JUMPING NUMBERS AND MULTIPLIER IDEALS ON ALGEBRAIC SURFACES

by

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CHAPTER I

Introduction

The singular or non-manifold points of a complex algebraic variety have subtle local structure, and detailing their properties – even in the study of smooth varieties – is a critical part of many investigations. For example, the seminal work [BCHM07] proves the existence of a distinguished birational modification or canonical model for every smooth complex projective variety (*cf.* [Siu06]). This model is produced via the so-called Minimal Model Program, wherein it is essential to control the singularities appearing in steps along the way. In this dissertation, we shall be concerned with certain invariants of singularities on complex algebraic varieties arising naturally in birational geometry.

To every sheaf of ideals $\mathfrak{a}$ on a complex algebraic variety X with mild singularities, one can associate its multiplier ideals $\mathcal{J}(X, \mathfrak{a}^{\lambda})$. Indexed by positive rational numbers λ, this family forms a nested sequence of ideals. These invariants can be thought to give a measure of the singularities of the pair $(X, \mathfrak{a})$, with deeper or smaller multiplier ideals corresponding to "worse" singularities. In recent years, multiplier ideals have found numerous applications in complex algebraic geometry and become a fundamental tool in the subject (*e.g.* [Dem93], [AS95], [EL99], [Siu98], [ELS01], [HM07], [Dem01], [Laz04]).

The values of λ where the multiplier ideals change are known as jumping numbers. These discrete numerical invariants were studied systematically in [ELSV04], after appearing indirectly in [Lib83], [LV90], [Vaq92], and [Vaq94]. Jumping numbers are known to encode both algebraic information about the ideal in question and geometric properties of the associated closed subscheme. Our main results address questions concerning multiplier ideals and jumping numbers on algebraic surfaces.

Multiplier ideals are automatically integrally closed (or complete) and have many noteworthy properties. These largely stem from their use in extending well-known vanishing statements for cohomology on smooth varieties through resolution of singularities. Thus, one might wonder: is every integrally closed ideal a multiplier ideal? Recently, a negative answer was given by Lazarsfeld and Lee [LL07], who found examples of integrally closed ideals on smooth varieties of dimension at least three which cannot be realized as multiplier ideals. The landscape in dimension two, however, is vastly different. Concurrently, [LW03] and [FJ05] have shown that every integrally closed ideal on a smooth surface is locally a multiplier ideal. While their proofs strongly use the theory of complete ideals specific to smooth surfaces, parts of this theory extend to surfaces with rational singularities. Thus it is natural to ask the following question (first posed in a slightly different form in [LLS08]):

Question I.1. *Suppose X is a complex algebraic surface with rational singularities. Locally on X, is every integrally closed ideal which is contained in $\mathcal{J}(X, \mathcal{O}_X)$ a multiplier ideal?*

Our first main result will address this question in the case of a surface with log terminal singularities by extending the methods of [LW03] (see Theorem IV.3 for a more detailed statement).

Theorem I.2. *If X is a complex algebraic surface with log terminal singularities, then locally every integrally closed ideal is a multiplier ideal.*

As the condition $\mathcal{J}(X, \mathcal{O}_X) = \mathcal{O}_X$ defines log terminal singularities, which are necessarily rational (see Theorem III.11 or Theorem 5.22 in [KM98]), Theorem I.2 gives a complete answer to the above question in this case. Furthermore, note that a similar result cannot hold on a surface with "worse" singularities than log terminal (*e.g.* log canonical) as the trivial ideal will then not be realized as a multiplier ideal.

Our second main result concerns the computation of jumping numbers on complex algebraic surfaces with rational singularities. In order to find the jumping numbers and multiplier ideals of a given ideal, one must first undertake the difficult task of resolving singularities. Even when a resolution is readily available, however, calculating jumping numbers can be problematic. In Chapter V, we will give an algorithm for computing jumping numbers from the numerical data of a fixed log resolution.

Theorem I.3. *Suppose $\pi \colon Y \to X$ is a log resolution of an ideal sheaf $\mathfrak{a}$ on a complex algebraic surface X with rational singularities. Then there is an effective procedure for calculating the jumping numbers of $(X, \mathfrak{a})$ using the intersection product for divisors on Y and their orders along $\mathfrak{a}$.*

The procedure is based upon identifying certain collections of "contributing exceptional divisors," building on the work of Smith and Thompson in [ST07]. Explicit instructions for computing the jumping numbers can be found in Section 5.5. Using this result, we are able to provide important new examples for the continuing study of jumping numbers, *e.g.* the jumping numbers of the maximal ideal at the singular point in a Du Val (Example V.16) or toric surface singularity (Example V.17).

Perhaps the most important application of our method, however, lies in finding the jumping numbers of an embedded curve on a *smooth* surface. While progress has been made along these lines in [Jär06], the algorithm we present is easy to use and original in that it applies to reducible curves. Furthermore, in Chapter VI, we show an alternative (and simpler) proof of the formula for the jumping numbers of the germ of an analytically irreducible plane curve – the main result of [Jär06]. In Example VI.17, two non-equisingular plane curves with the same jumping numbers will be given as well.

We now turn to a more detailed overview of the content of the proofs of the above theorems and the individual chapters. In Chapter II, we begin with a summary of the formalism and properties of divisors and $\mathbb{Q}$-divisors, a language which is central to our presentation throughout the dissertation. Since integral closure of ideals and Rees valuations play a central role in Theorem I.2 and Theorem I.3, we proceed to give a detailed overview of the theory. While this material can also be found in either [Laz04] or [HS06], our presentation is distinguished by an emphasis on the use of divisors throughout.

Multiplier ideals are defined (in all dimensions) in Chapter III. We refer the reader to [BL04] for a more complete introduction to these invariants. The standard reference for the properties of multiplier ideals is [Laz04]. However, many of the results we will need are only proved therein when the ambient variety is smooth. As such, we have opted here to give proofs of relevant results in a singular setting. These include local vanishing for multiplier ideals and Skoda's theorem. Furthermore, because a simple proof (avoiding unnecessary use of canonical covers) does not exist in the literature, a proof that log terminal singularities are rational will also be presented. Following this chapter, we will focus our attention to ideals on algebraic surfaces.

At the beginning of Chapter IV, we review the local restrictions on the minimal syzygies of multiplier ideals detailed in [LL07] and [LLS08]. These restrictions are the source of the examples (from [LL07]) of integrally closed ideals which are not multiplier ideals, and it follows from Theorem I.2 that integrally closed ideals on log terminal surfaces satisfy these restrictions. See Corollary IV.8 for a precise statement.

The remainder of Chapter IV is largely devoted to the proof of Theorem I.2. There are several difficulties in trying to extend the techniques used in [LW03]. One must show that successful choices can be made in the construction (specifically, the choice of ϵ and N in Lemma 2.2 of [LW03]). Here, it is essential that X has log terminal singularities. Further problems arise from the failure of unique factorization to hold for integrally closed ideals. As X is not necessarily factorial, we may no longer reduce to the finite colength case. In addition, the crucial contradiction argument which concludes the proof in [LW03] does not apply.

These nontrivial difficulties are overcome by using a relative numerical decomposition for divisors on a resolution over X, which will be developed during the course of the proof (see Section 4.2.1). This simple idea grew out the use of various well-known bases for the intersection lattice of the exceptional divisors. The relative numerical decomposition and associated bases also appear in our use of the Zariski-Lipman theory of complete ideals on a smooth surface in Section VI, as well as in our treatment of the proximity matrix of the resolution of a unibranched plane curve germ in Section 6.2.

The remainder of the dissertation – Chapters V and VI – concerns the aforementioned algorithm for computing jumping numbers and its applications. After reviewing rational surface singularities, the algorithm will be derived in Chapter V.

Let us preview the original techniques and terminology used therein. Fix a log resolution $\pi : Y \to X$ of the pair $(X, \mathfrak{a})$ with $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$ and relative canonical divisor K_π. With this notation, the multiplier ideal with coefficient $\lambda \in \mathbb{Q}_{>0}$ can be defined as $\mathcal{J}(X, \mathfrak{a}^\lambda) = \pi_*\mathcal{O}_Y(\lceil K_\pi - \lambda F\rceil)$. Varying λ causes changes in the expression $\lceil K_\pi - \lambda F\rceil$ at certain discrete values called candidate jumping numbers, and λ is a jumping number if $\mathcal{J}(X, \mathfrak{a}^{\lambda-\epsilon}) \neq \mathcal{J}(X, \mathfrak{a}^\lambda)$ for all $\epsilon > 0$.

Not every candidate jumping number is a jumping number (see Example V.3), and deciding when a candidate jumping number results in a jump in the multiplier ideal is a difficult and important question. We shall address this question and give a complete answer when X is a complex algebraic surface with a rational singularity. Our techniques build upon the work of Smith and Thompson in [ST07], which attempts to identify the divisorial conditions that are essential for the computations of multiplier ideals. Precisely, if G is a reduced subdivisor of F, we say $\lambda \in \mathbb{Q}_{>0}$ is a *candidate jumping number* for $G = E_1 + \cdots + E_k$ when $\mathrm{ord}_{E_i}(K_\pi - \lambda F)$ is an integer for all $i = 1, \ldots, k$. When a candidate jumping number λ for G is a jumping number, we say λ is *contributed* by G if

$$\mathcal{J}(X, \mathfrak{a}^\lambda) = \pi_*\mathcal{O}_Y(\lceil K_\pi - \lambda F\rceil) \neq \pi_*\mathcal{O}_Y(\lceil K_\pi - \lambda F\rceil + G).$$

This contribution is said to be *critical* if, in addition, no proper subdivisor of G contributes λ. The content of Theorems V.8 and V.10 is summarized below, showing how to identify the reduced exceptional divisors which critically contribute a jumping number.

Theorem I.4. *Suppose $\mathfrak{a}$ is an ideal sheaf on a complex surface X with an isolated rational singularity. Fix a log resolution $\pi : Y \to X$ with $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$, and a reduced divisor $G = E_1 + \cdots + E_k$ on Y with exceptional support.*

(i) The jumping numbers λ critically contributed by G are determined by the intersection numbers $\lceil K_\pi - \lambda F \rceil \cdot E_i$, for $i = 1, \ldots, k$.

(ii) If G critically contributes a jumping number, then it is necessarily a connected chain of smooth rational curves. The ends of G must either intersect three other prime divisors in the support of F, or correspond to a Rees valuation of $\mathfrak{a}$.

Again, we stress that these results are new and interesting even on smooth surfaces. As such, we will use plane curves in motivating examples throughout Chapter V. In fact, we hope our methods will lead to further discoveries about the information encoded in jumping numbers on smooth surfaces, as in the result below (see Proposition V.18).

Proposition I.5. *A complete finite colength ideal in the local ring of a smooth complex surface is simple if and only if it does not have* 1 *as a jumping number.*

Chapter VI is entirely devoted towards the calculation of the jumping numbers of the germ of a unibranch or analytically irreducible plane curve, first given in [Jär06]. We now briefly recall this formula. Let C be a unibranch plane curve and $\mathcal{O}_C$ the local ring of C at the origin. The normalization of $\mathcal{O}_C$ is a DVR, and we let $\operatorname{ord}_{\bar{C}}$ be its corresponding valuation. Following Zariski, let $\bar{\beta}_0, \ldots, \bar{\beta}_g$ be minimal generators for the semigroup $\operatorname{ord}_{\bar{C}}(\mathcal{O}_C)$ and put $e_i = \gcd(\bar{\beta}_0, \ldots, \bar{\beta}_i)$. The jumping numbers of a unibranch curve C are the union of the sets

$$\mathscr{H}_i = \left\{ \frac{r+1}{e_{i-1}} + \frac{s+1}{\bar{\beta}_i} + \frac{m}{e_i} \;\middle|\; r, s, m \in \mathbb{Z}_{\geq 0} \text{ with } \frac{r+1}{e_{i-1}} + \frac{s+1}{\bar{\beta}_i} \leq \frac{1}{e_i} \right\}$$

for $i = 1, \ldots, g$ together with $\mathbb{Z}_{\geq 0}$.

The use of our method in the calculation above has several advantages. For one, it is simpler and shorter than the original calculation. More importantly, however, it leads to new insights into the formula. First, the above decomposition of the jumping numbers (which appeared even in [Jär06]) is very natural from our point of view. The following result was first announced in [Tuc08], and an independent proof (using similar ideas) was later given by [Nai09].

Theorem I.6. *The set $\mathscr{H}_i$ is precisely the set of jumping numbers of C (critically) contributed by the prime exceptional divisor E_{ν_i} corresponding to the i-th star vertex of the dual graph of the minimal log resolution of C. In particular, all of the jumping numbers of C less than one are critically contributed by a prime exceptional divisor.*

Another advantage of our calculation is that we are able to use geometric arguments to simplify our computation, due in part to the following corollary of our methods.

Theorem I.7. *Two equisingular (i.e. topologically equivalent) plane curve germs have the same jumping numbers.*

From this, we are able to reduce the computation of the jumping numbers to the case of Fermat curve $y^e + x^b$ for $e, b \in \mathbb{Z}_{>0}$ with $\gcd(e, b) = 1$ (which are easily computed using ideas from toric geometry [How01]). If $C_1, \ldots, C_g$ are the approximate roots of C, then the strict transform of C_i becomes equisingular to a Fermat curve well in advance of the creation of the divisor $E_{\nu_{i-1}}$. After recalling the relationship between the equisingularity invariants of C and C_i, this leads to the following result.

Theorem I.8. *Let $C_1, \ldots, C_g$ be the approximate roots of C. Then ξ is a jumping numbers of C (critically) contributed by $E_{\nu_{i-1}}$ if and only if $e_{\nu_{i-1}}\xi$ is a jumping number of C_i (critically) contributed by $E_{\nu_{i-1}}$. In other words, $e_{\nu_{i-1}}\mathscr{H}_{i-1}^{C_i} = \mathscr{H}_{i-1}^{C}$. Furthermore, the jumping numbers of C_i (critically) contributed by $E_{\nu_{i-1}}$ are the same as the jumping numbers of the Fermat curve $y^{e_{i-1}} + x^{\bar{\beta}_i} = 0$.*

We conclude Chapter VI by giving a simpler version of another result from [Jär06], showing that the jumping numbers of a unibranch curve determine its equisingularity class. However, we also show that the converse to Theorem I.7 cannot hold in general, as Example VI.17 gives two non-equisingular plane curves with four analytic branches having the same jumping numbers. The construction of the example also shows that, even in dimension two, the jumping numbers of a monomial ideal do not determine the ideal up to reordering of the coordinates (*i.e.* switching x and y). It would be interesting to know if the jumping numbers of the germ of a plane curve with multiple branches determine the equisingularity class of each branch (see Question VI.18), as this is certainly the case for a unibranch curve and in Example VI.17.

CHAPTER II

Integral Closure of Ideals

2.1 Divisors on Algebraic Varieties

An *algebraic variety* is an integral separated scheme X of finite type over a field F. We are primarily interested in *complex algebraic varieties* and thus will assume $F = \mathbb{C}$ hereafter unless otherwise mentioned. However, the reader should be aware that much of the material we present is valid to varying degrees over other fields.

2.1.1 Weil and Cartier Divisors and $\mathbb{Q}$-Divisors

Definition II.1. If X is a normal complex algebraic variety, the group of *integral Weil divisors* or simply *divisors* on X is the free abelian group $\mathrm{Div}(X)$ on the set of closed subvarieties of X of codimension one. More generally, a $\mathbb{Q}$-*divisor* on X is an element of the rational vector space $\mathrm{Div}_{\mathbb{Q}}(X) = \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

A closed subvariety on X of codimension one is called a *prime divisor*, and a $\mathbb{Q}$-divisor D on X has the form $D = \sum a_E E$ where all but finitely many of the $a_E \in \mathbb{Q}$ vanish as E ranges over all of the prime divisors on X. For a fixed E, we write $\mathrm{ord}_E(D) = a_E$; the $\mathbb{Q}$-divisor D is integral if $\mathrm{ord}_E(D) \in \mathbb{Z}$ for all E. The support of a D is the union of the prime divisors E on X with $\mathrm{ord}_E \neq 0$.

A $\mathbb{Q}$-divisor D is effective if $\operatorname{ord}_E(D) \geq 0$ for all E, and we write $D_2 \geq D_1$ when $D_2 - D_1$ is effective. If $D = \sum a_E E$, the integer part of D is the integral divisor $\lfloor D \rfloor = \sum \lfloor a_E \rfloor E$, where $\lfloor \_\_ \rfloor : \mathbb{Q} \to \mathbb{Z}$ is the greatest integer function. The fractional part $\{D\} = D - \lfloor D \rfloor$ and round-up $\lceil D \rceil = -\lfloor -D \rfloor$ of D are defined similarly.

For each prime divisor E on a normal complex variety X, the local ring $\mathcal{O}_{X,E}$ at the generic point of E is a normal Noetherian domain of dimension one or discrete valuation ring (DVR). The fraction field of $\mathcal{O}_{X,E}$ is equal to the function field $\mathbb{C}(X)$, and we denote the associated valuation by $\operatorname{ord}_E \colon \mathbb{C}(X) \setminus \{0\} \to \mathbb{Z}$. These valuations allow one to define the divisor $\operatorname{div}(f)$ of zeroes and poles of a rational function $f \in \mathbb{C}(X) \setminus \{0\}$ by setting $\operatorname{ord}_E(\operatorname{div}(f)) = \operatorname{ord}_E(f)$ for all prime divisors E on X.[1] The divisor of a rational function $f \in \mathbb{C}(X) \setminus \{0\}$ may be used to test the regularity of f on an open subset $U \subseteq X$. Specifically, we have $f \in \mathbb{C}[U]$ if and only if $\operatorname{div}(f)|_U \geq 0$. When $U = \operatorname{Spec}(R)$ is affine, this fact reduces to the algebraic statement $R = \bigcap_{\substack{P \in \operatorname{Spec}(R) \\ \operatorname{ht}(P)=1}} R_P$.

Definition II.2. A *principal divisor* has the form $\operatorname{div}(f)$ for some $f \in \mathbb{C}(X) \setminus \{0\}$. A divisor C on X is a *Cartier divisor* if it is locally principal, *i.e.* for each $x \in X$ there is an open neighborhood U of x and a rational function $f_U \in \mathbb{C}(X) \setminus \{0\}$ such that $C|_U = \operatorname{div}_U(f_U)$. If D is a $\mathbb{Q}$-divisor, we say D is $\mathbb{Q}$-*Cartier* if there is an integer m such that mD is a Cartier divisor.

Note that a Cartier divisor is automatically integral, and recall that every integral Weil divisor on a smooth variety is automatically Cartier. If U is an open subset of

[1] To see that $\operatorname{div}(f)$ is well-defined, one must check that $\operatorname{ord}_E(f) \neq 0$ for at most finitely many E. To that end, suppose $U = \operatorname{Spec}(R)$ is an affine open subset of X. Writing $f = \frac{a}{b}$ for $a, b \in R \setminus \{0\}$, we have $\operatorname{div}(f) = \operatorname{div}(a) - \operatorname{div}(b)$. The desired finiteness on U now follows immediately from the following algebraic fact: the principal ideals $\langle a \rangle$ and $\langle b \rangle$ have only finitely many minimal prime ideals.

X and C is a Cartier divisor with $C|_U = \mathrm{div}_U(f_U)$ for $f_U \in C(X) \setminus \{0\}$, we say f_U is a local defining equation for C. In this case, f_U is uniquely determined up to an invertible regular function on U. Thus, if we consider $\mathbb{C}(X) \setminus \{0\}$ as a constant sheaf on X and denote by $\mathcal{O}_X^*$ the sheaf of invertible regular functions, a Cartier divisor can be equivalently defined as global section of the quotient sheaf $(\mathbb{C}(X) \setminus \{0\})/\mathcal{O}_X^*$.

Associated to any integral divisor D on a normal variety X is a subsheaf $\mathcal{O}_X(D)$ of the constant sheaf $\mathbb{C}(X)$. If $U \subseteq X$ is an open subset, the sections of $\mathcal{O}_X(D)$ are given by

$$H^0(U, \mathcal{O}_X(D)) = \{\, f \in \mathbb{C}(X) \mid \mathrm{div}_U(f) + D|_U \geq 0 \,\}.$$

If D is principal when restricted to an open subset U of X and $D|_U = \mathrm{div}_U(g)$ for $g \in \mathbb{C}(X)$, then $H^0(U, \mathcal{O}_X(D)) = \frac{1}{g} \cdot \mathbb{C}[U]$. Thus, for a Cartier divisor C, it is immediate that $\mathcal{O}_X(C)$ is an invertible sheaf. Furthermore, the first Chern class of the associated line bundle is equal to the class determined by C in $H^2(X, \mathbb{Z})$.

On a normal variety X, the complement of the smooth locus $U = X_{\mathrm{reg}}$ has codimension at least two. Thus if D is an integral divisor on X, $D|_U$ is Cartier even when D is not. It follows that $\mathcal{O}_X(D)|_U = \mathcal{O}_U(D|_U)$ is invertible and the sheaf $\mathcal{O}_X(D)$ has rank one. Furthermore, it is a reflexive sheaf with respect to the functor $(\_)^\vee = \mathscr{H}om_{\mathcal{O}_X}(\_, \mathcal{O}_X)$, *i.e.* we have $((\mathcal{O}_X(D))^\vee)^\vee \simeq \mathcal{O}_X(D)$. This property is particularly important in light of Proposition II.3 below, as it implies that the sheaves $\mathcal{O}_X(D)$ are determined by the invertible sheaves $\mathcal{O}_U(D|_U)$ on $U = X_{\mathrm{reg}}$.

Proposition II.3. *Suppose X is a normal variety and $\iota : U \to X$ is the inclusion of an open subvariety U where $X \setminus U$ has codimension at least two (e.g. $U = X_{\mathrm{reg}}$). If $\mathscr{N}$ is a reflexive coherent sheaf on U, then $\iota_*(\mathscr{N})$ is a reflexive coherent sheaf on X. Conversely, if $\mathscr{M}$ is a reflexive coherent sheaf on X, then $\mathscr{M}|_U$ is so on U and we* *have $\iota_*(\mathscr{M}|_U) \simeq \mathscr{M}$. In this manner, ι_* induces an equivalence of categories between reflexive coherent sheaves on U and reflexive coherent sheaves on X.*

Two divisors D_1 and D_2 on a normal variety X give rise to isomorphic coherent sheaves $\mathcal{O}_X(D_1) \simeq \mathcal{O}_X(D_2)$ if and only if they are linearly equivalent, *i.e.* $D_1 - D_2$ is a principal divisor. The class group of X is the group of divisors up to linear equivalence, *i.e.* $Div(X)$ modulo the subgroup of principal divisors. Though they will appear only in passing through our investigations, class groups are classically important objects of study in both algebraic geometry and number theory. A primary concern, however, is the generalization of linear equivalence to $\mathbb{Q}$-divisors: two $\mathbb{Q}$-divisors D_1 and D_2 are $\mathbb{Q}$-linearly equivalent if there is an integer m such that $m(D_1 - D_2)$ is principal.

2.1.2 Functorial Operations on Divisors

In many cases, one is able to associate operations on divisors to a morphism $\pi\colon Y \to X$ of normal varieties. First and foremost, if C is a Cartier divisor on X whose support does not contain $\pi(Y)$, the pullback $\pi^*(C)$ is a well-defined Cartier divisor on Y. Specifically, if $f_U \in \mathbb{C}(X)$ is a local defining equation for C on an open set U, then $f_U \circ \pi \in \mathbb{C}(Y)$ is a local defining equation for $\pi^*(C)$ on $\pi^{-1}(U)$. This operation is compatible with linear equivalence and also satisfies $\pi^*\mathcal{O}_X(C) \simeq \mathcal{O}_X(\pi^*(C))$, respecting the natural pullback of invertible sheaves. Pullback naturally extends to $\mathbb{Q}$-Cartier divisors by linearity: if mD is a Cartier divisor for some integer m, then $\pi^*(D) = \frac{1}{m}\pi^*(mD)$. The pullbacks of $\mathbb{Q}$-linearly equivalent $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors remain $\mathbb{Q}$-linearly equivalent.

Recall that a Cartier divisor on X is very ample if it is linearly equivalent to a hyperplane section of some embedding of X into projective space; a $\mathbb{Q}$-divisor is ample if some positive multiple of it is very ample. More generally, the divisor A on Y is relatively very ample for a morphism $\pi\colon Y \to X$ of normal varieties if for some n there is a factorization

$$\begin{tikzcd} Y \arrow[rr, "\iota"] \arrow[dr, "\pi"'] & & \mathbb{P}^n \times X \arrow[dl, "pr_2"] \\ & X & \end{tikzcd}$$

where ι is an embedding and $\iota^*(\mathcal{O}_{\mathbb{P}^n \times X}(1)) \simeq \mathcal{O}_Y(A)$. A $\mathbb{Q}$-divisor D on Y is π-ample if mD is relatively ample for some integer $m > 0$.

When $\eta\colon V \dashrightarrow U$ is a generically finite rational map of normal varieties, one can define the pushforward to U of arbitrary divisors or $\mathbb{Q}$-divisors on V. If $V' \subseteq V$ is the domain of definition of η, *i.e.* the largest open subset of V on which η is defined, recall that $V \setminus V'$ has codimension at least two in V. Consequently, every prime divisor E on Y has nonempty intersection with V'. If $W = \mathrm{cl}_X(\pi(E \cap V'))$, set

$$\pi_*(E) = \begin{cases} [\mathbb{C}(E) : \mathbb{C}(W)] \cdot W & \dim(E) = \dim(W) \\ 0 & \dim(E) > \dim(W) \end{cases}$$

and simply extend to $\mathrm{Div}(Y)$ and $\mathrm{Div}_{\mathbb{Q}}(Y)$ linearly. If $\eta\colon V \to U$ is in fact a proper generically finite morphism, then pushforward preserves linear or $\mathbb{Q}$-linear equivalence, respectively.

Recall that a (birational) model of a normal variety X is a normal variety Y together with a proper birational morphism $\pi\colon Y \to X$. The most important instances of pushforward of divisors we will need are those associated to a model $\pi\colon Y \to X$ and its rational inverse $\pi^{-1}\colon X \dashrightarrow Y$. A prime divisor E on Y is said to be exceptional for π if it is contracted to a subvariety of higher codimension, *i.e.* $\pi(E)$ has codimension at least two in X. More generally, the domain of definition X' of π^{-1}

is the largest open set over which π is an isomorphism, and $\operatorname{Exc}(\pi) = Y \setminus \pi^{-1}(X')$ is called the exceptional locus of π. If E is a prime divisor on Y and W is a prime divisor on X, the definitions of $\pi_*(E)$ and $\pi_*^{-1}(W) = (\pi^{-1})_*(W)$ simplify to

$$\begin{aligned}
\pi_*(E) &= \begin{cases} \pi(E) & \pi(E) \text{ is a prime divisor on } X \\ 0 & E \text{ is exceptional for } \pi \end{cases} \\
\pi_*^{-1}(W) &= \operatorname{cl}_Y(\pi^{-1}(W \cap X')).
\end{aligned}$$

Note that, for any $\mathbb{Q}$-divisor D on X, we have $\pi_*(\pi_*^{-1}(D)) = D$. When D is also $\mathbb{Q}$-Cartier, $\pi^*(D) - \pi_*^{-1}(D)$ is exceptionally supported and $\pi_*(\pi^*(D)) = D$. However, if F is a $\mathbb{Q}$-divisor on Y, $\pi_*^{-1}(\pi_*(F)) - F$ is generally nonzero and exceptionally supported.

For a model $\pi\colon Y \to X$, all of the operations π^*, π_*^{-1}, and π_* preserve the property of being effective. However, special care must be taken when using the rounding operations $\lfloor \_\_ \rfloor$, $\lceil \_\_ \rceil$, and $\{\_\_\}$ on $\mathbb{Q}$-divisors defined above: both π_* and π_*^{-1} commute with rounding operations, while π^* in general does not. This maxim is particularly important when computing intersection numbers with curves. Recall that, when $Z \subseteq Y$ is an irreducible projective curve and C is a Cartier divisor on Y, the intersection number $C \cdot Z$ is simply $\deg_Z(\mathcal{O}_Y(C)|_Z)$. This pairing can be extended to $\mathbb{Q}$-Cartier divisors by linearity, as well as formal $\mathbb{Z}$-linear or $\mathbb{Q}$-linear combinations of irreducible projective curves. We say a $\mathbb{Q}$-Cartier divisor D on Y is nef[2] if $D \cdot Z \geq 0$ for all irreducible projective curves Z on X. Similarly, for a model $\pi\colon Y \to X$, we say D is nef if $D \cdot Z \geq 0$ for all irreducible projective curves Z on Y which are contracted to a point by π. An easy consequence of the projection formula gives the

[2]The term nef was originally meant to suggest either 'numerically effective' or 'numerically eventually free.'

relation $\pi^*(H) \cdot Z = H \cdot \pi_* Z$, where H is any $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X and

$$\pi_*(Z) = \begin{cases} [\mathbb{C}(Z) : \mathbb{C}(\pi(Z))] \cdot \pi(Z) & \pi(Z) \text{ is a curve} \\ 0 & \pi(Z) \text{ is a point} \end{cases}.$$

Because a model $\pi \colon Y \to X$ is a birational morphism, the pullback of rational functions $f \in \mathbb{C}(X) \mapsto f \circ \pi \in \mathbb{C}(Y)$ identifies the function fields of X and Y with one another. In particular, the discrete valuation $\mathrm{ord}_E \colon \mathbb{C}(Y) \setminus \{0\} \to \mathbb{Z}$ associated to a prime divisor E on Y gives rise to a valuation $\mathbb{C}(X) \setminus \{0\} \to \mathbb{Z}$ centered on X. This valuation – somewhat abusively – will also be denoted ord_E and is given explicitly by $f \in \mathbb{C}(X) \setminus \{0\} \longmapsto \mathrm{ord}_E(f \circ \pi)$. Valuations on $\mathbb{C}(X) \setminus \{0\}$ arising in this manner are called divisorial valuations and will be central to our investigations. If $\pi' \colon Y' \to X$ is a model dominating Y, *i.e.* factoring as

$$\begin{tikzcd} Y' \arrow[rr, "\theta"] \arrow[dr, "\pi'"'] & & Y \arrow[dl, "\pi"] \\ & X & \end{tikzcd},$$

then the valuations ord_E and $\mathrm{ord}_{\theta_*^{-1}(E)}$ on $\mathbb{C}(X) \setminus \{0\}$ coincide. Motivated by this equality, we will sometimes find it convenient to conflate E and $\theta_*^{-1}(E)$ to eschew overly cumbersome notation. However, special attention will be paid to avoid confusion throughout.

2.2 Integral Closure of Ideals

Integral closure of ideals is an operation described in terms of certain divisorial valuations (called Rees valuations) and will serve as an important source of intuition when manipulating multiplier ideals later on. We begin by reviewing normalized blowups of ideals.

2.2.1 Normalized Blowups of Ideal Sheaves

Lemma II.4. *Suppose $\pi\colon Y \to X$ is a morphism of normal varieties and A is a relatively ample Cartier divisor on Y. Then $\bigoplus_{m\geq 0} \pi_*\mathcal{O}_Y(mA)$ is a coherent sheaf of normal graded $\mathcal{O}_X$-algebras with $\mathbf{Proj}_{\mathcal{O}_X} \bigoplus_{m\geq 0} \pi_*\mathcal{O}_Y(mA) \simeq Y$.*

Proof. We verify normality, and refer the reader to [Har77] for the remainder. Without loss of generality, we may assume X is affine. Thus, we need to show the graded domain $S = \bigoplus_{m\geq 0} S_m$ where $S_m = H^0(Y, \mathcal{O}_Y(mA))$ is normal. To make it easier to keep track of the grading, we can introduce an indeterminate t and view S as the subring

$$S = S_0 + S_1 t + S_2 t^2 + \cdots + S_m t^m + \cdots$$

of $\mathbb{C}(Y)[t]$. Let $\bar{S}$ be the integral closure of S in $\mathbb{C}(Y)[t]$. Since $\mathbb{C}(Y)[t]$ is normal, it suffices to show $S = \bar{S}$.

