Taking into account Eq. (15), one determines the integration constants $\bar{C}_i (i=1,\ldots,4)$ as

$$\bar{C}_1 = \frac{2\omega k_1 s_1 s_3 - 2\omega a s_2 s_3 + a(b^2+k_2)s_3^2 - (4\omega^2 - k_2 - b^2)s_1 s_2}{4\omega k_1 a(a^2+b^2)s_3}$$

$$\bar{C}_2 = \frac{-2\omega k_1 s_1 s_3 - 2\omega a s_2 s_3 + a(b^2+k_2)s_3^2 + (4\omega^2 - k_2 - b^2)s_1 s_2}{2\omega k_1 a(a^2+b^2)s_3}$$

$$\bar{C}_3 = \frac{2\omega s_2 + (a^2 - k_2)s_3}{2\omega k_1 (a^2+b^2)}$$

$$\bar{C}_4 = -\frac{2\omega k_1 s_1 s_3 - (4\omega^2 - k_2 + a^2)s_1 s_2}{2\omega k_1 b(a^2+b^2)s_3} \tag{17}$$

With the notation $\gamma_i = \alpha_i (u^1)^* + \beta_i (u^2)^*$, Eq. (15) becomes

$$\Sigma \bar{C}_k x_{ik}(\tau) = x_i - \gamma_i \tag{18}$$

with the unknowns $e^{a\tau}$, $e^{-a\tau}$, $\sin b\tau$, and $\cos b\tau$. Performing the calculations, we obtain the analytical expression of the pursuit time on the optimal trajectory

$$\begin{aligned}\tau(x) = (1/d)\ln\{ & [2\omega k_1 (x_1 - \gamma_1) + 2\omega a (x_2 - \gamma_2) \\ & + a(b^2+k_2)(x_3 - \gamma_3) + b(4\omega^2 - k_2 - b^2)(x_4 - \gamma_4)]s_3\} \\ & \div [2\omega k_1 s_1 s_3 - 2\omega a s_2 s_3 + a(b^2+k^2)s_3^2 \\ & - (4\omega^2 - k_2 - b^2)s_1 s_2]\end{aligned} \tag{19}$$

An analysis of the motion for the entire time devoted to the pursuit is made by taking into account the expressions for the control functions. For the analyzed motion, the values of the physical coordinates x_i $(i=1,\ldots,4)$ have to fulfill condition $x_1 x_2 + x_3 x_4 < 0$. After the determination of the domain D, the set of the admissible states for the pursuit problem coincides with the domain of the states for the auxiliary problem. A particular case of the analyzed problem is obtained when the two vehicles meet on the terminal surface for $s_1 = s_3 = 0$. The domain of the admissible states for this case is analogous to the one above.

Appendix

Expressions for the terms of Eq. (15) are as follows:

$$\alpha_1 = \frac{b^2(4\omega^2 - k_2 - b^2) - a^2(a^2+b^2)}{k_1[4\omega^2(k_2 - a^2) + 2a^2(b^2+k_2) + b^4 - k_2^2]}$$

$$\beta_1 = -\frac{(a^2+b^2)(4\omega^2/b - k_2 + a^2)(4\omega^2 - k_2 - b^2 - 1)}{4\omega a k_1[4\omega^2(k_2 - a^2) + 2a^2(b^2+k_2) + b^4 - k_2^2]}$$

$$\alpha_2 = \frac{b^2(a^2 - k_2)}{a[4\omega^2(k_2 - a^2) + 2a^2(b^2+k_2) + b^4 - k_2^2]}$$

$$\begin{aligned}\beta_2 = \{ & (a^2+b^2)[2a^2(b^2+k_2)(4\omega^2 - k_2 + a^2) \\ & - b^2(a^2 - k_2)(4\omega^2 - k_2 - b^2 + 1)\} \\ & \div 4\omega a^2 b^2 [4\omega^2(k_2 - a^2) + 2a^2(b^2+k_2) + b^4 - k_2^2]\end{aligned}$$

