhere for every open set $U \subset X$.*

Proof. If there is a locally p-integrable p-gradient for F, then it follows from the definition that $(\rho_F)|_U$ is a p-gradient for F_U, so that $\rho_{F|_U} \leq (\rho_F)|_U$. Conversely, suppose that for every $x \in X$ there is a p-integrable p-gradient for $F|_U$ for some neighborhood U of x. By separability, and the paracompactness of metric spaces, there is a countable, locally finite collection $\{U_i\}$ of such sets that covers X. We define $\rho\colon X \to \mathbb{R}$ by the equation

$$\rho(x) = \sup_{U_i \ni x} \rho_{F|_{U_i}}(x).$$

By local finiteness, ρ is locally p-integrable. Moreover, if Γ is the family of curves along which inequality (4.12) fails, and $\Gamma_i \subset \Gamma$ the subfamily of those supported on U_i, then by compactness of the interval, every curve in Γ has a subcurve in some Γ_i, so by Lemma 4.8, (4.17) holds. Finally, to prove the last statement, we must show $\rho_{F|_U} \geq (\rho_F)|_U$ for every open set $U \subset X$. Define $\rho_U\colon X \to \mathbb{R}$ by

$$\rho_U(x) = \begin{cases} \rho_{F|_U}(x) & \text{if } x \in U \\ \rho_F(x) & \text{otherwise.} \end{cases}$$

It is enough to show that ρ_U is a p-gradient of F, since if this is true, $\rho_{F|_U} = \rho_U|_U \geq (\rho_F)|_U$. Note that since F has a p-integrable upper gradient, F is absolutely continuous on almost every curve. Moreover, by the remarks at the beginning of this proof, $\rho_U \leq \rho_F$ almost everywhere, and hence ρ_U is also p-integrable. By Lemma 4.19 and Proposition 4.14, it is enough to show that for almost every curve γ, inequality 4.20 holds for almost every $s \in [a, b]$. Since ρ_F is a p-gradient, on almost every curve γ, inequality (4.20) holds for almost every $s \in \gamma^{-1}(X \setminus U)$, again by Lemma 4.19. Therefore it is enough to show that for p-almost every γ, inequality (4.20) holds for almost every $s \in \gamma^{-1}(U)$.

Let Γ be the family of curves such that (4.20) fails on a set of positive measure in $\gamma^{-1}(U)$, and let $\Gamma' \subset \Gamma$ consist of those curves supported in U. For a given curve γ, $\gamma^{-1}(U)$ is open, and can therefore be written as a countable disjoint union of intervals. Thus every curve in Γ has a subcurve in Γ'. Since $\rho_U|_U = \rho_{F|_U}$, Lemma 4.8 and Lemma 4.19 imply that $\mathrm{Mod}_p(\Gamma) \leq \mathrm{Mod}_p(\Gamma') = 0$. □

A celebrated result of Heinonen and Koskela [23] relates the notions of curve modulus and upper gradients. They show that a generalization of the classical Poincaré inequality, formulated in terms of upper gradients, is equivalent to lower bounds on the modulus of certain curve families.

The following theorem gives another connection between modulus and upper gradients.

Theorem 4.29. *Let $F \colon X \to Y$ be continuous on p-almost every curve γ. Then F has a p-integrable p-gradient if and only if*

$$\lim_{n\to\infty} 2^{-n} (\mathrm{Mod}_p(\Gamma_n))^{1/p} < \infty,$$

where $\Gamma_n = \Gamma_n^F = \{\gamma \in \Gamma_c(X) : \gamma$ joins x and y, and $|F(y) - F(x)| \geq \frac{1}{2^n}\}$. Moreover, if this is the case, then

$$\|\rho_F\|_p = \lim_{n\to\infty} 2^{-n} (\mathrm{Mod}_p(\Gamma_n))^{1/p}. \tag{4.30}$$

Proof. Throughout the proof, we will say that a function ρ is *almost admissible* for a curve family Γ if ρ is admissible for some subfamily $\Gamma'_n \subset \Gamma_n$ with $\mathrm{Mod}_p(\Gamma_n \backslash \Gamma'_n) = 0$. Note that in this situation we have

$$\mathrm{Mod}_p(\Gamma_n) = \mathrm{Mod}_p(\Gamma'_n) \leq \int_X \rho^p,$$

so that from the point of view of estimating modulus, almost admissible functions work as well as admissible ones.

