HOW MUCH TO STOCK TO MEET A STREAM OF DEMANDS

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Abstract

We consider a cyclic inventory system in which replenishment and selling capacities may be uncertain. Moreover, in the system, selling revenues, inventory costs and demand distributions are assumed to vary seasonally. The objective is to decide optimal selling and replenishment strategy maximizing the expected revenue of the system. The structure of the optimal policy is shown to have a single critical number in any period. We also show the relationship between the critical numbers in two consecutive periods.

1 Introduction

In many inventory systems, prices, costs and demands vary seasonally. With this seasonal variation major problems arise, such as how to replenish commodities, how to manage stocks and how to meet demands. Moreover, in recent years, these problems become more and more complicated in involving uncertain replenishment and selling capacities due to increasingly sophisticated market requirements. These uncertainties are influenced by many factors such as machine breakdowns, insufficiency of labor, limitations of transportation capabilities, etc.

In this paper we restrict our attention to the systems dealing with a single commodity. Karlin (1960a,b) treats this problem with a number of periods of equal duration. In his model, demands are allowed to vary periodically, but with identical costs. At the beginning of each period, an ordering decision is made *before* observing the demand. The ordering capacity is always "*perfect*", i.e., the ordering output quantity is always equal to ordering request. Then, the demand is satisfied as long as stock is available, i.e., there is no control on stock after ordering. This implies that the selling capacity is also *perfect*, i.e., selling a stock to satisfy an existing demand is always possible. He shows that the optimal ordering policy has a single critical number. Zipkin (1989) extends Karlin's results to the case of cyclic costs as well as demands.

Karlin's and Zipkin's models are very useful for managing many inventory systems; however, the ordering (or producing) capability of some products, such as corn, is only available once in certain periods, e.g., 12 months. Moreover, the producing and selling capacities in some systems are not certain. For example, the uncertainties in pumped-storage hydroelectric systems may appear in both pumping (producing) and generating (selling) modes due to leaking tunnels, pump/turbine breakdowns, shortage of water resource,..., and so on. In this paper, we consider such a system involving uncertain capacities and seasonal demands as well as costs. In our model, there is only one chance to replenish the inventory in certain periods instead of every period in Karlin's and in Zipkin's models. Moreover, under uncertain producing and selling capacities, two decisions are made *after* observing demand: *a*) how much to order in certain periods, *b*) how much demand to meet, i.e., how much stock to be leftover, in every period. In the next section we will describe and formulate this problem in an infinite horizon case with certain capacities. Then, we also show that the form of the optimal decision(s) is based on a single critical number in each period. In Section 3 we discuss the model dealing with uncertain capacities and show that a single-critical-number policy is still optimal for both multi-cycle and infinite horizon cases.

2 Perfect Capacities

Here, we discuss an inventory system in which demands, selling prices and holding costs of a certain type of commodity repeat every n periods. We consider any consecutive n periods as a cycle. Within a cycle, period n is the first period and period 1 is the last. In each cycle there is only one chance to replenish the inventory through an ordering process. The ordering and selling capacities are assumed to be *perfect.* Without loss of generality, let the purchasing period be the first period in a cycle.

Under the cyclic behavior of demand, price and cost structures, we may define all system parameters within a cycle assuming the optimal strategy also repeats every cycle. Later in this section we will verify this assumption. Now, let I_k be the inventory level at the beginning of period k. In period k, let $0 < \alpha_k < 1$ be the discount factor and let $Q_k(\xi_k)$ be the demand distribution with p.d.f. $q_k(\xi_k)$. The demand distributions in different periods are mutually independent.

At the beginning of period k, a planned initial inventory level, u_k, for the next period has to be decided upon based on the observation of ξ_k and I_k. In other words, at period $k \neq n$, u_k is the planned leftover stocks, and at period n, $(u_n - \xi_n - I_n)^+$ is the purchasing quantity where $(d)^+ = \max\{0, d\}$ for any real value d. Notice that the quantity of the actual leftover stock is $\max\{u_k, I_k - \xi_k\}$. Let u_k^* denote the optimal value of u_k.

Several costs and revenues are incurred during each cycle: *a*) a selling revenue, π_k, is associated with each unit of the satisfied demand in period k (all the unsatisfied demands are presupposed to be lost), *b*) a purchasing cost, w $(= \pi_0)$, is associated with each unit of the replenished inventory in period n, and *c*) a holding cost, h_k, is related to each unsold unit of the stock at the period k. All the costs and revenues are assumed to be non-negative. Naturally, the purchasing cost w must be smaller than any selling revenue π_k. Let Δ_k represent the marginal revenue within a cycle if we *always* can sell a product in period k instead of in the next period, i.e., $\Delta_k = \pi_k + h_k - \alpha_k \pi_{k-1}$ for $k \neq n$ and $\Delta_n = w + h_n - \alpha_n \pi_{n-1}$.

Now, we should recursively define the revenue functions within a cycle. (For convenience, we shall neglect the demand in the purchasing period. Then, let $\xi_n = 0$ with probability 1. Later in this section, we will discuss how to incorporate this demand.) Let $R_k(I_k, \xi_k)$ be the expected revenue of selling I_k items optimally from period k through period 1, based on the observation of the current demand ξ_k and the inventory level I_k at period k. Then, $R_n(I_n, \xi_n)$ represents the maximum expected revenue to operate a cycle, given initial inventory I_n at the beginning of the cycle. Therefore, $I_n \leq u_n$ in period n and $(I_k - \xi_k)^+ \leq u_k \leq I_k$ in period $k \neq n$. Therefore, $R_k(I_k, \xi_k)$ satisfies the functional equations:

$$R_k(I_k, \xi_k) \quad = \max_{(I_k - \xi_k)^+ \leq u_k \leq I_k} \gamma_k(I_k, u_k), \qquad k \neq n$$

$$R_n(I_n, \xi_n) = \max_{I_n \le u_n} \gamma_n(I_n, u_n),$$

where

$$\gamma_1(I_1, u_1) = \pi_1(I_1 - u_1) - h_1 u_1 + \alpha_1 w u_1,$$

$$\gamma_k(I_k, u_k) = \pi_k(I_k - u_k) - h_k u_k + \alpha_k E_{\xi_{k-1}}[R_{k-1}(u_k, \xi_{k-1})], \qquad k = 2, ..., n-1, \tag{1}$$

$$\gamma_n(I_n, u_n) = -w(u_n - I_n) - h_n u_n + \alpha_n E_{\xi_{n-1}}[R_{n-1}(u_n, \xi_{n-1})]. \tag{2}$$

