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Solving Optimal Continuous Thrust Rendezvous Problems with Generating Functions

Chandeok Park,* Vincent Guibout,† and Daniel J. Scheeres‡
University of Michigan, Ann Arbor, Michigan 48109

The optimal control of a spacecraft as it transitions between specified states using continuous thrust in a fixed amount of time is studied using a recently developed technique based on Hamilton–Jacobi theory. Started from the first-order necessary conditions for optimality, a Hamiltonian system is derived for the state and adjoints with split boundary conditions. Then, with recognition of the two-point boundary-value problem as a canonical transformation, generating functions are employed to find the optimal feedback control, as well as the optimal trajectory. Although the optimal control problem is formulated in the context of the necessary conditions for optimality, our closed-loop solution also formally satisfies the sufficient conditions for optimality via the fundamental connection between the optimal cost function and generating functions. A solution procedure for these generating functions is posed and numerically tested on a nonlinear optimal rendezvous problem in the vicinity of a circular orbit. Generating functions are developed as series expansions, and the optimal trajectories obtained from them are compared favorably with those of a numerical solution to the two-point boundary-value problem using a forward-shooting method.

Nomenclature

A	=	linearized system dynamics about the circular reference trajectory
$\arg\min_{(\cdot)}$	=	argument minimum with respect to the variable $(\cdot)$
B	=	linearized system dynamics about the circular reference trajectory
$\boldsymbol{F}$	=	nongravitational control force vector
F_1, F_2, F_3, F_4	=	principal kinds of generating functions
$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t)$	=	system dynamics
$H(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{u}, t)$	=	Hamiltonian of the system
I	=	identity matrix
$\boldsymbol{i}$	=	radial unit vector in the rotating coordinate frame
J	=	performance index to be minimized
$\boldsymbol{j}$	=	tangential unit vector in the rotating coordinate frame
$\boldsymbol{k}$	=	normal unit vector in the rotating coordinate frame
$L(\boldsymbol{x}, \boldsymbol{u}, t)$	=	full time performance index or Lagrangian
m	=	mass of the spacecraft, assumed constant in the current application
$\boldsymbol{p}$	=	adjoint vector
$\boldsymbol{R}$	=	position vector of the origin of the rotating frame from the central body, $R\boldsymbol{i}$
$\boldsymbol{r}$	=	position vector of the spacecraft from the center of gravity
r	=	$\lvert\boldsymbol{r}\rvert$
t	=	time
$\boldsymbol{u}$	=	control vector
u_x	=	radial $\boldsymbol{i}$ component of control acceleration
u_y	=	tangential $\boldsymbol{j}$ component of control acceleration
u_z	=	normal $\boldsymbol{k}$ component of control acceleration
$\boldsymbol{x}$	=	state vector
x	=	radial $\boldsymbol{i}$ component of $\delta\boldsymbol{r}$
y	=	tangential $\boldsymbol{j}$ component of $\delta\boldsymbol{r}$
z	=	normal $\boldsymbol{k}$ component of $\delta\boldsymbol{r}$
$\delta\boldsymbol{r}$	=	position vector of the spacecraft from the origin of the rotating frame, $[x \;\; y \;\; z]^T$
μ	=	gravitational parameter of the central body
ω	=	$\omega k = \sqrt{(\mu/R^3)}k$, constant angular velocity vector
$\Phi(t, t_0)$	=	state transition matrix
$\lvert\cdot\rvert$	=	magnitude of the vector quantity

Subscripts

f	=	terminal value of the variable
0	=	initial value of the variable

Superscripts

$\cdot$	=	time derivative
$*$	=	optimized or minimized variable with respect to $\boldsymbol{u}$

Introduction

THIS paper presents a novel approach to evaluating optimal continuous thrust trajectories and feedback control laws for a spacecraft subject to a general gravity field. This approach is derived by relying on the Hamiltonian nature of the necessary conditions associated with optimal control and by using certain properties of generating functions and canonical transformations. In particular, we show that certain solutions to the Hamilton–Jacobi (HJ) equation, associated with canonical transformations of Hamiltonian systems, can directly yield optimal control laws for a general system. Typically, application of Pontryagin's principle changes the nonlinear optimal rendezvous problem to a two-point boundary value problem (TPBVP), for which one generally requires an initial estimate for the initial (or final) adjoint variables followed by an iterative solution procedure. Our approach provides an algorithm to compute the initial (or final) values of the adjoints without requiring an initial estimate and, for arbitrary boundary conditions, simply by algebraic manipulations of the generating function. Our approach not only satisfies the TPBVP found from the necessary conditions, by

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*Ph.D. Candidate, Department of Aerospace Engineering; chandeok@umich.edu.

†Research Assistant, Department of Aerospace Engineering; guibout@umich.edu.

‡Associate Professor of Aerospace Engineering; scheeres@umich.edu. Associate Fellow AIAA.

definition, but also satisfies the Hamilton–Jacobi–Bellman equation for the optimal cost, which is a sufficient condition for optimality. Most important, we have derived and applied a general solution procedure for this problem to a nonlinear dynamic system of interest to astrodynamics. To develop and apply this algorithm requires certain conditions on the dynamics and cost function, which we detail in this paper.

Since Lawden[1] initially introduced primer vector theory, the problem of continuous thrust optimal rendezvous has been a topic of continual interest. Much work has been done on this topic; here we only give a brief review of work that has direct relation to analytical work on the optimal rendezvous problem for space trajectories. Billik[2] applied differential game theory to rendezvous problems subject to linearized dynamics. London[3] and Antony and Sasaki[4] studied the uncontrolled motion subject to second-order approximation. Euler[5] considered low-thrust optimal rendezvous maneuver in the vicinity of an elliptical orbit. Jezewski and Stoolz[6] considered minimum-time problems subject to the inverse square field and evaluated an analytic solution under highly restricted assumptions. Later, Marec[7] extended Lawden's primer vector theory graphically with the Contensou principle. Various types of continuous thrust optimal rendezvous problems subject to a linearized gravity field have been extensively explored by Carter,[8] Carter and Humi,[9] Humi,[10] Carter and Paridis,[11] and others. Also, Lembeck and Prussing[12] solved a combined problem of impulse intercept and continuous-thrust rendezvous subject to linearized dynamics.

As is seen, except for the very basic works of London,[3] Antony and Sasaki,[4] and Jezewski and Stoolz,[6] all of the cited researchers consider linearized dynamics, which clearly limits the applicability and utility of this problem. Thus, it is desirable to find the optimal trajectory subject to the original nonlinear dynamics. However, to do so in general requires one to solve the TPBVP for the adjoints for each boundary condition of interest, a challenging problem. Additionally, it is even more difficult to find a nonlinear optimal feedback control, generally found by solving the Hamilton–Jacobi–Bellman (HJB) equation for the optimal cost function.

Recently, using generating functions appearing in HJ theory, Guibout and Scheeres[13,14] suggested a new methodology to solve TPBVPs for Hamiltonian systems, including optimal control systems defined by Pontryagin's necessary condition. Based on these works, Park and Scheeres[15] devised a new algorithm to solve optimal feedback control problem. Evaluating generating functions, they computed the initial adjoints (without guess) and obtained optimal feedback control for a special type of boundary conditions (hard constraint problem). Then Scheeres et al.[16] applied their method to the nonlinear optimal rendezvous of a spacecraft and demonstrated that their higher-order control law is superior to the linear control law. Later, in Refs. 17 and 18, Park and Scheeres extended the applicability of their method for general boundary conditions.

This document is a continuation and extension of Ref. 16 and demonstrates a direct application of this algorithm to continuous thrust optimal rendezvous problems subject to inverse-square central gravity fields. The discussion is structured as follows. First, we give a brief review of classical optimal control theory as applied to a specific class of problems, then motivate our current approach and show how it satisfies the necessary conditions by default, as well as how it can be used to derive an optimal feedback control law. Then, we formulate the continuous thrust optimal control of transferring from one state to another, using nonlinear dynamics relative to a circular orbit. A detailed solution procedure follows, and the trajectories based on higher-order control law are compared with those based on linear control. Finally, we discuss the uniqueness of our solutions and contrast the current results with previous works.

General Solution of the Optimal Control Problem

Classical Necessary Conditions for Optimal Control

Assume we have a dynamic system stipulated as $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t)$. The goal is to transfer from an initial state to a final state in a specified time span while minimizing some cost function. The application envisioned is for a spacecraft in a specified state (consisting of a specific orbit, hence, position and velocity) to transition to another state while minimizing the Lagrange-type cost function of the form

$$J = \int_{t_0}^{t_f} L(\boldsymbol{x}, \boldsymbol{u}, t)\,\mathrm{d}t \tag{1}$$

For a comprehensive introduction to the theory of optimal control of space trajectories, we cite Refs. 1 and 7. The Hamiltonian of the system can be stated as

$$\mathcal{H}(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{u}, t) = L(\boldsymbol{x}, \boldsymbol{u}, t) + \boldsymbol{p}(t) \cdot \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t) \tag{2}$$

Applying the Pontryagin principle, we find the optimal control,

$$\boldsymbol{u}^* = \arg\min_{u} \mathcal{H}(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{u}, t) \tag{3}$$

Substituting this control back into $\mathcal{H}$ leads to the new Hamiltonian $\mathcal{H}^*(\boldsymbol{x}, \boldsymbol{p}, t)$ and to a necessary condition for the optimal control system,

$$\dot{\boldsymbol{x}} = \frac{\partial \mathcal{H}^*}{\partial \boldsymbol{p}} \tag{4}$$

$$\dot{\boldsymbol{p}} = -\frac{\partial \mathcal{H}^*}{\partial \boldsymbol{x}} \tag{5}$$

For the associated boundary conditions, note that the initial and terminal states should be completely specified to reflect the rendezvous condition, that is,

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0, \qquad \boldsymbol{x}(t_f) = \boldsymbol{x}_f \tag{6}$$

The fundamental difficulty in this approach is, as is well known, finding the initial or final adjoints, $\boldsymbol{p}_0$ or $\boldsymbol{p}_f$, satisfying this boundary-value problem. Once we find $\boldsymbol{p}_0$ (or $\boldsymbol{p}_f$), we can directly integrate the associated differential equations forward (or backward), along with the specified initial (or terminal) states, solving for the optimal control from the Pontryagin's principle at each point along the trajectory.

