Studies in Radar Cross-Sections – III Scattering by a Cone

by K. M. Siegel and H. A. Alperin

Project MIRO

Contract No. AF 30 (602) -9

Expenditure Order Numbers:
RDO 166-17 R161-43
RDO 166-17 AD-4
RDO 166-17 AD-7

The research reported in this document has been made possible through support and sponsorship extended by The Rome Air Development Center under Contract No. AF 30 (602)-9. It is published for technical information only and does not represent recommendations or conclusions of the sponsoring agency.

Willow Run Research Center
Engineering Research Institute
University of Michigan

UMM-87. January 1952

TABLE OF CONTENTS

PREFACE

The most important parameter characterizing the scattering from any body is the radar cross-section. Unfortunately, theoretical radar cross-sections of general shapes have not been found, and experimental data still lack precision.

Exact theoretical solutions have been obtained for certain three-dimensional shapes including the sphere, prolate spheroid, paraboloid and semi-infinite cone (with source on extension of axis of the cone). The radar cross-section of a sphere has been determined and numerical answers obtained over a range of several parameters (Ref. 1, 2, and 3). For the prolate spheroid (Ref. 4) and the paraboloid (Ref. 5) theoretical solutions have been formulated but only one numerical answer exists for the former and none exists for the latter. The exact solution for the semi-infinite cone has been found from electromagnetic theory by Hansen and Schiff (Ref. 6). In order to compute numerical values from this solution, it is necessary to determine the zeros of certain associated Legendre functions. However, as shown in Reference 7 the tables currently in use which list the values of these zeros are inaccurate.

The present report is the third in a series of reports which will discuss the problem of the scattering of microwaves by various shaped objects. In this over-all program, the numerical values for the cross-sections of certain simple shapes are being computed for a range of physical parameters which is either realistic at present or will be realistic in the near future.

The first of these reports (Ref. 4) described the scattering by a prolate spheroid.

The second report (Ref. 7) discussed the properties of the associated Legendre functions of non-integral degree.

The present report reviews the solutions to the cone problems which are obtained from physical and geometrical optics and electromagnetic theory. New results presented for the first time in this report are as follows:

1. A quick method is used to obtain solutions to the scalar wave equation for plane waves scattered from a body when the solution is already known for the case of a point source placed a finite distance from the body.

2. Carslaw's method for a point source is extended to include the solution for plane waves.

3. The scalar differential scattering cross-section is obtained for the scattering of plane waves by a semi-infinite cone.

4. The relationship between radar cross-section and differential scattering cross-section is derived. (Although the authors know of no reference in which this derivation appears, they feel this result is well known).

5. A comparison is made between the sound-theory (scalar) and electromagnetic (vector) solutions to the cone problem.

The next report in this series (Ref. 23), will discuss the Hansen and Schiff solution (Ref. 6) for a semi-infinite cone more fully. An analysis will be made which shows how to re-interpret the Hansen and Schiff solution to yield a correct conclusion. A cross-section for the cone is then obtained from this solution and compared with experiment.

NOMENCLATURE

$\vec{A}$ = vector potential

$\vec{E}$ = electric field vector

$\vec{H}$ = magnetic field vector

$H^{(2)}_{i+\frac{1}{2}}(k\rho)$ = Hankel function of the 2nd order, degree i + 1/2, argument $k\rho$

$J_{n_i+\frac{1}{2}}(kr)$ = cylindrical Bessel function of degree n_i + 1/2 and argument kr

$\vec{K}$ = surface current density

$$K_{n_i+\frac{1}{2}}(jk\rho) \equiv \frac{\pi}{2 \sin (n_i + \frac{1}{2})\pi} e^{-\frac{1}{2}(n_i+\frac{1}{2})j\pi} \times \left[J_{-n_i-\frac{1}{2}}(k\rho) - e^{(n_i+\frac{1}{2})j\pi} J_{n_i+\frac{1}{2}}(k\rho) \right]$$

$N_{n_i+\frac{1}{2}}(kr)$ = cylindrical Neumann function of degree n_i + 1/2 and argument kr

$P^m_{n_i}(\mu)$ = associated Legendre function of the 1st kind, order m, degree n_i, argument μ

$Q^m_{n_i}(\mu)$ = associated Legendre function of the 2nd kind, order m, degree n_i, argument μ

c = velocity of light

g $\equiv \int_A e^{-2jkz} \, dA$ = equivalent flat plate area

i = real integer

NOMENCLATURE (Continued)

j	$\equiv \sqrt{-1}$
$j_{n_i}(kr)$	= spherical Bessel function $\equiv \sqrt{\frac{\pi}{2kr}}\, J_{n_i+\frac{1}{2}}(kr)$
k	$= \frac{2\pi}{\lambda}$
ℓ	$\equiv n + 1/2$
m_i	= real number defined by $\frac{dP'_{m_i}(\mu_o)}{d\mu} = 0$
n_i	= real number defined by $\frac{dP_{n_i}(\mu_o)}{d\mu} = 0$
$n_{n_i}(kr)$	= spherical Neumann function $\equiv \sqrt{\frac{\pi}{2kr}}\, N_{n_{i+1/2}}(kr)$
r	= radial distance in spherical coordinates
r_1	= radial distance in cylindrical coordinates
y, y'	= solutions of Legendre's equation
α, β	= real numbers defined by $n \equiv \alpha + j\beta$
ϵ	= dielectric constant (ϵ_o = dielectric constant of free space)
ϵ'	= a real number $0 < \epsilon' < 1$
η_o	= intrinsic impedance of free space
θ	= angle measured from z-axis in spherical coordinates
θ_1	= angle in cylindrical coordinates
θ_o	= 1/2 total cone angle
λ	= wave length

NOMENCLATURE (Continued)

μ = $\cos\theta$ $\quad (\mu_o = \cos\theta_o)$

μ' = magnetic permeability (μ'_o = magnetic permeability of free space)

$\Pi(x)$ = $\Gamma(x+1)$ where Γ is the Gamma function

ρ = distance measured along axis of cone from vertex to source of radiation

$\sigma(0)$ = radar cross-section

$\sigma_D(\theta) = \sigma_D$ = differential scattering cross-section

$\sigma_D(0)$ = differential back scattering cross-section

ω = angular frequency = $\frac{2\pi c}{\lambda}$

ϕ = azimuthal angle in spherical coordinates

Miscellaneous

$*$ = complex conjugate

$\mathcal{Re}(x)$ = the real part of x

$\hat{i}, \hat{j}, \hat{k}$ = unit vectors along the rectangular x, y, z, axes respectively (Note: the symbol $\wedge$ will always denote a unit vector).

$$\frac{dP_{n_i}(\mu_o)}{d\mu} = \left.\frac{dP_{n_i}(\mu)}{d\mu}\right|_{\mu=\mu_o}$$

$$\frac{\partial^2 P_{n_i}(\mu_o)}{\partial n_i \partial\mu} = \left.\frac{\partial^2 P_{n_i}(\mu)}{\partial n_i \partial\mu}\right|_{\mu=\mu_o}$$

NOMENCLATURE (Continued)

Miscellaneous (Cont'd)

W, u, u_o = eigenfunctions of the operator $\nabla^2 + k^2$ with the eigenvalue 0.

$\mathcal{O}(x)$ = terms of the order of x

I

RADAR CROSS-SECTION

The radar cross-section $\sigma(0)$ is defined as the area intercepting that amount of power which, when scattered isotropically, produces an echo at the source of radiation equal to that observed from the target. That is, $\overline{S^s} = \frac{\overline{S^i}\,\sigma(0)}{4\pi r^2}$ where r is the distance from the antenna to the target and $\overline{S^s}$ and $\overline{S^i}$ are the average scattered and incident Poynting vectors respectively. $\sigma(0)$ indicates that backscattering takes place along $\theta = 0$ where θ is the usual angle in spherical coordinates measured down from the z-axis. It is assumed that the incident wave is linearly polarized and along the z-axis and that only the component of the scattered wave with polarization similar to the incident wave is used in determining $\sigma(0)$.

The radar cross-section described above is related to what is commonly known as the differential scattering cross-section, σ_D. The latter can be described in the following manner. Consider a plane wave incident upon the scattering body; then physically, at large distances from the body, the solution to the wave equation (either vector or scalar) must have the character of an incoming plane wave plus an outgoing spherical wave which appears to diverge from the scattering body. If spherical coordinates are employed, this condition can be expressed mathematically as

$$u = e^{jkr\cos\theta} + \frac{e^{-jkr}}{r}\left[f(\theta,\phi)\right]$$

where, for scalar fields, u is the solution to the scalar wave equation, and for the vector fields, u represents one of the three components, u_1, u_2, u_3, of the solution to the vector wave equation. The differential scattering cross-section σ_D is then defined for the scalar case as $\sigma_D = \left|f(\theta,\phi)\right|^2$

and for the vector case as $\sigma_D = \left|f_1(\theta,\phi)\right|^2 + \left|f_2(\theta,\phi)\right|^2 + \left|f_3(\theta,\phi)\right|^2$.

Although the radar cross-section is strictly defined only for the backward direction, the differential scattering cross-section gives the complete angular distribution of the scattered wave. For the backward direction we call the latter the differential backscattering cross-section or, simply, the backscattering cross-section.

In Appendix B it is demonstrated that the radar and backscattering cross-sections are related in simple fashion, namely, that $\sigma(0) = 4\pi\,\sigma_D(0)$.

In general, in order to calculate the cross-section of any object one must determine the electric and magnetic field intensities at all points in space. That is, for a charge-free, homogeneous medium, a solution to the vector wave equation $\nabla^2\vec{C} + k^2\vec{C} = 0$ is required, subject to Maxwell's equations and to the proper boundary conditions at the surface of the scattering body. Here, $\vec{C}$ stands for any of the pertinent vector quantities, $\vec{E}$, $\vec{H}$, $\vec{B}$, $\vec{D}$, or $\vec{A}$.

The solution of the vector wave equation for any body presents great difficulties, both mathematical and theoretical. One widely known method developed by Hansen (Ref. 9, p. 392) constructs solutions to the vector wave equation from known solutions to the corresponding scalar wave equation. But the scattering body must be the contour surface of a coordinate system in which the scalar wave equation is separable and for even the simpler types of separable bodies the problem involved in obtaining a numerical answer can become quite formidable (see Ref. 4 and 6).

II

APPROXIMATE METHODS

Because of the difficulties mentioned above and in order that some numerical answers may be obtained (even if they are only approximate) the methods of physical and geometrical optics are employed. The regions for which these methods are applicable are as follows: when the wave length of the incoming radiation is small compared to the dimensions of the scattering body the methods of physical optics may be used. In the limit of vanishing wave length the methods of physical optics reduce to those of geometrical optics.* When the wave length of the radiation is of the same order as the dimensions of the scatterer, then electromagnetic theory must be used, i.e., a direct solution to the vector wave equation must be obtained. When the wave length is much greater than the dimensions of the scattering body, the Rayleigh Law is applicable.

The basis of both physical and geometrical optics is the Kirchhoff-Huygens principle, originally applied to scalar fields but later extended (see for example Ref. 9, Sec. 8.14) to electromagnetic fields. This principle states that if the value of a field quantity is known at every point on any closed surface surrounding a source-free region, each elementary unit of surface can be considered as a radiating source, and the total field at any interior point is given by integrating the contributions of all the elements over the surface.

A. PHYSICAL OPTICS

In order to apply the methods of physical optics, certain simplifying assumptions will be made in this report. They are.

---

*For a proof of the statement that the diffuse boundary of the shadow in diffraction phenomena (i.e., physical optics) becomes the sharp shadow of geometrical optics as the wave tends to zero, (see Ref. 8, p. 79).

1. Only back scattering is calculated.

2. The scattering body is assumed to be a perfect electrical conductor.

3. The dimensions of the scattering surface are large compared to λ, the wave length of the incoming radiation.

4. The scattering surface is assumed to be smooth, containing no abrupt corners except possibly at its extreme edges.

5. The distance from the radiation source to the target is large compared with the dimensions of the target.

Because of Assumption 2, all fields inside the body are zero and the tangential component of $\vec{E}$ and the normal component of $\vec{H}$ are zero on the surface. Because of Assumption 3, the surface of the body is so large in terms of wave length that $\vec{H}_t$ (the total magnetic field on the surface) has the same value that it would have if the surface were infinite in extent. On the back side the field is then zero, and on the front side $\vec{H}_t$ is twice the tangential component of the incident plane wave. Thus, if the incident wave is traveling along the z-axis with its magnetic vector of amplitude H_o in the direction $\hat{a}$, perpendicular to the z-axis, then on the surface the magnetic field is given by

$$\vec{H}_t = \vec{i}_t \, 2 H_o \, e^{-jkz} \qquad \text{(II-1)}$$

where $\vec{i}_t$ is a vector tangent to the surface and giving the direction of the total surface field.

$$\vec{i}_t = \hat{a} - (\hat{a} \cdot \hat{n}) \, \hat{n} \qquad \text{(II-2)}$$

and $\hat{n}$ is the outward normal to the surface. Stated in another way, since no tangential electric field can exist on the target surface, a reflected tangential field must be set up which is equal in magnitude but opposite in phase to the incident tangential electric field at every element of the surface. Hence, a reflected tangential magnetic field is induced which is of the same magnitude and phase as the incident tangential magnetic field, and the total tangential magnetic field is equal to twice the incident magnetic field. This situation is illustrated in Figure II-1 below. $\hat{n}_o$ is a unit vector in the direction of the incident wave.

