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by

N. X. VINH
The University of Michigan
Ann Arbor, Michigan

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-- NOTES --

RECURRENCE FORMULAE FOR THE HANSEN'S DEVELOPMENTS*

N. X. Vinh
Associate Professor, Department of Aerospace Engineering
The University of Michigan
Ann Arbor, Michigan

Abstract

In this paper we derive some recurrence formulae which can be used to calculate the Fourier expansions of the functions $(r/a)^n \cos mv$ and $(r/a)^n \sin mv$ in terms of the eccentric anomaly E or the mean anomaly M. We also establish a recurrence process for computing the series expansions for all n and m when the expansions of two basic series are known. These basic series were given in explicit form in the classical literature. The recurrence formulae are linear in the functions involved and thus make very simple the computation of the series.

1. Introduction

It has long been recognized that digital computers are capable of formal manipulation of literal expansions in Celestial Mechanics. Thus it is now easy to extend Cayley's tables of the expansions in elliptic motion(1) to include higher powers in the eccentricity. Using various computing schemes several authors have been successful in obtaining analytical expansions of functions which arise in Celestial Mechanics(2-6). It may, however, still be of interest if we can derive some exact formulae relating these expansions. The new mathematical relations not only provide material for teaching elliptic motion expansions but at the same time they can be used to check the accuracy of the different algorithms already formulated.

In this paper we shall consider the developments in terms of the eccentric anomaly E or the mean anomaly M of the functions

$$\Phi^{n,m} = \left(\frac{r}{a}\right)^n \cos mv \quad \text{and} \quad \Psi^{n,m} = \left(\frac{r}{a}\right)^n \sin mv \tag{1.1}$$

where a is the semi major-axis, r the radial distance and v the true anomaly in elliptic motion. The functions were first considered by Hansen in his Fundamenta(7). For each specific pair of values of n and m where n is a positive or negative integer and m is a positive integer, after a series of transformations he arrived to express $\Phi^{n,m}$ and $\Psi^{n,m}$ in terms of the expansions of $(r/a)^2$ and $(r/a)^{-2}$ and their derivatives with respect to the eccentricity e.

In general we have

$$\left(\frac{r}{a}\right)^n \cos mv = A_0^{n,m} + A_1^{n,m}\cos M + A_2^{n,m}\cos 2M + \cdots$$
$$\left(\frac{r}{a}\right)^n \sin mv = B_1^{n,m}\sin M + B_2^{n,m}\sin 2M + \cdots \tag{1.2}$$

The coefficients $A_0^{n,m}$, $A_1^{n,m}$, ..., $B_1^{n,m}$, ..., called the Hansen's coefficients, are functions of e. The approach of LeShack and Sconzo in computing these coefficients(3) is through the use of the Cauchy's numbers(8) while the key of Deprit and Rom's procedure is the application of Poincaré's method of continuation in the integration of a differential equation with $\rho = \cos E$ as dependent variable and the mean anomaly M as independent variable.(2) In this paper we shall construct a homogeneous linear differential equation of the second order which has the functions (1.1) as independent solutions. From the governing equation we can derive some recurrence formulae and at the same time establish a recurrence process which can be used to generate tables of the expansion of the functions (1.1) in any of the three anomalies to the desired order in the eccentricity and with a minimum number of computations involved.

2. Differential Equations

Consider the vector equation

$$\dot{X} = A(t)X \tag{2.1}$$

where A is the 2×2 matrix

$$A(t) = \begin{bmatrix} f_1(t) & \alpha(f_1 - f_2) \\ \beta(f_1 - f_2) & f_2(t) \end{bmatrix} \tag{2.2}$$

where $f_1(t)$ and $f_2(t)$ are two arbitrary functions of t of the class C^1, and α and β are two arbitrary constants. It can be shown that A(t) is the most general 2×2 matrix which commutes with its integral. For this paper it suffices to prove the following theorem.

Theorem. The equation (2.1) where A(t) is given by (2.2) can be transformed into a homogeneous linear equation with constant coefficient.