Motivation of the Proposed Method

The drawback of the approach just described is that solution procedures for the TPBVP generally require an initial estimate for the adjoints, which usually have no physical interpretations. Furthermore, we must repetitively solve the TPBVP for each boundary condition of interest, which is time consuming, lacks definiteness, and is subject to numerical divergence. The conventional alternative method is to solve the HJB equation for the optimal cost function and, thus, to evaluate the optimal cost and the corresponding optimal control law. However, the HJB is a first-order partial differential equation and is extremely difficult to solve in general. Furthermore, for the boundary condition we are considering here, the HJB cost function has a singularity at the terminal condition, which makes the problem even more difficult. (This singularity is discussed in more detail in Ref. 15.)

In an attempt to overcome these disadvantages, we suggest an alternative method, which specifically utilizes the theory of canonical transformations and their associated generating functions. This method provides a way to compute the initial (or final) adjoints as a function of known initial and/or final states and, thus, to evaluate the optimal trajectory by simple forward (or backward) integration. It also enables us to construct systematically the optimal feedback control, even with the fundamental singularity prevailing in the HJB equation at its terminal condition. The next section is dedicated to the discussion of our approach.

Solution of the Boundary-Value Problem Using Generating Functions

Recall the theory of canonical transformations and generating functions in Hamiltonian dynamics (c.f., Ref. 19). In addition to generating canonical transformations between Hamiltonian systems, generating functions also solve boundary-value problems between Hamiltonian coordinate and momentum states for a single flow field. See the Appendix for a more detailed derivation of the results we present in the following paragraphs.

In the case where the initial and terminal states are explicitly given, the generating function $F_1(\boldsymbol{x}_0, \boldsymbol{x}_f, t_0, t_f)$ can be directly used to find the initial and final momentum vectors from the relationship

$$\boldsymbol{p}_0 = -\frac{\partial F_1(\boldsymbol{x}_0, \boldsymbol{x}_f, t_0, t_f)}{\partial \boldsymbol{x}_0}, \qquad \boldsymbol{p}_f = \frac{\partial F_1(\boldsymbol{x}_0, \boldsymbol{x}_f, t_0, t_f)}{\partial \boldsymbol{x}_f} \tag{7}$$

The generating function F_1 also satisfies a HJ partial differential equation,

$$\frac{\partial F_1}{\partial t_f} + \mathcal{H}^*\left(\boldsymbol{x}_f, \frac{\partial F_1}{\partial \boldsymbol{x}_f}, t_f\right) = 0 \tag{8}$$

Analogous relations and definitions exist for the generating functions $F_2(\boldsymbol{x}_f, \boldsymbol{p}_0, t_0, t_f)$, $F_3(\boldsymbol{x}_0, \boldsymbol{p}_f, t_0, t_f)$, and $F_4(\boldsymbol{p}_0, \boldsymbol{p}_f, t_0, t_f)$ with results

$$\boldsymbol{x}_0 = \frac{\partial F_2}{\partial \boldsymbol{p}_0}, \qquad \boldsymbol{p}_f = \frac{\partial F_2}{\partial \boldsymbol{x}_f} \tag{9}$$

$$\boldsymbol{p}_0 = -\frac{\partial F_3}{\partial \boldsymbol{x}_0}, \qquad \boldsymbol{x}_f = -\frac{\partial F_3}{\partial \boldsymbol{p}_f} \tag{10}$$

$$\boldsymbol{x}_0 = -\frac{\partial F_4}{\partial \boldsymbol{p}_0}, \qquad \boldsymbol{x}_f = \frac{\partial F_4}{\partial \boldsymbol{p}_f} \tag{11}$$

These functions all solve their own version of the HJ equation,

$$\frac{\partial F_2}{\partial t_f} + \mathcal{H}^*\left(\boldsymbol{x}_f, \frac{\partial F_2}{\partial \boldsymbol{x}_f}, t_f\right) = 0 \tag{12}$$

$$\frac{\partial F_3}{\partial t_f} + \mathcal{H}^*\left(-\frac{\partial F_3}{\partial \boldsymbol{p}_f}, \boldsymbol{p}_f, t_f\right) = 0 \tag{13}$$

$$\frac{\partial F_4}{\partial t_f} + \mathcal{H}^*\left(-\frac{\partial F_4}{\partial \boldsymbol{p}_f}, \boldsymbol{p}_f, t_f\right) = 0 \tag{14}$$

A final property of the generating functions is that they can be transformed into each other via the Legendre transformation. Specifically, we find the following relations between the generating functions:

$$F_2(\boldsymbol{x}_0, \boldsymbol{p}_f, t_0, t_f) = F_1(\boldsymbol{x}_0, \boldsymbol{x}_f, t_0, t_f) + \boldsymbol{p}_f \cdot \boldsymbol{x}_f \tag{15}$$

$$F_3(\boldsymbol{p}_0, \boldsymbol{x}_f, t_0, t_f) = F_1(\boldsymbol{x}_0, \boldsymbol{x}_f, t_0, t_f) - \boldsymbol{p}_0 \cdot \boldsymbol{x}_0 \tag{16}$$

$$F_4(\boldsymbol{p}_0, \boldsymbol{p}_f, t_0, t_f) = F_2(\boldsymbol{x}_0, \boldsymbol{p}_f, t_0, t_f) - \boldsymbol{p}_0 \cdot \boldsymbol{x}_0 \tag{17}$$

The key observation we make is that solving for F_1 solves the boundary-value problem and, hence, the optimal control problem. Suppose there exists an analytical form for F_1 such that we can find it. Then, we can directly take its partial derivatives, specify $\boldsymbol{x}_0$ and $\boldsymbol{x}_f$, and find the appropriate momentum to generate the optimal control for rendezvous. Also it has been shown that $J = -F_1$ satisfies the HJB equation and, thus, is the optimal cost function and satisfies the sufficient condition for optimality (see Ref. 18). Finally, using control (3) and the desired F_1 function, we can define a feedback control law:

$$\boldsymbol{u}^* = \arg\min_{u} \mathcal{H}\left[\boldsymbol{x}, -\frac{\partial F_1(\boldsymbol{x}, \boldsymbol{x}_f, t, t_f)}{\partial \boldsymbol{x}}, \boldsymbol{u}, t\right] \tag{18}$$

where we fix the terminal boundary condition $\boldsymbol{x}_f$ and allow the initial condition to equal the current state. Note that we have replaced $\boldsymbol{x}_0$ and t_0 with $\boldsymbol{x}$ and t to stress the arbitrariness of initial states and, thus, the feedback nature of the control law.

Implementing a Solution for F_1

The difficulty, of course, is in finding the generating function F_1. This problem is directly addressed by Guibout and Scheeres,[14] who show that the generating functions, if they exist in analytical form, can be solved as power series expansions in their respective arguments. The coefficients of these power series satisfy a set of ordinary differential equations derived from the HJ equations for generating functions.

To carry out this method, however, requires that some restrictions be placed on the system dynamics and cost function. The approach developed in Ref. 14 is based on constructing local solutions to the generating functions, that is, expanding them as a Taylor series about a nominal trajectory that is known. This implies that a solution to the optimal control problem has already been found and that our specific method operates in the vicinity of this solution. Note that this includes the case of no control, that is, if the dynamics of the system carry a state between two points, $\boldsymbol{x}_0$–$\boldsymbol{x}_f$, then the optimal control for this transition is simply stated as $\boldsymbol{u} \equiv 0$ and our method can be used on such a system.

Thus, to apply formally that method to the current system requires that the system has zero equilibrium point and to satisfy $\boldsymbol{f}(\boldsymbol{x} = \boldsymbol{0}, \boldsymbol{u} = \boldsymbol{0}, t) = \boldsymbol{0}$. Furthermore, as we expand the Hamiltonian as a Taylor series of states and adjoints about a nominal solution, we also require analyticity of the Hamiltonian. This, in turn, places a requirement on the analyticity of L in Eq. (1) (because this becomes part of the Hamiltonian function through the Pontryagin principle). Finally, we assume the control u is unbounded also for the sake of analyticity. (Note that even with all of these strong assumptions about analyticity, the convergence of the series solution of generating function is not always guaranteed. For some special cases including resonance phenomenon, the convergence of the series solution may be suspect as time evolves, in which case our series-based method should be used with caution.)

