Engineering Notes

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Sampled Data Control of Flexible Structures Using Constant Gain Velocity Feedback

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Models for Sampled Data Controlled Flexible Structures

A SAMPLED data-controlled flexible structure can be defined as a distributed parameter system, in which the structure input is the output of an ideal zero order hold and the structure output is sampled. Although distributed parameter models typically involve infinite dimensional variables, our analysis is based on the finite dimensional model

$$M\ddot{x}+Kx=Bu \tag{1}$$

$$y=C\dot{x} \tag{2}$$

For simplicity in the subsequent development, no structural damping is included. The structural displacement vector $x=(x^1,...,x^n)$, the force input vector $u=(u^1,...,u^m)$, and the velocity output vector $y=(y^1,...,y^m)$. The mass matrix M and the structural stiffness matrix K are assumed symmetric and positive definite. The input influence matrix B and the output influence matrix C are assumed to be dimensionless.

The structure input u is defined in terms of the input sequence u_k by the ideal zero order hold relation $u(t)=u_k$, $kT\leq t<kT+T$; the output sequence y_k is defined in terms of the structure output y by the ideal sampling relation $y_k=y(kT)$, $k=0,1,...$ The fixed value $T>0$ is the constant sampling time. This open-loop sampled data-controlled structure can be viewed as a discrete time system with input sequence u_k and output sequence y_k, where $k=0,1,...$

Let ϕ be nxn nonsingular modal matrix and let Ω^2 be nxn diagonal modal frequency matrix[1] satisfying

$$\phi^TM\phi=I,\quad \phi^TK\phi=\Omega^2=\mathrm{diag}(\omega_1^2,...,\omega_n^2) \tag{3}$$

Introduce the coordinate change $x=\phi\eta$ so that Eqs. (1) and (2) can be written in modal coordinates as

$$\ddot{\eta}+\Omega^2\eta=\phi^TBu \tag{4}$$

$$y=C\phi\dot{\eta} \tag{5}$$

It is an easy task to solve vector Eq. (4), using the sampling constraint, to obtain first order recursions in $\eta_k=\eta(kT)$ and $\dot{\eta}_k=\dot{\eta}(kT)$.[2] Although the first order recursive equations could be used, it is more convenient for our purposes to make use of a second order recursive equation in $\dot{\eta}_k$ alone. It is easily shown that $\dot{\eta}_k$ satisfies

$$\dot{\eta}_{k+1}-2\cos\Omega T\dot{\eta}_k+\dot{\eta}_{k-1}=\Omega^{-1}\sin\Omega T\phi^TB(u_k-u_{k-1}) \tag{6}$$

$$y_k=C\phi\dot{\eta}_k \tag{7}$$

where

$$\cos\Omega T=\mathrm{diag}(\cos\omega_1T,...,\cos\omega_nT)$$

$$\sin\Omega T=\mathrm{diag}(\sin\omega_1T,...,\sin\omega_nT)$$

Modal Eqs. (6) and (7) form the basis for our subsequent analysis.

It should be noted that Eqs. (6) and (7) involve no numerical approximation; they are valid for any sampling time $T>0$.

Constant gain output velocity feedback has been studied extensively for analog-controlled structures. Our interest is in use of constant gain output velocity feedback for sampled data controlled structures.

Consider the closed-loop sampled data-controlled structure defined by Eqs. (6) and (7) using the control input sequence

$$u_k=-Gy_k \tag{8}$$

where G is a constant mxm feedback gain matrix. Substituting Eq. (8) into Eqs. (6) and (7), a closed-loop recursive equation is obtained

$$\dot{\eta}_{k+1}-[2\cos\Omega T-\Omega^{-1}\sin\Omega T\phi^TBGC\phi]\dot{\eta}_k$$
$$+[I-\Omega^{-1}\sin\Omega T\phi^TBGC\phi]\dot{\eta}_{k-1}=0 \tag{9}$$

The closed-loop characteristic equation can be written as

$$d(T,z)=\det[z^2I-(2\cos\Omega T-\Omega^{-1}\sin\Omega T\phi^TBGC\phi)z$$
$$+(I-\Omega^{-1}\sin\Omega T\phi^TBGC\phi)]=0 \tag{10}$$

The objective of constant gain velocity feedback control is to make the closed loop as described by Eq. (9) geometrically stable, i.e., to make the closed-loop characteristic zeros lie inside the unit disk.