$$\alpha_3 = \frac{2\omega a}{[4\omega^2(k_2 - a^2) + 2a^2(b^2+k_2) + b^4 - k_2^2]}$$

$$\beta_3 = \frac{(a^2+b^2)[b^2(4\omega^2 - k_2 - b^2 + 1) + 2a^2(4\omega^2 - k_2 + a^2)]}{2\omega a^2 b^2 [4\omega^2(k_2 - a^2) + 2a^2(b^2+k_2) + b^4 - k_2^2]}$$

$$\alpha_4 = -\frac{2\omega b^2}{[4\omega^2(k_2 - a^2) + 2a^2(b^2+k_2) + b^4 - k_2^2]}$$

$$\beta_4 = \frac{(a^2+b^2)(4\omega^2 - k_2 - b^2 - 1)}{2a[4\omega^2(k_2 - a^2) + 2a^2(b^2+k_2) + b^4 - k_2^2]}$$

References

[1]Rozenberg, G.S., "Construction of Optimal Pursuit Trajectories," *Automatica i Telemekhanica,* Vol. 26, 1965.

[2]Simakov, E.N., "Plane Pursuit Problem," *Automatica i Telemekhanica,* Vol. 28, 1967.

[3]Burrow, J.W. and Rishel, R.W., The Strange Extremals of a Normal Acceleration Control Problem," *"IEEE Transactions on Automatic Control,* Vol. AC-10, 1965.

[4]Cockayne, E., "Plane Pursuit with Curvature Constraints," *SIAM Journal on Applied Mathematics,* Vol. 15, No. 6, 1967.

[5]Popescu, M., "Optimal Pursuit on Lagrangian Orbits," IAF Paper 351 presented at 35th International Astronautical Congress, Lausanne, 1984.

[6]Popescu, M., "Optimal Transfer from the Lagrangian Points," *Acta Astronautica,* Vol. 12, No. 4, 1985.

[7]Popescu, M., "The Study of Some Classes of Extremals Concerning the Optimal Transfer from the Equilateral Libration Points," IAF Paper 313 presented at 33rd International Astronautical Congress, Paris, 1982.

Comparison of Local Pole Assignment Methods

P.T. Kabamba* and K.C. Shin†

University of Michigan, Ann Arbor, Michigan

I. Introduction

CONSIDER the following standard time-invariant, minimal linear dynamical system

$$\dot{x} = Ax + Bu \tag{1}$$

$$y = Cx \tag{2}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector; $y \mathbb{R}^r$ is the output vector; A, B, and C are matrices of appropriate dimensions. System (1,2) could be controlled by state feedback of the form $u = Fx$, but such a control would be unfeasible if the state vector is not accessible. This difficulty can be circumvented by including an estimator in the controller and feeding back the state vector estimate, but this would require a possibly cumbersome controller. It has been suggested to control system (1,2) by direct output feedback of the form $u = Py$, in which case the closed loop satisfies

$$\dot{x} = (A + BPC)x \tag{3}$$

The dynamical behavior of Eq. (3) will depend on the location of the eigenvalues of $A + BPC$. The problem of pole assignment by direct-output feedback has received substantial

Received Sept. 16, 1985; revision received Jan. 14, 1986.

*Department of Aerospace Engineering.

†Department of Mechanical Engineering.

contributions.[1-13] However, a simple criterion for complete pole assignability is still to be discovered.