Now, we shall prove that there exists a sequence of critical numbers $S_1, S_2, ..., S_n$ such that the system is operating optimally. Furthermore, the form of the optimal policy is stock-up-to at period $k \neq n$ (see Figure 1):

$$u_k^* = \begin{cases} I_k & \text{if } I_k < S_k \\ S_k & \text{if } S_k \le I_k \le \xi_k + S_k \\ I_k - \xi_k & \text{if } I_k > \xi_k + S_k, \end{cases} \tag{3}$$

and order-up-to at period n:

$$u_n^* = \begin{cases} S_n & \text{if } I_n < S_n \\ I_n & \text{otherwise.} \end{cases} \tag{4}$$

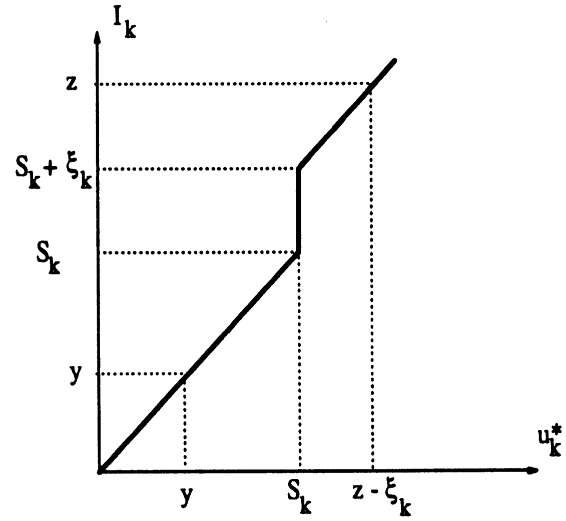

Figure 1: Optimal Policy at Period k

The optimal policy in ordering period says that if the inventory level drops below the critical number S_n, then the inventory level should be filled up to the number; otherwise, no order should be placed. The optimal policy in the other periods says that a) if the inventory level I_k drops below the critical number S_k, then keep all the stock to the next period, i.e., do not satisfy any demand, b) if I_k is in between S_k and S_k plus observed demand ξ_k, then keep exact S_k units of inventory to the next period, i.e., satisfy $I_k - S_k$ units of demand, and c) if stock on hand is more than the critical number plus observed demand, then satisfy all the demand, i.e., keep $I_k - \xi_k$ units of inventory to the next period.

In order to show the optimal policies with the form (3) and (4), we shall prove that all the revenue functions are concave by induction. Hence, consider the last period in a cycle. Taking the first derivative of $\gamma_1(I_1, u_1)$ with respect to u_1, we have

$$\frac{\partial \gamma_1(I_1, u_1)}{\partial u_1} = -\Delta_1 < 0.$$

Since marginal revenue is decreasing in holding any extra stock, we can define $S_1 = 0$, i.e., we are not trying to leave over anything to the next cycle. Therefore, the optimal policy is (3). As a result, we have

$$R_1(I_1, \xi_1) = \begin{cases} \gamma_1(I_1, S_1) & \text{if } S_1 \le I_1 \le \xi_1 + S_1 \\ \gamma_1(I_1, I_1 - \xi_1) & \text{if } I_1 > \xi_1 + S_1. \end{cases}$$

Hence, differentiating $R_1(I_1, \xi_1)$ with respect to I_1, we have

$$\frac{\partial R_1(I_1, \xi_1)}{\partial I_1} = \begin{cases} \pi_1 & \text{if } S_1 \le I_1 \le \xi_1 + S_1 \\ -h_1 + \alpha_1 w & \text{if } I_1 > \xi_1 + S_1. \end{cases}$$

Clearly, $R_1(I_1, \xi_1)$ is concave in I_1.

Trying to show the concavity of revenue function $\gamma_k(I_k, u_k)$ in decision variable and optimal revenue function $R_k(I_k, \xi_k)$ in inventory for all periods, let us recall a useful lemma from Iglehart (1965):

LEMMA 1 *Let*

$$N(x) = \min_{y \in D(x)} \{M(x, y)\} = M[x, y_0(x)] \qquad y_0(x) \in D(x) \text{ and } x \in C,$$

where x and y are k-dimensional vectors, $N(\cdot)$ and $M(\cdot,\cdot)$ are real-valued functions, and $D(x)$ is some domain in k-dimensional space which depends on x. If $M(x, y)$ is a convex function of the

2k-dimensional vector $z = (x, y)$, C is convex, and the set $\{(x, y) : y \in D(x)\}$ is convex, then $N(x)$ is convex in $x \in C$.