The process derived in Ref. 14 consists of expanding the Hamiltonian function as a Taylor series in the states and adjoints and the F_1 generating function as a Taylor series in the initial and final states. Then the series for F_1 is substituted into the HJ equation (8) and a balancing technique is used to equate all like powers of the states to zero. This defines a set of differential equations for the coefficients of the F_1 Taylor series expansion. A major problem in this approach, however, is that initial conditions for F_1 at time $t_0 = t_f$ and $\boldsymbol{x}_0 \neq \boldsymbol{x}_f$ are undefined, making it impossible to initiate the integration of the coefficients. Furthermore, these coefficients are not known a priori at any other time. This problem can be circumvented, however, by solving for a different generating function and then transforming back to the F_1 function, using the Legendre transformation, at some later moment when F_1 is well defined. (In general, it happens that all generating functions may suffer from singularities, but at different moments. In this case it is impossible to solve for one generating function for the entire time span of interest. Instead, we initiate the time evolution for one generating function, then before a singularity occurs, we jump to another generating function via the Legendre transformation to reinitiate the time evolution. However, it will be shown later that F_1 and F_2 in our formulation do not suffer from such multiple singularities, but have only one inherent singularity for F_1. For those who are more interested in this singularity issue, we cite Guibout and Scheeres.[13]) Given a power series expansion for a given $F_i, i = 1 \sim 4$, it is always possible to transform to a different generating function using the transformations in Eqs. (15–17) along with the fundamental results given in Eqs. (7) and (9–11).

The F_2 generating function, it turns out, can be solved using our initial-value approach, in contrast to F_1. This is because F_2 can generate the identity transformation when $t_0 = t_f$. (Refer to Refs. 13 and 18 for more detailed arguments.) Thus, we solve for the F_2 generating function as a function of time by integrating the differential equations for the coefficients and, when needed, transform to the F_1 function via the Legendre transformation to solve the boundary-value problem, which in turn solves the optimal control problem.

In this paper, we investigate the application of this approach to the optimal control of a spacecraft in the vicinity of a nominal trajectory, incorporating dynamic nonlinearities. For definiteness, we will develop and apply this approach for a specific example.

Specific Formulation of the Optimal Rendezvous Problem

Consider a spacecraft subject to a central gravity field. Its equations of motion, in the inertial frame with the origin located at the center of gravity, are given by

$$\ddot{\boldsymbol{r}} = -(\mu/r^3)\boldsymbol{r} + \boldsymbol{F}/m \tag{19}$$

We introduce another coordinate frame that is rotating along a circular orbit at a constant angular velocity. Then represented in the rotating coordinate frame, the position, velocity, and acceleration vectors are, respectively,

$$\boldsymbol{r} = \boldsymbol{R} + \delta\boldsymbol{r} = (R + x)\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \tag{20}$$

$$\Rightarrow \dot{\boldsymbol{r}} = (\dot{x} - \omega y)\boldsymbol{i} + [\dot{y} + \omega(R + x)]\boldsymbol{j} + \dot{z}\boldsymbol{k} \tag{21}$$

$$\Rightarrow \ddot{\boldsymbol{r}} = [\ddot{x} - 2\omega\dot{y} - \omega^2(R + x)]\boldsymbol{i} + (\ddot{y} + 2\omega\dot{x} - \omega^2 y)\boldsymbol{j} + \ddot{z}\boldsymbol{k} \tag{22}$$

From Newton's law, we obtain the following componentwise equations of motion in the rotating frame:

$$\ddot{x} - 2\omega\dot{y} - \omega^2(R + x) = -(\mu/r^3)(R + x) + u_x \tag{23}$$

$$\ddot{y} + 2\omega\dot{x} - \omega^2 y = -(\mu/r^3)y + u_y \tag{24}$$

$$\ddot{z} = -(\mu/r^3)z + u_z \tag{25}$$

where $r = \sqrt{[(R + x)^2 + y^2 + z^2]}$. If nondimensionalized with reference length R and reference time $1/\omega$, they are simplified as

$$\ddot{x} - 2\dot{y} + (1 + x)(1/r^3 - 1) = u_x \tag{26}$$

$$\ddot{y} + 2\dot{x} + y(1/r^3 - 1) = u_y \tag{27}$$

$$\ddot{z} + (1/r^3)z = u_z \tag{28}$$

where now $r = \sqrt{[(x + 1)^2 + y^2 + z^2]}$. For simplicity's sake, we consider planar motion henceforth. Defining the states as $\boldsymbol{x} = [x_1\ \ x_2\ \ x_3\ \ x_4]^T = [x\ \ y\ \ \dot{x}\ \ \dot{y}]^T$ and control as $\boldsymbol{u} = [u_1\ \ u_2]^T = [u_x\ \ u_y]^T$, we can construct the equations of planar motion in state space form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ 2x_4 - (1 + x_1)(1/r^3 - 1) + u_1 \\ -2x_3 - x_2(1/r^3 - 1) + u_2 \end{bmatrix} \tag{29}$$

where $r = \sqrt{[(x_1 + 1)^2 + x_2^2]}$. Note that linearization about the circular reference trajectory leads to the in-plane dynamics of the well-known Clohessy–Wiltshire equation,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{30}$$

$$\Leftrightarrow \dot{x} = Ax + Bu$$

Furthermore, the right-hand side of Eq. (29) is analytic and can be expanded as a Taylor series about the circular reference trajectory,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ 3x_1 + 2x_4 - 3x_1^2 + 1.5x_2^2 + 4x_1^3 - 6x_1x_2^2 + \cdots + u_1 \\ -2x_3 + 3x_1x_2 - 6x_1^2x_2 + 1.5x_2^3 + \cdots + u_2 \end{bmatrix} \tag{31}$$

a result which will be used later.

Finally, the objective is to minimize the quadratic performance index

$$J = \frac{1}{2}\int_{t_0}^{t_f} \boldsymbol{u}^T(t)\boldsymbol{u}(t)\,\mathrm{d}t \tag{32}$$

subject to the nonlinear dynamics (29), satisfying given boundary conditions. Note that the integrand of J, i.e., $L = \boldsymbol{u}^T\boldsymbol{u}/2 = (u_1^2 + u_2^2)/2$, is analytic with respect to its arguments.

Nonlinear Analytical Solution to the Optimal Rendezvous Problem

Derivation of the Optimal Solution in Series Form

To summarize the preceding discussion, let us consider minimization of the quadratic cost

$$J = \frac{1}{2}\int_{t_0}^{t_f} \boldsymbol{u}^T(t)\boldsymbol{u}(t)\,\mathrm{d}t \tag{33}$$

subject to the system dynamics in central gravity field

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ 3x_1 + 2x_4 - 3x_1^2 + 1.5x_2^2 + 4x_1^3 - 6x_1x_2^2 + \cdots + u_1 \\ -2x_3 + 3x_1x_2 - 6x_1^2x_2 + 1.5x_2^3 + \cdots + u_2 \end{bmatrix} \tag{34}$$

With the Hamiltonian $\mathcal{H}$ defined as

$$\mathcal{H} = \tfrac{1}{2}\boldsymbol{u}^T\boldsymbol{u} + \boldsymbol{p}^T\dot{\boldsymbol{x}} \tag{35}$$

which can be expanded as a Taylor series to find

$$\begin{aligned} \mathcal{H} = {} & \tfrac{1}{2}\left(u_1^2 + u_2^2\right) + p_1x_3 + p_2x_4 \\ & + p_3\left(3x_1 + 2x_4 - 3x_1^2 + 1.5x_2^2 + 4x_1^3 - 6x_1x_2^2 + \cdots + u_1\right) \\ & + p_4\left(-2x_3 + 3x_1x_2 - 6x_1^2x_2 + 1.5x_2^3 + \cdots + u_2\right) \end{aligned} \tag{36}$$

the corresponding costate equations are

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \\ \dot{p}_4 \end{bmatrix} = \begin{bmatrix} -3p_3 + 6x_1p_3 - 3x_2p_4 - 12x_1^2p_3 + 6x_2^2p_3 + 12x_1x_2p_4 \cdots \\ -3x_2p_3 - 3x_1p_4 + 12x_1x_2p_3 + 6x_1^2p_4 - 4.5x_2^2p_4 \cdots \\ -p_1 + 2p_4 \\ -p_2 - 2p_3 \end{bmatrix} \tag{37}$$

From the optimality condition $\mathcal{H}_u = 0$, we find the optimal control law

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -p_3 \\ -p_4 \end{bmatrix} \tag{38}$$

Introducing Eq. (38) into Eq. (36) yields the Hamiltonian as a function of states and adjoints,

$$\begin{aligned} \mathcal{H}(\boldsymbol{x}, \boldsymbol{p}, t) = {} & \tfrac{1}{2}\left(p_3^2 + p_4^2\right) + p_1x_3 + p_2x_4 \\ & + p_3\left(3x_1 + 2x_4 - 3x_1^2 + 1.5x_2^2 + 4x_1^3 - 6x_1x_2^2 + \cdots - p_3\right) \\ & + p_4\left(-2x_3 + 3x_1x_2 - 6x_1^2x_2 + 1.5x_2^3 + \cdots - p_4\right) \end{aligned} \tag{39}$$