Most necessary or sufficient conditions available in the literature are *global* in nature. In contrast, this paper focuses on *local* methods for pole assignment. Such methods are based on an idea similar to that of continuation methods,[15,16] and can be understood as follows: Let $N \geq n$, $\zeta: C^N \to C^n$: $\xi \to \zeta = \zeta(\xi)$ be differentiable, and suppose we want to solve the equation $\zeta(\xi) = 0$. Starting from an arbitrary guess ξ_0, solve the linear (differential) equation $d\zeta = (\partial\zeta/\partial\xi)^T d\xi$ for $d\xi$. If ζ is locally assignable (see definition below), an increment $d\xi$ can be computed to cause a given increment $d\zeta$. The numerical solution is thus obtained iteratively by successive increment of ξ until the value $\zeta = 0$ is attained. If $N > n$, the differential equation has many solutions, indicating that the final solution is not unique. If along the integration path ζ fails to be locally assignable, Sard's theorem[15,16] can be used to step away from the singularity.

This paper presents three local methods for pole assignment. One is based on assigning the eigenvalues of $A + BPC$ directly; the second on assigning the coefficients of the characteristic polynomial of $A + BPC$; and the third on assigning the traces of successive powers of $A + BPC$. Sec. II gives gradient formulas and local assignability criteria for the three methods. The numerical merits of these equivalent methods are compared in Sec. III. It is found that in general, the method based on eigenvalue assignment is more recommendable. An example is presented in Sec. IV.

We use the following notations and definitions: $a \in \mathbb{R}^n$ is the vector of coefficients of the characteristic polynomial, i.e., $d(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n$; $d(s)$ is the characteristic polynomial, i.e., $d(s) = \det[sI_n - A - BPC]$; $d'(s)$ is the derivative of $d(s)$; $H(s)$ is the closed loop transfer function i.e., $H(s) = C[sI_n - A - BPC]^{-1}B$; $I_n \in \mathbb{R}^{n \times n}$ is the unit matrix; N is the number of degrees of freedom, i.e., $N = mr$; $\mathrm{tr}(M)$ denotes the trace of the square matrix M; vec: $\mathbb{R}^{m \times r} \to \mathbb{R}^N$ is the operator which transforms a matrix into a vector containing the columns of the matrix ordered sequentially; $p = \mathrm{vec}(P)$; $\lambda \in \mathbb{C}^n$ is the vector of eigenvalues of $A + BPC$, i.e., $d(s) = (s - \lambda_1)(s - \lambda_2)\ldots(s - \lambda_n)$; $\tau \in \mathbb{R}^n$, $\tau_k = \mathrm{tr}[A + BPC]^k = \Sigma_{i=1}^n \lambda_i^k$, $1 \leq k \leq n$; $\|M\|_F$ is the Frobenius norm of the complex matrix M, i.e., $\|M\|_F = (\mathrm{tr}(M^H M))^{1/2}$; superscripts T and H denote matrix transpose and complex conjugate transpose respectively; $M^\sim$ denotes the cofactor matrix of the square matrix M, such that if $\det M \neq 0$, $M^{-1} = M^{\sim T}/\det M$. If $y: \mathbb{C}^{\alpha \times \beta} \to \mathbb{C}: X \to y = y(X)$, then $\partial y/\partial x \in \mathbb{C}^{\alpha \times \beta}$ and $(\partial y/\partial X)_{ij} = \partial y/\partial X_{ij}$. If $y: \mathbb{C}^\alpha \to \mathbb{C}^\beta$: $x \to y(x)$, then $\partial y/\partial x \in \mathbb{C}^{\alpha \times \beta}$ and $(\partial y/\partial X)_{ij} = \partial y_j/\partial x_i$. If $N \geq n$ and $\zeta: \mathbb{C}^N \to \mathbb{C}^n: \xi \to \zeta = \zeta(\xi)$ is differentiable, then ζ is *locally assignable* at ξ_0 if $\partial\zeta/\partial\xi$ has full rank at ξ_0 [Ref. 14, Theorem 0.5]. If $H_i \in \mathbb{C}^{\alpha \times \beta}$, $1 \leq i \leq n$, then H_i are *linearly independent* if for any $\alpha \in \mathbb{C}^n$, $\Sigma_{i=1}^n H_i \alpha_i = 0$ implies $\alpha = 0$.

