UNCLASSIFIED

UNIVERSITY OF MICHIGAN

WRRC-24-J
16 February 1955

THE NUMERICAL DETERMINATION OF

THE RADAR CROSS-SECTION OF A PROLATE SPHEROID*

K. M. Siegel, B. H. Gere,** F. B. Sleator

Willow Run Research Center
of the
Engineering Research Institute
UNIVERSITY OF MICHIGAN

The exact curve is found for the nose-on radar cross-section of a perfectly conducting prolate spheroid whose ratio of major to minor axis is 10/1, for values of π times the major axis divided by the wavelength less than three. The exact acoustical cross-section is also found for the same range of parameters.

\- - - - - -

Introduction

Analysis of the numerical work required to obtain exact scattering cross-section curves by the method of Mie or Hansen, even when the scatterer is a coordinate surface of a separable system, quickly suggests that there should be a better method. Much work has been done in the effort to find better methods[1,2] and an approximate method for the spheroid problem has been developed

*This article is a shortened version of "Studies in Radar Cross-Sections XI--The Numerical Determination of the Radar Cross-Section of a Prolate Spheroid", K. M. Siegel, B. H. Gere, I. Marx and F. B. Sleator (UMM-126, December 1953).

**Now Chairman, Department of Mathematics, Hamilton College, Clinton, N. Y.

1. W. Franz and K. Deppermann, "The Creeping Wave in the Theory of Diffraction", McGill Symposium on Microwave Optics, June 1953.
2. M. Kline, "Asymptotic Solutions of Linear Partial Diffraction Equations and the WKB Method", ibid.

UNCLASSIFIED

by Stevenson [3], but up to the present the results of References 1 and 2 have been applied only to cases where the exact solutions were already known.

Thus, although it was known a priori that the magnitude of the computations involved would be enormous, the classical solution for the spheroid problem was carried out, and with the aid of the Mark III Digital Computer and its excellent staff at the Naval Proving Ground at Dahlgren, Va., the numerical results presented here were obtained. This problem required roughly five times the capacity of the Mark III. To obtain 22 values of the radar cross-section it was necessary to run the computer for ten weeks, part of the time on a 24 hour seven-day-a-week basis.

Since it was expected that for ratios of semi-major axis a to semi-minor axis b near unity, the curve of nose-on radar cross-section σ vs. wavelength λ would approximate that of a sphere, it was decided that more information would be obtained by examining a case with a larger value of a/b. Thus a value of 10 was chosen for this ratio.

Furthermore, analysis revealed that for a fixed amount of machine time the number of values of cross-section of a given body which could be computed to a given accuracy decreased sharply as the wavelength decreased.

3. A. F. Stevenson, "Solution of Electromagnetic Scattering Problems as Power Series in the Ratio (Dimension of Scatterer) / wavelength. Application to Scattering by an Ellipsoid", Journal of Applied Physics, Vol. 24, p. 1143, (1953). The present paper contains some of the numerical work referred to in Stevenson's "note added in proof".

Since the Rayleigh solution was immediately available for the region of large wavelength, it was felt that the present computations would most profitably be concentrated in the region of the first maximum.

Previous analysis of the sphere problem indicated by analogy that the first maximum for the prolate spheroid should occur at a larger wavelength than that predicted by physical optics, and also that the Rayleigh solution would form an upper bound on the curve in its region of validity. Furthermore it was felt that the 10/1 spheroid should behave electromagnetically rather like a thin wire, and accordingly the abscissas of the successive maxima as predicted by the thin wire theory of Van Vleck, Bloch, and Hamermesh[4] were computed. On the basis of this information a set of values of $2\pi a/\lambda$ was chosen to span the region in which the first maximum might occur.

In addition to computing the radar cross-sections, the Mark III recorded enough intermediate information so that the exact acoustical answers were easily obtainable by hand computation.

---

4. J. H. Van Vleck, F. Bloch, and M. Hamermesh, "Theory of Radar Reflection from Wires or Thin Metallic Strips", Journal of Applied Physics, Vol. 18, p. 274, (1947).