Kerr (Ref. 1, Ch. 6) derives the physical optics approximation to the radar cross-section by making use of Stratton's expression for the magnetic field which comes from a vector extension of the Kirchhoff-Huygens principle. Another method of obtaining the physical optics expression is to calculate the surface currents, the resulting vector potential and then the field from the vector potential. This method will be discussed in Appendix C. The treatment given below, however, is due to Kerr.

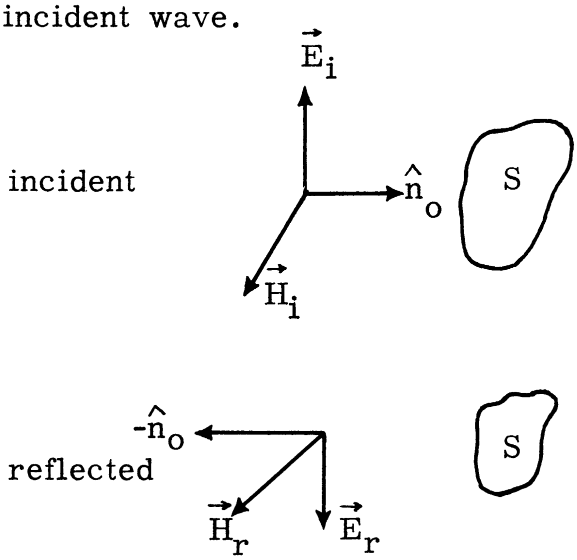

Figure II-1.

If a plane wave falls on an object of arbitrary surface S, the object scatters the incident wave, and the current and charges within the object may be considered the source of the scattered wave.

Stratton's Equation (20) (Ref. 9, p. 466) may be adapted to the present case and becomes, by virtue of Assumption 2, (superscript s denotes scattered wave)

$$\vec{H}^s = -\frac{1}{4\pi}\int_S \left[(\hat{n} \times \vec{H}_t) \times \nabla\left(\frac{e^{-jkr}}{r}\right)\right] dS. \qquad \text{(II-3)}$$

The surface of integration is the surface of the object and a closed surface at infinity (the integral over this surface is zero). Because of Assumptions 1, 3, and 5, we may write

$$\nabla\left(\frac{e^{-jkr}}{r}\right) \simeq -\hat{n}_o\left(jk\,\frac{e^{-jkr}}{r}\right)$$

where r is the distance from any element of area dS to the point of observation. Using the usual physical optics approximation (removing r from under the integral sign where it affects only the amplitude and may be treated as a constant, but retaining it where it has an effect on the phase), Equation (II-3) becomes

$$\vec{H}^s = -\frac{jk}{4\pi r}\int_S \hat{n}_o \times (\hat{n} \times \vec{H}_t)\, e^{-jkr} dS\,. \qquad \text{(II-4)}$$

Let us now consider the case of the incident plane wave traveling along the z-axis. Then Equations (II-1) and (II-2) apply and we find that

$$\hat{n}_o \times (\hat{n} \times \vec{H}_t) = -2\hat{a}H_o\, e^{-jkz}\,(\hat{n}_o \cdot \hat{n})\,.^*$$

If dS is the element of area on the scattering surface and dA is the projection of dS on the plane normal to the line-of-sight, then dS and dA are related by $dA = -(\hat{n}_o \cdot \hat{n})\, dS$. Then for back-scattering, Equation (II-4) becomes

$$\vec{H}^s = -\frac{jkH_o\,\hat{a}}{2\pi r}\int_A e^{-2jkz}\, dA\,. \qquad \text{(II-5)}$$

The cross-section is given by

$$\sigma = 4\pi r^2 \frac{\bar{S}^s}{\bar{S}^i} = 4\pi r^2 \left|\frac{H^s}{H_o}\right|^2.$$

*Kerr's third equation (Ref. 1, p. 463) is in error and should read $\hat{n}_o \times (\hat{n} \times \vec{i}_t) = -\hat{a}\,(\hat{n}_o \cdot \hat{n})$. Also equation (55) on page 463 should have a minus sign.

Using Equation (II-5) and remembering that $k = \frac{2\pi}{\lambda}$, we finally have

$$\sigma = \frac{4\pi}{\lambda^2} \left| \int_A e^{-2jkz} \, dA \right|^2 = \frac{4\pi}{\lambda^2} \left| g \right|^2. \qquad \text{(II-6)}$$

The term $\int_A e^{-2jkz} \, dA$ is called g, the equivalent flat-plate area, by Spencer (Ref. 10). It is equal to the summation (of the projection on a plane normal to the line-of-sight) of each elementary area dS of the target multiplied by a phase factor e^{-2jkz} which assigns to each area a phase depending upon its distance from the transmitting antenna of the radar. Thus Spencer notes that the amplitude and plane of polarization are constant and independent of orientation as long as the projection of the scatterer in the direction of the line-of-sight is constant. Note that for a closed body integration is from the tip to the edge of the geometrical shadow. Thus the assumption is made that the geometrical shadow region does not contribute to the scattering. This point will be discussed later in this section.

We now turn to the actual calculation of the backscattering cross-section of a finite cone (aligned along the z-axis as shown in Figure II-2) using the method of physical optics. The equivalent flat-plate area, g, must first be calculated. For a cone, $r_1^2 = z^2 \tan^2 \theta_1$ and the cross-sectional area $A = \pi r_1^2 = \pi z^2 \tan^2 \theta_1$.

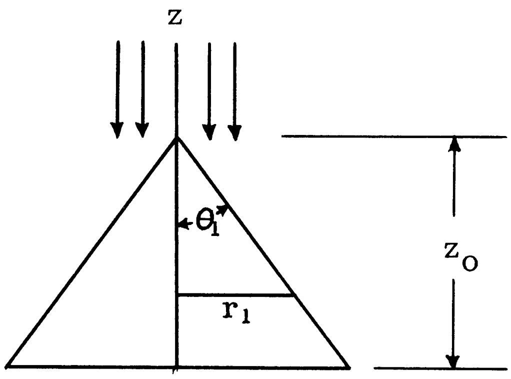

Figure II-2.

Then
$$g = 2 \tan^2 \theta_1 \pi \int_0^{z_o} e^{-2jkz} \, z dz$$
$$= 2\pi \tan^2 \theta_1 \left[\frac{z e^{-2jkz}}{-2jk} - \frac{e^{-2jkz}}{(-2jk)^2} \right]_o^{z_o}$$
$$= \frac{2\pi \tan^2 \theta_1}{(2jk)^2} \left[-2jkz_o e^{-2jkz_o} - e^{-2jkz_o} + 1 \right].$$

Using Equation (II-6) and $k = \frac{2\pi}{\lambda}$ we have

$$\sigma = \frac{\tan^4 \theta_1 \lambda^2}{16\pi} \left| -2jkz_o e^{-2jkz_o} - e^{-2jkz_o} + 1 \right|^2 .$$

If we restrict our consideration to microwaves ($\lambda < 30$ cm and $k > \frac{1}{5}$ cm^{-1}), then $kz_o >> 1$ for any object of dimensions $z_o >> 5$ cm. Using this approximation, the scattering from the base of the cone reduces to $\sigma = \pi z_o^2 \tan^4 \theta_1$, while for scattering from the tip,

$$\sigma = \frac{\lambda^2}{16\pi} \tan^4 \theta_1 . \qquad \text{(II-7)}$$

As can be seen, with the assumption $kz_o >> 1$, if the base of the cone is sharply defined, the scattering from it will outweigh the contribution from the tip of the cone. However, if the base of the cone is cut very irregularly or the cone curves gradually so that it becomes parallel to its axis, the scattering from the tip will become dominant. Hansen and Schiff (Ref. 11) discuss this point in some detail. They investigate the contribution to the scattering of a pointed body of revolution from the penumbra region, i.e., the region immediately behind the edge of the geometrical shadow, and conclude that under the assumption of short wavelengths ($kz_o >> 1$) the contribution from the tip is the dominant one. This conclusion is very important for it enables us to approximate the back-scattering from certain important practical shapes, such as the ogive, by considering only the scattering which would result from the tip of the semi-infinite cone with the same angular opening as the front end of the missile.

B. GEOMETRICAL OPTICS

In the limit of vanishing wavelength, the methods of geometrical optics are applicable. Polarization and phase are now neglected, and the scattering of the energy depends entirely on the target and antenna geometry, since the energy flow is directed now in straight lines.

The method is applicable only to smooth surfaces, or to those portions of a body which are smooth. Geometrical optics does not take into account diffraction effects which arise because of the wave nature of electromagnetic radiation. The method is most easily utilized for quadric surfaces with two finite principal radii of curvature, but may be applied by suitable refinements to such surfaces as flat plates and cylinders where one of the radii of curvature is infinite or very large compared to the other.

The scattering from a semi-infinite cone or from a pointed body of revolution viewed head-on may not be calculated by this method since the effect of the conical point cannot be determined. However, formula (II-7), derived previously from physical optics under the assumption $kz_o >> 1$, may be thought of as a geometrical optics formula since it is the physical optics formula in the limit of very small wave length.

Given below is the geometrical optics derivation for the radar cross-section valid for smooth, curved surfaces with two finite principal radii of curvature. This derivation is due to R. C. Spencer (Ref. 12).

The cross-section of a body is now defined as the cross-section of an isotropic scatterer which would scatter the same energy in any given direction as the actual body. It is assumed that the plane waves of intensity I_o, arrive at the point of reflection P on the body. An isotropic scatterer intercepts an amount of power equal to $I_o\sigma$ and thus, the scattered power dP^s per unit solid angle $d\omega$ is

$$\frac{dP^s}{d\omega} = \frac{I_o\sigma}{4\pi} \qquad \text{and} \qquad \sigma = \frac{4\pi\, dP^s}{I_o\, d\omega} .$$

Note that the differential solid angle $d\omega$ encloses the reflected ray. Making use of the law of reflection for a smooth body (namely that the angle of incidence θ is equal to the angle of reflection), it is now possible to express the cross-section in terms of the differential solid angle $d\omega_o$ enclosing the normal to the surface. This is shown in Figure II-3, below, by the solid angles $d\omega$ and $d\omega_o$ on the unit sphere, where the points T, P, N, and R, are respectively in the direction of the transmitter, the point of reflection, the normal to the scattering surface at the point of reflection, and the receiver.

$$d\omega_o = \sin\theta\, d\phi\, d\theta$$

$$d\omega = \sin(2\theta) d\phi\, d(2\theta) = 4\cos\theta\, d\omega_o$$

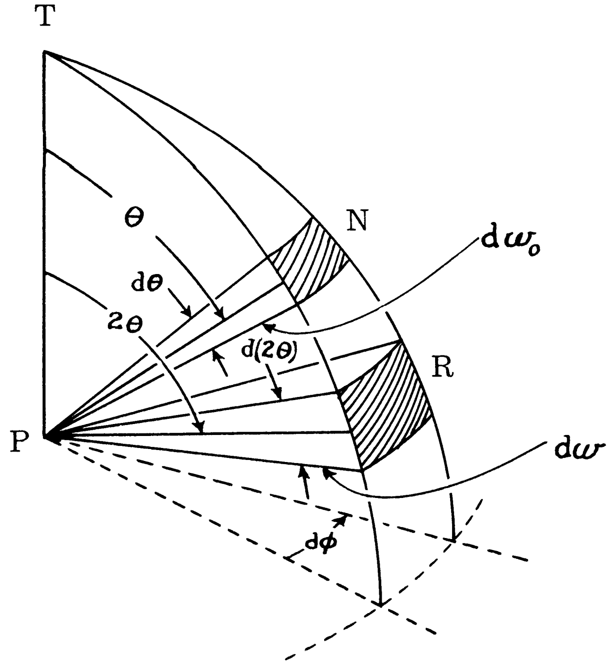

Figure II-3

The power incident on any surface area dS of the body is $dP^I = I_o \cos \theta \, dS$ so that σ becomes $\sigma = \frac{\pi \, dS}{d\omega_o}$ since $dP^S = dP^I$.

For a quadric surface with principal radii of curvature R_1 and R_2 the solid angle enclosed by the normal to the surface element dS is

$$d\omega_o = \frac{dS}{R_1 R_2} .$$

This simply says that the expression $\frac{dS}{R_1 R_2}$ is equal to the equivalent solid angle $d\omega_o$ on a unit sphere enclosing the normal to dS. This is shown in Figure II-4, where the unit sphere may be enclosed within the body or may completely surround the body and the center of the sphere is on the normal to dS.