THEOREM **2** *For all periods,*

1. *$\gamma_k(I_k, u_k)$ is concave in u_k.*

2. *$R_k(I_k, \xi_k)$ is concave in I_k.*

Proof: We have shown that $\gamma_1(I_1, u_1)$ is concave in u_1 and $R_1(I_1, \xi_1)$ is concave in I_1. Notice that the first two terms in the right-hand-sides of (1) and (2) are linear in u_k. By induction, suppose that $R_{k-1}(I_{k-1}, \xi_{k-1})$ is concave in I_{k-1}. Then, $\gamma_k(I_k, u_k)$ is concave in u_k. Hence, $R_k(I_k, \xi_k)$ is concave in I_k by Lemma 1. Similarly, $R_n(I_n, \xi_n)$ is also concave in I_n. ■

By concavity, define S_k such that $\frac{\partial \gamma_k(I_k, u_k = S_k)}{\partial u_k} = 0$ for $k = 2$ to n, i.e., for given I_k, $\gamma_k(I_k, u_k)$ has the maximum at $u_k = S_k$. Then, we shall show that the critical number S_k is independent of inventory level.

THEOREM **3** *The critical number S_k is independent of I_k for all k.*

Proof: We know that $S_1 = 0$ for all I_1. For $k = 2$ to n, it is clear from (1) and (2) that $\frac{\partial \gamma_k(I_k, u_k)}{\partial u_k}$ is not a function of I_k. QED. ■

Since cost function $\gamma_k(I_k, u_k)$ is concave in decision variable with maximum at $u_k = S_k$, the optimal policy can be described by the maximal S_k. Moreover, in considering the restriction of the decision variable, the form of the optimal policy is (3) and (4) since S_k is independent of the inventory level.

The optimal policy in ordering period indicates that we will never order if and only if $S_n < I_n$. Suppose that $I_n > S_n$. Then, within a finite number of cycles, I_n will be less than S_n by satisfying cyclic demand streams. Consequently, in an infinite horizon problem, the initial inventory I_{n-1} at the beginning of period $n - 1$ is always equal to S_n and represents a renew point in the long run. Now, recall the case where a demand ξ_n occurs in period n. Because the purchasing capacity is unlimited, the inventory level still can be raised up to S_n after satisfying the demand, i.e., ξ_n does not affect the optimal decision.

3 Uncertain Capacities

In the previous section we have shown that the optimal policy for the problem with perfect capacities has a single critical number in each period. First of all, in this section we shall discuss and formulate the problem dealing with uncertain production and selling capacities in a multi-cycle case. Then, in section 3.1 we will prove that the optimal policy is also described by a single critical number in this multi-cycle uncertain capacity problem. Moreover, we have illustrated the relationship between the critical numbers in two consecutive periods. Finally, in section 3.2 we will show that the optimal policy in an infinite horizon case has a cyclic sequence of critical numbers.

Now, consider an N-cycle problem. Cycle N is the first cycle and cycle 1 is the last. Let $G_k(x_k)$ represent the selling capacity distribution with p.d.f. $g_k(x_k)$ and let $F(y)$ be the ordering (or producing) capacity distribution with p.d.f. $F(y)$. All distributions are assumed to be mutually independent. Two issues allow us to neglect the demands occurring in ordering periods: a) producing and selling processes can not be executed at the same time, for example, pump/turbines in hydroelectric plants allow either pumping or generating phase at a moment, b) the leadtime, the time between placing and receiving an order, is negligible. For convenience, let $\xi_n = 0$ with probability 1 and let $k-1=n$ when $k=1$. Due to uncertain capacities, it is not for sure that in ordering periods, the inventory level can be always raised up to a certain level (which is the critical number S_n in the previous perfect capacity problem). Therefore, we should formulate this uncertain capacity problem over whole planning horizon instead of over a single cycle in the perfect capacity problem. Then let the notations $I_{j,k}$, $u_{j,k}$, $u^*_{j,k}$ and $S_{j,k}$ represent respectively, I_k, u_k, u^*_k and S_k in cycle j. Now, we are in a position to define recursively the discounted revenue functions which form the basis of the analysis. Let $R_{j,k}(I_{j,k}, \xi_k)$ be the expected revenue of selling $I_{j,k}$ items optimally from period k in cycle j through period 1 in cycle 1, based on the observation of the demand ξ_k and the inventory level $I_{j,k}$ at period k. $R_{N,n}(I_{N,n}, \xi_n)$ represents the maximum total revenue to operate N cycles, given initial inventory $I_{N,n}$ at the beginning of cycle N. Then, the constraints of decision variables during cycle j become $I_{j,n} \le u_{j,n}$ in period n and $(I_{j,k} - \xi_k)^+ \le u_{j,k} \le I_{j,k}$ in period $k \ne n$. Therefore, $R_{j,k}(I_{j,k}, \xi_k)$ with a boundary condition $R_{0,n}(I_{0,n}, \xi_n) = 0$ satisfies the functional equations:

$$R_{j,k}(I_{j,k}, \xi_k) = \max_{(I_{j,k}-\xi_k)^+ \le u_{j,k} \le I_{j,k}} \gamma_{j,k}(I_{j,k}, u_{j,k}), \quad k \ne n,$$

$$R_{j,n}(I_{j,n},\xi_n) = \max_{I_{j,n}\le u_{j,n}} \gamma_{j,n}(I_{j,n},u_{j,n}),$$

