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Towards High-Order Fluctuation-Splitting Schemes for Navier-Stokes Equations

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This paper reports progress towards high-order fluctuation-splitting schemes for the Navier-Stokes Equations. High-order schemes we examined previously are all based on gradient reconstruction, which may result in undesired mesh-dependency problem due to the somewhat ambiguous gradient reconstruction procedures. Here, we consider schemes for P_2 elements in order to eliminate the need for such gradient reconstruction. For pure advection, a P_2 version of the LDA scheme is derived from a constrained least-squares minimization. This scheme is fourth-order accurate. For pure diffusion, a P_2 Galerkin scheme is derived from a minimization principle, which turns out to be equivalent to applying Richardson's extrapolation technique to the standard second-order Galerkin scheme. This scheme is again fourth-order accurate. Finally, strategies for integrating P_2 advection and diffusion schemes to develop uniformly accurate P_2 schemes are discussed.

I. Introduction

Accurate heat transfer prediction in hypersonic flows requires a CFD code to have excellent performance in resolving various types of flows: strong shocks, almost incompressible flows near stagnation, and viscous layers, especially on unstructured grids. Because of dissatisfaction with current finite-volume codes for this type of problem,[1] we are trying to develop a radically new methodology for solving the compressible Navier-Stokes equations. We have been focusing on the fluctuation-splitting schemes as a new method because they are based on multidimensional upwinding and have a better performance on unstructured grids than the conventional finite-volume method. In the previous work,[2] taking the two-dimensional advection-diffusion equation,

$$u_t + a\,u_x + b\,u_y = \nu\,(u_{xx} + u_{yy}). \tag{1}$$

as a model equation, we examined the fluctuation-splitting schemes. We suggested reduction to an equivalent first-order system and unified treatment of the advective and the diffusive fluctuations, and then developed third and fourth order schemes based on gradient reconstruction. Such schemes can be readily extended to the Navier-Stokes equations, but there is a concern about the mesh-dependency of the schemes because of the irregular nature of the gradient recovery methods. To eliminate such a concern, we consider here an alternative way to achieve high-order accuracy without reconstructing the solution gradients. The alternative is the use of P_2 elements which introduce additional degrees of freedom at the midpoints of edges. P_2 elements have just enough degrees of freedom to represent a solution by a quadratic variation, thereby we do not have to reconstruct anything to better represent the solution. But as the price of this merit, we now need to devise a scheme to compute the solution at the midpoints. In this paper, we present such schemes for a pure advection and a pure diffusion equation. We will show that it is possible to achieve fourth-order accuracy with P_2 elements without any kind of reconstruction.

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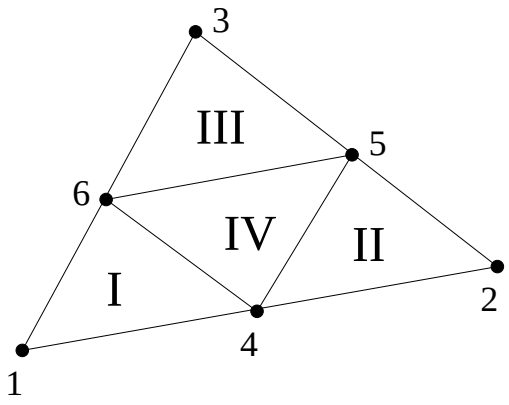

Figure 1. A typical P_2 element

II. Fluctuation-Splitting Schemes and Reconstruction

Fluctuation-splitting schemes are based on nodal variables, leading to cell-based residuals (fluctuations) that are distributed to the nodes. Consider, for example, solving a pure advection equation,

$$u_t + a\,u_x + b\,u_y = 0 \tag{2}$$

in a domain divided into a set of triangles $\{T\}$. In a loop over the elements, we compute the fluctuation ϕ_T for each triangle T,

$$\phi^T = \int_T u_t\,dxdy = \int_T \left[-(a\,u_x + b\,u_y)\right]\,dxdy \tag{3}$$

assuming a piecewise linear variation of u, and then distribute it to the nodes of the triangle, thus accumulating the changes at every node. This results in the following update formula for the solution at node j,