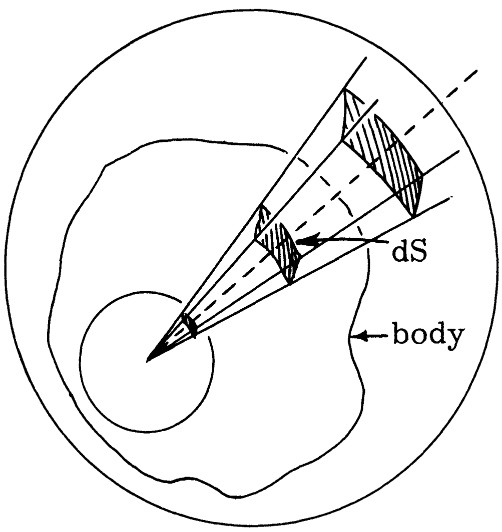

Figure II-4

Thus, the $d\omega_o$ shown in Figure II-3 is of the same magnitude as the $d\omega_o$ occurring in the expression $d\omega_o = \frac{dS}{R_1 R_2}$ since they both are the differential solid angles on a unit sphere enclosing the normal to the surface dS of the scattering body. Finally then,

$$\sigma = \pi R_1 R_2 .$$

Note that σ is independent of θ; hence, it can properly be called the radar cross-section since the formula is valid for $\theta = 0$.

III

SCALAR SCATTERING BY A SEMI-INFINITE CONE

In this section, the scattering from the semi-infinite cone is solved by the scalar wave equation, making use of a particularly elegant technique. Two other techniques are exhibited in Appendix A: the method of separation of variables; and the method of contour integration. Solution of the scalar cone problem by three different methods is justified by the wide interest shown in it (see, for example, Ref. 13, 14, 15, 16, and 17). Furthermore, the auxiliary information obtained by these different methods has been valuable in obtaining relationships between special functions which have proved useful in our other scattering work.

Carslaw (Ref. 17) solved the following "sound" problem. He placed a point source on the extension of the axis of symmetry of a semi-infinite cone at a distance ρ from the tip. He solved the problem for the scalar wave equation with $e^{j\omega t}$ type time dependence and insisted that the normal derivative of the solution go to zero at the boundary and that there should be no reflection from infinity. This is stated mathematically

$$\nabla^2 u + k^2 u = 0. \qquad k = \frac{2\pi}{\lambda}$$

$$\left(\frac{\partial u}{\partial \, \text{normal}}\right) = 0 \text{ at body}.$$

As r approaches ∞, u cannot have e^{jkr} type dependence.

The solution to his problem was split into 2 parts. He obtained one part for $\rho > r$ and one for $\rho < r$. For $\rho > r$

$$u = \frac{4}{\sqrt{r\rho}} \sum_i e^{j\frac{\pi}{2}(n_i+\frac{1}{2})} (n_i+\tfrac{1}{2}) K_{n_i+\frac{1}{2}}(jk\rho) \frac{P_{n_i}(\mu)\, J_{n_i+\frac{1}{2}}(kr)}{(1-\mu_o^2)P_{n_i}(\mu_o)\dfrac{d^2P_{n_i}(\mu_o)}{d_{n_i}\, d\mu}} \tag{III-1}$$

where the n_i are the zeros of $P^1_{n_i}(\mu_o)$, $j = \sqrt{-1}$, $\mu = \cos\theta$ and θ_o = cone half angle.

Now a point source at infinity appears to an observer as a plane wave. The cone problem of interest to us is the scattering of a plane wave by a semi-infinite cone. (Physically the plane wave is supposed to represent, much later in our analysis, a radar beam emitted by a parabolic antenna.)

Since Carslaw's solution already is a solution of the correct differential equation with the correct boundary condition at the body, we must only find a method of moving the point source at $(\rho, 0, 0)$ to infinity.

We thus go on a "classical tour" to portray mathematically the physics of a point source moving to infinity and becoming a plane wave. The expansion for a point source (Ref. 18) is

$$\frac{e^{-jkR}}{R} = -\frac{\pi}{2}\sqrt{\frac{-1}{r\rho}} \sum_{i=0}^{\infty} (2i+1)\, J_{i+\frac{1}{2}}(kr)\, H^{(2)}_{i+\frac{1}{2}}(k\rho)\, P_i(\mu) \qquad \text{for } \rho > r$$

where $R = \sqrt{r^2 + \rho^2 - 2r\rho\mu} = \rho\sqrt{1 - \frac{2r\mu}{\rho} + \frac{r^2}{\rho^2}}$.

Now for large ρ

$$R = \rho\left(1 - \frac{r}{\rho}\mu\right)$$

$$H^{(2)}_{i+\frac{1}{2}}(k\rho) = \sqrt{\frac{2}{\pi k\rho}}\,(-1)^{\frac{i+1}{2}}\, e^{-jk\rho}$$

$$\frac{e^{-jk(\rho - r\mu)}}{\rho - r\mu} = \sqrt{\frac{\pi}{2kr}} \sum_i (-1)^{i/2}(2i+1)\, J_{i+\frac{1}{2}}(kr)\, \frac{e^{-jk\rho}}{\rho}\, P_i(\mu)$$

$$\frac{e^{jkr\mu}}{1 - \frac{r\mu}{\rho}} = \sqrt{\frac{\pi}{2kr}} \sum_i (-1)^{i/2}(2i+1)\, J_{i+\frac{1}{2}}(kr)\, P_i(\mu)$$

Now taking the limit as $\rho \to \infty$

$$e^{jkr\mu} = \sqrt{\frac{\pi}{2kr}} \sum_i (-1)^{i/2} (2i + 1) J_{i+\frac{1}{2}} (kr) P_i(\mu)$$

which is the well known expansion for a plane wave.

We went on this classical tour to observe that

$$\lim_{\rho \to \infty} \left[\frac{e^{-jkR}}{R} \cdot \frac{\rho}{e^{-jk\rho}} \right] = e^{jkr\mu} .$$

Since the sources are really fundamental solutions to the boundary value problems they represent, the transformation between sources is representative of the transformation between total boundary value problem solutions. Thus, our method of attack is to take Carslaw's solution* for a point source at $(\rho, 0, 0)$ and transform it in exactly the same way in which we transformed a point source into a plane wave.

Let $\lim_{\rho \to \infty} \left[u_{Carslaw} \cdot \frac{\rho}{e^{-jk\rho}} \right] = w$, where $u_{Carslaw}$ is Equation (III-1).

Now for large ρ (Ref. 17, p. 138)

$$K_{n_i}(jk\rho) = \sqrt{\frac{\pi}{2k\rho}} \; e^{-j(k\rho + \frac{\pi}{4})} .$$

*All sums will now be considered to be finite sums. This assumption is justified in Reference 23.

$$W = \lim_{\rho \to \infty} \left[\frac{4}{\sqrt{r\rho}} \; \frac{\rho}{e^{-jk\rho}} \; \sqrt{\frac{\pi}{2k\rho}} \; e^{-j(k\rho + \frac{\pi}{4})} \right.$$

$$\left. \times \sum_{i=1} \frac{e^{\frac{j\pi}{2}(n_i + \frac{1}{2})} (n_i + \frac{1}{2}) J_{n_i + \frac{1}{2}}(kr) P_{n_i}(\mu)}{(1 - \mu_o^2) P_{n_i}(\mu_o) \dfrac{\partial^2 P_{n_i}(\mu_o)}{\partial n_i \partial \mu}} \right]$$

$$W = 4 \sqrt{\frac{\pi}{2kr}} \sum_i \frac{e^{j\frac{n_i \pi}{2}} (n_i + \frac{1}{2}) J_{n_i + \frac{1}{2}}(kr) P_{n_i}(\mu)}{(1 - \mu_o^2) P_{n_i}(\mu_o) \dfrac{\partial^2 P_{n_i}(\mu_o)}{\partial n_i \partial \mu}} \qquad \text{(III-2)}$$

where the n_i are the zeros of $\dfrac{dP_{n_i}(\mu_o)}{d\mu}$.

Ordinarily we would now prove that the boundary conditions were satisfied; however, this will be shown in Appendix A, where the same answer is derived by two other methods.

In Appendix A we derive the differential scattering coefficient from Equation (III-2). The differential coefficient is

$$\sigma_D(\theta) = \frac{1}{k^2} \left| \sum_i \frac{(2n_i + 1) e^{jn_i \pi} P_{n_i}(\mu)}{P_{n_i}(\mu_o) (1 - \mu_o^2) \dfrac{\partial^2 P_{n_i}(\mu_o)}{\partial n_i \partial \mu}} \right|^2$$

In Appendix D we show that

$$\int_{\mu_o}^{1} \left[P_{n_i}(\mu) \right]^2 d\mu = \frac{1 - \mu_o^2}{2n_i + 1} P_{n_i}(\mu_o) \frac{\partial^2 P_{n_i}(\mu_o)}{\partial n_i \partial \mu}$$

when the n_i's are the zeros of $P_{n_i}^1(\mu_o)$.

Thus $$\sigma_D(\theta) = \frac{1}{k^2} \left| \sum_i \frac{e^{jn_i \pi} P_{n_i}(\mu)}{\int_{\mu_o}^{1} \left[P_{n_i}(\mu) \right]^2 d\mu} \right|^2 .$$

For axially symmetric back scattering

$$\sigma_D(0) = \frac{1}{k^2} \left| \sum_i \frac{e^{jn_i\pi}}{\int_{\mu_o}^{1} \left[P_{n_i}(\mu) \right]^2 d\mu} \right|^2$$

as $P_{n_i}(1) = 1$.

IV

HANSEN AND SCHIFF SOLUTION TO VECTOR SCATTERING BY A SEMI-INFINITE CONE

Hansen and Schiff (Ref. 6), using a method due to Hansen (Ref. 9, p. 393), solve the semi-infinite cone problem for electromagnetic scattering. For axially symmetric back-scattering from a semi-infinite cone Hansen and Schiff show that

$$\sigma_D'(0) = \frac{1}{k^2} \left| \sum_i \frac{n_i(n_i + 1)\, e^{-jn_i\pi}}{2B_{n_i}} - \frac{m_i(m_i + 1)\, e^{-jm_i\pi}}{2B_{m_i}} \right|^2$$

where

$$B_{n_i} = \int_{\mu_o}^{1} \left[P^1_{n_i}(\mu)\right]^2 d\mu$$

$$B_{m_i} = \int_{\mu_o}^{1} \left[P^1_{m_i}(\mu)\right]^2 d\mu \,.$$

The n_i's are the zeros of $P^1_{n_i}(\mu_o)$ and the m_i's are the zeros of $\frac{dP^1_{m_i}(\mu_o)}{d\mu}$.

V

COMPARISON BETWEEN SCALAR AND VECTOR SCATTERING FOR A SEMI-INFINITE CONE

Much has been said in the literature about the use of differential cross-sections obtained from scalar theory to approximate scattering cross-sections obtained from vector theory. In this section we point out a relationship that arises when a comparison is made between the $\sigma'_D(0)$ of the vector cone problem and the $\sigma_D(0)$ of the scalar case.

For the scalar case we observed

$$\sigma_D(0) = \frac{1}{k^2}\left|\sum_i \frac{e^{jn_i\pi}}{\int_{\mu_o}^{1}\left[P_{n_i}(\mu)\right]^2 d\mu}\right|^2 \qquad \text{(V-1)}$$

where the n_i's are zeros of $P^1_{n_i}(\mu_o)$.

Hansen and Schiff found for the vector case that

$$\sigma'_D(0) = \frac{1}{k^2}\left|\sum_i\left(\frac{n_i(n_i+1)\,e^{-jn_i\pi}}{2\int_{\mu_o}^{1}\left[P^1_{n_i}(\mu)\right]^2 d\mu} - \frac{m_i(m_i+1)\,e^{-jm_i\pi}}{2\int_{\mu_o}^{1}\left[P^1_{m_i}(\mu)\right]^2 d\mu}\right)\right|^2 \qquad \text{(V-2)}$$

where the n_i's are defined as in the scalar case and the m_i's are zeros of $\dfrac{dP^1_{m_i}(\mu_o)}{d\mu}$

In order to make the form of Equation (V-2) more like Equation (V-1) replace n_i by $-n_i-1$ and m_i by $-m_i-1$.