II. Local Assignability Criteria

Proposition 1: Suppose λ_i, $1 \leq i \leq n$ are simple distinct eigenvalues of $A + BPC$ with right and left eigenvectors x_i and y_i respectively, normalized so that $y_i^T x_i = 1$. Then

$$\frac{\partial \lambda_i}{\partial P} = B^T y_i x_i^T C^T \tag{4}$$

The vector λ is locally assignable if the matrices [Eq. (4)] are linearly independent, $1 \leq i \leq n$.

Proof: Equation (4) is a straightforward extension of the theory of Ref. 17, Sec. 2.9. The assignability criterion follows from the definitions above.

Proposition 2. The matrices $\partial a_i/\partial P$, $1 \leq i \leq n$ satisfy the identity

$$s^{n-1}\frac{\partial a_1}{\partial P} + s^{n-2}\frac{\partial_a 2_n}{\partial P} + \ldots + \frac{\partial a_n}{\partial P} = -B^T[sI_n - A - BPC]^\sim C^T, \forall s \in C \tag{5}$$

$$= -H^T(s)\,d(s), \qquad \text{if } d(s) \neq 0 \tag{6}$$

The vector a is locally assignable if any of the following criteria is met:

1) For any choice of distinct complex numbers s_i, $1 \leq i \leq n$, the matrices $B^T[s_i I_n - A - BPC]^\sim C^T$, $1 \leq i \leq n$, are linearly independent.

2) For any choice of distinct complex numbers s_i, $1 \leq i \leq n$, none of which is eigenvalue of $A + BPC$, the matrices $H(s_i)$, $1 \leq i \leq n$ are linearly independent.

Proof: For $(M, X, U, Y, V) \in \mathbb{C}^{\alpha \times \alpha} x \mathbb{C}^{\alpha \times \alpha} x \mathbb{C}^{\alpha \times \beta} x \mathbb{C}^{\beta \times \gamma} x \mathbb{C}^{\gamma \times \alpha}$ and $M = X + UYV$, we have

$$\frac{\partial \det M}{\partial Y} = U^T M^\sim V^T \tag{7}$$

which is easily obtained by using the development of a determinant as function of minors of rows or columns. Apply Eq. (7) with $X = sI_n - A$, $U = -B$, $V = C$, $Y = P$ to obtain Eq. (5). Furthermore, if $d(s) \neq 0$, then Eq. (5) implies Eq. (6). Apply the operator $\mathrm{vec}(\cdot)$ to Eq. (5) taken at arbitrary distinct points $s_i \in C$, $1 \leq i \leq n$ to obtain

$$\left(\frac{\partial a}{\partial p}\right)\begin{bmatrix} s_1^{n-1} & s_2^{n-1} & \ldots & s_n^{n-1} \\ s_1^{n-2} & s_2^{n-2} & \ldots & s_n^{n-2} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ 1 & 1 & \ldots & 1 \end{bmatrix} = D \tag{8}$$

where $D \in \mathbb{C}^{N \times m}$, the ith column of D being $\mathrm{vec}[-B^T(s_i I_n - A - BPC)^\sim C^T]$. Since the Vandermonde matrix in Eq. (8) is nonsingular, $\partial a/\partial p$ has full rank if D has full rank, which establishes criterion 1. Criterion 2 follows after dividing by $d(s_i)$.

Proposition 3[9]. The matrices $\partial\tau_i/\partial P$, $1 \leq i \leq n$, satisfy the identity

$$\frac{\partial \tau_i}{\partial P} = iB^T(A + BPC)^{(T}_{i-1)}C^T, \qquad 1 \leq i \leq n \tag{9}$$

The vector τ is locally assignable if the matrices (9) are linearly independent, $1 \leq i \leq n$.