We first claim that a p-integrable function ρ is a p-gradient of F if and only if for every n, $2^n\rho$ is almost admissible for Γ_n. Indeed, if ρ is a p-gradient, then on almost every $\gamma \in \Gamma_n$, if γ is a path from x to y, we have

$$\int_\gamma 2^n \rho \geq 2^n |F(y) - F(x)| \geq 1.$$

Conversely, suppose that for every n, $2^n\rho$ is almost admissible for Γ_n, and thus is admissible for $\Gamma_n \backslash \Delta_n$, for some family Δ_n with $\mathrm{Mod}_p(\Delta_n) = 0$. Let Δ be the family

$$\Delta = \{\gamma \in \Gamma_c(X) : \text{ there is a subcurve } \gamma' \text{ of } \gamma \text{ such that } \gamma' \in \Delta_n \text{ for some } n.\}.$$

Then the family Δ minorizes $\bigcup_{n=1}^\infty \Delta_n$, and so by Lemma 4.8, along with the countable subadditivity of modulus, we have

$$\mathrm{Mod}_p(\Delta) \leq \mathrm{Mod}_p\left(\bigcup_{n=1}^\infty \Delta_n\right) = 0.$$

Now for each $\gamma \notin \Delta$, each n, and every subcurve γ_0 of γ joining z and w such that $|F(z) - F(w)| \geq 2^{-n}$, we have $\int_\gamma \rho = 2^{-n} \int_\gamma 2^n \rho \, ds \geq 2^{-n}$, by the admissibility of $2^n\rho$ for $\Gamma_n \backslash \Delta_n$. If, in addition, F is continuous on γ, and γ is parametrized by $[a, b]$, then there is a (possibly infinite) partition $a = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq \cdots$ with $\lim_{t_i \to \infty} = b$, such that $|F(\gamma(t_{i+1})) - F(\gamma(t_i))| = 2^{-m_i}$ for some integer m_i, so inequality (4.12) holds for every subcurve $\gamma|_{[t_i, t_{i+1}]}$, and hence, by the triangle inequality, holds for γ as well. Since, by assumption, F is continuous along almost every curve, the claim is proved.

Now, suppose F has a p-integrable p-gradient. Then by the claim, $2^n\rho_F$ is almost admissible for Γ_n. Thus

$$2^{-n}(\mathrm{Mod}_p(\Gamma_n))^{1/p} \leq 2^{-n}\left(\int_X (2^n\rho_F)^p\right)^{1/p} = ||\rho_F||_p,$$

and so inequality (4.29) holds, and the left hand side of equation (4.30) is greater than the right-hand side.

Conversely, suppose (4.29) is satisfied. To complete the proof, it is enough to construct a p-integrable function ρ such that for every n, $2^n\rho$ is almost admissible for Γ_n, and such that

$$||\rho||_p \leq \lim_{n\to\infty} 2^{-n}(\mathrm{Mod}_p(\Gamma_n))^{1/p}. \tag{4.31}$$

From the definition of modulus, we may choose p-integrable Borel functions ρ_n, with $2^n\rho_n$ admissible for Γ_n, such that

$$||\rho_n||_p \leq 2^{-n}\,\mathrm{Mod}_p(\gamma_n)^{1/p} + \epsilon/n.$$

We note that for all $m \leq n$, $2^m\rho_n$ is almost admissible for Γ_m. Indeed, consider the set $\Gamma_m^c \subset \Gamma_m$ consisting of those curves in Γ_m along which F is continuous. By the continuity of F, for every curve $\gamma \in \Gamma_{n-1}^c$ defined on the interval $[a,b]$, there is a point $t \in [a,b]$ such that $|F(\gamma(t)) - F(\gamma(a))| \geq 2^{-n}$ and $|F(\gamma(b)) - F(\gamma(t))| \geq 2^{-n}$. Thus $2^{n-1}\rho_n$ is admissible for Γ_{n-1}^c. Since by our assumption, F is continuous on almost every curve, $2^{n-1}\rho_n$ is almost admissible for Γ_{n-1}. Applying this argument successively we see that $2^m\rho_n$ is almost admissible for Γ_m.

We now use a fairly standard line of argument to construct ρ. By the finiteness condition (4.29), the functions ρ_n are uniformly bounded in the p-norm. Thus by the reflexivity of $L^p(X,\mu)$, there is a subsequence ρ_{k_n} that converges weakly to some function $\rho \in L^p(X,\mu)$. By Mazur's Lemma, there is a sequence of convex combinations $\omega_n = \sum_{i=1}^{m_n} \lambda^{n_i}\rho_{n_i}$ that converges (strongly) in the p-norm to ρ, such that for all n and all $i \leq m_n$, we have $n_i \geq n$. By Fuglede's Theorem, there is yet another subsequence ω_{k_n} such that $\int_\gamma \omega_{k_n}$ converges to $\int_\gamma \rho$ on almost every curve γ.