We first show $\bar{S}$ is a graded subring of $\mathbb{C}(Y)[t]$. If $\lambda \in \mathbb{C} \setminus \{0\}$, the substitution $t \mapsto \lambda t$ gives a ring automorphism of $\mathbb{C}(Y)[t]$ preserving S. Thus, if a polynomial $\bar{s}(t) \in \mathbb{C}(Y)[t]$ satisfies an equation of integral dependence over S, so also does $\bar{s}(\lambda t)$. Write

$$\bar{s}(t) = \bar{s}_h t^h + \bar{s}_{h+1} t^{h+1} + \cdots + \bar{s}_{h+d} t^{h+d}$$

for $\bar{s}_h, \ldots, \bar{s}_{h+d} \in \mathbb{C}(Y)$, and choose distinct constants $\lambda_0, \ldots, \lambda_d \in \mathbb{C} \setminus \{0\}$. We have

$$\begin{pmatrix} \lambda_0^h & \cdots & \lambda_0^{h+d} \\ \vdots & \ddots & \vdots \\ \lambda_d^h & \cdots & \lambda_d^{h+d} \end{pmatrix} \begin{pmatrix} \bar{s}_h t^h \\ \vdots \\ \bar{s}_{h+d} t^{h+d} \end{pmatrix} = \begin{pmatrix} \bar{s}(\lambda_0 t) \\ \vdots \\ \bar{s}(\lambda_d t) \end{pmatrix}.$$

Now, the the leftmost matrix is invertible as its determinant is given by

$$\lambda_0^h \cdots \lambda_d^h \det \begin{pmatrix} 1 & \cdots & \lambda_0^d \\ \vdots & \ddots & \vdots \\ 1 & \cdots & \lambda_d^d \end{pmatrix} = \lambda_0^h \cdots \lambda_d^h \prod_{0 \le i < j \le d} (\lambda_j - \lambda_i) \neq 0$$

according to the formula for the Vandermonde determinant. It follows immediately that $\bar{s}_h t^h, \ldots, \bar{s}_{h+d} t^{h+d} \in \bar{S}$. Thus $\bar{S}$ is graded, and we can write

$$\bar{S} = \bar{S}_0 + \bar{S}_1 t + \cdots + \bar{S}_m t^m + \cdots$$

for some S_0-submodules of $\mathbb{C}(Y)$. It suffices to show $\bar{S}_m = S_m$.

Every $\bar{s}_m \in \bar{S}_m \setminus \{0\}$ satisfies an equation of the form

$$(\bar{s}_m t^m)^k + a_1(t) \cdot (\bar{s}_m t^m)^{k-1} + a_2(t) \cdot (\bar{s}_m t^m)^{k-2} + \cdots + a_k(t) = 0 \tag{2.1}$$

for $a_1(t), \ldots, a_k(t) \in S$. If $a_{i,j} \in S_j$ is the coefficient of t^j in $a_i(t)$, taking the coefficient of t^{mk} in (2.1) gives

$$\bar{s}_m^k + a_{1,m} \bar{s}_m^{k-1} + a_{2,2m} \bar{s}_m^{k-2} + \cdots + a_{k,mk} = 0. \tag{2.2}$$

Suppose, by way of contradiction, there is a prime divisor E on Y such that $\operatorname{ord}_E(\bar{s}_m) < -m \operatorname{ord}_E(A)$. Since $\operatorname{ord}_E(a_{i,j}) \geq -j \operatorname{ord}_E(A)$, we have

$$\operatorname{ord}_E(a_{i,mi} \bar{s}_m^{k-i}) \geq -mi \operatorname{ord}_E(A) + (k-i) \operatorname{ord}_E(\bar{s}_m) > k \operatorname{ord}_E(\bar{s}_m)$$

for $i = 1, \ldots, k$. Hence

$$\operatorname{ord}_E(\bar{s}_m^k + a_{1,m} \bar{s}_m^{k-1} + a_{2,2m} \bar{s}_m^{k-2} + \cdots + a_{k,mk}) = k \operatorname{ord}_E(\bar{s}_m) \neq \infty,$$

contradicting (2.2). It follows that $\operatorname{ord}_E(\bar{s}_m) \geq -m \operatorname{ord}_E(A)$ and hence we conclude $\operatorname{div}_Y(\bar{s}_m) + mA \geq 0$ or $\bar{s}_m \in H^0(Y, \mathcal{O}_Y(mA)) = S_m$. $\square$

When $\mathfrak{a} \subseteq \mathcal{O}_X$ is a coherent sheaf of ideals on an algebraic variety (not necessarily normal) X defining a closed subscheme Z, the Rees algebra of $\mathfrak{a}$ is the graded sheaf of $\mathcal{O}_X$-algebras $\bigoplus_{m\geq 0} \mathfrak{a}^m$. Recall that the blowup of X along $\mathfrak{a}$ (alternatively, the blow-up of X along Z) is the variety $\operatorname{Bl}_{\mathfrak{a}} X = \operatorname{Bl}_Z X = \mathbf{Proj}_{\mathcal{O}_X} \bigoplus_{m\geq 0} \mathfrak{a}^m$ along with its birational projection morphism to X. On $\operatorname{Bl}_{\mathfrak{a}} X$, one has that the inverse ideal sheaf $\mathfrak{a}\mathcal{O}_{\operatorname{Bl}_{\mathfrak{a}} X} = \mathcal{O}_{\operatorname{Bl}_{\mathfrak{a}} X}(1)$ is invertible, *i.e.* $\mathfrak{a}\mathcal{O}_{\operatorname{Bl}_{\mathfrak{a}} X}$ is a sheaf of locally principal ideals. Furthermore, any morphism of varieties $W \to X$ such that $\mathfrak{a}\mathcal{O}_W$ is invertible factors uniquely through $\operatorname{Bl}_{\mathfrak{a}} X$ as

$$
\begin{tikzcd}
W \arrow[rr] \arrow[dr] & & \operatorname{Bl}_{\mathfrak{a}} X . \arrow[dl] \\
& X &
\end{tikzcd}
$$

This universal property determines $\operatorname{Bl}_{\mathfrak{a}} X$ up to canonical isomorphism.

In the case X is affine, we shall frequently fail to distinguish between a sheaf of ideals $\mathfrak{a} \subseteq \mathcal{O}_X$ and its global sections $\mathfrak{a} \subseteq \mathbb{C}[X]$. Choosing a set of generators $a_0, \ldots, a_N$ for $\mathfrak{a} \subseteq \mathbb{C}[X]$, the blowup has the following description: $\operatorname{Bl}_{\mathfrak{a}} X \subseteq X \times \mathbb{P}^N$ is simply the graph of the rational map $X \dashrightarrow \mathbb{P}^N$ given by

$$x \mapsto [a_0(x) : a_1(x) : \cdots : a_N(x)].$$

In particular, it is covered by affine open patches $U_i = \operatorname{Spec}(\mathbb{C}[X][\frac{a_0}{a_i}, \ldots, \frac{a_N}{a_i}])$ for $0 \leq i \leq N$. It is easy to see that $\mathfrak{a}\mathcal{O}_{\operatorname{Bl}_{\mathfrak{a}} X}$ is the restriction of $\mathcal{O}_{X\times\mathbb{P}^N}(1)$ and is thus relatively very ample.

Even when X is a normal variety, it may happen that the blowup $\operatorname{Bl}_{\mathfrak{a}} X$ of an ideal sheaf $\mathfrak{a}$ is not normal. In this case, however, one easily arrives at a model of X by taking the normalization of the blowup. We shall use the notation

$$\pi_{\mathfrak{a}} \colon Y_{\mathfrak{a}} = \overline{\operatorname{Bl}_{\mathfrak{a}} X} \to X$$

to denote the normalized blowup of X along $\mathfrak{a}$. As before, the ideal sheaf $\mathfrak{a}\mathcal{O}_{Y_{\mathfrak{a}}}$ is locally principal and hence (since $Y_{\mathfrak{a}}$ is normal) cuts out an effective Cartier divisor $F_{\mathfrak{a}}$ on the normalized blowup: equivalently, we have $\mathfrak{a}\mathcal{O}_{Y_{\mathfrak{a}}} = \mathcal{O}_{Y_{\mathfrak{a}}}(-F_{\mathfrak{a}})$. While $Y_{\mathfrak{a}}$ does not generally have a simple local description as for the blowup above, it is characterized up to canonical isomorphism by a similar universal property. If $\mu\colon W \to X$ is a morphism of normal varieties such that $\mathfrak{a}\mathcal{O}_W = \mathcal{O}_W(-F)$ is locally principal and defines effective Cartier divisor F, there is a unique morphism $\theta\colon W \to Y_{\mathfrak{a}}$ such that the diagram

$$\begin{tikzcd} W \arrow[rr, "\theta"] \arrow[dr, "\mu"'] & & Y_{\mathfrak{a}} \arrow[dl, "\pi_{\mathfrak{a}}"] \\ & X & \end{tikzcd}$$

is commutative. Furthermore, in this case, $\theta(W)$ is not contained in the support of F_α and one has $\theta^*(F_{\mathfrak{a}}) = F$.

2.2.2 Definition and Geometric Properties

Definition II.5. Suppose $\mathfrak{a}$ is an ideal sheaf on a normal variety X. Denote by $\pi_{\mathfrak{a}}\colon Y_{\mathfrak{a}} \to X$ the normalized blowup of X along $\mathfrak{a}$, and suppose $\mathfrak{a}\mathcal{O}_{Y_{\mathfrak{a}}} = \mathcal{O}_{Y_{\mathfrak{a}}}(-F_{\mathfrak{a}})$ cuts out an effective Cartier divisor $F_{\mathfrak{a}}$. Then the *Rees valuations of* $\mathfrak{a}$ are the divisorial valuations ord_E on $\mathbb{C}(X) \setminus \{0\}$ corresponding to the prime divisors E on Y in the support of $F_{\mathfrak{a}}$, and the *integral closure of* $\mathfrak{a}$ is simply the ideal sheaf $\bar{\mathfrak{a}} = \pi_{\mathfrak{a},*}\mathcal{O}_{Y_{\mathfrak{a}}}(-F_{\mathfrak{a}})$. Thus, when X is affine, we have

$$\begin{aligned} \bar{\mathfrak{a}} &= \left\{ f \in \mathbb{C}[X] \;\middle|\; \mathrm{ord}_E(f) \geq \mathrm{ord}_E(\mathfrak{a}) \quad \begin{array}{c}\text{for all prime}\\ \text{divisors } E \text{ on } Y_\alpha\end{array} \right\} \\ &= \left\{ f \in \mathbb{C}[X] \;\middle|\; \mathrm{ord}_E(f) \geq \mathrm{ord}_E(\mathfrak{a}) \quad \begin{array}{c}\text{for all Rees}\\ \text{valuations } \mathrm{ord}_E \text{ of } \mathfrak{a}\end{array} \right\}. \end{aligned}$$

When $\mathfrak{a} = \bar{\mathfrak{a}}$, we say the ideal sheaf $\mathfrak{a}$ is *integrally closed* or (more classically) *complete*. If $\tau \subseteq \mathfrak{a}$ is another ideal sheaf, then τ is a *reduction* of $\mathfrak{a}$ if $\bar{\tau} = \bar{\mathfrak{a}}$.

Proposition II.6. *Let $\mathfrak{a}$ be a nonzero ideal sheaf on a normal variety X of dimension n.*

(i) If $\mathfrak{a}$ is locally principal, then $\mathfrak{a} = \bar{\mathfrak{a}}$ is integrally closed.

(ii) If $\mu\colon W \to X$ is a morphism of normal varieties such that $\mathfrak{a}\mathcal{O}_W \neq 0$ (i.e. $\mu(W)$ is not contained in the closed subset of X determined by $\mathfrak{a}$), then $\bar{\mathfrak{a}}\mathcal{O}_W \subseteq \overline{\mathfrak{a}\mathcal{O}_W}$.

(iii) If $\pi\colon Y \to X$ is any model such that $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$ is locally principal, then $\pi_*\mathcal{O}_Y(-F) = \bar{\mathfrak{a}}$ and $\mathfrak{a}\mathcal{O}_Y = \bar{\mathfrak{a}}\mathcal{O}_Y$. In particular, if $\mathfrak{b}$ is another ideal sheaf on X, then $\bar{\mathfrak{b}} = \bar{\mathfrak{a}}$ if and only if $\mathfrak{b}\mathcal{O}_Y = \mathcal{O}_Y(-F)$. Also, $\overline{(\bar{\mathfrak{a}})} = \bar{\mathfrak{a}}$.

(iv) Locally on X, $\mathfrak{a}$ has a reduction generated by at most n elements.

Proof. Each statement is local on X, and thus we may assume without loss of generality that X is affine. For *(i)*, simply note blowing up a locally principal ideal sheaf has no effect, hence $Y_{\mathfrak{a}} = X$ and $\pi_{\mathfrak{a}}$ is the identity map. Next, suppose $\mu\colon W \to X$ is a morphism with $\mathfrak{a}\mathcal{O}_W \neq 0$ whose source W is normal. Let $\nu_{\mathfrak{a}}\colon V_{\mathfrak{a}} \to W$ be the normalized blowup of W along $\mathfrak{a}\mathcal{O}_W$ with $\mathfrak{a}\mathcal{O}_{W_{\mathfrak{a}}} = \mathcal{O}_{W_{\mathfrak{a}}}(-G_{\mathfrak{a}})$ for a Cartier divisor $G_{\mathfrak{a}}$ on $W_{\mathfrak{a}}$. Thus, $\mu \circ \nu_{\mathfrak{a}}$ factors uniquely through the normalized blowup $Y_{\mathfrak{a}}$of X along $\mathfrak{a}$, so that there is a morphism $\theta\colon V_{\mathfrak{a}} \to Y_{\mathfrak{a}}$ with $\theta^*(F_{\mathfrak{a}}) = G_{\mathfrak{a}}$ and a commuting diagram

$$\begin{tikzcd} V_{\mathfrak{a}} \arrow[r, "\theta"] \arrow[d, "\nu_{\mathfrak{a}}"'] & Y_{\mathfrak{a}} \arrow[d, "\pi_{\mathfrak{a}}"] \\ W \arrow[r, "\mu"'] & X \end{tikzcd} .$$

If $f \in \bar{\mathfrak{a}} \subseteq \mathbb{C}[X]$ does not vanish entirely on $\mu(W)$, then since $\mathrm{div}_{Y_{\mathfrak{a}}}(f \circ \pi_{\mathfrak{a}}) \geq F_{\mathfrak{a}}$ it follows

$$\nu_{\mathfrak{a}}^* \operatorname{div}_{W_{\mathfrak{a}}}(f \circ \mu) = \theta^* \operatorname{div}_{Y_{\mathfrak{a}}}(f \circ \pi_{\mathfrak{a}}) \geq \theta^* F_{\mathfrak{a}} = G_{\mathfrak{a}}.$$

This shows *(ii)*. Using the notation in *(iii)*, again we know there is a morphism $\delta\colon Y \to Y_{\mathfrak{a}}$ such that $\pi = \pi_{\mathfrak{a}} \circ \delta$ and $\delta^*(F_{\mathfrak{a}}) = F$. Thus, by the projection formula, we have

$$\pi_* \mathcal{O}_Y(-F) = \pi_{\mathfrak{a},*}(\delta_* \delta^* \mathcal{O}_{Y_{\mathfrak{a}}}(-F_{\mathfrak{a}})) = \pi_{\mathfrak{a},*}(-F_{\mathfrak{a}}) = \bar{\mathfrak{a}}.$$

Furthermore, from *(i)* and *(ii)*, it follows

$$\mathcal{O}_Y(-F) = \mathfrak{a}O_Y \subseteq \bar{\mathfrak{a}}\mathcal{O}_Y \subseteq \overline{\mathfrak{a}\mathcal{O}_Y} = \mathcal{O}_Y(-F)$$

and so we must have equality throughout. In particular, if $\bar{b} = \bar{\mathfrak{a}}$, we see $\mathfrak{b}\mathcal{O}_Y = \bar{\mathfrak{b}}\mathcal{O}_Y = \bar{\mathfrak{a}}\mathcal{O}_Y = \mathcal{O}_Y(-F)$ and also $\overline{(\bar{\mathfrak{a}})} = \bar{\mathfrak{a}}$. (The converse statement is trivial.) This shows *(iii)*. Lastly, suppose $\mathfrak{a} \subseteq \mathbb{C}[X]$ is generated by $a_1, \ldots, a_m \in \mathbb{C}[X]$ so that the invertible sheaf $\mathfrak{a}\mathcal{O}_{Y_{\mathfrak{a}}}$ is globally generated by the sections $a_1, \ldots, a_m$. Since $\pi_{\mathfrak{a}}^{-1}(\{p\})$ has dimension at most $n-1$ for each point $p \in X$, we can find n generic $\mathbb{C}$-linear combinations of these sections which globally generate $\mathfrak{a}\mathcal{O}_{Y_{\alpha}}$ over $\pi_{\mathfrak{a}}^{-1}(\{p\})$. If we call $\tau \subseteq \mathbb{C}[X]$ the ideal generated by these combinations, it follows immediately from *(iii)* that τ is a reduction of $\mathfrak{a}$ in a neighborhood of p. $\square$

Proposition II.7. *Let $\mathfrak{a}$ be an ideal sheaf on a normal affine variety X.*

(i)
$$\bar{\mathfrak{a}} = \left\{ f \in \mathbb{C}[X] \;\middle|\; \operatorname{ord}_E(f) \geq \operatorname{ord}_E(\mathfrak{a}) \quad \begin{array}{c}\text{for all divisorial}\\ \text{valuations } \operatorname{ord}_E \text{ on}\\ \mathbb{C}(X)\setminus\{0\}\end{array} \right\}.$$

(ii) $\mathfrak{a}$ is integrally closed if and only if there exists a set $\{\operatorname{ord}_{E_i}\}_{i\in I}$ of divisorial valuations on $\mathbb{C}(X)\setminus\{0\}$ and positive integers $\{\alpha_i\}_{i\in I}$ such that

$$\mathfrak{a} = \left\{ f \in \mathbb{C}[X] \;\middle|\; \operatorname{ord}_{E_i}(f) \geq \alpha_i \quad \text{for } i \in I \right\}.$$

Proof. Since the Rees valuations of $\mathfrak{a}$ are a subset of all of the divisorial valuations, the containment $\supseteq$ in *(i)* is clear. For the opposite inclusion, suppose $f \in \bar{\mathfrak{a}}$ and $\mathrm{ord}_E \colon \mathbb{C}(X) \setminus \{0\}$ is the divisorial valuation corresponding to a prime divisor E on a model $\pi \colon Y \to X$. Let $\theta \colon Y' \to Y$ be the normalized blowup of Y along $\mathfrak{a}\mathcal{O}_Y$ and set $\pi' = \pi \circ \theta$. On Y', we have $\mathfrak{a}\mathcal{O}_{Y'} = \mathcal{O}_{Y'}(-F')$ for an effective Cartier divisor F'. If $E' = \theta_*^{-1}E$ is the strict transform of E on Y', the divisorial valuations ord_E and $\mathrm{ord}_{E'}$ agree on $\mathbb{C}(X) \setminus \{0\}$. By Proposition II.6 *(iii)* we have $f \circ \pi' \in \mathfrak{a}\mathcal{O}_{Y'} = \mathcal{O}_{Y'}(-F')$ and thus

$$\mathrm{ord}_E(f) = \mathrm{ord}_{E'}(f) = \mathrm{ord}_{E'}(f \circ \pi') \geq \mathrm{ord}_{E'}(F') = \mathrm{ord}_{E'}(\mathfrak{a}) = \mathrm{ord}_E(\mathfrak{a}),$$

completing the proof of *(i)*.

For *(ii)*, one direction is immediate from the definition of integral closure. Indeed, if $\mathfrak{a} = \bar{\mathfrak{a}}$, then membership in $\mathfrak{a}$ can be verified by checking for the appropriate order of vanishing along the Rees valuations of $\mathfrak{a}$ (alternatively, we can also use *(i)*). Conversely, suppose

$$\mathfrak{a} \quad = \quad \left\{ \; f \in \mathbb{C}[X] \;\middle|\; \mathrm{ord}_{E_i}(f) \geq \alpha_i \quad \text{for } i \in I \quad \right\}$$

for a set $\{\mathrm{ord}_{E_i}\}_{i \in I}$ of divisorial valuations. Without loss of generality, we may assume $\alpha_i = \mathrm{ord}_{E_i}(\mathfrak{a})$ for each $i \in I$. But then from *(i)*, it follows that $\bar{\mathfrak{a}} \subseteq \mathfrak{a}$, whence $\bar{\mathfrak{a}} = \mathfrak{a}$. ◻

Proposition II.8. *Let X be a normal variety.*

(i) If $\mathfrak{a} \subseteq \mathfrak{b}$ are ideal sheaves on X, then $\bar{\mathfrak{a}} \subseteq \bar{\mathfrak{b}}$.

(ii) The (arbitrary) intersection of integrally closed ideal sheaves on X is integrally closed.

(iii) Radical ideal sheaves on X are integrally closed. In particular, for any ideal sheaf $\mathfrak{a}$ we have $\mathfrak{a} \subseteq \bar{\mathfrak{a}} \subseteq \sqrt{\mathfrak{a}}$.

(iv) If C is a Cartier divisor on a model $\pi\colon Y \to X$ such that π_*C is effective, then $\mathfrak{a} = \pi_*\mathcal{O}_Y(-C)$ is an integrally closed ideal on X.

Proof. Without loss of generality, we may assume X is affine. Then *(i)* follows immediately from Proposition II.7 *(i)* as $\operatorname{ord}_E(\mathfrak{a}) \geq \operatorname{ord}_E(\mathfrak{b})$ for all divisorial valuations ord_E on $\mathbb{C}(X) \setminus \{0\}$. Similarly, if $\{\mathfrak{a}_i\}_{i \in I}$ is a collection of integrally closed ideals on X, then

$$\bigcap_{i \in I} \mathfrak{a}_i \quad = \quad \left\{ f \in \mathbb{C}[X] \;\middle|\; \operatorname{ord}_E(f) \geq \alpha_E \quad \begin{array}{c} \text{for all divisorial} \\ \text{valuations } \operatorname{ord}_E \text{ on} \\ \mathbb{C}(X) \setminus \{0\} \end{array} \right\}$$

where $\alpha_E = \min\{ \operatorname{ord}_E(\alpha_i) \mid i \in I \}$. Thus, *(ii)* follows from Proposition II.7 *(ii)*.

To see *(iii)*, it now suffices to show that a prime ideal $\mathfrak{p} \subseteq \mathbb{C}[X]$ is integrally closed. Let Z be the irreducible closed subset of X determined by $\mathfrak{p}$. Thus, if $f \in \mathbb{C}[X]$, we have $f \in \mathfrak{p}$ if and only if $f|_Z = 0$. Consider the normalized blowup $\pi_{\mathfrak{p}}\colon Y_{\mathfrak{p}} \to X$ of X along $\mathfrak{p}$ with $\mathfrak{p}\mathcal{O}_{Y_{\mathfrak{p}}} = \mathcal{O}_{Y_{\mathfrak{p}}}(-F_{\mathfrak{p}})$. Since $\pi_{\mathfrak{p}}(F_{\mathfrak{p}}) = Z$, we can find a prime divisor E_Z on $Y_{\mathfrak{p}}$ with $\pi(E_Z) = Z$.[3] In particular, in order for the pullback $f \circ \pi_{\mathfrak{p}}$ of a regular function $f \in \mathbb{C}[X]$ to vanish along E_Z, it is necessary and sufficient for f to vanish along Z itself. Thus, we have

$$\mathfrak{p} \quad = \quad \{\, f \in \mathbb{C}[X] \;\mid\; \operatorname{ord}_{E_Z}(f) \geq 1 \,\}$$

is integrally closed. Lastly, if C is a Cartier divisor on a model $\pi\colon Y \to X$ such that

[3]Note that, when Z is not contained in the singular locus of X, the prime divisor E_Z is uniquely determined.

$\pi_* C$ is effective, it is immediate that $\pi_*\mathcal{O}_Y(-C) \subseteq \mathcal{O}_X$. Thus,

$$\pi_*\mathcal{O}_Y(-C) \;=\; \left\{ f \in \mathbb{C}[X] \;\middle|\; \operatorname{ord}_E(f) \geq \operatorname{ord}_E(C) \quad \begin{array}{c}\text{for all prime}\\ \text{divisors } E \text{ on } Y\end{array} \right\}$$

is an integrally closed ideal sheaf on X. □

2.2.3 Algebraic Properties and $\mathbb{Q}$-Coefficients

Proposition II.9. *Suppose $\mathfrak{a}$ is an ideal sheaf on a normal variety X. Then the normalization of the Rees algebra of $\mathfrak{a}$ equals $\overline{\bigoplus_{m\geq 0}\mathfrak{a}^m} = \bigoplus_{m\geq 0}\overline{\mathfrak{a}^m}$.*

Proof. The statement is local on X, and thus we may assume that X is affine. Note that $\bigoplus_{m\geq 0}\mathfrak{a}^m \subseteq \bigoplus_{m\geq 0}\overline{\mathfrak{a}^m}$, and again we will introduce an indeterminate t to view this as an inclusion of subrings of $\mathbb{C}(X)[t]$. Thus, if

$$\begin{aligned} R &= \mathbb{C}[X] + \mathfrak{a}t + \mathfrak{a}^2t^2 + \cdots + \mathfrak{a}^m t^m + \cdots \\ S &= \mathbb{C}[X] + \overline{\mathfrak{a}^1}t + \overline{\mathfrak{a}^2}t^2 + \cdots + \overline{\mathfrak{a}^m}t^m + \cdots \end{aligned} \tag{2.3}$$

we need to show the normalization $\bar{R}$ of R is actually equal to S. It follows from Lemma II.4 that S is normal and hence $\bar{R} \subseteq S$: furthermore, the methods used therein also show this is a graded inclusion of rings (as $\bar{R}$ is again closed under the action of $\lambda \in \mathbb{C} \setminus \{0\}$ on $\mathbb{C}(X)[t]$). But then it is clear that both $\mathbf{Proj}_X \bar{R}$ and $\mathbf{Proj}_X S$ are equal to the normalized blowup $Y_{\mathfrak{a}}$ of X along $\mathfrak{a}$, so $\bar{R}$ and S must agree in sufficiently large degree. In particular, S is a finitely generated $\bar{R}$-module with the same fraction field $\mathbb{C}(X)(t)$. As $\bar{R}$ is normal, we must have $\bar{R} = S$ as desired. □

Corollary II.10. *Suppose X is a normal affine variety and $\mathfrak{a} \subseteq \mathbb{C}[X]$ is an ideal.*

(i) If $f \in \mathbb{C}[X]$, then $f \in \bar{\mathfrak{a}}$ if and only if it satisfies an equation of the form

$$f^n + a_1 f^{n-1} + a_2 f^{n-2} + \cdots + a_n = 0$$

where $a_i \in \mathfrak{a}^i$ *for* $i = 1, \ldots, n$. *Alternatively,* $f \in \bar{\mathfrak{a}}$ *if and only if for some* $c \in \mathbb{C}[X] \setminus \{0\}$ *one has* $cf^l \in \mathfrak{a}^l$ *for infinitely many* $l \geq 0$.[4]

(ii) If $\mathfrak{b} \subseteq \mathbb{C}[X]$ *is another ideal with* $\mathfrak{a} \subseteq \mathfrak{b}$*, then* $\mathfrak{b} \subseteq \bar{\mathfrak{a}}$ *if and only if there is a positive integer* k *such that* $\mathfrak{b}^{k+1} = \mathfrak{a}\mathfrak{b}^k$. *More generally,* $\mathfrak{b} \subseteq \bar{\mathfrak{a}}$ *if and only if there is a finitely generated faithful* $\mathbb{C}[X]$*-module* M *such that* $\mathfrak{b}M = \mathfrak{a}M$.

Proof. Using the notation from (2.3), we have for $f \in \mathbb{C}[X]$ that $f \in \bar{\mathfrak{a}}$ if and only if $ft \in S$ satisfies an equation of integral dependence over R. If

$$f^n + a_1 f^{n-1} + a_2 f^{n-2} + \cdots + a_n = 0 \tag{2.4}$$

where $a_i \in \mathfrak{a}^i$ for $i = 1, \ldots, n$, then

$$(ft)^n + (a_1 t)(ft)^{n-1} + (a_2 t^2)(ft)^{n-2} + \cdots + (a_n t^n) = 0$$

is such an equation. Conversely, if (ft) satisfies an equation of integral dependence over R of degree n, the subsequent vanishing of the coefficient of t^n gives a relation of the form (2.4). For the second characterization, if there exists a $c \in \mathbb{C}[X] \setminus \{0\}$ with $cf^l \in \mathfrak{a}^l$ for infinitely many $l \geq 0$, then

$$\operatorname{ord}_E(c) + l \operatorname{ord}_E(f) \geq l \operatorname{ord}_E(\mathfrak{a})$$

for infinitely many $l \geq 0$ and all divisorial valuations ord_E on $\mathbb{C}(X) \setminus \{0\}$. It follows that $\operatorname{ord}_E(f) \geq \operatorname{ord}_E(\mathfrak{a})$ and thus $f \in \bar{\mathfrak{a}}$. For the other direction, if $f \in \bar{\mathfrak{a}}$ and satisfies (2.4), let $c \in \mathfrak{a}^n \setminus \{0\}$. Then for $l > n$ we have

$$cf^l = -(a_1 cf^{l-1} + a_2 cf^{l-2} + \cdots + a_n cf^{l-n})$$

and it follows by induction on l that $cf^l \in \mathfrak{a}^l$ for all $l \geq 0$.

[4] In fact, the proof below shows it is equivalent to require $cf^l \in \mathfrak{a}^l$ for all $l \geq 0$.

For *(ii)*, suppose first $b \subseteq \bar{\mathfrak{a}}$. Choose a set of generators $b_1, \ldots, b_m \in \mathbb{C}[X]$ for $\mathfrak{b}$. Suppose b_i satisfies an equation of the form (2.4) of degree n_i, so that $(\mathfrak{a} + \langle b_i \rangle)^{n_i} \subseteq \mathfrak{a}(\mathfrak{a} + \langle b_i \rangle)^{n_i - 1}$. Let $k = n_1 n_2 \cdots n_m$. Now, $\mathfrak{b}^{k+1}$ is generated by all of the monomials in $b_1, \ldots, b_m$ of degree $k+1$, and each of these monomials is divisible by $b_i^{n_i}$ for some i. Since $b_i^{n_i} \in \mathfrak{a}(\mathfrak{a} + \langle b_i \rangle)^{n_i - 1}$, each of these monomials lies in $\mathfrak{a}\mathfrak{b}^k$ and thus we have $\mathfrak{b}^{k+1} = \mathfrak{a}\mathfrak{b}^k$. Note that $\mathfrak{b}^k$ is certainly a finitely generated faithful $\mathbb{C}[X]$-module. Conversely, suppose M is a finitely generated faithful $\mathbb{C}[X]$-module with $\mathfrak{b}M = \mathfrak{a}M$. Choose generators $m_1, \ldots, m_n$ for M, and suppose $f \in \mathfrak{b}$. For each $i = 1, \ldots, n$, we can write

$$f m_i = a_{i,1} m_1 + a_{i,2} m_2 + \cdots + a_{i,n} m_n$$

for some $a_{i,j} \in \mathfrak{a}$. Let $A = (a_{i,j})$ be the associated matrix, and put $m = (m_j)$ to be the column vector given by the generators. If $\mathbb{1}_n$ is the $n \times n$ identity matrix, we have that $(f \cdot \mathbb{1}_n - A)$ kills m. Multiplying by the adjoint of this matrix, it follows that $\det(f \cdot \mathbb{1}_n - A)\mathbb{1}_n$ also kills m, and thus $\det(f \cdot \mathbb{1}_n - A)$ kills M. Since M is faithful, we must have $\det(f \cdot \mathbb{1}_n - A) = 0$, which is an equation of the form (2.4) for f. Thus, we conclude $f \in \bar{\mathfrak{a}}$ and it follows $\mathfrak{b} \subseteq \bar{\mathfrak{a}}$. □

Definition II.11 (Integral Closure with $\mathbb{Q}$-Coefficients)**.** Suppose $\mathfrak{a}$ is an ideal sheaf on a normal variety X and $\lambda \in \mathbb{Q}_{>0}$. Then one can define an ideal sheaf $\overline{\mathfrak{a}^\lambda}$ called the *integral closure of* $\mathfrak{a}$ *with coefficient* λ as follows. Write $\lambda = \frac{p}{q}$ with p, q positive integers. On an open set $U \subseteq X$ with $f \in \mathbb{C}[U]$, we have $f \in H^0(U, \overline{\mathfrak{a}^\lambda})$ if and only if $f^q \in H^0(U, \overline{\mathfrak{a}^p})$. We leave it as an exercise for the reader to check that $\overline{\mathfrak{a}^\lambda}$ is independent of the choice of p, q and that this definition agrees with the previously defined $\overline{\mathfrak{a}^m}$ for all positive integers m.