Thus, Equation (V-2) becomes

$$\sigma'_D(0) = \frac{1}{k^2} \left| \sum_i \left(\frac{n_i (n_i + 1) e^{jn_i\pi}}{2\int_{\mu_o}^1 \left[P^1_{n_i}(\mu)\right]^2 d\mu} - \frac{m_i(m_i + 1) e^{jm_i\pi}}{2\int_{\mu_o}^1 \left[P^1_{m_i}(\mu)\right]^2 d\mu} \right) \right|^2 . \qquad \text{(V-3)}$$

By Appendix D

$$\int_{\mu_o}^1 \left[P^1_{n_i}(\mu)\right]^2 d\mu = -\frac{1 - \mu_o^2}{2n_i + 1} \left(\frac{dP^1_{n_i}(\mu_o)}{d\mu} \right) \left(\frac{dP^1_{n_i}(\mu_o)}{d\, n_i} \right) \qquad \text{(V-4)}$$

and

$$\int_{\mu_o}^1 \left[P_{n_i}(\mu)\right]^2 d\mu = \frac{1 - \mu_o^2}{2n_i + 1} \, \frac{\partial^2 P_{n_i}(\mu_o)}{\partial\mu \partial n_i} \, P_{n_i}(\mu_o) . \qquad \text{(V-5)}$$

In Appendix D (Eq. D-11) it is shown that when the boundary condition is $\frac{dP^1_{m_i}(\mu_o)}{d\mu} = 0$, one obtains

$$\int_{\mu_o}^1 \left[P^1_{m_i}(\mu)\right]^2 d\mu = \frac{1 - \mu_o^2}{2m_i + 1} \, \frac{\partial^2 P^1_{m_i}(\mu_o)}{\partial\mu \partial m_i} \, P^1_{m_i}(\mu_o). \qquad \text{(V-6)}$$

Applying a recursion formula plus the boundary condition we find

$$(1 - \mu_o^2) \frac{dP^1_{n_i}(\mu_o)}{d\mu} = n_i(n_i + 1) \sqrt{1 - \mu_o^2} \; P_{n_i}(\mu_o). \qquad \text{(V-7)}$$

The definition of the associated Legendre function is

$$P^1_{n_i}(\mu) = -\sqrt{1 - \mu^2} \, \frac{dP_{n_i}(\mu)}{d\mu} . \qquad \text{(V-8)}$$

Applying Equation (V-8) and (V-7), to Equation (V-4) we obtain

$$\int_{\mu_o}^{1} \left[P^1_{n_i}(\mu)\right]^2 d\mu = +\frac{n_i(n_i + 1)}{2n_i + 1}(1 - \mu_o^2)\, P_{n_i}(\mu_o)\frac{\partial^2 P_{n_i}(\mu_o)}{\partial n_i \partial \mu}. \quad \text{(V-9)}$$

Now combining Equations (V-9) and (V-5) we obtain

$$\int_{\mu_o}^{1} \left[P^1_{n_i}(\mu)\right]^2 d\mu = + n_i(n_i + 1)\int_{\mu_o}^{1} \left[P_{n_i}(\mu)\right]^2 d\mu. \quad \text{(V-10)}$$

Using recursion relationships with Equation (V-6), remembering $\frac{dP^1_{m_i}(\mu_o)}{d\mu} = 0$, and using the m_i form of Equation (V-8) we obtain

$$\int_{\mu_o}^{1} \left[P^1_{m_i}(\mu)\right]^2 d\mu = m_i(m_i + 1)(\mu_o^2 - 1)\, P_{m_i}(\mu_o)$$

$$\times \left\{ \frac{P_{m_i}(\mu_o)}{\mu_o} + \frac{m_i(m_i + 1)}{\mu_o(2m_i + 1)} \frac{dP_{m_i}(\mu_o)}{dm_i} - \frac{1}{2m_i + 1} \frac{\partial^2 P_{m_i}(\mu_o)}{\partial \mu \partial m_i} \right\}. \quad \text{(V-11)}$$

We have now evaluated algebraically all the terms in Equations (V-1) and (V-2). Now by Equations (V-10) and (V-3) we observe

$$\sigma'_D(0) = \frac{1}{k^2}\left| \frac{1}{2}\sum_i \frac{e^{jn_i\pi}}{\int_{\mu_o}^{1}\left[P_{n_i}(\mu)\right]^2 d\mu} - \frac{1}{2}\sum_i \frac{m_i(m_i + 1)\, e^{jm_i\pi}}{\int_{\mu_o}^{1}\left[P^1_{m_i}(\mu)\right]^2 d\mu}\right|^2. \quad \text{(V-12)}$$

Thus Equation (V-1) for the scalar case and Equation (V-12) for the vector case will be identical if

$$\sum_i \frac{e^{jn_i\pi}}{\int_{\mu_o}^{1}\left[P_{n_i}(\mu)\right]^2 d\mu} = - \sum_i \frac{m_i(m_i+1)\, e^{jm_i\pi}}{\int_{\mu_o}^{1}\left[P^1_{m_i}(\mu)\right]^2 d\mu} \qquad \text{(V-13)}$$

and then the cross-section for the scalar case would be identical with the vector case.

It will be shown in Reference 23 that Equation (V-13) cannot be true and thus the scalar and vector cross-sections for a cone cannot be equal. However, as pointed out in Reference 23, for many bodies the scalar cross-section may be a first approximation to the vector answer.

VI

CONCLUSIONS

In this report an attempt was made to lay the ground work of our physical optics and geometrical optics approximations for radar cross-sections. We have also presented a cone solution to the scalar or "sound" problem. We presented the Hansen and Schiff cone cross-section solution to the vector problem to point out some mathematical similarities between the solutions to vector and scalar problems.

APPENDIX A

SCALAR SCATTERING BY A SEMI-INFINITE CONE

1. CONTOUR INTEGRATION METHOD

This section will be divided into four parts:

a. The expansion of the source (a plane wave) in Bessel and Legendre functions.

b. The expression of the source function as a contour integral.

c. The total solution (made up of the plane wave plus the scattering due to the cone).

d. The boundary conditions.

The physical picture is shown in Figure A-1 below:

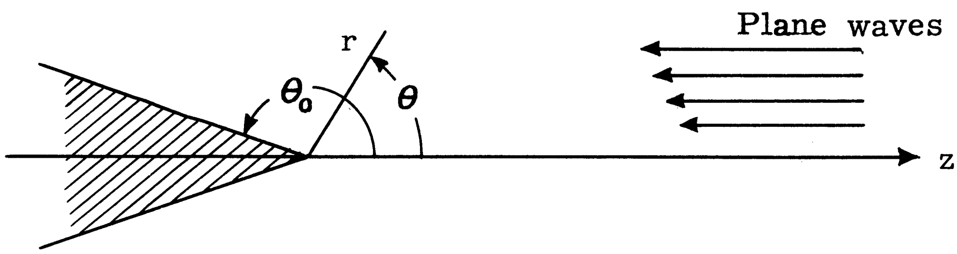

Figure A-1

The differential equation to be satisfied is:

$$\nabla^2 u + k^2 u = 0 .$$

a. The expansion of the source.

For a plane wave traveling along the z axis as shown,

$$u = e^{jk(ct+z)} .$$

The time independent part $u_o = e^{jkz} = e^{jkr\cos\theta}$ is a solution of

$$\nabla^2 u_o + k^2 u_o = 0.$$

We know a general solution of this differential equation to be:

$$u_o = \frac{1}{(kr)^{1/2}} \sum_{i=0}^{\infty} A_i \, J_{i+1/2}(kr) \, P_i(\cos\theta).$$

Therefore

$$e^{jkr\cos\theta} = \frac{1}{(kr)^{1/2}} \sum_{i=0}^{\infty} A_i \, J_{i+1/2}(kr) \, P_i(\cos\theta).$$

We may now compare powers of kr cos θ on both sides of the above equations. Note first, however, that

$$J_{i+1/2}(kr) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(i + 3/2 + m)} \left(\frac{kr}{2}\right)^{i+1/2+2m}$$

and

$$P_i(\cos\theta) = \frac{(2i)!}{2^i\,(i!)^2} \left[\cos\theta\right]^i \, {}_2F_1\left(-\frac{i}{2}, \frac{1-i}{2}, \frac{1}{2} - i; \frac{1}{\cos^2\theta}\right).$$

Then taking the first terms of $J_{i+1/2}(kr)$ and $P_i(\cos\theta)$

we have

$$\frac{(j)^i\,(kr\cos\theta)^i}{i!} = \frac{A_i}{(kr)^{1/2}} \, \frac{(kr)^{i+\frac{1}{2}}}{2^{i+\frac{1}{2}}\,\Gamma(i+\frac{3}{2})} \, \frac{(2i)!\,[\cos\theta]^i}{2^i\,(i!)^2} . \qquad \text{(A-1)}$$

Now using the following relations (Ref. 18, p. 1)

$$\Gamma(2i) = \frac{1}{\sqrt{2\pi}} 2^{-\frac{1}{2}} 2^{2i} \Gamma(i) \Gamma(i+\tfrac{1}{2})$$

$$(2i)! = \Gamma(2i+1) = 2i \Gamma(2i)$$

equation (A-1) becomes

$$(j)^i = \frac{A_i 2i \, 2^{2i} \Gamma(i) \Gamma(i+\frac{1}{2})}{2^{i+\frac{1}{2}} \Gamma(i+\frac{3}{2}) \sqrt{2\pi} \; 2^{\frac{1}{2}} \; 2^i \; i!} .$$

But

$$i\Gamma(i) = \Gamma(i+1) = i! \qquad \text{and} \qquad \Gamma(i+\tfrac{3}{2}) = (i+\tfrac{1}{2}) \Gamma(i+\tfrac{1}{2}) .$$

Then $(j)^i = \dfrac{A_i}{(i+\frac{1}{2})\sqrt{2\pi}}$ or $A_i = (i+\frac{1}{2}) j^i \sqrt{2\pi}$.

Finally

$$u_o = e^{jkr \cos\theta} = \left(\frac{2\pi}{kr}\right)^{\frac{1}{2}} \sum_{i=0}^{\infty} j^i (i+\tfrac{1}{2}) J_{i+\frac{1}{2}}(kr) P_i(\cos\theta). \quad \text{(A-2)}$$

b. <u>Expressing the source as a contour integral</u>

This section follows the methods of Carslaw (Ref. 17). Consider the following integral over the infinite path shown in the plane of the complex variable n (See Figure A-2).

$$\frac{1}{2\pi j} \int \frac{(n+\frac{1}{2}) e^{\frac{n\pi j}{2}} J_{n+\frac{1}{2}}(kr) \; P_n(-\cos\theta)}{\sin n\pi} dn \qquad \text{(A-3)}$$

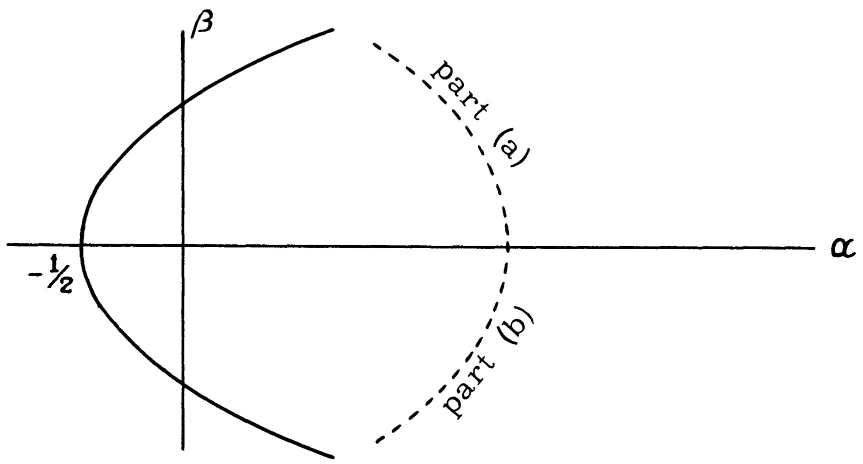

Figure A-2

We wish to use Cauchy's theorem to evaluate the integral, taking as the closed path, the solid curve plus a circle of infinite radius connecting the ends of the solid curve. It will now be shown that the integral over the dotted portion will vanish. To do this, it is sufficient to show that the integrand times $|n|$ vanishes over the dotted portion (i.e., as $|n|$ becomes large).