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{V_j} \sum_{T\in\{T_j\}} \phi_i^T \tag{4}$$

where

$$\phi_i^T = \beta_j^T\,\phi^T \tag{5}$$

and $\{T_j\}$ is a set of triangles that shares the node j, $\Delta t = t^{n+1} - t^n$ is a (local or global) timestep, V_j is the volume of the median dual control volume, and β_j^T is the distribution coefficient that assigns the fraction of the fluctuation sent to the node j within the triangle T.

In whatever way the fluctuation is distributed, the method is at most second-order accurate because of the assumption of the linear variation (except for some very special cases). To achieve higher-order, we need to establish high-order representation of the solution within each element and then evaluate the fluctuation with it. Consider a triangular element with midpoints introduced (cf. Figure 1). To evaluate the fluctuation, we convert the integral to a contour integral

$$\phi_T^H = \int_T (a\,u_x + b\,u_y)\,dxdy = \oint_{\partial T} (au\,dy - bu\,dx) \tag{6}$$

and apply Simpson's rule along each edge. Note that this is fourth-order accurate. Now, instead of storing the solutions at the midpoints as unknowns, we may interpolate the solution. To achieve higher-order, such an interpolation must be better than the linear interpolation. To this end, we recover the gradients at nodes by a least-squares reconstruction technique, and use them to construct the Hermite interpolation along each edge. This gives the midpoint values with sufficient accuracy, and completely determines the high-order fluctuation.

Note that the midpoints have no further roles. This high-order fluctuation can then be distributed to the three nodes of the triangle. The method just described above is exactly the one proposed by Caraeni and Fuchs[3] which they claim third-order accurate. Later, Nishikawa, Rad and Roe[4] rearranged their method so that it can be implemented as adding a high-order correction to existing second-order code. In the previous work,[2] we confirmed that this type of high-order fluctuation-splitting scheme achieved third-order accuracy with the Green-Gauss gradient reconstruction and fourth-order accuracy with the quadratic least-squares reconstruction. However, these reconstruction schemes do not appear to be grid independent because of the irregularity of the stencil for reconstruction, especially with the quadratic least-squares reconstruction which involves nodes that are not immediate neighbor. In this paper, we consider the alternative, i.e. treat the midpoint values as unknowns and develop schemes directly applied to the quadratic elements. Such schemes are expected to have better accuracy on unstructured grids partly because they preserve polynomial solutions of higher order on arbitrary grids.

III. P_2 Advection Scheme

We first consider the advection limit of (1),

$$u_t + a\,u_x + b\,u_y = 0. \tag{7}$$

Triangular elements with midpoints are now regarded as P_2 elements (See Figure 1). With 6 degrees of freedom $\{u_i\}$ available, we can represent the solution with quadratic variation,

$$u = \sum_{i=1}^{6} u_i N_i \tag{8}$$

where

$$\begin{aligned} N_1 &= (2L_1 - 1)L_1 && (9)\\ N_2 &= (2L_2 - 1)L_2 && (10)\\ N_3 &= (2L_3 - 1)L_3 && (11)\\ N_4 &= 4L_1L_2 && (12)\\ N_5 &= 4L_2L_3 && (13)\\ N_6 &= 4L_3L_1 && (14) \end{aligned}$$

and L_1, L_2, and L_3 are the local area coordinates of the triangle. To solve the advection equation, we define the high-order fluctuation as

$$\phi_T^H = \int_T (a\,u_x + b\,u_y)\,dxdy = \oint_{\partial T} (au\,dy - bu\,dx). \tag{15}$$

