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SECOND-ORDER ANALYTIC SOLUTIONS FOR RE-ENTRY TRAJECTORIES

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Abstract

With the development of aeroassist technology, either for near-earth operations or for planetary aero-capture, it is of interest to have accurate analytic solutions for the speed, flight path angle and altitude during the atmospheric passage. For a future aerospace plane which uses the accumulated kinetic energy to glide for a long range, explicit relations among the main state variables are also useful for guidance purposes. In this paper we have used normalization to put the equations of motion for planar entry around a non-rotating planet into a form which is suitable for an analytic integration. Explicit and accurate solutions are then obtained for ballistic fly-through trajectories, lifting skip trajectories and equilibrium glide trajectories.

Introduction

With the development of aeroassist technology, either for near-earth orbital transfer or for planetary aero-capture, it is of interest to have accurate analytic solutions for re-entry trajectories. In this paper, it is proposed to obtain the relationships between the altitude, speed, flight path angle and distance traveled during entry for the three typical entry modes.

Since the early development of the theory of planetary entry, the first-order solutions developed by Allen and Eggers[1], and by Chapman's analysis[2] have always been considered as the main results to analyze the variations of the deceleration and the various forms of aerodynamic heating during entry. Although these classical formulas correctly predict the occurrence of the peak deceleration and the peak heating rate, they are not sufficiently accurate for guidance purposes. A second-order theory has been proposed by Loh[3] but his integration of the equations of motion involves an empirical step which is not justified mathematically. A complete account of the known results has been presented in Ref. 4.

In the present analysis, starting with the equations of motion for planar entry around a non-rotating planet, we use normalization to put the equations into a form which is suitable for an analytic integration. Then, depending on the type of constant angle-of-attack entry trajectories, namely the ballistic entry, the skip entry and the equilibrium glide, with the proper simplification, we easily obtain the first-order solutions. By analytic continuation, the second-order, or even the third-order solutions are obtained and compared with the exact solutions from numerical integration of the non-linear equations. We believe that up to the present time our solutions are the most accurate and they are in the form appropriate for guidance.implementation and also for accurate prediction of the peak deceleration and the various peak heating rates during entry.

Dimensionless equations of motion

For planar entry of a non-thrusting, lifting vehicle we have the equations

$$\frac{dr}{dt} = V \sin\gamma \tag{1a}$$

$$\frac{d\theta}{dt} = \frac{V}{r}\cos\gamma \tag{1b}$$

$$\frac{dV}{dt} = -\frac{\rho S C_D V^2}{2m} - g\sin\gamma \tag{1c}$$

$$V\frac{d\gamma}{dt} = \frac{\rho S C_L V^2}{2m} - \left(g - \frac{V^2}{r}\right)\cos\gamma \tag{1d}$$

where r is the radial distance, θ the range angle, V the airspeed, and γ is the flight path angle, that is, the angle between the local horizontal and the velocity vector, measured positively upward.

We use the initial altitude as the reference altitude for a central Newtonian gravitational attraction and an exponential atmosphere, that is, we consider

$$\frac{g}{g_0} = \frac{{r_0}^2}{r^2} \tag{2}$$

and

$$\frac{\rho}{\rho_0} = e^{-\beta(r-r_0)} \tag{3}$$

where β is the constant inverse of the scale height, appropriately selected for agreement with the atmosphere. For the lift-drag relationship we use the standard parabolic polar

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$$C_D = C_{D_0} + K C_L{}^2 \tag{4}$$

with constant C_{D_0} and K for hypersonic flow. The altitude and speed variables are non-dimensionalized through the definition

$$h = \frac{r - r_0}{r_0} \tag{5}$$

$$u = \frac{V^2}{g_0 r_0} \tag{6}$$

Then we have the equations of motion, with the time eliminated in favor of the range angle

$$\frac{dh}{d\theta} = (1+h)\tan\gamma \tag{7a}$$

$$\frac{du}{d\theta} = -\frac{B(1+h)(1+\lambda^2)u\,e^{-\beta r_0 h}}{E^* \cos\gamma} - \frac{2}{(1+h)}\tan\gamma \tag{7b}$$

$$\frac{d\gamma}{d\theta} = \frac{B(1+h)\lambda\, e^{-\beta r_0 h}}{\cos\gamma} - \frac{1}{u(1+h)} + 1 \tag{7c}$$

where E^* is the maximum lift-to-drag ratio and B is a small parameter specifying the entry altitude and physical characteristics of the vehicle.

$$E^* = \frac{1}{2\sqrt{K C_{D_0}}} \tag{8}$$

$$B = \frac{\rho_0 S r_0}{2m}\sqrt{\frac{C_{D_0}}{K}} \tag{9}$$

The equations are integrated from the initial entry point at $\theta = 0$ with $h = 0$, $u = u_e$, $\gamma = \gamma_e$. The atmosphere is here specified by the so-called Chapman's parameter βr_0, while E^* is the performance characteristic of the vehicle. The other physical characteristics of the vehicle such as its mass and size are contained in the dimensionless parameter B which also serves as an indication of the entry altitude.

The system (7) constitutes the exact equations for re-entry, even with lift modulation. The lift control here is the normalized lift coefficient

$$\lambda = \frac{C_L}{\sqrt{C_{D_0}/K}} \tag{10}$$

which is defined such that when $\lambda = 1$ the flight is at maximum lift-to-drag ratio. We shall develop second-order analytic solutions for entry at constant λ, with the limiting case of $\lambda = 0$ for ballistic entry.

To perform the integration we introduce new dimensionless variables with y for altitude, v for speed, and ϕ for flight path angle such that

$$y = e^{-\beta r_0 h} = \frac{\rho}{\rho_0} \tag{11}$$

$$v = \frac{1}{\eta}\log\frac{V_e{}^2}{V^2} \tag{12}$$

$$\phi = -\sqrt{\beta r_0}\sin\gamma \tag{13}$$

where V_e is the entry speed, and η is a very small parameter defined as

$$\eta = \frac{B}{E^*\sqrt{\beta r_0}} = \frac{\rho_0 S C_{D_0}}{m}\sqrt{\frac{r_0}{\beta}} \tag{14}$$

For a typical entry vehicle at 100 km, η is of the order of 10^{-4}. In addition to the injection of this small parameter into the equations of motion, the new variables defined above are such that we always have at entry $y(0) = 1$ and $v(0) = 0$. The system (7) is now transformed into

$$\frac{dy}{d\tau} = \frac{(1+h)}{\cos\gamma} y\phi \tag{15a}$$

$$\frac{dv}{d\tau} = \frac{(1+h)}{\cos\gamma}\left(1+\lambda^2\right)y - \frac{1}{(1+h)\cos\gamma} k\alpha\phi\, e^{\eta v} \tag{15b}$$

$$\frac{d\phi}{d\tau} = -(1+h)B\lambda y + \frac{\cos\gamma}{(1+h)}\alpha e^{\eta v} - \cos\gamma \tag{15c}$$

where $\tau = \sqrt{\beta r_0}\,\theta$ is the new independent variable and

$$k = \frac{2E^*}{\sqrt{\beta r_0}\,B} \tag{16}$$

$$\alpha = \frac{g_0 r_0}{V_e{}^2} \tag{17}$$

The system (15) is the exact system, but the selection of the dimensionless variables, coupled with the fact that, for entry trajectories, $\cos\gamma \approx 1$, $1 + h \approx 1$, leads to the simpler system

$$\frac{dy}{d\tau} = y\phi \tag{18a}$$

$$\frac{dv}{d\tau} = \left(1+\lambda^2\right)y - k\alpha\phi\, e^{\eta v} \tag{18b}$$

$$\frac{d\phi}{d\tau} = -B\lambda y + \alpha e^{\eta v} - 1 \tag{18c}$$

Extensive numerical integration has been performed for several cases of entry trajectories and we are satisfied that the solutions from (18) are nearly the same as the solutions from (15).