As discussed earlier, we evaluate $F_2(\boldsymbol{x}, \boldsymbol{p}_0, t; t_0)$ as a power series instead of $F_1(\boldsymbol{x}, \boldsymbol{x}_0, t; t_0)$. For illustration, in the following paragraphs, we only derive the equations to the linear order for the control law; in the actual analysis, we kept higher orders terms (HOT) using symbolic manipulators. The Hamiltonian is reduced to

$$\mathcal{H}(\boldsymbol{x}, \boldsymbol{p}, t) = \frac{1}{2}\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{p} \end{bmatrix}^T \begin{bmatrix} 0 & A^T \\ A & -BB^T \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{p} \end{bmatrix} + \text{HOT} \tag{40}$$

In keeping with this quadratic form of the Hamiltonian, we also expand F_2 in a quadratic form,

$$F_2(\boldsymbol{x}, \boldsymbol{p}_0, t; t_0) = \frac{1}{2}\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{p}_0\end{bmatrix}^T \begin{bmatrix}F_{xx}(t;t_0) & F_{xp_0}(t;t_0)\\ F_{p_0x}(t;t_0) & F_{p_0p_0}(t;t_0)\end{bmatrix}\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{p}_0\end{bmatrix} + \text{HOT} \tag{41}$$

Now recalling the relation

$$\boldsymbol{p} = \frac{\partial F_2}{\partial \boldsymbol{x}} = \begin{bmatrix}F_{xx} & F_{xp_0}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{p}_0\end{bmatrix} + \text{HOT} \tag{42}$$

we can express the Hamiltonian as

$$\mathcal{H} = \frac{1}{2}\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{p}_0\end{bmatrix}^T \begin{bmatrix}I & F_{xx}\\ 0 & F_{p_0x}\end{bmatrix}\begin{bmatrix}0 & A^T\\ A & -BB^T\end{bmatrix}\begin{bmatrix}I & 0\\ F_{xx} & F_{xp_0}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{p}_0\end{bmatrix} + \text{HOT} \tag{43}$$

Introduction of Eqs. (41) and (43) into the HJ equation (12) yields the following set of differential equations for $F_{xx}(t;t_0)$, $F_{xp_0}(t;t_0) = F^T_{p_0x}(t;t_0)$, and $F_{p_0p_0}(t;t_0)$:

$$0 = \dot{F}_{xx} + F_{xx}A + A^T F_{xx} - F_{xx}BB^T F_{xx}$$
$$0 = \dot{F}_{xp_0} + A^T F_{xp_0} - F_{xx}BB^T F_{xp_0}$$
$$0 = \dot{F}_{p_0p_0} - F_{p_0x}BB^T F_{xp_0} \tag{44}$$

Also, the corresponding initial conditions are derived from the identity transformation, $F_2(\boldsymbol{x}, \boldsymbol{p}_0, t = t_0; t_0) = \boldsymbol{x}\cdot\boldsymbol{p}_0$, as

$$F_{xx}(t_0;t_0) = 0, \qquad F_{xp_0}(t_0;t_0) = I = F_{p_0x}(t_0;t_0)$$
$$F_{p_0p_0}(t_0;t_0) = 0 \tag{45}$$

Again, refer to Refs. 13 and 18 for more detailed arguments.

Note that $F_{xx} \equiv 0$ due to the zero initial conditions; it satisfies the corresponding differential equation and the given initial condition. Generalizing this method, we can solve recursively for the remaining HOT. We do not show the HOT here, due to space limitations. The symbolic and numerical computations and results reported here have been carried out using MATLAB® and Mathematica®.

Once this system of differential equations is solved up to as high an order as desired, we can construct F_2. Then, rearranging the second equation of Eq. (9) for $\boldsymbol{p}_f = \boldsymbol{p}_f(\boldsymbol{x}_0, \boldsymbol{x}_f)$ using series inversion and introducing into the Legendre transformation (15) leads to $F_1(\boldsymbol{x}_0, \boldsymbol{x}_f, t_0, t_f)$,

$$F_1 = \frac{1}{2}\begin{bmatrix}\boldsymbol{x}_f\\ \boldsymbol{x}_0\end{bmatrix}^T \times \begin{bmatrix}\left(F_{xx} - F_{xp_0}F^{-1}_{p_0p_0}F_{p_0x}\right)(t_f,t_0) & \left(F_{xp_0}F^{-1}_{p_0p_0}\right)(t_f,t_0)\\ \left(F^{-1}_{p_0p_0}F_{p_0x}\right)(t_f,t_0) & -\left(F^{-1}_{p_0p_0}\right)(t_f,t_0)\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_f\\ \boldsymbol{x}_0\end{bmatrix} + \frac{1}{3!}\sum_i^{2n}\sum_j^{2n}\sum_k^{2n} f_{ijk}(t_f,t_0)y_iy_jy_k + \cdots \tag{46}$$

where $\boldsymbol{y} = [y_1\ y_2\ \cdots\ y_{2n}]^T = [x_{f1}\ \cdots\ x_{fn}\ x_{01}\ \cdots\ x_{0n}]^T$ and the last term indicates the HOT expressed with a tensor notation. (Refer to [13] for a rigorous derivation of HOT of F_1.) Note that the optimal cost function, from the HJB theory, simply equals $J(\boldsymbol{x}, t; \boldsymbol{x}_f, t_f) = -F_1(\boldsymbol{x}, \boldsymbol{x}_f, t, t_f)$, where we fix the terminal condition and take the initial condition as a moving coordinate $[\boldsymbol{x}_0 \to \boldsymbol{x}(t)]$. Also $\boldsymbol{p}_0$ can be computed from Eq. (7), which enables us to evaluate the optimal trajectory by simple forward integration.

Finally, after some algebraic manipulations, the optimal feedback control can be obtained from Eq. (18),

$$\boldsymbol{u}^* = -B^T\left[F^{-1}_{p_0p_0}(t_f,t)\left[\boldsymbol{x}(t) - F_{p_0x}(t_f,t)\boldsymbol{x}(t_f)\right] + \frac{\partial}{\partial \boldsymbol{x}}\frac{1}{3!}\sum_i^{2n}\sum_j^{2n}\sum_k^{2n} f_{ijk}(t_f,t)y_iy_jy_k + \cdots\right] \tag{47}$$

Here $\boldsymbol{y} = [y_1\ y_2\ \cdots\ y_{2n}]^T = [x_{f1}\ \cdots\ x_{fn}\ x_1\ \cdots\ x_n]^T$ and the partial differentiation $\partial/\partial\boldsymbol{x}$ is with respect to the initial variables $\boldsymbol{x} = [x_1\ \cdots\ x_n]^T = [y_{n+1}\ \cdots\ y_{2n}]^T$. Note that we only compute the coefficients for F_1 once as a function of time, then we have complete freedom to change initial and final conditions and time span.

Remark: The preceding series solutions for F_1 and F_2 formally satisfy their respective HJ equations. Although our numerical comparisons with the reference trajectories are highly suggestive of the convergence of our series solution to the reference solution, we have not proven the convergence of our series solution and, hence, have not proven the existence of the solution. In Refs. 17 and 18, it is proven that the feedback control law derived from F_1 satisfies the sufficient conditions for optimality. Thus, if the convergence of our series solution for F_1 is proven, then we will have satisfied the sufficient conditions. This is a topic of future research.

Numerical Example

Before the discussion of specific numerical examples, recall that we have used nondimensionalized equations of motion. Given the appropriate scale factor, we can analyze any circular reference orbits of arbitrary altitude. Here, as an example, we consider a geosynchronous orbit where the reference frequency is $\omega = 2\pi/1\ \text{day} = 7.27\times10^{-5}$ rad/s, the reference time is $\bar{t} = 1/\omega = 1.38\times10^4$ s, and the reference length is $R = \sqrt[3]{(\mu/\omega^2)} = 4.23\times10^4$ km. Also for the control inputs, one nondimensional unit corresponds to 0.222 m/s^2.

Figures 1–9 show the optimal trajectory and control history for three specific examples. Example 1 (Figs. 1–3) represents the result for a general offset in initial conditions of [0.2, 0.2, 0.1, 0.1] in position and velocity, transitioning to the origin [0, 0, 0, 0] in one unit of time. This rather general condition has been chosen to test the validity of our algorithm. Example 2 (Figs. 4–6) shows the optimal trajectory starting from a circular orbit displaced in downtrack direction with an offset of 0.1 units, that is, [0, 0.1, 0, 0], and then transitioning to a circular orbit at the coordinate frame origin in one unit of time. Example 3 (Figs. 7–9) shows a similar result from an offset of 0.003 units, that is, [0, 0.003, 0, 0], in the downtrack direction to the origin in 2π units of time, that is, one orbital period. As the time of the transfer increases, more terms are needed in the control to approximate accurately the true solution. Here the optimal controls are developed up to fourth order.

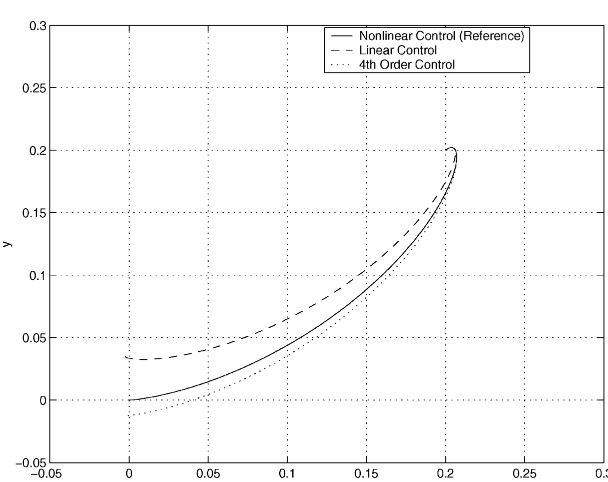

Fig. 1 Radial and tangential positions (example 1).