Proof: Let

$$X = \exp\left(\tfrac{1}{2}\int (f_1 + f_2)dt\right)Z \tag{2.3}$$

$$\dot{X} = \tfrac{1}{2}(f_1 + f_2)\exp(\ )Z + \exp(\ )\dot{Z} = \exp(\ )AZ$$

Dividing out by exp ( )

$$\dot{Z} = [A - \tfrac{1}{2}(f_1 + f_2)I]Z$$

or

$$\dot{Z} = (f_1 - f_2)BZ \tag{2.4}$$

where B is the constant matrix

$$B = \begin{bmatrix} -\frac{1}{2} & \alpha \\ \beta & -\frac{1}{2} \end{bmatrix} \tag{2.5}$$

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By using the new independent variable s such that

$$s = \int (f_1 - f_2)dt \tag{2.6}$$

we have the required equation

$$\frac{dZ}{ds} = BZ \tag{2.7}$$

Now, the equivalent second order differential equation of the system (2.1) is

$$\ddot{x} - \left[f_1 + f_2 + \frac{\dot{f}_1 - \dot{f}_2}{f_1 - f_2}\right]\dot{x}$$

$$- \left[\frac{f_1 \dot{f}_2 - \dot{f}_1 f_2}{f_1 - f_2} + \alpha\beta(f_1 - f_2)^2 - f_1 f_2\right] x = 0 \tag{2.8}$$

where x is any of the two components of X. Since the characteristic equation of the system (2.7) is

$$\lambda^2 = \alpha\beta + \frac{1}{4} \tag{2.9}$$

we immediately have, through the changes of variables (2.3) and (2.6), for the general solution of (2.8)

If $\lambda^2 = \alpha\beta + \frac{1}{4} > 0$

$$x(t) = \exp\left(\frac{1}{2}\int(f_1 + f_2)dt\right)$$

$$\cdot \left[C_1 \exp(\lambda\int(f_1 - f_2)dt) + C_2 \exp(-\lambda\int(f_1 - f_2)dt)\right]$$

If $\alpha\beta + \frac{1}{4} < 0$

$$x(t) = \exp\left(\frac{1}{2}\int(f_1 + f_2)dt\right)$$

$$\cdot \left[C_1 \cos\sqrt{-\lambda^2}\int(f_1 - f_2)dt + C_2 \sin\sqrt{-\lambda^2}\int(f_1 - f_2)dt\right] \tag{2.10}$$

If $\alpha\beta + \frac{1}{4} = 0$

$$x(t) = \exp\left(\frac{1}{2}\int(f_1 + f_2)dt\right)\left[C_1\int(f_1 - f_2)dt + C_2\right]$$

We observe that the functions $(r/a)^n \cos mv$ and $(r/a)^n \sin mv$ are special cases of the second solution in (2.10) with t = v.

Let

$$-\lambda^2 = -(\alpha\beta + \frac{1}{4}) = m^2$$

$$\exp\left(\frac{1}{2}\int(f_1 + f_2)dv\right) = \left(\frac{r}{a}\right)^n = \frac{(1 - e^2)^n}{(1 + e\cos v)^n}$$

$$\int(f_1 - f_2)dv = v$$

we can deduce

$$f_1(v) = \frac{n e \sin v}{1 + e\cos v} + \frac{1}{2}, \quad f_2(v) = \frac{n e \sin v}{1 + e\cos v} - \frac{1}{2} \tag{2.11}$$

By substituting into (2.8) we have the differential equation which is satisfied by (1.1)

$$\frac{d^2x}{dv^2} - \frac{2n e \sin v}{1 + e\cos v}\frac{dx}{dv}$$

$$+ \left[m^2 + \frac{n^2 e^2 \sin^2 v}{(1 + e\cos v)^2} - \frac{ne(e + \cos v)}{(1 + e\cos v)^2}\right] x = 0 \tag{2.12}$$

If we consider $\Phi^{n,m}$ and $\Psi^{n,m}$ as functions of the eccentric anomaly E, then by the change of variables

$$\sin v = \frac{\sqrt{1 - e^2}\sin E}{1 - e\cos E}$$

$$\cos v = \frac{\cos E - e}{1 - e\cos E} \tag{2.13}$$

$$\frac{dE}{dv} = \frac{1}{\sqrt{1 - e^2}}(1 - e\cos E)$$

we have the differential equation with E as the independent variable

$$(1 - e\cos E)^2 \frac{d^2x}{dE^2} + (1 - 2n)e\sin E(1 - e\cos E)\frac{dx}{dE}$$

$$+ [(1 - e^2)(m^2 - n^2) + n(2n - 1)(1 - e\cos E)$$

$$-n(n - 1)(1 - e\cos E)^2]x = 0 \tag{2.14}$$

Finally if we consider $\Phi^{n,m}$ and $\Psi^{n,m}$ as functions of the mean anomaly M, then by the transformation

$$M = E - e\sin E$$

$$\frac{dM}{dE} = 1 - e\cos E \tag{2.15}$$

we have the differential equation which is satisfied by $\Phi^{n,m}$ and $\Psi^{n,m}$, considered as functions of M.