In summary, the dimensionless equations of motion for re-entry are governed by the system (18) with y being the density ratio, used as the altitude variable, v the speed variable and ϕ the flight path angle variable, while τ is proportional to the distance traveled. At $\tau = 0$, we always have $y(0) = 1$ and $v(0) = 0$. The initial entry speed is now specified by the dimensionless parameter α with $\alpha = 1$ for circular entry and $\alpha = 0.5$ for parabolic entry. The planetary atmosphere is specified by Chapman's characteristic parameter in the form of the product βr_0, with the value $\beta r_0 = 900$ for the earth's atmosphere. The vehicle physical characteristics are specified by the maximum lift-to-drag ratio E^*, and the dimensionless parameter B, which also includes the effective entry altitude. We have then reduced to a minimum the number of physical parameters required for the analysis. It suffices finally to select an entry angle γ_e, to evaluate the initial value $\phi_e = -\sqrt{\beta r_0} \sin \gamma_e$, and to specify the flight program in terms of the lift coefficient λ, to generate the trajectory.

Ballistic skip trajectories

For ballistic entry, the normalized lift coefficient λ is zero and, from system (18), we have

$$\frac{dy}{d\tau} = y\phi \tag{19a}$$

$$\frac{dv}{d\tau} = y - k\alpha\phi \, e^{\eta v} \tag{19b}$$

$$\frac{d\phi}{d\tau} = \alpha \, e^{\eta v} - 1 \tag{19c}$$

with the initial conditions at $\tau = 0$,

$$y(0) = 1, \quad v(0) = 0, \quad \phi(0) = c \tag{20}$$

Beside the value $c = -\sqrt{\beta r_0} \sin \gamma_e > 0$ for the entry angle, we specify the entry speed by selecting a value for α. The vehicle physical characteristics and the effective entry altitude are contained in the two parameters k and η. It is seen that for ballistic entry it suffices to specify

$$\overline{B} = \frac{B}{E^*} = \frac{\rho_0 S C_{D_0} r_0}{m} \tag{21}$$

and we have both k and η since $\beta r_0 = 900$ is known for the earth's atmosphere. Therefore a ballistic entry trajectory is governed by the entry speed, the entry angle and the ballistic factor $\overline{B}$ as defined in Eq. (21).

The ballistic entry mode at subcircular speed and at moderate entry angle is well studied, and accurate analytic solutions have been derived previously[5]. For aeroassisted orbital transfer and planetary aero-capture, we may use the ballistic mode. With supercircular entry speed, at a low entry angle, the vehicle will skip out and the accurate prediction of the exit speed and exit angle using explicit analytic formulas is obviously of interest to mission planning.

Higher-order analytic solutions are obtained by observing that the parameter η is small, of the order of 10^{-4}. Hence, we seek series solutions of the form

$$\begin{aligned} y &= y_0 + \eta y_1 + \eta^2 y_2 + \cdots \\ v &= v_0 + \eta v_1 + \eta^2 v_2 + \cdots \\ \phi &= \phi_0 + \eta \phi_1 + \eta^2 \phi_2 + \cdots \end{aligned} \tag{22}$$

with the initial conditions

$$y_0(0) = 1, \quad v_0(0) = 0, \quad \phi_0(0) = c \tag{23}$$

and

$$y_i(0) = 0, \quad v_i(0) = 0, \quad \phi_i(0) = 0 \quad i = 1, 2, \cdots \tag{24}$$

By substituting (22) into the system (19) and equating terms of like powers in η we obtain the following systems of various orders.

$$\begin{aligned} \frac{dy_0}{d\tau} &= y_0 \phi_0 \\ \frac{dv_0}{d\tau} &= y_0 - k\alpha\phi_0 \\ \frac{d\phi_0}{d\tau} &= -(1-\alpha) \end{aligned} \tag{25}$$

$$\begin{aligned} \frac{dy_1}{d\tau} &= y_0 \phi_1 + y_1 \phi_0 \\ \frac{dv_1}{d\tau} &= y_1 - k\alpha(\phi_0 v_0 + \phi_1) \\ \frac{d\phi_1}{d\tau} &= \alpha v_0 \end{aligned} \tag{26}$$

$$\begin{aligned} \frac{dy_2}{d\tau} &= y_0 \phi_2 + y_1 \phi_1 + y_2 \phi_0 \\ \frac{dv_2}{d\tau} &= y_2 - k\alpha\left(\phi_0 v_1 + \frac{1}{2}\phi_0 v_0^2 + \phi_1 v_0 + \phi_2\right) \end{aligned} \tag{27}$$

$$\frac{d\phi_2}{d\tau} = \alpha\left(v_1 + \frac{1}{2} v_0{}^2 \right)$$

Since the solution for ϕ_0 is linear in τ, it is convenient to use this function as the new independent variable. For convenience of notation, let

$$x = \phi_0 \tag{28}$$

$$\delta = 2(1-\alpha) \tag{29}$$

Then, we have the system of equations for the first-order solution

$$\frac{dy_0}{dx} = -\frac{2}{\delta} y_0 x \tag{30a}$$

$$\frac{dv_0}{dx} = -\frac{2}{\delta} y_0 + \frac{2k\alpha}{\delta} x \tag{30b}$$

With the initial value of $x(0) = c$, we easily obtain the solution

$$y_0 = \exp\left(\frac{c^2 - x^2}{\delta} \right) \tag{31}$$

and

$$v_0 = -\frac{k\alpha}{\delta}\left(c^2 - x^2\right) + \sqrt{\frac{\pi}{\delta}} \exp\left(\frac{c^2}{\delta}\right)\left[\operatorname{erf}\left(\frac{c}{\sqrt{\delta}}\right) - \operatorname{erf}\left(\frac{x}{\sqrt{\delta}}\right)\right] \tag{32}$$

where the error function is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du \tag{33}$$

Notice that although $\phi_0 = -\sqrt{\beta r_0} \sin\gamma$ is the first-order solution for the flight path angle, we use it in the higher-order solution simply as the independent variable which, from the last equation in system (25) is related to the distance traveled by

$$\phi_0 = -\frac{\delta}{2}\tau + c \tag{34}$$

With $x = \phi_0$ as the new independent variable, we rewrite the systems (26) and (27).

$$\frac{d\phi_1}{dx} = -\frac{2\alpha}{\delta} v_0 \tag{35a}$$

$$\frac{dy_1}{dx} = -\frac{2}{\delta} y_0 \phi_1 - \frac{2}{\delta} y_1 x \tag{35b}$$

$$\frac{dv_1}{dx} = -\frac{2}{\delta} y_1 + \frac{2k\alpha}{\delta} \phi_1 + \frac{2k\alpha}{\delta} v_0 x \tag{35c}$$

$$\frac{d\phi_2}{dx} = -\frac{2\alpha}{\delta} v_1 - \frac{\alpha}{\delta} v_0{}^2 \tag{36a}$$

$$\frac{dy_2}{dx} = -\frac{2}{\delta} y_0 \phi_2 - \frac{2}{\delta} y_1 \phi_1 - \frac{2}{\delta} y_2 x \tag{36b}$$

$$\frac{dv_2}{dx} = -\frac{2}{\delta} y_2 + \frac{2k\alpha}{\delta} \phi_2 + \frac{2k\alpha}{\delta} \phi_1 v_0 + \frac{2k\alpha}{\delta} v_1 x + \frac{k\alpha}{\delta} v_0{}^2 x \tag{36c}$$

We obtain the second-order solutions by integrating the system (35) with null initial values for all the variables, and then, upon substituting the first- and second-order solutions in the system (36), we proceed in the same way to obtain the third-order solutions.