Proposition II.12. *Suppose* $\mathfrak{a}$ *is an ideal sheaf on a normal variety* X *defining* *a closed subscheme Z, $\lambda \in \mathbb{Q}_{>0}$, and $\pi_{\mathfrak{a}}\colon Y_{\mathfrak{a}} \to X$ is the normalized blowup of X along $\mathfrak{a}$ with $\mathfrak{a}\mathcal{O}_{Y_{\mathfrak{a}}} = \mathcal{O}_{Y_{\mathfrak{a}}}(-F_\alpha)$. Then $\overline{\mathfrak{a}^\lambda} = \pi_{\mathfrak{a},*}\mathcal{O}_{Y_{\mathfrak{a}}}(\lfloor -\lambda F_\alpha \rfloor)$. In particular, $\overline{\mathfrak{a}^\lambda}$ is integrally closed. Furthermore, for all sufficiently small rational numbers $\epsilon > 0$, we have $\overline{\mathfrak{a}^{\lambda-\epsilon}} = \overline{\mathfrak{a}^\lambda}$ and $\overline{\mathfrak{a}^\epsilon} = \sqrt{\mathfrak{a}}$.*

Proof. Write $\lambda = \frac{p}{q}$ for p, q positive integers. Without loss of generality, we may assume X is affine. For $f \in \mathbb{C}[X]$, we have

$$
\begin{aligned}
f^q \in \overline{\mathfrak{a}^p} \quad &\iff \quad q \operatorname{ord}_E(f) \geq p \operatorname{ord}_E(\mathfrak{a}) && \text{for all Rees valuations } ord_E \text{ of } \mathfrak{a} \\
&\iff \quad \operatorname{ord}_E(f) \geq \lceil \tfrac{p}{q} \operatorname{ord}_E(\mathfrak{a}) \rceil && \text{for all Rees valuations } ord_E \text{ of } \mathfrak{a} \\
&\iff \quad \operatorname{ord}_E(f) + \lfloor -\lambda \operatorname{ord}_E(\mathfrak{a}) \rfloor \geq 0 && \text{for all Rees valuations } \operatorname{ord}_E \text{ of } \mathfrak{a} \\
&\iff \quad f \in H^0(X, \pi_{\mathfrak{a},*}\mathcal{O}_{Y_{\mathfrak{a}}}(\lfloor -\lambda F_{\mathfrak{a}} \rfloor)).
\end{aligned}
$$

Thus, we see that $\overline{\mathfrak{a}^\lambda} = \pi_{\mathfrak{a},*}\mathcal{O}_{Y_{\mathfrak{a}}}(\lfloor -\lambda F_{\mathfrak{a}} \rfloor)$. If $0 < \epsilon << 1$ is sufficiently small, we have $\operatorname{ord}_E(\lfloor -\epsilon F_{\mathfrak{a}} \rfloor) = -1$ for all Rees valuations ord_E of $\mathfrak{a}$. In this case, we have

$$
\begin{aligned}
\overline{\mathfrak{a}^\epsilon} &= \left\{ f \in \mathbb{C}[X] \;\middle|\; \operatorname{ord}_E(f) \geq 1 \quad \begin{array}{c}\text{for all Rees} \\ \text{valuations } \operatorname{ord}_E \text{ of } \mathfrak{a}\end{array} \right\} \\
&= \left\{ f \in \mathbb{C}[X] \;\middle|\; f|_{\pi_{\mathfrak{a}}(E)} = 0 \quad \begin{array}{c}\text{for all Rees} \\ \text{valuations } \operatorname{ord}_E \text{ of } \mathfrak{a}\end{array} \right\} \\
&= \{ f \in \mathbb{C}[X] \mid f|_Z = 0 \} = \sqrt{\mathfrak{a}}
\end{aligned}
$$

as claimed. $\square$

CHAPTER III

Multiplier Ideals

3.1 First Properties

3.1.1 Log Resolutions

If $\mathfrak{a}$ is an ideal sheaf on a normal variety X (defining a closed subscheme Z), recall that a model $\pi\colon Y \to X$ is said to be a log resolution of the pair $(X, \mathfrak{a})$ (or of the pair (X, Z)) when:

(i) Y is smooth, and $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$ is the locally principal ideal sheaf of an effective Cartier divisor F;

(ii) The prime divisors which are either exceptional or appear in the support of F are smooth and intersect transversely.

The second condition has the following interpretation: on some neighborhood of each point $y \in Y$ there are local analytic coordinates $z_1, \ldots, z_n$ (centered at y) such that any divisor appearing in *(ii)* and passing through y is given locally by $z_j = 0$ for some j. A divisor on a smooth variety whose support satisfies this condition is said to have simple normal crossings. Note that the individual prime components of a simple normal crossings divisor are required to be smooth and thus cannot locally have multiple analytic branches or "self intersections."

In case X is an affine variety, one can interpret a log resolution $\pi\colon Y \to X$ of an ideal $\mathfrak{a} \subseteq \mathbb{C}[X]$ as a "separating" log resolution of the divisors of general members of $\mathfrak{a}$. Precisely, recall that a generic $\mathbb{C}$-linear combination $g = \lambda_1 g_1 + \lambda_2 g_2 + \cdots + \lambda_k g_k$ of generators $g_1, \ldots, g_k$ is called a general element of $\mathfrak{a}$ (with respect to this choice of generators). If $C = \operatorname{div}(g)$, then $\pi\colon Y \to X$ is also a log resolution of (X, C). Furthermore, if we write $\pi^*(C) = F + C_Y$ where $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$, then the divisor C_Y is smooth (and in particular reduced). Log resolutions are "separating" in the following sense: when g' is another element of $\mathfrak{a}$ with $C' = \operatorname{div}(g')$ and $\pi^*(C') = F + C'_Y$, it follows that C'_Y and C_Y have no irreducible components in common. Indeed, all of these facts follow by showing that

$$(\pi^* \operatorname{div}(g_1) - F), (\pi^* \operatorname{div}(g_2) - F), \ldots, (\pi^* \operatorname{div}(g_k) - F)$$

generate a base-point free linear series on Y; see Section 9.1 in [Laz04] for further details.

Because we are working in characteristic zero, log resolutions always exist according to a fundamental result of Hironaka [Hir64]. Yet log resolutions are far from unique: for example, additional blowups along smooth centers will produce larger resolutions. Any two log resolutions $\pi\colon Y \to X$ and $\pi'\colon Y' \to X'$, however, are always dominated by a third $\pi''\colon Y'' \to X$. Precisely, this means there are proper birational morphisms $\theta\colon Y'' \to Y$ and $\theta'\colon Y'' \to Y'$ such that $\pi'' = \pi' \circ \theta' = \pi \circ \theta$, *i.e.*

$$\begin{tikzcd} & Y'' \arrow[dl, "\theta"'] \arrow[dr, "\theta'"] \arrow[dd, "\pi''"] & \\ Y \arrow[dr, "\pi"'] & & Y' \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$

is a commutative diagram.

When X is smooth, one can find a log resolution which is a composition of blowups along smooth centers. In fact, any log resolution is dominated by another of this form. However, the process of finding a log resolution can be an extremely difficult and complicated in practice. To give a flavor for this procedure in a case of primary interest for the coming chapters, we sketch a proof of the existence of log resolutions for curves on smooth projective surfaces.

Proposition III.1. *If X is a smooth projective surface and C is an effective Cartier divisor on X, then $(X, \mathcal{O}_X(-C))$ has a log resolution which is a composition of point blowups.*

Proof. Suppose first that C is an irreducible curve (*i.e.* a prime divisor) on X. Suppose that $c \in C$ is a singular point with multiplicity $m > 1$. Denote by $p_a(C) = 1 - \chi(\mathcal{O}_C)$ the arithmetic genus of C. Consider the blowup $\pi' \colon X' \to X$ of X at c, and let $E' = \pi'^{-1}(c)$ be the unique exceptional divisor. Thus, we have $E' \simeq \mathbb{P}^1$ and $E' \cdot E' = -1$, and $\pi'^*(C) = C' + mE$ where $C' = \pi_*^{-1}$ is the strict transform of C. The adjunction formula tells us that

$$(K_X + C) \cdot C = 2p_a(C) - 2.$$

Recall also that $K_X = \pi'^* K_X + E'$. Thus, we have

$$\begin{aligned} 2p_a(C) - 2 &= (K_X + C) \cdot C \;=\; \pi'^*(K_X + C) \cdot \pi'^*(C) \\ &= \pi'^*(K_X + C) \cdot (C' + E) \;=\; \pi'^*(K_X + C) \cdot C' \\ &= (K_Y + C') \cdot C' + (m-1)E \cdot C' \\ &> (K_Y + C') \cdot C' \;=\; 2p_a(C') - 2 \end{aligned}$$

which implies $p_a(C') < p_a(C)$. Since the arithmetic genus is at least zero and cannot continue to drop indefinitely, blowing up the singular points of C repeatedly will eventually result in a model on which the strict transform of C must be smooth.

To finish the proof, we may now assume that C has smooth irreducible components. Suppose we have distinct prime divisors C_1, C_2 in the support of C with $C_1 \cdot C_2 > 1$. Let $c \in C_1 \cap C_2$ be an intersection point, and again let $\pi' \colon X' \to X$ be the blowup of X at c with $\pi'^{-1}(c) = E'$ the exceptional divisor. Then if C_1' and C_2' are the strict transforms of C_1 and C_2, respectively, we compute

$$C_1' \cdot E' = C_2' \cdot E' = 1$$

$$C_1 \cdot C_2 = \pi'^*(C_1) \cdot \pi'^*(C_2) = (C_1' + E) \cdot (C_2' + E) = C_1' \cdot C_2' + 1$$

so again $C_1 \cdot C_2 > C_1' \cdot C_2'$. We conclude that, after a sequence of blowups, we will have that all the components of the pullback of C will be smooth and have pairwise intersection either zero or one. In other word, the pullback of C will be a simple normal crossings divisor, and we have produced the desired log resolution. $\square$

3.1.2 Relative Canonical Divisors

As on a smooth variety, a normal variety X of dimension n has a well-defined canonical sheaf. Specifically, on $U = X_{\text{reg}}$, the sheaf of regular n-forms $\omega_U = \bigwedge^n \Omega_U$ is invertible. If $\iota \colon U \to X$ is the natural inclusion, the canonical sheaf $\omega_X = \iota_* \omega_U$ is a rank one invertible sheaf according to Proposition II.3. An integral Weil divisor whose associated subsheaf of $\mathbb{C}(X)$ is isomorphic to ω_X is called a canonical divisor, and the associated linear equivalence class is called the canonical class. When the canonical class is $\mathbb{Q}$-Cartier, we say that X is $\mathbb{Q}$-Gorenstein.

While the above approach to defining the canonical sheaf ω_X of a normal variety X may be the most natural from a geometric perspective, alternative constructions arising algebraically or via duality theory are also important. Recall that on X we can consider the bounded derived category $D^b_{coh}(X)$. Thus, the objects of $D^b_{coh}(X)$

are represented by bounded complexes of $\mathcal{O}_X$-modules with coherent cohomology up to quasi-isomorphism. In other words, two complexes $F^\bullet$ and $G^\bullet$ of $\mathcal{O}_X$-modules give rise the same object in $D^b_{coh}(X)$ when they are connected by a map of complexes that induces an isomorphism on cohomology; see [Har66]. We shall denote by $\omega_X^\bullet$ a normalized dualizing complex, and the canonical sheaf ω_X can also be characterized as the cohomology sheaf of $\omega_X^\bullet$ in degree $-n$. In fact, X is Cohen-Macaulay if and only if the cohomology sheaves of $\omega_X^\bullet$ in all other degrees vanish. The use of the formalism of derived categories and dualizing complexes in this thesis will be confined to the background material presented in this chapter.

When $\pi\colon Y \to X$ is a model of X, we may choose a representative K_X of the canonical class on X by setting $K_X = \pi_* K_Y$ where K_Y is a representative of the canonical divisor on Y. We shall always assume compatible choices of K_Y and K_X in this manner for a model $\pi\colon Y \to X$ without explicit mention. Suppose now additionally that X is $\mathbb{Q}$-Gorenstein, *i.e.* there is an integer $m > 0$ such that mK_X is a Cartier divisor. Then $\pi^* K_X = \frac{1}{m}\pi^*(mK_X)$ is a well-defined $\mathbb{Q}$-divisor on Y. By construction, there is an exceptionally supported $\mathbb{Q}$-divisor K_π such that

$$K_Y = \pi^* K_X + K_\pi.$$

We refer to K_π as the relative canonical divisor, and one checks that K_π is independent of the choice of canonical divisor on Y. In particular, whereas a canonical divisor is specified only up to linear equivalence, the relative canonical divisor is a uniquely determined $\mathbb{Q}$-divisor on Y. In general, K_π is neither integral nor effective; however, when X and Y are smooth, K_π is both as it is defined by the Jacobian determinant of π.

3.1.3 Definitions and Relation to Integral Closure

Definition III.2. Suppose X is a $\mathbb{Q}$-Gorenstein normal variety and $\mathfrak{a} \subseteq \mathcal{O}_X$ is an ideal sheaf. The *multiplier ideal* of the pair $(X, \mathfrak{a})$ with coefficient $\lambda \in \mathbb{Q}_{>0}$ is the ideal sheaf

$$\mathcal{J}(X, \mathfrak{a}^\lambda) = \pi_* \mathcal{O}_Y(\lceil K_\pi - \lambda F \rceil)$$

where $\pi \colon Y \to X$ is any log resolution of $(X, \mathfrak{a})$. Thus, when X is affine, we have

$$\mathcal{J}(X, \mathfrak{a}^\lambda) \quad = \quad \left\{ f \in \mathbb{C}[X] \;\middle|\; \operatorname{ord}_E(f) \geq \operatorname{ord}_E(\lfloor \lambda F - K_\pi \rfloor) \quad \begin{array}{c}\text{for all prime} \\ \text{divisors } E \text{ on } Y\end{array} \right\}.$$

For a more extensive introduction (in the smooth case) than will be provided herein, we refer the reader to [BL04]. A detailed account of the properties of multiplier ideals, applications, and further references, may be found in [Laz04]. One immediately checks that Definition III.2 is independent of the choice of log resolution.[1] For the sake of completeness, we sketch the argument here. Since any two log resolutions are dominated by a third, it suffices to verify

$$\pi'_* \mathcal{O}_{Y'}(\lceil K_{\pi'} - \lambda F' \rceil) = \pi_* \mathcal{O}_Y(\lceil K_\pi - \lambda F \rceil)$$

where $\mathfrak{a}\mathcal{O}_{Y'} = \mathcal{O}_{Y'}(F')$ for another log resolution $\pi' \colon Y' \to X$ is another log resolution admitting a morphism $\theta \colon Y' \to Y$ and a commuting diagram

$$\begin{tikzcd} Y' \arrow[rr, "\theta"] \arrow[dr, "\pi'"'] & & Y \arrow[dl, "\pi"] \\ & X & \end{tikzcd} .$$

Because $\pi'_* \mathcal{O}_{Y'}(\lceil K_{\pi'} - \lambda F' \rceil) = \pi_* \theta_* \mathcal{O}_{Y'}(K_\theta + \lceil \theta^*(K_\pi - \lambda F) \rceil)$, we can simply check

[1] In the case of rational surface singularities, the results of this thesis are often strongest when applied to the minimal resolution (see [Lip69]), i.e. the unique log resolution through which all others must factor.

that

$$\theta_* \mathcal{O}_{Y'}(K_\theta + \lceil D \rceil) = \mathcal{O}_Y(\lceil D \rceil)$$

for a $\mathbb{Q}$-divisor D on Y such that D and $\theta^*(D)$ are both simple normal crossings divisors. This fact reduces to an easy calculation in local analytic coordinates, and we refer the reader to Lemma 9.2.19 of [Laz04] for a complete proof.

Multiplier ideals were first described analytically. If X is a smooth affine variety and $\mathfrak{a} = \langle g_1, \ldots, g_k \rangle \subseteq \mathbb{C}[X]$, then one can check

$$\mathcal{J}(X, \mathfrak{a}^\lambda) \;=\; \left\{ f \in \mathbb{C}[X] \;\middle|\; \frac{|f|^2}{\sum_{i=1}^k |g_i|^2} \quad \begin{array}{l} \text{is a locally integrable} \\ \text{function } (i.e. \in L^1_{\text{loc}}) \end{array} \right\}.$$

Many properties which are immediate from Definition III.2 are unclear from this perspective (*e.g.* independence of choice of generators, or that $\mathcal{J}(X, \mathfrak{a}^\lambda)$ is a coherent algebraic sheaf). Nevertheless, the analytic description of multiplier ideals is particularly important as a source of intuition. The idea is that, when $g_1, \ldots, g_k$ define a subscheme Z with very bad singularities, they must vanish to high order and consequently $\frac{1}{\sum_{i=1}^k |g_i|^2}$ grows rapidly near Z. A function in the multiplier ideal must vanish enough to control the explosion of this kernel, and for this reason deeper or smaller multiplier ideals should be thought to correspond to "worse" singularities.

Proposition III.3. *Suppose X is a $\mathbb{Q}$-Gorenstein normal variety and $\mathfrak{a} \subseteq \mathcal{O}_X$ is an ideal sheaf. Then the multiplier ideal $\mathcal{J}(X, \mathfrak{a}^\lambda)$ is integrally closed. If m is any positive integer, then $\mathcal{J}(X, \mathfrak{a}^\lambda) = \mathcal{J}(X, (\mathfrak{a}^m)^{\frac{\lambda}{m}}) = \mathcal{J}(X, (\overline{\mathfrak{a}^m})^{\frac{\lambda}{m}})$ (in particular,*

$\mathcal{J}(X,\mathfrak{a})=\mathcal{J}(X,\bar{\mathfrak{a}})$). *Furthermore, if X is affine, we have*[2]

$$\mathcal{J}(X,\mathfrak{a}^{\lambda}) \;=\; \left\{ f\in\mathbb{C}[X] \;\middle|\; \operatorname{ord}_E(f)\geq \operatorname{ord}_E(\lfloor \lambda F-K_{\pi}\rfloor) \;\begin{array}{c}\text{for all divisorial}\\ \text{valuations } \operatorname{ord}_E \text{ on}\\ \mathbb{C}(X)\setminus\{0\}\end{array} \right\}.$$

Proof. It follows from Proposition II.8 *(iv)* that multiplier ideals are integrally closed. Furthermore, if $\pi\colon Y\to X$ is a log resolution of $(X,\mathfrak{a})$ with $\mathfrak{a}\mathcal{O}_Y=\mathcal{O}_Y(-F)$, by Proposition II.6 *(iii)* we have $\overline{\mathfrak{a}^m}\mathcal{O}_Y=\mathfrak{a}^m\mathcal{O}_Y=\mathcal{O}_Y(-mF)$. In particular, $\pi\colon Y\to X$ is also a log resolution of $(X,\overline{\mathfrak{a}^m})$ and $(X,(\mathfrak{a}^m))$, and $\mathcal{J}(X,\mathfrak{a}^{\lambda})=\mathcal{J}(X,(\mathfrak{a}^m)^{\frac{\lambda}{m}})=\mathcal{J}(X,(\overline{\mathfrak{a}^m})^{\frac{\lambda}{m}})$ from the definition of multiplier ideals. Lastly, if $\mu\colon X'\to X$ is any model, let $\theta\colon Y'\to X'$ be a log resolution of $(X',\mathfrak{a}\mathcal{O}_{X'})$ and set $\pi'=\mu\circ\theta$. It is easily seen that $\pi'\colon Y'\to X$ is a log resolution of $(X,\mathfrak{a})$. Since the divisorial valuation of $\mathbb{C}(X)\setminus\{0\}$ associated to any prime divisor E' on X' is the same as that arising from $\theta_*^{-1}(E')$, the remaining statement is clear. $\square$

Definition III.4. Suppose X is a $\mathbb{Q}$-Gorenstein normal variety and $\mathfrak{a}\subseteq\mathcal{O}_X$ is an ideal sheaf. If $\lambda\in\mathbb{Q}_{>0}$, we say the pair $(X,\mathfrak{a}^{\lambda})$ has *log terminal singularities* if $\mathcal{J}(X,\mathfrak{a}^{\lambda})=\mathcal{O}_X$. From Proposition III.3, we see that this is equivalent to

$$\lambda\operatorname{ord}_E(\mathfrak{a})-\operatorname{ord}_E(K_{\pi})<1$$

for all divisorial valuations ord_E on $\mathbb{C}(X)\setminus\{0\}$ corresponding to a prime divisor E living on a model $\pi\colon Y\to X$. If instead we have

$$\lambda\operatorname{ord}_E(\mathfrak{a})-\operatorname{ord}_E(K_{\pi})\leq 1$$

or all divisorial valuations ord_E on $\mathbb{C}(X)\setminus\{0\}$, we say $(X,\mathfrak{a}^{\lambda})$ has *log canonical singularities*. When the ideal under consideration is the trivial ideal, it will often be

[2]We remark that this characterization of multiplier ideals can be taken as the definition over a field of positive characteristic, where log resolutions are not known to exist in dimension greater than two.

omitted from the notation. In this case, we set $\mathcal{J}(X) = \mathcal{J}(X, \mathcal{O}_X)$ and say simply that X is log terminal or log canonical as appropriate. If X is log terminal, the *log canonical threshold* of $(X, \mathfrak{a})$ is

$$\sup \left\{ c \in \mathbb{Q}_{>0} \;\middle|\; (X, \mathfrak{a}^c) \begin{array}{c} \text{is log canonical} \\ \text{(or log terminal)} \end{array} \right\} = \sup \{ \, c \in \mathbb{Q}_{>0} \mid \mathcal{J}(X, \mathfrak{a}^c) = \mathcal{O}_X \, \}$$

Thus, the log canonical threshold of $(X, \mathfrak{a})$ is simply the infemum of the values

$$\frac{\operatorname{ord}_E(K_\pi) + 1}{\operatorname{ord}_E(\mathfrak{a})}$$

over all divisorial valuations ord_E on $\mathbb{C}(X)\backslash\{0\}$ which are positive along $\mathfrak{a}$. Note that one could also restrict attention to only those divisorial valuations corresponding to prime divisors on a single log resolution of $(X, \mathfrak{a})$. We will explore similar ideas more fully in Chapter V.

Proposition III.5. *Suppose X is a $\mathbb{Q}$-Gorenstein normal variety with log terminal singularities. If $\mathfrak{a}$ is any ideal and $\lambda \in \mathbb{Q}_{>0}$, we have $\overline{\mathfrak{a}^\lambda} \subseteq \mathcal{J}(X, \mathfrak{a}^\lambda)$.*

Proof. Let $\pi \colon Y \to X$ be a log resolution of $(X, \mathfrak{a})$ with $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$. We have $\overline{\mathfrak{a}^\lambda} = \pi_*\mathcal{O}_Y(\lfloor -\lambda F \rfloor)$ and $\mathcal{J}(X, \mathfrak{a}^\lambda) = \pi_*\mathcal{O}_Y(\lceil K_\pi - \lambda F \rceil)$, so it suffices to check

$$\lceil K_\pi - \lambda F \rceil \geq \lfloor -\lambda F \rfloor$$

on Y. Since X is log terminal, we have $\operatorname{ord}_E(K_\pi) > -1$ for all prime divisors E on Y. Thus, we have

$$\operatorname{ord}_E(K_\pi - \lambda F) > -1 + \operatorname{ord}_E(-\lambda F) \geq -1 + \operatorname{ord}_E(\lfloor -\lambda F \rfloor)$$

and it follows that $\operatorname{ord}_E(\lceil K_\pi - \lambda F \rceil) \geq \operatorname{ord}_E(\lfloor -\lambda F \rfloor)$ as desired. $\square$

There are many variations on the definition of a multiplier ideal. In the analytic setting, one can associate a multiplier ideal to any plurisubharmonic function on a complex manifold. We mention here one algebraic variant which will be useful later on. Consider now an effective $\mathbb{Q}$-divisors Δ on a normal $\mathbb{Q}$-Gorenstein variety X. We can find a positive integer m such that $m\Delta$ is an integral effective Weil divisor, and if we set

$$\mathcal{J}(X, \Delta) = \mathcal{J}(X, (\mathcal{O}_X(-m\Delta))^{\frac{1}{m}})$$

follows from Proposition III.3 that our definition is independent of the choice of integer m. When Δ is a $\mathbb{Q}$-Cartier divisor and $\pi \colon Y \to X$ is a log resolution of $(X, \mathcal{O}_X(-m\Delta))$, we have

$$\mathcal{J}(X, \Delta) = \pi_* \mathcal{O}_Y(\lceil K_\pi - \pi^*(\Delta) \rceil) \ .$$

In fact, the next proposition shows that every multiplier ideal is given locally as the multiplier ideal of a $\mathbb{Q}$-divisor.

Proposition III.6. *Suppose X is an affine $\mathbb{Q}$-Gorenstein normal variety, $\mathfrak{a} \subseteq \mathbb{C}[X]$ is an ideal sheaf, and λ is a positive rational number. Let $k > \lambda$ be a positive integer, and choose general elements $f_1, \ldots, f_k \in \mathfrak{a}$ (i.e. each f_i is a generic $\mathbb{C}$-linear combinations of a given set of generators for $\mathfrak{a}$). If*

$$\Delta = \lambda \cdot \frac{1}{k} \left(\operatorname{div}(f_1) + \operatorname{div}(f_2) + \cdots + \operatorname{div}(f_k) \right) \ ,$$

then $\mathcal{J}(X, \mathfrak{a}^\lambda) = \mathcal{J}(X, \Delta)$. In particular, if $\lambda < 1$ and C is the divisor of a general element of $\mathfrak{a}$, we have $\mathcal{J}(X, \mathfrak{a}^\lambda) = \mathcal{J}(X, \lambda C)$.

Proof. Before beginning, we remark that the main idea is essentially contained in the following fact: given any finite set of (divisorial) valuations, the general elements of an ideal (with respect to any set of generators) can be chosen so that they agree with the ideal along those valuations.

Turning towards a more detailed argument, for each i let $C_i = \mathrm{div}(f_i)$ and set $C_i^Y = \pi^*(C_i) - F$ where $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$. Now, the divisors

$$C_1^Y, C_2^Y, \ldots, C_k^Y$$

are all reduced (even smooth), and have no components in common with each other or with either K_π or F. Thus, it follows that

$$\begin{aligned} \lceil K_\pi - \pi^*\Delta \rceil &= \left\lceil K_\pi - (\lambda \cdot \tfrac{1}{k})(\textstyle\sum_{i=1}^{k}(F + C_i^Y)\right\rceil \\ &= \lceil K_\pi - \lambda F \rceil + \textstyle\sum_{i=1}^{k} \lceil -\tfrac{\lambda}{k} C_i^Y \rceil \quad = \quad \lceil K_\pi - \lambda F \rceil \end{aligned}$$

since $\lambda < k$. It follows at once that $\mathcal{J}(X, \mathfrak{a}^\lambda) = \mathcal{J}(X, \Delta)$. □

Note that there is an obvious obstruction to extending the definition of multiplier ideals to normal varieties X which are not $\mathbb{Q}$-Gorenstein; namely, there is no definitive way[3] to make sense of the relative canonical divisor.[4] However, if Δ is a $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, we can still define a multiplier ideal $\mathcal{J}(X, \Delta)$ as $\pi_*\mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta)\rceil)$ for a log resolution $\pi \colon Y \to X$ of (X, Δ). See Section 9.4.G of [Laz04] for further details as well as many other generalizations.

3.2 Local Vanishing and Applications

In this section, we wish to highlight some of the properties and applications of multiplier ideals which will be important later on. A more detailed account along with further references may be found in [Laz04].

[3]For surfaces, one may use numerical pullback to define the relative canonical divisor. We will return to this point at the beginning of the next chapter.

[4]See [DH09] for recent developments.

3.2.1 Local Vanishing for Multiplier Ideals

Recall that, given a sufficiently positive Cartier divisor on a smooth variety Y, one often has vanishing statements for the cohomology of certain invertible sheaves. Perhaps the most famous is the vanishing theorem of Kodaira: if A is ample, then $H^i(Y, \mathcal{O}_Y(K_Y + A)) = 0$ for all $i > 0$. An extremely powerful generalization of this statement is given below.

Theorem III.7 (Kawamata-Viehweg Vanishing). *Let Y be a smooth projective variety. Suppose the Cartier divisor D on Y is $\mathbb{Q}$-linearly equivalent to $B + \Phi$, where*

- *B is a big and nef $\mathbb{Q}$-divisor, and*
- *Φ is an effective divisor with simple normal crossings support satisfying $\lfloor \Phi \rfloor = 0$.*

Then $H^i(Y, \mathcal{O}_Y(D)) = 0$ for all $i > 0$.

The basic idea of the proof of Theorem III.7 is to use so-called "covering tricks" and resolution of singularities to reduce to the classical statement of Kodaira vanishing given above. We remark that Theorem III.7 has largely been the driving force behind the widespread use of $\mathbb{Q}$-divisors in birational algebraic geometry. The following two theorems should be thought of as local variants of Theorem III.7; the first theorem underlies many of the remarkable properties of multiplier ideals.

Theorem III.8 (Local Vanishing for Multiplier Ideals). *Suppose $\pi\colon Y \to X$ is a log resolution of the ideal sheaf $\mathfrak{a}$ on a normal $\mathbb{Q}$-Gorenstein variety X with $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$. Then $R^i\pi_*\mathcal{O}_Y(\lceil K_\pi - \lambda F \rceil) = 0$ for all $i > 0$ and $\lambda \in \mathbb{Q}_{>0}$.*

Theorem III.9 (Grauert-Riemenschneider Vanishing). *If $\pi\colon Y \to X$ is any resolution of singularities, then $R^i\pi_*\omega_Y = 0$ for all $i > 0$.*

The proofs of Theorems III.8 and III.9 proceed along a standard method of producing local variants of global vanishing statements via the following lemma.