$$(1 - e\cos E)^2 \frac{d^2x}{dM^2} + 2(1 - n)e\sin E\frac{dx}{dM} \tag{2.16}$$

$$+ \frac{1}{(1 - e\cos E)^2}[(1 - e^2)(m^2 - n^2) + n(2n - 1)(1 - e\cos E)$$

$$-n(n - 1)(1 - e\cos E)^2]x = 0$$

In the last equation the coefficients are to be expressed in terms of M using the Kepler's equation (2.15).

The differential equations (2.12), (2.14) and (2.16) with respectively the true anomaly v, the eccentric anomaly E, and the mean anomaly M as independent variable will serve as basic equations in the derivation of the recurrence formulae for the series expansions of $\Phi^{n,m}$ and $\Psi^{n,m}$ in each of the three anomalies. In the following we shall consider the expansions in E and in M.

3. Fourier Expansions in Terms of E

Let

$$X^{n,m} = \Phi^{n,m} + i\Psi^{n,m} = \left(\frac{r}{a}\right)^n \exp(imv) \tag{3.1}$$

We have seen that $X^{n,m}$, considered as function of the eccentric anomaly E, satisfies the differential equation

$$(1-e\cos E)^2\frac{d^2X^{n,m}}{dE^2}$$

$$+(1-2n)e\sin E(1-e\cos E)\frac{dX^{n,m}}{dE}$$

$$+[(1-e^2)(m^2-n^2)+n(2n-1)(1-e\cos E)$$

$$-n(n-1)(1-e\cos E)^2]X^{n,m}=0 \qquad (3.2)$$

From (3.1)

$$\frac{dX^{n,m}}{dE}=n\left(\frac{r}{a}\right)^{n-1}\exp(imv)\frac{d}{dE}\left(\frac{r}{a}\right)$$

$$+im\left(\frac{r}{a}\right)^{n}\exp(imv)\frac{dv}{dE}$$

Since

$$\frac{r}{a}=1-e\cos E$$

$$\frac{d}{dE}\left(\frac{r}{a}\right)=e\sin E$$

Also

$$\frac{dv}{dE}=\frac{\sqrt{1-e^2}}{1-e\cos E}=\sqrt{1-e^2}\left(\frac{r}{a}\right)^{-1}$$

Therefore

$$e\sin E(1-e\cos E)\frac{dX^{n,m}}{dE}$$

$$=ne^2\sin^2E\,X^{n,m}+im\sqrt{1-e^2}\,e\sin E\,X^{n,m}$$

Using the relations

$$e^2\sin^2E=e^2-e^2\cos^2E=-(1-e^2)+2\left(\frac{r}{a}\right)-\left(\frac{r}{a}\right)^2$$

$$\sqrt{1-e^2}\sin E=\left(\frac{r}{a}\right)\sin v$$

$$i\sin v=\exp(iv)-\cos v=\exp(iv)+\frac{1}{e}-\frac{(1-e^2)}{e}\left(\frac{r}{a}\right)^{-1}$$

we have

$$e\sin E(1-e\cos E)\frac{dX^{n,m}}{dE}=m\,eX^{n+1,m+1}-nX^{n+2,m}$$

$$+(m+2n)X^{n+1,m}$$

$$-(m+n)(1-e^2)X^{n,m}$$

By substituting into Eq. (3.2) we have the recurrence formula

$$m(2n-1)eX^{n-1,m+1}=\frac{d^2X^{n,m}}{dE^2}+n^2X^{n,m}$$

$$-(m+n)(2n-1)X^{n-1,m} \qquad (3.3)$$

$$+(m+n)(m+n-1)(1-e^2)X^{n-2,m}$$

where X can be Φ or Ψ. This formula can be used to go from cos m v (or sin m v) to cos (m + 1) v (or *sin (m + 1)v). Changing m into -m and noticing* that $X^{n,-m}=\overline{X}^{n,m}$ where $\overline{X}^{n,m}$ is the complex conjugate of $X^{n,m}$ we have

