Stability of Plane Poiseuille Flows of Viscoelastic Liquids: An Asymptotic Solution

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The Orr-Sommerfeld equation, modified to include viscoelastic effects, is solved asymptotically. The method of inner and outer expansions is used to determine the characteristic equation. The present results are found to be in fairly good agreement with the results obtained numerically by Chun and Schwarz. However, the present results do not agree with those obtained in an earlier asymptotic solution by Chan Man Fong and Walters. It is shown that the effect of the viscoelasticity is stabilizing.

Introduction

As a result of the increased interest in non-Newtonian flows, there have been two recent studies of the stability of plane Poiseuille flow of viscoelastic liquids. In the first of these studies, Walters,[1] using an integral constitutive equation of the type proposed by Oldroyd,[2] developed a modified version of the Orr-Sommerfeld equation, and later Chan Man Fong and Walters[3] published two asymptotic solutions. In the second study, Chun and Schwarz,[4] using the differential constitutive equation of a second-order liquid derived by Coleman and Noll,[5] also developed a modified version of the Orr-Sommerfeld equation and, in the same paper, published a numerical solution. The two versions of the stability equation are the same (see the Appendix), and in both solutions the same boundary conditions and primary velocity profile were used. Thus, it would be interesting to compare the two sets of results directly; however, this is not possible for reasons to be given later. In fact, the two asymptotic solutions of Chan Man Fong and Walters cannot be quantitatively compared.

The purpose of this paper is to present an asymptotic solution to the same problem for which the results can be compared quantitatively with those of Chun and Schwarz as well as with one set of the results of Chan Man Fong and Walters.

Although the present asymptotic solution is essentially a perturbation of the solutions obtained by Tollmien[6] and Lin,[7] the characteristic equation is derived in a somewhat different manner. The approach used here is due, with slight modifications, to Graebel[8] and it explains the roles of the viscous and inviscid solutions in terms of inner and outer expansions. Moreover, the approximations for both the Newtonian as well as the non-Newtonian case are precisely established in the sense that the method is general and could be used to take higher-order terms into consideration.

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Stability Problem

For the remainder of this paper, the flow is taken to be in the x-direction and the plates to be located at $y = \pm 1$.

Walters, as well as Chun and Schwarz, found the stability of infinitesimal, two-dimensional,[9] in-plane disturbances in parallel flows to be governed by

$$(U - c)(D^2 - \alpha^2)\phi - (D^2U)\phi = \{[1/i\alpha - \beta(U - c)] \times (D^2 - \alpha^2)^2\phi + \beta(D^4U)\phi\}/Re \quad (1)$$

in which Re is a Reynolds number based on the speed at the midpoint; β a non-Newtonian parameter (see the Appendix); $U(y)$ the primary flow speed; D indicates differentiation in the y direction; and α is the (real) wave number, c the (complex) wave speed and $\phi(y)$ the amplitude of the disturbance. For the details of the derivation of Eq. (1) and any further explanation of the symbols, the reader is referred to the papers by Walters, Chun, and Schwarz. It turns out that

$$U = 1 - y^2 \quad (2)$$

as in the case of flows of Navier-Stokes liquids. Even with the addition of the non-Newtonian term to the stability equation, ϕ can still be divided into independent even and odd parts, and here, as in the Navier-Stokes case, it is assumed that only the even disturbances need to be considered. Thus, the boundary conditions are

$$D\phi(0) = 0,\ D^3\phi(0) = 0,\ \phi(-1) = 0,\ D\phi(-1) = 0 \quad (3)$$

When a set of four independent solutions to Eq. (1) are substituted into Eq. (3), one can obtain a characteristic equation of the form $F(\alpha,Re,c,\beta) = 0$. For given values of β and c, this equation provides a relationship between α and Re. Of paramount interest is the relationship for $c_i = 0$, the graphical representation being the so-called neutral stability curve.

Determination of the Asymptotic Form of the Characteristic Equation

Since the procedure used to determine the characteristic equation, with slight modifications, is due to Graebel, some of the results are just stated, and in these instances the reader is referred to Graebel's paper.

The explanation is believed to be simplest when the characteristic equation for $\beta = 0$ is obtained first and then the modification that is necessary when β is not zero is discussed.