We evaluate this by applying Simpson's rule along each edge. The result can be simplified and written as a sum of second-order fluctuations

$$\phi_T^H = \frac{1}{3}\phi_T + \frac{8}{3}\phi_{T_{IV}} \tag{16}$$

where ϕ_T and $\phi_{T_{IV}}$ are the fluctuations over triangles T and T_{IV} assuming a piecewise linear variation of u within each triangle, i.e.

$$\begin{aligned} \phi_T &= \sum_{i=1}^{3} k_i^T u_i = \sum_{i=1}^{3} \left[\frac{1}{2}(a,b)\cdot \mathbf{n}_i^T\right] u_i && (17)\\ \phi_{T_{IV}} &= \sum_{i=4}^{6} k_i^{T_{IV}} u_i = \sum_{i=4}^{6} \left[\frac{1}{2}(a,b)\cdot \mathbf{n}_i^{T_{IV}}\right] u_i && (18) \end{aligned}$$

where $\mathbf{n}_i^T$ is a scaled inward normal vector of the edge opposite to the node i in triangle T, and similarly for $\mathbf{n}_i^{T_{IV}}$. To simplify the notation, we write

$$\phi_T^H = \sum_{i=1}^{6} k_i u_i \tag{19}$$

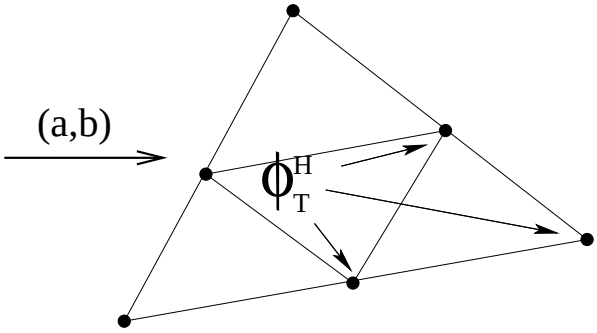

Figure 2. P_2 **LDA Scheme, Case 1**

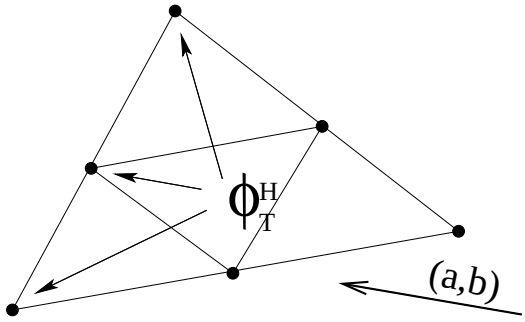

Figure 3. P_2 **LDA Scheme, Case 2**

where

$$k_i = \begin{cases} \frac{1}{3}k_i^T & i = 1,2,3 \\ \frac{8}{3}k_i^{T_{IV}} & i = 4,5,6 \end{cases} \tag{20}$$

To derive a distribution coefficient, we consider minimizing the functional defined by

$$\mathcal{F} = \frac{1}{2}\sum_{T\in\{T\}} \left(\phi_T^H\right)^2. \tag{21}$$

Employing the steepest descent method, we obtain the following partial fluctuation ϕ_j, which is distributed to node j within triangle T,

$$\phi_j = -\frac{\partial \phi_T^H}{\partial u_j}\,\phi_T^H = -k_j\,\phi_T^H. \tag{22}$$

Unfortunately, this scheme is extremely slow to converge although it is indeed fourth-order accurate. To improve the convergence, we introduce an upwinding constraint and devise the following scheme,

$$\phi_j^* = \beta_j^T\,\phi_T^H \tag{23}$$

where

$$\beta_j^T = -\frac{\max(0,\,k_j)}{\sum_{i=1}^{6}\max(0,\,k_i)} \tag{24}$$

i.e., only the downwind nodes get updated (see Figures 2 and 3). Note that the denominator in (24) has been introduced for conservation, i.e. $\sum_i \beta_i^T = 1$, and therefore $\sum_i \phi_i^* = \phi_T^H$. It can be easily shown that exactly the same derivation results in the LDA scheme for P_1 elements. For this reason, we may call the scheme (24) a P_2 LDA scheme. Note that the extension to three dimensions is straightforward. Our truncation analysis has confirmed the fourth-order accuracy of this scheme on a regular triangular grid.