First, by using the solution (32) for v_0 in Eq. (35a), $\phi_1(x)$ is obtained by quadrature. It has been found that by direct integration by parts with an efficient use of the existing differential relations we can always cast the integration of a long expression into some simple integrals. For example we have

$$\begin{aligned} \phi_1 &= -\frac{2\alpha}{\delta} \int v_0 dx \\ &= -\frac{2\alpha}{\delta} v_0 x + \frac{2\alpha}{\delta} \int x\, dv_0 \\ &= -\frac{2\alpha}{\delta} v_0 x + \frac{2\alpha}{\delta} \int x\left(-\frac{2}{\delta} y_0 + \frac{2k\alpha}{\delta} x \right) dx \\ &= -\frac{2\alpha}{\delta} v_0 x + \frac{4k\alpha^2}{3\delta^2} x^3 + \frac{2\alpha}{\delta} \int dy_0 \\ &= -\frac{2\alpha}{\delta} v_0 x + \frac{4k\alpha^2}{3\delta^2} x^3 + \frac{2\alpha}{\delta} y_0 + C \end{aligned} \tag{37}$$

where C is a constant. Then, with the initial condition $\phi_1(0) = 0$, we have

$$\phi_1 = \frac{2\alpha}{\delta}\left(y_0 - v_0 x - 1\right) - \frac{4k\alpha^2}{3\delta^2}\left(c^3 - x^3\right) \tag{38}$$

Next, since the homogeneous equation for y_1 (and later on for y_2) is of the same form as the equation for y_0, we put

$$y_i = y_0 z_i \tag{39}$$

Then, Eq. (35b) for y_1 becomes

$$\frac{dz_1}{dx} = -\frac{2}{\delta}\phi_1 \tag{40}$$

with the initial condition $z_1(0) = 0$. We have the same remark as above when we obtain ϕ_1 by integration by parts with successive transformation of the integrals involved. The final result for y_1, after using this process to obtain z_1, is

$$y_1 = -\frac{2\alpha}{\delta^2} y_0 x\left[y_0 - v_0 x - 1 - \frac{4k\alpha}{3\delta}\left(c^3 - x^3\right)\right]$$
$$-\frac{k\alpha}{\delta^2} y_0\left[\frac{\alpha}{\delta}\left(c^4 - x^4\right) - \left(c^2 - x^2\right)\right] \tag{41}$$
$$+\frac{\alpha}{\delta} y_0\left[v_0 - \frac{2}{\delta}(c - x)\right]$$

By substituting v_0, ϕ_1 and y_1 in Eq. (35c) we obtain v_1 by quadrature using the same technique of integration by parts.

$$v_1 = -\frac{2\alpha}{\delta^2} y_0\left[y_0 - v_0 x - 1 - \frac{4k\alpha}{3\delta}\left(c^3 - x^3\right)\right]$$
$$+\frac{4k\alpha^2}{\delta^2} x\left[y_0 - v_0 x - 1 - \frac{2k\alpha}{3\delta}\left(c^3 - x^3\right)\right]$$
$$+\frac{2\alpha}{\delta^2} y_0\left[\frac{k\alpha}{2\delta}x^3 - \frac{k(4-\alpha)}{4}x + 1\right]$$
$$-\frac{2\alpha}{\delta^2}\left[\frac{k\alpha}{2\delta}c^3 - \frac{k(4-\alpha)}{4}c + 1\right]$$
$$-\frac{2\alpha}{\delta}\left[\frac{k\alpha}{2\delta^2}c^4 - \frac{k\alpha}{2\delta}c^2 + \frac{c}{\delta} + \frac{k(4-\alpha)}{8}\right] \tag{42}$$
$$\times\left[v_0 + \frac{k\alpha}{\delta}\left(c^2 - x^2\right)\right]$$
$$+\frac{\alpha}{\delta}v_0^2 + \frac{k\alpha}{\delta}v_0 x^2 + \frac{k^2\alpha^2}{2\delta^2}\left(c^4 - x^4\right)$$

The second-order solution for the speed is fairly accurate. To improve the accuracy we give below the third-order solution for the flight path angle and the altitude. We have

$$\phi_2 = \frac{2\alpha}{\delta}\left(y_1 - v_1 x - \phi_1 v_0 + \frac{2k\alpha}{\delta}\phi_1 x^2\right) - \frac{2k\alpha^2}{3\delta}\left(y_0 - 1\right)$$
$$+\frac{8k\alpha^2(2+\alpha)}{3\delta^3}\left[\frac{2k\alpha}{5\delta}\left(c^5 - x^5\right) + \left(c^2 - y_0 x^2\right) + v_0 x^3\right]$$
$$-\frac{4\alpha(1+\alpha)}{\delta^2}\sqrt{\frac{\pi}{2\delta}}\exp\left(\frac{2c^2}{\delta}\right)\left[\operatorname{erf}\left(\sqrt{\frac{2}{\delta}}c\right) - \operatorname{erf}\left(\sqrt{\frac{2}{\delta}}x\right)\right]$$
$$+\frac{2\alpha(1+\alpha)}{\delta^2}\left(2v_0 y_0 - v_0^2 x\right) \tag{43}$$

and

$$y_2 = \frac{1}{2}y_1 z_1 + \frac{k\alpha(1+\alpha)}{\delta}y_1 + \frac{\alpha}{\delta}y_0 v_1$$
$$-\frac{2\alpha}{\delta^2}y_0 x\left[y_1 - v_1 x - 2\phi_1 v_0 + \frac{k(2+5\alpha)}{2\delta}\phi_1 x^2\right]$$
$$-\frac{16k\alpha^2(2+\alpha)}{3\delta^4}y_0 x\left[\frac{2k\alpha}{5\delta}\left(c^5 - x^5\right) + \left(c^2 - y_0 x^2\right)\right]$$
$$-\frac{4k\alpha^2}{3\delta^2}y_0 x\left[\frac{4(2+\alpha)}{\delta^2}v_0 x^3 + 1\right]$$
$$+\frac{\alpha}{\delta^2}y_0 x\left[\frac{k(1+\alpha)}{\delta}\phi_1 - v_0^2 x + 2v_0 y_0 + \frac{k\alpha(2-5\alpha)}{3\delta}y_0\right]$$
$$-\frac{\alpha}{\delta^2}y_0\left[2\phi_1 y_1 - v_0^2 + y_0^2 - 1 - \frac{k\alpha(2-\alpha)}{\delta}c\right]$$
$$-\frac{k^2\alpha^3}{18\delta^3}y_0\left[\frac{4(6-5\alpha)}{\delta^2}\left(c^6 - x^6\right) - 3\left(c^4 - x^4\right)\right]$$
$$-\frac{k\alpha^3}{2\delta^2}y_0\left[v_0 + \frac{k\alpha}{\delta}\left(c^2 - x^2\right)\right]$$
$$-\frac{4\alpha(1+\alpha)}{\delta^3}y_0 x\left(2v_0 y_0 - v_0^2 x\right)$$
$$+\frac{8\alpha(1+\alpha)}{\delta^3}\sqrt{\frac{\pi}{2\delta}}\exp\left(\frac{2c^2}{\delta}\right)$$
$$\times y_0 x\left[\operatorname{erf}\left(\sqrt{\frac{2}{\delta}}c\right) - \operatorname{erf}\left(\sqrt{\frac{2}{\delta}}x\right)\right] \tag{44}$$

To test the accuracy of the solution, we consider the ballistic skip trajectory of a vehicle with $\overline{B} = B/E^* = 0.005/0.75$. Again, we emphasize the fact that it is sufficient to just provide a value for $\overline{B}$ as defined in Eq. (21). Table 1 shows the results for parabolic entry $u_e = 2$ with various entry angles, while in Table 2 we have the results for an entry angle of $\gamma_e = -3°$ but with various values of entry speed. Notice that $u_e = 1.733$ corresponds to an entry speed of a return from a geosynchronous orbit. When u_e tends to circular speed, the accuracy decreases. For pure ballistic entry without skip, we refer to Ref. 5 for the analytic solutions.

Table 1 Parabolic entry with ballistic factor $\bar{B} = (2/3)\times10^{-2}$.

		Entry Angle γ_e °		
		$\gamma_e = -2°$	$\gamma_e = -3°$	$\gamma_e = -4°$
Exit Arc Length θ_f	Numerical	0.139573	0.209516	0.283273
	1st-Order	0.139598	0.209344	0.279026
	2nd-Order	0.139590	0.209519	0.282674
	3rd-Order	0.139573	0.209516	0.283202
Exit Angle γ_f °	Numerical	1.998470	2.988717	3.880639
	1st-Order	2.000000	3.000000	4.000000
	2nd-Order	1.998960	2.988850	3.873140
	3rd-Order	1.998470	2.988710	3.879080
Exit Speed V_f/V_c	Numerical	1.412778	1.407836	1.369582
	1st-Order	1.142780	1.407850	1.370590
	2nd-Order	1.412780	1.407840	1.369650

Table 2 Entry with $\bar{B} = (2/3)\times10^{-2}$ and $\gamma_e = -3°$ with various entry speeds.

