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Analytical Solution for Horizontal Gliding Flight

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This paper presents the analytical solutions for atmospheric flight in the horizontal plane. The flight conditions considered include maximum range flight, maximum endurance flight, chattering flight, and constant lift turn flight. They enable us to have a wide understanding of the characteristics of flight in the horizontal plane.

I. Introduction

In general, there are five state variables and three control variables for flight in the horizontal plane. The five state variables are two components of the position vector, two components of the velocity vector, and mass. The three control variables are two components of the thrust vector and the combination of flight and bank angle in order to keep the flight at constant altitude. For gliding flight, the thrust is zero and the problem is reduced to four state variables and one control variable. This problem had been studied extensively in two eminent books [1,2] and many published papers.[3-11] The purpose of this paper is twofold: first of all, to summarize the analytical solutions obtained before, and secondly to elaborate possible extensions.

II. Equations of Motion

For gliding flight in the horizontal plane, the equations of motion are[1,2]

$$\frac{dX}{dt} = V\cos\psi \tag{1a}$$

$$\frac{dY}{dt} = V\sin\psi \tag{1b}$$

$$\frac{dV}{dt} = -\frac{\rho S V^2 C_D}{2m} \tag{1c}$$

$$\frac{d\psi}{dt} = \frac{\rho S V C_L \sin\sigma}{2m} \tag{1d}$$

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where X is down range, Y is lateral range, V is speed, ψ is heading angle, ρ is atmospheric density, S is reference area of vehicle, C_D is drag coefficient, C_L is lift coefficient, m is mass of vehicle, σ is bank angle, and t is the time. In order to keep the flight in the horizontal plane, a constraining relation between C_L and σ is

$$\frac{1}{2}\rho V^2 S C_L \cos\sigma = mg = W \tag{2}$$

where W is weight of vehicle. The drag coefficient is modeled by the parabolic polar

$$C_D = C_{D_0} + KC_L^2 \tag{3}$$

where C_{D_0} is zero lift drag coefficient and K is induced drag factor. By introducing the dimensionless variables

$$x = \frac{gX}{V_0^2},\quad y = \frac{gY}{V_0^2},\quad u = \frac{V}{V_0},\quad \theta = \frac{gt}{V_0},\quad \omega = \frac{2W}{\rho S V_0^2 C_L^*} \tag{4}$$

where the subscript 0 denotes initial condition, g is gravitational acceleration, and C_L^* is lift coefficient for maximum lift-to-drag ratio, we obtain the dimensionless equations of motion[2]

$$x' = u\cos\psi \tag{5a}$$

$$y' = u\sin\psi \tag{5b}$$

$$u' = -\frac{u^2}{2E^*\omega}\left(1 + \frac{\omega^2}{u^4\cos^2\sigma}\right) \tag{5c}$$

$$\psi' = \frac{\tan\sigma}{u} \tag{5d}$$

where the prime denotes the derivative taken with respect to the dimensionless time θ, and E^* is maximum lift-to-drag ratio. The constraining relation in Eq. (2) becomes

$$\cos\sigma = \frac{\omega}{\lambda u^2} \tag{6}$$

where $\lambda = C_L / C_L^*$. In Eqs. (5), the four state variables are x, y, u and ψ, and σ is the sole control variable. The required lift coefficient can be calculated from Eq. (6) and C_L^*. In this paper we shall assume that both C_{D_0} and K are constant and therefore $C_L^* = (C_{D_0} / K)^{1/2}$ is also constant. The load factor is defined to be

$$n = \frac{L}{W} = \frac{1}{\cos\sigma} \tag{7}$$

It is clear that the dimensionless constraining relation described by Eq. (6) has the limitations

$$-1 \le \cos\sigma = \frac{\omega}{\lambda u^2} \le 1 \tag{8}$$

Therefore, if the maximum and minimum values of λ are assumed to be

$$\lambda_{max} = 2 \tag{9a}$$

$$\lambda_{min} = -2 \tag{9b}$$

we have the maximum maneuverable domain as shown in Fig. 1.