Remark 1. Equation (5) suggests using Faddeev's algorithm[18] for the computation of $\partial a/\partial p$. This algorithm computes simultaneously the a_i and the coefficients of the polynomial matrix $[sI_n - (A + BPC)]^\sim$. However, Faddeev's algorithm is notoriously ill-conditioned. We have therefore preferred using Eq. (8) and computing $\partial a/\partial p$ by interpolation at s_i, $1 \leq i \leq n$. We have chosen the s_i equally spaced on the unit circle. In this case the Vandermonde matrix of Eq. (8) becomes a well-conditioned scalar multiple of a unitary matrix, as is easily shown.

Remark 2. The local assignability criteria of Propositions 1-3 yield controllability and observability criteria as special cases for $C = I_n$ and $B = I_n$, respectively.[9]

III. Comparison of the Three Methods

The vectors λ, a, and τ being related by one-to-one maps, we can choose to assign any one of them. We should, if possible, select among λ, a, and τ the most sensitive vector, for best accuracy. We introduce the notion of *relative sensitivity*: If $\zeta: \mathbb{C}^\alpha \to \mathbb{C}^\beta: \xi \to \zeta = \zeta(\xi)$ is differentiable, the sensitivity of ζ relative to ξ is defined as $S(\zeta, \xi) = \|\partial\zeta/\partial\xi\|_F$. The relative sensitivity $S(\zeta, \xi)$ measures the average ratio of magnitudes of differential increments of ζ and ξ. More precisely, if ξ undergoes a random differential increment of zero-mean and unit covariance, the expected magnitude of the increment of ζ will be $S(\zeta, \xi)$. The sensitivities of λ, a, and τ relative to one another will be used to assess the merit of choosing either one.

Proposition 4. Suppose λ_i, $1 \le i \le n$ are distinct. Then we have

$$\left(\frac{\partial \tau}{\partial \lambda}\right)^T = \text{diag}[1,2,\ldots,n] \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_n \\ \vdots & & & \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \ldots & \lambda_n^{n-1} \end{bmatrix} \quad (10)$$

$$\left(\frac{\partial \lambda}{\partial a}\right)^T = -\text{diag}[d'(\lambda_1), d'(\lambda_2), \ldots, d'(\lambda_n)]^{-1}$$

$$\times \begin{bmatrix} \lambda_1^{n-1} & \lambda_1^{n-2} & \ldots & 1 \\ \lambda_2^{n-1} & \lambda_2^{n-2} & \ldots & 1 \\ \vdots & & & \\ \lambda_n^{n-1} & \lambda_n^{n-2} & \ldots & 1 \end{bmatrix} \quad (11)$$

$$\left(\frac{\partial \tau}{\partial a}\right)^T = \begin{bmatrix} -1 & 0 & 0 & \ldots \\ 2a_1 & -2 & 0 & \ldots \\ -3a_1^2 & 3a_1 & -3 & \ldots \\ \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \ddots \; -n \end{bmatrix} \quad (12)$$

Proof: Equation (10) is obtained from the definition of τ. Equation (11) is obtained by differentiating $d(s)$ with respect to a_i, $1 \le i \le n$, and letting s tend to λ_j, $1 \le j \le n$, yielding $\lambda_j^{n-i} = -d'(\lambda_j)\partial\lambda_j/\partial a_i$, thus Eq. (11). Equation (12) is obtained from Newton's formulas[19]:

$$\tau_1 + a_1 = 0$$

$$\tau_2 + a_1\tau_1 + 2a_2 = 0$$

$$\tau_3 + a_1\tau_2 + a_2\tau_1 + 3a_3 = 0$$

$$\vdots$$

$$\tau_{n-1} + a_1\tau_{n-2} + a_2\tau_{n-3} + \ldots + a_{n-2}\tau_1 + (n-1)a_{n-1} = 0$$

$$\tau_n + a_1\tau_{n-1} + a_2\tau_{n-2} + \ldots + a_{n-1}\tau_1 + a_n = 0 \quad (13)$$

Equation (10) shows that if all the λ_i are smaller than 1 in magnitude, τ will be insensitive to changes in λ; therefore, assigning λ is preferrable to assigning τ. If some λ_i are greater than 1 in norm, τ will be very sensitive to changes in λ; but in this case, τ will often have numerically overflowing values, as shown by its definition.