We use Eq. (9) as the basis for our subsequent analysis of the closed loop. If $\sin\Omega T$ is nonsingular, the following implications hold: if $\dot{\eta}_k\to0$ as $k\to\infty$, then necessarily $u_k\to0$ as $k\to\infty$ and $\eta_k\to0$ as $k\to\infty$; consequently $\dot{x}(kT)\to0$ and $x(kT)\to0$ as $k\to\infty$. Throughout, we assume that the sampling time satisfies the nondegeneracy condition that $T\neq j\pi/\omega_i$, $i=1,...,n$; $j=0,1,...$ so that $\sin\Omega T$ is nonsingular. Of course this condition is satisfied for sufficiently small sampling time satisfying $T<\pi/\omega_i$, $i=1,...,n$.

Conditions for Closed-Loop Stabilization

Recall the following results for constant gain output velocity analog feedback control where $u=-Gy$. If co-located force actuators and velocity sensors are selected so that $C=B^T$, then the closed-loop, analog-controlled structure is asymptotically stable if G is *any* symmetric, positive

Submitted June 13, 1983; revision received Sept. 6, 1983.

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definite matrix, and if a certain controllability assumption is satisfied.[3-5] Moreover, this result does not depend on the particular values of the modal frequencies and modal functions.

We first mention a rather obvious result that if the sampled data feedback control is chosen according to the analog feedback theory the closed lop is stable for sufficiently small sampling time. The brief proof is included for completeness; it also serves as an introduction to our subsequent development.

Theorem 1

Assume that

a) $C=B^T$

b) the matrix pair Ω^2, $\phi^T B$ is completely controllable

c) G is symmetric and positive definite.

Closed-loop Eq. (9) is geometrically stable for sufficiently small sampling time $T>0$.

Proof: Consider the associated polynomial

$$p(T,w)=\det\Big[2(I+\cos\Omega T-\Omega^{-1}\sin\Omega T\phi^T BGC\phi)\frac{T^2w^2}{4}$$

$$+2\Omega^{-1}\sin\Omega T\phi^T BGC\phi)\frac{Tw}{2}+2(I-\cos\Omega T)\Big] \tag{11}$$

Using the results in REf. 3, the polynomial defined by

$$\lim_{T-0} p(T,w)\frac{1}{T^2}$$

has all zeros in the left-half plane; hence there is $\bar{T}>0$ such that $p(T,w)$ has all zeros in the left-half plane for $0<T<\bar{T}$. Using the bilinear transformation

$$z=1+\frac{Tw}{2}\Big/1-\frac{Tw}{2} \tag{12}$$

it follows that the zeros of $d(T,z)$ are necessarily in the unit disk for $0<T<\bar{T}$; hence Eq. (9) is stable for $0<T<\bar{T}$.

This result has limited application, since there is no indication of the range of values of the sampling times, relative to the feedback gain matrix, required for closed-loop stability. In Ref. 6, conditions are developed which, in principle, characterize a range of values of the sampling time for which the closed loop is stable. Unfortunately, the conditions depend on an a priori computable bound on an exponential matrix; computation of such a bound, in analytical terms, is not considered in Ref. 6. Of course one could perform a numerical search, based on the characteristic polynomial $d(T,z)$, or equivalently on $p(T,w)$, for a specific case, to determine a range of values of the sampling time for which the closed loop is stable. But, for the case of co-located velocity feedback there are no known explicit conditions, in terms of the sampling time and feedback gain matrix, which gurantee stability of the closed-loop sampled data system.

We now present the main result of the paper: a set of explicit conditions on the input and output influence matrices, the sampling time and the feedback gain matrix for which the closed-loop sampled data-controlled structure is stable. The key idea is to modify the assumptions suitably so that the approach used in the proof of Theorem 1 can be followed.

Theorem 2

Assume that $\sin\Omega T$ is nonsingular and a) the matrix pair $[I+\cos\Omega T]^{-1}$ $[I-\cos\Omega T]$, $\phi^T B$ is completely controllable, b) the matrix $\Omega^{-1}\sin\Omega T\phi^T BGC\phi$ is symmetric and positive definite, c) the matrix $I+\cos\Omega T-\Omega^{-1}\sin\Omega T\phi^T BGC\phi$ is symmetric and positive definite.

Closed-loop Eq. (9) is geometrically stable.

Proof: The assumptions, as in the proof of Theorem 1, guarantee that the zeros of $p(T,w)$ defined in Eq. (11) are in the left-half plane. The bilinear transformation defined in Eq. (12) guarantees that the zeros of $d(T,z)$ are necessarily inside the unit disk in the complex plane. Hence, Eq. (9) is stable.