where

$$\begin{aligned}\gamma_{j,1}(I_{j,1},u_{j,1}) &= \bar{G}_1(I_{j,1}-u_{j,1})\{\pi_1(I_{j,1}-u_{j,1})-h_1u_{j,1}+\alpha_1E_{\xi_n}[R_{j-1,n}(u_{j,1},\xi_n)]\}\\ &+\int_0^{I_{j,1}-u_{j,1}}\{\pi_1x_1-h_1(I_{j,1}-x_1)+\alpha_1E_{\xi_n}[R_{j-1,n}(I_{j,1}-x_1,\xi_n)]\}\,dG_1(x_1), \quad (5)\\ \gamma_{j,k}(I_{j,k},u_{j,k}) &= \bar{G}_k(I_{j,k}-u_{j,k})\{\pi_k(I_{j,k}-u_{j,k})-h_ku_{j,k}+\alpha_kE_{\xi_{k-1}}[R_{j,k-1}(u_{j,k},\xi_{k-1})]\}\\ &+\int_0^{I_{j,k}-u_{j,k}}\{\pi_kx_k-h_k(I_{j,k}-x_k)+\alpha_kE_{\xi_{k-1}}[R_{j,k-1}(I_{j,k}-x_k,\xi_{k-1})]\}\,dG_k(x_k),\\ &\qquad k=2,\dots,n-1, \quad (6)\\ \gamma_{j,n}(I_{j,n},u_{j,n}) &= \bar{F}(u_{j,n}-I_{j,n})\{-w(u_{j,n}-I_{j,n})-h_nu_{j,n}+\alpha_nE_{\xi_{n-1}}[R_{j,n-1}(u_{j,n},\xi_{n-1})]\}\\ &+\int_0^{u_{j,n}-I_{j,n}}\{-wy-h_n(y+I_{j,n})+\alpha_nE_{\xi_{n-1}}[R_{j,n-1}(y+I_{j,n},\xi_{n-1})]\}\,dF(y). \quad (7)\end{aligned}$$

3.1 The Multi-Cycle Problem

Here, we shall show that there exists a sequence of critical numbers $\{S_{j,k}\}$ for all j, k such that the system is operating optimally. Similar to the perfect capacity problem, the form of the optimal policy is

$$u^*_{j,k} = \begin{cases} I_{j,k} & \text{if } I_{j,k} < S_{j,k} \\ S_{j,k} & \text{if } S_{j,k} \le I_{j,k} \le \xi_k + S_{j,k} \qquad k \ne n, \\ I_{j,k}-\xi_k & \text{if } I_{j,k} > \xi_k + S_{j,k} \end{cases} \quad (8)$$

and

$$u^*_{j,n} = \begin{cases} S_{j,n} & \text{if } I_{j,n} < S_{j,n} \\ I_{j,n} & \text{otherwise.} \end{cases} \quad (9)$$

However, in the following discussion, we have found that the revenue function $\gamma_{j,k}(I_{j,k},u_{j,k})$ is no longer a concave function of decision variable $u_{j,k}$. The non-concave property essentially complicates in analyzing the problem. But, we are able to show that the optimal revenue function $R_{j,k}(I_{j,k},\xi_k)$ is still concave in inventory level through induction.

In order to characterize the behavior of the revenue functions, let us consider the revenue function in the last period of the last cycle. Taking the first and the second derivatives of the revenue function with respect to decision variable, we have respectively

$$\frac{\partial\gamma_{1,1}(I_{1,1},u_{1,1})}{\partial u_{1,1}} = -\bar{G}_1(I_{1,1}-u_{1,1})(\pi_1+h_1) < 0,$$

and

$$\frac{\partial^2 \gamma_{1,1}(I_{1,1}, u_{1,1})}{\partial u_{1,1}^2} = -g_1(I_{1,1} - u_{1,1})(\pi_1 + h_1) < 0.$$

Since the revenue function is decreasing in decision variable, we can define $S_{1,1} = 0$, i.e., we are not trying to leave over any stock. Then, the form of the optimal policy is (8) since $\gamma_{1,1}(I_{1,1}, u_{1,1})$ is also concave in $u_{1,1}$. As a result, the optimal revenue function becomes

$$R_{1,1}(I_{1,1}, \xi_1) = \begin{cases} \gamma_{1,1}(I_{1,1}, S_{1,1}) & \text{if } S_{1,1} \leq I_{1,1} \leq \xi_1 + S_{1,1} \\ \gamma_{1,1}(I_{1,1}, I_{1,1} - \xi_1) & \text{if } I_{1,1} > \xi_1 + S_{1,1}. \end{cases}$$

Hence, taking the first and the second derivatives of $R_{1,1}(I_{1,1}, \xi_1)$ with respect to $I_{1,1}$, we have respectively

$$\frac{\partial R_{1,1}(I_{1,1}, \xi_1)}{\partial I_{1,1}} = \begin{cases} \pi_1 \bar{G}_1(I_{1,1} - S_{1,1}) & \text{if } S_{1,1} \leq I_{1,1} \leq \xi_1 + S_{1,1} \\ -h_1 & \text{if } I_{1,1} > \xi_1 + S_{1,1}, \end{cases}$$

and

$$\frac{\partial^2 R_{1,1}(I_{1,1}, \xi_1)}{\partial I_{1,1}^2} = \begin{cases} -\pi_1 g_1(I_{1,1} - S_{1,1}) & \text{if } S_{1,1} \leq I_{1,1} \leq \xi_1 + S_{1,1} \\ 0 & \text{if } I_{1,1} > \xi_1 + S_{1,1}. \end{cases}$$

Clearly, $R_{1,1}(I_{1,1}, \xi_1)$ is concave in $I_{1,1}$.

Now, we have proved the concave property of $R_{1,1}(I_{1,1}, \xi_1)$ in $I_{1,1}$. Then, we need to show that if the optimal revenue function in a period is concave in inventory level, then the optimal policy in the previous period is described by a single critical number.