Equation (11) is difficult to interpret because of the factor $\text{diag}[d'(\lambda_i)]^{-1}$. However, Eq. (12) shows that τ is in general very sensitive to changes in a; therefore, assigning τ is preferable to assigning a.

The overall suggestion of Eqs. (10-12) is that, in general, it may be better to assign λ directly rather than a or τ, for reasons of sensitivity and possible numerical overflow of τ and even a. This suggestion strongly corroborates our empirical findings, based on the use of the three methods in various examples.

IV. Example

Consider $n=8$, $m=r=4$

$$A = \begin{bmatrix} 1. & 2. & 1. & 3. & 1. & 4 & 1. & 5. \\ 2. & -2. & 1. & 2. & 3. & 4. & 5. & 6. \\ 2.5 & -3.5 & 4.5 & 5.5 & 6.5 & 0. & 0. & 0. \\ -3.3 & -4.4 & -2.2 & -3.1 & -2.1 & -4.8 & 3. & 1. \\ -5. & -3. & 21. & 3. & 2. & 8. & 3. & 2. \\ 0. & 3. & 5. & 1. & -2. & -3. & -4.3 & -3.2 \\ -2.2 & -9.9 & 10. & 10. & 3. & 2. & 2. & 1. \\ -9. & -2. & -3. & -4. & -5. & -6. & 1. & 1. \end{bmatrix}$$

$$B = \begin{bmatrix} 1. & 2. & 3. & 4. \\ 5. & 6. & 7. & 8. \\ 9. & 1. & 2. & 3. \\ 4. & 5. & -2. & -3. \\ -4. & 2. & 1. & -4. \\ -2. & -3. & 4. & -5. \\ 2. & -2. & 0. & -5. \\ -2. & 1. & 2. & 1. \end{bmatrix}$$

$$C = \begin{bmatrix} 2. & 1. & 1. & 4. & 5. & -1. & -2. & 1. \\ 2. & 1. & 3. & 4. & 2. & 5. & 6. & 2. \\ 3. & 4. & -1. & -2. & -3. & -4. & -5. & -1. \\ -2. & -3. & -2. & -3. & 1. & 3. & 1. & -2. \end{bmatrix}$$

The open-loop eigenvalues are 11.8, $4.8 \pm j3.8$, 2.4, -2.2, $-4.5 \pm j7.3$, -10.1. Suppose we want the closed-loop eigenvalues $-1.$, $-2.$, $-2. \pm j2.$, $-4 \pm j4.$, $-5.$, -10. Using direct eigenvalue assignment and starting with $P=0$, it takes 16 iterations to obtain a direct-output feedback gain

$$P = \begin{bmatrix} -0.1819 & 0.03415 & -0.01277 & 0.05046 \\ -0.12121 & 0.02716 & -0.06953 & 0.01293 \\ -0.00101 & -0.1144 & -0.0739 & 0.02756 \\ 0.13625 & -0.01556 & -0.19773 & -0.09637 \end{bmatrix}$$

which performs the pole assignment.

In this example of relatively small order, the magnitudes of the initial and desired vectors λ are of the order of 10, whereas those of the vectors a and τ are already larger than 10^5 and 10^9, respectively. It is our experience that for $n \ge 12$, the vectors a and λ often have numerically overflowing values. This situation is not alleviated by scaling the matrices A,B and C, because in that case, a and τ become insensitive to changes in λ.