		Entry Speed u_e		
		$u_e = 2$	$u_e = 1.733$	$u_e = 1.36$
Exit Arc Length θ_f	Numerical	0.209516	0.248154	0.437479
	1st-Order	0.209344	0.247471	0.395427
	2nd-Order	0.209519	0.248148	0.415114
	3rd-Order	0.209516	0.248153	0.425422
Exit Angle γ_f °	Numerical	2.988717	2.972743	2.431356
	1st-Order	3.000000	3.000000	3.000000
	2nd-Order	2.988850	2.972830	2.340330
	3rd-Order	2.988710	2.972720	2.341270
Exit Speed V_f/V_c	Numerical	1.407836	1.306212	1.093156
	1st-Order	1.407850	1.306250	1.101740
	2nd-Order	1.407840	1.306210	1.095760

In planning a fly-through trajectory with a ballistic mode, either for a parabolic aero-capture or in a simple aeroassisted transfer, it is of interest to follow the deceleration and the heating build-up.

We consider the deceleration due to the drag force

$$\frac{a}{g} = -\frac{1}{g}\frac{dV}{dt} = \frac{\rho S C_D V^2}{2mg} \tag{45}$$

With $g \approx g_0$, we have, with the dimensionless variables selected,

$$\frac{a}{g_0} = \frac{\bar{B}}{2\alpha} y\, e^{-\eta v} = \frac{\bar{B}}{2} yu \tag{46}$$

By maximizing this quantity, we are led to the equation

$$\phi = \eta\left(y - k\alpha\phi\, e^{\eta v}\right) \tag{47}$$

Since the solutions at various orders are expressed in terms of $x = \phi_0$, we can solve for this value, and then use it to evaluate the critical value for the flight path angle $\phi_*(x)$, the altitude $y_*(x)$ and the speed $u_*(x)$ where the maximum deceleration occurs. A good approximation is $\phi_0 \approx 0$. Hence we can write Eq. (47) as

$$\phi_0 + \eta\phi_1 = \eta y_0 + \eta^2 y_1 \tag{48}$$

and use $x = 0$ in evaluating y_0, ϕ_1 and y_1. The result is

$$\begin{aligned}\phi_0 = &\frac{2\eta}{\delta}\left[(1-2\alpha)\exp\left(\frac{c^2}{\delta}\right) + \frac{2k\alpha^2}{3\delta}c^3 + \alpha\right] \\ &-\frac{\alpha\eta^2}{\delta}\left(\frac{k\alpha}{\delta^2}c^4 - \frac{k}{2}c^2 + \frac{2}{\delta}c\right)\exp\left(\frac{c^2}{\delta}\right) \\ &+\frac{\alpha\eta^2}{\delta}\sqrt{\frac{\pi}{\delta}}\exp\left(\frac{2c^2}{\delta}\right)\operatorname{erf}\left(\frac{c}{\sqrt{\delta}}\right)\end{aligned} \tag{49}$$

With this improved solution for ϕ_0, we recalculate ϕ_*, y_*, u_* and $(a/g_0)_{\max}$ to the second order. The results are shown in Table 3 for a parabolic entry at $\gamma_e = -4°$. The critical altitude as computed by the explicit analytical solutions differs from the numerical solution by a few meters.

Similarly, we consider the dimensionless time rate of average heat input per unit area as proportional to the density and to the cube of the speed, that is

$$\bar{q}_{av} = y\exp(-3\eta v/2) \tag{50}$$

On the other hand, the dimensionless heating rate at a stagnation point is assumed to be proportional to the square root of the density and to the cube of the speed, that is

$$\bar{q}_s = y^{1/2}\exp(-3\eta v/2) \tag{51}$$

By using the same approximation, with $\phi_0 \approx 0$, we obtain the improved solution from

$$\phi_0 = -\eta\phi_1 + \frac{3}{2}\eta y_0 + \frac{3}{2}\eta^2 y_1 \quad \text{for } (\bar{q}_{av})_{\max} \tag{52}$$

and

$$\phi_0 = -\eta\phi_1 + 3\eta y_0 + 3\eta^2 y_1 \quad \text{for } (\bar{q}_s)_{\max} \tag{53}$$

where $x=0$ has been used to evaluate y_0, ϕ_1 and y_1. We have the results summarized in Table 3, with the same accuracy as for the peak deceleration.

Table 3 Maximum values of a/g_0, $\bar{q}_{av}$ and $\bar{q}_s$ and the critical values for altitude, speed and flight path angle where they occur for parabolic and ballistic entry with $\bar{B}=(2/3)\times 10^{-2}$, $\gamma_e=-4^\circ$

		a/g_0	$\bar{q}_{av}$	$\bar{q}_s$
Max. Values	Numerical	0.520436	77.055576	8.612519
	Analytic	0.519438	76.908000	8.604290
s_*	Numerical	0.139033	0.138424	0.136605
	Analytic	0.139042	0.138450	0.136672
h_*	Numerical	-0.004871	-0.004870	-0.004868
	Analytic	-0.004869	-0.004868	-0.004866
$\sqrt{u_*}$	Numerical	1.395807	1.396034	1.396710
	Analytic	1.395800	1.396020	1.396690
$\gamma_*{}^\circ$	Numerical	-0.034012	-0.050997	-0.101771
	Analytic	-0.033598	-0.050133	-0.099768

Lifting skip trajectories

If the vehicle has some lifting capability, even with subcircular entry speed, it can negotiate a turn and exit from the atmosphere. A three-dimensional skip with a change in the heading has promising applications in aeroassisted orbital transfer with plane change. Again, here we restrict our analysis to the planar case, with the full set of equations given in system (18).

We recall that for the integration of the equations it is necessary to specify the maximum lift-to-drag ratio E^*, which is a performance characteristic, and the parameter B, which represents the entry altitude and vehicle characteristics. The entry speed is represented by the parameter α and the entry angle is specified by the value c. The flight program is specified by the normalized lift coefficient λ. The equations have been designed for the general case of lift modulation in which λ may vary as a function of the time.

From Eqs. (18) and by expanding the exponential function $e^{\eta v}$, we first neglect the effect of the variation in the speed to obtain

$$\frac{dy}{d\tau}=y\phi \tag{54a}$$

$$\frac{dv}{d\tau}=\left(1+\lambda^2\right)y-k\alpha\phi \tag{54b}$$

$$\frac{d\phi}{d\tau}=-k_1 y-(1-\alpha) \tag{54c}$$

where we define

$$k_1=B\lambda \tag{55}$$

By dividing the third equation by the first equation, we obtain the equation for ϕ^2 with the altitude as the independent variable.

$$\frac{d\phi^2}{dy}=-2k_1-2(1-\alpha)\frac{1}{y} \tag{56}$$

Upon integrating from $y(0)=1$, $\phi(0)=c$, we have the solution

$$\phi^2=c^2-2k_1\left(e^x-1\right)-2(1-\alpha)x \tag{57}$$

where, in this solution, we define the independent variable x as the linear variation of the altitude.

$$x=\log y=-\beta r_0 h \tag{58}$$

With this definition, Eq. (54a) and the solution (57) result in

$$d\tau=\frac{dx}{\phi}=\frac{dx}{\pm\sqrt{c^2-2k_1\left(e^x-1\right)-2(1-\alpha)x}} \tag{59}$$

To perform this quadrature, we approximate ϕ^2 as a trinomial in x, that is, by putting $\phi^2\approx\phi_a{}^2$, where

$$\phi_a{}^2(x)=a_1x^2+a_2x+a_3 \tag{60}$$

where the coefficients a_1, a_2 and a_3 are to be determined through a proper approximation scheme.
The solution can be obtained as

$$x=-\frac{a_2}{2a_1}-\frac{\sqrt{a_2{}^2-4a_1a_3}}{2a_1}\sin\left(\sqrt{-a_1}\,\tau-\beta\right) \tag{61}$$

where, for this solution, β is defined as

$$\sin\beta=\frac{a_2}{\sqrt{a_2{}^2-4a_1a_3}} \tag{62}$$

We have then obtained the first-order solution for the flight path angle as a function of the altitude variation in Eq. (57), and the solution for the altitude as a function of the range angle in Eq. (61). During the skip, x varies from zero to a maximum x_1, at the bottom of the trajectory, and this value is obtained by setting $\phi^2 = 0$ in Eq. (57). After passing through this maximum, x returns to the value zero at the exit.