It is anticipated that αR will be large. Consequently, from Eq. (1) it appears that ϕ approximately satisfies

$$(U - c)(D^2 - \alpha^2)\phi - D^2U\phi = 0 \quad (4)$$

everywhere except near the points where $U = c$, the so-called critical points. (It should be noted that interest here is focused on the neutral stability condition for which $c_i = 0$.) That a critical point lies in or near the flowfield is inferred from the fact that one cannot, in general, satisfy all four conditions in Eq. (3) with any pair of linearly independent solutions to Eq. (4). In the following development, the critical points are assumed to lie in the flowfield near the plates. The previous presumptions, as in any such approximate technique, are considered correct if an asymptotic solution can be developed that is entirely consistent with them.

The flow region is divided into an inner and an outer region. The inner region is a strip that includes the rigid bottom boundary at $y = -1$ and the critical point, and the outer region extends from the edge of the inner region to the centerline between the plates at $y = 0$. It is assumed that the two regions overlap. The procedure now is to obtain a solution valid in the inner region and a solution valid in the outer region and then to match the inner with the outer solution where the regions overlap. The inner solution is then used to satisfy the boundary conditions at $y = -1$ and the outer solution is used at $y = 0$.

In the inner region, the approximate form of Eq. (1) is obtained by introducing the change in variable (coordinate stretching)

$$\eta = (y - yc)/\mu = z/\mu \quad (5)$$

and by putting

$$\phi(y) = \chi(\eta;\mu) = \chi^{(0)}(\eta) + \mu\chi^{(1)}(\eta) + \ldots \quad (6)$$

where μ is a function of αR, unknown at this point, but expected to be small. (Henceforth, at the critical point $y = yc$, $DU = DUc$, etc.) When Eqs. (5) and (6) are substituted into Eq. (1) and the coefficients of like powers of μ set equal to zero, it appears that the least-degenerate forms of the equations that are to be used to determine $\chi^{(0)}$, $\chi^{(1)}$, etc. are

$$d^4\chi^{(0)}/d\eta^4 - iDUcd^2\chi^{(0)}/d\eta^2 = 0 \quad (7)$$

$$d^4\chi^{(1)}/d\eta^4 - iDUc\eta d^2\chi^{(1)}/d\eta^2 = i[\eta^2 d^2\chi^{(0)}/d\eta^2 + 2\chi^{(0)}], \text{ etc.} \quad (8)$$

corresponding to the choice $\mu = (\alpha R)^{-1/3}$. Then it follows from Eq. (7)

$$\chi_1^{(0)} = \eta \quad (9)$$

$$\chi_2^{(0)} = 1 \quad (10)$$

$$\chi_3^{(0)} = \int_{-\infty}^{\eta} d\eta \int_{-\infty}^{\eta} d\eta h_1[i(DUc)^{1/3}i\eta]$$
$$\chi_4^{(0)} = \int_{+\infty}^{\eta} d\eta \int_{+\infty}^{\eta} d\eta h_2[i(DUc)^{1/3}\eta] \quad (11)$$

where $h_1,h_2(x) = \frac{2}{3}x^{3/2}H_{1/3}{}^{(1),(2)}(\frac{2}{3}x^{3/2})$ and $H_{1/3}{}^{(1),(2)}$ are Hankel functions of order one-third. The h_1 and h_2 are tabulated.[10]

Because $H_{1/3}{}^{(2)}$ increases exponentially with large positive η, $\chi_4^{(0)}$ cannot be matched with the outer solution and, thus, is discarded at this point.

Putting Eq. (9) into Eq. (8) leads to

$$\chi_1^{(1)} = -\eta^2/DUc \quad (12)$$

From Eqs. (10) and (8), Graebel obtained the following exact solution for $\chi_2^{(1)}$:

$$\chi_2^{(1)} = \frac{2}{(iDUc)^{2/3}}\int_0^{\eta} d\eta \int_0^{\eta} d\eta \times \int_0^{\eta} ds \exp\{-i[(iDUc)^{1/3}\eta s + s^3/3]\} \quad (13)$$

where the path of integration for s is chosen so that the integral does not grow exponentially as $|\eta|$ approaches infinity. For large η,

$$\chi_2^{(1)} \sim -(2/DUc)\eta \ln\eta + 2i/[3(DUc)^2\eta^2] + \ldots \quad (14)$$

where $-7\pi/6 < \arg\eta < \pi/6$. This determines the proper branch of $\ln \eta$ to be used.