For large values of $|n|$ we have the following approximations (Ref. 17, p. 135):

$$J_{n+\frac{1}{2}}(kr) = \frac{(kr)^{n+\frac{1}{2}}}{2^{n+\frac{1}{2}}\Pi(n+\frac{1}{2})}$$

$$P_n(-\mu) = \sqrt{\frac{2}{n\pi\sin\theta}}\ \sin\left[(n+\tfrac{1}{2})(\pi-\theta)+\frac{\pi}{4}\right].$$

Substituting the asymptotic values of $J_{n+\frac{1}{2}}(kr)$ and $P_n(-\mu)$ into Equation (A-3) we have:

$$\frac{1}{2\pi j}\int (n+\tfrac{1}{2})\ \frac{e^{\frac{n\pi j}{2}}(kr)^{n+\frac{1}{2}}}{2^{n+\frac{1}{2}}\Pi\ (n+\frac{1}{2})}\sqrt{\frac{2}{n\pi\sin\theta}}\ \frac{\sin\left[(n+\frac{1}{2})(\pi-\theta)+\frac{\pi}{4}\right]}{\sin n\pi}\ dn\ .$$

Let us first examine the expression:

$$|n| \left| e^{\frac{n\pi j}{2}} \frac{(n+\frac{1}{2})}{\sqrt{n}} \frac{(kr/2)^{n+\frac{1}{2}}}{\Pi(n+\frac{1}{2})} \right| \qquad \text{as } |n| \text{ approaches } \infty.$$

Let $n + 1/2 = \ell$, $kr/2 = a$, $\Pi(\ell) = \Gamma(\ell + 1)$.
Writing $\Gamma(\ell + 1) = (\ell + 1)\ell\Gamma(\ell - 1)$ we have

$$\left| \frac{\ell a^{\ell} (\ell - \frac{1}{2})^{\frac{1}{2}}}{\Gamma(\ell + 1)} e^{(\ell - \frac{1}{2})\frac{\pi j}{2}} \right| < \left| \frac{a^{\ell}}{\Gamma(\ell - 1)} \right|$$

Now use Stirling's formula for the gamma function (Ref. 18, p. 3); valid for large values of the argument.

$$\ln\Gamma(\ell + 1) = (\ell + \tfrac{1}{2}) \ln(\ell + 1) - (\ell + 1) + \tfrac{1}{2} \ln 2\pi$$

$$+ \sum_{m=1}^{N-1} \frac{(-1)^{m-1} B_{2m}(\ell + 1)^{1-2m}}{2m\,(2m - 1)} + R_N(\ell + 1)$$

For large ℓ we may write

$$\ln\Gamma(\ell - 1) \simeq \frac{1}{k} (\ell - \tfrac{3}{2}) \ln(\ell - 1) \qquad \text{where } 1.1 > k > 1$$

so that

$$\left| \frac{a^{\ell}}{\Gamma(\ell - 1)} \right| \simeq \left| \frac{a^{\ell}}{(\ell - 1)^{\frac{\ell - 3/2}{k}}} \right| = \left| \left(\frac{a}{(\ell - 1)^{\frac{1}{k} - \frac{3}{2\ell k}}} \right)^{\ell} \right|$$

which is seen to approach 0 as $|\ell|$ approaches ∞.

Therefore, we have shown that

$$\lim_{|n|\to\infty}\left[|n|\,\frac{e^{\frac{n\pi j}{2}}\,(n+\frac{1}{2})}{\sqrt{n}}\cdot\frac{\left(\frac{kr}{2}\right)^{n+\frac{1}{2}}}{\Pi(n+\frac{1}{2})}\right]=0\,.$$

We now examine the remaining terms over the infinite path in the complex n-plane.

The remaining terms are $\dfrac{\sin\left[(n+1/2)(\pi-\theta)+\pi/4\right]}{\sin n\pi}$

Now:

$$\frac{\sin\left[(n+1/2)(\pi-\theta)+\pi/4\right]}{\sin n\pi}=\frac{e^{j\left[(n+1/2)(\pi-\theta)+\pi/4\right]}}{e^{n\pi j}-e^{-n\pi j}}$$

$$-\frac{e^{-j\left[(n+1/2)(\pi-\theta)+\pi/4\right]}}{e^{n\pi j}-e^{-n\pi j}}$$

Let $n = \alpha + j\beta$. For $\beta < 0$, i.e., over the part (a) of the path we may write the above expression in the form

$$\frac{e^{\frac{\pi j}{4}}\,e^{-n\pi j}}{1-e^{-2n\pi j}}\,e^{j\left[(n+\frac{1}{2})(\pi-\theta)\right]}-\frac{e^{-\frac{\pi j}{4}}\,e^{-n\pi j}}{1-\frac{1}{e^{2n\pi j}}}\,e^{-j\left[(n+\frac{1}{2})(\pi-\theta)\right]}$$

$$=\frac{e^{\pi j/4}\,e^{j\left[\frac{1}{2}(\pi-\theta)-n\theta\right]}}{1-\frac{1}{e^{2n\pi j}}}-\frac{e^{-\pi j/4}\,e^{-j\left[n(2\pi-\theta)+\frac{1}{2}(\pi-\theta)\right]}}{1-\frac{1}{e^{2n\pi j}}}\,.$$

Then for β approaching $-\infty$ the expression approaches 0 since $0 < \theta < \pi$. For $\beta > 0$, i.e., over the part (b) of the path we may write the expression in the form

$$\frac{e^{n\pi j}\, e^{\pi j/4}}{e^{2n\pi j} - 1}\, e^{j\left[(n+\frac{1}{2})(\pi-\theta)\right]} - \frac{e^{-\pi j/4}}{e^{2n\pi j} - 1}\, e^{-j\left[-n\theta + \frac{1}{2}(\pi-\theta)\right]}.$$

Then for β approaching $+\infty$ this expression approaches 0. Thus, we have shown that the integral vanishes over the dotted portion of the path.

Then finally by the Cauchy residue theorem

$$\frac{1}{2\pi j}\int \frac{(n+\frac{1}{2})\, e^{n\pi j/2}}{\sin n\pi}\, J_{n+\frac{1}{2}}(kr)\, P_n(-\mu)\, dn$$

$$= \sum_{n=0}^{\infty} \frac{(n+\frac{1}{2})\, e^{\frac{n\pi j}{2}}}{\pi \cos n\pi}\, J_{n+\frac{1}{2}}(kr)\, P_n(-\mu).$$

But for $n = i =$ integer $P_i(-\mu) = (-1)^i P_i(\mu)$

$$\therefore \frac{1}{2\pi j}\int \frac{(n+\frac{1}{2})\, e^{\frac{n\pi j}{2}}}{\sin n\pi}\, J_{n+\frac{1}{2}}(kr)\, P_n(-\mu)\, dn$$

$$= \frac{1}{\pi}\sum_{i=0}^{\infty} (i+\tfrac{1}{2})\, e^{\frac{i\pi j}{2}}\, J_{i+\frac{1}{2}}(kr)\, P_i(\mu)$$

Noting, Equation (A-2) then

$$u_o = \left(\frac{2\pi}{kr}\right)^{\frac{1}{2}} \frac{1}{2j} \int \frac{(n+\frac{1}{2})\, e^{\frac{n\pi j}{2}}}{\sin n\pi} J_{n+\frac{1}{2}}(kr)\, P_n(-\mu)\, dn$$

$$u_o = \left(\frac{\pi}{2kr}\right)^{\frac{1}{2}} e^{-\pi j/2} \int \frac{(n+\frac{1}{2})\, e^{\frac{n\pi j}{2}}}{\sin n\pi} J_{n+\frac{1}{2}}(kr)\, P_n(-\mu)\, dn. \tag{A-4}$$

c. <u>The Total Solution:</u>

This section too, follows closely the methods used in Reference 17. We now add to the source u_o the part due to the scattering by the cone. Consider the integral

$$u = \left(\frac{\pi}{2kr}\right)^{\frac{1}{2}} e^{-\frac{\pi j}{2}} \int \frac{(n+\frac{1}{2})\, e^{\frac{n\pi j}{2}}}{\sin n\pi} J_{n+\frac{1}{2}}(kr) \left\{ \frac{P_n(-\mu)\frac{d}{d\mu}P_n(\mu_o) - P_n(\mu)\frac{d}{d\mu}P_n(-\mu_o)}{\frac{d}{d\mu}P_n(\mu_o)} \right\} dn \tag{A-5}$$

over the same path as before. We show, as before, that the integral vanishes over the dotted portion of the path (see Fig. A-2).

We had for large $|n|$

$$P_n(-\mu) = \sqrt{\frac{2}{n\pi \sin\theta}} \sin\left[(n+\tfrac{1}{2})(\pi-\theta) + \frac{\pi}{4}\right].$$

Then

$$\frac{d}{d\mu} P_n(-\mu) = \sqrt{\frac{2}{n\pi \sin\theta}} \left\{\cos\left[(n+\tfrac{1}{2})(\pi-\theta) + \frac{\pi}{4}\right]\right\} \frac{(n+\frac{1}{2})}{\sin\theta}$$

$$+ \frac{1}{\sqrt{\frac{2}{n\pi\sin\theta}}} \left(\frac{\cos\theta}{n\pi\sin^3\theta}\right) \sin\left[(n+\tfrac{1}{2})(\pi-\theta) + \frac{\pi}{4}\right]$$

$$\frac{d}{d\mu} P_n(-\mu) \simeq \sqrt{\frac{2}{n\pi\sin^3\theta}} \left\{\cos\left[(n+\tfrac{1}{2})(\pi-\theta) + \frac{\pi}{4}\right]\right\}(n+\tfrac{1}{2})$$

for large $|n|$.

Substituting $\pi - \theta$ for θ in the above equation we get

$$\frac{d}{d\mu} P_n(\mu) = -\sqrt{\frac{2}{n\pi\sin^3\theta}} \left\{\cos\left[(n+\tfrac{1}{2})\theta + \pi/4\right]\right\}(n+\tfrac{1}{2})$$

$$P_n(\mu) = \sqrt{\frac{2}{n\pi\sin\theta}} \sin\left[(n+\tfrac{1}{2})\theta + \pi/4\right].$$

The first part of the integrand of equation (A-5) is just the u_o we examined before. The second part is:

$$-\int \frac{(n+\frac{1}{2})\, e^{n\pi j/2}}{\sin n\pi} J_{n+\frac{1}{2}}(kr) \frac{P_n(\mu)\frac{d}{d\mu}P_n(-\mu)}{\frac{d}{d\mu}P_n(\mu_o)} \, dn .$$

Substituting the appropriate approximate expression when $|n|$ is large we have:

$$\int \left\{ \frac{2\,(n+\frac{1}{2})\, e^{n\pi j/2}\,(kr)^{n+\frac{1}{2}}}{\sin n\pi \sqrt{n\pi \sin\theta}\;\; 2^{n+\frac{1}{2}}\,\Pi(n+\frac{1}{2})} \;\; \frac{\sin\left[(n+\frac{1}{2})\theta + \frac{\pi}{4}\right] \cos\left[(n+\frac{1}{2})(\pi-\theta_0)+\frac{\pi}{4}\right]}{\cos\left[(n+\frac{1}{2})\;\theta_0 + \frac{\pi}{4}\right]} \right\} dn \,.$$

Again, we desire the integrand to vanish over the infinite portion of the path. We have already examined the terms:

$(n+\frac{1}{2})e^{n\pi j/2} \dfrac{(kr)^{n+\frac{1}{2}}}{2^{n+\frac{1}{2}}\,\Pi(n+\frac{1}{2})}$ and shown that this expression

vanishes over the infinite (dotted) part of the path. Also

$\dfrac{\sin\left[(n+\frac{1}{2})\;\theta\; + \frac{\pi}{4}\right]}{\sin\; n\pi}$ will vanish exactly like $\dfrac{\sin\left[(n+\frac{1}{2})(\pi-\theta)+\frac{\pi}{4}\right]}{\sin\; n\pi}$

did previously since both θ and $\pi - \theta$ are positive quantities. Thus, all that remains is

$$\frac{\cos\left[(n+\frac{1}{2})(\pi-\theta_0)+\frac{\pi}{4}\right]}{\cos\left[(n+\frac{1}{2})\;\theta_0 + \frac{\pi}{4}\right]} =$$

$$\frac{e^{j\left[(n+\frac{1}{2})(\pi-\theta_0)+\frac{\pi}{4}\right]} + e^{-j\left[(n+\frac{1}{2})(\pi-\theta_0)+\frac{\pi}{4}\right]}}{e^{j\left[(n+\frac{1}{2})\;\theta_0+\frac{\pi}{4}\right]} + e^{-j\left[(n+\frac{1}{2})\;\theta_0+\frac{\pi}{4}\right]}} \,.$$

For the part (b) of the path, ($\beta > 0$) we write:

$$\frac{\cos\left[(n+\frac{1}{2})(\pi-\theta_o)+\frac{\pi}{4}\right]}{\cos\left[(n+\frac{1}{2})\;\theta_o+\frac{\pi}{4}\right]} =$$

$$\frac{e^{j\left[(n+\frac{1}{2})(\pi-\theta_o)+\frac{\pi}{4}\right]}}{e^{j\left[(n+\frac{1}{2})\;\theta_o+\frac{\pi}{4}\right]}+e^{-j\left[(n+\frac{1}{2})\,\theta_o+\frac{\pi}{4}\right]}} + \frac{1}{e^{j\pi(n+1)}+e^{-j\left[(n+\frac{1}{2})(2\theta_o-\pi)\right]}}\;.$$

This will approach 0 as β approaches ∞ if $\pi < 2\theta_o$. This condition is satisfied for the configuration under discussion.

For the part (a) of the path ($\beta < 0$) we write:

$$\frac{\cos\left[(n+\frac{1}{2})(\pi-\theta_o)+\frac{\pi}{4}\right]}{\cos\left[(n+\frac{1}{2})\;\theta_o+\frac{\pi}{4}\right]} =$$

$$\frac{1}{e^{j\left[(n+\frac{1}{2})(2\theta_o-\pi)\right]}+e^{-j\pi(n+1)}} + \frac{e^{-j\left[(n+\frac{1}{2})(\pi-\theta_o)+\frac{\pi}{4}\right]}}{e^{j\left[(n+\frac{1}{2})\,\theta_o+\frac{\pi}{4}\right]}+e^{-j\left[(n+\frac{1}{2})\,\theta_o+\frac{\pi}{4}\right]}}$$

which will approach 0 as β approaches $-\infty$.