This general result gives sufficient conditions on the influence matrices B and C, the feedback gain matrix G, and the sampling time T, for which the closed-loop system is stable.

A few general statements can be made regarding satisfaction of conditions b) and c) of Theorem 2. First, note that it is the matrix product BGC which appears in the conditions; in general, this matrix product is required to be neither symmetric nor positive definite. Also, for fixed influence matrices B and C, conditions b) and c) of Theorem 2 can be viewed as characterizing the relation between the feedback gain matrix G and the sampling time T for which the closed loop is stable. Informally, note that condition c) implies that if the sampling time T is "small," then the feedback gain matrix G may be "large," while if the sampling time is "large," the feedback gain matrix must be "small." In addition as the sampling time satisfies $T\rightarrow 0$ condition c) becomes trivally satisfied and condition b) implies that BGC tends toward a symmetric matrix, just as required by Theorem 1. Satisfaction of the conditions in Theorem 2 does require explicit knowledge of the modal data. Note also that for fixed influence matrices B and C, e.g., $C=B^T$, and a fixed sampling time T there is no guarantee that there is a feedback gain matrix G which satisfies the above stability conditions.

There are two special cases where the existence of a stabilizing feedback gain matrix can be guaranteed. These two cases are indicated in the following two corollaries.

Corollary 1

Assume that $\sin\Omega T$ is nonsingular and a) the matrix pair $[I+\cos\Omega T]^{-1}$ $[I-\cos\Omega T]$, $\phi^T B$ is completely controllable, b) the influence matrices C and B satisfy

$$C=B^T\phi\sin\Omega T\Omega^{-1}T^{-1}\phi^{-1}$$

Then there exists a feedback gain matrix G satisfying c) the gain matrix G is symmetric and positive definite, d) the matrix $I+\cos\Omega T-T\phi^T C^T GC\phi$ is symmetric and positive definite, such that closed-loop Eq. (9) is geometrically stable.

Although Corollary 1 gives sufficient conditions for closed-loop stability, it is not clear where force actuators and velocity sensors should be located on a structure so that the influence matrices B and C satisfy condition b). This question is not addressed here, but the actuators and sensors would generally *not* be co-locatied.

Corollary 2

Assume that $\sin\Omega T$ is nonsingular and a) rank $B=$ rank $C=n$. Then there exists a feedback gain matrix G satisfying b) the matrix $\Omega^{-1}\sin\Omega T\phi^T BGC\phi$ is symmetric and positive definite, c) the matrix $I+\cos\Omega T-\Omega^{-1}\sin\Omega T\phi^T BGC\phi$ is symmetric and positive definite, such that closed-loop Eq. (9) is geometrically stable.

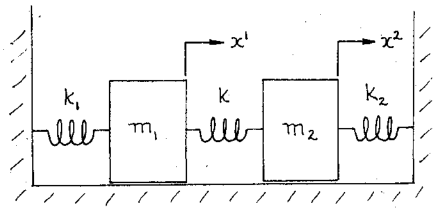

Fig. 1 Controlled spring-mass example.

In this case, with at least as many actuators and sensors as there are modes to be controlled, there is no need for an explicit condition on the influence matrices. The feedback gain matrix can always be suitably chosen to satisfy the stabilization conditions. But in general, the feedback gain matrix would be neither symmetric nor positive definite.

These several sufficient conditions for stability of sampled data-controlled flexible structures indicate the importance of the sampling constraint. These results are now illustrated with an example.

An Example

Consider the two mass and three spring connections indicated in Fig. 1, with notation given in Fig. 1. This is the same example studied in Ref. 7, where analog feedback was used to stabilize the closed loop. Our objective is to use sampled data feedback to achieve stabilization of the spring and mass system.

For simplicity, the numerical values for the masses and spring stiffnesses are chosen as $m_1=m_2=1$, $k_1=1$, $k_2=4$, $k=2$ so that

$$M=\begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix} \quad K=\begin{bmatrix} 3.0 & -2.0 \\ -2.0 & 6.0 \end{bmatrix}$$

Suppose that the control forces u^1 on m_1 and u^2 on m_2 are given by the analog feedback relations

$$u^1=-g_1\dot{x}^1-g(\dot{x}^1-\dot{x}^2)$$

$$u^2=-g(\dot{x}^2-\dot{x}^1)-g_2\dot{x}^2$$

so that g_1, g_2, and g can be viewed as damping parameters for three analog dampers. From the results in Ref. 7 the system is stable if

$$g_1\geq 0, \qquad g_2\geq 0, \qquad g\geq 0$$

with at least one strict inequality. Further, this conclusion does not depend on the particular numerical values of the masses and spring stiffnesses considered.