From the recursion relation,

$$d^{4+n}\chi^{(0)}/d\eta^{4+n} - iDUc\eta d^{2+n}\chi^{(0)}/d\eta^{2+}N = inDUcd^{1+n}\chi^{(0)}/d\eta^{1+n} \quad (15)$$

it follows that

$$\chi_3^{(1)} = 2i\left[\frac{\eta}{10(DUc)^2}\frac{d^3\chi^{(0)}}{d\eta^3} - \frac{9}{10(DUc)^2}\frac{d^2\chi_3^{(0)}}{d\eta^2} + \frac{i\eta}{2DUc}\chi_3^{(0)}\right] + \gamma\chi_2^{(1)} \quad (16)$$

where $\gamma = 0.67830/DUc - 0.39099 + i(0.39160/DUc - 0.67896)$.

Since η is negative at the bottom plate, $\ln \eta$ has a complex value there, and it appears that χ_2 may not be adequately described there unless the $\chi_2^{(1)}$ term is included. In fact, Graebel obtained a characteristic equation, for which $\chi_2^{(1)}$ was neglected, that did not yield the typical loop of a neutral stability curve. Consequently, in the inner region the solution is taken to be

$$\phi(y) \sim c_1(\eta - \mu\eta^2/DUc) + c_2[1 - (2\mu/DUc)\eta \ln\eta] + c_3\chi_3^{(0)}(\eta) \quad (17)$$

The following two additional presumptions were made: first, $|\eta|$ is large enough to justify using the asymptotic expansion for $\chi_2^{(1)}$ [in a more refined approximation, Eq. (13) would be used in place of Eq. (14)] and second, $\chi_3^{(1)}$ may be safely neglected. The $\chi_1^{(1)}$ term is included since it actually aids in the merger with the outer expansion.

In the outer region, the following expansion for ϕ is assumed:

$$\phi(y;\mu) = \epsilon_0(\mu)u^{(0)}(y) + \epsilon_1(\mu)u^{(1)}(y) + \ldots \quad (18)$$

where the functions $\epsilon(\mu)$ are to be determined from the matching. Substituting Eq. (18) into Eq. (1) gives

$$(U - c)(D^2 - \alpha^2)u^{(0)} + 2u^{(0)} = 0 \quad (19)$$

and $u^{(0)} = u^{(n)}$ for all n such that $\epsilon_n < 0(\mu^3)$. The solutions to Eq. (19) used in this paper are

$$u_1^{(0)}(y) = F_1(z) = \sum_{n=0}^{\infty} A_n z^{n+1} \quad (20)$$

$$u_2^{(0)}(y) = F_2(z) = F_1 \ln z + \sum_{n=0}^{\infty} B_n z^n \quad (21)$$

where $A_0 = 1$, $A_1 = -1/DUc$, $A_2 = \alpha^2/6$, and $A_{n+3} = [(n + 1)(n + 4)A_{n+2} + 2DUcA_{n+1} - 2A_n]/(n + 3)(n + 4)DUc$; $B_0 = -DUc/2$, $B_1 = 0$, $B_2 = 2/DUc - \alpha^2DUc/4$, and $B_{n+3} = [n(n + 3)B_{n+2} - DUc(2n + 5)A_{n+2} + (2n + 3)A_{n+1} + 2DUcB_{n+1} - 2B_n]/(n + 2)(n + 3)DUc$.

Thus, the outer expansion has the form

$$\phi(y;\mu) \sim \sum_{n=0}^{N} \epsilon_n(\mu)(e_1^{(n)}F_1 + e_2^{(n)}F_2) + 0(\mu^3)$$

