# Technical Notes

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## Asymptotic Solution to the Problem of Optimal Low-Thrust Energy Increase

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### Introduction

THE problem to be considered is that of optimal ascent from an initial circular planetary orbit to some specified final energy level by a spacecraft equipped with a low-thrust engine. Optimal will be defined as minimum time; and since it will be assumed that the engine produces continuous thrust with constant thrust-acceleration, fuel expenditure is minimized.

Analytical solutions to this problem have previously been found by Lawden,[1] and by Breakwell and Rauch.[2] Until now, Lawden's was the only analysis which used a small parameter perturbation approach; and his results failed to predict the oscillatory nature of the optimal control program. Breakwell and Rauch's work was directed primarily toward the analytical representation of a nominal trajectory and guidance coefficients for a neighboring optimal guidance scheme. Their solution is basically a series solution in the radial distance, but also contains some additional correction terms which were found by developing a set of defining differential equations, assuming a periodic solution with variable coefficients, and employing the method of averaging to determine those coefficients. The solution correctly, represents the characteristics of the control program and trajectory and, for at least eight revolutions, matches a numerically generated solution to within 1%. The use of the radial distance as independent variable and the series form of the solution, however, make an analysis of the motion and a comparison with other spiral trajectories difficult.

The purpose of this Note is to present an accurate small parameter perturbation solution to the problem. In addition, the optimal trajectory is analyzed and compared with a tangential thrust trajectory.

### Analysis

The problem is formulated using the equations of motion

$$dr/d\tau = v \sin\gamma \tag{1}$$

$$dv/d\tau = -(1/r^2)\sin\gamma + \varepsilon\cos\varphi \tag{2}$$

$$d\gamma/d\tau = (1/v)[(v^2/r) - (1/r^2)]\cos\gamma + (\varepsilon/v)\sin\varphi \tag{3}$$

with boundary conditions

$$r(0) = 1, \quad v(0) = 1, \quad \gamma(0) = 0, \quad \tfrac{1}{2}v^2(\tau_f) - 1/r(\tau_f) = E_f \tag{4}$$

where $r$ is the radial distance, $v$ the total velocity, $\gamma$ the flight path angle, $\varepsilon$ the thrust-acceleration, $\varphi$ the thrust direction angle away from the velocity vector, $E_f$ the specified final energy level, and $\tau$ the time. Equations (1–4) have been nondimensionalized with respect to the initial circular orbit conditions. The minimum time control program is obtained by maximizing the following Hamiltonian as a function of $\varphi$

$$H = \lambda_r v\sin\gamma - \lambda_v(1/r^2)\sin\gamma + \lambda_\gamma(v/r - 1/r^2v)\cos\gamma + \varepsilon[\lambda_v\cos\varphi + \lambda_\gamma(1/v)\sin\varphi] \tag{5}$$

The resulting optimal control is

$$\sin\varphi = \lambda_\gamma(\lambda_\gamma{}^2 + \lambda_v{}^2v^2)^{-1/2}, \quad \cos\varphi = \lambda_v v(\lambda_\gamma{}^2 + \lambda_v{}^2v^2)^{-1/2} \tag{6}$$

where the multipliers are defined by the differential equations

$$d\lambda_r/d\tau = -\partial H/\partial r, \quad d\lambda_v/d\tau = -\partial H/\partial v, \quad d\lambda_\gamma/d\tau = -\partial H/\partial\gamma \tag{7}$$

with boundary conditions given by the transversality conditions

$$\lambda_r(\tau_f) - [r^2(\tau_f)v(\tau_f)]^{-1} = 0, \quad \lambda_v(\tau_f) - 1 = 0, \quad \lambda_\gamma(\tau_f) = 0 \tag{8}$$