Lemma III.10. *Suppose $\pi \colon Y \to X$ is a proper morphism of varieties, $\mathscr{F}$ is a coherent sheaf on Y, and A is a sufficiently ample divisor on X. Then $R^i\pi_*\mathscr{F} = 0$ for all $i > 0$ if and only if $H^i(Y, \mathscr{F} \otimes \mathcal{O}_Y(\pi^*(A))) = 0$ for all $i > 0$.*

Proof. Assume A is sufficiently ample that the coherent sheaves

$$(R^j\pi_*\mathscr{F}) \otimes_{\mathcal{O}_X} \mathcal{O}_X(A) = R^j\pi_*(\mathscr{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(\pi^*(A))) = R^j\pi_*\mathscr{F}(\pi^*(A))$$

for $j \geq 0$ are all globally generated and also satisfy $H^i(X, R^j\pi_*\mathscr{F}(\pi^*(A))) = 0$ for $i > 0$. Thus, we have

$$R^j\pi_*\mathscr{F} = 0 \quad \Longleftrightarrow \quad R^j\pi_*\mathscr{F}(\pi^*(A)) = 0 \quad \Longleftrightarrow \quad H^0(X, R^j\pi_*\mathscr{F}(\pi^*(A))) = 0\ .$$

Let $\mathscr{I}^\bullet$ be a bounded below complex of injectives representing $R\pi_*\mathscr{F}(\pi^*(A))$ in $D_{\mathrm{coh}}(X)$ (*i.e.* quasi-isomorphic to $R\pi_*\mathscr{F}(\pi^*(A))$). If $h^i(\_\_)$ denotes the i-the cohomology of a complex, it is easy to see $\Gamma(X, h^i(\mathscr{I}^\bullet)) = h^i(\Gamma(X, \mathscr{I}^\bullet))$ so that

$$\begin{aligned} H^0(X, R^j\pi_*\mathscr{F}(\pi^*(A))) &= \Gamma(X, h^i(\mathscr{I}^\bullet)) = h^i(\Gamma(X, \mathscr{I}^\bullet)) \\ &= h^i\left((R(\Gamma(X, \_\_) \circ R\pi_*)\left(\mathscr{F}(\pi^*(A))\right)\right) \\ &= h^i\left((R\Gamma(Y, \_\_)\left(\mathscr{F}(\pi^*(A))\right)\right) \\ &= H^i(Y, \mathscr{F}(\pi^*(A))) \end{aligned}$$

and the desired equivalence follows immediately. $\square$

Proof of Theorem III.9. The statement is local on X, and so we may assume X is affine. Let $\overline{X}$ be a compactification of X. We claim there is a resolution of singularities $\bar{\pi} \colon \overline{Y} \to \overline{X}$ such that $\bar{\pi}^{-1}(X) = Y$ and $\bar{\pi}|_Y = \pi$. Indeed, start by taking any compactification of Y. By taking the normalization of graph of the rational map induced by π, it is clear that there is a model over X which restricts to π over X. Now one can produce $\bar{\pi}\colon \overline{Y} \to \overline{X}$ by simply resolving the singularities of this model while preserving the smooth locus.

It now suffices to show $R^i\bar{\pi}_*\omega_{\overline{Y}} = 0$ for $i > 0$. Suppose A is a sufficiently ample Cartier divisor on $\overline{X}$. Then $\bar{\pi}^*(A)$ is a big and nef, and thus we have $H^i(Y, \mathcal{O}_Y(K_Y + \pi^*(A))) = 0$ for $i > 0$ by Theorem III.7. Our conclusion now follows from Lemma III.10. $\square$

Proof of Theorem III.8. Again, the statement is local on X, and so we may begin by assuming X is affine. Choose Δ as in the proof of Proposition III.6, so that

$$\lceil K_\pi - \lambda F \rceil = \lceil K_Y - \pi^*(K_X + \Delta) \rceil \ .$$

Shrinking X further as necessary, we may also assume there is a positive integer m and a rational function $f \in \mathbb{C}(X) \setminus \{0\}$ such that $\mathrm{div}_X(f) = m(K_X + \Delta)$.

As before, we need to compactify X, but in a much more careful manner than in the previous proof to preserve $\mathbb{Q}$-Cartier assumptions. Fix an embedding of X in $\mathbb{A}^N$, and view $\mathbb{A}^N = \mathbb{P}^N \setminus H$ as the complement of a hyperplane H in $\mathbb{P}^N$. Let $\overline{X}$ be the normalization of the (Zariski) closure of X in $\mathbb{P}^N$, and let D be the pullback of H to $\overline{X}$. Thus, we have $X \subseteq \overline{X}$ is an open subset, D is an effective Cartier divisor on $\overline{X}$, and the support of D is equal to $\overline{X} \setminus X$. We will assume $K_{\overline{X}}|_X = K_X$ (otherwise, simply replace K_X by $K_{\overline{X}}|_X$). Put

$$\bar{\Delta} = \frac{1}{m}\,\mathrm{div}_{\overline{X}}(f) - K_{\overline{X}} + nD$$

where n is a positive integer ensuring that $\bar{\Delta}$ is effective. Thus, $\bar{\Delta}$ is an effective $\mathbb{Q}$-divisor on $\overline{X}$ such that $(K_{\overline{X}} - \bar{\Delta})|_X = K_X + \Delta$ and $K_{\overline{X}} + \bar{\Delta}$ is $\mathbb{Q}$-Cartier.

Now, proceeding as in the proof of Theorem III.9, we can find a log resolution $\bar{\pi}\colon \overline{Y} \to \overline{X}$ of $(X, \bar{\Delta})$ such that $\bar{\pi}^{-1}(X) = Y$ and $\bar{\pi}|_Y = \pi$. Start by taking any compactification of Y. Taking the normalization of graph of the rational map induced by π, there is a model over X which restricts to π over X. Now one can produce $\bar{\pi}\colon \overline{Y} \to \overline{X}$ by simply resolving singularities while preserving Y, which is possible as Y is smooth and $\pi^*(\Delta)$ has simple normal crossings.

It now suffices to show $R^i\bar{\pi}_*(\lceil K_{\overline{Y}} - \pi^*(K_{\overline{X}} + \bar{\Delta})\rceil) = 0$ for $i > 0$. Suppose A is a sufficiently ample Cartier divisor on $\overline{X}$ with $(A - K_{\overline{X}} + \bar{\Delta})$ also ample. Then $B = \bar{\pi}^*(A - K_{\overline{X}} + \bar{\Delta})$ is a big and nef, and $\Phi = \{\bar{\pi}^*(K_{\overline{X}} + \bar{\Delta}) - K_{\overline{Y}}\}$ is an effective divisor with $\lfloor \Phi \rfloor = 0$ and simple normal crossings support. Our conclusion now follows from Lemma III.10 and Theorem III.7 as $\lceil K_Y - \bar{\pi}^*(K_{\overline{X}} + \bar{\Delta})\rceil + \pi^*(A)) = K_{\overline{Y}} + B + \Phi$ and thus $H^i(\overline{Y}, \mathcal{O}_Y(\lceil K_Y - \bar{\pi}^*(K_{\overline{X}} + \bar{\Delta})\rceil + \pi^*(A))) = 0$ for $i > 0$. $\square$

Our goal now is to highlight a pair of applications which underscore the importance of local vanishing for multiplier ideals. First, however, we need review a definition which will be very important in Chapter V. Recall that a normal variety X is said to have rational singularities if it satisfies any of the following equivalent conditions:

(1.) Some (equivalently any) resolution of singularities $\pi\colon Y \to X$ satisfies $R^i\pi_*\mathcal{O}_Y = 0$ for all $i > 0$.

(2.) X is Cohen-Macaulay, and some (equivalently any) resolution of singularities $\pi\colon Y \to X$ satisfies $\pi_*\omega_Y = \omega_X$.

(3.) For some (equivalently any) resolution of singularities $\pi\colon Y \to X$, the natural map $\mathcal{O}_X \to R\pi_*\mathcal{O}_Y$ in $D^b_{\mathrm{coh}}(X)$ has a splitting (*i.e.* a left inverse $R\pi_*\mathcal{O}_Y \to \mathcal{O}_X$

in the derived category).

Let us briefly review the equivalence of these conditions. [(1.) $\Longrightarrow$ (3.)] Since X is normal, we have that $\pi_*\mathcal{O}_Y = \mathcal{O}_X$ and thus by (1.) the natural map $\mathcal{O}_X \to R\pi_*\mathcal{O}_Y$ is an isomorphism in $D^b_{\mathrm{coh}}(X)$ (*i.e.* a quasi-isomorphism of complexes). [(3.) $\Longrightarrow$ (2.)] By assumption, we can find a composition

$$\mathcal{O}_X \to R\pi_*\mathcal{O}_Y \to \mathcal{O}_X$$

which is an isomorphism in $D^b_{\mathrm{coh}}(X)$. Applying $R\mathscr{H}om(\_\_, \omega_X^\bullet)$ and using Grothendieck duality, we get that

$$\omega_X^\bullet \leftarrow R\pi_*\omega_Y^\bullet \leftarrow \omega_X^\bullet$$

is also an isomorphism in $D^b_{\mathrm{coh}}(X)$. Since $R^i\pi_*\omega_Y = 0$ for $i > 0$ by Grauert-Riemenschneider Vanishing (Theorem III.9) and $\omega_Y^\bullet = \omega_Y[-n]$, we have $R\pi_*\omega_Y^\bullet = \pi_*\omega_Y[-n]$. In particular, the cohomology of $\omega_X^\bullet$ is concentrated in degree $-n$ and thus X is Cohen-Macaulay. Furthermore, the natural inclusion $\pi_*\omega_Y \to \omega_X$ is also surjective and so $\pi_*\omega_Y = \omega_X$. [(2.) $\Longrightarrow$ (1.)] Again, using Grauert-Riemenschneider Vanishing and (2.), we see $R\pi_*\omega_Y^\bullet = \omega_X^\bullet$ in $D^b_{\mathrm{coh}}(X)$. Applying $R\mathscr{H}om(\_\_, \omega_X^\bullet)$ and Grothendieck duality gives $R\pi_*\mathcal{O}_Y = \mathcal{O}_X$ in $D^b_{\mathrm{coh}}(X)$ yielding (1.).

3.2.2 Log Terminal Singularities are Rational and Skoda's Theorem

In Chapter V, we will focus our attention on varieties with rational singularities in dimension two. More specific information about surfaces with rational singularities will be given at that time. However, more immediately, we will be concerned with log terminal surfaces in the next chapter. Using local vanishing for multiplier ideals, we can easily see the relationship between log terminal and rational singularities.

Theorem III.11. *Suppose $(X, \mathfrak{a}^\lambda)$ has log terminal singularities, where X is a $\mathbb{Q}$-Gorenstein normal variety, $\mathfrak{a} \subseteq \mathcal{O}_X$ is an ideal sheaf, and $\lambda \in \mathbb{Q}_{>0}$. Then X must have rational singularities.*

Proof. We seek to show that X satisfies characterization (3.) of rational singularities. Let $\pi\colon Y \to X$ be a log resolution of $(X, \mathfrak{a})$ with $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$. Since $(X, \mathfrak{a}^\lambda)$ is log terminal, we have $\lceil K_\pi - \lambda F \rceil \geq 0$. Consider the inclusion

$$\mathcal{O}_Y \to \mathcal{O}_Y(\lceil K_\pi - \lambda F \rceil) \ . \tag{3.1}$$

By local vanishing for multiplier ideals, we have that

$$R\pi_*\mathcal{O}_Y(\lceil K_\pi - \lambda F \rceil) = \mathcal{J}(X, \mathfrak{a}^\lambda) = \mathcal{O}_X$$

in $D^b_{\mathrm{coh}}(X)$. Thus, applying $R\pi_*(\_\_)$ to (3.1) gives a map

$$R\pi_*\mathcal{O}_Y \to R\pi_*\mathcal{O}_Y(\lceil K_\pi - \lambda F \rceil) = \mathcal{O}_X$$

in $D^b_{\mathrm{coh}}(X)$ which is easily seen to split the natural inclusion of $\mathcal{O}_X \to R\pi_*\mathcal{O}_Y$. $\square$

We end this chapter with one final application of local vanishing. This result should be thought of as a kind of periodicity statement for multiplier ideals.

Theorem III.12 (Skoda's Theorem)**.** *Let $\mathfrak{a}$ be an ideal sheaf on a $\mathbb{Q}$-Gorenstein normal variety X of dimension n. Suppose that, locally on X, one can always find a reduction of $\mathfrak{a}$ which is generated by at most k elements (in particular, we can always take $k \leq n$). Then $\mathcal{J}(X, \mathfrak{a}^\lambda) = \mathfrak{a} \cdot \mathcal{J}(X, \mathfrak{a}^{(\lambda-1)})$ for all $\lambda \geq k$.*

Proof. We may assume that X is affine and that $\tau = \langle g_1, \ldots, g_k \rangle \subseteq \mathfrak{a} \subseteq \mathbb{C}[X]$ generate a reduction of $\mathfrak{a}$. Consider a log resolution $\pi\colon Y \to X$ with $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$.

Note that $\mathfrak{a} \cdot \mathcal{J}(X, \mathfrak{a}^{(\lambda-1)}) \subseteq \mathcal{J}(X, \mathfrak{a}^{\lambda})$ since

$$\begin{aligned} \operatorname{ord}_E(\mathfrak{a} \cdot \mathcal{J}(X, \mathfrak{a}^{(\lambda-1)})) &= \operatorname{ord}_E(\mathfrak{a}) + \operatorname{ord}_E(\mathcal{J}(X, \mathfrak{a}^{(\lambda-1)})) \\ &\geq \operatorname{ord}_E(F) + \operatorname{ord}_E(\lfloor (\lambda-1)F - K_\pi \rfloor) \\ &= \operatorname{ord}_E(\lfloor \lambda F - K_\pi \rfloor)\ . \end{aligned}$$

Since $\tau \mathcal{O}_Y = \mathcal{O}_Y(-F)$, the line bundle $\mathcal{O}_Y(-F)$ is globally generated by the sections $g_1, \ldots, g_k$ (after identifying $C(X)$ and $\mathbb{C}(Y)$). In particular, the Koszul complex $G_\bullet$ determined by these sections is necessarily exact. Recall that this complex has

$$G_i = \bigwedge^{k-i} \left(\mathcal{O}_Y(F)^{\oplus k} \right) \simeq \mathcal{O}_Y((k-i)F)^{\oplus \binom{k}{i}}$$

and the maps $G_i \to G_{i+1}$ are simply contraction with the section $g_1 \oplus g_2 \oplus \cdots \oplus g_k$ of $\mathcal{O}_Y(-F)^{\oplus k}$. When we tensor this complex by the invertible sheaf $\mathcal{O}_Y(\lceil K_\pi - \lambda F \rceil)$, it remains exact and the individual terms become

$$\mathcal{O}_Y(\lceil K_\pi - (\lambda - k + i)F \rceil)^{\oplus \binom{k}{i}}\ .$$

Since $R^j \pi_* \mathcal{O}_Y(\lceil K_\pi - (\lambda - k + i)F \rceil) = 0$ for $j > 0$ and all $0 \leq i \leq k$ by the local vanishing theorem for multiplier ideals (Theorem III.8), it follows that the pushforward of this tensored complex remains exact. In particular, at the k-th spot of the complex we have that

$$\mathcal{J}(X, \mathfrak{a}^{(\lambda-1)})^{\oplus k} \xrightarrow{(g_j)} \mathcal{J}(X, \mathfrak{a}^{\lambda}) \longrightarrow 0$$

is exact, and thus we see $\tau \cdot \mathcal{J}(X, \mathfrak{a}^{(\lambda-1)}) = \mathcal{J}(X, \mathfrak{a}^{\lambda})$. Since $\tau \subseteq \mathfrak{a}$, it follows immediately that $\mathfrak{a} \cdot \mathcal{J}(X, \mathfrak{a}^{(\lambda-1)}) = \mathcal{J}(X, \mathfrak{a}^{\lambda})$ as desired. $\square$

We would be remiss not to mention the following consequence.

Corollary III.13 (Briançon-Skoda)**.** *If X is a $\mathbb{Q}$-Gorenstein normal variety with log terminal singularities, then for any ideal sheaf $\mathfrak{a}$ we have $\overline{\mathfrak{a}^n} \subseteq \mathfrak{a}$.*

Proof. Since X is log terminal, it follows from Proposition III.5 that

$$\overline{\mathfrak{a}^n} \subseteq \mathcal{J}(X, \mathfrak{a}^n) = \mathfrak{a} \cdot \mathcal{J}(X, \mathfrak{a}^{n-1}) \subseteq \mathfrak{a} \ .$$

$\square$

Example III.14. Suppose n is a positive integer and $f_1, \ldots, f_{n+1} \in \mathbb{C}[x_1, \ldots, x_n]$ are polynomials in n variables. Then $f_1^n f_2^n \cdots f_{n+1}^n \in \overline{\langle f_1^{n+1}, f_2^{n+1}, \ldots, f_{n+1}^{n+1} \rangle^n}$, since

$$(f_1^n f_2^n \cdots f_{n+1}^n)^{n+1} \in \langle f_1^{n+1}, f_2^{n+1}, \ldots, f_{n+1}^{n+1} \rangle^{n(n+1)} \ .$$

By Corollary III.13, it follows that $f_1^n f_2^n \cdots f_{n+1}^n \in \langle f_1^{n+1}, f_2^{n+1}, \ldots, f_{n+1}^{n+1} \rangle$. For example, when $n = 2$, we have the elementary statement that $f^2 g^2 h^2 \in \langle f^3, g^3, h^3 \rangle$ for $f, g, h \in \mathbb{C}[x, y]$. The reader is challenged to give an elementary proof.

CHAPTER IV

Integrally Closed Ideals on Log Terminal Surfaces are Multiplier Ideals

4.1 Local Syzygies of Multiplier ideals

From this chapter onward, we shall be concerned only with local properties and constructions. As such, we shall adhere to the following notational shift. We will consider a scheme $X = \mathrm{Spec}(\mathcal{O}_X)$ where $\mathcal{O}_X$ is the local ring at a point on a normal complex variety. Equivalently, $\mathcal{O}_X$ is simply a local normal domain essentially of finite type over $\mathbb{C}$. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_X$ and set $k = \mathcal{O}_X/\mathfrak{m}$.

If M is an $\mathcal{O}_X$-module, recall that a free resolution $F^\bullet \to M$ is said to be minimal if each of the maps $F_i \to F_{i-1}$ vanishes after applying the functor $(\_\_ \otimes k)$. Alternatively, if we choose bases and represent $F_i \to F_{i-1}$ by a matrix, that matrix has entries in $\mathfrak{m}$. A minimal i-th syzygy of M is a nonzero element of the module $\mathrm{Syz}_i(M) = \mathrm{image}(F_i \to F_{i-1}) \subseteq F_{i-1}$ (called the i-th syzygy module of M) which is part of a minimal set of generators for $\mathrm{Syz}_i(M)$. In [LL07] and [LLS08], restrictions were found on the minimal syzygies of multiplier ideals.

Theorem IV.1. *Suppose X is $\mathbb{Q}$-Gorenstein of dimension d, $\lambda \in \mathbb{Q}_{>0}$, and $\mathfrak{a} \subseteq \mathcal{O}_X$ is an ideal.*

(i.) *If $\mathcal{O}_X$ is Cohen-Macaulay with system of parameters $z_1, \ldots, z_d$, then no minimal first syzygy of $\mathcal{J}(X, \mathfrak{a}^\lambda)$ vanishes modulo $(z_1, \ldots, z_d)^d$.*

(ii.) *If $\mathcal{O}_X$ is regular and $i \geq 1$, then no minimal i-th syzygy of $\mathcal{J}(X, \mathfrak{a}^\lambda)$ vanishes modulo $\mathfrak{m}^{d+1-i}$.*

We refer the reader to the original papers for the proofs of these results. Lazarsfeld and Lee used Theorem IV.1 *(ii.)* to show that, when the dimension d is at least three, smooth varieties have integrally closed ideal sheaves which cannot be realized as multiplier ideals. In fact, consider two general homogeneous cubic equations $f, g \in \mathbb{C}[x, y, z]$ (*e.g.* the defining equations of two general cubics in $\mathbb{P}^2$) and let $\mathcal{O}_X = \mathbb{C}[x, y, z]_{\langle x,y,z \rangle}$. One can show $\mathfrak{b} = \langle f, g \rangle + \mathfrak{m}^7 \subseteq \mathcal{O}_X$ is an integrally closed ideal, and that the Koszul syzygy $gf - fg = 0$ is a minimal first syzygy of $\mathfrak{b}$. Since this syzygy vanishes modulo $\mathfrak{m}^3$, it follows that $\mathfrak{b}$ cannot be realized as a multiplier ideal.

When X has dimension two, however, the story is very different. Concurrently, [LW03] and [FJ05] show that every integrally closed ideal on a smooth surface is a multiplier ideal. This lead [LLS08] to ask whether every integrally closed ideal closed ideal on a surface with rational singularities can be realized as a multiplier ideal. More precisely, one should ask:

Question IV.2. *Consider a scheme $X = \operatorname{Spec} \mathcal{O}_X$, where $\mathcal{O}_X$ is a two-dimensional local normal domain essentially of finite type over $\mathbb{C}$. If X has a rational singularity, is every integrally closed ideal which is contained in $\mathcal{J}(X, \mathcal{O}_X)$ a multiplier ideal?*

The remainder of this chapter is devoted to generalizing the methods of [LW03] and [FJ05] in order to prove the following:

Theorem IV.3. *Consider a scheme $X = \operatorname{Spec} \mathcal{O}_X$, where $\mathcal{O}_X$ is a two-dimensional local normal domain essentially of finite type over $\mathbb{C}$. Suppose X has log terminal singularities. Then every integrally closed ideal is a multiplier ideal.*

Recall from the previous chapter that log terminal singularities satisfy $\mathcal{J}(X, \mathcal{O}_X) = \mathcal{O}_X$ by definition and are necessarily rational. Thus, Theorem IV.3 gives a complete answer to the above question in this case.

4.2 Proof of Theorem IV.3

4.2.1 Relative Numerical Decomposition

Let $x \in X$ be the unique closed point, and suppose $f \colon Y \to X$ is a projective birational morphism such that Y is regular and $f^{-1}(x)$ is a simple normal crossing divisor. Let $E_1, \ldots, E_u$ be the irreducible components of $f^{-1}(x)$, and $\Lambda = \oplus_i \mathbb{Z} E_i \subset \operatorname{Div}(Y)$ the lattice they generate.

The intersection pairing $\operatorname{Div}(Y) \times \Lambda \to \mathbb{Z}$ induces a negative definite $\mathbb{Q}$-bilinear form on $\Lambda_\mathbb{Q}$ (see [Art66] for an elementary proof). Consequently, there is a dual basis $\check{E}_1, \ldots, \check{E}_u$ for $\Lambda_\mathbb{Q}$ defined by the property that

$$\check{E}_i \cdot E_j = -\delta_{ij} = \begin{cases} -1 & i = j \\ 0 & i \neq j \end{cases}.$$

Recall that a divisor $D \in \operatorname{Div}_\mathbb{Q}(Y)$ is said to be f-antinef if $D \cdot E_i \leq 0$ for all $i = 1, \ldots, u$. In this case, D is effective if and only if $f_* D$ is effective (see Lemma 3.39 in [KM98]). In particular, $\check{E}_1, \ldots, \check{E}_u$ are effective.

If $C \in \operatorname{Div}_\mathbb{Q}(X)$, we define the numerical pullback of C to be the unique $\mathbb{Q}$-divisor $f^* C$ on Y such that $f_* f^* C = C$ and $f^* C \cdot E_i = 0$ for all $i = 1, \ldots, u$. Note that, when C is Cartier or even $\mathbb{Q}$-Cartier, this agrees with the standard pullback of C. If

$D \in \mathrm{Div}_{\mathbb{Q}}(Y)$, we have

$$D = f^* f_* D + \sum_i (-D \cdot E_i)\check{E}_i. \tag{4.1}$$

We shall refer to this as a relative numerical decomposition for D. Note that, even when D is integral, both $f^* f_* D$ and $\check{E}_1, \ldots, \check{E}_u$ are likely non-integral. The fact that $f^* f_* D$ and $\check{E}_1, \ldots, \check{E}_u$ are always integral divisors when X is smooth and D is integral is equivalent to the unique factorization of integrally closed ideals. See [Lip69] for further discussion.

4.2.2 Antinef Closures and Global Sections

Suppose now that $D' = \sum_E a'_E E$ and $D'' = \sum_E a''_E E$ are f-antinef divisors, where the sums range over the prime divisors E on Y. It is easy to check that $D' \wedge D'' = \sum_E \min\{a'_E, a''_E\} E$ is also f-antinef. Further, any integral $D \in \mathrm{Div}(Y)$ is dominated by some integral f-antinef divisor (e.g. $(f^{-1}{}_*) f_* D + M(\check{E}_1 + \cdots + \check{E}_u)$ for sufficiently large and divisible M). In particular, there is a unique smallest integral f-antinef divisor $D^\sim$, called the f-antinef closure of D, such that $D^\sim \geq D$. One can verify that $f_* D = f_* D^\sim$, and in addition the following important lemma holds (see Lemma 1.2 of [LW03]). The proof also gives an effective algorithm for computing f-antinef closures.

Lemma IV.4. *For any $D \in \mathrm{Div}(Y)$, we have $f_* \mathcal{O}_Y(-D) = f_* \mathcal{O}_Y(-D^\sim)$.*

Proof. Let $s_D \in \mathbb{N}$ be the sum of the coefficients of $D^\sim - D$ when written in terms of $E_1, \ldots, E_u$. If $s_D = 0$, then $D = D^\sim$ is f-antinef and the statement follows trivially. Else, there is an index i such that $D \cdot E_i > 0$. As $E_i \cdot E_j \geq 0$ for $j \neq i$, we must have

$$D \leq D + E_i \leq D^\sim = (D + E_i)^\sim.$$

Thus, $s_{D+E_i} = s_D - 1$. By induction, we may assume

$$f_*\mathcal{O}_Y(-(D+E_i)) = f_*\mathcal{O}_Y(-(D+E_i)^{\sim}) = f_*\mathcal{O}_Y(-D^{\sim})$$

and it is enough to show $f_*\mathcal{O}_Y(-D) = f_*\mathcal{O}_Y(-(D+E_i))$. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-(D+E_i)) \longrightarrow \mathcal{O}_Y(-D) \longrightarrow \mathcal{O}_{E_i}(-D) \longrightarrow 0.$$

Since $\deg(\mathcal{O}_{E_i}(-D)) = -D \cdot E_i < 0$, we have $f_*\mathcal{O}_{E_i}(-D) = 0$; applying f_* yields the desired result. $\square$

4.2.3 Generic Sequences of Blowups

In the proof of Theorem IV.3, we will make use of the following auxiliary construction. Suppose $x^{(i)}$ is a closed point of E_i with $x^{(i)} \notin E_j$ for $j \neq i$. A generic sequence of n-blowups over $x^{(i)}$ is:

$$Y = Y_0 \xleftarrow{\sigma_1} Y_1 \xleftarrow{\sigma_2} \cdots \xleftarrow{\sigma_{n-1}} Y_{n-1} \xleftarrow{\sigma_n} Y_n$$

where $\sigma_1 \colon Y_1 \to Y_0$ is the blowup of $Y_0 = Y$ at $x_1 := x^{(i)}$, and $\sigma_k \colon Y_k \to Y_{k-1}$ is the blowup of Y_{k-1} at a generic closed point x_k of $(\sigma_{k-1})^{-1}(x_{k-1})$ for $k = 2, \ldots, n$. Let $\sigma \colon Y_n \to Y$ be the composition $\sigma_n \circ \cdots \circ \sigma_1$. We will denote by $E(1), \ldots, E(u)$ the strict transforms of $E_1, \ldots, E_u$ on Y_n. Also, let $E(i, x^{(i)}, k)$, $k = 1, \ldots, n$, be the strict transforms of the n new σ-exceptional divisors created by the blowups $\sigma_1, \ldots, \sigma_n$, respectively.

Lemma IV.5. *(a.) Let $\sigma \colon Y_n \to Y$ be a generic sequence of blowups over $x^{(i)} \in E_i$. Then one has*

$$\check{E}(i) \leq \check{E}(i, x^{(i)}, 1) \leq \cdots \leq \check{E}(i, x^{(i)}, n).$$

(b.) Suppose $D \in \mathrm{Div}(Y_n)$ is an integral $(f \circ \sigma)$-antinef divisor such that E_i is the unique component of $\sigma_* D$ containing $x^{(i)}$. If $\mathrm{ord}_{E(i)} D = a_0$ and $\mathrm{ord}_{E(i,x^{(i)},k)} D =$

a_k for $k = 1, \ldots, n$, then

$$a_0 \leq a_1 \leq \cdots \leq a_n.$$

Further, $a_0 < a_n$ if and only if

$$\left(\sum_{k=1}^{n} (-D \cdot E(i, x^{(i)}, k)) \check{E}(i, x^{(i)}, k) \right) \geq \check{E}(i).$$

Proof. If $n = 1$, we have

$$\check{E}(i, x^{(i)}, 1) = \left(\sigma^* \check{E}_i + E(i, x^{(i)}, 1) \right) \geq \sigma^* \check{E}_i = \check{E}(i)$$

$$D = \sigma^* \sigma_* D + (-D \cdot E(i, x^{(i)}, 1)) \check{E}(i, x^{(i)}, 1).$$

The general case of both statments follows easily by induction. □

4.2.4 Numerical Log Terminal Singularities and Multiplier Ideals

Once more, suppose $x \in X$ is the unique closed point and $f \colon Y \to X$ is a projective birational morphism such that Y is regular and $f^{-1}(x)$ is a simple normal crossing divisor. Let $E_1, \ldots, E_u$ be the irreducible components of $f^{-1}(x)$, and let K_Y be a canonical divisor on Y. Then $K_X := f_* K_Y$ is a canonical divisor on X. If we write the relative canonical divisor as

$$K_f := K_Y - f^* K_X = \sum_i b_i E_i$$

then X has numerically log terminal singularities if and only if $b_i > -1$ for all $i = 1, \ldots, u$. In this case, as we are working over $\mathbb{C}$, X is automatically $\mathbb{Q}$-factorial (see Proposition 4.11 in [KM98], as well as [DH09] for recent developments). Thus, a numerically log terminal surface is in fact log terminal in the sense of Definition III.4.

If $\mathfrak{a} \subseteq \mathcal{O}$ is an ideal and $f \colon Y \to X$ is as above and also a log resolution of $\mathfrak{a}$ with $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-G)$ for an effective divisor G. Thus, $\mathrm{Ex}(f) \cup \mathrm{Supp}(G)$ has simple normal crossings. In this case, we can define the (numerical) multiplier ideal of $(X, \mathfrak{a})$ with coefficient $\lambda \in \mathbb{Q}_{>0}$ as

$$\mathcal{J}(X, \mathfrak{a}^{\lambda}) = f_* \mathcal{O}_Y(\lceil K_f - \lambda G \rceil).$$

4.2.5 Choosing $\mathfrak{a}$ and λ

We now begin the proof of Theorem IV.3. For the remainder, assume X is log terminal, and let $I \subseteq \mathcal{O}_X$ be an integrally closed ideal. In this section, we construct another ideal $\mathfrak{a} \subseteq \mathcal{O}_X$ along with a coefficient $\lambda \in \mathbb{Q}_{>0}$; and in the following section it will be shown that $\mathcal{J}(X, \mathfrak{a}^{\lambda}) = I$. Let $f \colon Y \to X$ a log resolution of I with exceptional divisors $E_1, \ldots, E_u$. Suppose $I\mathcal{O}_Y = \mathcal{O}_Y(-F^0)$, and write

$$K_f = \sum_{i=1}^{u} b_i E_i$$

$$F^0 = (f^{-1}{}_*) f_*(F^0) + \sum_{i=1}^{u} a_i E_i.$$

Choose $0 < \epsilon < 1/2$ such that $\lfloor \epsilon (f^{-1}{}_*) f_*(F^0) \rfloor = 0$ and

$$\epsilon(a_i + 1) < 1 + b_i$$

for $i = 1, \ldots, u$. Note that, since X is log terminal, $1 + b_i > 0$ and any sufficiently small $\epsilon > 0$ will do. Let $n_i := \lfloor \frac{1+b_i}{\epsilon} - (a_i + 1) \rfloor \geq 0$, and $e_i := (-F^0 \cdot E_i)$. Choose e_i distinct closed points $x_1^{(i)}, \ldots, x_{e_i}^{(i)}$ on E_i such that $x_j^{(i)} \notin \operatorname{Supp}\big((f^{-1}{}_*) f_*(F^0)\big)$ and $x_j^{(i)} \notin E_l$ for $l \neq i$. Denote by $g : Z \to Y$ the composition of n_i generic blowups at each of the points $x_j^{(i)}$ for $j = 1, \ldots, e_i$ and $i = 1, \ldots, u$. As in Section 4.2.3, denote by $E(1), \ldots, E(u)$ the strict transforms of $E_1, \ldots, E_u$, and $E(i, x_j^{(i)}, 1), \ldots, E(i, x_j^{(i)}, n_i)$ the strict transforms of the n_i exceptional divisors over $x_j^{(i)}$.