THEOREM 4 *Assume that $R_{j,k-1}(I_{j,k-1}, \xi_{k-1})$ is concave in $I_{j,k-1}$. Then, the form of the optimal policy in period k of cycle j is (8) when $k \neq n$ and (9) when $k = n$.*

Proof: For $k \neq n$, we have the first and the second derivatives of $\gamma_{j,k}(I_{j,k}, u_{j,k})$ with respect to $u_{j,k}$ as follows,

$$\frac{\partial \gamma_{j,k}(I_{j,k}, u_{j,k})}{\partial u_{j,k}} = \bar{G}_k(I_{j,k} - u_{j,k}) p_{j,k}(u_{j,k}),$$

and

$$\frac{\partial^2 \gamma_{j,k}(I_{j,k}, u_{j,k})}{\partial u_{j,k}^2} = \bar{G}_k(I_{j,k} - u_{j,k}) p'_{j,k}(u_{j,k}) + g_k(I_{j,k} - u_{j,k}) p_{j,k}(u_{j,k}),$$

where

$$p_{j,k}(u_{j,k}) = -\pi_k - h_k + \alpha_k E_{\xi_{k-1}}[\frac{\partial R_{j,k-1}(u_{j,k}, \xi_{k-1})}{\partial u_{j,k}}],$$
$$p'_{j,k}(u_{j,k}) = \alpha_k E_{\xi_{k-1}}[\frac{\partial^2 R_{j,k-1}(u_{j,k}, \xi_{k-1})}{\partial u^2_{j,k}}].$$

Define $S_{j,k}$ such that $p_{j,k}(S_{j,k}) = 0$. Clearly, $S_{j,k}$ is independent of $I_{j,k}$. Since $R_{j,k-1}(u_{j,k-1}, \xi_{k-1})$ is concave in $u_{j,k-1}$, $p_{j,k}(\cdot)$ is decreasing. Therefore, $\gamma_{j,k}(I_{j,k}, u_{j,k})$ is decreasing in $u_{j,k} > S_{j,k}$, and is increasing and concave in $u_{j,k} \leq S_{j,k}$. Hence, the minimum at $u_{j,k} = S_{j,k}$ is the global minimum. Therefore, the form of the optimal policy is (8).

For $k = n$, taking the first and the second derivatives of $\gamma_{j,n}(I_{j,n}, u_{j,n})$ with respect to $u_{j,n}$, we have respectively,

$$\frac{\partial \gamma_{j,n}(I_{j,n}, u_{j,n})}{\partial u_{j,n}} = \bar{F}(u_{j,n} - I_{j,n}) p_{j,n}(u_{j,n}),$$

and

$$\frac{\partial^2 \gamma_{j,n}(I_{j,n}, u_{j,n})}{\partial u^2_{j,n}} = \bar{F}(u_{j,n} - I_{j,n}) p'_j(u_{j,n}) - f(u_{j,n} - I_{j,n}) p_{j,n}(u_{j,n}),$$

where

$$p_{j,n}(u_{j,n}) = -w - h_n + \alpha_n E_{\xi_{n-1}}[\frac{\partial R_{j,n-1}(u_{j,n}, \xi_{n-1})}{\partial u_{j,n}}],$$
$$p'_{j,n}(u_{j,n}) = \alpha_n E_{\xi_{n-1}}[\frac{\partial^2 R_{j,n-1}(u_{j,n}, \xi_{n-1})}{\partial u^2_{j,n}}].$$

Define $S_{j,n}$ such that $p_{j,n}(S_{j,n}) = 0$. Clearly, $S_{j,n}$ is independent of $I_{j,n}$. Since $R_{j,n-1}(u_{j,n}, \xi_{n-1})$ is concave in $u_{j,n}$, $p_{j,n}(\cdot)$ is decreasing. Therefore, $\gamma_{j,n}(I_{j,n}, u_{j,n})$ is decreasing in $u_{j,n} > S_{j,n}$, and is increasing and concave in $u_{j,n} \leq S_{j,n}$. Hence, the minimum at $u_{j,n} = S_{j,n}$ is the global minimum. As a result, the form of the optimal policy is (9). ∎

In order to perform the proof by induction, it is necessary to show the following theorem before hand.

THEOREM 5 *If* $R_{j,k-1}(I_{j,k-1}, \xi_{k-1})$ *is concave in* $I_{j,k-1}$*, then* $R_{j,k}(I_{j,k}, \xi_k)$ *is also concave in* $I_{j,k}$*.*

Proof: For $k \neq n$, since $R_{j,k-1}(I_{j,k-1}, \xi_{k-1})$ is concave in $I_{j,k-1}$, the form of the optimal policy for period k in cycle j is (8) by Theorem 4. Therefore, we have

$$R_{j,k}(I_{j,k}, \xi_k) = \begin{cases} \gamma_{j,k}(I_{j,k}, I_{j,k}) & \text{if } I_{j,k} < S_{j,k} \\ \gamma_{j,k}(I_{j,k}, S_{j,k}) & \text{if } S_{j,k} \leq I_{j,k} \leq \xi_k + S_{j,k} \\ \gamma_{j,k}(I_{j,k}, I_{j,k} - \xi_k) & \text{if } I_{j,k} > \xi_k + S_{j,k}. \end{cases}$$

Taking the first derivative of $R_{j,k}(I_{j,k}, \xi_k)$ with respect to $I_{j,k}$, we have

$$\frac{\partial R_{j,k}(I_{j,k}, \xi_k)}{\partial I_{j,k}} =$$

$$\begin{cases} -h_k + \alpha_k E_{\xi_{k-1}}[\frac{\partial R_{j,k-1}(I_{j,k}, \xi_{k-1})}{\partial I_{j,k}}] & \text{if } I_{j,k} < S_{j,k} \\ \\ \alpha_k \int_0^{I_k - S_{j,k}} E_{\xi_{k-1}}[\frac{\partial R_{j,k-1}(I_{j,k} - x_k, \xi_{k-1})}{\partial I_{j,k}}]\, dG_k(x_k) & \\ \quad + \pi_k \bar{G}_k(I_{j,k} - S_{j,k}) - h_k G_k(I_k - S_{j,k}) & \text{if } S_{j,k} \leq I_{j,k} \leq \xi_k + S_{j,k} \\ \\ \alpha_k \int_0^{\xi_k} E_{\xi_{k-1}}[\frac{\partial R_{j,k-1}(I_{j,k} - x_k, \xi_{k-1})}{\partial I_{j,k}}]\, dG_k(x_k) & \\ \quad -h_k + \alpha_k \bar{G}_k(\xi_{j,k}) E_{\xi_{k-1}}[\frac{\partial R_{j,k-1}(I_{j,k} - \xi_k, \xi_{k-1})}{\partial I_{j,k}}] & \text{if } I_{j,k} > \xi_k + S_{j,k}. \end{cases} \tag{10}$$