Although the two point boundary value problem presented previously cannot be solved analytically, its solution can be approximated by use of the two variable expansion procedure.[3,4] The state variables and multipliers are assumed to be represented by asymptotic expansions in powers of the small parameter $\varepsilon$. In addition, these expansions are taken to be functions of the two new independent variables

$$\tau_1 = \varepsilon\tau, \quad \tau_2 = \int_0^\tau \omega_o(\varepsilon s)\, ds \tag{9}$$

where $\omega_o(\cdot)$ is a function to be defined in the course of the problem solution. Note that in view of (9), derivatives with respect to $\tau$ become

$$d(\cdot)/d\tau = \omega_o(\tau_1)[\partial(\cdot)/\partial\tau_2] + \varepsilon[\partial(\cdot)/\partial\tau_1] \tag{10}$$

The terms in the asymptotic expansions are determined by solving the equations which result when the expansions are substituted into (1–4, 6–8) and relation 10 is used to represent derivatives. To first order the approximate solution obtained in this manner is

$$r(\tau_1,\tau_2) = \omega_o{}^{-2/3}(\tau_1) + \varepsilon\omega_o{}^{-2/3}(\tau_1)[B_1(\tau_1)\sin\tau_2 - A_1(\tau_1)\cos\tau_2] \tag{11}$$

$$v(\tau_1,\tau_2) = \omega_o{}^{1/3}(\tau_1) + \varepsilon\omega_o{}^{1/3}(\tau_1)[A_1(\tau_1)\cos\tau_2 - B_1(\tau_1)\sin\tau_2] \tag{12}$$

$$\gamma(\tau_1,\tau_2) = \varepsilon[2\omega_o{}^{-4/3}(\tau_1) + A_1(\tau_1)\sin\tau_2 + B_1(\tau_1)\cos\tau_2] \tag{13}$$

$$\lambda_r(\tau_1,\tau_2) = \omega_o(\tau_1) + \varepsilon\{\omega_o(\tau_1)D_2(\tau_1) + \omega_o{}^{2/3}(\tau_1)D_1(\tau_1)\cos[\tau_2 + \beta(\tau_1)] + 2\omega_o(\tau_1)[A_1(\tau_1)\cos\tau_2 - B_1(\tau_1)\sin\tau_2]\} \tag{14}$$

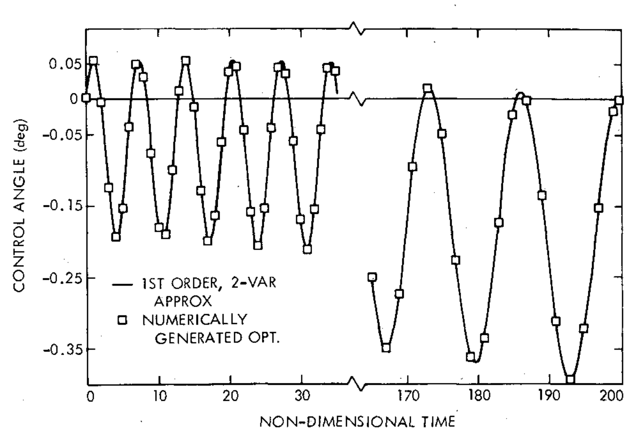


**Fig. 1 Time histories of the control angle.**

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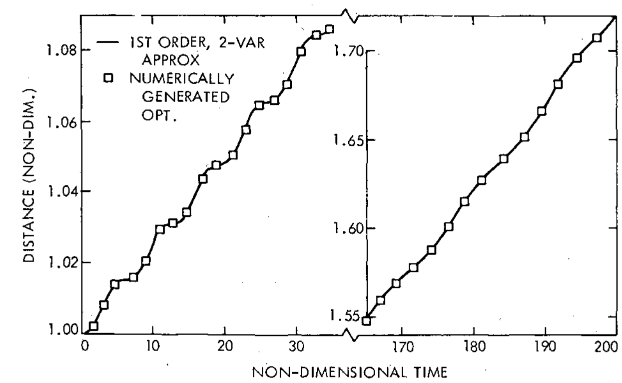