Let $h := f \circ g$, $F = g^*(F^0)$, and choose an effective h-exceptional integral divisor

A on Z such that $-A$ is h-ample. It is easy to see that

$$K_g = \sum_{i=1}^{u} \sum_{j=1}^{e_i} \sum_{k=1}^{n_i} k\, E(i, x_j^{(i)}, k)$$

and one checks

$$K_g \cdot E(i) = e_i \qquad K_g \cdot E(i, x_j^{(i)}, k) = \begin{cases} 0 & k \neq n_i \\ -1 & k = n_i \end{cases} .$$

It follows immediately that $F + K_g$ is h-antinef. Choose $\mu > 0$ sufficiently small that

$$\lfloor (1+\epsilon)(F + K_g + \mu A) - K_h \rfloor = \lfloor (1+\epsilon)(F + K_g) - K_h \rfloor. \tag{4.2}$$

As $-(F + K_g + \mu A)$ is h-ample, there exists $N >> 0$ such that $G := N(F + K_g + \mu A)$ is integral and $-G$ is relatively globally generated.[1] In other words, $\mathfrak{a} := h_* \mathcal{O}_Z(-G)$ is an integrally closed ideal such that $\mathfrak{a}\mathcal{O}_Z = \mathcal{O}_Z(-G)$. Set $\lambda = \frac{1+\epsilon}{N}$.

4.2.6 Conclusion of Proof

Here, we will show $\mathcal{J}(X, \mathfrak{a}^\lambda) = I = h_* \mathcal{O}_Z(-F)$. Since

$$\mathcal{J}(X, \mathfrak{a}^\lambda) = h_* \mathcal{O}_Z(\lceil K_h - \lambda G \rceil) = h_* \mathcal{O}_Z(-\lfloor \lambda G - K_h \rfloor),$$

by Lemma IV.4, it suffices to show $F' := \lfloor \lambda G - K_h \rfloor^{\sim} = F$. In particular, we have reduced to showing a purely numerical statement.

Lemma IV.6. *We have $F' \leq F$ and $h_* F' = h_* F$. In addition, for $i = 1, \ldots, u$ and $j = 1, \ldots, e_i$,*

$$\operatorname{ord}_{E(i, x_j^{(i)}, n_i)}(F') = \operatorname{ord}_{E(i, x_j^{(i)}, n_i)}(F) = \operatorname{ord}_{E(i)}(F).$$

[1] Over $\mathbb{C}$, as X is log terminal, it also has rational singularities and by Theorem 12.1 of [Lip69] it follows that $-(F + K_g)$ is already globally generated without the addition of $-A$. However, the above approach seems more elementary, and avoids unnecessary reference to these nontrivial results.

Proof. Since $F' = \lfloor \lambda G - K_h \rfloor^{\sim}$ and F is h-antinef ($-F$ is relatively globally generated), it suffices to show these statements with $\lfloor \lambda G - K_h \rfloor$ in place of F'. By (4.2), we have

$$\begin{aligned} \lfloor \lambda G - K_h \rfloor &= \lfloor (1+\epsilon)(F + K_g) - K_h \rfloor \\ &= F + \lfloor \epsilon(F + K_g) - g^* K_f \rfloor. \end{aligned}$$

Since $\lfloor \epsilon(f^{-1}{}_*) f_* F^0 \rfloor = 0$, it follows immediately that $h_* \lfloor \lambda G - K_h \rfloor = h_* F$. For the remaining two statements, consider the coefficients of $\epsilon(F + K_g) - g^* K_f$. Along $E(i)$, we have $\epsilon a_i - b_i$, which is less than one by choice of ϵ. Along $E(i, x_j^{(i)}, k)$, we have $\epsilon(a_i + k) - b_i$. This expression is greatest when $k = n_i$, where our choice of n_i guarantees

$$0 \le \epsilon(a_i + n_i) - b_i < 1.$$

It follows that $\lfloor \lambda G - K_h \rfloor \le F$, with equality along $E(i, x_j^{(i)}, n_i)$. $\square$

Lemma IV.7. *For each $i = 1, \ldots, u$,*

$$(-F' \cdot E(i))\check{E}(i) + \sum_{j=1}^{e_i} \sum_{k=1}^{n_i} (-F' \cdot E(i, x_j^{(i)}, k))\check{E}(i, x_j^{(i)}, k) \quad \ge \quad (-F \cdot E(i))\check{E}(i).$$

Proof. If $\operatorname{ord}_{E(i)} F' = \operatorname{ord}_{E(i)} F$, as $F' \le F$ we have $F' \cdot E(i) \le F \cdot E(i)$ and the conclusion follows as $\check{E}(i)$ and $\check{E}(i, x_j^{(i)}, k)$ are effective and F' is h-antinef. Otherwise, if $\operatorname{ord}_{E(i)} F' < \operatorname{ord}_{E(i)} F = \operatorname{ord}_{E(i, x_j^{(i)}, n_i)} F'$, then for each $j = 1, \ldots, e_i$ we saw in Lemma IV.5(b) that

$$\sum_{k=1}^{n_i} (-F' \cdot E(i, x_j^{(i)}, k))\check{E}(i, x_j^{(i)}, k) \quad \ge \quad \check{E}(i).$$

Summing over all j gives the desired conclusion. $\square$

We now finish the proof by showing that $F' \ge F$. Using the relative numerical decomposition (4.1) and the previous two Lemmas, we compute

$$
\begin{aligned}
F' &= h^*h_*F' + \sum_{i=1}^{u}(-F'\cdot E(i))\check{E}(i) + \sum_{i=1}^{u}\sum_{j=1}^{e_i}\sum_{k=1}^{n_i}(-F'\cdot E(i,x_j^{(i)},k))\check{E}(i,x_j^{(i)},k) \\
&= h^*(h_*F) + \sum_{i=1}^{u}\left((-F'\cdot E(i))\check{E}(i) + \sum_{j=1}^{e_i}\sum_{k=1}^{n_i}(-F'\cdot E(i,x_j^{(i)},k))\check{E}(i,x_j^{(i)},k)\right) \\
&\geq h^*h_*F + \sum_{i=1}^{u}(-F\cdot E(i))\check{E}(i) = F.
\end{aligned}
$$

This concludes the proof of Theorem IV.3.

Corollary IV.8. *Consider a scheme $X = \operatorname{Spec}\mathcal{O}_X$, where $\mathcal{O}_X$ is a two-dimensional local normal domain essentially of finite type over $\mathbb{C}$. Suppose X has log terminal singularities and z_1, z_2 are a system of parameters for $\mathcal{O}_X$. If $\mathfrak{a} \subseteq \mathcal{O}_X$ is any integrally closed ideal, then no minimal first syzygy of $\mathfrak{a}$ vanishes modulo $\langle z_1, z_2\rangle^2$.*

CHAPTER V

Jumping Number Contribution on Algebraic Surfaces with Rational Singularities

5.1 Multiplier Ideals on Rational Surface Singularities

Again, we will consider a scheme $X = \operatorname{Spec}(\mathcal{O}_X)$ where $\mathcal{O}_X$ is the local ring at a point on a normal complex variety of dimension two. Recall that X is said to have a rational singularity if there exists a resolution of singularities $\pi : Y \to X$ such that $H^1(Y, \mathcal{O}_Y) = 0$. The theory of rational singularities of algebraic surfaces was first developed by Artin in [Art66] and [Art62], and studied extensively by Lipman in [Lip69]. We shall need various facts proved therein, and cite them without proof as necessary.

Suppose now that $\pi \colon Y \to X$ is a log resolution of an ideal sheaf $\mathfrak{a}$ on X with $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$. To check whether a function $f \in \mathcal{O}_X$ is in $\mathcal{J}(X, \mathfrak{a}^\lambda)$, one must show for all such E that

$$\operatorname{ord}_E f \geq \operatorname{ord}_E(\lfloor \lambda F - K_\pi \rfloor). \tag{5.1}$$

Consider what happens as one varies λ. Increasing λ slightly does not change (5.1), since the right side will remain the same. Thus, $\mathcal{J}(X, \mathfrak{a}^\lambda) = \mathcal{J}(X, \mathfrak{a}^{\lambda+\epsilon})$ for sufficiently small $\epsilon > 0$. However, continuing to increase λ further will cause the coefficient of E in $\lfloor \lambda F - K_\pi \rfloor$ to change, precisely when $\operatorname{ord}_E(\lfloor \lambda F - K_\pi \rfloor)$ is an integer. This change sometimes results in a jump in the mutliplier ideals $\mathcal{J}(X, \mathfrak{a}^{\lambda})$, and motivates the following definition.

Definition V.1. We say that $\lambda \in \mathbb{Q}_{>0}$ is a *candidate jumping number* for a prime divisor E appearing in F if $\mathrm{ord}_E(\lambda F - K_\pi)$ is an integer. If G is a reduced divisor on Y, a candidate jumping number for G is a common candidate jumping number for the prime divisors in its support. The coefficient $\lambda \in \mathbb{Q}_{>0}$ is a *jumping number* if $\mathcal{J}(X, \mathfrak{a}^{\lambda-\epsilon}) \neq \mathcal{J}(X, \mathfrak{a}^{\lambda})$ for all $\epsilon > 0$. Note that the smallest jumping number is the log canonical threshold of the pair $(X, \mathfrak{a})$ (*cf.* Definition III.4).

Since X is normal, note that condition (5.1) is trivial for $\mathrm{ord}_E(\lfloor \lambda F - K_\pi \rfloor) \leq 0$. We see explicitly that the nontrivial candidate jumping numbers for E are $\{\frac{\mathrm{ord}_E K_\pi + m}{\mathrm{ord}_E F} : m \in \mathbb{Z}_{>0}\}$. The jumping numbers of $(X, \mathfrak{a})$ are in general strictly contained in the union of the candidate jumping numbers of all of the prime divisors appearing in F. In particular, they form a discrete set of invariants. Furthermore, by Skoda's Theorem, the jumping numbers are eventually periodic; $\lambda > 2$ is a jumping number if and only if $\lambda - 1$ is a jumping number.

5.2 Jumping Numbers Contributed by Divisors

In order to compute the jumping numbers of $(X, \mathfrak{a})$ from a log resolution $\pi : Y \to X$, we must first understand the causes of the underlying jumps of the multiplier ideals. To this end, the following definitions allow us to attribute the appearance of a jumping number to certain reduced divisors on Y.

Definition V.2. Let G be a reduced divisor on Y whose support is contained in the support of F. We will say G *contributes* a candidate jumping number λ if

$$\mathcal{J}(X, \mathfrak{a}^{\lambda}) \subsetneq \pi_* \mathcal{O}_Y(\lceil K_\pi - \lambda F \rceil + G).$$

This contribution is said to be *critical* if, in addition, no proper subdivisor of G contributes λ, i.e.

$$\mathcal{J}(X, \mathfrak{a}^{\lambda}) = \pi_* \mathcal{O}_Y(\lceil K_\pi - \lambda F \rceil + G')$$

for all divisors G' on Y such that $0 \leq G' < G$.

Note that this is an extension of Definition 5 from [ST07], where Smith and Thompson introduced jumping number contribution for prime divisors. Further, if a jumping number is contributed by a prime divisor E, this contribution is automatically critical. It is easy to see that every jumping number is critically contributed by some reduced divisor on Y. The following example illustrates the original motivation for defining jumping number contribution.

Example V.3. Suppose R is the local ring at the origin in $\mathbb{A}^2$, and C is the germ of the analytically irreducible curve defined by the polynomial $x^{13} - y^5 = 0$. The minimal log resolution $\pi : Y \to X$ of C is a sequence of six blow-ups along closed points (there is a unique singular point on the transform of C for the first three blow-ups, after which it takes an additional three blow-ups to ensure normal crossings). If $E_1, \ldots, E_6$ are the exceptional divisors created, one checks

$$\pi^* C = C + 5E_1 + 10E_2 + 13E_3 + 25E_4 + 39E_5 + 65E_6$$

$$K_\pi = E_1 + 2E_2 + 3E_3 + 6E_4 + 10E_5 + 17E_6.$$

Thus, the nontrivial candidate jumping numbers of E_1 are $\{\frac{1+m}{5} : m \in \mathbb{Z}_{>0}\}$, whereas those for E_6 are $\{\frac{17+m}{65} : m \in \mathbb{Z}_{>0}\}$. One can compute[1] that the jumping numbers

[1] The polynomial $f(x, y) = x^{13} - y^5$ is nondegenerate with respect to its Newton polyhedron, and thus it is a theorem of Howald [How03] that the jumping numbers of f less than 1 coincide with those of its term ideal (x^{13}, y^5). One may then use the explicit formula [How01] for the jumping numbers of a monomial ideal to achieve the desired result. This argument is essentially repeated

of the pair $(\mathbb{A}^2, C)$ are precisely

$$\left\{\frac{13(r+1)+5(s+1)}{65}+t \;\;\middle|\;\; r,s,t\in\mathbb{Z}_{\geq 0} \text{ and } \frac{13(r+1)+5(s+1)}{65}<1\right\} \cup \ \mathbb{Z}_{>0}.$$

Note that the jumping numbers less than one are all candidate jumping numbers for E_6, but for no other E_i. Thus, for any jumping number $\lambda < 1$ and sufficiently small $\epsilon > 0$, we have

$$\mathcal{J}(X, \lambda C) \subsetneq \pi_* \mathcal{O}_Y(\lceil K_\pi - \lambda\pi^* C\rceil + E_6) = \mathcal{J}(X, (\lambda-\epsilon)C).$$

In other words, the jump in the multiplier ideal at λ is due solely to the change in condition (5.1) along E_6. According to Definition V.2, all of the jumping numbers less than one are contributed by E_6, and are not contributed by any other divisor.

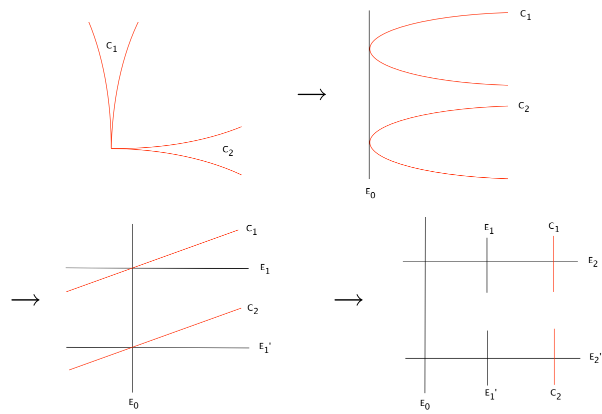
In general, however, the situation is often far less transparent. Distinct prime divisors often have common candidate jumping numbers. In some cases, as the next example from [ST07] shows, these prime divisors may separately contribute the same jumping number. In others, collections of these divisors may be needed to capture a jump in the multiplier ideals.

Example V.4. Suppose R is the local ring at the origin in $\mathbb{A}^2$, and C is the germ of the plane curve defined by the polynomial $(x^3 - y^2)(x^2 - y^3) = 0$ at the origin. The minimal log resolution π has five exceptional divisors: E_0 obtained from blowing up the origin; E_1 and E_1' obtained by blowing up the two intersections of E_0 with the transform of the curve C (both points of tangency); and E_2 (respectively E_2') obtained by blowing up the intersection of the three smooth curves C, E_0, and E_1

in Example 3.6 of [ELSV04], and discussed at greater length in Section 9.3.C of [Laz04]. Note that since this curve is analytically irreducible, the result also follows from [Jär06] or Chapter VI. It is also possible to use the numerical results of Section 5.4 to check this directly.

(respectively, the three smooth curves C, E_0, and E_1'). One checks

$$\pi^*C = C + 4E_0 + 5(E_1 + E_1') + 10(E_2 + E_2') \qquad K_\pi = E_0 + 2(E_1 + E_1') + 4(E_2 + E_2')$$



so that the log canonical threshold is $\frac{1}{2}$. Here, we have $\lceil K_\pi - \frac{1}{2}\pi^*C \rceil = -E_0 - E_2 - E_2'$, so that the three new conditions for membership in $\mathcal{J}(\frac{1}{2}C)$ are vanishing along E_0, E_2, E_2'. However, and herein lies the problem in determining the precise cause of the jump in the multiplier ideal, these are not independent conditions. Requiring vanishing along any of these three divisors automatically guarantees vanishing along the others. Thus, instead of attributing the jump to any prime divisor, it seems natural to suggest that the collection $E_0 + E_2 + E_2'$ is responsible. According to Definition V.2, $E_0 + E_2 + E_2'$ critically contributes $\frac{1}{2}$. Further, it is shown in [**?**] that $\frac{9}{10}$ is a jumping number contributed by either E_2 or E_2'. One may even argue there is a sense in which the collection $E_2 + E_2'$ is responsible for this jump. Indeed, for sufficiently small $\epsilon > 0$, we have

$$\begin{aligned}
\mathcal{J}(X, (\frac{9}{10} - \epsilon)C) &\subsetneq \pi_*\mathcal{O}_X(\lceil K_\pi - \frac{9}{10}\pi^*C \rceil + E_2) \\
&\subsetneq \pi_*\mathcal{O}_X(\lceil K_\pi - \frac{9}{10}\pi^*C \rceil + E_2 + E_2') \\
&= \mathcal{J}(X, \frac{9}{10}C).
\end{aligned}$$

In this case, the jumping number $\frac{9}{10}$ is contributed by $E_2 + E_2'$; however, this contribution is not critical as either E_2 or E_2' also contribute $\frac{9}{10}$.

Remark V.5. Contribution and critical contribution are somewhat subtle to formulate valuatively. If $G = E_1 + \cdots + E_k$ critically contributes λ, one can show there is some $f \in R$ which is not in $\mathcal{J}(X, \mathfrak{a}^\lambda)$ because it fails to satisfy condition (5.1) precisely along $E_1, \ldots, E_k$, and G is a minimal collection with this property. This depends not only on the divisorial valuations appearing in G, but all those appearing in F. In particular, there is no reason to believe this is independent of the chosen resolution. However, when X is smooth, it is possible to formulate a notion of contribution which is model independent by considering all possible resolutions simultaneously. Explicitly, it is shown in [FJ04] that the dual graphs of all resolutions fit together in a nice way to give the so-called valuative tree, and a reduced effective divisor on Y corresponds in a natural way to a union of subtrees of the valuative tree. Similar ideas were explored in [FJ05].

5.3 Numerical Criterion for Critical Contribution

We now begin working towards a numerical test for jumping number contribution. The first step is to interpret contribution cohomologically.

Proposition V.6. *Suppose that λ is a candidate jumping number for the reduced divisor G. Then λ is realized as a jumping number for $(X, \mathfrak{a})$ contributed by G if and only if*

$$H^0(G, (\lceil K_\pi - \lambda F \rceil + G)|_G) \neq 0.$$

Furthermore, this contribution is critical if and only if we have

$$H^0(G', (\lceil K_\pi - \lambda F \rceil + G')|_{G'}) = 0$$

for all divisors G' on Y such that $0 \leq G' < G$.

Proof. Consider the short exact sequence

$$0 \to \mathcal{O}_Y(\lceil K_\pi - \lambda F \rceil) \to \mathcal{O}_Y(\lceil K_\pi - \lambda F \rceil + G) \to \mathcal{O}_G((\lceil K_\pi - \lambda F \rceil + G)|_G) \to 0.$$

on Y. Pushing down to X, we arrive at

$$0 \longrightarrow \mathcal{J}(X, \mathfrak{a}^\lambda) \longrightarrow \pi_*\mathcal{O}_Y(\lceil K_\pi - \lambda F \rceil + G) \longrightarrow \cdots$$

$$\cdots \longrightarrow \pi_*\mathcal{O}_G((\lceil K_\pi - \lambda F \rceil + G)|_G) \longrightarrow R^1\pi_*\mathcal{O}_Y(\lceil K_\pi - \lambda F \rceil) \longrightarrow$$

However, local vanishing for multiplier ideals guarantees $R^1\pi_*\mathcal{O}_Y(\lceil K_\pi - \lambda F \rceil) = 0$. In particular, we see that G contributes a common candidate jumping number λ for $E_1, \ldots, E_k$ if and only if $\pi_*\mathcal{O}_G((\lceil K_\pi - \lambda F \rceil + G)|_G) = H^0(G, (\lceil K_\pi - \lambda F \rceil + G)|_G) \neq 0$. This implies both statements of the proposition. □

Corollary V.7. *If G critically contributes a jumping number λ, then G is connected.*

Proof. By way of contradiction, suppose we may write $G = G' + G''$ giving a separation, where $0 < G', G'' < G$ and G', G'' are disjoint. Then we have

$$H^0(G, (\lceil K_\pi - \lambda F \rceil + G)|_G)$$

$$=$$

$$H^0(G', (\lceil K_\pi - \lambda F \rceil + G')|_{G'}) \oplus H^0(G'', (\lceil K_\pi - \lambda F \rceil + G'')|_{G''}).$$

Thus, if G contributes a jumping number λ, either G or G' must also contribute λ. In particular, G does not critically contribute λ. □

Suppose now that G is a reduced divisor on Y with exceptional support. The prime exceptional divisors of π are all smooth rational curves intersecting transversely, and there are no loops of exceptional divisors. When X is smooth, this statement can be shown by induction on the number of blow-ups in π. More generally, Proposition 1 of [Art66] states that rational singularities are equivalent to $p_a(Z) \leq 0$ for all effective exceptional divisors Z, where $p_a(Z) = 1 - \chi(Z)$ denotes the arithmetic genus. We therefore assume that $G = E_1 + \cdots + E_k$ is a nodal tree of smooth rational curves. A global section s of $\mathcal{O}_G((\lceil K_\pi - \lambda F \rceil + G)|_G)$ is equivalent to a collection of global sections s_j of $\mathcal{O}_{E_j}((\lceil K_\pi - \lambda F \rceil + G)|_{E_j})$ for $j = 1, \ldots, k$ which agree on the intersections. Indeed, this statement is easy verified for two rational curves intersecting transversely, and the general case follows by induction on k. Since the existence of nonzero global sections on smooth rational curves is equivalent to having non-negative degree, we now show critical contribution by reduced exceptional divisors can be checked numerically. When G is prime and X is smooth, this criterion was given in [ST07].

Theorem V.8. *Denote by R the local ring at an isolated rational singularity on a normal complex surface. Let $\mathfrak{a} \subseteq R$ be an ideal, and $\pi : Y \to X = \operatorname{Spec}(R)$ a log resolution of $(X, \mathfrak{a})$ such that $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$. Suppose that λ is a candidate jumping number for the reduced divisor G with connected exceptional support.*

- *If $G = E$ is prime, then λ is (critically) contributed by E to X if and only if*
$$\lceil K_\pi - \lambda F \rceil \cdot E \geq -E \cdot E.$$
- *If G is reducible, then λ is critically contributed by G if and only if*
$$\lceil K_\pi - \lambda F \rceil \cdot E = -G \cdot E$$
for all prime divisors E in the support of G.

Proof. Suppose first $G = E$ is a single prime exceptional divisor. Then λ is contributed by E if and only if $H^0(E, (\lceil K_\pi - \lambda F \rceil + E)|_E) \neq 0$. Since $E \cong \mathbb{P}^1$, it is equivalent that this line bundle have non-negative degree, i.e. $\lceil K_\pi - \lambda F\rceil \cdot E \geq -E \cdot E$.

Thus, we assume $G = E_1 + \cdots + E_k$ is reducible. Theorem 1.7 of [Art62] concludes that the isomorphism class of a line bundle on G is determined by the degrees of its restrictions to $E_1, \ldots, E_k$. It follows that the numerical conditions given are sufficient. They are equivalent to saying $\mathcal{O}_G((\lceil K_\pi - \lambda F\rceil + G)|_G)$ restricts to the trivial bundle on each of $E_1, \ldots, E_k$, hence must be the trivial bundle on G. In particular, $H^0(G, (\lceil K_\pi - \lambda F\rceil + G)|_G) \neq 0$, and G contributes λ. To see this contribution is critical, note that if $0 \leq G' < G$, then the degree of $\mathcal{O}_{G'}((\lceil K_\pi - \lambda F\rceil + G')|_{G'})|_{E_i}$ along E_i is $-E_i \cdot (G - G')$. In particular, the sections of $\mathcal{O}_{G'}((\lceil K_\pi - \lambda F\rceil + G')|_{G'})$ are identically zero when restricted to to any component E_i of G' which intersects $G - G'$, and are constant along any other component of G'. Since G was connected, one sees any global section must be identically zero.

Now, assume G critically contributes λ, and let $s \in H^0(G, (\lceil K_\pi - \lambda F\rceil + G)|_G)$ be nonzero. There is some E in $\{E_1, \ldots, E_k\}$ such that $s|_E$ is nonzero. In particular, we see that the restriction of $\mathcal{O}_G((\lceil K_\pi - \lambda F\rceil + G)|_G)$ to E has non-negative degree. Suppose, by way of contradiction, its degree is strictly positive. Partition $G - E$ into its connected components, i.e. write $G - E = B_1 + \cdots + B_r$ where each B_i for $1 \leq i \leq r$ is the sum of all of the prime divisors in some connected component of $G - E$. Since G is a nodal tree, we have that $0 < B_i \leq G - E$ and $B_i \cdot E = 1$ for each $i = 1, \ldots, r$. Furthermore, observe that the supports of $B_1, \ldots, B_r$ are pairwise disjoint. Let $p_1, \ldots, p_r$ be the intersection points of $B_1, \ldots, B_r$ with E, respectively. Re-indexing if necessary, choose a point $q \in E \setminus \{p_2, \ldots, p_r\}$ such that $s(q) = 0$.

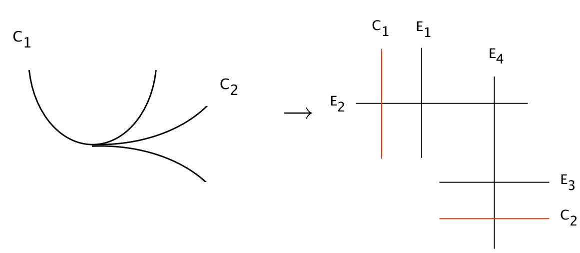
We will show that $G' = G - B_1$ contributes λ by proving

$$H^0(G', \mathcal{O}_{G'}((\lceil K_\pi - \lambda F\rceil + G')|_{G'}) \neq 0 \ .$$

For $i \neq 1$, we have $(\lceil K_\pi - \lambda F\rceil + G')|_{B_i} = (\lceil K_\pi - \lambda F\rceil + G)|_{B_i}$ since the supports of B_1 and B_i are disjoint. In particular, we may consider $s|_{B_i}$ as a global section of $\mathcal{O}_{B_i}((\lceil K_\pi - \lambda F\rceil + G')|_{B_i})$. Next, identify $s|_E$ with a nonzero homogeneous polynomial on $\mathbb{P}^1$ of strictly positive degree. Since $\deg(\mathcal{O}_E\ (\lceil K_\pi - \lambda F\rceil + G')|_E) = \deg(\mathcal{O}_E\ (\lceil K_\pi - \lambda F\rceil + G)|_E) - 1$, removing one of its linear factors corresponding to a zero at q yields a nonzero global section t of $\mathcal{O}_E((\lceil K_\pi - \lambda F\rceil + G')|_E)$. By construction, $t(p_i) \neq 0$ if and only if $s(p_i) \neq 0$ for $2 \leq i \leq r$. After scaling each $s|_{B_i}$ to agree with t at p_i, we may glue to obtain a nonzero global section of $\mathcal{O}_{G'}((\lceil K_\pi - \lambda F\rceil + G')|_{G'})$. But this is absurd, as it implies that G' contributes λ. Hence, we must have that $\deg(\mathcal{O}_E\ (\lceil K_\pi - \lambda F\rceil + G)|_E) = 0$. Furthermore, nonzero global sections of $\mathcal{O}_E\ (\lceil K_\pi - \lambda F\rceil + G)|_E$ never vanish. As s does not restrict to zero along any component of G which intersects E, the same arguments apply. Using that G is connected, the theorem now follows. □

Example V.9. Suppose R is the local ring at the origin in $\mathbb{A}^2$, and C is the germ of the plane curve defined by the polynomial $(y - x^2)(y^2 - x^5) = 0$. The minimal log resolution π is a sequence of four blow-ups along closed points (there is a unique singular point on the transform of C for the first two blow-ups, after which it takes an additional two blowups to ensure normal crossings), and is pictured below. If $E_1, \ldots, E_4$ are the exceptional divisors created, one checks

$$\pi^* C = C + 3E_1 + 6E_2 + 7E_3 + 14E_4 \qquad K_\pi = E_1 + 2E_2 + 3E_3 + 6E_4$$



The only candidate jumping number less than one shared by both E_2 and E_4 is $\frac{1}{2}$. One now computes directly that $\lceil K_\pi - \frac{1}{2}\pi^* C \rceil = -E_2 - E_4$, and Theorem V.8 now implies that $E_2 + E_4$ critically contributes the jumping number $\frac{1}{2}$. In Section 5.5, we will discuss how the numerical criteria in Theorem V.8 give an algorithm for numerically computing all of the jumping numbers in such examples. However, we postpone further discussion until after we have examined which collections of exceptional divisors have the potential to critically contribute jumping numbers.

5.4 Geometry of Contributing Collections

If $\mathcal{O}_X$ is the local ring at an isolated rational singularity of a normal complex surface, and $\mathfrak{a}$ is an ideal of $\mathcal{O}_X$, the Rees valuations of $\mathfrak{a}$ have a useful numerical description. If $\pi\colon Y \to X$ is a log resolution with $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$, since $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$ is globally generated, so is $\mathcal{O}_E(-F|_E)$ for any prime exceptional divisor E. In particular, we have $F \cdot E \leq 0$. Lemma 21.2 of [Lip69] shows that $F \cdot E < 0$ if and only if E corresponds to a Rees valuation of $\mathfrak{a}$.

In [ST07], it was shown that a prime exceptional divisor on the minimal resolution of a curve on a smooth surface contributes a jumping number if and only if it intersects at least three other components of the support of the pull-back of the curve. The following theorem gives analogous restrictions to critically contributing collections in our setting.

Theorem V.10. *Suppose $\mathcal{O}_X$ is the local ring at an isolated rational singularity of a normal complex surface. Let $\mathfrak{a} \subseteq \mathcal{O}_X$ be an ideal, and $\pi : Y \to X = \operatorname{Spec}(\mathcal{O}_X)$ a log resolution of $(X, \mathfrak{a})$ such that $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$. If the reduced divisor G with exceptional support critically contributes the jumping number λ to the pair $(X, \mathfrak{a})$,* *then G is a connected chain. The ends E of G must either:*

- *intersect at least three other prime divisors in the support of either F or K_π, or;*
- *correspond to a Rees valuation of $\mathfrak{a}$.*

Furthermore, the non-ends of G can intersect only those components of the support of F that also have λ as a candidate jumping number, and never correspond to a Rees valuation of $\mathfrak{a}$.