Now, by using y in Eq. (54b) as obtained from Eq. (54c) and ϕ as obtained from Eq. (59), we have

$$\frac{dv}{d\tau} = -\frac{\left(1+\lambda^2\right)}{k_1}\frac{d\phi}{d\tau} - \frac{(1-\alpha)\left(1+\lambda^2\right)}{k_1} - k\alpha\frac{dx}{d\tau} \quad (63)$$

This can be easily integrated to give the first-order solution for the speed.

$$v = \frac{\left(1+\lambda^2\right)}{k_1}(c-\phi) - k\alpha x - \frac{(1-\alpha)\left(1+\lambda^2\right)}{k_1}\tau \quad (64)$$

Now, to improve the accuracy of the solution in the original system (18), we keep the η term in the expansion of the exponential function $e^{\eta v}$ to obtain

$$\frac{dy}{d\tau} = y\phi \quad (65a)$$

$$\frac{dv}{d\tau} = \left(1+\lambda^2\right)y - k\alpha\phi - \eta k\alpha\phi v \quad (65b)$$

$$\frac{d\phi}{d\tau} = -k_1 y - (1-\alpha) + \eta\alpha v \quad (65c)$$

Then, as before, by dividing Eq. (65c) by Eq. (65a) and using x instead of y as the independent variable, we have the equation for ϕ^2

$$\frac{d\phi^2}{dx} = -2k_1 e^x - 2(1-\alpha) + 2\eta\alpha v \quad (66)$$

If we use the solution (64) for v for the perturbation which is the last term in Eq. (66), we can integrate this equation term by term to obtain the improved solution for ϕ^2. We have

$$\phi^2 = c^2 - 2k_1\left(e^x - 1\right) - 2(1-\alpha)x + 2\eta\alpha I(x) \quad (67)$$

where the integral $I(x)$ is defined as

$$\begin{aligned} I(x) &= \int_0^x v\,dx \\ &= \frac{\left(1+\lambda^2\right)}{k_1}cx - \frac{1}{2}k\alpha x^2 - \frac{(1-\alpha)\left(1+\lambda^2\right)}{k_1}\tau x \end{aligned} \quad (68)$$

$$-\frac{\left(1+\lambda^2\right)}{k_1}I_1(x) + \frac{(1-\alpha)\left(1+\lambda^2\right)}{k_1}I_2(x)$$

and the integrals $I_1(x)$ and $I_2(x)$ are respectively defined as

$$\begin{aligned} I_1(x) &= \int_0^x \phi\,dx = \int_0^x \phi_a\,dx \\ &= \frac{1}{2}\phi_a x - \frac{a_2}{4a_1}\left(c-\phi_a\right) + \frac{\left(4a_1a_3 - a_2{}^2\right)}{8a_1}\tau \end{aligned} \quad (69)$$

and

$$\begin{aligned} I_2(x) &= \int_0^x \frac{x\,dx}{\phi} \approx \int_0^x \frac{x\,dx}{\phi_a} \\ &= -\frac{1}{a_1}\left(c-\phi_a\right) - \frac{a_2}{2a_1}\tau \end{aligned} \quad (70)$$

The solution (67), with the integrals as defined, is the second-order solution for the flight path angle ϕ.

We proceed likewise with Eq. (65b) for the speed variable v by replacing on its right-hand side ϕ and y obtained from Eq. (65a) and (65c) respectively. We have then

$$\begin{aligned} \frac{dv}{d\tau} &= -\frac{\left(1+\lambda^2\right)}{k_1}\frac{d\phi}{d\tau} - \frac{(1-\alpha)\left(1+\lambda^2\right)}{k_1} \\ &+ \frac{\eta\alpha\left(1+\lambda^2\right)}{k_1}v - k\alpha\frac{dx}{d\tau} - \eta k\alpha v\frac{dx}{d\tau} \end{aligned} \quad (71)$$

On the right-hand side, the terms with the factor ηv are the perturbations. Hence, it is appropriate to use the solution (64) for these terms, and then the second-order solution for the speed variable v is obtained by integration of Eq. (71) term by term. We have

$$\begin{aligned} v &= \frac{\left(1+\lambda^2\right)}{k_1}(c-\phi) - k\alpha x - \frac{(1-\alpha)\left(1+\lambda^2\right)}{k_1}\tau \\ &- \eta k\alpha I(x) + \frac{\eta\alpha\left(1+\lambda^2\right)}{k_1}J(x) \end{aligned} \quad (72)$$

where the integral $J(x)$ is defined as

$$\begin{aligned} J(x) &= \int_0^\tau v\,d\tau \\ &= \frac{\left(1+\lambda^2\right)}{k_1}c\tau - \frac{\left(1+\lambda^2\right)}{k_1}x \end{aligned} \quad (73)$$

$$-\frac{(1-\alpha)\left(1+\lambda^2\right)}{2k_1}\tau^2 - k\alpha I_2(x)$$

and $I(x)$ and $I_2(x)$ are defined in Eqs. (68) - (70) above.

To complete this section, we recall that in obtaining τ and the integrals $I(x)$ and $J(x)$ we have approximated the exact expression for the first-order solution in ϕ^2, namely

$$\phi_0{}^2(x) = c^2 - 2k_1\left(e^x - 1\right) - 2(1-\alpha)x \qquad (74)$$

as the trinomial (60). In the diagrams plotting $\phi_0{}^2(x)$ and $\phi_a{}^2(x)$ from $x = 0$ to its maximum value x_1, we match the end-points by setting $\phi_0{}^2(0) = c^2 = \phi_a{}^2(0)$, and $\phi_0{}^2(x_1) = 0 = \phi_a{}^2(x_1)$. Furthermore, we equate the areas under the curves by setting

$$\int_0^{x_1} \phi_0{}^2(x)\,dx = \int_0^{x_1} \phi_a{}^2(x)\,dx \qquad (75)$$

This leads to the solution for the coefficient a_i of the approximate trinomial

$$a_1 = \frac{3}{x_1}\left[2(1-\alpha) - \frac{\left(c^2+4k_1\right)}{x_1} + \frac{4k_1\left(e^{x_1}-1\right)}{x_1{}^2}\right]$$

$$a_2 = -6(1-\alpha) + \frac{2\left(c^2+6k_1\right)}{x_1} - \frac{12k_1\left(e^{x_1}-1\right)}{x_1{}^2} \qquad (76)$$

$$a_3 = c^2$$

where, by definition, we compute x_1 from

$$c^2 - 2k_1\left(e^{x_1}-1\right) - 2(1-\alpha)x_1 = 0 \qquad (77)$$

For high lift-to-drag ratio skip trajectories, the analytic solutions with the approximation used for the integrals are fairly accurate. But a small problem arises for entry with low C_L/C_D. This is because, in the first-order solution, x_1 is the maximum altitude drop when $\phi_0{}^2(x_1) = 0$, and it is slightly smaller than the more accurate value x_2, when the second-order solution is considered. Since the positive function $\phi_a{}^2(x)$ also vanishes with x_1, it becomes negative when $x > x_1$. This defect is remedied as follows. In the second-order solution (67) for ϕ^2, we hold the perturbed function $2\eta\alpha I(x)$ constant at its values at x_1, for $x > x_1$, and compute x_2 approximately from the equation

$$\begin{gathered} c^2 - 2k_1\left(e^{x_2}-1\right) - 2(1-\alpha)x_2 \\ +2\eta\alpha I\left(x_1, \tau(x_1), \phi_a(x_1)\right) = 0 \end{gathered} \qquad (78)$$

With this improved value x_2, we recalculate the coefficients a_i of the trinomial exactly as described above. Therefore we have the system (76) to calculate the new values a_i by simply replacing x_1 by x_2.