**Fig. 2 Time histories of the radial distance.**

$$\lambda_v(\tau_1, \tau_2) = 1 + \varepsilon\{D_2(\tau_1) + 2\omega_o^{-1/3}(\tau_1)D_1(\tau_1)\cos[\tau_2 + \beta(\tau_1)] + A_1(\tau_1)\cos\tau_2 - B_1(\tau_1)\sin\tau_2\} \quad (15)$$

$$\lambda_\gamma(\tau_1, \tau_2) = \varepsilon\{-\omega_0^{-1}(\tau_1) + D_1(\tau_1)\sin[\tau_2 + \beta(\tau_1)]\} \quad (16)$$

where

$$\omega_o(\tau_1) = (1 - \tau_1)^3$$

$$A_1(\tau_1) = -2[c_1(p_1 + 1)(1 - \tau_1)^{p_1 - 1} + c_2(p_2 + 1)(1 - \tau_1)^{p_2 - 1}]$$

$$B_1(\tau_1) = -2[c_3(p_1 + 1)(1 - \tau_1)^{p_1 - 1} + c_4(p_2 + 1)(1 - \tau_1)^{p_2 - 1}]$$

$$D_1(\tau_1) = [(c_1^2 + c_3^2)(1 - \tau_1)^{2p_1} + 2(c_1c_2 + c_3c_4)(1 - \tau_1)^{p_1 + p_2} + (c_2^2 + c_4^2)(1 - \tau_1)^{2p_2}]^{1/2}$$

$$\beta(\tau_1) = \tan^{-1}\{[c_3(1 - \tau_1)^{p_1} + c_4(1 - \tau_1)^{p_2}]/[c_1(1 - \tau_1)^{p_1} + c_2(1 - \tau_1)^{p_2}]\}$$

$$D_2(\tau_1) = -2(c_1 + c_2)(1 - \tau_1)^3 + 2[(p_1 + 1)/(p_1 - 1)](c_1 - c_3) \cdot [(1 - \tau_1)^{p_1 - 1} - (1 - \tau_{1f})^{p_1 - 1}] + 2[(p_2 + 1)/(p_2 - 1)](c_2 - c_4)[(1 - \tau_1)^{p_2 - 1} - (1 - \tau_{1f})^{p_2 - 1}]$$

with

$$p_1 = \tfrac{1}{2}[1 + (10)^{1/2}], \quad p_2 = \tfrac{1}{2}[1 - (10)^{1/2}]$$

$$c_1 = -(1/c_5)(p_2 + 1)(1 - \tau_{1f})^{-3}\sin\tau_{2f}$$

$$c_2 = (1/c_5)(p_1 + 1)(1 - \tau_{1f})^{-3}\sin\tau_{2f}$$

$$c_3 = -(1/c_5)[(p_2 + 1)(1 - \tau_{1f})^{-3}\cos\tau_{2f} - (1 - \tau_{1f})^{p_2}]$$

$$c_4 = (1/c_5)[(p_1 + 1)(1 - \tau_{1f})^{-3}\cos\tau_{2f} - (1 - \tau_{1f})^{p_1}]$$

$$c_5 = (p_1 + 1)(1 - \tau_{1f})^{p_2} - (p_2 + 1)(1 - \tau_{1f})^{p_1}$$

Also evaluation of the integral in (9) gives

$$\tau_2 = (1 - \tfrac{3}{2}\varepsilon\tau + \varepsilon^2\tau^2 - \tfrac{1}{4}\varepsilon^3\tau^3)\tau = \omega_1(\tau_1)\tau \quad (17)$$

Lastly, the optimal control program is

$$\tan\varphi = -\varepsilon\omega_o^{-1/3}[\omega_o^{-1} - D_1\sin(\tau_2 + \beta)] \quad (18)$$

In order to ascertain the accuracy of the solution, a comparison was made with a numerically generated optimal energy increase trajectory. With a specified terminal energy of $E_f = -0.2904134$, a minimum time of $\tau_f = 200$ was obtained for an $\varepsilon = 1.189409$