The P_2 LDA scheme was applied to a test problem in a square domain $(0 < x < 1, 0 < y < 1)$ with the exact solution,

$$u(x,y) = -\cos(2\pi(bx - ay)) \tag{25}$$

and $(a,b) = (5,1)$. The solutions were computed for a series of structured and unstructured grids with the number of triangles 800, 3200, 12800 in both cases, corresponding to the structured grids of sizes 20×20, 40×40, 80×80 (see Figures 4 and 5). Structured grids were constructed by inserting a diagonal (bottom-left to upper-right) in a Cartesian cell. The errors are given in Table 1. It is remarkable that the scheme performs very well on unstructured grids, showing the same (or even better) error level especially on fine grids. We remark that, to our knowledge, this is the first fluctuation-splitting scheme for the linear advection equation

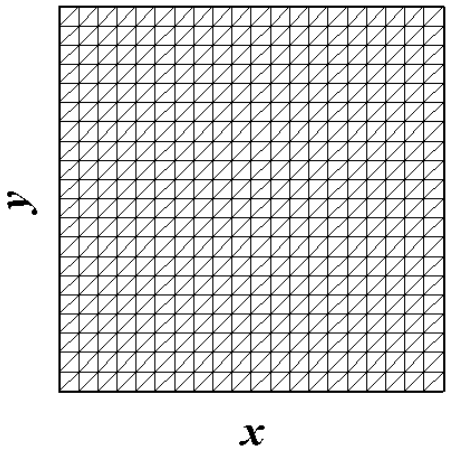

Figure 4. Structured grid (800 triangles)

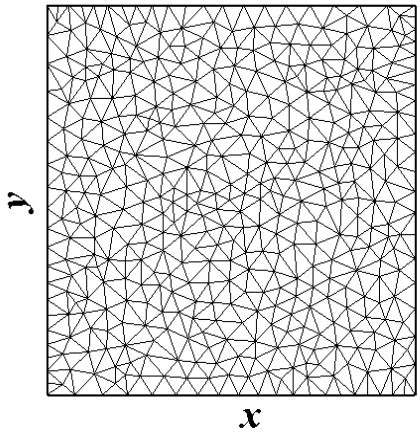

Figure 5. Unstructured grid (800 triangles)

	Structured	Order	Unstructured	Order
20×20	6.40E-04		1.00E-03	
40×40	6.78E-05	3.24	7.33E-05	3.77
80×80	5.52E-06	3.62	4.39E-06	4.06

Table 1. P_2 LDA Scheme: L_2 errors and the order of convergence on structured and unstructured grids.

that has achieved fourth-order accuracy on P_2 elements. Abgrall and Roe[5] developed P_2 schemes based on high-order fluctuations defined on subtriangles, and also Villedieu et. al.[6] developed P_2 scheme based on the Petrov-Galerkin formulation of the fluctuation-splitting scheme. But they are all third-order accurate.

It is also possible to devise a scheme by applying the LDA scheme for P_1 elements to each subtriangle with the high-order fluctuation restricted on that subtriangle. However, we found from a truncation error analysis that it is third-order accurate only when $(a, b) = (2, 1)$ (for a particular regular triangular grids mentioned above) and in general only second-order accurate. In fact, the LDA scheme for P_1 elements also has the same superconvergent property. Therefore, there is basically no improvement with this strategy. We believe that we should distribute the whole high-order fluctuation to each node as all the nodes equally affect the fluctuation. Also, it is important to develop limiters for the P_2 scheme to achieve both high-order and monotonicity. A type of limiter that is applied to the split fluctuations and hence suitable for P_2 schemes has been developed by Abgrall and Roe.[5]