Note that by our definition of the functions ρ_n, ρ must satisfy inequality (4.31). All that remains, then, is to check that $2^m\rho$ is almost admissible for Γ_m.

Now for each function ρ_{n_i} in the definition of ω_n, $n_i \geq n$, so by our previous remark, $2^m\rho_{n_i}$ is almost admissible for Γ_m whenever $m \leq n$. The admissibility condition is preserved under convex combinations, and so $2^m\omega_n$ is also almost admissible for Γ_m when $m \leq n$. Passing to the limit as n approaches infinity, and fixing m, we see that on almost every curve in Γ_m,

$$\int_\gamma 2^m\rho\, ds = \lim_{n\to\infty}\int_\gamma 2^m\omega_{k_n}\, ds \geq 1.$$

□

Remark 4.32. As discussed in the introduction, there is a conceptual link between Theorem 4.29 and the theory of currents.

Note that if γ is an arc (that is, a curve that is a 1 to 1 mapping) in X, then, as in [42], define the current $T_\gamma \in \mathbf{M}_1(X)$ by $T_\gamma(f\,dg) = \int_0^{l(\gamma)} f\circ\hat{\gamma}\cdot(g\circ\hat{\gamma})'\,dt$. Then the mass measure $||T_\gamma||$ of T_γ is exactly the arc-length measure, i.e.,

$$\int_X f\, d||T_\gamma|| = \int_\gamma f\, ds. \tag{4.33}$$

Equation (4.33) suggests a natural definition of the current modulus of a family $\Gamma \subset \mathbf{M}_1(X)\backslash\{0\}$, namely the same as Definition 4.4, but with the admissibility condition (4.5) replaced by

$$\int_X \rho \, d||T|| \geq 1. \tag{4.34}$$

Similarly, by equation (4.33), the upper gradient inequality (4.12) becomes

$$\int_X \rho \, d||T|| = \int_\gamma \rho \, ds \geq |f(x_2) - f(x_1)| = |\partial T(f)| = T(df) \tag{4.35}$$

whenever $f \in \mathrm{Lip}_c(X)$, and $T = T_\gamma$. Inspired by this, we might define ρ to be an upper gradient "in the sense of currents" if the inequality (4.35) holds for *every* current $T \in \mathbf{M}_1(X) \backslash \{0\}$.

Clearly inequality (4.35) holds if and only if it holds whenever $|T(df)| > 0$, and by the scale invariance of the inequality, we see then that it holds if and only if it holds whenever $|T(df)| = 1$. Thus ρ is an upper gradient of f (always in the sense of currents, for the remainder of this discussion) if and only if

$$\int_X \rho \, d||T|| \geq 1$$

whenever $|T(df)| = 1$ (or equivalently, whenever $|T(df)| \geq 1$). Equivalently, ρ is an upper gradient of f if and only if ρ is admissible for the family $\Gamma^f = \{T \in \mathbf{M}_1(X) : |T(df)| \geq 1\}$. Thus we have proved a much simpler, easier, and more elegant version of Theorem 4.29.

Theorem 4.36. *Let* $f \in \mathrm{Lip}_c(X)$. *Then* f *has a* p*-integrable upper gradient if and only if* $\mathrm{Mod}_p(\Gamma^f)$ *is finite. If this is the case, then*

$$||\rho_f||_p = \mathrm{Mod}_p(\Gamma^f)^{\frac{1}{p}}.$$

Of course, it is not clear at all whether there are applications to Theorem 4.36. The rescaling argument removes some of the geometry from the situation. A more serious issue is that metric currents are not so well suited to the theory of quasiconformal mappings in metric spaces, since pushing currents forward along a map requires that the map be Lipschitz. On the other hand, the integrability of the local Lipschitz constant L_F of a quasisymmetric map $F\colon X \to Y$ (see, e.g., [20], Section 7) implies, by standard arguments using the theorems of Lusin and Egorov, that X has a measurable decomposition into subsets on which F is Lipschitz, which suggests some potential compatibility between the two theories.

4.3 Geometric vs. analytic quasiconformality.

We are now ready to prove Theorem 1.12.