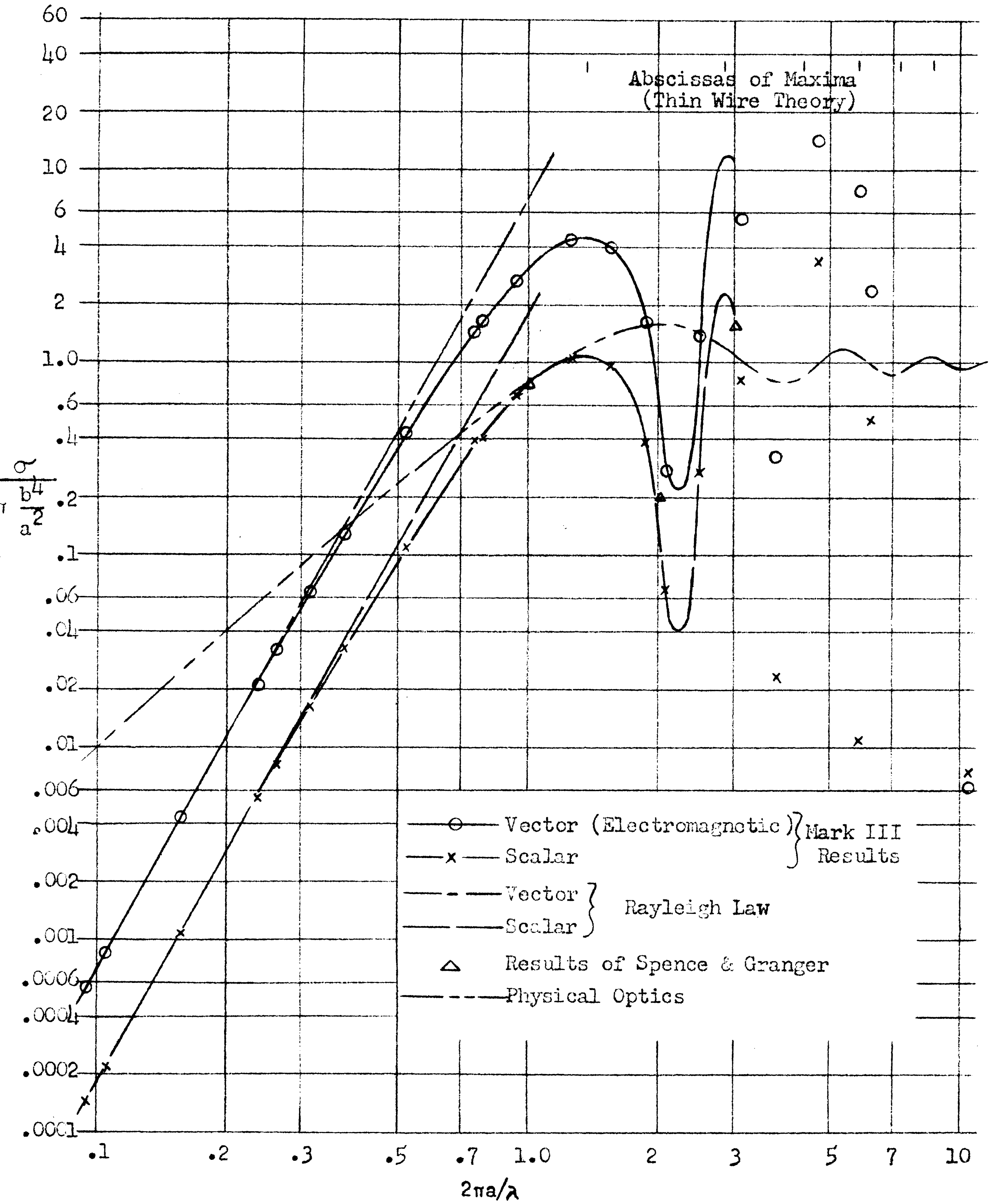

Fig. 1 - Back-Scattering From a Prolate Spheroid

UNCLASSIFIED

Results

The results shown in Fig. 1 were obtained from the following formulas:

The geometric optics solution is

$$\sigma_{G.O.} = \pi b^4/a^2 \ .$$

The physical optics solution is

$$\sigma_{P.O.} = \frac{\pi b^4}{a^2}\left[1 - 2\left[\cos ka\right]\left(\frac{\sin ka}{ka}\right) + \left(\frac{\sin ka}{ka}\right)^2\right].$$