V. Conclusions

Three local methods for pole assignment by direct output feedback have been presented. They are based on assigning the eigenvalues, the coefficients of the characteristic polynomial, and the traces of successive powers of the closed-loop state matrix. Gradient formulas are given together with local assignability criteria, allowing iterative solution of the pole assignment problem. The three methods are compared by

discussing the relative sensitivities of the eigenvalues, the characteristic coefficients, and the traces. The method based on assigning the eigenvalues is found the most worthy of recommendation for reasons of sensitivity and possible numerical overflow. This conclusion strongly corroborates our experience based on using these methods on various examples.

References

[1]Tarokh, M., "On Output Feedback Stabilization and Pole Assignment," *International Journal of Control,* Vol. 31, 1980, pp. 339-408.

[2]Anderson, B.D.O., Bose, N.K., and Jury, E.I., "Output Feedback Stabilization and Related Problem Solutions via Decision Methods," *IEEE Transactions on Automatic Control,* Vol. AC-20, Feb. 1975, pp. 53-66.

[3]Brasch, F.M. and Pearson, J.B., "Pole Placement Using Dynamic Compensation," *IEEE Transactions on Automatic Control,* Vol. AC-15, Feb. 1970, pp. 34-43.

[4]Davison, E.J. and Wang, S.H., "On Pole Assignment in Linear Multi-Variable Systems, Using Output Feedback," *IEEE Transactions on Automatic Control,* Vol. AC-20, Aug. 1975, pp. 516-518.

[5]Kimura, H., "Pole Assignment by Gain Ouput Feedback," *IEEE Transactions on Automatic Control,* Vol. AC-20, Aug. 1975, pp. 509-516.

[6]Kimura, H., "A Further Result on the Problem of Pole Assignment by Output Feedback," *IEEE Transactions on Automatic Control,* Vol. AC-22, June 1977, pp. 458-463.

[7]Davison, E.J. and Wang, S.H., "Properties of Linear Time-Invariant Multivariable Systems Subject to Arbitrary Output and State Feedback," *IEEE Transactions on Automatic Control,* Vol. AC-15, Feb. 1973, pp. 24-32.

[8]Davison, E.J., "On Pole Assignment in Linear Systems with Incomplete State Feedback," *IEEE Transactions on Automatic Control,* Vol. AC-15, June 1970, pp. 348-351.

[9]Kabamba, P.T. and Longman, R.W., "Exact Pole Assignment Using Direct or Dynamic Output Feedback," *IEEE Transactions on Automatic Control,* Vol. AC-27, Dec. 1982, pp. 1244-1246.

[10]Byrnes, C.I. and Anderson B.D.O., "Output Feedback and Generic Stabilizability," *SIAM Journal of Control and Optimization,* Vol. 11, May 1984, pp. 362-380.

[11]Paraskevopoulos, P.N., "General Solution to the Output Feedback Eigenvalue-Assignment Problem," *International Journal of Control,* Vol. 27, 1976, pp. 509-528.

[12]Shapiro, E.Y., Fredricks, D.A., Rooney, R.H., and Barmish, B.R., "Pole Placement with Output Feedback," *Journal of Guidance and Control,* Vol. 4, July 1981, pp. 441-442.

[13]Godbout, F.G. and Jordan, D., "Gradient Matrices for Output Feedback Systems," *International Journal of Control,* Vol. 32, 1980, pp. 411-433.

[14]Palis, J. and deMelo, W., *Geometric Theory of Dynamic Systems,* Springer-Verlag, New York, 1982.

[15]Garcia, C.B. and Zangill, W.I., *Pathways to Solutions, Fixed Points and Equilibria,* Prentice-Hall, 1981.

[16]Richter, S.L. and DeCarlo, R.A., "Continuation Methods: Theory and Applications," *IEEE Transactions on Circuits and Systems,* Vol. CAS-30, No. 6, June 1983, pp. 347-352.

[17]Wilkinson, J.H., *The Algebraic Eigenvalue Problem,* Oxford, 1965.