$$m(1-2n)e\overline{X}^{n-1,m-1}=\frac{d^2\overline{X}^{n,m}}{dE^2}+n^2\overline{X}^{n,m}$$

$$+(n-m)(1-2n)\overline{X}^{n-1,m} \qquad (3.4)$$

$$+(n-m)(n-m-1)(1-e^2)\overline{X}^{n-2,m}$$

where $\overline{X}$ can be Φ or Ψ. This formula can be used to go from cos (m + 1)v (or sin (m + 1)v) to cos m v (or sin m v).

Combining the Eqs. (3.3) and (3.4) we easily obtain

$$e[X^{n,m+1}+X^{n,m-1}]=2(1-e^2)X^{n-1,m}-2X^{n,m} \qquad (3.5)$$

where X can be Φ or Ψ. This last relation can be derived directly from the polar equation of elliptic orbit.

The process for constructing tables of the expansions of $\Phi^{n,m}$ and $\Psi^{n,m}$ is as follows.

Expansions of $(r/a)^n$ cos m v

First step

Let m = 0 in (3.3) and we have

$$\frac{d^2}{dE^2}\left(\frac{r}{a}\right)^n+n^2\left(\frac{r}{a}\right)^n-n(2n-1)\left(\frac{r}{a}\right)^{n-1}$$

$$+n(n-1)(1-e^2)\left(\frac{r}{a}\right)^{n-2}=0 \qquad (3.6)$$

This recurrence formula can be used to calculate the series for $(r/a)^n$ for all values of n when those for n = -1, and n = -2 have been obtained.

Let

$$\left(\frac{r}{a}\right)^n=\sum_{p=0,1,2...}A_p^n\cos pE \qquad (3.7)$$

Then we have the recurrence formula for the coefficients A_p^n

$$(n^2-p^2)A_p^n-n(2n-1)A_p^{n-1}+n(n-1)(1-e^2)A_p^{n-2}=0 \quad (3.8)$$

When n is negative the series is infinite. When n ≥ 0 the series terminates at the term cos n E. In *this case the last coefficient cannot be calculated by formula (3.8) but by setting E = 0 in Eq. (3.7) it* can readily be seen that

$$A_n^n=(1-e)^n-\sum_{p=0}^{n-1}A_p^n \qquad (3.9)$$

The starting series for n = -1 and n = -2 can be easily calculated by the classical methods. We have[9]

$$\left(\frac{r}{a}\right)^{-1}=\frac{1}{\sqrt{1-e^2}}(1+2\beta\cos E+2\beta^2\cos 2E+\cdots) \quad (3.10)$$

where β is given by

$$\beta = \frac{1 - \sqrt{1 - e^2}}{e} \tag{3.11}$$

and in general

$$\beta^p = \left(\frac{e}{2}\right)^p + p\left(\frac{e}{2}\right)^{p+2} + \frac{p}{2!}(p+3)\left(\frac{e}{2}\right)^{p+4} + \frac{p}{3!}(p+4)(p+5)\left(\frac{e}{2}\right)^{p+6} + \frac{p}{4!}(p+5)(p+6)(p+7)\left(\frac{e}{2}\right)^{p+8} + \cdots \tag{3.12}$$

Also

$$\left(\frac{r}{a}\right)^{-2} = (1 - e^2)^{-\frac{3}{2}} + (1 - e^2)^{-\frac{3}{2}} \sum_{p=1}^{\infty} 2\beta^p (1 + p\sqrt{1 - e^2}) \cos pE \tag{3.13}$$

Second Step

Calculate all $(r/a)^n \cos v$ by taking $m = 0$ in the recurrence formula (3.5). Explicitly we have

$$\left(\frac{r}{a}\right)^n \cos v = \frac{(1 - e^2)}{e}\left(\frac{r}{a}\right)^{n-1} - \frac{1}{e}\left(\frac{r}{a}\right)^n \tag{3.14}$$

Third Step

Use (3.5) again with $m = 1$ to calculate all $(r/a)^n \cos 2v$

$$\left(\frac{r}{a}\right)^n \cos 2v = \frac{2(1 - e^2)}{e}\left(\frac{r}{a}\right)^{n-1} \cos v - \frac{2}{e}\left(\frac{r}{a}\right)^n \cos v - \left(\frac{r}{a}\right)^n \tag{3.15}$$

The process continues by successive applications of (3.5) to calculate $(r/a)^n \cos mv$.