We note that

$$\lim_{k \to \infty} \sigma_{P.O.} = \sigma_{G.O.} \qquad (k = 2\pi/\lambda)$$

We also note that

$$\lim_{k \to 0} \sigma_{P.O.} \approx \frac{\pi b^4}{a^2}\left[ka\right]^2.$$

The Rayleigh electromagnetic answer is

$$\sigma_{Ray.} = \frac{64\pi^3 T^2 k^4}{N^2(4\pi - N)^2},$$

where

$$T = 4\pi a b^2/3,$$

and

$$N = \frac{2\pi a^2}{a^2 - b^2}\left\{1 - \frac{b^2}{2a(a^2-b^2)^{1/2}} \log \frac{a + (a^2-b^2)^{1/2}}{a - (a^2-b^2)^{1/2}}\right\}.$$

For $a = 10b$

$$\sigma_{Ray.} \approx \frac{64\pi b^4}{9a^2}\left(ka\right)^4.$$

The Rayleigh acoustical answer is

$$\sigma_{a.Ray.} = \frac{4\pi^3 T^2}{\lambda^4}\left(\frac{2 - L}{1 - L}\right)^2,$$

UNCLASSIFIED

where

$$L = \frac{b^2}{a^2-b^2}\left[\frac{a}{2(a^2-b^2)^{1/2}}\log\frac{a+(a^2-b^2)^{1/2}}{a-(a^2-b^2)^{1/2}} - 1\right].$$

For $a/b = 10$

$$\sigma_{a,\,Ray.} \approx \frac{16}{9}\,\frac{\pi b^4}{a^2}(ka)^4$$

$$\approx \frac{1}{4}\,\sigma_{Ray.}$$

For ka $\approx$ 1, 2, 3, one may use the work of Spence and Granger [5] to obtain the nose-on acoustical cross section.

As stated previously, if one considers a 10/1 prolate spheroid as a thin wire one could use the work of Ref. 4 to predict the abscissas of the successive maxima.

The appropriate formula is

$$\frac{\pi}{4}\left\{2\log_e\frac{\lambda}{\pi b} + \frac{1}{2}\log_e ka - 1.87\right\}^{-1} = \begin{cases}\cot ka & \text{for odd maxima,}\\ -\tan ka & \text{for even maxima.}\end{cases}$$

---

5. R. D. Spence and S. Granger, "Scattering of Sound from a Prolate Spheroid", Journal of the Acoustical Society of America, Vol. 23, No. 6, p. 701, (1951).

When a/b = 10, the left hand side of this expressions reduces to

$$\frac{\pi}{4}\left\{4.12 - 1.5 \log_e ka\right\} .$$

On the basis of similar analyses for the sphere it was felt that four terms in the field expansions would be sufficient to guarantee accuracy of two significant figures in the results for ka $\leq$ 3. Consequently the exact curve is drawn out to ka = 3, and beyond this the computed points are plotted on the chance that the fourth order results for slightly higher values of ka might be of some value. At several values of ka both third and fourth order results were obtained in order to give some idea of the magnitude of the errors involved in truncating the infinite determinants. The differences between third and fourth order solutions are small for ka < 3.

In order to show that the truncation error was small at ka = 3 and that a fourth order solution was sufficiently accurate, the acoustical cross-section was computed for all orders up to 9. The results are tabulated here:

n	0	1	2	3	4	5	6	7	8
$\frac{\sigma}{\pi b^4/a^2}$	17.7	3.22	7.95	2.04	1.88	1.88	1.88	1.88	1.88

Details of the numerical analysis used in obtaining the exact answers are given in Appendix A. The exact cross-section formulas are presented in Appendix B, while a list of the quantities which were tabulated by the Mark III is given in Appendix C.

Conclusions

The abscissa of the first maximum of the nose-on cross-section curve for a thin prolate spheroid can be obtained quite accurately from thin wire theory. The ordinate of the first maximum for a 10/1 prolate spheroid is only slightly higher than that for a sphere. This suggests that for all prolate spheroids such that $1 < a/b \leq 10$, the ordinate of the first maximum could probably be predicted to two significant figures by linear interpolation between the sphere solution and that of the 10/1 spheroid.

The striking difference between the sphere and the prolate spheroid solutions in both the electromagnetic and acoustical cases is that for the spheroid the second maximum has a larger ordinate than the first. This is the first such problem solved in either electromagnetic or acoustical theory in which the first maximum is not the greatest.