[18]Gantmacher, F.R., *The Theory of Matrices,* Chelsea, 1977.

[19]Bocher, M., *Introduction to Higher Algebra,* Macmillan, New York, 1924.

Efficient Modal Analysis of Damped Large Space Structures

Trevor Williams*
Kingston Polytechnic
Kingston upon Thames, England

Introduction

ATTITUDE and shape control of large space structures (LSS) is a problem made extremely difficult by the dynamic properties typical of such vehicles. The most important of these are the very low inherent damping of their flexible modes and the fact that they generally have a large number of low, closely spaced natural frequencies.

A very attractive way of simplifying the overall control problem somewhat is to initially increase the structural damping in some manner. This could be achieved passively, either by means of dampers distributed over the structure or by careful selection of the materials used in its construction[1] (this should allow[2] damping up to about 10% of critical, as opposed to about 0.5% for "standard" LSS). Alternatively, active damping could be introduced by applying feedback between linear or angular velocity sensors and force or torque actuators (not necessarily colocated) positioned throughout the structure. Similar application of position feedback can also be very useful for raising the structure's natural frequencies.

The damped natural frequencies and damping ratios produced by any of these schemes are very expensive to compute, being given from the eigenvalues of a matrix of large dimension. This has prompted various authors to consider the comparatively small amount of damping applied as a perturbation to the original system dynamics: low-order approximations should then give reasonably accurate estimates for the actual damped eigenstructure. In particular, Refs. 3 and 4 implement this for velocity feedback only, while Ref. 5 also allows circulatory effects (a special case of position feedback) and Ref. 6 considers a general "low-authority controller" feedback structure. These methods are far more efficient than a full eigenstructure calculation: however, they were not in the main specifically developed with the LSS case in mind, and tend to suffer from accuracy problems when applied to such a structure with its closely spaced undamped natural frequencies.

This Note presents a new eigenstructure perturbation technique valid for general feedback which minimizes such numerical difficulties by making exclusive use of *unitary* transformations[7] throughout. Note that these complex matrices (or their real subclass, the *orthogonal* matrices) are basic to nearly all of the numerically reliable algorithms developed in control theory in recent years.[8,9] A further practical advantage of the new method is that it gives directly the order of error anticipated in its eigenvalue and eigenvector estimates, as opposed to the incomplete and/or somewhat complicated results of Refs. 3-6.

Problem Formulation

Consider the n-mode undamped model

$$M\ddot{q}+Kq=u \tag{1}$$

for the structural dynamics of an LSS, where q is the vector of generalized coordinates, u the vector of generalized applied forces, and M and K the system mass and stiffness matrices, respectively. Let ω_i be the ith natural frequency of this structure and ϕ_i the corresponding natural mode: $\{\phi_i\}$ can then be normalized so that $\Phi=(\phi_1,\ldots,\phi_n)$ satisfies $\Phi^TM\Phi=I$ and $\Phi^TK\Phi=\text{diag}(\omega_i^2)$. Defining the modal amplitude vector $\hat{q}$ by $q=\Phi\hat{q}$ and, similarly, $\hat{u}=\Phi^Tu$, Eq. (1), in modal form, becomes

$$\ddot{\hat{q}}+\text{diag}(\omega_i^2)\hat{q}=\hat{u} \tag{2}$$

Velocity plus position feedback corresponds to an applied force

$$u=-C\dot{q}-Dq \tag{3}$$

We consider general matrices C and D here: in the terminology of Ref. 5, such a C corresponds to an arbitrary combination of damping and gyroscopic terms, while D is the result of general stiffness and circulatory contributions. Note that for the special case of direct velocity feedback (DVFB), of great practical interest due to its guaranteed absence[10] of "spillover" instabilities, we must have $C=C^T\geq 0$, $D=0$, and colocated sensors and actuators.

Received April 11, 1985; revision received April 4, 1986.

*Senior Research Associate, School of Computing. Member AIAA.