IV. P_2 Diffusion Scheme

We now turn our attention to the diffusion term and consider solving the Laplace equation with an appropriate time derivative,

$$u_t = u_{xx} + u_{yy} \tag{26}$$

using the P_2 elements. It is easy to show by integration by parts that the solution to the Laplace equation minimizes the functional,

$$\mathcal{F} = \frac{1}{2} \int \left(u_x^2 + u_y^2 \right) \, dxdy. \tag{27}$$

In fact, the standard P_1 Galerkin method can be derived by minimizing this on the assumption of the piecewise linear variation of the solution over a triangle. We follow this approach to derive a scheme for P_2 elements. Using the quadratic variation, we find

$$\mathcal{F} = \frac{1}{2}\int \left(u_x^2 + u_y^2\right)\, dxdy \tag{28}$$

$$= \frac{1}{6}\sum_{T\in\{T\}} \left[(u_x^q)^2 + (u_y^q)^2\right] S_T \tag{29}$$

where $(u_x^q)^2$ denotes a value expressed by the constant gradients defined in the subtriangles assuming the piecewise linear variation, $u_x^{T_I}$, $u_x^{T_{II}}$, $u_x^{T_{III}}$, $u_x^{T_{IV}}$,

$$(u_x^q)^2 = \frac{1}{4}\left\{(u_x^{T_I} - u_x^{T_{II}})^2 + (u_x^{T_{II}} - u_x^{T_{III}})^2 + (u_x^{T_{III}} - u_x^{T_I})^2\right. \tag{30}$$

$$\left. + \; (u_x^{T_I} + u_x^{T_{IV}})^2 + (u_x^{T_{II}} + u_x^{T_{IV}})^2 + (u_x^{T_{III}} + u_x^{T_{IV}})^2\right\} > 0, \tag{31}$$

similarly for $(u_y^q)^2$. After further rearrangement, we find that it is possible to rewrite this functional as

$$\mathcal{F} = \sum_{T\in\{T\}} \left\{\frac{4}{3}\sum_{T_\xi\in\{T_\xi\}} F_{T_\xi} - \frac{1}{3}F_T\right\} \tag{32}$$

$$= \frac{4}{3}\sum_{T\in\{T\}}\sum_{T_\xi\in\{T_\xi\}} F_{T_\xi} - \frac{1}{3}\sum_{T\in\{T\}} F_T \tag{33}$$

where $\{T_\xi\} = \{T_I, T_{II}, T_{III}, T_{IV}\}$, and

$$F_T = \frac{1}{2}\left[(u_x^T)^2 + (u_y^T)^2\right] S_T \tag{34}$$

$$F_{T_\xi} = \frac{1}{2}\left[(u_x^{T_\xi})^2 + (u_y^{T_\xi})^2\right] S_{T_\xi} \tag{35}$$

where (u_x^T, u_y^T) is the constant gradient on the triangle T. It is clear now that the minimizing F_T gives the second-order Galerkin scheme for the triangle T while minimizing F_{T_ξ} gives the second-order Galerkin scheme for the subtriangles. We thus obtain the following scheme from minimization: four-third of the second-order update on each subtriangle,

$$u_j^{n+1} = u_j^n - \frac{4}{3}\left\{\frac{\omega}{2}\left(u_x^{T_\xi}, u_y^{T_\xi}\right)\cdot \mathbf{n}_j^{T_\xi}\right\} \quad j\in\{i_{T_\xi}\} \tag{36}$$

followed by subtracting one-third of the second-order update on the original triangle,

$$u_j^{n+1} = u_j^n + \frac{1}{3}\left\{\frac{\omega}{2}\left(u_x^T, u_y^T\right)\cdot \mathbf{n}_j^T\right\} \quad j\in\{i_T\} \tag{37}$$

where ω is a small constant and $\{i_{T_\xi}\}$ is the set of nodes of the subtriangle T_ξ. Note that this is nothing but the Richardson extrapolation applied to the 2nd-order Galerkin method. In fact, the weights, $4/3$ and $-1/3$ could have been found readily from this viewpoint. It is then immediate from this that the resulting scheme is in general third-order accurate. However, the Galerkin scheme does not contain odd order terms in its truncation error at least on regular grids due to the symmetry of its stencil, and so in such a case the method becomes fourth-order accurate. Naturally, we may call this scheme a P_2 Galerkin scheme.