III. Variational Formulation

For variational formulation, the Hamiltonian can be formed as[2,11]

$$H = p_x u\cos\psi + p_y u\sin\psi - p_u\left(\frac{u^2}{2E^*\omega}\right)\left[1+\frac{\omega^2}{u^4\cos^2\sigma}\right] + p_\psi\frac{\tan\sigma}{u} \tag{10}$$

where p_x, p_y, p_u, and p_ψ are adjoint variables corresponding to x, y, u, and ψ, respectively. The variational problem has the integrals[2]

$$H = C_0 \tag{11a}$$

$$p_x = C_1 \tag{11b}$$

$$p_y = C_2 \tag{11c}$$

$$p_\psi = -C_2 x + C_1 y + C_3 \tag{11d}$$

where C_0, C_1, C_2, and C_3 are constants of integration. When the optimal bank control is interior, we have $\partial H/\partial\sigma = 0$ and

$$\tan\sigma = \left(\frac{p_\psi}{p_u}\right)\left(\frac{E^* u}{\omega}\right) \tag{12}$$

The first integral $H = C_0$ comes from the fact that $\partial H/\partial\theta = 0$ since H is not an explicit function of the independent variable θ. In the cases the final time θ_f is free, in other words, it is neither specified nor being extremized, we have $H_f = C_0 = 0$. Also, we shall have $C_1 = 0$ if x_f is free, and $C_2 = 0$ if y_f is free. It happens in many cases the final heading ψ_f is free, and the transversality condition may result that $p_{\psi_f} = -C_2 x_f + C_1 y_f + C_3 = 0$. Actually, one of the integrals can be eliminated as long as it is not zero. For example when $C_0 \ne 0$, by letting $k_1 = C_1/C_0$, $k_2 = C_2/C_0$, and $k_3 = C_3/C_0$, the following binomial equation for the bank angle can be derived

$$(-k_2 x + k_1 y + k_3)\tan^2\sigma + 2u[1 + u(k_1\cos\psi + k_2\sin\psi)]\tan\sigma - \left(\frac{-k_2 x + k_1 y + k_3}{\omega^2}\right)(\omega^2 + u^4) = 0 \tag{13}$$

IV. Rectilinear Flight

A. Maximum Range and Maximum Endurance Glide

For rectilinear motion in the horizontal plane, we have $\sigma = 0,\ \psi = 0$ and the equations of motion are simply

$$x' = u \tag{14a}$$

$$u' = -\frac{u^2}{2E^*\omega}(1+\frac{\omega^2}{u^4}) \tag{14b}$$

The two equations (14a) and (14b) can be combined to give

$$\frac{dx}{du} = -\frac{2E^*\omega u^3}{u^4+\omega^2} \tag{15}$$

If we specify the final speed u_f to be the stall speed which occurs at λ_{max}, then from Eq. (6) we have

$$u_f = \sqrt{\frac{\omega}{\lambda_{max}}}$$

The integration from $x_0 = 0$ and $u_0 = 1$ to $x_f = x_{max}$ and $u_f = \sqrt{\omega/\lambda_{max}}$ gives the maximum range for rectilinear flight[2,11]

$$x_{max} = \frac{1}{2}E^*\omega\log[\frac{(1+\omega^2)\lambda_{max}^2}{\omega^2(1+\lambda_{max}^2)}] \tag{16}$$

The inverse of Eq. (14b) is

$$\frac{d\theta}{du} = -2E^*\omega(\frac{u^2}{u^4+\omega^2}) \tag{17}$$

Its integration from $\theta_0 = 0$ and $u_0 = 1$ to $\theta_f = \theta_{max}$ and $u_f = \sqrt{\omega/\lambda_{max}}$ gives the maximum endurance for rectilinear flight

$$\theta_{max} = \frac{E^*\sqrt{\omega}}{2\sqrt{2}}[\log(\frac{1+\sqrt{2\lambda_{max}}+\lambda_{max}}{1-\sqrt{2\lambda_{max}}+\lambda_{max}})+2\tan^{-1}(\frac{\sqrt{2\lambda_{max}}}{1-\lambda_{max}})-\log(\frac{1+\sqrt{2\omega}+\omega}{1-\sqrt{2\omega}+\omega})-2\tan^{-1}(\frac{\sqrt{2\omega}}{1-\omega})] \tag{18}$$