Proof of Theorem 1.12. We first note that the analytic inequality (1.13) is equivalent, by the definition of the Radon-Nikodym derivative and the push-forward measure (see Section 1.3) to the condition

$$\int_U \rho_F^Q \, d\mu \leq K\nu_a(F(U)),$$

or equivalently

$$||\rho_F|_U||_Q \leq (K\nu_a(F(U)))^{1/Q},$$

for all Borel sets $U \subset X$. By the singularity of the measure $\nu_s = \nu - \mu_a$ with respect to μ, this further is equivalent to the inequality

$$||\rho_F|_U||_Q \leq (K\nu(F(U)))^{1/Q}. \tag{4.37}$$

Moreover, by the Borel regularity of μ and ν, it suffices to verify the inequality (4.37) whenever U is open.

$\Rightarrow$. By Lemma 4.28, it suffices to show that if $\nu(F(U)) < \infty$, then $F|_U$ has a Q-integrable Q-gradient, and

$$||\rho_{F|_U}||_Q \leq (K\nu(F(U)))^{1/Q}.$$

Fix such a U, and let $I_{F(U)}\colon F(U) \to F(U)$ be the identity map. Using the terminology of Proposition 4.29, we have $\Gamma_n^{I_{F(U)}} = F(\Gamma_n^{F|_U})$. Moreover, every curve in $\Gamma_n^{I_{F(U)}}$ has length at least $2^{-n}n$, so $\rho_n = 2^n$ is admissible for $\Gamma_n^{I_{F(U)}}$, and so $\mathrm{Mod}_Q(\Gamma_n^{I_{F(U)}}) \leq 2^{nQ}\nu(F(U))$. Thus

$$\begin{aligned}
\lim_{n\to\infty} 2^{-n}(\mathrm{Mod}_Q(\Gamma_n^{F|_U}))^{1/Q} &\leq K^{1/Q}\lim_{n\to\infty} 2^{-n}(\mathrm{Mod}_Q(F(\Gamma_n^{F|_U})))^{1/Q} \\
&= K^{1/Q}\lim_{n\to\infty} 2^{-n}(\mathrm{Mod}_Q(\Gamma_n^{I_{F(U)}}))^{1/Q} \\
&\leq K^{1/Q}\lim_{n\to\infty} 2^{-n}(2^{nQ}\nu(F(U)))^{1/Q} \\
&= (K\nu(F(U)))^{1/Q} \\
&< \infty.
\end{aligned}$$

The result now follows from Proposition 4.29.

$\Leftarrow$. Suppose F has a locally p-integrable p-gradient, and that ρ_F satisfies inequality (4.37), and let $\Gamma \subset \Gamma_c(X)$.

First, we consider the special case where there is an open set $U \subset \Gamma$, with $\nu(F(U)) < \infty$, such that every curve $\gamma \in \Gamma$ is supported in U. Let $\omega\colon Y \to \mathbb{R}$ be admissible for $F(\Gamma)$. Without loss of generality, we assume that $\omega \equiv 0$ on $Y\backslash F(U)$. Then whenever F is absolutely continuous along a curve $\gamma \in \Gamma$, we conclude from equation (4.2) and inequality 4.22 that

$$\int_\gamma \rho_F \cdot \omega \circ F\, ds = \int_0^{l(\gamma)} \rho_F \circ \hat{\gamma} \cdot \omega \circ F \circ \hat{\gamma}\, dt \geq \int_0^{l(\gamma)} v'_{F\circ\hat{\gamma}} \cdot \omega \circ F \circ \hat{\gamma}\, dt$$
$$= \int_{F\circ\hat{\gamma}} \omega\, ds = \int_{F\circ\gamma} \omega\, ds \geq 1.$$

Since, by Proposition 4.14, F is absolutely continuous on p-almost every curve, it follows that $\rho_F \cdot \omega \circ F$ is almost admissible for Γ, and so

$$\mathrm{Mod}_Q(\Gamma) \leq \int_X \rho_F^Q \cdot \omega^Q \circ F\, d\mu \leq K \int_X \omega^Q\, d\nu.$$

Since this holds for all all admissible ω, we have

$$\mathrm{Mod}_Q(\Gamma) \leq K\, \mathrm{Mod}_Q(F(\Gamma)).$$

To complete the proof, we let $U_1 \subset U_2 \subset \cdots$ be an increasing sequence of open sets that covers X, with $\nu(F(U_i)) < \infty$ for each i. Then $\Gamma = \bigcup_i \Gamma_i$, where Γ_i is the family of curves in Γ supported in U_i. We now have

$$\mathrm{Mod}_Q(\Gamma) = \lim_{i\to\infty} \mathrm{Mod}_Q(\Gamma_i) \leq K \lim_{i\to\infty} \mathrm{Mod}_Q(F(\Gamma_i)) = K\, \mathrm{Mod}_Q(F(\Gamma)).$$

$\square$

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