As numerical example, we take the value $B = 0.005$ with various maximum lift-to-drag ratios E^*. For the lift coefficient, we take $\lambda = 1$. Then for this constant angle-of-attack skip entry, it is the same as for the skip with E^* as the lift-to-drag ratio. In Fig. 1 we plot the variation of the altitude as function of the flight path angle for an entry at $u_e = 1.2$ and for two values $\gamma_e = -4°$ and $\gamma_e = -6°$, using a vehicle with $E^* = 0.75$. We have the same two cases in Fig. 2 plotting the altitude versus the speed ratio. The legend is obvious. The continuous lines denote the numerical solution while the triangles represent the first-order solution and the diamonds are for the second-order solution.

In Figs. 3 and 4, we have $E^* = 0.75$ and $\gamma_e = -4°$ but with three values of the entry speed. For clarity, only the second-order solutions are plotted for comparison with the numerical solutions.

Finally in Figs. 5 and 6, we consider the cases of skip entry $u_e = 1.2$ and $\gamma_e = -4°$ but with several values of lift-to-drag ratio.

As shown in the figures, the analytic solution can be used for a good prediction of the exit speed and the exit angle. Since the solutions are expressed explicitly in terms of the variable $x = \log(\rho/\rho_0)$, it suffices to put $x = x_f = 0$ to have the final values at exit.

As for the ballistic entry, the analytic solutions for skip entry can be used to compute the deceleration and the various heating rates with excellent agreement as compared to the numerical solutions. Let us simply consider the tangential deceleration due to the drag force

$$\frac{a}{g_0} = \frac{\left(1+\lambda^2\right)B}{2E^*} yu \qquad (79)$$

Considered as a function of x, we have

$$\frac{a}{g_0} = \frac{\left(1+\lambda^2\right)B}{2\alpha E^*} \exp(x - \eta v) \qquad (80)$$

where of course v is a function of x. By maximizing this function and using Eqs. (54), we obtain the equation

$$(1+\eta k\alpha)\phi = \eta(1+\lambda^2)e^x \tag{81}$$

Since $\phi(x)$ is known, this equation can be solved for the critical value x_* of the altitude where the peak deceleration occurs. Then we also have the speed $v_*(x)$ and hence $(a/g_0)_{max}$. As a display of the accuracy of the solution, since the peak deceleration occurs during the descending phase and near the bottom of the trajectory, we consider approximately

$$\phi_* = \frac{\eta(1+\lambda^2)}{(1+\eta k\alpha)}e^{x_1} \tag{82}$$

where x_1 is the value which makes $\phi_0(x)$ vanish. For the case where $E^* = 0.75$, $u_e = 1.2$ and $\gamma_e = -4°$, this gives $\gamma_* = -0.215472°$, as compared with the exact value of -0.221187°. Therefore the value $x = x_1$ is a good starting value for the iteration to obtain the critical value x_*.

Equilibrium glide trajectories

If the entry speed is nearly circular, with a very small starting flight path angle, the forces along the radial line may be in near perfect equilibrium and the vehicle follows the curvature of the earth with nearly zero flight path angle. This is the equilibrium glide trajectory and it is conceived to achieve long range.

From the system (18) with $k_1 = B\lambda$, we use the first equation to eliminate the arc length τ.

$$\frac{dv}{dy} = \frac{(1+\lambda^2)}{\phi} - \frac{k\alpha e^{\eta v}}{y}$$
$$\frac{d\phi}{dy} = -\frac{k_1}{\phi} + \frac{(\alpha e^{\eta v}-1)}{y\phi} \tag{83}$$

Next, we use the relation $\alpha u = \exp(-\eta v)$ to write the equation for the speed as

$$\frac{\eta(1+\lambda^2)u}{\phi} = -\frac{1}{y'} + \frac{\eta k}{y} \tag{84}$$

where the prime denotes the derivative with respect to u. Similarly, by noticing that $d\phi/dy = \phi'/y'$, we can write the equations for the flight path angle, using Eq. (84) for y' as

$$y\left[k_1 - \eta(1+\lambda^2)u\phi'\right] = \frac{(1-u)}{u} - \eta k\phi\phi' \tag{85}$$

The last two equations constitute a system of two non-linear equations for y and ϕ with u as independent variable. We choose the iterative technique to construct higher order solutions for glide trajectories.

First, in Eq. (85), by setting $\phi' \approx 0$, we have the first-order solution for the altitude density ratio as function of the speed, within the assumption of equilibrium glide.

$$y_0 = \frac{(1-u)}{k_1 u} \tag{86}$$

Next, by using y_0 and $y_0' = -1/k_1u^2$ on the right-hand-side of Eq. (84) in place of y and y', we have the first-order solution for the flight path angle.

$$\phi_0 = \frac{2(1-u)}{E\sqrt{\beta r_0}\left[\frac{2}{\beta r_0} + u(1-u)\right]} \tag{87}$$

In the derivation, from the previous definitions of the constant parameters, we have used the following relations

$$\eta(1+\lambda^2) = \frac{B}{E^*\sqrt{\beta r_0}} \times \frac{2C_D}{C_D^*} = \frac{2k_1}{E\sqrt{\beta r_0}} \tag{88}$$

and

$$\eta k = \frac{B}{E^*\sqrt{\beta r_0}} \times \frac{2E^*}{\sqrt{\beta r_0}B} = \frac{2}{\beta r_0} \tag{89}$$

Hence, in the solution, E is simply the lift-to-drag ratio used for the glide.

Again, we use ϕ_0 and ϕ_0' in place of ϕ and ϕ' in Eq. (85) to solve for the second-order solution in y. We obtain

$$y_1 = y_0\frac{(1+AB)}{(1+B)} \tag{90}$$

where, by definition,

$$w = \frac{2}{\beta r_0 u(1-u)} \tag{91}$$

$$A = \frac{w}{1+w} \tag{92}$$

$$B = \frac{2w[1-(1-w)u]}{E^2(1+w)^2} \tag{93}$$

The solution for y_1 is displayed in the form to show that it is a small perturbation away from y_0 since the function w of the speed is small. This is because $\beta r_0 = 900$ and hence although $(1-u)$ is small at entry and u is small at the end, the term w is never large and always remains small throughout. For example, if we take $u_e = 0.995536$, w would vary from 0.5 at entry to a minimum of 0.0089 when $u = 0.5$ and return to 0.5 at $u_f = 0.004464$, a final speed value which corresponds to a 1.54 Mach number. If the solution (90) for y_1 and its derivative y_1' are substituted on the right-hand-side of Eq. (84) in place of y and y', we can finally solve for ϕ to give

$$\phi_1 = \phi_0 \frac{(1+AB)}{(1+B)} \frac{(1+w)}{D} \tag{94}$$

where

$$D = w + \frac{1}{1 + \frac{\left[\begin{array}{l} B(1-2u)(3A+AB-2) \\ + \frac{2A(A-1)}{E^2}(1-u)(3u-1-wu) \end{array}\right]}{(1+B)(1+AB)(1+w)}} \tag{95}$$

The equilibrium glide mode was first investigated by Sänger and Bredt[6] and it is now conceived as the flight program for the future aerospace plane to achieve long range. To calculate the distance traveled, we consider the equation for the speed, with the variable u

$$\frac{du}{d\tau} = -\eta\left(1+\lambda^2\right)yu + \eta k\phi \tag{96}$$

By using the first-order solutions (86) and (87) for y and ϕ, this gives

$$\frac{du}{d\tau} = -\frac{2u(1-u)^2}{E\sqrt{\beta r_0}\left[\eta k + u(1-u)\right]} \tag{97}$$

We obtain τ and hence the range angle by simple quadrature

$$\theta = \frac{1}{2}E\left[\begin{array}{l} \log\frac{(1-u)}{(1-u_e)} + \frac{2}{\beta r_0}\log\frac{u_e}{(1-u_e)}\frac{(1-u)}{u} \\ + \frac{2}{\beta r_0}\left(\frac{1}{1-u_e} - \frac{1}{1-u}\right) \end{array}\right] \tag{98}$$

In this expression, the first term is the classical solution[6] while the other terms with the coefficient $2/\beta r_0$ are the correctional terms in the present theory.