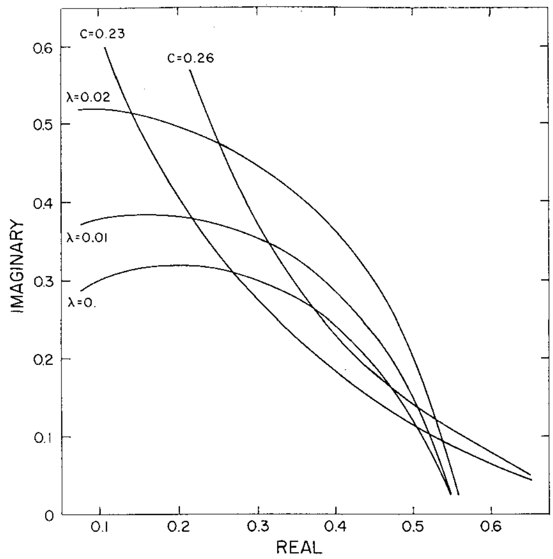

Fig. 1 A plot of the real against the imaginary part of Eq. (29).

Replacing z by $\mu\eta$ and expanding for small μ leads to

$$\phi \sim \epsilon_0[\mu(\eta - 2/DUc + \ldots)e_1^{(0)} + (\mu\eta \ln\mu\eta - DUc/2 + \ldots)e_2^{(0)}] + \ldots$$

Thus, for the inner and outer expansions to merge $\epsilon_0 = 1/\mu$, $e_1^{(0)} = c_1$, $e_2^{(0)} = 0$; $\epsilon_1 = \ln\mu$, $e_1^{(1)} = 2c_2/DUc$, $e_2^{(1)} = 0$; and $\epsilon_2 = 1$, $e_1^{(2)} = 0$, $e_2^{(2)} = -2c_2/DUc$. Then in the outer region,

$$\phi(y;\mu) \sim (c_1/\mu)F_1 + \ln\mu(2c_2/DUc)F_1 - (2c_2/DUc)F_2 + 0(\mu \ln\mu) \quad (22)$$

Now putting Eqs. (17) and (22) into Eq. (3) leads to

$$c_1\chi_1(\eta_1) + c_2\chi_2(\eta_1) + c_3\chi_3(\eta_1) = 0 \quad (23a)$$

$$c_1d\chi_1(\eta_1)/d\eta + c_2d\chi_2(\eta_1)/d\eta + c_3d\chi_3(\eta_1)/d\eta = 0 \quad (23b)$$

$$(c_1/\mu)DF_1(z_0) - (2c_2/DUc)DF_2(z_0) = 0 \quad (23c)$$

The subscripts zero and one correspond to the values of y at the points where the quantities are to be evaluated. Note that in the outer region where ϕ satisfies Eq. (19) it follows that $D\phi(0)$ being zero implies $D^3\phi(0)$ is zero also. For a nontrivial set of c_1, c_2, and c_3 to exist it follows from Eq. (23) that

$$\frac{\chi_3^{(0)}(\eta_1)}{D\chi_3^{(0)}(\eta_1)} = \frac{F_1(z_1)DF_2(z_0) - F_2(z_1)DF_1(z_0)}{DF_1(z_1)DF_2(z_0) - DF_2(z_1)DF_1(z_0)} \quad (25)$$

In Eq. (25), $\chi_1(\eta_1)$ and $\chi_2(\eta_1)$ have been approximated by $F_1(z_1)/\mu$ and $-2F_2(z_1)/DUc$, respectively. Equation (25) is essentially the characteristic equation used by Lin, although he elected to express the outer expansion in powers of α^2.

When β is not zero, the characteristic equation is modified as follows: since β is small the outer solution remains unchanged. In the inner region, however, the simplest expression for $\chi^{(0)}$ that contains the viscoelastic term is

$$(1 - i\epsilon DUc)d^4\chi^{(0)}/d\eta^4 - iDUc\eta d^2\chi^{(0)}/d\eta^2 = 0 \quad (26)$$

where $\epsilon = \mu\alpha\beta$ and all other terms are defined as previously. Since ϵ multiplies only the fourth derivative, χ_1 and χ_2 remain unchanged. The change in χ_3 may be no more than what would result from keeping $\chi_3^{(1)}$; hence, the results should be interpreted more as indicating a trend than as providing an accurate solution. For small ϵ (i.e., slight viscoelasticity), χ_3 can be approximated by setting

$$\chi_3 = \psi_0 + \epsilon\psi_1 \quad (27)$$

where $d^4\psi_0/d\eta^4 - iDUcd^2\psi_0/d\eta^2 = 0$ and $d^4\psi_1/d\eta^4 - iDUcd\psi_1/d\eta^2 = iDUcd^4\psi_0/d\eta^4$. From the recursion relation (15), it follows that