Taking the second derivative of $R_{j,k}(I_{j,k}, \xi_k)$ with respect to $I_{j,k}$, we have

$$\frac{\partial^2 R_{j,k}(I_{j,k}, \xi_k)}{\partial I_{j,k}^2} =$$

$$\begin{cases} \alpha_k E_{\xi_{k-1}}[\frac{\partial^2 R_{j,k-1}(I_{j,k}, \xi_{k-1})}{\partial I_{j,k}^2}] & \text{if } I_{j,k} < S_{j,k} \\ \\ \alpha_k \int_0^{I_k - S_{j,k}} E_{\xi_{k-1}}[\frac{\partial^2 R_{j,k-1}(I_{j,k} - x_k, \xi_{k-1})}{\partial I_{j,k}^2}]\, dG_k(x_k) & \\ \quad -(\pi_k + h_k) g_k(I_{j,k} - S_{j,k}) & \text{if } S_{j,k} \leq I_{j,k} \leq \xi_k + S_{j,k} \\ \\ \alpha_k \int_0^{\xi_k} E_{\xi_{k-1}}[\frac{\partial^2 R_{j,k-1}(I_{j,k} - x_k, \xi_{k-1})}{\partial I_{j,k}^2}]\, dG_k(x_k) & \\ \quad + \alpha_k \bar{G}_k(\xi_{j,k}) E_{\xi_{k-1}}[\frac{\partial^2 R_{j,k-1}(I_{j,k} - \xi_k, \xi_{k-1})}{\partial I_{j,k}^2}] & \text{if } I_{j,k} > \xi_k + S_{j,k}. \end{cases}$$

Since $R_{j,k-1}(I_{j,k}, \xi_{k-1})$ is concave in $I_{j,k}$, $R_{j,k}(I_{j,k}, \xi_k)$ is concave in $I_{j,k}$ belonging to three intervals, $(0, S_{j,k})$, $(S_{j,k}, \xi_k + S_{j,k})$ and $(\xi_k + S_{j,k}, \infty)$ respectively. Moreover, by using the first order condition $p_{j,k}(S_{j,k}) = 0$, it is easily to show that the limitations of $\frac{\partial R_{j,k}(I_{j,k}, \xi_k)}{\partial I_{j,k}}$ at $I_{j,k}$ equal to $S_{j,k}$ and equal to $\xi_k + S_{j,k}$ respectively, exist. This guarantees the concavity of $R_{j,k}(I_{j,k}, \xi_k)$ in all $I_{j,k}$.

For $k = n$, since $R_{j,n}(I_{j,n-1}, \xi_{n-1})$ is concave in $I_{j,n-1}$, the form of the optimal policy for period n in cycle j is (9) by Theorem 4. Then,

$$R_{j,n}(I_{j,n}, \xi_n) = \begin{cases} \gamma_{j,n}(I_{j,n}, S_{j,n}) & \text{if } I_{j,n} < S_{j,n} \\ \gamma_{j,n}(I_{j,n}, I_{j,n}) & \text{otherwise.} \end{cases}$$

Taking the first derivative of $R_{j,n}(I_{j,n}, \xi_n)$, we have

$$\frac{\partial R_{j,n}(I_{j,n}, \xi_n)}{\partial I_{j,n}} = \begin{cases} \alpha_n \int_0^{S_{j,n}-I_{j,n}} E_{\xi_{n-1}}[\frac{\partial R_{j,n-1}(y+I_{j,n},\xi_{n-1})}{\partial I_{j,n}}]\, dF(y) & \\ \quad +w\bar{F}(S_{j,n} - I_{j,n}) & \text{if } I_{j,n} < S_{j,n} \\ \alpha_n E_{\xi_{n-1}}[\frac{\partial R_{j,n-1}(I_{j,n},\xi_{n-1})}{\partial I_{j,n}}] - h_n & \text{otherwise.} \end{cases} \tag{11}$$

Taking the second derivative of $R_{j,n}(I_{j,n}, \xi_n)$ and then by $p_{j,n}(S_{j,n}) = 0$, we have

$$\frac{\partial^2 R_{j,n}(I_{j,n}, \xi_n)}{\partial I_{j,n}^2} = \begin{cases} \alpha_n \int_0^{S_{j,n}-I_{j,n}} E_{\xi_{n-1}}[\frac{\partial^2 R_{j,n-1}(y+I_{j,n},\xi_{n-1})}{\partial I_{j,n}^2}]\, dF(y) & \text{if } I_{j,n} < S_{j,n} \\ \alpha_n E_{\xi_{n-1}}[\frac{\partial^2 R_{j,n-1}(I_{j,n},\xi_{n-1})}{\partial I_{j,n}^2}] & \text{otherwise.} \end{cases}$$

Since $R_{j,n-1}(I_{j,n-1}, \xi_{n-1})$ is concave in $I_{j,n-1}$, $R_{j,n}(I_{j,n}, \xi_n)$ is concave in $I_{j,n} \in (0, S_{j,n})$ and $I_{j,n} \in (S_{j,n}, \infty)$. Again, we can show that the limitation of $\frac{\partial R_{j,n}(I_{j,n},\xi_n)}{\partial I_{j,n}}$ at $I_{j,n} = S_{j,n}$ exists. Therefore, $R_{j,n}(I_{j,n}, \xi_n)$ is concave in in $I_{j,n}$. ■

By induction, the following corollary is ready to perceive.