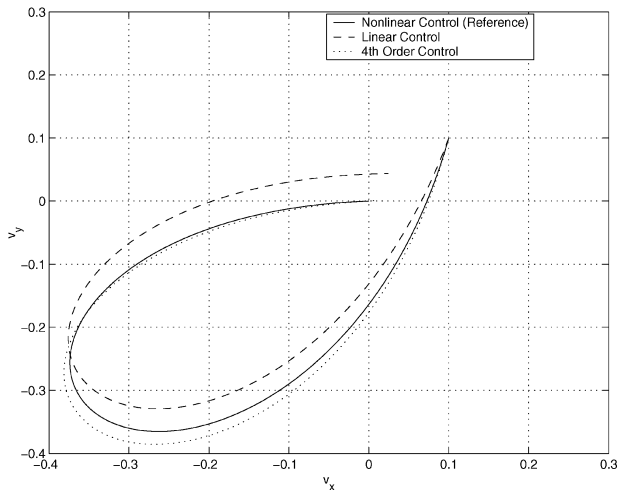

Fig. 2 Radial and tangential velocities (example 1).

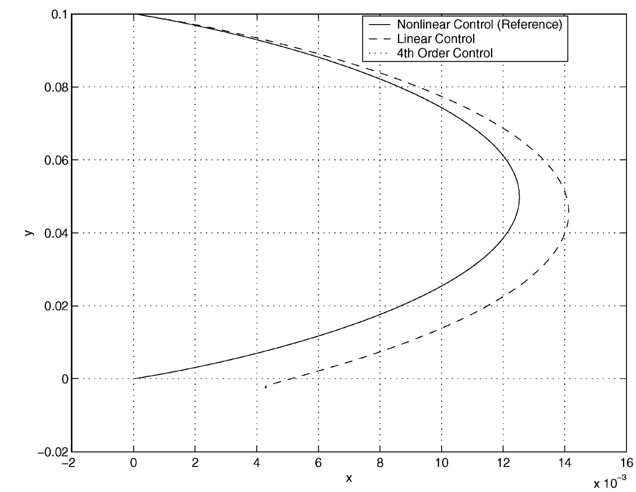

Fig. 4 Radial and tangential positions (example 2).

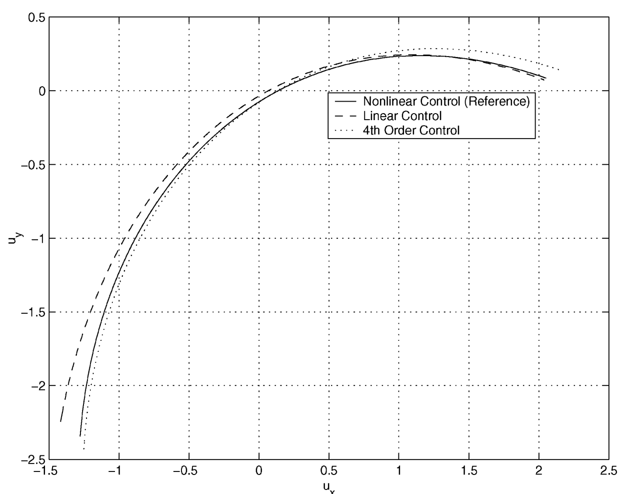

Fig. 3 Radial and tangential controls (example 1).

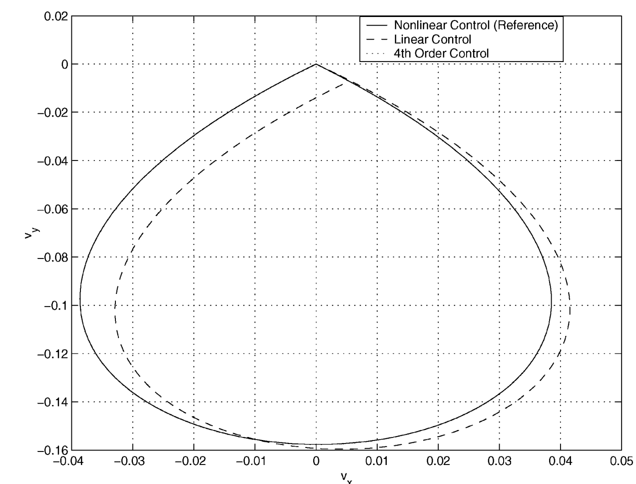

Fig. 5 Radial and tangential velocities (example 2).

If converted into real dimensions, example 1 represents the transition from the initial condition of [846 km, 846 km, 0.307 km/s, 0.307 km/s] to the origin in about 3.83 h. Similarly, examples 2 and 3, respectively, show the optimal transition from an initial offset of 4.23×10^3 and 127 km in the downtrack direction to a circular orbit at the coordinate frame origin in 3.83 h and 1 day, respectively.

For the control phase flows (Figs. 3, 6, and 9), the solid line, dashed line, and dotted line indicate optimal trajectories computed from the original nonlinear systems, linearized systems about the reference orbit, and the fourth-order approximated systems expanded as Taylor series about the reference orbit, respectively. The reference nonlinear solution (solid line) has been evaluated for comparison by solving the TPBVP numerically using a forward-shooting method. The linear optimal control has been evaluated from the quadratic expansion of F_1 generating function, which also coincides with the solution from the Ricatti transformation method (or sweep method) in Ref. 20.

For the state trajectories (Figs. 1, 2, 4, 5, 7, and 8), the solid line, dashed line, and dotted line represent the application of nonlinear, linear, and fourth-order control scheme to the original nonlinear system. It is clear that the fourth-order control yields better approximation than the linear control. This observation also holds as long as the boundary condition is close enough to the reference trajectory.

Here note that F_1 and the associated feedback control law is only computed once and that each optimal trajectory is determined algebraically by introducing the appropriate numeric boundary condition, whereas the reference solution must be determined by solving TPBVP repetitively for the varying boundary conditions. (In fact, this favorable property suggests the following: Given the same system we can obtain the optimal feedback control law for different types of boundary conditions without resolving HJ partial differential equations. From the given F_1, we can compute feedback control law for other types of boundary conditions only by partial differentiation and series inversion. Refer to Ref. 17 for more details.) By introducing additional HOT in the system dynamics, we can approximate the original system to as high an order as desired. The current implementation is limited only by computer memory constraints.

Figures 10 and 11 show the offset of the terminal boundary condition from the origin (that is, the true boundary condition to be satisfied) by the linear (dotted line) and fourth-order control scheme (solid line). Here the initial conditions are chosen such that the initial positions are located along the circle of radius 0.15 and initial velocities are identically zero (that is, the initial conditions are $[0.15\cos\theta \;\; 0.15\sin\theta \;\; 0 \;\; 0]$, with θ varying from 0 to 2π rad). It is clear that for all phase angles the fourth-order control scheme shows better convergent properties than the linear control scheme. Figures 12 and 13 show the phase trajectory of position variables and velocity variables for the same initial conditions. Finally, Fig. 14 shows the magnitude of control history. Again note that the initial conditions for all of these results are obtained from the F_1 generating function, which we only computed once; we have not solved the TPBVP numerically and repetitively.

Singularities of Generating Functions

Thus far, we have demonstrated a step-by-step procedure for evaluating optimal trajectory, as well as optimal feedback control via generating functions. Also it has been shown that, once one kind

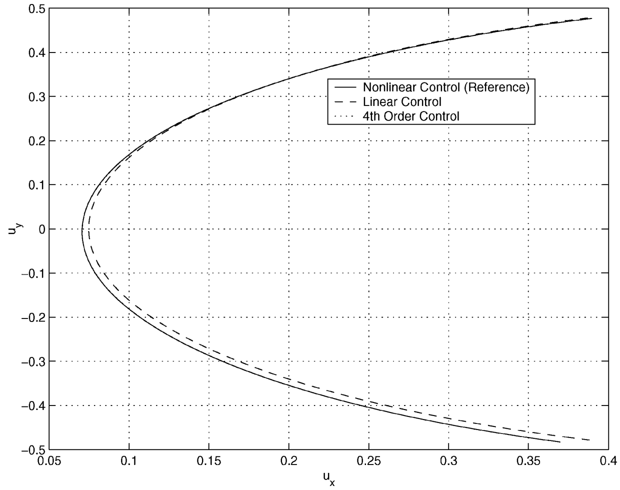

Fig. 6 Radial and tangential controls (example 2).

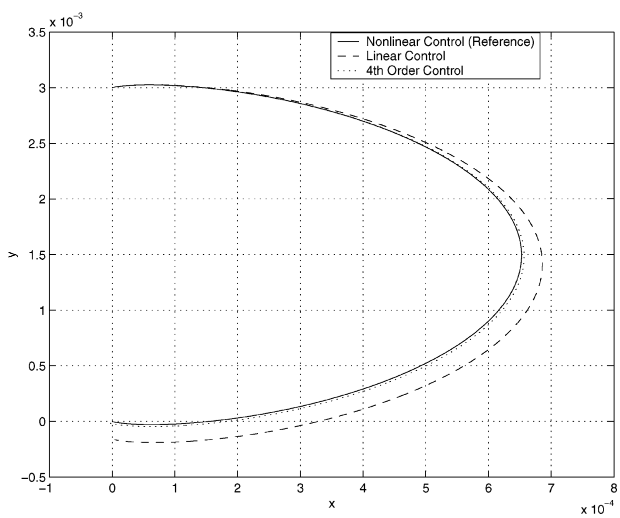

Fig. 7 Radial and tangential positions (example 3).