We have thus, finally shown that the integral representation of u (the total solution) will vanish over the dotted part of the path of integration. We may then take the complete path and evaluate the integral by the Cauchy residue theorem.

In examining the integrand of Equation (A-5) for poles we should note that the integrand has no poles when n = integer. For, when n = i = integer we may use the relation

$P_i(\mu) = (-1)^i P_i(-\mu)$

Then the term $P_i(-\mu)\frac{d}{d\mu}P_i(\mu_o) - P_i(\mu)\frac{d}{d\mu}P_i(-\mu_o)$

becomes

$$(-1)^i P_i(\mu)\frac{d}{d\mu}P_i(\mu_o) - (-1)^i P_i(\mu)\frac{d}{d\mu}P_i(\mu_o) = 0$$

The denominator also becomes 0. Evaluating by L'Hospital's rule, the integrand then becomes an analytic function (whose value is 0) for n = integer. The only poles then occur for the zeros of $\frac{d}{d\mu}P_{n_i}(\mu_o)$ when n_i is not an integer.

Thus:

$$u = -\left(\frac{\pi}{2kr}\right)^{\frac{1}{2}} e^{-\pi j/2}\, 2\pi j \sum_i \frac{(n_i+\frac{1}{2})e^{n_i\pi j/2} J_{n_i+\frac{1}{2}}(kr)}{\sin n_i\pi} \frac{P_{n_i}(\mu)\frac{d}{d\mu}P_{n_i}(-\mu_o)}{\frac{\partial^2}{\partial n_i\partial\mu}P_{n_i}(\mu_o)} \tag{A-6}$$

The summation is taken over all zeros of $\frac{dP_{n_i}(\mu_o)}{d\mu}$ which are greater than -1/2.

The above result may be simplified by the following relation:

$$(1-\mu_o^2)P_{n_i}(\mu_o)\frac{d}{d\mu}P_{n_i}(-\mu_o) = -\frac{2\sin n_i\pi}{\pi}.$$

This may be derived by considering two relations to be found on page 63 of Reference 18 and making use of the fact that $\frac{dP_{n_i}(\mu_o)}{d\mu} = 0$.

The first relation is $P_{n_i}(\mu_o)\frac{d\,Q_{n_i}(\mu_o)}{d\mu} = \frac{1}{1-\mu_o^2}$

and the second is $\dfrac{dP_{n_i}(-\mu_o)}{d\mu} = -\dfrac{2}{\pi} \sin n_i\pi \dfrac{dQ_{n_i}(\mu_o)}{d\mu}$.

Multiply the second by $P_{n_i}(\mu_o)$ and substitute into the first relation. We then obtain the result that was to have been derived.

Equation (A-6) then becomes

$$u = \left(\frac{\pi}{2kr}\right)^{\frac{1}{2}} e^{-\pi j/2}\, 2\pi j \sum_i \frac{(n_i + \frac{1}{2})\, e^{n_i\pi j/2}\, J_{n_i+\frac{1}{2}}(kr)\, P_{n_i}(\mu)\, (2/\pi)}{P_{n_i}(\mu_o)\,(1-\mu_o^2)\, \frac{\partial^2}{\partial n_i \partial\mu} P_{n_i}(\mu_o)} .$$

$$u = \left(\frac{\pi}{2kr}\right)^{\frac{1}{2}} \sum_i (n_i + \tfrac{1}{2})\, e^{n_i\pi j/2} \frac{J_{n_i+\frac{1}{2}}(kr)\, P_{n_i}(\mu)}{P_{n_i}(\mu_o)\,(1-\mu_o^2)\, \frac{\partial^2}{\partial n_i \partial\mu} P_{n_i}(\mu_o)} \qquad \text{(A-7)}$$

where the n_i's are the zeros of $\dfrac{dP_{n_i}(\mu_o)}{d\mu}$ greater than -1/2.

d. Boundary Value Problem

Our solution, Equation (A-7) must satisfy the following boundary value problem.

1. $\nabla^2 u + k^2 u = 0$.
2. $\dfrac{\partial u}{\partial n'} = 0$ at the surface of the cone ($\theta = \theta_o$; $\mu = \mu_o$) where n' is the outward drawn normal to the surface.
3. Sommerfeld's 'Ausstrahlungsbedingung' (radiation condition).

It can very easily be shown by separation of variables that in spherical coordinates a solution to (1) is of the form $\left(\frac{1}{r}\right)^{\frac{1}{2}} \sum_i A_i J_{n_i+\frac{1}{2}}(kr)\, P_{n_i}(\cos\theta)$. It is seen that Equation (A-7) is in the required form and therefore condition (1) is satisfied. Condition (2) is obviously fulfilled when we remember that the n_i's were determined by the relation $\left[\frac{dP_{n_i}(\mu)}{d\mu}\right]_{\mu=\mu_o} = 0.$

Condition (3) warrants a little more discussion. This discussion however, will be deferred until we come to part 2 of this appendix, where the radiation condition is stated, and it will be shown just how it is satisfied.

2. SEPARATION OF VARIABLES METHOD

The method of solution by means of the separation of variables technique is one which is perhaps a little clearer than the methods previously employed. It is the classic, time-honored way in which to solve boundary value problems and because of this it will be much easier to compare this method for the scalar solution to the cone problem with the work that has already been done on vector scattering. In addition, it is felt that the physics is displayed in a more lucid manner when separation of variables can be employed. For example, the differential scattering cross-section is obtained directly.

The problem to be solved is the scalar scattering of plane waves by a semi-infinite cone. The scalar potential satisfies Pockel's equation subject to the following boundary conditions:

1. $\frac{\partial u}{\partial n'} = 0$ at the surface of the cone where n' is the outward normal to the surface.

2. u must be finite everywhere in the region of interest.

3. At large distances from the scattering body the solution must have the character of an incoming plane wave plus a diverging spherical wave which appears to originate at the scattering body. The plane wave is denoted by $e^{jk(ct + z)}$ and $\theta = \theta_o$ denotes the surface of the cone.

Now the solution obtained by separation of variables is

$$u(r,\theta,\phi) = e^{jm\phi} \begin{pmatrix} P_{n_i}^m (\cos\theta) \\ Q_{n_i}^m (\cos\theta) \end{pmatrix} \begin{pmatrix} j_{n_i}(kr) \\ n_{n_i}(kr) \end{pmatrix}$$

where $j_{n_i} = \sqrt{\frac{\pi}{2kr}}\, J_{n_i+\frac{1}{2}}(kr)$ and $n_{n_i} = \sqrt{\frac{\pi}{2kr}}\, N_{n_i+\frac{1}{2}}(kr)$ are the spherical Bessel and Neumann functions, respectively. The function $n_{n_i}(kr)$ cannot be used since it will cause u to become infinite at $r = 0$. Similarly, $Q_{n_i}^m(\cos\theta)$ may not be used since it has singularities at $\theta = 0$ and π. $P_{n_i}^m(\cos\theta)$ has a singularity at $\theta = \pi$ when $n_i \neq$ integer. However, this occurs, as can be seen, wholly within the cone and thus does not effect the region of physical interest; namely, all of space up to the surface of the cone. In addition, since symmetry about the z axis prevails, derivatives of u must be independent of ϕ and so m must be taken equal to zero, as $C_1\phi$ $(C_1 \neq 0)$ is not a solution of $\frac{d^2u}{d\phi^2} = -m^2u$. Finally, then, the solution will be expressible in the form

$$u(r,\theta) = \sum_{i=1}^{\infty} A_i\, j_{n_i}(kr)\, P_{n_i}(\mu) \tag{A-8}$$

where $\mu = \cos\theta$

Applying boundary condition (1) we have

$$\left.\frac{\partial u}{\partial n'}\right|_{\mu_o} = \sum_i A_i\, j_{n_i}(kr)\, \frac{\partial P_{n_i}(\mu_o)}{\partial n'} = 0$$ Thus, we choose

$$\frac{\partial P_{n_i}(\mu_o)}{\partial n'} = 0.$$

This can be written $\frac{dP_{n_i}(\mu_o)}{d\mu} = 0$. This condition will determine the value of the n_i's which in this case turn out to be non-integral. According to boundary condition (3) for very large r we must identify the incoming part of the solution with that of the incoming plane wave; therefore, the first step is to expand the plane wave in terms of the non-integral n_i's.

$$e^{jkr\cos\theta} = \sum_i f_i(kr)\, P_{n_i}(\mu) \qquad (\theta_o < \theta \leq 0)$$

Multiply both sides of the above equation by $P_{n'_i}(\mu)$ and integrate from μ_o to 1.

$$\int_{\mu_o}^{1} e^{jkr\cos\theta}\, P_{n'_i}(\mu)\, d\mu = \sum_i f_i(kr) \int_{\mu_o}^{1} P_{n'_i}(\mu)\, P_{n_i}(\mu)\, d\mu$$

The orthogonality property of $P_{n_i}(\mu)$ may now be used (see Appendix D).

$$\int_{\mu_o}^{1} P_{n_i}(\mu)\, P_{n'_i}(\mu)\, d\mu = \begin{cases} 0 & \text{for } n_i \neq n'_i . \\ \dfrac{(1-\mu_o^2)}{2n_i+1}\, P_{n_i}(\mu_o)\, \dfrac{\partial^2 P_{n_i}(\mu_o)}{\partial n_i\, \partial\mu} & \text{for } n_i = n'_i . \end{cases}$$

Thus

$$f_i(kr) = \frac{2n_i + 1}{(1-\mu_o^2)\, P_{n_i}(\mu_o)\, \dfrac{\partial^2 P_{n_i}(\mu_o)}{\partial n_i\, \partial\mu}} \int_{\mu_o}^{1} e^{jkr\mu}\, P_{n_i}(\mu)\, d\mu .$$

Now $\displaystyle\int_{\mu_o}^{1} e^{jkr\mu}\, P_{n_i}(\mu)\, d\mu = -j\,\frac{e^{jkr}}{kr}$ (see Appendix E) so that we finally have*

$$e^{jkr\cos\theta} = -\frac{2j}{k}\sum_i \frac{(n_i+\frac{1}{2})\,P_{n_i}(\mu)}{(1-\mu_o^2)\,P'_{n_i}(\mu_o)\,\dfrac{\partial^2 P_{n_i}(\mu_o)}{\partial n_i\,\partial\mu}}\left[\frac{e^{jkr}}{r}\right].$$

The asymptotic form of $u(r,\theta)$ for large r is now required. This is easily arrived at by making use of the asymptotic form of the spherical Bessel function. For large r, (Ref. 1, p. 449).

$$j_{n_i}(kr) \simeq \frac{1}{kr}\cos\left[kr - \frac{n_i+1}{2}\pi\right]. \qquad \text{(A-9)}$$

Substituting into equation (A-8) we have,

$$\underset{r\to\infty}{u(r,\theta)} \simeq \sum_i A_i\,\frac{1}{2kr}\left[e^{j\left(kr-\frac{n_i+1}{2}\pi\right)} + e^{-j\left(kr-\frac{n_i+1}{2}\pi\right)}\right]P_{n_i}(\mu).$$

Then, upon identifying the incoming part $\frac{e^{jkr}}{r}$ with the plane wave expansion, the coefficient A_i satisfies the relation

$$\frac{A_i}{2}\,e^{-j\left(\frac{n_i+1}{2}\pi\right)} = \frac{-2j\,(n_i+\frac{1}{2})}{(1-\mu_o^2)\,P_{n_i}(\mu_o)\,\dfrac{\partial^2 P_{n_i}(\mu_o)}{\partial n_i\,\partial\mu}}$$

and

$$A_i = \frac{4e^{\frac{n_i\pi j}{2}}\,(n_i+\frac{1}{2})}{(1-\mu_o^2)\,P_{n_i}(\mu_o)\,\dfrac{\partial^2 P_{n_i}(\mu_o)}{\partial n_i\,\partial\mu}}.$$

*All sums will now be considered finite sums. The reduction of Equation (A-8) to a finite number of terms and the details of the subsequent substitution of (A-9) into (A-8) is justified in Reference 23.

The final solution then becomes

$$u(r,\theta) = 4 \sum_i \frac{(n_i + \frac{1}{2}) e^{\frac{n_i \pi j}{2}} j_{n_i}(kr) P_{n_i}(\mu)}{(1 - \mu_o^2) P_{n_i}(\mu_o) \frac{\partial^2 P_{n_i}(\mu_o)}{\partial n_i \partial \mu}}$$

where the n_i's satisfy the condition $\frac{dP_{n_i}(\mu_o)}{d\mu} = 0$. This solution, of course, agrees exactly with the solution obtained by the method of contour integration (Equation A-7). (Note $j = \sqrt{-1}$ but j_{n_i} is the spherical Bessel function.)