As numerical application, we have selected a typical hypersonic glider with $E^* = 1.5$ and $B = 0.005$.

Figure 7 plots the variation of the altitude versus the speed ratio. The dashed line presents the case of the ideal equilibrium glide. This is defined as such that we have exact equilibrium of the forces at the entry point, and hence we have $\phi'(u_e) = 0$. This leads to using the following initial conditions

$$u_e = \frac{1}{1+k_1} \tag{99}$$

and

$$\sin\gamma_e = -\frac{2k_1(1+k_1)}{E\left[2(1+k_1)^2 + \beta r_0 k_1\right]} \tag{100}$$

With our example vehicle, we have then $u_e = 0.995025$ and $\gamma_e = -0.058877°$. With these initial conditions, the numerical solutions for h and γ and their analytical solutions are nearly the same. For example, in Fig. 7, in checking the final point where $V_f = 0.1\ V_c$, which corresponds to a speed of 2.3 Mach number we have the approximate altitude drop of $h_f = -0.010882$ as compared to the exact value of $h_f = -0.010894$. With a value r_0 of 6478 km, this represents an altitude deviation of only 78 meters. Figure 8 plots the variation of the flight path angle versus the speed ratio. At the final point, the approximate solution (94) gives a flight path angle of $\gamma_f = -6.07819°$ as compared to the value from numerical integration of $\gamma_f = -6.143445°$ Within the scales used in Figs. 7 and 8, the dashed lines also represent the numerical solutions for the case of the perfect equilibrium of the forces at the starting point. It is known that if the initial speed is higher, the vehicle will skip out. If the entry speed u_e is smaller than the ideal speed for equilibrium glide, the trajectory exhibits oscillations as shown in the solid lines in Figs. 7 and 8 for the case where $u_e = 0.95$. In this case, the equilibrium glide theory only produces the average solution.

Finally, we have plotted in Fig. 9, the variation of the speed versus the range angle. The three curves are respectively, the classical solution[6], the present solution, and the numerical solution. A check at the final point shows an approximate range angle of $\theta_f = 4.31975$ as compared to the exact solution of $\theta_f = 4.36003$. The deviation is 256 km, for a glide of more than half the circumference of the earth.

Conclusions

In this paper, accurate analytical solutions for the three typical entry trajectories have been obtained. Starting with the equations of motion for planar entry around a non-rotating planet, we have used normalization to put the equations into a form which is suitable for an analytic integration.

The solutions for ballistic fly-through trajectories, which are of special importance in aeroassisted transfer are obtained to the third order. The case of lifting skip trajectories are then analyzed in detail and accurate analytical solutions are obtained. Finally, an improved theory for the equilibrium glide of a future aerospace plane is proposed. In each of the problems examined, explicit relationships between the altitude, speed, flight path angle and distance traveled are obtained and they are in a form suitable for guidance purposes.

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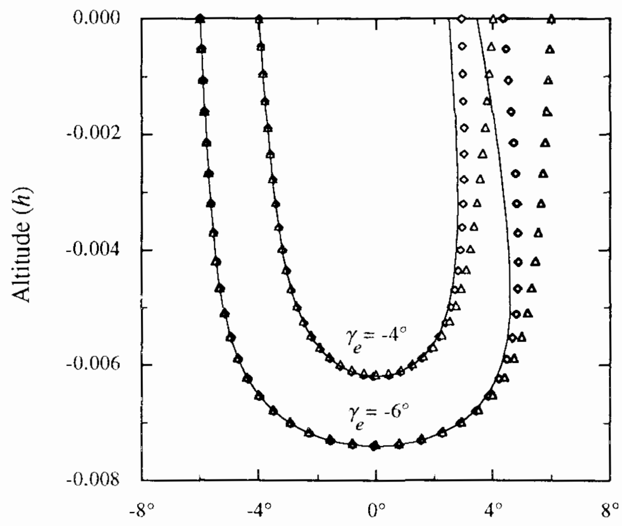

Fig. 1 Altitude versus flight path angle for skip entry at $u_e = 1.2$ with $E^* = 0.75$.

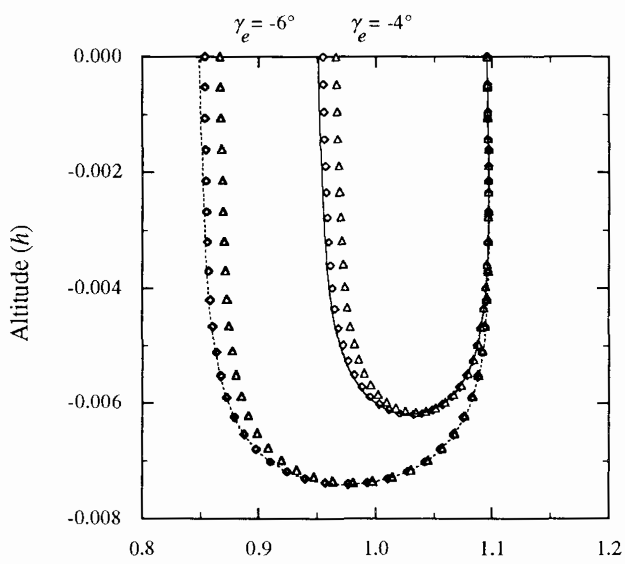

Fig. 2 Altitude versus speed for skip entry at $u_e = 1.2$ with $E^* = 0.75$.

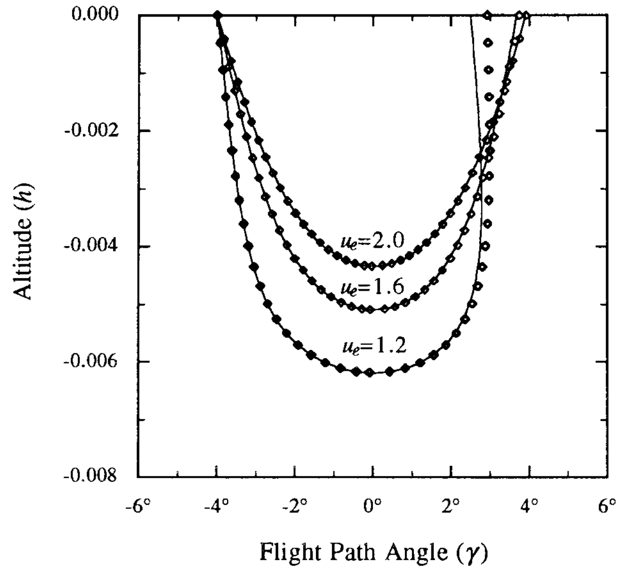

Fig. 3 Altitude versus flight path angle for skip entry at $\gamma_e = -4°$ with $E^* = 0.75$.

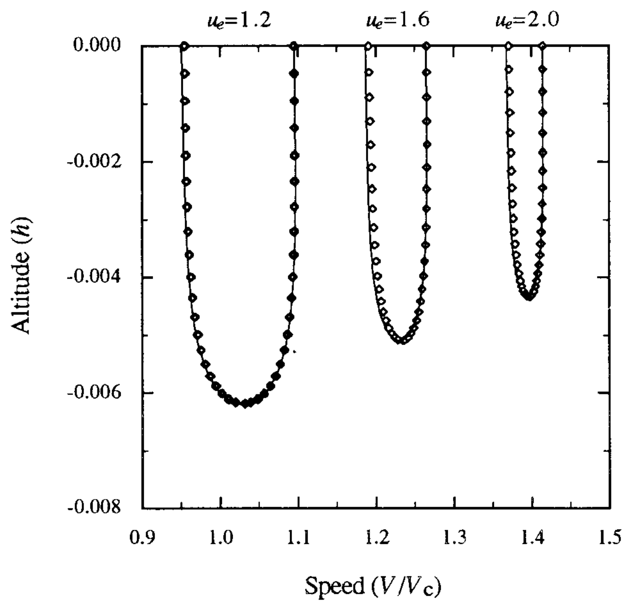

Fig. 4 Altitude versus speed for skip entry at $\gamma_e = -4°$ with $E^* = 0.75$.