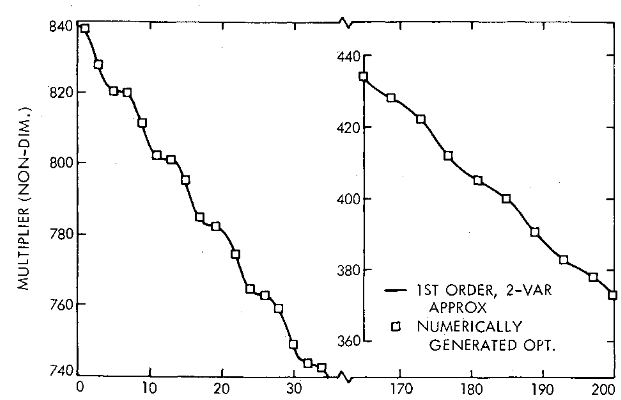


**Fig. 3 Time histories of the Lagrange multiplier—$\lambda_r$.**

**Table 1 Energy integrals**

| Case | $\tau_f = 100$ | $\tau_f = 200$ | $\tau_f = 300$ | $\tau_f = 400$ |
|---|---|---|---|---|
| Optimal | 2.98492 | 2.67723 | 4.79314 | 6.95066 |
| Tangential | 2.90058 | 1.75120 | 4.49649 | 4.22972 |

$\times 10^{-3}$; the optimal trajectory made slightly more than 22 revolutions about the planet. The two variable and numerical solutions agreed to three significant digits, thus confirming the validity of the approximate solution. Figures 1, 2, and 3 show the time histories of the optimal control angle, radial distance, and multiplier $\lambda_r$. Note how well the approximate solution matches the changing amplitude and period of the trajectory and control oscillations.

In Eqs. (11–13) the two basic components of the spiral are clearly shown. The zero-order terms correspond to the circular asymptote commonly encountered in connection with energy increase trajectories, i.e., the condition $r(\tau_1, \tau_2)v^2(\tau_1, \tau_2) = 1$ holds to zero order.[5] Moreover the representation

$$r(\tau) \sim \omega_o^{-2/3}(\tau_1) = (1 - \varepsilon\tau)^{-2} \quad (19)$$

is a well-known zero-order approximation to spirals.[6] The first-order terms bring out the trajectory's oscillatory character and show the manner in which the amplitude and frequency of the oscillations vary as the vehicle spirals outward. It is in this first order short period motion that the optimal differs from the near optimal tangential thrust trajectory. This difference can be clearly shown by use of approximate solutions because the tangential trajectory may be presented by Eq. (11–13) by defining $A_1(\tau_1) = 0$, and $B_1(\tau_1) = -2\omega_o^{1/3}(\tau_1)$.

The optimal reaches a given energy level faster than the tangential because of the oscillatory character noted previously. From the energy rate equation

$$dE/d\tau = \varepsilon v\cos\varphi \quad (20)$$

and the expansion solutions, the energy change for the optimal and for the tangential is

$$E_f - E_0 = \tfrac{1}{2}[1 - (1 - \varepsilon\tau_f)^2] + \varepsilon^2\int_0^{\tau_f}(1 - \tau_1)[A_1\cos\tau_2 - B_1\sin\tau_2]\,d\tau + 0(\varepsilon^3) \quad (21)$$

where the difference between the two trajectories is determined by their respective values of $A_1$ and $B_1$. For various values of $\tau_f$, the integral in (21) is given in Table 1 (for $\varepsilon = 10^{-3}$).

Since for a given $\tau_f$ the first term in (21) is the same for both trajectories, it follows from Table 1 that the energy level reached by the optimal is always largest. Consequently, the tangential will require additional time to attain a given energy. A closer comparison of the tangential and optimal reveals that due to the oscillation differences, the optimal has a higher average velocity. Since for both trajectories $\cos\varphi = 1$ to $0(\varepsilon^2)$, is is clear from (20) that the optimal has a higher average rate of energy increase; and this accounts for its ability to reach the desired energy in less time.

## References


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