$$\psi_1 = (\eta d^3\psi_0/d\eta^3 - 4d^2\psi_0/d\eta^2)/5 \quad (28)$$

Now combining Eqs. (28), (27), and (10) with Eq. (25), setting $\zeta = (DUc)^{1/3}\eta$ and $\lambda = \beta(\alpha DUc)^{2/3}/[5(Re)^{1/3}]$ lead to the following explicit relationship

$$\left\{\int_\infty^{\zeta_1} d\zeta \int_\infty^{\zeta_1} d\zeta h_1(i\zeta) + \lambda\,[\zeta_1 dh_1(i\zeta_1)/d\zeta - 4h_1(i\zeta_1)]\right\} \Big/ \left\{\zeta_1 \int_\infty^{\zeta_1} d\zeta h_1(i\zeta) + \lambda\,[i\zeta_1^2 h_1(i\zeta_1) - 3dh_1(i\zeta_1)/d\zeta]\right\} = \frac{1}{1+y_c}\,\frac{F_1(z_1)DF_2(z_0) - F_2(z_1)DF_1(z_0)}{DF_1(z_1)DF_2(z_0) - DF_2(z_1)DF_1(z_0)} \quad (29)$$

Equation (29) is the characteristic equation used to determine the neutral stability curves in this paper.

Results and Conclusions

For the integration indicated in Eq. (29), the series given in Ref. 10 were integrated term by term for $|\zeta| < 6$. For $|\zeta| > 6$, the integration was continued numerically using Simpson's rule and the asymptotic expression for h_1.

At this point, one can either determine neutral stability curves on which λ is constant or curves on which β is constant. Chan Man Fong and Walters calculated curves on which λ is constant, and on the other hand, Chun and Schwarz calculated curves on which β is constant. Thus, a direct quantitative comparison is impossible. The present results contain curves of constant λ as well as curves of constant β.

For the curves of constant λ the left side of Eq. (29) is a function of ζ only, and for a given value of c the right side is a function of α only. Thus, Eq. (29) can be solved graphically by plotting the real against the imaginary parts of the left (curves of constant λ) and right (curves of constant c) sides on the same graph. Figure 1 shows several typical curves.

For curves of constant β, the solution is not so amenable to graphical methods and so an iterative procedure was used. Values were assigned to β and c and then α, ζ, and Re were obtained. From the definitions of μ and ζ, the needed third relationship was found to be $\zeta_1 = (y_1 - y_c)[(DUc/\alpha Re)]^{1/3}$.

In Fig. 2 and in Table 1, the results of Chun and Schwarz are compared with the present results. It should be noted that β in the present paper is the negative of the β used by Chun and Schwarz. The results are all based on the present definition of β.

In Fig. 3 the results of Chan Man Fong and Walters are compared with the present results. The λ used in Fig. 3 is that defined in this paper, which is $\frac{1}{5}$ of that defined by Chan Man Fong and Walters. The disagreement in the two results is not understood; however, it is noted that their results for $\lambda = 0$ differ considerably from the accepted values of Lin. On this basis, the present results are believed to be the more accurate.

The differences in the expressions used for the inviscid solutions may explain the slight discrepancy for $\alpha > 1$ between the present results and those of Lin.

In all the results, the trend is clear and consistent. Both Chun and Schwarz as well as Chan Man Fong and Walters have stated that the effect of the viscoelasticity is destabilizing. A conclusion that apparently contradicts observations made for boundary-layer flows of very dilute polymer solutions. This contradiction can be explained. Markovitz and Coleman[11] have indicated that on the basis of thermodynamic intuition the material constant that determines β must be negative, and they even give an experimental value to substantiate their statement. It turns out that β must

Table 1 Critical values of α, c, and Re for various values of β, comparing the present results with those of Chun and Schwarz

	Re		α		c	
β	C&S	Present	C&S	Present	C&S	Present
−0.1	...	5623	...	1.012	...	0.2636
0.0	5775	5399	1.026	1.022	0.2646	0.2672
0.1	5537	5170	1.026	1.033	0.2668	0.2708
0.5	4620	4186	1.08	1.088	0.2840	0.2886
1.0	3630	2662	1.16	1.226	0.3065	0.3296

also be negative. Thus, the results indicate, since the trend is reversed when the sign of β (or, equivalently, λ) is changed, that the influence of the viscoelasticity is stabilizing.