Since the recurrence formulae are linear in the functions involved, the calculation process is extremely simple. But while this process is very simple, it suffers from two unavoidable defects:

The Hansen's coefficients behave like the Bessel's coefficients. Each time we apply formula (3.5) the order in e of the coefficients is decreased by one unit. Therefore to compute tables up to $\cos mv$ to the order of e^p we need to compute the basic series for $(r/a)^{-1}$, and $(r/a)^{-2}$ to the order of e^{p+m}. Since these series are given in explicit forms the defect does not create any real handicap.

When n is a negative integer, to compute $(r/a)^n \cos(m + 1)v$, it involves the expansion of $(r/a)^{n-1} \cos mv$. Therefore to compute tables down to $(r/a)^n \cos mv$, n being a negative integer, in the first step mentioned above we should compute down to the expansion of $(r/a)^{n-m}$. This defect again does not create any serious problem since by the recurrence formula (3.8) we can easily calculate $(r/a)^n$ for any negative n.

Expansions of $(r/a)^n \sin mv$

We first compute all $(r/a)^n \sin v$ by the relation

$$\left(\frac{r}{a}\right)^n \sin v = \frac{\sqrt{1 - e^2}}{ne} \frac{d}{dE}\left(\frac{r}{a}\right)^n \tag{3.16}$$

When $n = 0$ we can calculate the expansion of sin v by

$$\frac{d}{dE}(\sin v) = \sqrt{1 - e^2}\left(\frac{r}{a}\right)^{-1} \cos v = \frac{(1 - e^2)^{\frac{3}{2}}}{e}\left(\frac{r}{a}\right)^{-2} - \frac{(1 - e^2)^{\frac{1}{2}}}{e}\left(\frac{r}{a}\right)^{-1} \tag{3.17}$$

or directly by

$$\sin v = (1 - \beta^2) \sum_{p=1}^{\infty} \beta^{p-1} \sin pE \tag{3.18}$$

Next we can successively apply the recurrence formula (3.5) to calculate all $(r/a)^n \sin 2v$, and so on.

4. Fourier Expansions in Terms of M

First we notice that these developments can be deduced from those in terms of E by using the classical series expansions

$$\sin mE = m\sum_{p=1}^{\infty} \frac{\sin pM}{p}[J_{p-m}(pe) + J_{p+m}(pe)] \tag{4.1}$$

$$\cos mE = A_0 + m\sum_{p=1}^{\infty} \frac{\cos pM}{p}[J_{p-m}(pe) - J_{p+m}(pe)]$$

$A_0 = 1$ if $m = 0$

$= -\frac{1}{2}e$ if $m = 1$

$= 0$ if $m > 1$

where $J_k(pe)$ is the Bessel's coefficient of order k and argument pe.

If direct computation is desired, we can start with the differential equation (2.16) and, by using the same type of derivation as in the preceding section, we obtain the recurrence formula

$$2m(n-1)eX^{n-3,m+1} = \frac{d^2X^{n,m}}{dM^2} + n(n-1)X^{n-2,m} - (2n^2 + 2mn - 3n - 2m)X^{n-3,m} + (m+n)(m+n-2)(1-e^2)X^{n-4,m} \tag{4.2}$$

where X can be Φ or Ψ.

By taking $n = 1$ we have

$$\frac{d^2}{dM^2}\left(\frac{r}{a}\right)\begin{matrix}\cos mv\\ \sin mv\end{matrix} + \left(\frac{r}{a}\right)^{-2}\begin{matrix}\cos mv\\ \sin mv\end{matrix} + (m^2 - 1)(1 - e^2)\left(\frac{r}{a}\right)^{-3}\begin{matrix}\cos mv\\ \sin mv\end{matrix} = 0 \tag{4.3}$$

By further taking $m = 1$ we have the classical formula

$$\frac{d^2}{dM^2}\left(\frac{r}{a}\right)\begin{matrix}\cos v\\ \sin v\end{matrix} + \left(\frac{r}{a}\right)^{-2}\begin{matrix}\cos v\\ \sin v\end{matrix} = 0 \qquad (4.4)$$