Some remarks about the Sommerfeld radiation condition are now in order. The radiation condition for the three-dimensional problem is written

$$\lim_{r \to \infty} r \left\{ \frac{\partial u}{\partial r} + jku \right\} = 0 \qquad \text{(A-10)}$$

where it will be noted that the minus sign occurring in Sommerfeld (Reference 20, p. 193) is replaced here by a plus sign. This is due to the fact that Sommerfeld uses the time dependence $e^{-j\omega t}$ while we use $e^{+j\omega t}$. The physical meaning of the radiation condition is best expressed in a direct translation from Sommerfeld (Reference 20, p. 189):

> ".... the sources must be sources, not sinks of energy. The energy which is radiated from the sources must scatter to infinity; no energy may be radiated from infinity into the prescribed singularities of the field. (Plane waves are excluded since for them even the condition u = 0 fails to hold at infinity.)"

Thus, we may conclude that the scattered part alone of our solution must satisfy the radiation condition. For large r, making use of equation (A-9) we can write $u(r, \theta)$ as

$$\underset{r \to \infty}{u(r,\theta)} = \frac{-2j}{k} \sum_i \frac{(n_i + \frac{1}{2}) P_{n_i}(\mu)}{P_{n_i}(\mu_o)(1 - \mu_o^2) \frac{\partial^2 P_{n_i}(\mu_o)}{\partial n_i \partial \mu}} \left[\frac{e^{jkr}}{r} \right]$$

$$+ \frac{2j}{k} \sum_i \frac{(n_i + \frac{1}{2}) e^{n_i \pi j} P_{n_i}(\mu)}{P_{n_i}(\mu_o)(1 - \mu_o^2) \frac{\partial^2 P_{n_i}(\mu_o)}{\partial n_i \partial \mu}} \left[\frac{e^{-jkr}}{r} \right]$$

where the first term is the incoming plane wave and the second term the scattered wave. It is seen that the scattered wave obeys the radiation condition (Equation A-10).

The differential scattering cross-section, $\sigma_D(\theta)$ which gives the angular distribution of the scattered power is usually taken as the absolute value squared of the coefficient of $\frac{e^{-jkr}}{r}$ (the scattered wave). Thus,

$$\sigma_D(\theta) = \frac{4}{k^2} \left| \sum_i \frac{(n_i + \frac{1}{2}) e^{n_i \pi j} P_{n_i}(\mu)}{P_{n_i}(\mu_o)(1 - \mu_o^2) \frac{\partial^2 P_{n_i}(\mu_o)}{\partial n_i \partial \mu}} \right|^2$$

and for backscattering ($\theta = 0$)

$$\sigma_D(0) = \frac{4}{k^2} \left| \sum_i \frac{(n_i + \frac{1}{2}) e^{n_i \pi j}}{P_{n_i}(\mu_o)(1 - \mu_o^2) \frac{\partial^2 P_{n_i}(\mu_o)}{\partial n_i \partial \mu}} \right|^2 .$$

APPENDIX B

EQUIVALENCE OF RADAR AND DIFFERENTIAL SCATTERING CROSS-SECTION

The radar cross-section $\sigma(0) = \dfrac{4\pi r^2 \overline{S}^s_r}{\overline{S}^i_z}$.

$$\overline{\overline{S}} = 1/2 \; \mathcal{Re}\,(\vec{E} \times \vec{H}^*) \text{ (using the notation of Reference 1)}$$

$$\overline{S}_r = 1/2 \; \mathcal{Re}\,(E_\theta H^*_\phi - E_\phi H^*_\theta)$$

$$E_{\theta,\phi} = E^i_{\theta,\phi} + E^s_{\theta,\phi} \; ; \quad H_{\theta,\phi} = H^i_{\theta,\phi} + H^s_{\theta,\phi}$$

$$\overline{S}_r = 1/2 \; \mathcal{Re}\left[(E^i_\theta + E^s_\theta)(H^{i*}_\phi + H^{s*}_\phi) - (E^i_\phi + E^s_\phi)(H^{i*}_\theta + H^{s*}_\theta)\right]$$

$$= 1/2 \; \mathcal{Re}\left[(E^i_\theta H^{i*}_\phi - E^i_\phi H^{i*}_\theta) + (E^s_\theta H^{s*}_\phi - E^s_\phi H^{s*}_\theta) + (E^i_\theta H^{s*}_\phi - E^i_\phi H^{s*}_\theta + E^s_\theta H^{i*}_\phi - E^s_\phi H^{i*}_\theta)\right]$$

$$\overline{S}^s_r = 1/2 \; \mathcal{Re}\,(E^s_\theta H^{s*}_\phi - E^s_\phi H^{s*}_\theta)$$

Now we are considering the radial component of the Poynting vector, therefore, the direction of propagation for the scattered wave is $\hat{i}_r$.

Now $\hat{i}_r \times \vec{E}^s = \eta_o \vec{H}^s$ where η_o is the impedance of free space (Reference 9, p. 289).

Then $\eta_o H^s_\theta = -E^s_\phi$ and $\eta_o H^s_\phi = E^s_\theta$.

Then $\overline{S}^s_r = (1/2)\dfrac{1}{\eta_o}\left[\left|E^s_\theta\right|^2 + \left|E^s_\phi\right|^2\right]$.

This is computed for large r where we can write

$$E^s_\theta = \frac{E_o e^{-jkr}}{r} f_1(\theta)$$

and
$$E_{\phi}^{s} = \frac{E_o\, e^{-jkr}}{r} f_2(\theta).$$

Then
$$\overline{S}_r^s = \frac{1}{2\,\eta_o}\left[\left|\frac{E_o\, e^{-jkr}}{r} f_1(\theta)\right|^2 + \left|\frac{E_o\, e^{-jkr}}{r} f_2(\theta)\right|^2\right]$$

$$\overline{S}_r^s = \frac{E_o^2}{2\,\eta_o\, r^2}\left[\left|f_1(\theta)\right|^2 + \left|f_2(\theta)\right|^2\right].$$

Now $\overline{S}_z^i = \dfrac{E_o^2}{2\,\eta_o}$. (Reference 9, p. 284)

$$\sigma(0) = 4\pi\left[\left|f_1(0)\right|^2 + \left|f_2(0)\right|^2\right]$$

This differs by a factor 4π from the differential scattering cross-section defined as the absolute value-squared of the angular part of the scattered wave (i.e., the coefficient of $\frac{e^{-jkr}}{r}$).

APPENDIX C

THE PHYSICAL OPTICS FORMULA DERIVED BY CONSIDERING THE SURFACE CURRENTS

The plan of attack is as follows:

Assuming we know the power transmitted by the antenna we may then calculate the power incident on the scattering surface and hence the incident electric and magnetic fields. Making suitable assumptions about the surface, we may then determine the surface current density which in turn yields the vector potential. From this, the reflected fields; and hence, the received power can be found. Upon using the radar range equation which relates the received power, the transmitted power and the radar cross-section, an expression for the radar cross-section may be determined.

Consider the antenna at the origin of a rectangular coordinate system with unit vectors $\hat{i}$, $\hat{j}$, $\hat{k}$. Let r be the distance from the origin to the scattering surfaces which is oriented along the z-axis (r large compared to the dimensions of the target and the antenna). For a radar which transmits an average power P_T, we may consider the field at the target to be a plane wave and the power incident on the target is $P_O = \frac{P_T G_T}{4\pi r^2}$ (Ref. 21, p. 21) where G_T is the gain of the transmitting antenna.

Let the incident electric and magnetic fields, $\vec{E}_i$ and $\vec{H}_i$ be directed in the positive x and y directions respectively. Then

$P_O = \frac{1}{2\eta_o}\left|E_i\right|^2$ (Ref. 9, p. 284), and

$$\vec{E}_i = \hat{i}\left|\vec{E}_i\right| e^{j(\omega t-kz)} = \hat{i}\frac{1}{r}\sqrt{\frac{\eta_o P_T G_T}{2\pi}}\, e^{j(\omega t-kz)}.$$

From Maxwell's equations we have

$$\vec{H}_i = \hat{k} \times \sqrt{\frac{\epsilon}{\mu'}}\ \vec{E}_i = \hat{j} \sqrt{\frac{\epsilon}{\mu'}}\ \frac{1}{r} \sqrt{\frac{\eta_o P_T G_T}{2\pi}}\ e^{j(\omega t - kz)} .$$

We now invoke all the assumptions stated on page 3 where the physical optics formula was derived from the Kirchoff-Huygens principle. These assumptions postulating a perfectly conducting surface, etc., then enable us to represent the surface current density by

$$\vec{K} = \vec{n} \times \vec{H} = \vec{n} \times (2\vec{H}_i) = \hat{i} \left[-2n_o \sqrt{\frac{\epsilon}{\mu'}}\ \frac{1}{r} \sqrt{\frac{\eta_o P_T G_T}{2\pi}}\ e^{j(\omega t - kz)} \right]$$

where n_o is $\left| \vec{n} \cdot \hat{k} \right|$. The k component is neglected since only those components of current flowing normal to the line of sight will contribute to the backscattering.

The vector potential is now represented in terms of the retarded potential

$$\vec{A} = \frac{\mu'}{4\pi} \int_s \frac{\vec{K}(t')}{r}\, ds$$ where t' is the retarded time $t' = t - \frac{r}{c}$ and

where s is taken to be the surface visible to the radar observer since it is assumed that only that part has current flowing upon it.

The reflected field, $\vec{E}_r$, is

$$\vec{E}_r = - \frac{\partial \vec{A}}{\partial t} = - \frac{\mu'}{4\pi} \int_s \frac{\partial \vec{K}(t')}{\partial t}\ \frac{ds}{r} .$$

$$\frac{\partial \vec{K}(t')}{\partial t} = \hat{i} \left[-2j\omega n_o \sqrt{\frac{\epsilon}{\mu'}}\ \frac{1}{r} \sqrt{\frac{\eta_o P_T G_T}{2\pi}}\ e^{j(\omega t - kz)} \right]$$

But $t' = t - \frac{r}{c}$ and $\omega = kc$ so that

$$\omega t' = \omega t - kr .$$

$z \simeq r \simeq$ constant

Then $$\vec{E}_r = \hat{i}\left[\frac{i\omega\sqrt{\epsilon\mu'}}{2\pi r^2}\sqrt{\frac{\eta_o P_T G_T}{2\pi}}\, e^{j\omega t}\int_s e^{-2jkr}\, n_o\, ds\right].$$

The received power, P_r, is

$$P_r = \frac{A_r}{2\eta_o}\left|\vec{E}_r\right|^2 = \frac{\omega^2\epsilon\mu' \; P_T G_T}{16\pi^3 r^4}\,|g|^2\, A_r$$

where $g = \int_s e^{-2jkr}\, n_o\, ds$ and A_r is the effective area of the receiving antenna. Now $A_r = G_r \frac{\lambda^2}{4\pi}$ (Ref. 21, p. 20) where G_r is the gain of the receiving antenna, and $C^2 = \frac{1}{\epsilon_o \mu_o'}$. Then since we are dealing with free space $P_r = \frac{P_T G_T G_r}{(4\pi)^2 r^4}|g|^2$.

But the radar range equation states (Ref. 21, p. 21) that

$$P_r = \left(\frac{P_T G_T}{4\pi r^2}\right)\left(\frac{\sigma(0)}{4\pi r^2}\right)\left(\frac{G_r \lambda^2}{4\pi}\right)$$

so we have $\sigma(0) = \frac{4\pi}{\lambda^2}|g|^2$.

APPENDIX D

DERIVATION OF THE ORTHOGONALITY AND THE NORMALIZING FACTOR FOR THE ASSOCIATED LEGENDRE FUNCTIONS

Legendre's differential equation is

$$\frac{d}{d\mu}\left\{\left[1-\mu^2\right]\frac{dy}{d\mu}\right\}+\left\{n_i(n_i+1)-\frac{m'^2}{1-\mu^2}\right\}y=0 \qquad \text{(D-1)}$$

$$\frac{d}{d\mu}\left\{\left[1-\mu^2\right]\frac{dy'}{d\mu}\right\}+\left\{n_i'(n_i'+1)-\frac{m'^2}{1-\mu^2}\right\}y'=0. \qquad \text{(D-2)}$$

Multiply Equation (D-1) by y' and Equation (D-2) by y and subtract. We obtain

$$yy'=\frac{d}{d\mu}\frac{\left[(1-\mu^2)\left(-y'\frac{dy}{d\mu}+y\frac{dy'}{d\mu}\right)\right]}{(n_i-n_i')(n_i+n_i'+1)} \qquad \text{(D-3)}$$

Now integrate both sides from 1 to μ_o (the region of physical interest).

$$\int_1^{\mu_o} yy'\,d\mu=\left[\frac{(1-\mu)^2\left(-y'\frac{dy}{d\mu}+y\frac{dy'}{d\mu}\right)}{(n_i-n_i')(n_i+n_i'+1)}\right]_1^{\mu_o} \qquad \text{(D-4)}$$

Now if we consider $y=P_{n_i}(\mu)$ and $y'=P_{n_i'}(\mu)$ we observe if $n_i \neq n_i'$ that the right side vanishes because at the lower limit $(1-\mu^2)=0$ and at the upper limit $\frac{dP_{n_i}(\mu)}{d\mu}$ and $\frac{dP_{n_i'}(\mu)}{d\mu}$ vanish by virtue of the boundary condition $\frac{dP_{n_i}(\mu_o)}{d\mu}=0$. If we consider $y=P^1_{n_i}(\mu)$ and $y'=P^1_{n_i'}(\mu)$ the right side again vanishes under the same boundary condition as now for the upper limit y and y' are zero. This proves that the $P_{n_i}(\mu)$ are orthogonal to each other and also that the $P^1_{n_i}(\mu)$ are also orthogonal.