Proof. We will use the numerical criteria for critical contribution given in Theorem V.8. These are stated in terms of intersections with $\lceil K_\pi - \lambda F \rceil$, which we manipulate into the following form

$$\lceil K_\pi - \lambda F \rceil = -\lfloor \lambda F - K_\pi \rfloor = K_\pi - \lambda F + \{\lambda F - K_\pi\}.$$

Suppose first $G = E$ is a prime exceptional divisor, and E is not a Rees valuation of $\mathfrak{a}$. Then by Theorem V.8, since E contributes λ, we have that $\lceil K_\pi - \lambda F \rceil \cdot E \geq -E \cdot E$. Using the preceding equation and that $F \cdot E = 0$, we have

$$\{\lambda F - K_\pi\} \cdot E \geq 2,$$

where we have made use of the adjunction formula

$$-\deg \left.(K_\pi + E)\right|_E = -\deg K_E = 2 \tag{5.2}$$

applied to $E \cong \mathbb{P}^1$. Since λ is necessarily a candidate jumping number for E, it does not appear in $\{\lambda F - K_\pi\}$, which is an effective divisor with coefficients strictly less than one. As the divisors in K_π and F intersect transversely, at least three of them must intersect E in order for the above inequality to hold.

Assume now G is reducible. Since λ is critically contributed by G, we have that G is connected and $\lceil K_\pi - \lambda F \rceil \cdot E = -G \cdot E$ for all E in the support of G. As above, this gives

$$\{\lambda F - K_\pi\} \cdot E - \lambda F \cdot E = 2 - (G - E) \cdot E, \tag{5.3}$$

where we have made use of the adjunction formula (5.2) once more. Since $F \cdot E \leq 0$ and λ is a candidate jumping number for E, the left side of equation (5.3) is non-negative. Hence, we must have that $(G - E) \cdot E \leq 2$. As G is connected, in fact, $(G - E) \cdot E$ is either 1 or 2, so G is in fact a chain. If E is an end of G so that $(G - E) \cdot E = 1$ and E does not correspond to a Rees valuation of $\mathfrak{a}$, then

$$\{\lambda F - K_\pi\} \cdot E = 1.$$

It follows that E must intersect at least two components of F or K_π which do not have λ as a candidate jumping number. As it also intersects a component of G, all of which have λ as a candidate jumping number, the desired conclusion follows. On the other hand, if E is not an end of G so that $(G - E) \cdot E = 2$, we have

$$\{\lambda F - K_\pi\} \cdot E - \lambda F \cdot E = 0.$$

Thus, both terms on the left must vanish. In particular, $F \cdot E = 0$ so E does not correspond to a Rees valuation of E, and E can only intersect those components of F which also have λ as a candidate jumping number. $\square$

Remark V.11. Recently, Schwede and Takagi [ST08] have made use of *multiplier submodules*[2] in studying rational singularities of pairs. The multiplier submodules $\mathcal{J}(\omega_X, \mathfrak{a}^\lambda) = \pi_* \mathcal{O}_Y(\lceil K_Y - \lambda F \rceil)$ are indexed by the positive rational numbers, and

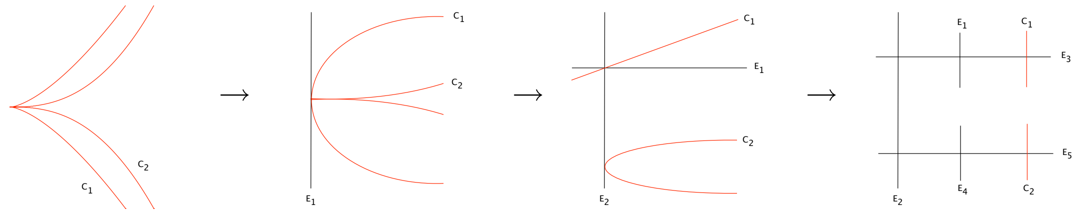
[2] These objects were also called *adjoint modules* in [HS03].

form nested sequence of submodules of the canonical module ω_X. These behave in a manner similar to multiplier ideals, and one can use them to define the rational threshold and rational jumping numbers of the pair $(X, \mathfrak{a})$. Since multiplier submodules satisfy the analogue of local vanishing, the same methods used above apply and give similar results for critical contribution of rational jumping numbers.

Remark V.12. Suppose momentarily that R is the local ring at a point on a smooth surface, and $\pi : Y \to X = \operatorname{Spec}(R)$ is the minimal resolution of the divisor C on X. In [ST07], it was shown that an exceptional divisor E which intersects three other prime divisors in the support of π^*C contributes a jumping number less than one. However, as the next example shows, a chain of exceptional divisors G in the minimal resolution of a plane curve C, where the ends E of G intersect at least three other prime divisors in the support of π^*C, may or may not critically contribute to the jumping numbers of the embedded curve. It remains unclear if additional geometric information would guarantee that G contributes a jumping number. A similar situation is found in [VV], where Van Proeyen and Veys are concerned with the poles of the topological zeta function. To determine whether or not a candidate pole is a pole, they also rely on both geometric and numerical data.

Example V.13. Suppose C is the germ of the plane curve defined by the polynomial $(y^2 - x^5)(y^2 - x^3) = 0$. It takes two blow-ups to separate the two components of C, creating divisors E_1 and E_2. At this point these components are both smooth. To ensure normal crossings, one must blow-up an additional point on the transform of the first component, and two additional points on the second, creating divisors E_3, E_4, and E_5, respectively. One checks

$$\pi^*C = C + 4E_1 + 7E_2 + 12E_3 + 8E_4 + 16E_5 \qquad K_\pi = E_1 + 2E_2 + 4E_3 + 3E_4 + 6E_5$$



By the Theorem V.10, the only possible chain of length greater than one that can contribute a jumping number is $E_2 + E_3 + E_5$. However, these three divisors do not share a common candidate jumping number less than one; hence, they cannot critically contribute any jumping number less than one. Notice the similarity between the exceptional divisors here and those in Example V.4. Despite the fact that the corresponding chains ($E_2 + E_3 + E_5$ here, and $E_0 + E_2 + E_2'$ in Example V.4) intersect their complements the same number of times, one chain contributes a jumping number while the other does not.

Proposition V.14. *Let $\mathfrak{a} \subseteq \mathcal{O}_X$ be an ideal, and $\pi\colon Y \to X$ a log resolution of $(X, \mathfrak{a})$ with $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$. Consider a reduced divisor G on Y with exceptional support, and suppose $\theta\colon Y' \to Y$ is such that $\pi' = \pi \circ \theta\colon Y' \to Y$ is also a log resolution of $(X, \mathfrak{a})$. Then a jumping number λ of $(X, \mathfrak{a})$ is critically contributed by G if and only if it is critically contributed by the unique chain G' containing $\theta_*^{-1}G$.*

Proof. Since Y is smooth, θ can be written as a composition of point blowups. Thus, we may assume without loss of generality that θ is in fact the blowup of a single closed point p on Y. Let E' be the exceptional divisor of θ.

A candidate jumping number fore G' is automatically a candidate jumping number for G since $\theta_*^{-1}G$ is a subdivisor of G'. If p lies on at most one component of G, then $G' = \theta_*^{-1}G$ and the converse statement is also clear. The only other possibility is for p to be the intersection point of two components E_1 and E_2 of G, in which case

$G' = \theta_*^{-1}G + E'$. Since

$$\lambda\theta^* F - K_{\pi'} = \theta^*(\lambda F - K_\pi) - E'$$

$$\mathrm{ord}_{E'}(\lambda\theta^* F - K_{\pi'}) = \mathrm{ord}_{E_1}(\lambda F - K_\pi) + \mathrm{ord}_{E_2}(\lambda F - K_\pi) - 1$$

we conclude the candidate jumping numbers for G and $G'$are always the same. Note also that, in this case,

Let us now verify the following claim: if D is a $\mathbb{Q}$-divisor on Y with simple normal crossings support disjoint from a π-exceptional divisor E, then $\{D\} \cdot E = \{\theta^* D\} \cdot \theta_*^{-1}E$. When $p \notin E$, the statement is clear as θ is an isomorphism over a neighborhood of E. Thus we suppose $p \in E$, let $E_1, \ldots, E_k$ be the components of the support of D which intersect E, and set p_i to be the intersection point of E_i with E. If $d_i = \mathrm{ord}_{E_i} D$, then $\{D\}\cdot E = \{d_1\}+\{d_2\}+\cdots+\{d_k\}$. Suppose first $p \neq p_i$ for any i. Then the components of the support of $\theta^* D$ intersecting $\theta_*^{-1}E$ are $\theta_*^{-1}E_1, \ldots, \theta_*^{-1}E_k$ and E'. We have that $\mathrm{ord}_E(\theta^* D) = \mathrm{ord}_E(D) \in \mathbb{Z}$, *i.e.* E' is not in the support of $\{\theta^* D\}$, and so the desired equality follows from $\mathrm{ord}_{\theta_*^{-1}E_i}(\theta^* D) = \mathrm{ord}_{E_i}(D) = d_i$. Next, assume $p = p_1$. Then the components of the support of $\theta^* D$ intersecting $\theta_*^{-1}E$ are $\theta_*^{-1}E_2, \ldots, \theta_*^{-1}E_k$ and E'. As $\mathrm{ord}_{E'}(\theta^* D) = \mathrm{ord}_{E_1}(D) = d_1$, the claim now follows.

Assume λ is a candidate jumping number for G and G'. From the claim, it follows that $\{\lambda F - K_\pi\} \cdot E = \{\lambda\theta^* F - K_{\pi'}\} \cdot \theta_*^{-1}E$ for all E in G. We now argue that

$$(\lceil K_{\pi'} - \lambda\theta^* F\rceil + G') \cdot \pi_*^{-1}E = (\lceil K_\pi - \lambda F\rceil + G) \cdot E \tag{5.4}$$

for all E in G. In case $p \notin G$, this follows from $G' = \pi^* G$ and $E' \cdot \pi_*^{-1}E = 0$ since $K_{\pi'} = \theta^* K_\pi + E'$ and $\lceil K_\pi - \lambda F\rceil = K_\pi - \lambda F + \{\lambda F - K_\pi\}$. When $p \in G$, it follows similarly as $G' = \theta^* G - E'$.

The proposition now follows from Theorem V.8. When p is on at most one component of G, (5.4) gives the equivalence of all of the necessary numerical conditions. If p is an intersection point of two components E_1 and E_2 of G, one must verify in addition $\lceil K_{\pi'} - \lambda\theta^* F\rceil \cdot E' = -1$. But this follows as

$$\lceil K_{\pi'} - \lambda\theta^* F\rceil = E' \cdot E' + \{\theta^*(\lambda F - K_\pi)\} \cdot E'$$

and none of $\theta_*^{-1}E_1, \theta_*^{-1}E_2, E'$ appear in $\{\theta^*(\lambda F - K_\pi)\}$ as λ is a candidate jumping number for G'. $\square$

Corollary V.15. *Suppose $\pi\colon Y \to X$ and $\pi'\colon Y' \to X$ are any two log resolutions of $(X, \mathfrak{a})$. Then there is a bijection between the critically contributing collections of exceptional divisors of on Y and Y' preserving the jumping numbers they contribute.*

5.5 Jumping Number Algorithm and Computations

We now describe an algorithm for computing the jumping numbers of $(X, \mathfrak{a})$ from a log resolution $\pi : Y \to X$. Let F be the effective divisor F on Y such that $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$, and $E_1, \ldots, E_r$ the prime divisors appearing in K_π or F.

Step 1. Compute the coefficients of the divisors $K_\pi = \sum_{i=1}^r b_i E_i$ and $F = \sum_{i=1}^r a_i E_i$, and use these to find the nontrivial candidate jumping numbers $\{\, \frac{b_i+m}{a_i} \,|\, m \in \mathbb{Z}_{>0} \,\}$ for each E_i which are at most equal to two.

Step 2. Next, we must determine those E_i which correspond to Rees valuations of $\mathfrak{a}$. The E_i which are not exceptional, i.e. the strict transforms of divisorial components of the subscheme defined by $\mathfrak{a}$, always correspond to Rees valuations. The prime exceptional divisors E_i corresponding to Rees valuations are characterized by the property that $E_i \cdot F < 0$. Also determine which E_i intersect at least three other E_j, for $j \neq i$.

Step 3. For each candidate jumping number $\lambda \leq 2$ appearing in the first step, perform the following series of checks to determine if λ is realized as an actual jumping number.

(i) If λ is a candidate jumping number for an E_i which is not exceptional and corresponds to a Rees valuation of $\mathfrak{a}$, then λ is realized as a jumping number contributed by E_i. Proceed to check the next candidate jumping number. Otherwise, continue to (ii).

(ii) Using the necessary geometric conditions from Theorem V.10, determine all of the connected chains of prime exceptional divisors which may critically contribute λ. Specifically, these are the connected exceptional chains $G = E_{i_1} + \cdots E_{i_k}$ such that λ is a candidate jumping number for each E_{i_j}, and the ends of G either correspond to Rees valuations or intersect at least three other E_i.

(iii) For each chain G from (ii), use the numerical criteria of Theorem V.8 to determine if λ is realized as a jumping number critically contributed by G. Specifically,

- If $G = E_{i_1}$ is prime, then λ is (critically) contributed by E_{i_1} to X if and only if

$$\lceil K_\pi - \lambda F \rceil \cdot E_{i_1} \geq -E_{i_1} \cdot E_{i_1}.$$

- If G is reducible, then λ is critically contributed by G if and only if

$$\lceil K_\pi - \lambda F \rceil \cdot E_{i_j} = -G \cdot E_{i_j}$$

for each of the prime divisors E_{i_j} in the support of G.

(iv) If we are not in the situation of (i) and λ were realized as a jumping number, it would be critically contributed by some collection of exceptional divisors. Indeed, the sum of all the exceptional divisors in F which share this candidate jumping number would contribute, and a minimal contributing collection would critically contribute. Thus, if (i) and (iii) have produced only negative answers, we deduce that λ cannot be a jumping number.

Step 4. From above, we now know all of the jumping numbers which are at most two. To determine the remaining jumping numbers, recall that the jumping numbers are eventually periodic; $\lambda > 2$ is a jumping number if and only if $\lambda - 1$ is also a jumping number. This concludes the algorithm for computing the jumping numbers of $(X, \mathfrak{a})$.

The remainder of this section focuses on a general scenario to which this method applies. We begin by altering our notation slightly. Assume $\mathcal{O}_X$ is the local ring at a rational singularity of a complex surface which is not smooth. Let $\pi\colon Y \to X$ be the minimal resolution of singularities of X, and $\mathfrak{m}$ the maximal ideal of $\mathcal{O}_X$. Since π is a composition of closed point blow-ups, and X is singular, it must begin with a blow-up along this singular point. Thus, π is also a minimal log resolution of $\mathfrak{m}$. In this case, the effective divisor Z cut out by the principal ideal sheaf $\mathfrak{m}\mathcal{O}_Y$ is called the *fundamental cycle* of X.

The fundamental cycle of X was first introduced by Artin in [Art66], where it was characterized numerically. We now recover this description while reproducing a summary from [Lip94] of results found in [Lip69]. Recall that a divisor D on Y is said to be *antinef* if $D \cdot E \leq 0$ for all prime exceptional divisors E on Y. By a fundamental result of Lipman, Theorem 12.1 in [Lip69], a divisor D on Y is antinef if and only if $\mathcal{O}_Y(-D)$ is globally generated. In particular, an antinef divisor is effective. It follows immediately that there is a bijective correspondence between complete ideals $I \subseteq \mathcal{O}_X$ such that $I\mathcal{O}_Y$ is invertible, and antinef divisors D on Y. Given a complete ideal $I \subseteq \mathcal{O}_X$, the principal ideal sheaf $I\mathcal{O}_Y$ cuts out an antinef divisor D. In other words, we have that $I\mathcal{O}_Y = \mathcal{O}_Y(-D)$ where D is antinef. Conversely, if D on Y is antinef, then $\pi_*\mathcal{O}_Y(-D) = H^0(Y, \mathcal{O}_Y(-D))$ is a complete ideal of $\mathcal{O}_X$. This correspondence is inclusion reversing, i.e. larger antinef divisors correspond to smaller ideals, and $\mathfrak{m}$-primary or *finite colength* ideals correspond to exceptionally supported antinef divisors. Since $\mathfrak{m}$ is the largest finite colength ideal of $\mathcal{O}_X$, Z is the unique smallest exceptionally supported antinef divisor on Y. In [Art66], it is shown that $-Z \cdot Z$ is the multiplicity of $\mathcal{O}_X$, and $-Z \cdot Z + 1$ is its embedding dimension. To compute Z, one may proceed as follows. Start with the reduced sum of all of the prime exceptional divisors on Y. Add an additional prime exceptional divisor E only if the intersection of E with this sum is positive, and repeat this process with the new sum of exceptional divisors. After finitely many iterations of this procedure, the corresponding sum will be antinef and must necessarily be equal to Z.

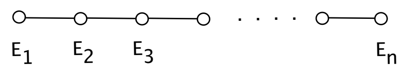
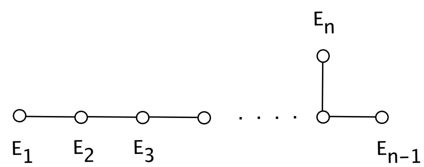
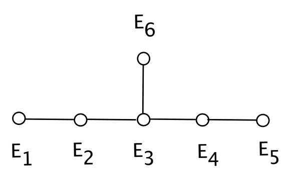
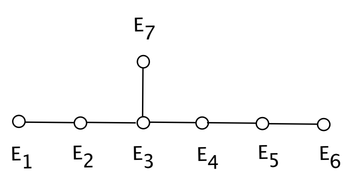
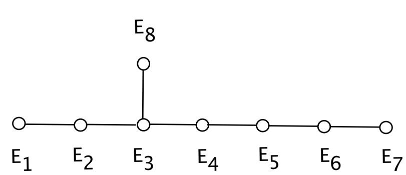
Once Z has been found, in order to compute the jumping numbers of $\mathfrak{m}$, we first need the relative canonical divisor K_π. Recall[3] that the restriction of the intersection product to the exceptional locus is negative definite. Thus, to compute K_π, it suffices to specify its intersection with any prime exceptional divisor E. Since $E \cong \mathbb{P}^1$, the adjunction formula once more gives $K_\pi \cdot E = -2 - E \cdot E$. Using the algorithm for finding jumping numbers described above, this shows how to compute the jumping numbers of $\mathfrak{m}$ starting from intersection matrix of the prime exceptional divisors on Y.

[3]Artin[Art66] attributes this fact to Mumford [Mum61], while Lipman [Lip69] gives credit to Du Val.

Example V.16 (Du Val Singularities)**.** In Figure 5.1, we give the results of applying the above techniques to the various types of *Du Val* singularities. In this case, the relative canonical divisor of the minimal resolution is zero, and all of the prime exceptional divisors have self-intersection -2. The dual graph corresponding to the exceptional locus is given by one of the Dynkin diagrams of type A, D, or E. See Section 4.3 of [Sha94] for a full description. Recall that $\lambda > 2$ is a jumping number if and only if $\lambda - 1$ is also a jumping number.

A_n $(n \geq 1)$	The fundamental cycle is $Z = E_1 + \cdots + E_n$, and both E_1 and E_n are Rees valuations of the maximal ideal. The log canonical threshold 1 is critically contributed by $E_1 + \cdots + E_n$, while all of the other jumping numbers are contributed by either E_1 or E_n.
D_n $(n \geq 4)$	The fundamental cycle is $Z = E_1 + E_n + E_{n-1} + 2E_2 + \cdots + 2E_{n-2}$, and E_2 is the only Rees valuation of the maximal ideal. The log canonical threshold $\frac{1}{2}$ is critically contributed by $E_2 + \cdots + E_{n-2}$, while all other jumping numbers are contributed by E_2.
E_6	The fundamental cycle is $Z = E_1 + 2E_2 + 3E_3 + 2E_4 + E_5 + 2E_6$, and E_6 is the only Rees valuation of the maximal ideal. The jumping numbers $\{\frac{1}{3} + \mathbb{Z}_{\geq 0}\}$ are contributed by E_3, while all other jumping numbers $\{\frac{3}{2} + \frac{1}{2}\mathbb{Z}_{\geq 0}\}$ are contributed by E_6.
E_7	The fundamental cycle is $Z = 2E_1 + 3E_2 + 4E_3 + 3E_4 + 2E_5 + E_6 + 2E_7$, and E_1 is the only Rees valuation of the maximal ideal. The jumping numbers $\{\frac{1}{4} + \mathbb{Z}_{\geq 0}\}$ are contributed by E_3, while all other jumping numbers $\{\frac{3}{2} + \frac{1}{2}\mathbb{Z}_{\geq 0}\}$ are contributed by E_1.

Figure 5.1: Jumping Numbers in Du Val Singularities

<table>
<tr><th>Type</th><th>Dual Graph</th><th>Jumping Numbers
of the Maximal Ideal</th></tr>
<tr><td>A_n $(n \geq 1)$</td><td>
</td><td>$\{1, 2, \ldots\}$</td></tr>
<tr><td>D_n $(n \geq 4)$</td><td>
</td><td>$\{\frac{1}{2}, \frac{3}{2}, 2, \ldots\}$</td></tr>
<tr><td>E_6</td><td>
</td><td>$\{\frac{1}{3}, \frac{4}{3}, \frac{3}{2}, 2, \ldots\}$</td></tr>
<tr><td>E_7</td><td>
</td><td>$\{\frac{1}{4}, \frac{5}{4}, \frac{3}{2}, 2, \ldots\}$</td></tr>
<tr><td>E_8</td><td>
</td><td>$\{\frac{1}{6}, \frac{7}{6}, \frac{3}{2}, 2, \ldots\}$</td></tr>
</table>

E_8 The fundamental cycle is $Z = 2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E_6 + 2E_7 + 3E_8$, and E_7 is the only Rees valuation of the maximal ideal. The jumping numbers $\{\frac{1}{6} + \mathbb{Z}_{\geq 0}\}$ are contributed by E_3, while all other jumping numbers $\{\frac{3}{2} + \frac{1}{2}\mathbb{Z}_{\geq 0}\}$ are contributed by E_7.

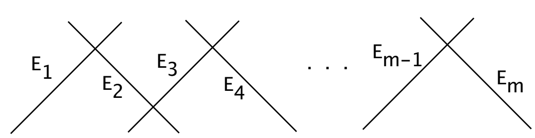
Example V.17 (Cyclic Quotient Surface Singularities)**.** Consider the action of the cyclic group of order n on $\mathbb{A}^2 = \operatorname{Spec}\mathbb{C}[x,y]$ given by

$$x \mapsto \zeta_n x \qquad y \mapsto (\zeta_n)^k y$$

where ζ_n is a primitive n-th root of unity, and $n > k$ are relatively prime positive integers. The quotient is a toric surface with a rational singularity. See [Ful93], Section 2.6, for a complete discription. Let R be the local ring at the singular point, and set $X = \operatorname{Spec}(R)$. Consider the Hirzebruch-Jung continued fraction

$$\frac{n}{k} = a_1 - \cfrac{1}{a_2 - \frac{1}{\cdots - \frac{1}{a_m}}}$$

of $\frac{n}{k}$, with integers $a_1, \ldots, a_m \geq 2$. The exceptional set of the minimal resolution $\pi\colon Y \mapsto X$ is a chain of m rational curves



where $E_i \cdot E_i = -a_i$ for $i = 1, \ldots, m$. The fundamental cycle is $Z = E_1 + \cdots + E_m$. To find the candidate jumping numbers of E_i, set $j_0 = 1$ and $j_1 = \frac{k+1}{n}$. Define $j_2, \ldots, j_m$ recursively by

$$j_{i+1} = a_i j_i - j_{i-1}.$$

One can check the nontrivial candidate jumping numbers of E_i are precisely $\{j_i + \mathbb{Z}_{\geq 0}\}$. Using that each $a_i \geq 2$ and the recursive definition, it is easy to see there is some $1 \leq k_1 \leq k_2 \leq m$ such that we have the inequalities

$$j_1 > j_2 > \cdots > j_{k_1} \qquad j_{k_1} = j_{k_1+1} = \cdots = j_{k_2} \qquad j_{k_2} < j_{k_2+1} < \cdots < j_m$$

and $j_1, j_m \leq 1$. These relationships allow one to progressively check the numerical conditions given in Theorem V.8, and we find the jumping numbers of the maximal ideal are precisely

$$\min\{j_1, \ldots, j_m\} \cup \left(\bigcup_{i \in \mathscr{R}} \{j_i + \mathbb{Z}_{>0}\} \right).$$

where $\mathscr{R} = \{1, m\} \cup \{\, i : \ a_i \geq 3 \,\}$ is the set of indices of the E_i corresponding to Rees valuations of the maximal ideal. The log canonical threshold $\min\{j_1, \ldots, j_m\}$ is critically contributed by $E_{j_{k_1}} + \cdots E_{j_{k_2}}$, while the jumping numbers $\{j_i + \mathbb{Z}_{>0}\}$ for $i \in \mathscr{R}$ are contributed by E_i.

5.6 Zariski-Lipman Theory of Complete Ideals on Smooth Surfaces and a Criterion for Simplicity

Before we begin, it is first necessary to review some of the Zariski-Lipman theory of complete ideals in two dimensional regular local rings. A good summary of this theory can be found in the introduction to [Lip69], as well as [Jär06]. Assume now $\mathcal{O}_X$ is regular and local with dimension two, and $\pi \colon Y \to X$ is a smooth model of X. Let $E_1, \ldots, E_n$ be the prime exceptional divisors, and consider $\Lambda = \mathbb{Z}E_1 + \cdots \mathbb{Z}E_n$ the lattice they generate. We have already made use of the dual basis of $\mathbb{Q}$-divisors $\check{E}_1, \ldots, \check{E}_n$ defined by the property that $\check{E}_i \cdot \check{E}_j = -1$ and $\check{E}_i \cdot E_j = 0$ for $i \neq j$. In our current setting, however, these are in fact integral divisors[4] and give a second $\mathbb{Z}$-basis for Λ. Note that these divisors generate the semigroup of antinef divisors in Λ. Indeed, $D = \check{d}_1 \check{E}_1 + \cdots + \check{d}_n \check{E}_n$ is antinef if and only if $\check{d}_i = -D \cdot E_i \geq 0$ for all $i = 1, \ldots, n$. It is not hard to see that the corresponding complete finite colength

[4]See Section 6.2 for an easy proof.

ideals $P_i = \pi_* \mathcal{O}_Y(-\check{E}_i)$ are simple, i.e. cannot be written nontrivially as a product of ideals.

Suppose $I = \pi_* \mathcal{O}_Y(-D)$ is the complete finite colength ideal corresponding to the antinef divisor $D = \check{d}_1 \check{E}_1 + \cdots + \check{d}_n \check{E}_n \in \Lambda$. Then we see immediately $I = P_1^{\check{d}_1} \cdots P_n^{\check{d}_n}$, and this factorization is unique as $\check{E}_1, \ldots, \check{E}_n$ are a basis for Λ. Further, the valuations on $\mathcal{K}$ corresponding to those E_i such that $\check{d}_i$ are nonzero are precisely the Rees valuations of I. As any complete ideal can be written uniquely as the product of a principal ideal and a finite colength complete ideal,[5] unique factorization extends to all complete ideals of $\mathcal{O}_X$.

For the remainder of this section, we fix the following notation. Let $\mathcal{O}_X$ be the local ring at a point on a smooth complex surface, and $\pi : Y \to X$ the minimal resolution of a complete finite colength ideal $\mathfrak{a} \subseteq \mathcal{O}_X$ such that $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$. Note that the numerical criterion for critical contribution can be simplified using the adjunction formula. A single exceptional prime divisor E contributes a candidate jumping number λ if and only if $-\lfloor \lambda F \rfloor \cdot E \geq 2$; a reducible chain of exceptional divisors G with common candidate jumping number λ critically contributes λ if and only if the ends E of G satisfy $-\lfloor \lambda F \rfloor \cdot E = 1$, and the non-ends E' of G satisfy $-\lfloor \lambda F \rfloor \cdot E' = 0$.

Proposition V.18. *A complete finite colength ideal $\mathfrak{a}$ in the local ring of a smooth complex surface is simple if and only if 1 is not a jumping number of $(X, \mathfrak{a})$.*

Proof. If $\mathfrak{a}$ is simple, then $\mathfrak{a} = P_i$ for some i, and we have that $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-\check{E}_i)$. Suppose, by way of contradiction, 1 is a jumping number of $(X, \mathfrak{a})$. We may assume there is a reduced chain of exceptional divisors G which critically contributes 1. G

[5] Observe that $I = (I^{-1})^{-1} \cdot (II^{-1})$ shows how this is achieved.

cannot be a single prime divisor E since $-\check{E}_i \cdot E$ is either 0 or 1. Thus, G is reducible and must have two distinct ends satisfying $-\check{E}_i \cdot E = 1$. Since this only happens for $E = E_i$, 1 is not a jumping number of $(X, \mathfrak{a})$.

Alternatively, assume that $\mathfrak{a} = P_1^{\check{d}_1} \cdots P_n^{\check{d}_n}$ is the finite colength complete ideal corresponding to the antinef divisor $D = \check{d}_1\check{E}_1 + \cdots + \check{d}_n\check{E}_n$, and $\mathfrak{a}$ is not simple. Suppose first there is some i such that $\check{d}_i \geq 2$. In this case, $-D \cdot E_i = \check{d}_i \geq 2$ shows that 1 is a jumping number contributed by E_i. Otherwise, we may assume $\check{d}_i$ is 0 or 1 for each i, and at least two such are nonzero. In this case, we can find two of them $\check{d}_{i_1} = \check{d}_{i_2} = 1$ such that for any E_j in the unique chain of exceptional divisors G connecting E_{i_1} and E_{i_2} we have $\check{d}_j = 0$. Theorem V.10 now gives that 1 is a jumping number of $(X, \mathfrak{a})$ critically contributed by G. $\square$

Remark V.19. The technique used in Corollary V.18 also shows that every chain of exceptional divisors critically contributes a jumping number for some ideal $\mathfrak{a} \subset \mathcal{O}_X$ having π as a resolution. Indeed, if G is the chain connecting E_{i_1} and E_{i_2}, then G critically contributes 1 to the ideal $P_{i_1}P_{i_2}$. One can also use this method to produce examples where many intersecting and nonintersecting chains critically contribute the same jumping number to a given pair.

CHAPTER VI

Jumping Numbers of Analytically Irreducible Plane Curves

6.1 Multiplicity Sequence and Dual Graph

Consider the local ring $\mathcal{O} = \mathcal{O}_{\mathbb{A}^2,0}$ at the origin in $\mathbb{A}^2$ and set $X = \operatorname{Spec}(\mathcal{O})$. Throughout this chapter, we will be concerned with the germs of plane curves at the origin. As confusion seems unlikely, we shall hereafter refer to a reduced effective divisor C on X as simply a curve. In this case, the branches of C are simply its irreducible components when regarded as an analytic germ. Thus, the branches can be described algebraically in the following manner. Consider a local defining equation $f \in \mathcal{O}$ for C as the germ of a holomorphic function, *i.e.* a convergent power series in a neighborhood of the origin. Since the ring of convergent power series $\mathbb{C}\{\{x, y\}\}$ is a unique factorization domain, it follows that $f = f_1 \cdot f_2 \cdots f_k$ can be written as a product of irreducible convergent power series; the holomorphic germs f_i give rise to the branches of C.

Two curves C_1 and C_2 are said to be *topologically equivalent* or *equisingular* if there are sufficiently small Euclidean neighborhoods U_1 and U_2 of the origin in $\mathbb{A}^2$ such that each C_i is defined on U_i and there is a homeomorphism (in the Euclidean topology) from U_1 to U_2 mapping C_1 onto C_2. It is well-known that two curves are equisingular if and only if there is a bijection between their branches which preserves the pairwise local intersection numbers of each. As such, one is often reduced to consider curves with a single branch. Each branch itself is locally homeomorphic to an open Euclidean ball in $\mathbb{A}^1$, and since every analytic germ with an isolated singularity is algebraic, there is generally no harm in assuming each branch actually comes from a curve. Thus, we are led to the following definition.