Putting m = 0 in (4.2) we have Hansen's recurrence formula[2,7,9]

$$\frac{d^2}{dM^2}\left(\frac{r}{a}\right)^n + n(n-1)\left(\frac{r}{a}\right)^{n-2} - n(2n-3)\left(\frac{r}{a}\right)^{n-3} + n(n-2)(1-e^2)\left(\frac{r}{a}\right)^{n-4} = 0 \qquad (4.5)$$

We also have as before

$$e[X^{n,m+1} + X^{n,m-1}] = 2(1-e^2)X^{n-1,m} - 2X^{n,m} \qquad (4.6)$$

where X can be Φ or Ψ.

The process for computing tables of the expansions of $\Phi^{n,m}$ and $\Psi^{n,m}$ is as follows.

Expansions of $(r/a)^n \cos mv$

First Step

The recurrence formula (4.5) is used to calculate the series for $(r/a)^n$ for all values of n when those for certain values have been obtained.

Let

$$\left(\frac{r}{a}\right)^n = \sum_{p=0}^{\infty} A_p^n \cos pM \qquad (4.7)$$

Then we have the recurrence formula for the coefficients A_p^n

$$p^2 A_p^n = n(n-1)A_p^{n-2} - n(2n-3)A_p^{n-3} + n(n-2)(1-e^2)A_p^{n-4} \qquad (4.8)$$

In particular the constant term is given by

$$(n+1)(n+2)A_0^n - (n+2)(2n+1)A_0^{n-1} + n(n+2)(1-e^2)A_0^{n-2} = 0 \qquad (4.9)$$

Examination of the formulae reveals that the expansions for all values of n can be evaluated in terms of the expansions for n = 1, 2, -2 and -4. But the expansions for n = 1, and n = -4 can be evaluated in terms of the expansions for n = 2, -2 and -3 through the relations

$$\left(\frac{r}{a}\right) = (1-e^2) + \frac{e}{2}\frac{d}{de}\left(\frac{r}{a}\right)^2$$

$$(1-e^2)\left(\frac{r}{a}\right)^{-4} = \left(\frac{r}{a}\right)^{-3} + \frac{e}{2}\frac{d}{de}\left(\frac{r}{a}\right)^{-2} \qquad (4.10)$$

Hence we only need to compute the two basic starting series $(r/a)^2$ and $(r/a)^{-2}$ by the classical methods. These are given explicitly in the classical literature. We have

$$\left(\frac{r}{a}\right)^2 = 1 + \frac{3}{2}e^2 - \sum_{p=1}^{\infty}\frac{4}{p^2}J_p(pe)\cos pM \qquad (4.11)$$

Series expansion of $(r/a)^{-2}$ is more involved. Probably the simplest way is to use the series (3.13) and the transformation (4.1) with the knowledge that the constant term in the expansion is $(1-e^2)^{-\frac{1}{2}}$. Another simple way to have the series expansion of $(r/a)^{-2}$ is to use the relation

$$\left(\frac{r}{a}\right)^{-2} = \frac{1}{\sqrt{1-e^2}}\frac{dv}{dM} \qquad (4.12)$$

The expansion of v in terms of M has been calculated by Schubert as far as e^{20} [10].

Second Step

Once the expansions of $(r/a)^n$ for all values of n have been obtained we can successively use the recurrence formula (4.6) with m = 0, 1, ... to calculate the expansions of $(r/a)^n \cos v$, $(r/a)^n \cos 2v$,... as described in the preceding section.

Expansions of $(r/a)^n \sin mv$

We first compute all $(r/a)^n \sin v$ by the relation

$$\left(\frac{r}{a}\right)^n \sin v = \frac{\sqrt{1-e^2}}{(n+1)e}\frac{d}{dM}\left(\frac{r}{a}\right)^{n+1} \qquad (4.13)$$

When n = -1 we can use the relation

$$\frac{d}{dM}\left(\frac{r}{a}\right)^{-1}\sin v = \frac{\sqrt{1-e^2}}{e}\left(\frac{r}{a}\right)^{-2} - 3\,\frac{\sqrt{1-e^2}}{e}\left(\frac{r}{a}\right)^{-3} + 2\,\frac{(1-e^2)^{\frac{3}{2}}}{e}\left(\frac{r}{a}\right)^{-4} \qquad (4.14)$$

to calculate $(r/a)^{-1}\sin v$.