Now suppose that the control forces are given according to the sampled data feedback relations

$$u_k^1=-g_1\dot{x}_k^1-g(\dot{x}_k^1-\dot{x}_k^2)$$

$$u_k^2=-g(\dot{x}_k^2-\dot{x}_k^1)-g_2\dot{x}_k^2$$

where the parameters g_1, g_2, and g can be viewed as damping parameters for three digital dampers. Corollary 2 can be used to show that the sampled data controlled structure is stable if, in addition to the previous requirements for the analog feedback case, the feedback gains also satisfy the equality

$$2g_2-2g_1-3g=0$$

and the inequalities

$$1.+\cos\sqrt{2}T-\sin\sqrt{2}T(4g_1+g_2+g)/5\sqrt{2}>0$$

$$1.+\cos\sqrt{7}T-\sin\sqrt{7}T(g_1+4g_2+9g)/5\sqrt{7}>0$$

An illustration is now given for the case where a single digital damper is located between the two masses so that equal and opposite forces are applied to the two masses. Consider the sampled data feedback relation

$$u_k^1=-u_k^2=-g(c_1\dot{x}_k^1+c_2\dot{x}_k^2)$$

The output influence coefficients c_1 and c_2 can be chosen to satisfy condition b) of Corollary 1 to obtain

$$c_1=[2(\sin\sqrt{2}T)/\sqrt{2}T+3(\sin\sqrt{7}T)/\sqrt{7}T]/5$$

$$c_2=[(\sin\sqrt{2}T)/\sqrt{2}T-6(\sin\sqrt{7}T)/\sqrt{7}T]/5$$

The additional conditions of Corollary 1 require that the feedback gain satisfy $g>0$ and that the matrix

$$\begin{bmatrix} 1+\cos\sqrt{2}T & 0 \\ 0 & 1+\cos\sqrt{7}T \end{bmatrix}-\frac{g}{5T}$$

$$\begin{bmatrix} (\sin\sqrt{2}T)^2/2 & 3(\sin\sqrt{2}T\sin\sqrt{7}T)/\sqrt{14} \\ 3(\sin\sqrt{2}T\sin\sqrt{7}T)/\sqrt{14} & 9(\sin\sqrt{7}T)^2/7 \end{bmatrix}$$

be positive definite. Notice the important feature that the control force from the digital damper does not depend on the relative velocity $\dot{x}_k^1-\dot{x}_k^2$. But rather, to compensate for the sampling effects, the control force depends on the determined linear combination of the mass velocities.

The closed-loop characteristic polynomial, of the fourth degree with coefficients depending on the feedback gains and the sampling time, could in principle be used as a basis for stability analysis. However, the resulting necessary and sufficient conditions for stability have an exceedingly complicated dependence on the feedback gains and the sampling time. Although our conditions above are sufficient conditions for stability, they expose rather clearly the dependence on the feedback gains and the sampling time.

Conclusions

We have presented several results which can serve as guidelines for choice of feedback gains and sampling time to guarantee that a sampled data-controlled structure is stable. In each case, the dependence on the sampling time is made explicit. The complexity of these results, in comparison with the simple results for stabilization using analog velocity feedback, is due to the delay effects introduced by the samping constraint.

Acknowledgments

This work was supported in part by the Air Force Office of Scientific Research/AFSC under AFOSR Contract Number F49620-82-C-0089 and the Robot Systems Division of the Center for Robotics and Integrated Manufacturing at the University of Michigan. An early version of this work was prepared at the Charles Stark Draper Laboratory, Inc., Cambridge, Mass., where the author was on sabbatical leave in 1982.

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Modal Decoupling Conditions for Distributed Control of Flexible Structures

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Introduction

THE problem of the control of distributed parameter systems can be roughly divided into two approaches: 1) discretize the system in space and then use finite dimensional control theory; and 2) deal with the distributed model directly without discretizing. Recently, Meirovitch and Baruh[1] proposed a scheme for the optimal control of a certain class of conservative distributed parameter systems without resorting to discretization. In particular, they treated the control of self-adjoint conservative systems having known eigensolutions. It is the intent of this Note to point out that their results are applicable to a more general class of problems that includes nonconservative forces and to note that the necessary conditions are available for the existence of decoupling control laws. Decoupling control laws are defined to be those control laws dependent only on the modal state vector of the decoupled equation. This yields an infinite set of independent equations including the feedback control.