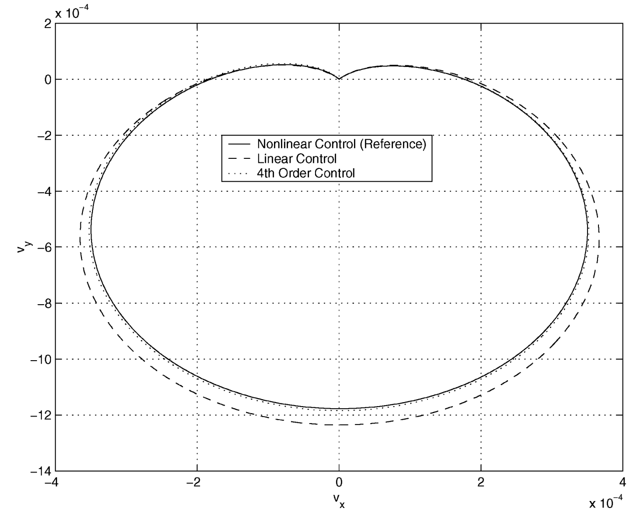

Fig. 8 Radial and tangential velocities (example 3).

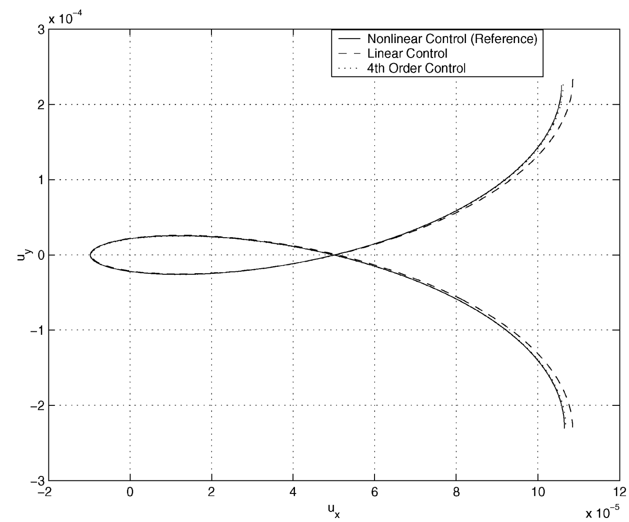

Fig. 9 Radial and tangential controls (example 3).

of generating function is computed, the others can also be obtained by applying a Legendre transformation. This section is dedicated to a discussion on the possibility of singularities in the generating functions (and how to avoid them, if any) and their relationship to optimal trajectories. This is a potentially important issue. In Ref. 14, it was found that all of the generating functions considered became singular at different times and that this formed a fundamental barrier to the construction of long-term solutions for the generating functions (which was ultimately overcome). Thus, it is of interest to consider the possibility of singularities in the generating functions we are computing here.

In terms of the boundary-value problem, the presence of singularities is usually associated with the existence of multiple solutions to the problem. In the case of Lambert-type problems in astrodynamics, a familiar situation where this arises concerns 180-deg transfers about a point mass in a fixed time because an infinity of possible transfer trajectories exist. The singularities arise in our approach as soon as there is more than one possible solution to the boundary-value problem because then the linear order terms in our expansion for F_i become degenerate and cannot represent the true solutions. This leads to a divergence in these linear terms and serves as a barrier for continued integration of the coefficients. Note, however, that not all of the generating functions can become singular at one time, and thus, it is always possible to transform, to a different generating function using the Legendre transformation and to continue computation of that generating function in time until the singularity in the other generating function has been passed. This has the drawback of complicating the solution procedure, however.

Fortunately, with our approach, singularities in the generating functions are easily identified because they are associated with singularities in the state transition matrix associated with the Hamiltonian system. Once the generating function F_2 is found, $\boldsymbol{p}$ and $\boldsymbol{x}_0$ can be derived from the necessary conditions associated with F_2,

$$\boldsymbol{p} = \frac{\partial F_2}{\partial \boldsymbol{x}} = F_{xx}\boldsymbol{x} + F_{xp0}\boldsymbol{p}_0 \tag{48}$$

$$\boldsymbol{x}_0 = \frac{\partial F_2}{\partial \boldsymbol{p}_0} = F_{p_0x}\boldsymbol{x} + F_{p_0p_0}\boldsymbol{p}_0 \tag{49}$$

Let us express $\boldsymbol{x}$ and $\boldsymbol{p}$ as a function of $\boldsymbol{x}_0$ and $\boldsymbol{p}_0$. From Eq. (49),

$$\boldsymbol{x} = F_{p_0x}^{-1}\boldsymbol{x}_0 - F_{p_0x}^{-1}F_{p_0p_0}\boldsymbol{p}_0 \tag{50}$$

When this expression is substituted into Eq. (48),

$$\boldsymbol{p} = F_{xx}F_{p_0x}^{-1}\boldsymbol{x}_0 + \left(F_{xp_0} - F_{xx}F_{p_0x}^{-1}F_{p_0p_0}\right)\boldsymbol{p}_0 \tag{51}$$

These two equations can be combined into matrix form,

$$\begin{bmatrix}\boldsymbol{x}(t)\\ \boldsymbol{p}(t)\end{bmatrix} = \begin{bmatrix} F_{p_0x}^{-1}(t,t_0) & -\left(F_{p_0x}^{-1}F_{p_0p_0}\right)(t,t_0)\\ \left(F_{xx}F_{p_0x}^{-1}\right)(t,t_0) & \left(F_{xp_0} - F_{xx}F_{p_0x}^{-1}F_{p_0p_0}\right)(t,t_0)\end{bmatrix} \times \begin{bmatrix}\boldsymbol{x}(t_0)\\ \boldsymbol{p}(t_0)\end{bmatrix} \tag{52}$$

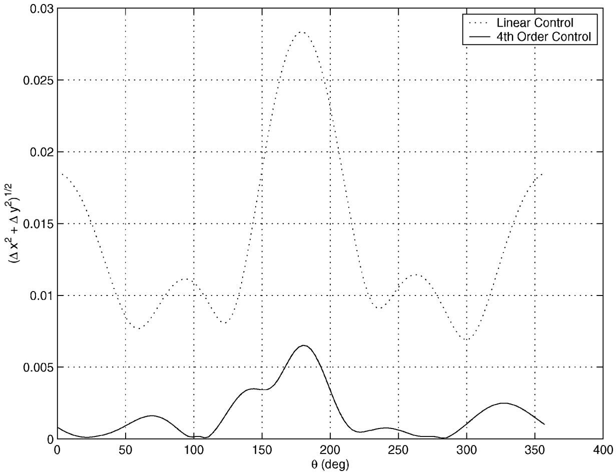

Fig. 10 Terminal position offset $x_0 = [r\cos\theta \;\; r\sin\theta \;\; 0 \;\; 0]$, $r = 0.15$, $0 \le \theta \le 360$ deg.

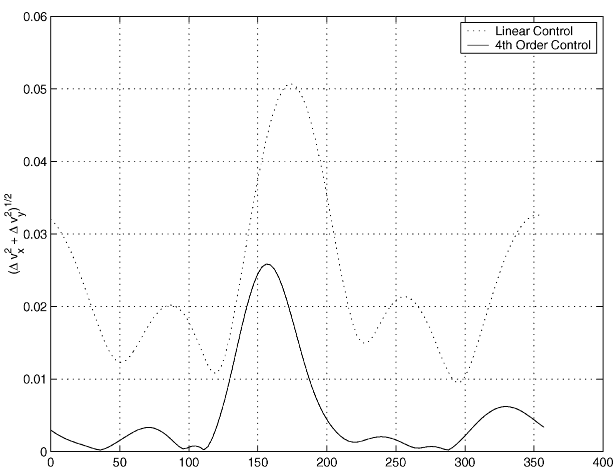

Fig. 11 Terminal velocity offset $x_0 = [r\cos\theta \;\; r\sin\theta \;\; 0 \;\; 0]$, $r = 0.15$, $0 \le \theta \le 360$ deg.

Also if we define the state transition matrix as

$$\Phi(t, t_0) = \begin{bmatrix} \phi_{xx}(t, t_0) & \phi_{xp}(t, t_0) \\ \phi_{px}(t, t_0) & \phi_{pp}(t, t_0) \end{bmatrix} \tag{53}$$

then the following expression holds:

$$\begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{p}(t) \end{bmatrix} = \begin{bmatrix} \phi_{xx}(t, t_0) & \phi_{xp}(t, t_0) \\ \phi_{px}(t, t_0) & \phi_{pp}(t, t_0) \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t_0) \\ \boldsymbol{p}(t_0) \end{bmatrix} \tag{54}$$

From the fact that Eqs. (52) and (54) should be equivalent, we can easily find the following relation between them,

$$F_{xx} = \phi_{px}\phi_{xx}^{-1}, \qquad F_{xp_0} = \phi_{xx}^{-T}$$
$$F_{p_0x} = \phi_{xx}^{-1}, \qquad F_{p_0p_0} = -\phi_{xx}^{-1}\phi_{xp} \tag{55}$$

These results indicate that F_2 is singular when ϕ_{xx} is singular. Also with the aid of Legendre transformation, it can be shown that F_1 is singular whenever ϕ_{xp} is singular. (Refer to Guibout and Scheeres[13] for a more comprehensive analysis of singularities.)