Definition VI.1. A curve C is said to be *analytically irreducible* or *unibranch* if one of the following equivalent conditions holds:

(1.) A local defining equation $f \in \mathcal{O}$ for C remains irreducible after passing to either *(i)* the ring of germs of holomorphic functions at the origin, or *(ii)* the formal completion of $\mathcal{O}$ at its maximal ideal.

(2.) For every sufficiently small Euclidean ball B_ϵ centered at p, the set $C \setminus \{0\} \cap B_\epsilon$ is connected in the Euclidean topology.

(3.) For every smooth model $\pi \colon Y \to X$, we have that $\pi_*^{-1}(C) \cap \pi^{-1}(\{0\})$ consists of a single point.

Note that, in particular, (3.) implies that the exceptional curve of the blowup of $\mathbb{A}^2$ at the origin intersects the strict transform of C in a single point. Thus, a unibranch curve C has a well defined tangent direction, *i.e.* the tangent cone of C at the origin consists of a single line. This fact also follows from (1.) using that the rings in *(i)* and *(ii)* are Henselian.

Consider now the minimal log resolution $\pi \colon Y \to X$ of unibranch curve C. We know π is a composition of point blowups, and we must begin with the blowup $X_1 \to X_0 = X$ at 0. Next, assuming we have not already found a log resolution, we are forced to blowup the unique point on the strict transform of C on X_1 over 0.

Continuing in this manner, we see $\pi\colon Y \to X$ is realized as a composition of point blowups

$$\pi\colon Y = X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = X$$

where each morphism $\theta_i\colon X_i \to X_{i-1}$ is the blowup at a closed point p_i lying over $p_1 = 0$. Let $E_i^{X_i}$ be the exceptional divisor of θ_i for $i = 1, \ldots, n$. For $j > i$, we will denote by $E_i^{X_j}$ the strict transform of $E_i^{X_i}$ on X_j and also set $E_i = E_i^{X_n}$.

We refer to the multiplicities of the strict transforms of C at the p_i's as the *multiplicity sequence* of C. Precisely, if $\pi_i\colon X_i \to X$ is the composition of the first i of these blowups, this is the sequence of positive integers

$$\operatorname{mult}_{p_1}(C) \quad , \quad \operatorname{mult}_{p_2}(\pi_{1,*}^{-1}C) \quad , \quad \operatorname{mult}_{p_3}(\pi_{2,*}^{-1}C) \quad , \quad \ldots \quad , \quad \operatorname{mult}_{p_n}(\pi_{n-1,*}^{-1}C)$$

and will generally be written as the row vector $\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix}$. More generally, if D is any curve, we will refer to the sequence of nonnegative integers

$$\operatorname{mult}_{p_1}(D) \quad , \quad \operatorname{mult}_{p_2}(\pi_{1,*}^{-1}D) \quad , \quad \operatorname{mult}_{p_3}(\pi_{2,*}^{-1}D) \quad , \quad \ldots \quad , \quad \operatorname{mult}_{p_n}(\pi_{n-1,*}^{-1}D)$$

as the multiplicity sequence of D along π. Thus, the multiplicity sequence of a unibranch curve C is the same as the multiplicity sequence of C along its minimal log resolution. It is a nontrivial result that the multiplicity sequence of C alone determines its equisingularity class.

From the proof of Proposition III.1, we know that the multiplicity sequence of a unibranch curve C is weakly decreasing, and also we must have $a_n = 1$. Further investigation will detail a very rich combinatorial structure in the multiplicity sequence encoding a vast amount of information about the numerics of its minimal log resolution.

Proposition VI.2. *Suppose C is an analytically irreducible curve on S through p with multiplicity sequence* $(\ a_1 \quad a_2 \quad a_3 \quad \cdots \quad a_n\ )$ *and minimal resolution* $\pi\colon Y \to X$ *as above. For each* $1 \le r < n$*, we have*

$$a_r = a_{r+1} + a_{r+2} + \cdots + a_{r+s} \tag{6.1}$$

where $s = -E_r \cdot E_r - 1$*. Furthermore, for* $1 \le t \le n - r$*, we have that* E_r *intersects* E_{r+t} *if and only if* $t = s$*, and*

$$a_{r+1} = \cdots = a_{r+s-1} \ge a_{r+s}\ . \tag{6.2}$$

Proof. Let s be the largest value of i for which the point p_{r+i} lies on $E_r^{X_{r+i-1}}$, and write $\theta\colon X_{r+s} \to X_{r-1}$. A simple calculation shows that the self-intersection number of the strict transform of a smooth projective curve on a smooth surface after the blowup of a point on the curve is one less than the original self-intersection number of the curve. Since none of the points $p_{r+s+1}, \ldots, p_n$ lie on the strict transform of $E_r^{X_r}$ and $E_r^{X_r} \cdot E_r^{X_r} = -1$, it follows immediately that $s = E_r^{X_{r+s}} \cdot E_r^{X_{r+s}} - 1 = -E_r \cdot E_r - 1$ and also that E_r intersects E_{r+t} if and only if $t = s$ for $1 \le t \le n - r$.

Suppose first that $s = 1$, *i.e.* on X_{r+1} we have that $\pi^{-1}_{r+1,*}(C)$ and $E_r^{X_{r+1}}$ are disjoint. We compute

$$\begin{aligned} \theta^*(\pi^{-1}_{r-1,*}C) &= \theta^*_{r+1}(\pi^{-1}_{r,*}C + a_r E_r^{X_r}) \\ &= \pi^{-1}_{r+1,*}C + a_r E_r^{X_{r+1}} + (a_r + a_{r+1})E_{r+1}^{X_{r+1}}\ . \end{aligned}$$

But then we also have $E_r^{X_{r+1}} \cdot \theta^*(\pi^{-1}_{r-1,*}C) = 0$, so it follows $-2a_r + a_r + a_{r+1} = 0$ or $a_r = a_{r+1}$, as desired.

Now assume $s \ge 2$. Then, for $2 \le i \le s$, we have that p_{r+i} is simply the single intersection point of $E_{r+i-1}^{X_{r+i-1}}$ with $E_r^{X_{r+i-1}}$ (through which $\pi^{-1}_{r+i-1,*}C$ also passes).

The calculation immediately preceding then implies

$$a_{r+1} = \cdots = a_{r+s-1} \geq a_{r+s} \ .$$

Furthermore, again we have $E_r^{X_{r+s}}$ and $\pi_{r+s,*}^{-1}C$ are disjoint. We compute

$$\begin{aligned}
\theta^*(\pi_{r-1,*}^{-1}C) &= (\theta_{r+s} \circ \cdots \circ \theta_{r+1})^*(\pi_{r,*}^{-1}C + a_r E_r^{X_r}) \\
&= (\theta_{r+s} \circ \cdots \circ \theta_{r+2})^*(\pi_{r+1,*}^{-1}C + a_r E_r^{X_{r+1}} + (a_r + a_{r+1})E_{r+1}^{X_{r+1}}) \\
&= \qquad \vdots \\
&= (\theta_{r+s} \circ \cdots \circ \theta_{r+j})^*(\pi_{r+j,*}^{-1}C + a_r E_r^{X_{r+j}} + \cdots \\
&\qquad \cdots + (ja_r + a_{r+1} + a_{r+2} + \cdots a_{r+j})E_{r+j}^{X_{r+j}}) \\
&= \qquad \vdots \\
&= \pi_{r+s,*}^{-1}C + a_r E_r^{X_{r+s}} + \cdots \\
&\qquad \cdots + (sa_r + a_{r+1} + a_{r+2} + \cdots a_{r+s})E_{r+s}^{X_{r+s}}) \ .
\end{aligned}$$

Thus, it follows from $E_r^{X_{r+s}} \cdot \theta^*(\pi_{r-1,*}^{-1}C) = 0$ that

$$(-s-1)a_r + (sa_r + a_{r+1} + a_{r+2} + \cdots a_{r+s}) = 0$$

or

$$a_r = a_{r+1} + a_{r+2} + \cdots + a_{r+s}$$

as desired. □

Perhaps the easiest way to visualize the geometric conclusions of Proposition VI.2 are in terms of the dual graph of the exceptional divisors of $\pi\colon Y \to X$ (also referenced in the previous Chapter). Recall that this graph is constructed in the following manner. There is a vertex $\bullet_{E_i}$ of the dual graph corresponding to each exceptional divisor $E_1, \ldots, E_n$ of π, and two vertices corresponding to E_i and E_j are adjacent for $i \neq j$ if and only if $E_i \cdot E_j = 1$ (*i.e.* E_i intersects E_j nontrivially). The vertex corresponding to E_i the weight $w_i = -E_i \cdot E_i$, denoted $\bullet_{E_i}^{(w_i)}$. Additionally, it will be convenient to include an unweighted vertex $\bullet_C$ corresponding to the strict transform of C on Y.

One may construct the dual graph of the minimal log resolution $\pi\colon Y \to X$ of an analytically irreducible curve C on X by "decorating" its multiplicity sequence

$$\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix}$$

as suggested by Proposition VI.2. Simply draw an arc connecting each grouping $a_r = a_{r+1} + \cdots + a_{r+s}$, and weight the leftmost entry by $(s+1)$

$$\overset{(s+1)}{a_r} \quad a_{r+1} \quad a_{r+2} \quad \cdots \quad a_{r+s} \ .$$

According to the proposition, this corresponds to an edge

$$\bullet_{E_r} \ \text{———} \ \bullet_{E_{r+s}}$$

in Γ_π. Furthermore, the weight of the vertex corresponding to E_r is $(s+1)$. Thus, after giving the vertex corresponding to E_n weight one, we have completely described the weighted dual graph.

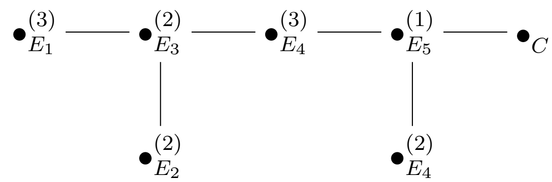
Example VI.3. Consider the analytically irreducible plane curve C parametrized by

$$x = t^4 \qquad y = t^6 + t^9$$

near the origin in $\mathbb{A}^2$. One can check that the multiplicity sequence of C is given by $\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 2 & 2 & 1 & 1 \end{pmatrix}$. We now "decorate" this sequence as described above.

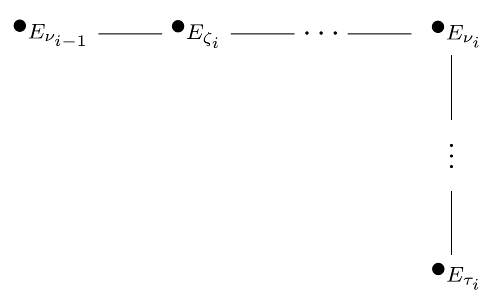
$$\overset{(3)}{4} \quad \underset{(2)}{2} \quad \overset{(2)}{2} \quad \overset{(3)}{2} \quad \underset{(2)}{1} \quad 1$$

Since $a_1 = a_2 + a_3$, draw an arc over these three entries in the sequence. Similarly, draw arcs grouping $a_2 = a_3$, $a_3 = a_4$, $a_4 = a_5 + a_6$, and $a_5 = a_6$. For each arc, weight the leftmost entry by the length of its span. According to Proposition VI.2, the dual graph of the minimal log resolution of C is shown below.



We will return to this example repeatedly as our discussion continues.

We now proceed to further describe the relationship between the dual graph and the multiplicity sequence of a unibranch curve C. Let $\nu_1, \ldots, \nu_g$ be the indices such that $\bullet_{E_{\nu_1}}, \ldots, \bullet_{E_{\nu_g}}$ are the star vertices of the dual graph, *i.e.* those whose valence [1] is at least three, and set $\nu_0 = 1$. Thus, in the previous example, we have $g = 2$ with $\nu_1 = 3$ and $\nu_2 = 5$. We will use the traditional notation $e_i = a_{\nu_i}$. The graph Γ_π is formed by joining together the pieces



where $\bullet_{\tau_i}$ is the vertex with valence one whose index τ_i immediately precedes ν_i, and $\bullet_{E_{\zeta_i}}$ is the unique vertex with $\zeta_i > \nu_{i-1}$ such that E_{ζ_i} intersects $E_{\nu_{i-1}}$ nontrivially. In the previous example, we have $\tau_1 = 2$, $\tau_2 = 4$, and $\zeta_1 = 3$, $\zeta_2 = 5$. Note that, in general, we have $\nu_{i-1} < \tau_i < \nu_{i-1}$, $\nu_{i-1} < \zeta_i \leq \nu_{i-1}$, and $\zeta_i \neq \tau_i$.

[1]It is important to remember that we have included a vertex corresponding to the strict transform of C in the dual graph when calculating the valence.

Let us now use Proposition VI.2 to give a very simple combinatorial description of the segment

$$a_{\nu_{i-1}} \quad a_{\nu_{i-1}+1} \quad \cdots \quad a_{\tau_i - 1} \quad a_{\tau_i} \quad \cdots \quad a_{\nu_i} \tag{6.3}$$

of the multiplicity sequence of C. First, we have that

$$a_{\nu_{i-1}} = a_{\nu_{i-1}+1} = \cdots = a_{\tau_i - 1} > a_{\tau_i} \ .$$

In fact, equality holds throughout (6.2) if and only if $r + s = \nu_i$ for some i. Let $\rho_i = a_{\nu_{i-1}} + a_{\nu_{i-1}+1} + \cdots + a_{\tau_i}$ and $\beta_i' = \frac{\rho_i}{e_{i-1}}$ for $i = 1, \ldots, g$. It follows immediately from (6.2) that the entries in (6.3) are given by unwinding the Euclidean algorithm for the integers ρ_i and e_{i-1}. More precisely, if we set $s_0 = \rho_i$ and $s_1 = e_{i-1}$ and the Euclidean algorithm gives

$$\begin{aligned}
(s_0) &= r_1 \cdot (s_1) + s_2 \\
(s_1) &= r_2 \cdot (s_2) + s_3 \\
&\vdots \\
(s_{k-2}) &= r_{k-1} \cdot (s_{k-1}) + s_k \\
(s_{k-1}) &= r_k \cdot (s_k) + 0
\end{aligned}$$

then the sequence of numbers in (6.3) is precisely

$$\underbrace{s_1 \ s_1 \ \cdots \ s_1}_{r_1\text{-times}} \ \underbrace{s_2 \ s_2 \ \cdots \ s_2}_{r_2\text{-times}} \ \cdots \ \underbrace{s_{k-1} \ s_{k-1} \ \cdots \ s_{k-1}}_{r_{k-1}\text{-times}} \ \underbrace{s_k \ s_k \ \cdots \ s_k}_{r_k\text{-times}} \ .$$

Note that $s_k = e_i = \gcd(\rho_i, e_{i-1})$. Since $\rho_i = \beta_i' \cdot e_{i-1}$ and $e_g = 1$, simply knowing the rational numbers $\beta_1', \ldots, \beta_g'$ is enough to reproduce the numbers $\rho_1, \ldots, \rho_g$ and $e_0, \ldots, e_g$. One may then give the entire multiplicity sequence by "gluing" the segments as in (6.3) together along their outermost entries. Before showing how this works in Example VI.3, we record the combinatorial fact

$$
\begin{aligned}
(6.4) \qquad a_{\nu_{i-1}}^2 + a_{\nu_{i-1}+1}^2 + \cdots + a_{\nu_i}^2 &= r_1 s_1^2 + r_2 s_2^2 + \cdots + r_{k-1} s_{k-1}^2 + r_k s_k^2 \\
&= (s_0 - s_2)s_1 + (s_1 - s_3)s_2 + \cdots \\
&\quad \cdots + (s_{k-2} - s_k)s_{k-1} + (s_{k-1} - 0)s_k \\
&= s_0 s_1 \;=\; \rho_i \cdot e_{i-1} \ .
\end{aligned}
$$

Example VI.4. In Example VI.3, starting from $\beta_2' = \frac{5}{2}$ written in lowest terms, we see $\rho_2 = 5$ and $e_1 = 2$. Perform the Euclidean algorithm for the integers 5 and 2. We have

$$
\begin{aligned}
(5) &= 2 \cdot (2) + 1 \\
(2) &= 2 \cdot (1) + 0
\end{aligned}
$$

and thus conclude that the second segment of the relevant multiplicity sequence consists of two 2's followed by two 1's, *i.e.* $\;2 \;\; 2 \;\; 1 \;\; 1\;$. Next, from $\beta_1' = \frac{3}{2}$ again written in lowest terms, we conclude $\rho_1 = 3 \cdot 2 = 6$ and $e_0 = 2 \cdot 2 = 4$. Perform the Euclidean algorithm for the 6 and 4 to get the sequence $\;4 \;\; 2 \;\; 2\;$. Now, simply match together the leftmost entry of the first sequence with the rightmost entry of the second, resulting in $\;4 \;\; 2 \;\; 2 \;\; 2 \;\; 1 \;\; 1\;$.

6.2 Proximity Matrix

Another way of encoding the information garnered about the multiplicity sequence of an analytically irreducible plane curve C via Proposition VI.2 is through the proximity matrix P (first introduced by Du Val) of its minimal log resolution π. As in Chapter IV, we will let $\Lambda = \mathbb{Z}E_1 + \cdots \mathbb{Z}E_n$ be the exceptional lattice. In Section 4.2.1, we made use of the two bases $E_1, \ldots, E_n$ and $\check{E}_1, \ldots, \check{E}_n$ for $\Lambda_{\mathbb{Q}}$. However, we now have access to an important third basis as well. Write $E_i^* = (\theta_n \circ \cdots \theta_{i+1})^*(E_i^{X_i})$ for each $i = 1, \dots, n$. Then $E_1^*, \dots, E_n^*$ form an integral basis for Λ, and furthermore we have

$$E_i^* \cdot E_j^* = -\delta_{ij} = \begin{cases} -1 & i = j \\ 0 & i \neq j \end{cases}.$$

As an immediate consequence, observe in particular that the intersection product on Λ is unimodular and thus the divisors $\check{E}_1, \dots, \check{E}_n$ must form an integral lattice basis.

By definition, the proximity matrix P is simply the change of basis matrix from the basis $E_1, \dots, E_n$ to $E_1^*, \dots, E_n^*$. It is easy to see that

$$E_r = E_r^* - E_{r+1}^* - \cdots - E_{r+s}^*$$

where s is as in Proposition VI.2, so that the proximity[2] matrix $P = (x_{ij})$ has the form

$$x_{ij} = \begin{cases} 0 & j > i \\ 1 & i = j \\ -1 & j > i \text{ and } p_j \text{ lies on } E_i^{X_{j-1}} \end{cases}.$$

Thus, we can interpret VI.2 as saying that the multiplicity sequence is simply the bottom row of the inverse to the proximity matrix. We will denote the rows of P^{-1} by $X_1, \dots, X_n$.

The proximity matrix can be used to relate any of the aforementioned bases of Λ to one another. Indeed, let $E = (E_j)$, $E^* = (E_j^*)$, and $\check{E} = \check{E}_j$ be the column vectors of these bases. By definition, $E = P^t \cdot E^*$. Note, in particular, this implies that the weighted incidence matrix of Γ_π (without $\bullet_C$) is given by $(E_i \cdot E_j) = E \cdot E^t = -P^t \cdot P$ as $E^* \cdot (E^*)^t = -\mathbb{1}_n$. Now, since $-\mathbb{1}_n = (\check{E}_i \cdot E_j) = \check{E} \cdot E^t = \check{E} \cdot (E^*)^t \cdot P$, it follows

[2] In standard terminology, j is said to be proximate to i (alternatively, E_j is proximate to E_i) when $j > i$ and p_j lies on $E_i^{X_{j-1}}$.

that $P \cdot \check{E} \cdot (E^*)^t = -\mathbb{1}_n$ as a matrix commutes with its inverse. Thus, we must have $P \cdot \check{E} = E^*$. Finally, we may also conclude $\check{E} = P^{-1} \cdot E^* = P^{-1} \cdot (P^{-1})^t \cdot E$. This last relation is quite important, as it gives

$$\mathrm{ord}_{E_j}(\check{E}_i) = \mathrm{ord}_{E_i}(\check{E}_j) = \langle X_i, X_j \rangle = X_i \cdot X_j^t \ .$$

This implies that the inner products of the rows of P^{-1} have an interpretation in terms of valuations. Using our deductions from Section 4.2.1, for any divisor $D \in \mathrm{Div}_{\mathbb{Q}}(X)$ we conclude from $\pi^* D = \pi_*^{-1} D + \sum_i (\pi_*^{-1} D \cdot E_i) \check{E}_i$ that

$$\mathrm{ord}_{E_j}(\pi^* D) = \sum_i (\pi_*^{-1} D \cdot E_i) \langle X_i, X_j \rangle \ .$$

In particular, we have $\mathrm{ord}_{E_i}(\pi^* C) = \langle X_i, X_n \rangle$ is simply the inner product of the multiplicity sequence of C with the i-th row of the inverse of the proximity matrix.

Calculating the i-th row of the proximity matrix is very easy when given the multiplicity sequence. First, we know that the i-th entry of X_i is 1 and every entry further to the right is 0. To complete the row, simply force the entries to satisfy exactly the same relations as in (6.1) satisfied by the multiplicity sequence. Furthermore, using this line of reasoning, various relations among the X_i can be deduced. For $1 \leq j \leq n$, let $X_i^{\leq j}$ denote the j-th truncation of the row vector X_i, so that

$$X_i^{\leq j} = \begin{pmatrix} x_{i,1} & x_{i,2} & \cdots & x_{i,k} \end{pmatrix} .$$

Consider $X_n^{\leq \nu_{i-1}}$ for some $1 \leq i \leq n$. It is easy to see each of the relations (6.1) for $r < \nu_{i-1}$ satisfies $r + s \leq \nu_{i-1}$. In particular, $X_n^{\leq \nu_{i-1}}$ and $X_{\nu_{i-1}}^{\leq \nu_{i-1}}$ can be constructed from exactly the same relations (6.1), except the former begins with rightmost entry e_{i-1} while the latter has rightmost entry 1. Thus, it follows that

$$X_n^{\leq \nu_{i-1}} = e_{i-1} X_{\nu_{i-1}}^{\leq \nu_{i-1}} \ . \tag{6.5}$$

Similar considerations show

$$
(6.6)\qquad
\begin{aligned}
X_{\tau_i} &= (\ \underbrace{X^{\leq \nu_{i-1}}_{\nu_{i-1}}}_{\nu_{i-1}\text{ entries}}\ \underbrace{1\ 1\ \cdots\ 1}_{(\tau_i-\nu_{i-1})-\text{times}}\ \underbrace{0\ 0\ \cdots\ 0}_{(n-\tau_i)-\text{times}}\ )\\
X_{\zeta_i} &= (\ \underbrace{sX^{\leq \nu_{i-1}}_{\nu_{i-1}}}_{\nu_{i-1}\text{ entries}}\ \underbrace{1\ 1\ \cdots\ 1}_{s-\text{times}}\ \underbrace{0\ 0\ \cdots\ 0}_{(n-\zeta_i)-\text{times}}\ )
\end{aligned}
$$

where we conventionally set $\nu_0 = 1$ and let $s = \zeta_i - \nu_{i-1} = -E_{\nu_{i-1}} \cdot E_{\nu_{i-1}} - 1$.

Let us point out one other important piece of information about X_i. It is easy to see that $K_\pi = E_1^* + \cdots + E_n^*$. Thus, if $\Sigma = (\ 1\ \ 1\ \ \cdots\ \ 1\ )$ is the all one's vector, we have $K_\pi = \Sigma \cdot E^* = \Sigma \cdot (P^{-1})^t \cdot E$. In other words, $\mathrm{ord}_{E_i}(K_\pi)$ is simply the sum of the entries of X_i. Note that we have now shown how to calculate π^*C and K_π, and are thus in a position to get at the candidate jumping numbers for each E_i starting from the multiplicity sequence.

Example VI.5. In Example VI.3, let us first calculate X_5. First, we know $x_{5,5} = 1$ and $x_{5,6} = 0$. Since $a_4 = a_5 + a_6$, we impose the condition $x_{5,4} = x_{5,5} + x_{5,6} = 1 + 0 = 1$. Now, $a_2 = a_3 = a_4$, so we also get $x_{5,2} = x_{5,3} = x_{5,4} = 1$. Lastly, we see $x_{5,1} = x_{5,2} + x_{5,3} = 1 + 1 = 2$. Thus, we have $X_5 = (\ 2\ \ 1\ \ 1\ \ 1\ \ 1\ \ 0\ )$. One can also verify

$$
P^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0 & 0\\ 2 & 1 & 1 & 0 & 0 & 0\\ 2 & 1 & 1 & 1 & 0 & 0\\ 2 & 1 & 1 & 1 & 1 & 0\\ 4 & 2 & 2 & 2 & 1 & 1 \end{pmatrix}
\qquad
P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ -1 & 1 & 0 & 0 & 0 & 0\\ -1 & -1 & 1 & 0 & 0 & 0\\ 0 & 0 & -1 & 1 & 0 & 0\\ 0 & 0 & 0 & -1 & 1 & 0\\ 0 & 0 & 0 & -1 & -1 & 1 \end{pmatrix}
$$

$$
\pi^*C = \pi_*^{-1}C + 4E_1 + 6E_2 + 12E_3 + 14E_4 + 15E_5 + 30E_6
$$

$$K_\pi = E_1 + 2E_2 + 4E_3 + 5E_4 + 6E_5 + 12E_6$$

We are now ready to prove the first ingredient in the computation of the jumping numbers of a unibranch curve C.

Theorem VI.6. *Every jumping number of a unibranch curve C is critically contributed by a prime divisor on its minimal resolution.*

Proof. Suppose, by way of contradiction, there is a jumping number $\xi < 1$ for the pair (X, C) which is not contributed by a prime divisor on its minimal log resolution $\pi\colon Y \to X$. Then we must have that ξ is critically contributed by a reduced chain of exceptional divisors $E_{i_1} + E_{i_2} + \cdots + E_{i_k}$ where $i_1 < i_2 < \cdots < i_k$. By Theorem V.10, we may assume $i_1 = \nu_{i-1}$ and $i_2 = \zeta_i$ for some $1 < i \leq g$. Let E_a and E_b be the two exceptional divisors intersecting $E_{\nu_{i-1}}$ nontrivially with $a \neq b$ and $a, b < \nu_{i-1}$, so that (up to switching a and b)

(6.7)
$$\begin{tikzcd} \cdots \arrow[r, dash] & \bullet_{E_a} \arrow[r, dash] & \bullet_{E_{\nu_{i-1}}} \arrow[r, dash] \arrow[d, dash] & \bullet_{E_{\zeta_i}} \arrow[r, dash] & \cdots \\ & & \bullet_{E_b} \arrow[d, dash] & & \\ & & \vdots & & \end{tikzcd}$$

appears in the dual graph. If $s = -E_{\nu_{i-1}} \cdot E_{\nu_{i-1}} - 1$, we have from (6.5) and (6.6) along with Proposition VI.2 that

(6.8)
$$\begin{aligned} \operatorname{ord}_{E_{\nu_{i-1}}}(\pi^* C) &= \langle X_{\nu_{i-1}}, X_n \rangle \\ &= e_{i-1} \langle X_{\nu_{i-1}}, X_{\nu_{i-1}} \rangle \\ \operatorname{ord}_{E_{\zeta_i}}(\pi^* C)) &= \langle X_{\zeta_i}, X_n \rangle \\ &= s e_{i-1} \langle X_{\nu_{i-1}}, X_{\nu_{i-1}} \rangle + a_{\nu_{i-1}+1} + a_{\nu_{i-1}+2} + \cdots + a_{\zeta_i} \\ &= s e_{i-1} \langle X_{\nu_{i-1}}, X_{\nu_{i-1}} \rangle + e_{i-1} \ . \end{aligned}$$

In particular, we conclude $\gcd(\mathrm{ord}_{E_{\nu_{i-1}}}(\pi^*C), \mathrm{ord}_{E_{\zeta_i}}(\pi^*C)) = e_{i-1}$.

Since ξ must be a common candidate jumping number for both $E_{\nu_{i-1}}$ and E_{ζ_i}, we must have $\mathrm{ord}_{E_{\nu_{i-1}}}(\pi^*C) \cdot \xi \in \mathbb{Z}$ and $\mathrm{ord}_{E_{\zeta_i}}(\pi^*C) \cdot \xi \in \mathbb{Z}$ and thus necessarily $\gcd(\mathrm{ord}_{E_{\nu_{i-1}}}(\pi^*C), \mathrm{ord}_{E_{\zeta_i}}(\pi^*C)) \cdot \xi = e_{i-1}\xi \in \mathbb{Z}$. According to Theorem V.8, we must have

$$- \lfloor \xi\pi^*C \rfloor \cdot E_{\nu_{i-1}} = \{\xi\pi^*C\} \cdot E_{\nu_{i-1}} = 1 \ . \tag{6.9}$$

Since $a, b \leq \nu_{i-1}$, it follows from (6.5) once more that

$$\begin{aligned} \mathrm{ord}_{E_a}(\pi^*C) &= \langle X_a, X_n \rangle = e_{i-1}\langle X_a, X_{\nu_{i-1}} \rangle \\ \mathrm{ord}_{E_b}(\pi^*C) &= \langle X_b, X_n \rangle = e_{i-1}\langle X_b, X_{\nu_{i-1}} \rangle \end{aligned} \tag{6.10}$$

are both divisible by e_{i-1}. Thus, the exceptionally supported $\mathbb{Q}$-divisor $\{\xi\pi^*C\}$ does not have any of $E_a, E_b, E_{\nu_{i-1}}, E_{\zeta_i}$ in its support. But then we must conclude $\{\xi\pi^*C\} \cdot E_{\nu_{i-1}} = 0$, contradicting (6.9). $\square$

6.3 Minimal Semigroup Generators, Characteristic Exponents, and Approximate Roots

We now wish to introduce some valuation theoretic invariants associated to the unibranch curve C. Consider a local defining equation $f \in \mathcal{O}$ for C. The ring $\mathcal{O}_C = \mathcal{O}/\langle f \rangle$ is a one dimensional local domain whose normalization $\overline{\mathcal{O}_C}$ is also a one dimensional local domain, and hence a DVR. Let $\mathrm{ord}_{\bar{C}}\colon \mathrm{Frac}(\mathcal{O}_C) \setminus \{0\} \to \mathbb{Z}$ be the corresponding valuation. Thus, for any $f' \in \mathcal{O}$ not divisible by f and defining a curve C', it is easy to see $\mathrm{ord}_{\bar{C}}(f' + \langle f \rangle) = \dim_{\mathbb{C}}(\mathcal{O}/\langle f, f' \rangle)$ is the local intersection number of C and C'. In particular, if $\pi_*^{-1}C'$ is disjoint from $\pi_*^{-1}C$, we have $\mathrm{ord}_{\bar{C}}(f' + \langle f \rangle) = \mathrm{ord}_{E_n}(\pi^*C) = \mathrm{ord}_{E_n}(f')$.

The set $\Gamma = \mathrm{ord}_{\bar{C}}(\mathcal{O}_C \setminus \{0\})$ forms an additive (sub)semigroup of $\mathbb{Z}_{\geq 0}$. Since $\mathrm{ord}_{\bar{C}}(\mathrm{Frac}(\mathcal{O}_C) \setminus \{0\})$ is surjective, it is easy to see that Γ contains all sufficiently large positive integers. One can choose a minimal set of generators $\bar{\beta}_0, \ldots, \bar{\beta}_g$ for Γ inductively as follows. First, let $\bar{\beta}_0$ be the smallest nonzero integer in Γ. Now, assuming $\bar{\beta}_0, \ldots, \bar{\beta}_i$ have been defined, simply let $\bar{\beta}_{i+1}$ be the smallest integer in Γ but not in the additive semigroup generated by $\bar{\beta}_0, \ldots, \bar{\beta}_i$. The invariants Γ or even $\bar{\beta}_0, \ldots, \bar{\beta}_g$ can be shown to be equivalent to the multiplicity sequence, and hence depend only on and completely determine the equisingularity class of C.