For numerical computation, we shall assume the following data for the aircraft model:

$$C_{D_0} = 0.0125,\ K = 0.05,\ C_D^* = 2C_{D_0} = 0.025,\ C_L^* = \sqrt{\frac{C_{D_0}}{K}} = 0.5,\ E^* = \frac{C_L^*}{C_D^*} = 20, \quad C_{L_{max}} = 1,\ \lambda_{max} = 2 \tag{19}$$

The maximum range and maximum endurance as functions of dimensional altitude are plotted in Fig. 2. It is seen from Fig. 2 that when the altitude is too high, the rectilinear flight in the horizontal plane does not exist. There is an altitude called ceiling where both x_{max} and θ_{max} are zero. From Eqs. (16) and (18), it is easy to find the ceiling altitude is

$$\omega_{ceiling} = \lambda_{max} = 2 \tag{20}$$

There is a global optimal altitude for global maximum range. At that point we have $\partial x_{max} / \partial \omega = 0$ and it can be derived, so that the $\omega_{maximum\ x}$ satisfies the relation

$$\frac{2}{1+\omega^2} = \log[\frac{(1+\omega^2)\lambda_{max}^2}{\omega^2(1+\lambda_{max}^2)}] \tag{21}$$

Eq. (21) provides the solution $\omega_{maximum\ x} = 0.411$. On the other hand, there is a $\omega_{maximum\ \theta}$ where the endurance is globally optimal. It satisfies $\partial \theta_{max} / \partial \omega = 0$ and we have the relation

$$\log(\frac{1+\sqrt{2\lambda_{max}}+\lambda_{max}}{1-\sqrt{2\lambda_{max}}+\lambda_{max}}) + 2\tan^{-1}(\frac{\sqrt{2\lambda_{max}}}{1-\lambda_{max}}) - \log(\frac{1+\sqrt{2\omega}+\omega}{1-\sqrt{2\omega}+\omega}) - 2\tan^{-1}(\frac{\sqrt{2\omega}}{1-\omega}) - \frac{4\sqrt{2\omega}}{1+\omega^2} = 0 \tag{22}$$

With some elaborative calculation, we obtain $\omega_{maximum\ \theta} = 0.27465$ from Eq. (22).

B. Chattering

The relation shown in Eq. (18) is unique. In other words, it takes the time θ_{max} to decelerate the aircraft from $u_0 = 1$ to $u_f = \sqrt{\omega / \lambda_{max}}$. Then, what is the θ_{min} value for the same speed reduction? To investigate this problem, we have to go back to Eq. (7). By letting $\sigma = 0$ in Eq. (6), we have

$$\omega = \lambda u^2 \tag{23}$$

Using the relation in Eq. (17) gives

$$\frac{d\theta}{du} = -2E^*(\frac{\lambda}{1+\lambda^2}) \tag{24}$$

It is apparent that the flight time will be smaller if the right hand side of Eq. (24) is less negative. Therefore, when the maximum lift is used, the aircraft will have maximum drag and need minimum time for aerobraking maneuver. However, the aircraft has to bank between $+\sigma_c$ and $-\sigma_c$ (where the subscript c denotes chattering) rapidly to keep the flight rectilinear. The value of σ_c can be calculated from the relation

$$\cos\sigma_c = \frac{\omega}{\lambda_{max} u^2} \tag{25}$$

With $\lambda = \lambda_{max}$ in Eq. (24), the minimum time required in simply

$$\theta_{min} = \frac{2E^* \lambda_{max}}{1+\lambda_{max}^2}(u_0 - u_f) \tag{26}$$