Appendix

Here the relationship between a constitutive equation of the type proposed by Oldroyd and the constitutive equation for a second-order liquid is established. The viscoelastic parameter β is also defined.

The material properties of viscoelastic liquids can be described, at least qualitatively, by constitutive equations of the form

$$\sigma_{ij} = -p\delta_{ij} + \int_{-\infty}^{t} \psi(t - \bar{t}) \frac{\partial \bar{x}_m}{\partial x_i} \frac{\partial \bar{x}_n}{\partial x_j} \bar{e}_{mn} d\bar{t} \qquad (A1)$$

and

$$\sigma_{ij} = -p\delta_{ij} + \mu e_{ij} + \eta e_{ik} e_{kj} + \gamma a_{ij} \qquad (A2)$$

in which t is the time; the σ_{ij} are the stress components; e_{ij} and $\bar{e}_{ij}$ the rate of deformation components, evaluated at t and $\bar{t}$, respectively; p the pressure; and δ_{ij} the Kronecker delta. In Eq. (A1) $\bar{x}_m$ represents the position at time $\bar{t}$ of the particle which is in the position represented by x_m at time t, and ψ is the relaxation function which is peculiar to the material. In Eq. (A2), μ, η, and γ are material constants and the a_{ij} are defined by

$$a_{ij} = e_{ik}\partial v_k/\partial x_i + e_{kj}\partial v_k/\partial x_j + De_{ij}/Dt \qquad (A3)$$

In Eq. (A3), the v_k are the components of the velocity and D/Dt represents the material derivative. Equations having generally the form of Eq. (A1) were proposed by Oldroyd and Eq. (A2) was proposed by Coleman and Noll.

In order to work with Eq. (A1), one must determine the $\bar{x}_i$ as a function of x_i, t, and $\bar{t}$. This relationship can be obtained by integrating

$$D\bar{x}_i/Dt = 0 \qquad (A4)$$

Since the $\bar{x}_i$ can serve as material coordinates, Eq. (A4) expresses the fact that the material coordinates of a given particle do not change with time.

The integral form of Eq. (A1) can be reduced to an equivalent differential form as follows: a formal series solution to Eq. (A4) is

$$\bar{x}_i = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{D^n x_i}{Dt^n}\right) (\bar{t} - t) \qquad (A5)$$

This is a Taylor series expansion backwards in time, holding the material coordinates constant. [Walters used only the first two terms in (A5) in deriving the stability equation (1).] Similarly, one can write

$$\bar{e}_{ij} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{D^n e_{ij}}{Dt^n}\right) (\bar{t} - t)^n \qquad (A6)$$

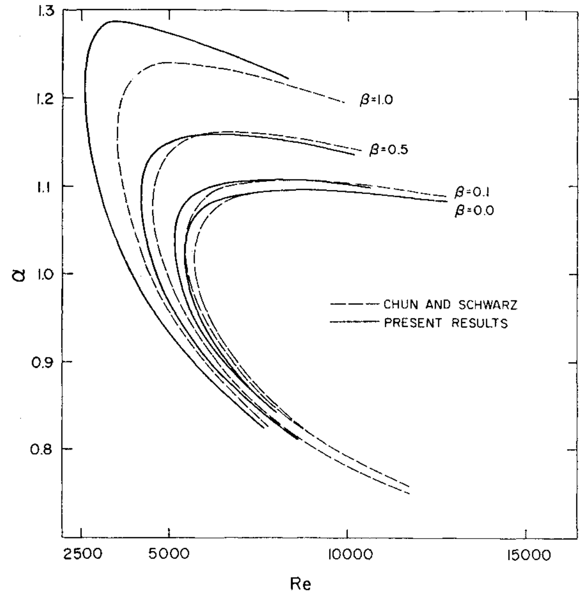

Fig. 2 Neutral stability curves for various values of λ, comparing the present results with those of Chun and Schwarz.