Among the many scientists who worked on the numerical aspects of obtaining the exact solution on the Mark III Electronic Calculator, the authors wish to single out W. Bauer, R. Beach, D. M. Brown, D. F. Eliezer, A. M. Fleishman, G. H. Gleissner, H. E. Hunter, K. Kozarsky, R. A. Niemann, L. M. Rauch, and I. Wyman for special acknowledgement.

UNCLASSIFIED

Appendix A

Computations

The essential mathematical formulation of the problem has been given by Schultz in the preceding article, and the expressions appearing there were used in the computations with no appreciable modification. For the range of parameters used in the present problem, however, it was necessary to compute the spheroidal coefficients d_k^{mn} which express the spheroidal wave functions in terms of spherical ones, and which Schultz assumed known. This was accomplished in the manner specified by Flammer[6].

A three-term recurrence relation is obtained by substituting the expansion in associated Legendre functions of the angular spheroidal function $S_{mn}^{(1)}(\eta)$ into its differential equation and then applying the differential equation and recurrence relations for the Legendre functions. This equation may be written

$$E_{k+2}^{m} d_{k+2}^{mn} + F_k^{mn} d_k^{mn} + G_{k-2}^{m} d_{k-2}^{mn} = 0 \qquad (1)$$

where
$$E_{k+2}^{m} = \frac{(2m+k+2)(2m+k+1)}{(2m+2k+3)(2m+2k+5)} \qquad (2)$$

$$F_k^{mn} = \frac{(m+k)(m+k+1)+A_{mn}}{c^2} + \frac{2(m+k)(m+k+1)-2m-1}{(2m+2k-1)(2m+2k+3)} \qquad (3)$$

---

6. C. Flammer, "Prolate Spheroidal Wave Functions", Technical Report No. 16, Stanford Research Institute, February 1951.

UNCLASSIFIED

$$G_{k-2}^{m} = \frac{k(k-1)}{(2m+2k-3)(2m+2k-1)}$$

and A_{mn} is the separation constant for the radial and angular spheroidal functions. Expansions of A_{mn} in positive or negative powers of c are given in Ref. 6. The number of coefficients given there, however, proved insufficient to give the necessary accuracy in most cases, with the result that when the recurrence relations for the spheriodal coefficients were used repeatedly, the errors built up indefinitely. Consequently an iteration scheme was used to refine the values of the A_{mn}. One such procedure is described by Bouwkamp[7], but this appeared unsuitable for programming on a digital machine, and a simpler though less direct modification of his technique was accordingly employed, the details are given below.

This iteration procedure and the tabulation of the spheroidal coefficients occupied a sizable fraction of the total computation time. The remainder of the operations were straightforward and presented no serious difficulties. However, as stated earlier, the volume of numbers was such that although the Mark III was the largest digital computer in use at the time, it was necessary to divide the problem into five successive runs, as illustrated in the schematic diagram shown in Figure 2.

7. C. J. Bouwkamp, "On Spheroidal Wave Functions of Order Zero", Journal of Mathematics and Physics, Vol. 26, p. 79, (1947).