In order to review the precise formulae relating $\bar{\beta}_0, \ldots, \bar{\beta}_g$ and the multiplicity sequence of C, however, we first need to recall another set of invariants associated to a parametrization of C. After a suitable change of coordinates, we may assume a local defining equation $f \in \mathcal{O}$ for C is in Weierstrass normal form

$$f = y^{e_0} + \mathbb{A}_1(x) y^{e_0 - 1} + \cdots + \mathbb{A}_{e_0}(x) \in \mathbb{C}((x))[y]$$

where $e_0 = \mathrm{mult}_0(C)$ and each $\mathbb{A}_i(x) \in \mathbb{C}((x))$ has order at least i in x. Since the field of Puiseux series $\mathbb{C}((x^{1/n} : n > 0))$ is the algebraic closure of $\mathbb{C}((x))$, it follows that y can be written as a Puiseux series in x on the curve C. In fact, this can be done explicitly starting from f: the Newton-Puiseux algorithm can be used to write y as a power series in x^{1/e_0} on C. In other words, we arrive at a parametrization (setting $t = x^{1/e_0}$) of the form

$$x = t^{e_0} \qquad y = \sum_{i \geq 0} \alpha_i t^i \ . \tag{6.11}$$

Following Zariski, we can inductively define invariants $\beta_1, \ldots, \beta_g$ called the characteristic exponents based on which terms appear in the power series for y above. Let β_1 be the smallest positive integer such that $\alpha_{\beta_1} \neq 0$. Assuming now $\beta_1, \ldots, \beta_i$ have been defined, set β_{i+1} to be the smallest positive integer with $\alpha_{\beta_{i+1}} \neq 0$ such that β_{i+1} is not in the additive semigroup generated by $e_0, \beta_1, \ldots, \beta_i$ (just in case such a

β_{i+1} exists). Our use of the index g here is consistent with our previous notation – one can show the algorithm stops after producing the same number of invariants as there are stars in the dual graph of the minimal log resolution of C.

The invariants $e_0, \beta_1, \ldots, \beta_g$ again uniquely determine the equisingularity class of C. This follows from the following relations between the $\beta_1, \ldots, \beta_g$ and previously defined invariants

$$e_i = \gcd(e_0, \beta_1, \ldots, \beta_i) \qquad \beta_1 = \rho_1 = e_0\beta_1'$$
$$\beta_i = \rho_i - e_{i-1} + \beta_{i-1} = (\beta_i' - 1)e_{i-1} + \beta_{i-1} \quad \text{for } i \geq 2\ .$$

As none of our technical arguments rely on these relations, we refer the reader to [Zariski's Book] for a proof of these facts. We can rewrite the power series for y in (6.11) as

$$\begin{aligned} y \;&=\; \alpha_{\beta_1}t^{\beta_1} + \sum_{0<j\leq h_1} \alpha_{\beta_1+je_1}t^{\beta_1+je_1} + \alpha_{\beta_2}t^{\beta_2} + \sum_{0<j\leq h_2} \alpha_{\beta_2+je_2}t^{\beta_2+je_2} + \cdots \\ &\quad \cdots + \alpha_{\beta_g}t^{\beta_g} + \sum_{j=1}^{\infty} \alpha_{\beta_g+j}t^{\beta_g+j} \end{aligned}$$

where $h_i = \lfloor \frac{\beta_{i+1}-\beta_i}{e_i} \rfloor$. For each $i = 1, \ldots, g$, let C_k be the curve given by the parametrization

$$\begin{aligned} x \;&=\; t^{e_0} \\ y \;&=\; \alpha_{\beta_1}t^{\beta_1} + \sum_{0<j\leq h_1} \alpha_{\beta_1+je_1}t^{\beta_1+je_1} + \alpha_{\beta_2}t^{\beta_2} + \sum_{0<j\leq h_2} \alpha_{\beta_2+je_2}t^{\beta_2+je_2} + \cdots \\ &\quad \cdots + \alpha_{\beta_{k-1}}t^{\beta_{i-1}} + \sum_{j=1}^{\infty} \alpha_{\beta_{k-1}+j}t^{\beta_{k-1}+j} \qquad . \end{aligned}$$

The reader should be forewarned that the parametrization for C_k given above is not primitive; the greatest common divisor of every power of t which appears is $e_{k-1} > 1$. We will refer to C_k as a *k-th approximate root* of C, as the k-th approximate root f_k of f gives a local defining equation for C_k.

The approximate roots of C will be very useful in our forthcoming calculations. However, the only truly essential facts we will need about C_i are the following:

(a.) For each $i = 1, \ldots, g$, we have $\bar{\beta}_i = \mathrm{ord}_{\bar{C}}(f_i) = \mathrm{ord}_{E_n}(\pi^* C_i)$.

(b.) The strict transform $\pi_*^{-1} C_k$ is smooth and satisfies $\pi_*^{-1} C_k \cdot E_j = \begin{cases} 1 & j = \tau_k \\ 0 & j \neq \tau_k \end{cases}$.

One can interpret (b.) in the following way: C_k behaves as if it were a general element of the simple finite colength ideal $\pi_* \mathcal{O}_Y(-\check{E}_{\tau_k})$. Thus, we have that $\pi^* C_k = \pi_*^{-1} C_k + \check{E}_{\tau_k}$; furthermore, while π is necessarily a log resolution of C_k, the truncation $\pi_{\nu_{k-1}} \colon X_{\nu_{k-1}} \to X$ is in fact the minimal log resolution of C_k. The reader who is unfamiliar with the many different invariants we have introduced in this section could simply hereafter regard the equation

$$\bar{\beta}_i = \mathrm{ord}_{E_n}(\pi^* C_i) = \mathrm{ord}_{E_n}(\check{E}_{\tau_i}) = \langle X_{\tau_i}, X_n \rangle$$

as the definition of the semigroup invariants, where C_i is a general element of P_{τ_i}.

Let us now show other two formulae characterizing the minimal semigroup generators. Both are based upon the observation using (6.6) that

$$\begin{aligned} \bar{\beta}_i &= \langle X_{\tau_i}, X_n \rangle = \langle X_{\nu_{i-1}}, X_n \rangle + a_{\nu_{i-1}+1} + a_{\nu_{i-1}+2} + \cdots + a_{\tau_i} \\ &= \langle X_{\nu_{i-1}}, X_n \rangle + \rho_i - e_{i-1} \end{aligned}$$

for $i = 1, \ldots, g$. First, we claim

$$\bar{\beta}_i = \frac{e_i}{e_{i-1}} \mathrm{ord}_{E_{\nu_i}}(\pi^* C) \ . \tag{6.12}$$

Indeed, using (6.4) and (6.5) we have

$$\begin{aligned} e_{i-1} \bar{\beta}_i &= e_{i-1}(\langle X_{\nu_{i-1}}, X_n \rangle + \rho_i - e_{i-1}) = \langle X_n^{\leq \nu_{i-1}}, X_n \rangle + e_{i-1}\rho_i - a_{\nu_{i-1}}^2 \\ &= \langle X_n^{\leq \nu_{i-1}}, X_n \rangle + (a_{\nu_{i-1}}^2 + \cdots + a_{\nu_i}^2) - a_{\nu_{i-1}}^2 \\ &= \langle X_n^{\leq \nu_i}, X_n \rangle = e_i \langle X_{\nu_i}, X_n \rangle = e_i \, \mathrm{ord}_{E_{\nu_i}}(\pi^* C) \end{aligned}$$

as desired. Second, we can use this to give a recursive formula for the semigroup generators in terms of $\beta'_1, \ldots, \beta'_g$. We have for $i = 2, \ldots, g$ that

$$(6.13)\qquad \begin{aligned} \bar{\beta}_i &= \operatorname{ord}_{E_{\nu_{i-1}}}(\pi^* C) + \rho_i - e_{i-1} \\ &= \tfrac{e_{i-2}}{e_{i-1}} \bar{\beta}_{i-1} + \rho_i - e_{i-1} \\ &= \tfrac{e_{i-2}}{e_{i-1}} \bar{\beta}_{i-1} + e_{i-1}(\beta'_i - 1)\ . \end{aligned}$$

Note that this equation also gives the relationship between the minimal semigroup generators and the characteristic exponents, namely

$$\beta_0 = \bar{\beta}_0 = e_0 = a_1 \qquad \beta_1 = \bar{\beta}_1 = \rho_1 = \beta'_1 e_0$$
$$\beta_i - \beta_{i-1} = \bar{\beta}_i - \tfrac{e_{i-2}}{e_{i-1}} \bar{\beta}_{i-1} = e_{i-1}(\beta'_i - 1) = \rho_i - e_{i-1} \quad \text{for } i \geq 2.$$

One can relate the many invariants of C_k to those of C, and to distinguish them from one other we will adorn the former with a superscript C_k. The relations all stem from the fact that X_{τ_k} is the multiplicity sequence of C_k along π, and also that $X_{\nu_{k-1}}^{\leq \nu_{k-1}}$ is the multiplicity sequence of C_k (along its minimal resolution $\pi_{\nu_{k-1}} \colon X_{\nu_{k-1}} \to X$). Thus, we have $g^{C_k} = k-1$ and from $X_n^{\leq \nu_{k-1}} = e_{\nu_{k-1}} X_{\nu_{k-1}}^{\leq \nu_{k-1}}$ it follows easily that

$$e_0 = e_{\nu_{k-1}} e_0^{C_k} \qquad e_{\nu_i} = e_{\nu_{k-1}} e_{\nu_i}^{C_k} \qquad \rho_i = e_{\nu_{k-1}} \rho_i^{C_k} \qquad \beta'_i = \beta_i'^{C_k}$$
$$\beta_0 = e_{\nu_{k-1}} \beta_0^{C_k} \qquad \beta_i = e_{\nu_{k-1}} \beta_i^{C_k} \qquad \bar{\beta}_0 = e_{\nu_{k-1}} \bar{\beta}_0^{C_k} \qquad \bar{\beta}_i = e_{\nu_{k-1}} \bar{\beta}_i^{C_k}$$

for $i = 1, \ldots, k-1$.

In light of Theorem VI.6, to compute the jumping numbers of a unibranch curve C it suffices to compute the sets $\mathscr{H}_i$ of jumping numbers (critically) contributed by each E_{ν_i} for $i = 1, \ldots, g$. Thus, following the above notation, $\mathscr{H}_{i-1}^{C_i}$ denotes the set of jumping numbers of C_i (critically) contributed by $E_{\nu_{i-1}}^{X_{\nu_{i-1}}}$ on the minimal log resolution $\pi_{\nu_{i-1}} \colon X_{\nu_{i-1}} \to X$ of C_i. By Proposition V.14, these are the same as the jumping numbers of C_i critically contributed by $E_{\nu_{i-1}}$.

Theorem VI.7. *Suppose C is a unibranch curve with approximate roots $C_1, \dots, C_g$. Then for each $i = 2, \dots, g$, $\xi \in \mathbb{Q}_{>0}$ is a jumping number of C (critically) contributed by $E_{\nu_{i-1}}$ if and only if $e_{\nu_{i-1}}\xi$ is a jumping number of C_i (critically) contributed by $E_{\nu_{i-1}}$. In other words, $\mathscr{H}_{i-1} = e_{\nu_{i-1}}\mathscr{H}^{C_i}_{i-1}$.*

Proof. First off, note that

$$\operatorname{ord}_{E_{\nu_{i-1}}}(\pi^* C) = e_{\nu_{i-1}} \operatorname{ord}_{E_{\nu_{i-1}}}(\pi^* C_i) = e_{\nu_{i-1}} \operatorname{ord}_{E_{\nu_{i-1}}^{X_{\nu_{i-1}}}}(\pi^*_{\nu_{i-1}} C_i) \ ,$$

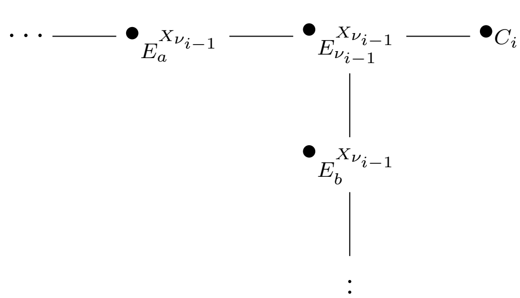
and thus ξ is a candidate jumping numbers for C along $E_{\nu_{i-1}}$ if and only if $e_{\nu_{i-1}}\xi$ is a candidate jumping number for C_i along $E_{\nu_{i-1}}$ (equivalently $E^{X_{\nu_{i-1}}}_{\nu_{i-1}}$). Now, we know ξ is a jumping number for C critically contributed by $E_{\nu_{i-1}}$ if and only if $-\lfloor \xi\pi^* C\rfloor \cdot E_{\nu_{i-1}} = \{\xi\pi^* C\} \cdot E_{\nu_{i-1}} \geq 2$, and similarly for the jumping numbers of C_i. We will use notation from the proof of Theorem VI.6. Let E_a and E_b be the two exceptional divisors intersecting $E_{\nu_{i-1}}$ nontrivially with $a \neq b$ and $a, b < \nu_{i-1}$, so that (up to switching a and b) the arrangement in (6.7) appears in the dual graph. Now, for every candidate jumping number ξ for C along $E_{\nu_{i-1}}$, we have

$$\{\xi\pi^* C\} \cdot E_{\nu_{i-1}} = \{\xi \operatorname{ord}_{E_a}(\pi^* C)\} + \{\xi \operatorname{ord}_{E_b}(\pi^* C)\} + \{\xi \operatorname{ord}_{E_{\zeta_i}}(\pi^* C)\} \ .$$

We see from (6.8) that $\{\xi \operatorname{ord}_{\zeta_i}(\pi^* C)\} = \{e_{\nu_{i-1}}\xi\}$ and from (6.10)

$$\{\xi \operatorname{ord}_{E_a}(\pi^* C)\} = \{e_{\nu_{i-1}}\xi \operatorname{ord}_{E_a}(\pi^* C_i)\} = \{e_{\nu_{i-1}}\xi \operatorname{ord}_{E_a^{X_{\nu_{i-1}}}}(\pi^*_{\nu_{i-1}} C_i)\}$$
$$\{\xi \operatorname{ord}_{E_b}(\pi^* C)\} = \{e_{\nu_{i-1}}\xi \operatorname{ord}_{E_b}(\pi^* C_i)\} = \{e_{\nu_{i-1}}\xi \operatorname{ord}_{E_b^{X_{\nu_{i-1}}}}(\pi^*_{\nu_{i-1}} C_i)\} \ .$$

Since the arrangement



appears in the dual graph to the minimal log resolution $\pi_{\nu_{i-1}}\colon X_{\nu_{i-1}} \to X$ of C_i, and thus

$$\begin{aligned}
\{e_{\nu_{i-1}}\xi\pi^*_{\nu_{i-1}}C_i\} \cdot E^{X_{\nu_{i-1}}}_{\nu_{i-1}} &= \{e_{\nu_{i-1}}\xi \operatorname{ord}_{E_a^{X_{\nu_{i-1}}}}(\pi^*_{\nu_{i-1}}C_i)\} \\
&\quad + \{e_{\nu_{i-1}}\xi \operatorname{ord}_{E_b^{X_{\nu_{i-1}}}}(\pi^*_{\nu_{i-1}}C_i)\} \\
&\quad + \{e_{\nu_{i-1}}\xi \operatorname{ord}_{\pi^{-1}_{\nu_{i-1},*}C_i}(\pi^*_{\nu_{i-1}}C_i)\} \\
&= \{\xi\pi^*C\} \cdot E_{\nu_{i-1}}
\end{aligned}$$

and the desired conclusion now follows. □

6.4 Computation of the Jumping Numbers of a Branch

Using Theorems VI.6 and VI.7, in order to calculate the jumping numbers of a unibranch curve C, it suffices to find those jumping numbers (critically) contributed by E_{ν_g}. To that end, let us first consider the case where $g = 1$. According to the following theorem, this reduces to the case of the Fermat curve $y^e + x^b$ for relatively prime positive integers e and b.

Theorem VI.8. *The jumping numbers of the germs of a plane curve depend only on its equisingularity class.*

Proof. This follows immediately from Theorem V.8, since the numerical data of the minimal log resolutions of two equisingular plane curves are the same [cite Brieskorn]. Note that we have already seen this fact explicitly in the case of equisingular unibranch curves. □

Corollary VI.9. *Suppose the dual graph to the minimal resolution of a unibranch curve C has a single star vertex, i.e. $g = 1$. Then the jumping numbers of C are*

$$\left\{ \frac{r+1}{e_0} + \frac{s+1}{\bar\beta_1} + m \;\middle|\; r, s, m \in \mathbb{Z}_{\geq 0} \text{ with } \frac{r+1}{e_0} + \frac{s+1}{\bar\beta_1} \leq 1 \right\}$$

and all of the jumping numbers less than one are (critically) contributed by E_{ν_1}.

Proof. If $g = 1$, the C is equisingular to the Fermat curve $y^{e_0} + x^{\bar{\beta}_1}$. Now, one can easily calculate the jumping numbers using toric methods. See [Laz04]. $\square$

To calculate the jumping numbers of a unibranch curve with $g \geq 2$, we will need the following generalizations of Theorem VI.8 and Corollary VI.9. This will allow us to reduce the computation in general to the case of a Fermat curve.

Lemma VI.10. *Suppose C and C' are two equisingular unibranch curves and L_1 and L_2 are smooth curves transverse to each other and C, C'. For a positive integers l_1 and l_2, consider the effective divisors $D = l_1L_1 + l_2L_2 + C$ and $D' = l_1L_1 + l_2L_2 + C'$. Then the jumping numbers of D (critically) contributed by $E^C_{\nu_g}$ on the minimal log resolution of C coincide with the jumping numbers of D' (critically) contributed by $E^{C'}_{\nu_g}$ on the minimal log resolution of C'.*

Proof. Again, it suffices to note that

$$\operatorname{ord}_{E^C_{\nu_g}}(\pi^{C,*}D) = \operatorname{ord}_{E^{C'}_{\nu_g}}(\pi^{C',*}D')$$

so the relevant candidate jumping numbers coincide, and also verify that

$$\{\lambda\pi^{C,*}D\} \cdot E^C_{\nu_g} = \{\lambda\pi^{C',*}D'\} \cdot E^{C'}_{\nu_g}$$

for each of these candidate jumping numbers λ. These follow immediately as the numerical data of the given resolutions of both divisors are identical. $\square$

Lemma VI.11. *The jumping numbers of a unibranch curve C (critically) contributed by E_n are the same as the jumping numbers of the germ of $\pi^*_{j-1}C$ at p_j contributed by E_n for all $j \leq n$.*

Proof. If $\theta = \theta_n \circ \cdots \circ \theta_j \colon Y \to X_n$, then θ is the minimal log resolution of the germs of $\pi_{j-1,*}^{-1}C$ or π_{j-1}^*C at p_j. Again, it suffices to note that

$$\operatorname{ord}_{E_n}(\pi^*C) = \operatorname{ord}_{E_n}(\theta^*(\pi_{j-1}^*C))$$

so the relevant candidate jumping numbers coincide, and also verify that

$$\{\lambda\pi^*C\} \cdot E_n = \{\lambda\theta^*(\pi_{j-1}^*C)\} \cdot E_n$$

for each of these candidate jumping numbers. But we have $\pi = \theta \circ \pi_{j-1}$, and furthermore it is easy to see $\lambda\pi^*C$ is determined in a neighborhood of E_n by the divisor π_{j-1}^*C. $\square$

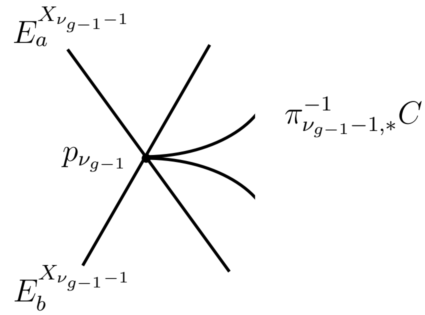
Lemma VI.12. *Suppose $e, b \in \mathbb{Z}_{>0}$ and $d \in \mathbb{Z}_{\geq 0}$ with $\gcd(e,b) = 1$ and $e < b$. Consider the unibranch curves $C = \operatorname{div}(y^e + x^{b+de})$ and $C' = \operatorname{div}(y^e + x^{b-e})$. Let $D = \operatorname{div}(x^{(d+1)e}(y^e + x^{b-e}))$. Then the jumping numbers of D (critically) contributed by $E_{\nu_1}^{C'}$ on the minimal resolution of C' coincide with those jumping numbers of C (critically) contributed by $E_{\nu_1}^{C}$ on the minimal resolution of C.*

Proof. By working explicitly in coordinates, it is easy to check that after the first $(d+1)$ blowups in the minimal resolution of the Fermat curve C, the corresponding germ π_{d+1}^*C at the singular point p_{d+2} on the strict transform of C has the form given by D. The statement now follows from Lemma VI.11. $\square$

Theorem VI.13. *The jumping numbers of a unibranch curve C (critically) contributed by E_{ν_g} are precisely*

$$\left\{ \frac{r+1}{e_{g-1}} + \frac{s+1}{\bar{\beta}_g} + m \;\middle|\; r, s, m \in \mathbb{Z}_{\geq 0} \text{ with } \frac{r+1}{e_{g-1}} + \frac{s+1}{\bar{\beta}_g} \leq 1 \right\}$$

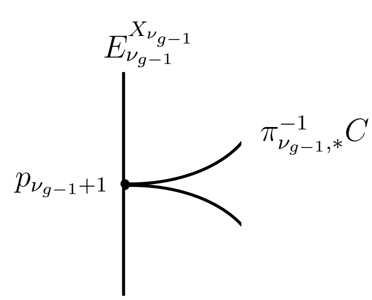
Proof. By Lemma VI.11, E_{ν_g} (critically) contributes the same jumping numbers to both C and the germ of $\pi^*_{\nu_{g-1}-1}C$ at $p_{\nu_{g-1}}$. Let E_a and E_b be the two exceptional divisors intersecting $E_{\nu_{i-1}}$ nontrivially with $a \neq b$ and $a, b < \nu_{i-1}$, so that (up to switching a and b) the arrangement (6.7) appears in the dual graph. Now, $\pi^*_{\nu_{g-1}-1}C$ has three components passing through $p_{\nu_{g-1}}$, namely $E_a^{X_{\nu_{g-1}-1}}$, $E_b^{X_{\nu_{g-1}-1}}$, and $\pi^{-1}_{\nu_{g-1}-1,*}C$. They are transverse to one another and the first two are smooth.



Let C' denote the germ of $\pi^{-1}_{\nu_{g-1}-1,*}C$ at $p_{\nu_{g-1}}$. Then C' is unibranch with multiplicity sequence

$$\begin{pmatrix} a_{\nu_{g-1}} & a_{\nu_{g-1}+1} & a_{\nu_{g-1}+2} & \cdots & a_{\nu_g} \end{pmatrix}$$

and thus we have $g^{C'} = 1$ with $e_0^{C'} = a_{\nu_{g-1}} = e_{g-1}$ and $\bar{\beta}_1^{C'} = \rho_g$. By Lemma VI.10, we may assume C' is the Fermat curve $y^{e_{g-1}} + x^{\rho_g}$ in some choice of coordinates. Consider now what happens when we blowup $p_{\nu_{g-1}}$. In local coordinates around $p_{\nu_{g-1}+1}$, the germ of $\theta^{-1}_{\nu_{g-1},*}C'$ has the form $y^{e_{g-1}} + x^{\rho_g - e_{g-1}}$ and the exceptional divisor $E_{\nu_{g-1}}^{X_{\nu_{g-1}}}$ is defined by x.



Thus, since $\operatorname{ord}_{E_{\nu_{g-1}}}(\pi^* C) = \frac{e_{g-2}}{e_{g-1}}\bar{\beta}_{g-1}$, the germ of $\pi^*_{\nu_{g-1}}C$ at $p_{\nu_{g-1}+1}$ is locally given in these coordinates by $x^{\frac{e_{g-2}}{e_{g-1}}\bar{\beta}_{g-1}}(y^{e_{g-1}} + x^{\rho_g - e_{g-1}})$. As e_{g-1} divides $\bar{\beta}_{g-1}$, $\frac{e_{g-2}}{e_{g-1}}\bar{\beta}_{g-1}$ is divisible by e_{g-1}, and it follows from Lemma VI.12 that jumping numbers (critically) contributed to C by E_{ν_g} are the same as the jumping numbers of the germ of the fermat curve $y^{e_{g-1}}+x^{\bar{\beta}_g}$ since $\bar{\beta}_g = \rho_g - e_{g-1} + \frac{e_{g-2}}{e_{g-1}}\bar{\beta}_{g-1}$. This completes the proof. □

Theorem VI.14. *The jumping numbers of a unibranch curve C are the union of the sets*

$$\mathscr{H}_i = \left\{ \frac{r+1}{e_{i-1}} + \frac{s+1}{\bar{\beta}_i} + \frac{m}{e_i} \,\middle|\, \quad r, s, m \in \mathbb{Z}_{\geq 0} \text{ with } \frac{r+1}{e_{i-1}} + \frac{s+1}{\bar{\beta}_i} \leq \frac{1}{e_i} \right\}$$

for $i = 1, \ldots, g$ together with $\mathbb{Z}_{\geq 0}$. The set $\mathscr{H}_i$ is precisely the jumping numbers (critically) contributed by the exceptional divisor E_{ν_i} corresponding to the i-th star vertex of the dual graph of the minimal log resolution of C.

Proof. The theorem now follows immediately from Theorems VI.6, VI.7, and VI.13. □

6.5 Jumping Numbers as Equisingularity Invariants

Lemma VI.15. *Suppose (e, b) and (e', b') are pairs of relatively prime positive integers with $e < b$ and $e' < b'$. If $\frac{e+b}{e'b'} = \frac{e'+b'}{eb}$, then $e = e'$ and $b = b'$.*

Proof. We have $eb(e' + b') = e'b'(e + b)$. Since e and b are relatively prime to $e + b$, it follows that eb divides $e'b'$. By symmetry, it follows that $eb = e'b'$. This, in turn, implies $e + b = e' + b'$. Thus,

$$e + \frac{e'b'}{e} = e' + b'$$

$$e^2 - e(e' + b') + e'b' = 0$$

$$(e - e')(e - b') = 0 \ .$$

Hence, we must have $e = e'$ from order considerations, and it follows that $b = b'$ as well. $\square$

Theorem VI.16. *The jumping numbers of a unibranch curve C determine its equisingularity class.*

Proof. Consider the sets $\mathscr{H}_i$ of jumping numbers detailed in Theorem VI.14. Since $\gcd(e_{i-1}, \bar{\beta}_i) = e_i$, it follows that every jumping number in $\mathscr{H}_i$ can be written with denominator $\frac{e_{i-1}\bar{\beta}_i}{e_i} = e_i \frac{e_{i-1}}{e_i}\frac{\bar{\beta}_i}{e_i}$. Furthermore, the smallest element ξ_i of $\mathscr{H}_i$ can be written as

$$\xi_i = \frac{\frac{e_{i-1}}{e_i} + \frac{\bar{\beta}_i}{e_i}}{e_i \frac{e_{i-1}}{e_i}\frac{\bar{\beta}_i}{e_i}}$$

in lowest terms, where $\frac{e_{i-1}}{e_i}$ and $\frac{\bar{\beta}_i}{e_i}$ are relatively prime positive integers.

Let us now argue that one can pick out $\xi_1, \ldots, \xi_g$ from the set of all of the jumping numbers. Using (6.13), we have $\bar{\beta}_i > \frac{e_{i-2}}{e_{i-1}}\bar{\beta}_{i-1}$ for $i \geq 2$. Thus, if we write the jumping numbers of C in lowest terms, the largest denominator which appears is $e_{g-1}\bar{\beta}_g$ (recall $e_g = 1$). Additionally, any jumping number in lowest terms having this denominator must be in $\mathscr{H}_g$, and the smallest such jumping number is ξ_g. Proceeding inductively, suppose we have identified ξ_i for some $i \geq 2$. The largest denominator which appears among the jumping numbers of C less than ξ_i written in lowest terms is $\frac{e_{i-2}\bar{\beta}_{i-1}}{e_{i-1}}$, and the smallest jumping number with this denominator is ξ_{i-1}

Finally, let us show how to recover $\bar{\beta}_0, \ldots, \bar{\beta}_g$ from $\xi_1, \ldots, \xi_g$. Now,

$$\xi_g = \frac{e_{g-1} + \bar{\beta}_g}{e_{g-1}\bar{\beta}_g}$$

where $e_{g-1} < \bar{\beta}_g$ are relatively prime integers. By Lemma VI.15, this allows us to recover e_{g-1} and $\bar{\beta}_g$. Proceeding inductively, assume we know e_{i-1} and $\bar{\beta}_i$ for some

$i \geq 2$. Then

$$e_{i-1}\xi_{i-1} = \frac{\frac{e_{i-2}}{e_{i-1}} + \frac{\bar{\beta}_{i-1}}{e_{i-1}}}{\frac{e_{i-2}}{e_{i-1}}\frac{\bar{\beta}_{i-1}}{e_{i-1}}}$$

and again using Lemma VI.15, we can recover $\frac{e_{i-2}}{e_{i-1}}$ and $\frac{\bar{\beta}_{i-1}}{e_{i-1}}$. Multiplying by e_{i-1} gives e_{i-2} and $\bar{\beta}_{i-1}$. □

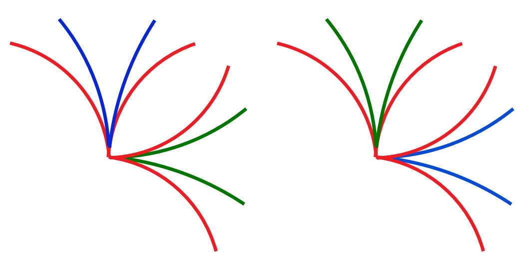
Example VI.17. Consider the monomial ideals

$$\mathfrak{a}_1 = \langle x^2, y^5 \rangle \cdot \langle x^2, y^3 \rangle \cdot \langle x^4, y^3 \rangle \cdot \langle x^7, y^2 \rangle$$

$$\mathfrak{a}_2 = \langle x^2, y^5 \rangle \cdot \langle x^3, y^2 \rangle \cdot \langle x^3, y^4 \rangle \cdot \langle x^7, y^2 \rangle$$

in $\mathcal{O}$. It is easy to see that the the jumping numbers of $\mathfrak{a}_1$ and $\mathfrak{a}_2$ coincide from the symmetry in their Newton polytopes.

It follows that a general element[3] C_1 of $\mathfrak{a}_1$ and C_2 of $\mathfrak{a}_2$ have the same jumping numbers. However, it is easy to see that C_1 and C_2 are not equisingular. Each of these curves has four branches, and there is a unique bijection between the branches which preserves their multiplicity sequences. However, this bijection does not preserve the pairwise local intersection numbers of the branches, so C_1 and C_2 are topologically distinct. The situation is pictured schematically in the diagram below.



[3]In fact, one may take

$$C_1 = \operatorname{div}\left((y^5 - x^2)(y^3 - x^2)(y^3 - x^4)(y^2 - x^7)\right)$$

$$C_2 = \operatorname{div}\left((y^5 - x^2)(x^3 - y^2)(x^3 - y^4)(y^2 - x^7)\right) \ .$$

Thus, as shown by the above example, the jumping numbers of a plane curve do not determine its equisingularity class when the curve has more than one branch. However, motivated by the example above and the case of a single branch, we conclude by asking:

Question VI.18. *Do the jumping numbers of the germ of a plane curve determine the equisingularity classes of its branches?*

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