To demonstrate its accuracy, we consider a test problem in the square domain $(0 < x < 1, 0 < y < 1)$ with the exact solution,

$$u(x,y) = \frac{\sinh(\pi x)\sin(\pi y) + \sinh(\pi y)\sin(\pi x)}{\sinh(\pi)}. \tag{38}$$

Both structured and unstructured grids were used with the number of triangles 200, 800, 3200, corresponding to the structured grids of sizes 10×10, 20×20, 40×40. Results are shown in Table 2. The method is fourth-order accurate on structured grids as we expected. For unstructured grids, it becomes third-order accurate (also as we expected), but the actual errors are comparable to those on structured grids.

	Structured	Order	Unstructured	Order
10×10	2.28E-05		3.22E-05	
20×20	1.43E-06	4.00	3.86E-06	3.10
40×40	8.87E-08	4.01	4.49E-07	3.11

Table 2. L_2 errors and the order of convergence on structured and unstructured grids.

V. Concluding Remarks

Obviously, simply adding the two schemes developed here will not yield a uniformly accurate scheme for the advection-diffusion equation: the scheme is fourth-order accurate in both advective and diffusive limits but not in between due to the incompatibility of the discretization methods.[2] Note in particular that there are no quantities corresponding the fluctuation in the P_2 Galerkin scheme as the updates sum up to zero over the nodes of the element. In the case of reconstruction schemes, we avoided this problem by solving the equivalent first-order system

$$u_t + a\,u_x + b\,u_y = \nu\,(p_x + q_y) \tag{39}$$

$$p - u_x = 0 \tag{40}$$

$$q - u_y = 0 \tag{41}$$

and achieve uniformly high-order accuracy. The same could be done for P_2 elements, of course. However, so far, we have not been able to do so due to the difficulty of solving (40) and (41) with sufficiently high order. Another possibility would be to recover the gradients at nodes by some high-order interpolation. This is based on the observation that we need a continuous representation of the gradient in order to define a conservative fluctuation for the diffusion term. Yet another possibility is to introduce the jump in the normal derivative on the element boundary to achieve the gradient continuity as typically done in the finite-element methods.

Further development of the P_2 advection scheme is also currently underway, focusing on monotonicity preservation.

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References

[1] "High Energy Flow Solver Synthesis (LAURA Benchmarks: Cylinder)," http://hefss.larc.nasa.gov/index.cfm?selection=laura&laura_type=benchmarks.

[2] Nishikawa, H. and Roe, P., "A Third-Order Fluctuation-Splitting Scheme That Preserves Potential Flow," *Third International Conference on Computational Fluid Dynamics*, Toronto, CA, 2004.

[3] Caraeni, D. and Fuchs, L., "Compact Third-Order Multidimensional Upwind Scheme for Navier-Stokes Simulations," *Theoretical and Computational Fluid Dynamics*, Vol. 15, 2002, pp. 373–401.

[4] Nishikawa, H., Rad, M., and Roe, P., "A Third-Order Fluctuation-Splitting Scheme That Preserves Potential Flow," *15th AIAA Computational Fluid Dynamics Conference*, AIAA Paper 01-2595, Anaheim, 2001.

[5] Abgrall, R. and Roe, P. L., "High-Order Fluctuation Schemes on Triangular Meshes," *Journal of Scientific Computing*, Vol. 19, No. 1-3, 2002, pp. 3–36.

[6] Villedieu, N., Ricchiutoa, M., and Deconinck, H., "Study of a 3rd Order RD Scheme for Advection-Diffusion Equation," *VKI Internal Report*, 2004.