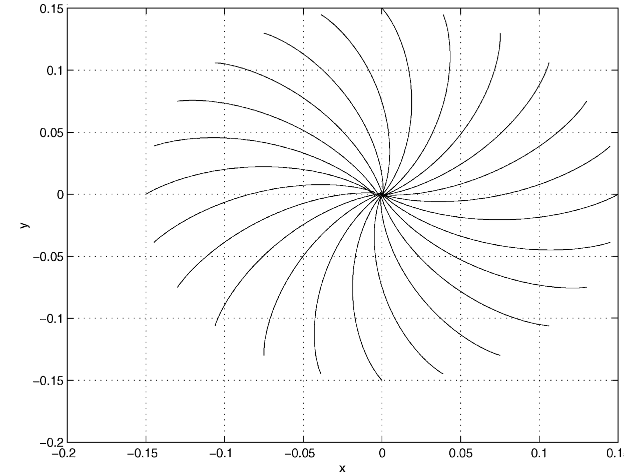

Fig. 12 Positional trajectory $x_0 = [r\cos\theta \;\; r\sin\theta \;\; 0 \;\; 0]$, $r = 0.15$, $0 \le \theta \le 360$ deg.

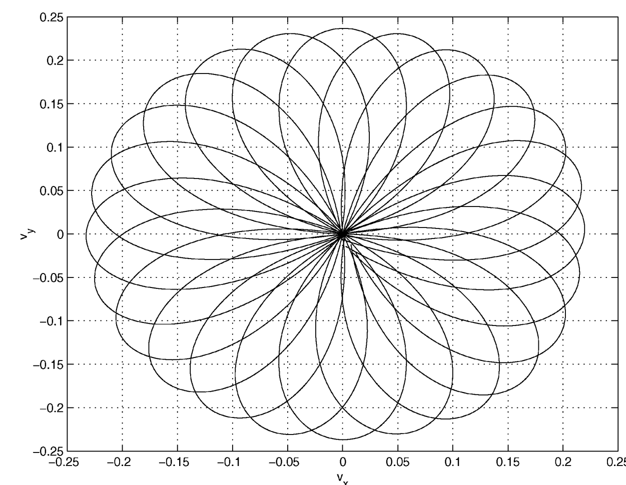

Fig. 13 Velocity trajectory $x_0 = [r\cos\theta \;\; r\sin\theta \;\; 0 \;\; 0]$, $r = 0.15$, $0 \le \theta \le 360$ deg.

Consider the linear dynamics for our specific system, from which we can derive the state transition matrix

$$\begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{p}} \end{bmatrix} = \begin{bmatrix} A & -BB^T \\ 0 & -A^T \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{p} \end{bmatrix} = \bar{A} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{p} \end{bmatrix} \tag{56}$$

where

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & -2 & 0 \end{bmatrix} \end{bmatrix} \tag{57}$$

Here we note that the $\bar{A}_{21}$ submatrix in $\bar{A}$ is always zero for the class of problems we consider here, namely, problems where the nominal solution is no control ($\boldsymbol{p} \equiv 0$ for $\boldsymbol{x} \equiv 0$). For expansions about an existing optimal control problem, the following observations are no longer true in general.

Now the state transition matrix is defined as

$$\Phi(t, t_0) = e^{A(t-t_0)} \tag{58}$$

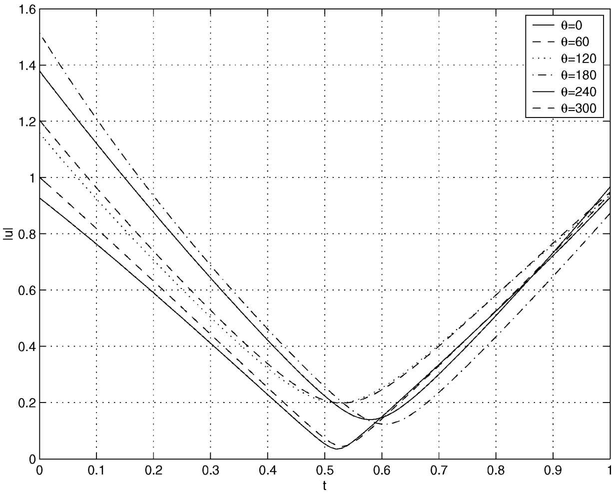

Fig. 14 Control history $x_0 = [r\cos\theta \;\; r\sin\theta \;\; 0 \;\; 0]$, $r = 0.15$, $0 \le \theta \le 360$ deg.

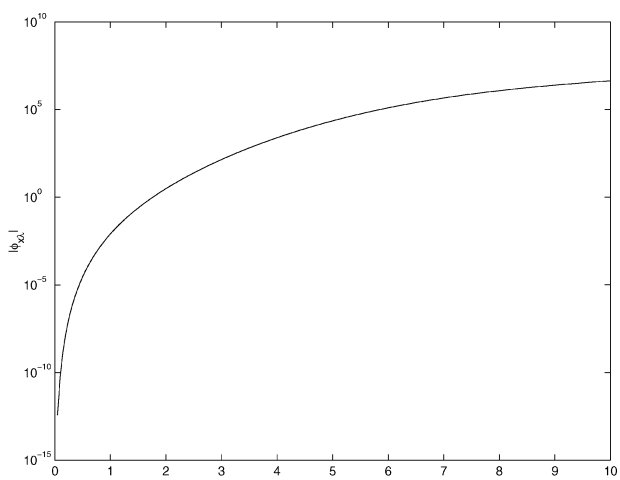

Fig. 15 Determinant of $\phi_{xp}(t)$.

Computing the determinants symbolically, we find

$$|\phi_{xx}(t)| \equiv \cos^2 t + \sin^2 t = 1 \tag{59}$$

$$\begin{aligned}|\phi_{xp}(t)| = {} & 1536 - 30t^4\cos t - 250.5t^2 - 2048\cos t - 912t\sin t \\ & - 18t^3\sin 2t + 456t\sin 2t + (27/32)t^4\cos 2t \\ & - 139.5t^2\cos 2t + (75/16)t^6 + 512\cos 2t \\ & + 400t^2\cos t + 222t^3\sin t - (1147/32)t^4\end{aligned} \tag{60}$$

$$|\phi_{px}(t)| \equiv 0 \tag{61}$$

$$|\phi_{pp}(t)| \equiv (\cos^2 t + \sin^2 t)^2 = 1 \tag{62}$$

For optimal control problems of this class, the ϕ_{xx} matrix is the state transition matrix of the dynamic system, and for well-defined dynamic systems, this matrix is never singular. We see this explicitly in the preceding equations. In fact, this should hold for all optimal control problems for which we expand the generating functions about a zero solution because the $\bar{A}_{21}$ submatrix will always be zero and allow the ϕ_{xx} submatrix dynamics to decouple from the other submatrices. For the applications in Ref. 14, the corresponding matrix was only a subelement of the state transition matrix and, hence, could be singular without violating singularity of the entire state transition matrix (and, indeed, was singular at certain times). Thus, we see that F_2 can never suffer this sort of singularity.

However, we are more interested in the occurrence of singularities of the matrix ϕ_{xp} because it affects whether F_1 becomes singular or not. (Note that it is F_1 that plays a key part in evaluating optimal feedback control and the optimal trajectory.) For that purpose, the time history of the determinant of ϕ_{xp} is shown in Fig. 15. From this, for our particular system, it is clear that ϕ_{xp} is never singular except the initial epoch where the singularity is inherent and, thus, that our optimal control is well defined and unique.

Conclusions

We have proposed a new method of evaluating an optimal trajectory as well as an optimal feedback control via generating functions, which has been successfully applied to the continuous thrust optimal rendezvous problem relative to a circular orbit. In contrast to the prevalent results in the literature based on linearized dynamics, we considered the nonlinear system by performing a Taylor series

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whose explicit solution is given by

$$\phi_{xx} = \begin{bmatrix} 4 - 3\cos t & 0 & \sin t & -2\cos t + 2 \\ 6\sin t - 6t & 1 & -2 + 2\cos t & -3t + 4\sin t \\ 3\sin t & 0 & \cos t & 2\sin t \\ 6\cos t - 6 & 0 & -2\sin t & -3 + 4\cos t \end{bmatrix}$$

$$\phi_{xp} = \begin{bmatrix} -2.5t\cos t + 6.5\sin t - 4t & -16\cos t - 5t\sin t + 16 - 3t^2 & \cdots \\ 16\cos t + 5t\sin t - 16 + 3t^2 & -10t\cos t + 1.5t^3 + 38\sin t - 28t & \cdots \\ 4\cos t - 4 + 2.5t\sin t & -5t\cos t + 11\sin t - 6t & \cdots \\ 5t\cos - 11\sin t + 6t & 28\cos t + 4.5t^2 + 10t\sin t - 28 & \cdots \\ \cdots & -4\cos t - 2.5t\sin t + 4 & 5t\cos t - 11\sin t + 6t \\ \cdots & -5t\cos t + 11\sin t - 6t & -4.5t^2 + 28 - 10t\sin t - 28\cos t \\ \cdots & 1.5\sin t - 2.5t\cos t & -6\cos t + 6 - 5t\sin t \\ \cdots & 6\cos t + 5t\sin t - 6 & -9t + 18\sin t - 10t\cos t \end{bmatrix}$$

$$\phi_{px} = 0_{4\times 4}, \qquad \phi_{pp} = \begin{bmatrix} 4 - 3\cos t & -6\sin t + 6t & -3\sin t & 6\cos t - 6 \\ 0 & 1 & 0 & 0 \\ -\sin t & -2 + 2\cos t & \cos t & 2\sin t \\ 2 - 2\cos t & 3t - 4\sin t & -2\sin t & -3 + 4\cos t \end{bmatrix}$$

expansion of the system dynamics and showed that the introduction of HOT results in numerical convergence to the (unapproximated) nonlinear solution. Finally, we considered the possibility of singularities existing in our control procedure and showed that they are absent in general for the particular application we are considering.