The range of chattering are x_c can be obtained by integrating Eq. (15) with $\omega = \lambda_{max} u^2$. It results that

$$x_c = \frac{E^* \lambda_{max}}{1+\lambda_{max}^2}(u_0^2 - u_f^2) \tag{27}$$

Theoretically, the chattering between $+\sigma_c$ and $-\sigma_c$ must be at the rate of infinity. However, it has been found that if the number of control switching is greater than 5, the flight path will be a quasi-straight line and the penalty on the flight time is only several tenth percent.[12]

In Fig. 3, the maximum range x_{max} and the chattering range x_c are plotted. Also shown are the flight times. When the final range is not specified, the solutions obtained are unique. It means that at a given altitude ω, θ_{max} corresponds to x_{max} and θ_{min} corresponds to x_c. In the case where the final range x_f is specified, further consideration is required. When $0 \le x_f < x_c$, the optimal trajectory for minimum time flight is a two-dimensional turning in the horizontal plane.[4] When $x_c \le x_f < x_{max}$, the optimal trajectory is simply a combination of chattering and gliding arcs. Let x_1 be the point where the two arcs join together. For minimum time flight, the aircraft glides from x_0 to x_1 in shortest time. Then from x_1 to x_f it chatters to reduce speed to u_f, also in shortest time. The time for glide from $x_0 = 0$ and $u_0 = 1$ to x_1 and u_1, denoted by θ_1, is

$$\theta_1 = \frac{E^*\sqrt{\omega}}{2\sqrt{2}}\left[\log\left(\frac{u_1^2+\sqrt{2\omega}u_1+\omega}{u_1^2-\sqrt{2\omega}u_1+\omega}\right)+2\tan^{-1}\left(\frac{\sqrt{2\omega}u_1}{u_1^2-\omega}\right)-\log\left(\frac{1+\sqrt{2\omega}+\omega}{1-\sqrt{2\omega}+\omega}\right)-2\tan^{-1}\left(\frac{\sqrt{2\omega}}{1-\omega}\right)\right] \tag{28}$$

The time for chattering from x_1 and u_1 to x_f and $u_f = \sqrt{\omega/\lambda_{max}}$, denoted by θ_2, is

$$\theta_2 = \frac{2E^* \lambda_{max}}{1+\lambda_{max}^2}(u_1 - u_f) \tag{29}$$

The total time is equal to the sum of θ_1 and θ_2, and is the minimum time for flight. We can call it a kind of interior-point boundary value problem (IPBVP). The total range of flight is the specified x_f. It is equal to the sum of the gliding part and the chattering part of the flight. The integration of Eq. (15) form u_0 to u_1 and then from u_1 to u_f gives

$$x_f = \frac{1}{2} E^* \omega \log\left(\frac{1+\omega^2}{u_1^4+\omega^2}\right) + \frac{E^* \lambda_{max}}{1+\lambda_{max}^2}(u_1^2 - u_f^2) \tag{30}$$

The only unknown in Eq. (30) is u_1. Therefore, this IPBVP has complet analytic solution.

As a numerical example, let $(\omega, x_f) = (1.0, 4.25)$. With the numerical values of $E^* = 20$, $\lambda_{max} = 2$, $\omega = 1$, $u_f = 0.7071$, and $x_f = 4.25$ insert in Eq. (30), it becomes

$$\log\left(\frac{2}{1+u_1^4}\right) + 0.8u_1^2 - 0.825 = 0 \tag{31}$$

The solution of Eq. (31) is $u_1 = 0.9344$. The minimum flight time can be calculated from Eqs. (28) and (29) and is $\theta_f = \theta_1 + \theta_2 = 1.3081 + 3.6368 = 4.9449$, which is 97% smaller than the pure glide flight time of 5.4729 calculated from Eq. (18). The first term of the right-hand side of Eq. (30) is x_1 and is calculated to be $x_1 = 1.2652$.