UNCLASSIFIED

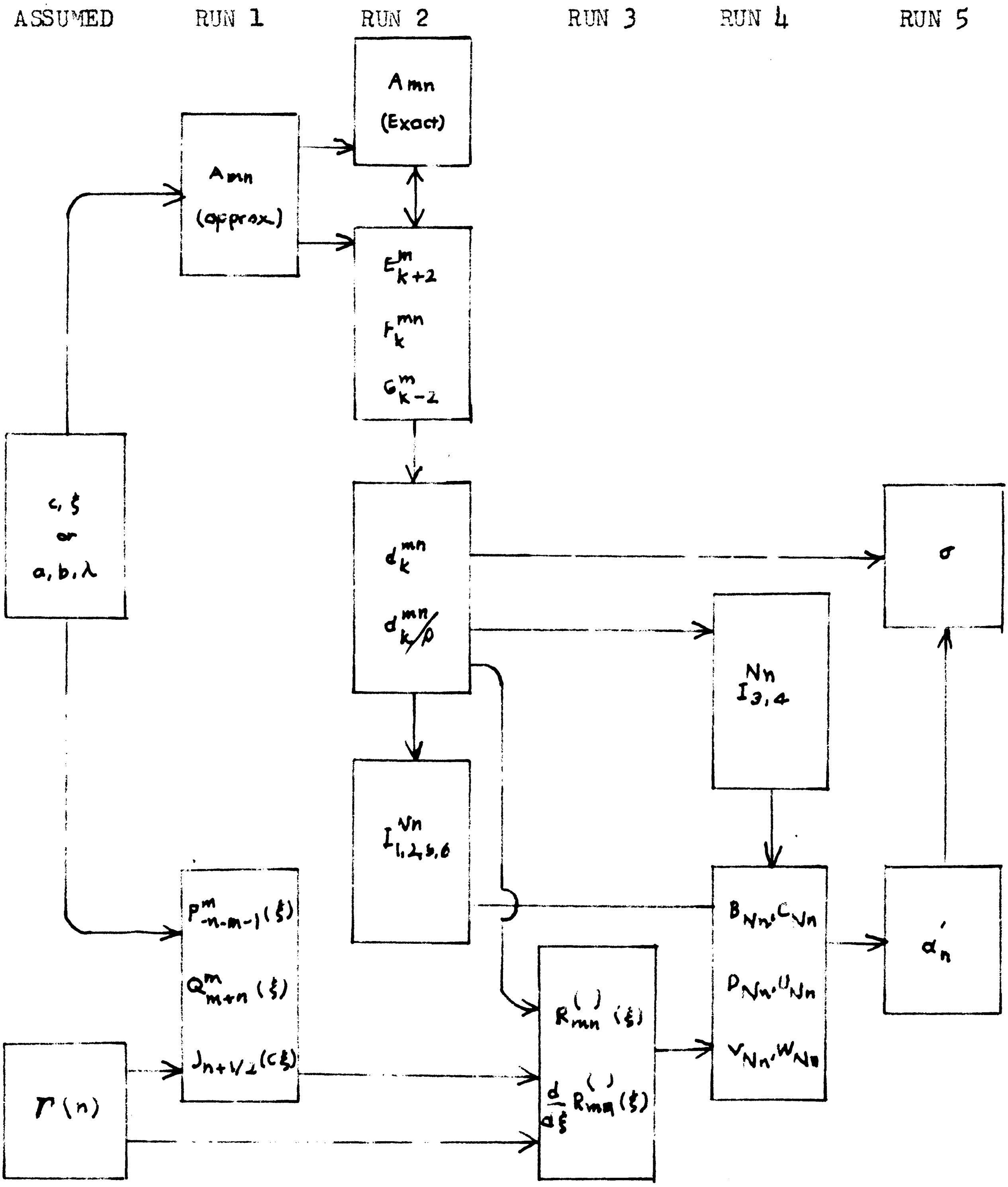

FIG. 2 LOGICAL AND SEQUENTIAL STRUCTURE OF COMPUTATIONS

UNCLASSIFIED

As indicated here, the first machine run included, in addition to the approximate values of the separation constants, computation of certain Bessel and Legendre functions, made necessary by the limitations of all previously existing tables. These were obtained from standard power series formulas.

The second run included the iteration procedure for the refinement of the A_{mn}, which may be outlined as follows, for each set of values of c, m, n:*

(1) The approximate value of A_{mn} is substituted in equation (3) to give an approximate value for F_k^{mn} and the other two coefficients E_k^m and G_k^m are computed exactly, as given by equations (2) and (4).

(2) The quantity $K_{14}^{mn} \equiv - G_{14}^m / F_{16}^{mn}$ is computed, on the assumption that $d_{18}^{mn} / d_{14}^{mn}$ is negligibly small.

(3) Values of $K_k^{mn} \equiv d_{k+2}^{mn} / d_k^{mn}$ are computed for $k = 12, 10, \ldots\ldots, 0$ using equation (1) in the form $1/K_{k-2}^{mn} = - \left(F_k^{mn} + E_{k+2}^m K_k^{mn}\right) / G_{k-2}^m$.