COROLLARY 6 *The form of the optimal policy for period k in cycle j is (8) when $k \neq n$ or (9) when $k = n$.*

So far, we have shown the structure of the optimal policy described by a sequence of critical numbers. Now, we shall show several properties of these critical numbers.

THEOREM 7 $S_{j,1} = 0$ *for all* j.

Proof: From (10) and (11), we have

$$\begin{aligned} p_{j,1}(0) &\leq -\pi_1 - h_1 + \alpha_1 E_{\xi_{k-1}}[\frac{\partial R_{j,k-1}(u_{j,k} = S_{j-1,n}, \xi_{k-1})}{\partial u_{j,k}}] \\ &= -\pi_1 - h_1 + \alpha_1 w \leq 0. \end{aligned}$$

The proof is done by decreasing property of $p_{j,1}(\cdot)$. QED. ∎

Theorem 7 indicates that to sell a product in the last period of a cycle is always more beneficial than to carry it over the next cycle.

In the following theorem and corollaries, we characterize the behavior of the critical numbers influenced by the marginal revenue Δ_k. If it is profitable to sell a product in period k instead of in the next period, then we try to keep less inventory in period k than in the coming period. On the other hand, if it is less beneficial to sell a product in period k instead of in period $k-1$, then we try to keep more inventory over the next period.

THEOREM 8 *If* $\Delta_k \le 0$, *then* $S_{j,k} \ge S_{j,k-1}$; *otherwise,* $S_{j,k} \le S_{j,k-1}$.

Proof: From (10), we have $\frac{\partial R_{j,k-1}(I_{j,k-1}=S_{j,k-1},\xi_{k-1})}{\partial I_{j,k-1}} = \pi_{k-1}$, by $p_{j,k-1}(S_{j,k-1}) = 0$ for all j, k. Hence, $p_{j,k}(S_{j,k-1}) = -\Delta_k$ for all k. Notice that $p_{j,k}(\cdot)$ is decreasing. Therefore, if $\Delta_k \le 0$, then $S_{j,k} \ge S_{j,k-1}$; if $\Delta_k \ge 0$, then $S_{j,k} \le S_{j,k-1}$ otherwise. ∎

Theorem 8 is very useful for searching critical numbers in reducing unnecessary computational burden. For example, if the marginal revenues in all periods are greater than zero, then all critical numbers are zero. Therefore, it is not a profitable problem since $S_{j,n} = 0$, i.e., it is not a benefit to order any product. On the other hand, if the marginal revenues in all periods are less than zero, then $S_{j,n} \ge S_{j,n-1} \ge \dots \ge S_{j,1}$ for all j.

3.2 The Infinite Horizon Problem

Let us consider an infinite horizon version of the uncertain capacity problem. It is clear from (5), (6) and (7) that in any cycle, the one period revenue is

$$\begin{aligned}\phi_{j,k}(I_{j,k},u_{j,k}) &= [\pi_k(I_{j,k}-u_{j,k}) - h_k u_{j,k}]\bar{G}_k(I_{j,k}-u_{j,k}) \\ &\quad + \int_0^{I_{j,k}-u_{j,k}} [\pi_k x_k - h_k(I_{j,k}-x_k)]\, dG_k(x_k), \qquad k \ne n,\end{aligned}$$

and

$$\begin{aligned}\phi_{j,n}(I_{j,n},u_{j,n}) &= -(wu_{j,n} - wI_{j,n} + h_n u_{j,n})\bar{F}(u_{j,n}-I_{j,n}) \\ &\quad - \int_0^{u_{j,n}-I_{j,n}} (wy + h_n y + h_n I_{j,n})\, dF(y).\end{aligned}$$

Then, the function to be minimized is given by

$$J_\mu(I_{N,n}) \;=\; \lim_{N\to\infty} \sum_{j=1}^{N} [(\prod_{l=1}^{n} \alpha_l)^j \sum_{m=1}^{n} \phi_{j,m}(u_{j,m})],$$

where $J_\mu(I_{N,n})$ denotes the revenue associated with an initial inventory $I_{N,n}$ and a policy $\mu = \{u_{N,n}, ..., u_{N,1}, ..., u_{1,n}, ..., u_{1,1}\}$.

Indeed, consider a stationary policy $\tilde{\mu} = \{\tilde{u}, \tilde{u}, ...\}$, where $\tilde{u}$ is defined by

$$\tilde{u}(I_{j,k}) \;=\; I_{j,k}, \qquad \forall\, j, k.$$

It follows that the inventory stock $I_{j,k}$ is always equal to $I_{N,n}$ when $\tilde{\mu}$ policy is used. Now, differentiating $\phi_{j,k}(I_{j,k}, u_{j,k})$ with respect to $u_{j,k}$, we have

$$\frac{\partial \phi_{j,k}(I_{j,k}, u_{j,k})}{\partial u_{j,k}} \;=\; -(\pi_k + h_k)\bar{G}_k(I_{j,k} - u_{j,k}) \;<\; 0, \qquad k \neq n,$$

and

$$\frac{\partial \phi_{j,n}(I_{j,n}, u_{j,n})}{\partial u_{j,n}} \;=\; -(w + h_n)\bar{F}(u_{j,n} - I_{j,n}) \;<\; 0,$$

i.e., $\phi_{j,k}(I_{j,k}, u_{j,k})$ is decreasing in $u_{j,k}$ for all j, k. Assume that the ordering capacity is bounded above and $E[Y] < \infty$. Furthermore, we have

$$\lim_{u_{j,n}\to\infty} \phi_{j,n}(I_{j,n}, u_{j,n}) \;=\; -(w + h_n)E[Y] - h_n I_{j,n}.$$