V. Constant Lift Turn

By using the relation of Eq. (6) in Eqs. (5c) and (5d), we have

$$\frac{du}{d\theta} = -\frac{u^2}{2E^*\omega}(1+\lambda^2) \tag{32}$$

$$\frac{d\psi}{d\theta} = \left(\frac{\lambda^2 u^2}{\omega^2} - \frac{1}{u^2}\right)^{1/2} \tag{33}$$

where λ is the control variable. It requires from Eq. (33) that $\omega^2 \le \lambda^2 u^4$ for the turn to be possible, as shown in Fig. 4 for positive λ. For constant lift turn, Eq. (32) can be integrated to give

$$\theta_f = \frac{2E^*\omega}{1+\lambda^2}\left(\frac{1}{u_f} - 1\right) \tag{34}$$

where $\theta_0 = 0$ and $u_0 = 1$ have been used. The combination of Eq. (32) and (33) gives the relation

$$\frac{d\psi}{du} = -\frac{2E^*}{1+\lambda^2}\frac{\sqrt{\lambda^2 u^4 - \omega^2}}{u^3}$$

or

$$d\psi = -\frac{E^*\lambda}{1+\lambda^2}\frac{\sqrt{\bar{u}^2 - (\frac{\omega}{\lambda})^2}}{\bar{u}^2} d\bar{u} \tag{35}$$

where $\bar{u} = u^2$. We can integrate Eq. (35) from the initial condition $u_0 = 1$ and $\psi_0 = 0$ to obtain

$$\psi = \frac{E^*}{1+\lambda^2}[\sqrt{\lambda^2 - \omega^2} + \lambda \log(1 + \frac{\sqrt{\lambda^2 - \omega^2}}{\lambda}) + \frac{\sqrt{\lambda^2 u^4 - \omega^2}}{u^2} - \lambda \log(u^2 + \frac{\sqrt{\lambda^2 u^4 - \omega^2}}{\lambda})] \quad (36)$$

When $\lambda = \lambda_{max}$, we have $u_f = \sqrt{\omega / \lambda_{max}}$ and

$$\psi_f |_{\lambda=\lambda_{max}} = \frac{\lambda_{max} E^*}{1+\lambda_{max}^2}[\frac{-1}{\lambda_{max}}\sqrt{\lambda_{max}^2 - \omega^2} + \log(1 + \frac{\sqrt{\lambda_{max}^2 - \omega^2}}{\lambda_{max}}) - \log(\frac{\omega}{\lambda_{max}})] \quad (37)$$

The variation of heading angle as a function of u for constant λ turning is shown in Fig. 5.

VI. Conclusions

It is very difficult to solve flight mechanics problems analytically. The analytic solutions are presented in this paper are focused on the cases of rectilinear flight cases. The maximum range and maximum endurance glide problems are solved completely and discussed extensively. Also solved are the chattering range and chattering endurance. This is a kind of flight that is not widely investigated. For the truning flight, the only problem solved is the constant lift turn. Nevertheless, this is still a research area that attracts many scientists and mathematicians. We believe that some more analytic solutions can be found somewhere at existing literature and also sometime in the future.

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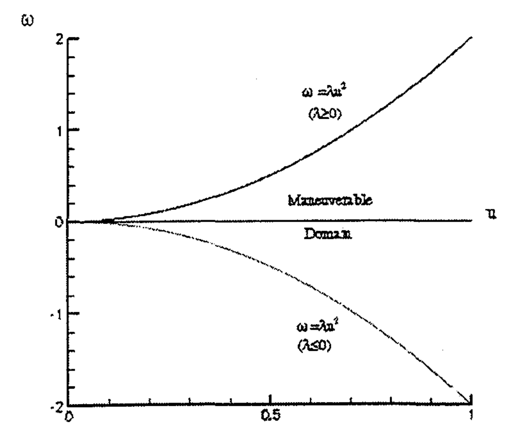