Next we can use the recurrence formula (4.6) to calculate all $(r/a)^n \sin 2v$, and so on.

As discussed before to compute tables for the expansions of $\Phi^{n,m}$ and $\Psi^{n,m}$ up to the value m and from a negative $-n (n > 0)$ to a positive n up to the order of e^p we should compute the basic series for $(r/a)^2$ and $(r/a)^{-2}$ to the order of e^{p+m}, and first calculate the series from $(r/a)^{-n-m}$ to $(r/a)^{n+1}$.

5. Conclusion

In this paper we have derived recurrence formulae to calculate the series expansions of $(r/a)^n \cos mv$ and $(r/a)^n \sin mv$ in terms of the eccentric anomaly E or the mean anomaly M. We also have established a recurrence process which can be used to compute the series expansions for all n and m when the expansions of two basic series are known. The expansions in terms of the true anomaly v are similar to those in terms of the eccentric anomaly E. By observing that

$$\left(\frac{r}{a}\right)^n = (1 - e\cos E)^n = \frac{(1-e^2)^n}{(1+e\cos v)^n} \qquad (5.1)$$

for the expansions in v we only need to change n into -n, e into -e, a into $a(1-e^2)$, and E into v in Eq. (3.6) and next change the sign of all the exponents to have

$$(1-e^2)\frac{d^2}{dv^2}\left(\frac{r}{a}\right)^n + n^2(1-e^2)\left(\frac{r}{a}\right)^n - n(2n+1)\left(\frac{r}{a}\right)^{n+1} + n(n+1)\left(\frac{r}{a}\right)^{n+2} = 0 \qquad (5.2)$$

In applying the recurrence formulae, each time we go to a next higher multiple anomaly the order in e in the Hansen's coefficients is decreased by one. This is caused by a property of the Hansen's coefficients, called the D'Alembert characteristic by E.W. Brown[11]; namely, the lowest order in e in the coefficient of cos pM (or sin pM) in the expansion of $(r/a)^n \cos m v$ (or $(r/a)^n \sin m v$) is $|p - m|$. This property is also true for the expansions in E and in v.

In his tables Cayley also gave the series expansion of log (r/a) in terms of M. Explicitly we have[11]

$$\log\left(\frac{r}{a}\right) = -\log(1+\beta^2) - 2\sum_{p=1}^{\infty}\frac{\beta^p}{p}\cos pE \qquad (5.3)$$

and

$$\log\left(\frac{r}{a}\right) = -\log(1+\beta^2) + e\beta - 2\sum_{s=1}^{\infty}\frac{1}{s}\sum_{s=1}^{\infty}\beta^p[J_{s-p}(se) - J_{s+p}(se)]\cos sM \qquad (5.4)$$

In our process we can have those expansions by integrating term by term the following relations

$$\frac{d}{dE}\log\frac{r}{a} = \frac{e}{\sqrt{1-e^2}}\sin v \qquad (5.5)$$

with the constant term in the integration being $\log (1+\sqrt{1-e^2})/2$ and

$$\frac{d}{dM}\log\frac{r}{a} = \frac{e}{\sqrt{1-e^2}}\left(\frac{r}{a}\right)^{-1}\sin v \qquad (5.6)$$

with the constant term being $\log(1+\sqrt{1-e^2})/2 + 1 - \sqrt{1-e^2}$.

The formulae we have derived are general and they may serve to add new dimension to the teaching of series expansions in elliptic motion. The differential equations in section 2 can be considered as general equations of motion of the two-body problem. For example, if we put n = 1, m = 0, $x = 1 - e\cos E = 1 - e\rho$ in Eq. (2.16) we have

$$D^2\rho + (\rho - e)(1 - e\rho)^{-3} = 0. \qquad (5.7)$$

where D denotes the differentiation with respect to M. Using binomial series expansion we have the differential equation considered by Deprit and Rom[2], and Moulton[12]

$$(D^2+1)\rho = \frac{1}{2}\sum_{p\geq 1}(p+1)[p-(p+2)\rho^2]\rho^{p-1}e^p \qquad (5.8)$$

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