Since $(I_{j,k} - \xi_k)^+ \leq u_{j,k} \leq I_{j,k}$ and $I_{j,n} \leq u_{j,n}$, we get

$$\begin{aligned}
-h_k I_{j,k} \;&\leq\; \phi_{j,k}(I_{j,k}, u_{j,k}) \\
&\leq\; [\pi_k(I_{j,k} - (I_{j,k} - \xi_k)^+) - h_k(I_{j,k} - \xi_k)^+]\bar{G}_k(I_{j,k} - (I_{j,k} - \xi_k)^+) \\
&\quad + \int_0^{I_{j,k} - (I_{j,k} - \xi_k)^+} [\pi_k x_k - h_k(I_{j,k} - x_k)]\, dG_k(x_k) \\
&=\; \begin{cases} \pi_k I_{j,k}\bar{G}_k(I_{j,k}) + \int_0^{I_{j,k}} [\pi_k x_k - h_k(I_{j,k} - x_k)]\, dG_k(x_k) & I_{j,k} \leq \xi_k \\ -[\pi_k \xi_k + h_k(I_{j,k} - \xi_k)]\bar{G}_k(\xi_k) + \int_0^{\xi_k} [\pi_k x_k - h_k(I_{j,k} - x_k)]\, dG_k(x_k) & I_{j,k} > \xi_k, \end{cases}
\end{aligned}$$

and

$$-(w + h_n)E[Y] - h_n I_{j,n} \;\leq\; \phi_{j,n}(I_{j,n}, u_{j,n}) \;\leq\; 0.$$

Hence, the revenue per period incurred when $\tilde{\mu}(I_{j,k})$ is used is bounded, and in view of the presence of the discount factor we have

$$\max_{\mu} J_{\mu}(I_{N,n}) \geq J_{\tilde{\mu}}(I_{N,n}) > -\infty.$$

Then, from Bertsekas (1987), the optimality equation reduces to the system of n equations:

$$\begin{aligned} \hat{R}_k(I_k,\xi_k) &= \max_{(I_k-\xi_k)^+ \leq u_k \leq I_k} \hat{\gamma}_k(I_k,u_k), \qquad k \neq n, \\ \hat{R}_n(I_n,\xi_n) &= \max_{I_n \leq u_n} \hat{\gamma}_n(I_n,u_n), \end{aligned}$$

where

$$\begin{aligned} \hat{\gamma}_k(I_k,u_k) &= \bar{G}_k(I_k-u_k)\{\pi_k(I_k-u_k) - h_k u_k + \alpha_k E_{\xi_{k-1}}[\hat{R}_{k-1}(u_k,\xi_{k-1})]\} \\ &\quad + \int_0^{I_k-u_k} \{\pi_k x_k - h_k(I_k-x_k) + \alpha_k E_{\xi_{k-1}}[\hat{R}_{k-1}(I_k-x_k,\xi_{k-1})]\}\, dG_k(x_k), \qquad k \neq n, \\ \hat{\gamma}_n(I_n,u_n) &= \bar{F}(u_n-I_n)\{-w(u_n-I_n) - h_n u_n + \alpha_n E_{\xi_{n-1}}[\hat{R}_{n-1}(u_n,\xi_{n-1})]\} \\ &\quad + \int_0^{u_n-I_n} \{-wy - h_n(y+I_n) + \alpha_n E_{\xi_{n-1}}[\hat{R}_{n-1}(y+I_n,\xi_{n-1})]\}\, dF(y). \end{aligned}$$

Furthermore, an optimal periodic policy, $\mu^* = \{\hat{u}_1^*(I_1),...,\hat{u}_n^*(I_n),...,\hat{u}_1^*(I_1),...,\hat{u}_n^*(I_n)\}$, is guaranteed to exist. Therefore, the optimal policy μ^* satisfies the following n first order conditions:

$$\bar{G}_k(I_k-\hat{u}_k^*(I_k))\{-\pi_k - h_k + \alpha_k E_{\xi_{k-1}}[\frac{\partial \hat{R}_{k-1}(u_k=\hat{u}_k^*(I_k),\xi_{k-1})}{\partial u_k}]\} = 0, \quad k \neq n, \tag{12}$$

$$\bar{F}(\hat{u}_n^*(I_n)-I_n)\{-w - h_n + \alpha_n E_{\xi_{n-1}}[\frac{\partial \hat{R}_{n-1}(u_n=\hat{u}_n^*(I_n),\xi_{n-1})}{\partial u_n}]\} = 0. \tag{13}$$

Clearly, the function inside the brackets of (12) and (13) are independent of inventory levels. Therefore, there exists a sequence of numbers $\{\hat{S}_1,\hat{S}_2,...,\hat{S}_n\}$ to satisfy these n first order conditions.

Since $R_{j,k}(I_{j,k},\xi_k)$ are concave in inventory level for all k, the limit functions $\hat{R}_k(I_k,\xi_k)$ is also concave in inventory level. Therefore, we can show that for all k, $\hat{\gamma}_k(I_k,u_k)$ is increasing in $u_k \in [0,\hat{S}_k]$ and is decreasing in $u_k \in [\hat{S}_k,\infty)$ through the similar analysis in Section 3.1. Furthermore, consider the decision variable constraints $(I_k-\xi_k)^+ \leq u_k \leq I_k$ for $k \neq n$ and $I_n \leq u_n$. As a result, the structure of the optimal policy is

$$\hat{u}_k^* = \begin{cases} I_k & \text{if } I_k < \hat{S}_k \\ \hat{S}_k & \text{if } \hat{S}_k \leq I_k \leq \xi_k + \hat{S}_k \\ I_k - \xi_k & \text{if } I_k > \xi_k + \hat{S}_k \end{cases} \qquad k \neq n,$$

and

$$\hat{u}_n^* = \begin{cases} \hat{S}_n & \text{if } I_n < \hat{S}_n \\ I_n & \text{otherwise.} \end{cases}$$

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