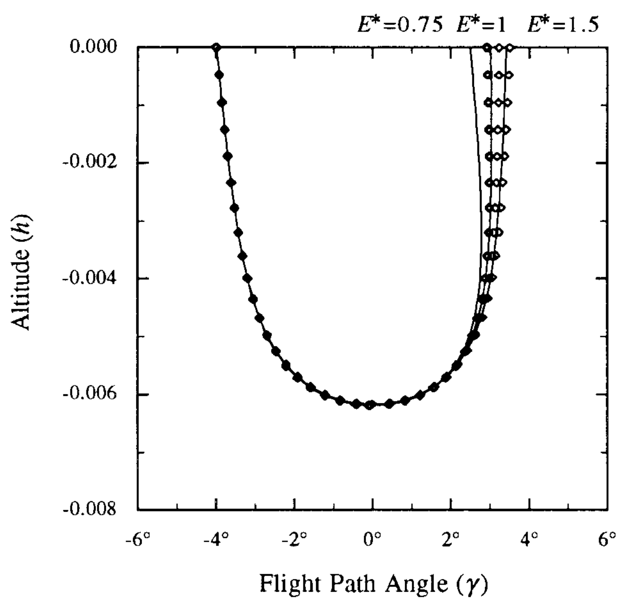

Fig. 5 Altitude versus flight path angle for skip entry at $u_e = 1.2$ and $\gamma_e = -4°$.

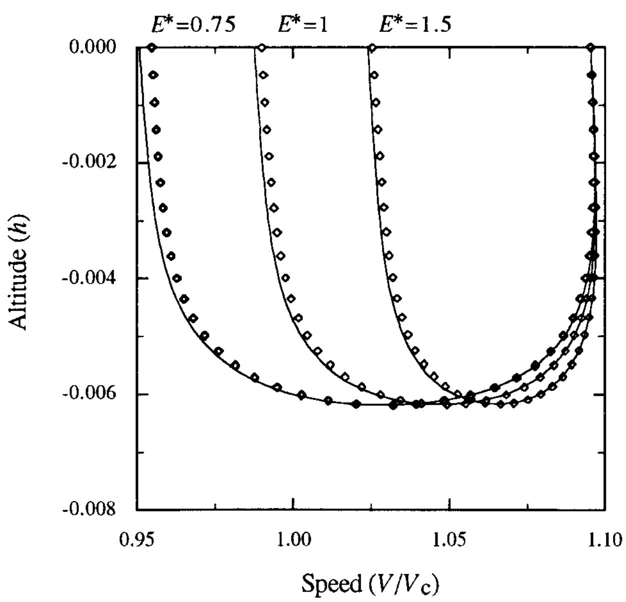

Fig. 6 Altitude versus speed for skip entry at $u_e = 1.2$ and $\gamma_e = -4°$.

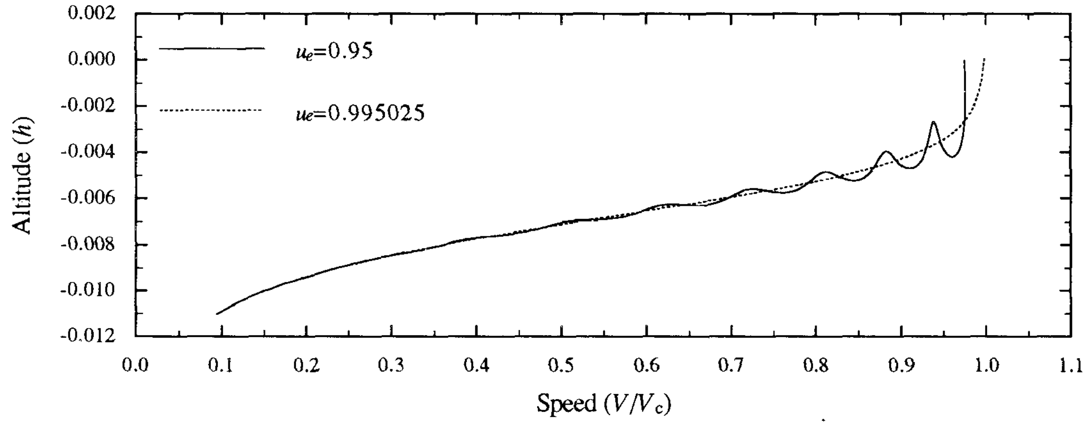

Fig. 7 Altitude versus speed for glide entry with $E^* = 1.5$ and $B = 0.005$.

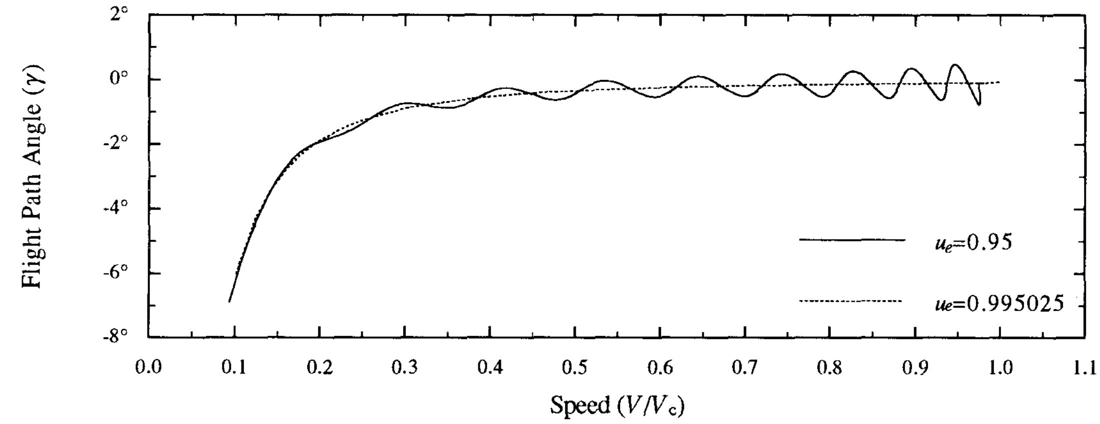

Fig. 8 Flight path angle versus speed for glide entry with $E^* = 1.5$ and $B = 0.005$.

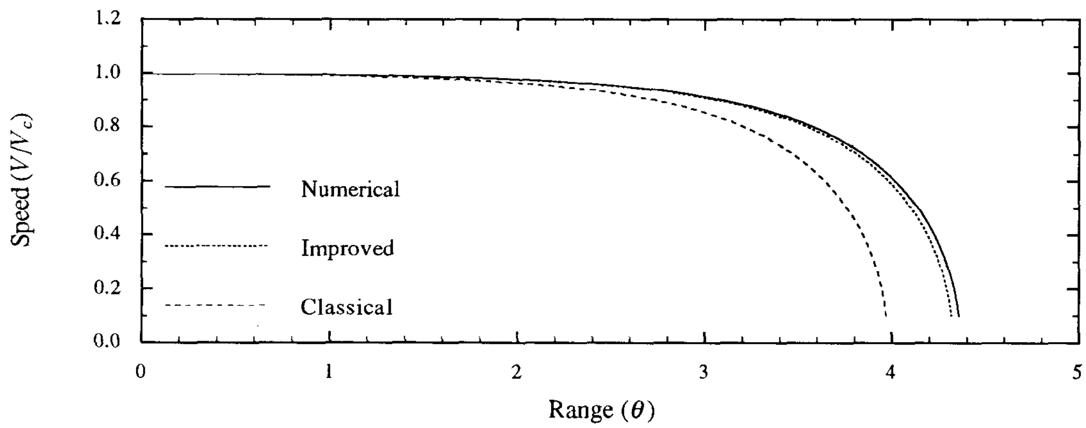

Fig. 9 Speed versus range for equilibrium glide with $E^* = 1.5$ and $B = 0.005$.