Our method has an advantage over the conventional numerical shooting method in the sense that it does not require that one should guess the initial or terminal adjoints. It also has an advantage over the method based on linearized dynamics in the sense that our higher-order solution enhances the numerical precision and the region of convergence to the nonlinear reference solution. All of these favorable results imply that our new method can be considered as an alternative and effective way of solving nonlinear optimal rendezvous problems. Furthermore, in addition to computing the optimal trajectory, our proposed optimal feedback control law can be used as an improved guidance law.

Appendix: Properties of Hamiltonian Systems and Their Application to Boundary Value Problems

Hamiltonian System and Canonical Transformation

This Appendix briefly reviews Hamiltonian dynamic systems. See Greenwood[19] for a more comprehensive discussion. Suppose we have a system whose equations of motion can be represented using Hamilton's canonical form,

$$\begin{bmatrix} \dot{\boldsymbol{q}}(t) \\ \dot{\boldsymbol{p}}(t) \end{bmatrix} = \begin{bmatrix} \dfrac{\partial H}{\partial p}[\boldsymbol{q}(t), \boldsymbol{p}(t), t] \\ -\dfrac{\partial H}{\partial q}[\boldsymbol{q}(t), \boldsymbol{p}(t), t] \end{bmatrix} \tag{A1}$$

where $H = H[q(t), p(t), t]$ is the Hamiltonian of the system, $\boldsymbol{q}(t) = [q_1(t) \;\; q_2(t) \;\; \cdots \;\; q_n(t)]^T$ is the generalized coordinate vector, and $\boldsymbol{p}(t) = [p_1(t) \;\; p_2(t) \;\; \cdots \;\; p_n(t)]^T$ is the generalized momentum vector conjugate to $\boldsymbol{q}(t)$. Suppose, furthermore, that there exists a canonical transformation from (q, p) to a new set of coordinate (Q, P) which is related by

$$Q = Q(q, p, t), \qquad P = P(q, p, t) \tag{A2}$$

Then there exists a Hamiltonian $K = K[Q(t), P(t), t]$ in the new set of coordinates such that the equations of motion is of the form

$$\begin{bmatrix} \dot{Q} \\ \dot{P} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial K}{\partial P}(Q, P, t) \\ -\dfrac{\partial K}{\partial Q}(Q, P, t) \end{bmatrix} \tag{A3}$$

To relate K with H, let us recall Hamilton's principle

$$\delta I = \delta \int_{t_0}^{t_f} L \, \mathrm{d}t = 0 \tag{A4}$$

where the Lagrangian L is defined as $L(q, \dot{q}, t) = p^T \dot{q} - H(q, p, t)$. From Eq. (A4), we have, both in old and new coordinates,

$$\delta \int_{t_0}^{t_f} [p^T \dot{q} - H(q, p, t)] \, \mathrm{d}t = \delta \int_{t_0}^{t_f} [P^T \dot{Q} - K(Q, P, t)] \, \mathrm{d}t = 0 \tag{A5}$$

which implies that the integrands of the two integrals differ at most by a total time derivative of an arbitrary function F, that is,

$$p^T \dot{q} - H(q, p, t) = P^T \dot{Q} - K(Q, P, t) + \frac{\mathrm{d}F}{\mathrm{d}t} \tag{A6}$$

Such a function is called a generating function and is a function of both old and new coordinates and time. However, from the $2n$ relations (A2) it turns out that F is a function of $2n + 1$ variables instead of $4n + 1$ variables. Let us assume that F is dependent on n old coordinates and n new coordinates. Then the generating function has one of the following forms[19]:

$$F_1(q, Q, t; t_0), \qquad F_2(q, P, t; t_0),$$
$$F_3(p, Q, t; t_0), \qquad F_4(p, P, t; t_0) \tag{A7}$$

If, for instance, q and Q are independent variables, then F_1 should be used. Expanding the total time derivative of F_1 yields

$$\frac{\mathrm{d}}{\mathrm{d}t} F_1(q, Q, t; t_0) = \frac{\partial F_1^T}{\partial q} \dot{q} + \frac{\partial F_1^T}{\partial Q} \dot{Q} + \frac{\partial F_1}{\partial t} \tag{A8}$$

Substitution of Eq. (A8) into Eq. (A6) leads to

$$\left(p - \frac{\partial F_1}{\partial q}\right)^T \dot{q} - H = \left(P + \frac{\partial F_1}{\partial Q}\right)^T \dot{Q} - K + \frac{\partial F_1}{\partial t} \tag{A9}$$

which is equivalent to

$$p = \frac{\partial F_1}{\partial q}(q, Q, t; t_0) \tag{A10}$$

$$P = -\frac{\partial F_1}{\partial Q}(q, Q, t; t_0) \tag{A11}$$

$$K(Q, P, t) = H(q, p, t) + \frac{\partial F_1}{\partial t}(q, Q, t; t_0) \tag{A12}$$

Similarly, if q and P are independent variables, then Eq. (A6) can be rewritten as a function of two independent variables q and P,

$$p^T \dot{q} - H(q, p, t) = -Q^T \dot{P} - K(Q, P, t) + \frac{\mathrm{d}F_2}{\mathrm{d}t} \tag{A13}$$

which yields

$$p = \frac{\partial F_2}{\partial q}(q, P, t; t_0) \tag{A14}$$

$$Q = \frac{\partial F_2}{\partial P}(q, P, t; t_0) \tag{A15}$$

$$K(Q, P, t) = H(q, p, t) + \frac{\partial F_2}{\partial t}(q, P, t; t_0) \tag{A16}$$

Furthermore, it can be verified that the Legendre transformation

$$F_2(q, P, t; t_0) = F_1(q, Q, t; t_0) + P^T Q \tag{A17}$$

relates F_1 with F_2. The same procedure leads to the similar results for $F_3(p, Q, t; t_0)$ and $F_4(p, P, t; t_0)$.

Application to Boundary-Value Problems

Consider again the canonical transformation (A2) where (q, p) and (Q, P) satisfy the canonical equations of motion (A1) and (A3) subject to the Hamiltonian $H = H(q, p, t)$ and $K = K(Q, P, t)$, respectively. Here the variables (Q, P) can be chosen to be constants by setting the new Hamiltonian $K \equiv 0$. That is,

$$\begin{bmatrix} \dot{Q} \\ \dot{P} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial K}{\partial P} \\ -\dfrac{\partial K}{\partial Q} \end{bmatrix} \equiv 0 \Rightarrow \begin{bmatrix} Q \\ P \end{bmatrix} \equiv \text{const} \tag{A18}$$

Equations (A12) and (A16) become

$$\frac{\partial F_1}{\partial t} + H\left(q, \frac{\partial F_1}{\partial q}, t\right) = 0 \tag{A19}$$

$$\frac{\partial F_2}{\partial t} + H\left(q, \frac{\partial F_2}{\partial q}, t\right) = 0 \tag{A20}$$

both of which are often referred to as the HJ equation. Indeed, they are equivalent; the only difference between the two is in their initial boundary conditions. This difference, however, leads to a very different time evolution and even leads to the functions becoming singular at different epochs.[14]

For an application to the boundary-value problem, let us simply choose the initial conditions of the trajectory to be the constants of motion and solve the HJ equation for the generating function. To solve the HJ equation, the value of the generating function needs to be specified at some epoch. At $t = 0$, both old and new coordinates are equal; therefore, the generating function must define an identity transformation. F_1 cannot generate such a transformation because the initial and final positions are equal and not independent at $t = t_0$; thus, F_1 is undefined at $t = t_0$. On the contrary, F_2 is well defined at $t = t_0$. In fact, if both H and F_2 are analytic, then

$$F_2(q, P, t = t_0; t_0) = q^T P \tag{A21}$$

is the unique possible expression and defines the identity transformation at $t = t_0$. Therefore, given the Hamiltonian of a system, we can solve the HJ equation for F_2 from the initial time. F_1 can only be solved if it is known at some epoch other than the initial time.

The main advantage of this approach is that once the generating function has been found, the unknown boundary conditions are simply evaluated from the algebraic manipulation of Eqs. (A10) and (A11) and (A14) and (A15) without solving a differential equation.

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