Then from Eqs. (A5) and (A6), it follows that

$$\frac{\partial \bar{x}_m}{\partial x_i} \frac{\partial \bar{x}_n}{\partial x_j} \bar{e}_{mn} = \left[\delta_{mi} + (\bar{t} - t)\frac{\partial v_m}{\partial x_i}(t) + \ldots\right] \times \left[\delta_{nj} + (\bar{t} - t)\frac{\partial v_m}{\partial x_j}(t) + \ldots\right] \times \left[e_{mn}(t) + (\bar{t} - t)\frac{De_{mn}}{Dt}(t) + \ldots\right] \qquad (A7)$$

Substituting Eq. (A7) into Eq. (A1) leads to

$$\sigma_{ij} = -p\delta_{ik} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \gamma\left(e_{in}\frac{\partial v_n}{\partial x_j} + e_{mj}\frac{\partial v_m}{\partial x_i} + \frac{De_{ij}}{Dt}\right) + \text{higher-order terms} \qquad (A8)$$

where

$$\mu = \int_{-\infty}^{t} \psi(t - \bar{t})d\bar{t} \text{ and } \gamma = \int_{-\infty}^{t} (\bar{t} - t)\psi(t - \bar{t})d\bar{t} \qquad (A9)$$

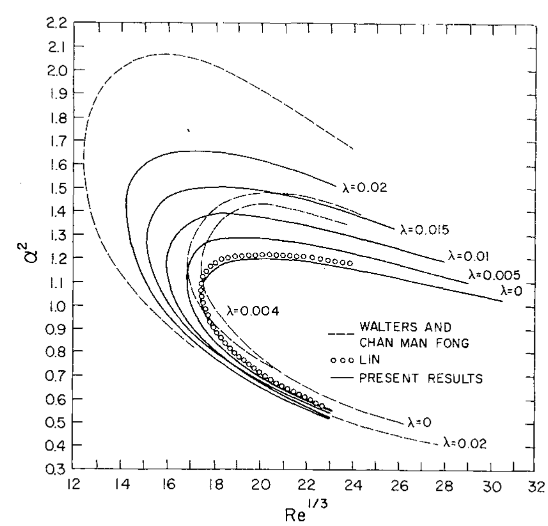

Fig. 3 Neutral stability curves for various values of β, comparing the present results with those of Chan Man Fong and Walters as well as with those of Lin.

The higher-order terms in Eq. (A8) contain factors of the form

$$\int_{-\infty}^{t} (\bar{t} - t)^n \psi(t - \bar{t}) d\bar{t}$$

where n, an integer, is greater than 1.

For materials for which the deformation history is important for only a short time, the higher-order terms can be neglected. In this case, Eq. (A8) differs from Eq. (A2) only in the term containing η. Thus, since there are no terms containing η in the stability equation for a second-order liquid (see Chun and Schwarz), this equation is also valid for liquids described by Eq. (A1).

In Eq. (1), β is defined by $\beta = \gamma Re/(\rho h^2)$. Re, the Reynolds number, is based on μ and h where $2h$ is the distance between the plates.

If μ is positive as thermodynamic considerations indicate, then from Eq. (A9), γ is apparently negative as Coleman and Markovitz have stated.

For the two-dimensional infinitesimal disturbances corresponding to velocity components of the form $[u,v] = [\bar{u}(y), \bar{v}(y)]E$, where $E = \exp [i\alpha(x - ct)]$, the series expression in Eq. (A5) can be replaced by

$$\bar{y} = y - [\bar{v}E(1 - F)/i\alpha(U - c)] \quad \text{(A10a)}$$

$$\bar{x} = x - U(t - \bar{t}) + \{[\bar{v}DU/\alpha^2 \times (U - c)^2] - [\bar{u}/i\alpha(U - c)]\} \times E(1 - F) + \bar{v}(DU)(EF)(t - \bar{t})/i\alpha(U - c) \quad \text{(A10b)}$$

where $F = \exp[i\alpha(U - c)(\bar{t} - t)]$.

Equations (A10) are valid for arbitrary finite time. Using Eqs. (A10) rather than only the first two terms in Eq. (A5), Mook and Graebel[12] derived a stability equation and showed that in the limit as the memory of the material fades this equation reduces to Eq. (1).

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