Fig1 Maneuverable domain for horizontal gliding flight

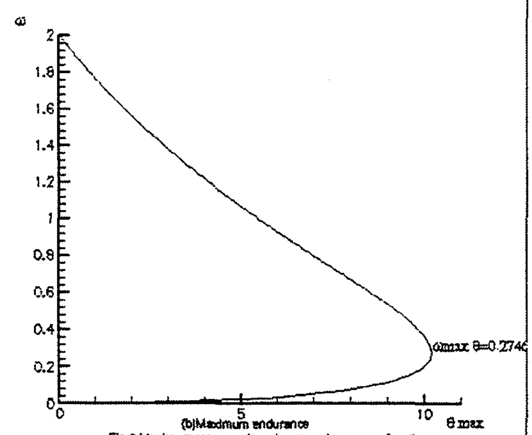


Fig 2 Maximum range and maximum endurance as function of dimensionless altitude

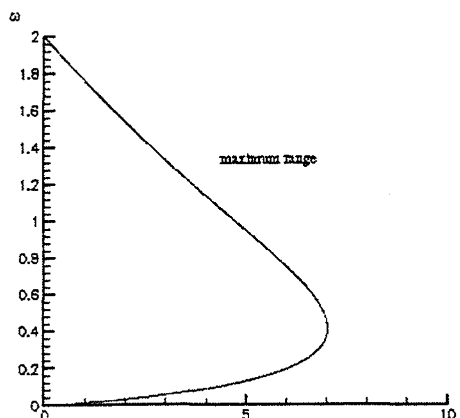


Fig 2 Maximum range and maximum endurance as functions of dimensionless altitude

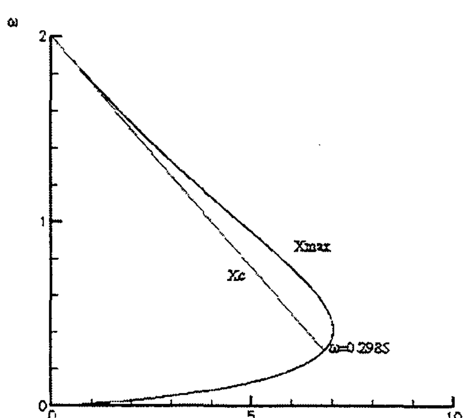
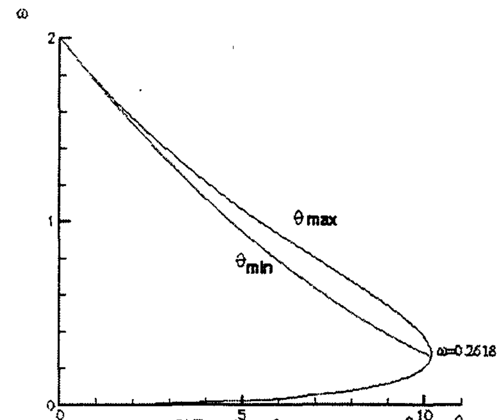


Fig3.(a) Chattering range and chattering endurance as function of dimensionless altitude



(b)Chattering endurance

Fig3.(b) Chattering range and chattering endurance as function of dimensionless altitude

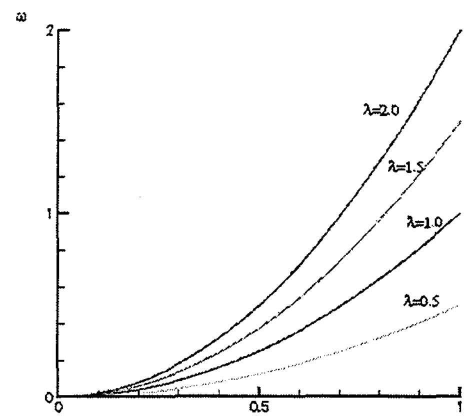


Fig4 Maneuverable domain for positive constant lift turn

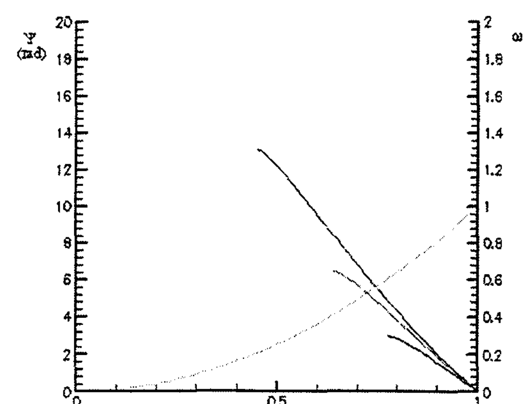


Fig5 Constant lift turn with λ=1