(4) The quantity $G_{-2}^m / K_{-2}^{mn} = -F_o^{mn} - E_2^m K_o^{mn}$ is computed. Since the exact value of G_{-2}^m is zero, the computed value

* The routine is given here for n even. The obvious modifications are applied to deal with odd n.

of G^{m}_{-2} / K^{mn}_{-2} is a measure of the error in the approximate value of A_{mn} used. If this does not exceed a certain empirically established tolerance, the values of K^{mn}_{k} obtained in step (3) are used to compute the required spheroidal coefficients d^{mn}_{k}.

(5) If the value obtained in (4) exceeds the tolerance, it is substituted back in equation (1) which then yields a second approximation to A_{mn}. An average of the first and second approximations may be used to repeat the procedure from step (1).

The remaining quantities appearing in Figure 2 have been discussed in the preceding article. The specific formula for the back-scattering cross-section, which is not given there, appears in Appendix B. Formulas used in computing the acoustical cross-section are also presented in this appendix.

UNCLASSIFIED

Appendix B

1. Formula for Radar Back-Scattering Cross-Section

If the value $\eta = 1$ is substituted into the expression for the scattering cross-section derived in the preceding paper (equation 79), the ϕ dependence disappears and the resulting formula for the back-scattering cross-section can be written

$$\sigma = 4\pi a^2 \left| \sum_{n=o}^{\infty} j^n \alpha'_n \sum_{k=o}^{\infty} d_k^{on} \right|^2 .$$

2. Acoustical Cross-Section

The problem of acoustical scattering by a prolate spheroid was solved by Spence and Granger. In the present terminology the expression they derived for the nose-on cross-section σ' may be written

$$\sigma' = \frac{4\pi a^2}{c^2 \xi_o^2} \cdot \sum_{n=o}^{\infty} j^{n+1} A_n H_n(\xi_o) S_{on}^{(1)}(\eta) \cdot \sum_{n=o}^{\infty} (-1)^{n+1} A_n^* H_n(\xi_o) S_{on}^{(1)}(\eta) ,$$

where

$$H_n(\xi) = \frac{dR_{on}^{(1)}}{d\xi}(\xi) \Big/ \frac{dR_{on}^{(4)}}{d\xi}(\xi) ,$$

$$A_n = 2 j^n S_{on}^{(1)}(1) / N_{on} ,$$

$$N_{on} = \int_{-1}^{1} \left[S_{on}^{(1)}(\eta) \right]^2 d\eta \equiv I_1^{Nn} ,$$

and A_n^* is the complex conjugate of A_n.

UNCLASSIFIED

UNCLASSIFIED

Appendix C

Quantities Tabulated in Machine Output

For each point shown on the graph of Figure 1 the following quantities were recorded in the course of the computations:

1. Associated Legendre functions $P^{m}_{-n-m-1}(\xi)$ and $Q^{m}_{m+n}(\xi)$ and their derivatives with respect to ξ.

   for $m = 0, 1$

   $n = -1, -2, \ldots \ldots -16.$

2. Bessel functions $J_{n+\frac{1}{2}}(c\xi)$ for $n = 1, 2, \ldots \ldots 17.$

3. Separation constants A_{mn},

   for $m = 0, 1$

   $n = 0, 1, 2, 3$

4. Spheroidal coefficients d^{mn}_{k} $(k > 0)$, and d^{mn}_{k}/ρ $(k \leq 0)$.

   for $m = 0, 1$

   $n = 0, 1, 2, 3$

   k = all required values in the range -16 to +16.

5. Radial spheroidal functions $R^{(1)}_{mn}(\xi)$ and $R^{(2)}_{mn}(\xi)$ and their derivatives with respect to ξ.

   for $\xi = 1.005$

   $m = 0, 1$

   $n = 0, 1, 2, 3.$

UNCLASSIFIED

6. Boundary integrals I_k^{Nn}

for k = 1, 2, 5, 6

all combinations of N and n in the range

N = 0, 1, 2, 3

n = 0, 1, 2, 3.

(I_3^{Nn} and I_4^{Nn} were not recorded.)

7. Determinantal elements B_{Nn}, C_{Nn}, D_{Nn}, U_{Nn}, V_{Nn}, W_{Nn},

for all combinations of N and n in the range

N = 0, 1, 2, 3,

n = 0, 1, 2, 3.

8. Radar cross-section σ.

With the exception of σ, all quantities are given to 15 significant figures. The values of σ are rounded off to 5 significant figures.

Particular values of any of these quantities may be obtained on request.