The use of decoupled controls allows the distributed parameter control problem to be solved by the independent modal-space control method.[1] This method allows each mode to be designed independently of other modes. As a result, the standard control problems involving optimal control and the regulator problem can be solved without difficulty. This method of control is not discussed in detail here, but is mentioned to supply motivation and application for the results that follow.

Class of Systems Considered

The class of flexible structures under consideration are those that may be successfully modeled by partial differential equations of the form

$$u_{tt}(x,t) + L_1 u_t(x,t) + L_2 u(x,t) = f(x,t) \text{ on } \Omega \qquad (1)$$

subject to boundary conditions of the form $Bu(x,t)=0$ on $\partial\Omega$ and the usual initial conditions. Here, $u(x,t)$ represents the displacement of the point x in the bounded open region Ω in R^n, $n=1,2,3$, at time $t>0$. The region Ω is bounded by the boundary $\partial\Omega$ and the operator B is a linear spatial differential operator expressing the usual boundary conditions. The subscript t indicates partial differentiation with respect to time and L_1 and L_2 are linear partial differential operators in spatial coordinates. In order to insure the existence of a series converging to the solution of Eq. (1), several assumptions on the time invariant operators L_1 and L_2 must be made. Fortunately these assumptions are not too harsh and Eq. (1), along with the various assumptions on the operators L_1 and L_2, adequately describes the linear vibration of strings, beams, membranes, plates, etc.

Let $L_2(\Omega)$ denote the Hilbert space of all real-valued squared integrable functions on the domain Ω in the Lebesgue sense with the usual inner product and norm defined by

$$\langle u,v\rangle = \int_\Omega u(x)v(x)\,\mathrm{d}\Omega$$

and

$$\|u(x)\| = \langle u,u\rangle^{1/2}$$

respectively. Let L_1 and L_2 be time-invariant partial differential operators of order n_1 and n_2, respectively. Let $D(L)$ be the set of all functions $u(x,t)$ such that $Bu=0$ on $\partial\Omega$ and $u(x,t)$ and all of its derivatives up to the order $K=\max(n_1,n_2)$ are in $L_2(\Omega)$. The assumptions required for $u(x,t)$ to be expressed in terms of a convergent series in orthonormal eigenfunctions may now be concisely stated as follows.[2] If L_1 and L_2 are self-adjoint on $D(L)$ such that $L_1L_2=L_2L_1$ for all functions in $D(L)$ and if each operator has a compact resolvent,[3] then the solution of Eq. (1) may be written as the uniformly convergent series

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x) \qquad (2)$$

where the set $\{\phi_n(x)\}_{n=1}^{\infty}$ are the orthonormal eigenfunctions ($\langle\phi_n,\phi_m\rangle=\delta_{mn}$, the Kronecker delta) of the operator L_2 that are identical to the eigenfunctions of L_1.[4] The temporal functions $a_n(t)$ are assumed to be at least twice differentiable.

The restriction of requiring the coefficient operators to have compact resolvents will be satisfied if the stiffness operator (L_2) of the vibration problem of interest has an inverse defined by a Green's function. The requirement that the operators L_1 and L_2 commute is tantamount to restricting the class of problems considered to the class that can be solved by separation of variables.

The independent modal control method focuses on controlling the temporal functions $a_n(t)$. The work presented here concerns itself with the nature of the function $f(x,t)$ in Eq. (1) considered as a distributed control and how to choose $f(x,t)$ in such a way as to allow the method of independent modal space control to be used.

Previous Theory

In Ref. 1, systems given by Eq. (1) with $L_1=0$ are discussed and an optimal control method is developed for proportional control. This method is based on substitution of Eq. (2) into Eq. (1), multiplying by $\phi_m(x)$ and integrating over the domain Ω. This yields an infinite number of decoupled second-order ordinary differential equations of motion given by

$$\ddot{a}_n(t) + \lambda_n^2 a_n(t) = f_n(t) \qquad (3a)$$

where λ_n^2 are the eigenvalues of L_2 associated with the eigenfunctions $\phi_n(x)$ and $f_n(t)$ is a generalized control force given by

$$f_n(t) = \int_\Omega \phi_n(x)f(x,t)\,\mathrm{d}\Omega \qquad (3b)$$

Submitted April 25, 1983; revision received Aug. 3, 1983.

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