We will now derive the values of the normalizing factors

$$\int_{\mu_o}^{1} \left[P_{n_i}(\mu)\right]^2 d\mu \qquad \text{and} \qquad \int_{\mu_o}^{1} \left[P^1_{n_i}(\mu)\right]^2 d\mu \, .$$

As we recall, the Taylor expansion takes the form

$$f(a + h) = f(a) + h\, f'(a) + \mathcal{O}(h^2). \tag{D-5}$$

Consider $a = n_i'$ and $h = n_i - n_i'$.

$$f(n_i) = f(n_i') + (n_i - n_i') \left[\frac{\partial f(n_i)}{\partial n_i}\right]_{n_i = n_i'} \tag{D-6}$$

Thus substituting in the Taylor expansion

$$P^{m'}_{n_i}(\mu) = P^{m'}_{n_i'}(\mu) + (n_i - n_i') \frac{\partial P^{m'}_{n_i'}(\mu)}{\partial n_i} \, . \tag{D-7}$$

Now letting m' = 1 or 0 we obtain by combining Equation (D-7) and Equation (D-4):

$$\int_{1}^{\mu_o} \left[P^{m'}_{n_i}(\mu)\right] \left[P^{m'}_{n_i'}(\mu)\right] d\mu =$$

$$\left[\frac{(1-\mu^2)\left[-P^{m'}_{n_i'}(\mu)\,\dfrac{\partial^2 P^{m'}_{n_i}(\mu)}{\partial\mu\,\partial n_i} + \dfrac{dP^{m'}_{n_i}(\mu)}{d\mu}\;\dfrac{dP^{m'}_{n_i}(\mu)}{d n_i} + (n_i - n_i')^2\right]}{n_i + n_i' + 1}\right]_1^{\mu_o} .$$

Now taking the limit as n_i' approaches n_i

$$\int_1^{\mu_o} \left[P_{n_i}^{m'}(\mu)\right]^2 d\mu =$$

$$\left[\frac{(1-\mu^2)}{2n_i+1}\left\{\frac{dP_{n_i}^{m'}(\mu)}{d\mu}\ \frac{dP_{n_i}^{m'}(\mu)}{dn_i} - \frac{\partial^2 P_{n_i}^{m'}(\mu)}{\partial\mu\partial n_i}\ P_{n_i}^{m'}(\mu)\right\}\right]_1^{\mu_o}$$

$$= \frac{1-\mu_o^2}{2n_i+1}\left\{\frac{dP_{n_i}^{m'}(\mu_o)}{d\mu}\ \frac{dP_{n_i}^{m'}(\mu_o)}{dn_i} - \frac{\partial^2 P_{n_i}^{m'}(\mu_o)}{\partial\mu\partial n_i}\ P_{n_i}^{m'}(\mu_o)\right\} \quad \text{(D-8)}$$

Now when $m' = 0$ we obtain

$$\int_1^{\mu_o}\left[P_{n_i}(\mu)\right]^2 d\mu = -\frac{1-\mu_o^2}{2n_i+1}\ P_{n_i}(\mu_o)\ \frac{\partial^2 P_{n_i}(\mu_o)}{\partial\mu\partial n_i}. \quad \text{(D-9)}$$

When $m' = 1$

$$\int_1^{\mu_o}\left[P_{n_i}^1(\mu)\right]^2 d\mu = \frac{1-\mu_o^2}{2n_i+1}\left\{\frac{dP_{n_i}^1(\mu_o)}{d\mu}\right\}\left\{\frac{dP_{n_i}^1(\mu_o)}{dn_i}\right\}. \quad \text{(D-10)}$$

If in Equation (D-8) n_i is replaced by m_i, $m' = 1$, and the normalizing integral is evaluated for the boundary condition

$\frac{dP_{m_i}^1(\mu_o)}{d\mu} = 0$ then one obtains

$$\int_{\mu_o}^{1}\left[P_{m_i}^1(\mu)\right]^2 d\mu = \frac{1-\mu_o^2}{2m_i+1}\ \frac{\partial^2 P_{m_i}^1(\mu_o)}{\partial\mu\partial m_i}\ P_{m_i}^1(\mu_o). \quad \text{(D-11)}$$

APPENDIX E

EVALUATION OF $\int_{\mu_o}^{1} e^{jkr\mu} P_{n_i}(\mu)\, d\mu$

This integral may be evaluated approximately for large r by making use of a limiting case of Bromwich's Theorem (Ref. 22, p. 230).

In order to apply the theorem we must first put our integral into suitable form. Thus, changing variables by setting $x = \mu - 1$ we obtain

$$-\int_{1}^{\mu_o} e^{jkr\mu} P_{n_i}(\mu)\, d\mu = -\int_{0}^{\mu_o-1} e^{jkr(x+1)} P_{n_i}(x+1)\, d\,x$$

$$= -\frac{e^{jkr}}{jkr}\left[j\int_{0}^{\mu_o-1} kr \cos krx\, P_{n_i}(x+1)\, dx - \int_{0}^{\mu_o-1} kr \sin krx\, P_{n_i}(x+1)\, dx \right]$$

$$= -\frac{e^{jkr}}{jkr}\left[j\int_{0}^{\mu_o-1} \lim_{\epsilon'\to 0}\left[(kr)^{1-\epsilon'}\, x^{-\epsilon'}\right] \cos krx\, P_{n_i}(x+1)\, dx \right.$$

$$\left. -\int_{0}^{\mu_o-1} \lim_{\epsilon'\to 0}\left[(kr)^{1-\epsilon'}\, x^{-\epsilon'}\right] \sin krx\, P_{n_i}(x+1)\, dx \right]$$

$$= -\frac{e^{jkr}}{jkr} \lim_{\epsilon'\to 0}\left[j(kr)^{1-\epsilon'} \int_{0}^{\mu_o-1} x^{-\epsilon'} \cos krx\, P_{n_i}(x+1)\, dx \right.$$

$$\left. -(kr)^{1-\epsilon'} \int_{0}^{\mu_o-1} x^{-\epsilon'} \sin krx\, P_{n_i}(x+1)\, dx \right].$$

Now Bromwich's Theorem states that for $\nu \longrightarrow \infty$

$$\nu^{\eta} \int_0^{\gamma} x^{\eta - 1} F(x) \sin \nu x \, dx \longrightarrow F(+0) \, \Gamma(\eta) \sin \frac{\eta\pi}{2}$$

if $-1 < \eta < 1$, $\lim_{\nu \to \infty} \nu\gamma \longrightarrow \infty$ and F(x) be of "limited total fluctuation" for $x \geq 0$ and

$$\nu^{\eta} \int_0^{\gamma} x^{\eta - 1} F(x) \cos \nu x \, dx \longrightarrow F(+0) \, \Gamma(\eta) \cos \frac{\eta\pi}{2}$$

where, in addition to the above conditions $0 < \eta < 1$. In our case, $\nu = kr$, $F(x) = P_{n_i}(x+1)$, $\eta = 1 - \epsilon'$, $\gamma = \mu_o - 1$.

Thus, for $kr \longrightarrow \infty$ (i.e., for $r \longrightarrow \infty$)

$$(kr)^{1-\epsilon'} \int_0^{\mu_o - 1} x^{-\epsilon'} \cos krx \, P_{n_i}(x+1) \, dx \longrightarrow P_{n_i}(1) \, \Gamma(1-\epsilon') \cos \frac{(1-\epsilon')\pi}{2}$$

$$= \Gamma(1-\epsilon') \cos \frac{(1-\epsilon')\pi}{2}$$

and $$(kr)^{1-\epsilon'} \int_0^{\mu_o - 1} x^{-\epsilon'} \sin krx \, P_{n_i}(x+1) \, dx \longrightarrow P_{n_i}(1) \, \Gamma(1-\epsilon') \sin \frac{(1-\epsilon')\pi}{2}$$

$$= \Gamma(1-\epsilon') \sin \frac{(1-\epsilon')\pi}{2} .$$

Finally then, for $r \longrightarrow \infty$

$$+\int_{\mu_o}^{1} e^{jkr\mu} P_{n_i}(\mu) \, d\mu \longrightarrow \frac{-e^{jkr}}{jkr} \lim_{\epsilon' \to 0} \left\{ \Gamma(1-\epsilon') \left[\cos \frac{(1-\epsilon')\pi}{2} - \sin \frac{(1-\epsilon')\pi}{2} \right] \right\}$$

$$= \frac{e^{jkr}}{jkr} .$$

REFERENCES

Number	Title
1	"Propagation of Short Waves", by D. E. Kerr, Vol. 13 in the MIT Radiation Laboratory Series, McGraw-Hill Book Co., 1951.
2	"Scattering From Spheres", Report No. 4, University of California, Department of Engineering.
3	"Electromagnetic Scattering From Metal and Water Spheres", by A. L. Aden, Technical Report No. 106, Cruft Laboratory, Harvard University.
4	UMM-42 -- "Studies in Radar Cross-sections - I, Scattering by a Prolate Spheroid", by F. V. Schultz, Willow Run Research Center, University of Michigan, March 1950.
5	"The Diffraction of a Plane Wave by a Paraboloid of Revolution" by F. C. Karal, Bumblebee Series, Report No. 139, Defense Research Laboratory of the University of Texas.
6	"Theoretical Study of Electromagnetic Waves Scattered From Shaped Metal Surfaces", by W. W. Hansen and L. I. Schiff, Quarterly Report No. 4, Stanford University, Microwave Laboratory.
7	UMM-82 -- "Studies in Radar Cross-sections -- II, The Zeros of the Associated Legendre Functions of Non-integral Degree", by K. Siegel, D. Brown, H. Hunter, H. Alperin, and C. Quillen, Willow Run Research Center, University of Michigan, April 1951.
8	"The Mathematical Theory of Huygen's Principle", by B. B. Baker and E. I. Copson, 2nd Edition, Oxford University Press, 1950.
9	"Electromagnetic Theory", by J. A. Stratton, 1st Edition, McGraw-Hill Book Co., 1941.

REFERENCES (Continued)

Number	Title
10	"Back Scattering From Conducting Surfaces", by R. C. Spencer, Cambridge Research Laboratories, CRL-R-E5070, April 1951.
11	"Theoretical Study of Electromagnetic Waves Scattered From Shaped Metal Surfaces", by W. W. Hansen and L. I. Schiff, Quarterly Report No. 3, Stanford University, Microwave Laboratory.
12	"Reflections From Smooth Curved Surfaces", by R. C. Spencer, MIT Radiation Laboratory, Report No. 661.
13	"Essay on Electricity and Magnetism", by G. Green, Mathematical Papers, p. 67, 1828.
14	"Cavendish Papers", by J. C. Maxwell, p. 385, 1879.
15	"On a Function Related to Spherical and Cylindrical Functions: Its Application in the Theory of Distribution of Electricity", by F. G. Mehler, Mathematische Annalen, Bd. XVIII, p. 161, 1881.
16	"Demonstration of Green's Formula for Electric Density Near the Vertex of a Right Cone", by H. M. McDonald, Cambridge Philosophic Transactions, Vol. 18, p. 292, 1900.
17	"The Scattering of Sound Waves by a Cone", by H. S. Carslaw, Mathematische Annalen, Bd. LXXV, 1914.
18	"Formulas on Theorems for the Special Functions of Mathematical Physics", by W. Magnus and F. Oberhetinger, Chelsea Publishing Company, 1949.
19	"Handbuch der Kugelfunctionen", Vol. I, by E. Heine, Berlin, 1878.

REFERENCES (Continued)

Number	Title
20	"Partial Differential Equations in Physics", by A. Sommerfeld, Academic Press, Inc., 1949.
21	"Radar System Engineering", by L. N. Ridenour, Vol. 1, MIT Radiation Laboratory Series, McGraw-Hill Book Co., 1947.
22	"A Treatise on the Theory of Bessel Function", by G. N. Watson, 2nd Edition, Cambridge University Press, 1948.
23	UMM-92 -- "Studies in Radar Cross-sections - IV, Comparison Between Theory and Experiment of the Differential Scattering Cross-section of a Semi-infinite Cone", by K. M. Siegel, H. A. Alperin, J. W. Crispin, R. E. Kleinman, and H. E. Hunter, Willow Run Research Center, to be published.

DISTRIBUTION

To be distributed in accordance with the terms of the contract.