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RADIATIVE ENTROPY PRODUCTION

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Abstract

In terms of the specular moments of the transfer equation, a radiative tensor is considered:

$$\Pi_{ij} = \sum_{n=o}^{\infty} \frac{\nabla^{2n-2}(2n\partial_i\partial_j+\nabla^2\delta_{ij})B}{\kappa_M^{2n}\ (2n+1)\ (2n+3)},$$

where ∇^2 is the usual Laplacian, κ_M is a mean absorption coefficient, $B = 4E_b$ and $E_b = \sigma T^4$ is the Stefan-Boltzmann Law. This tensor is related to the radiative stress τ_{ij}^R through the speed of light c,

$$\Pi_{ij} = c\ \tau_{ij}^R.$$

In terms of the Π_{ij}-tensor, the radiative entropy production is shown to be

$$s''' = \frac{4\sigma}{\kappa_R T^2} \sum_{n=o}^{\infty} \left(\frac{\nabla^2}{\kappa_M^2}\right)^n \frac{\partial T^4/\partial x_i}{(2n+3)} \left(\frac{\partial T}{\partial x_i}\right),$$

where κ_R is the Rosseland mean of the absorption coefficient.

1. Introduction

The theory of gas radiation dates back to Rayleigh's studies over a century ago on the illumination and polarization of the sunlit sky. Since then the theory has rapidly grown because of the efforts of astrophysicists and later those of applied scientists and engineers. However, the entropy production associated with radiation apparently remained untreated and is the motivation of this study.

As is well known, the entropy production results from dissipative processes (involving mass, species, momentum and/or heat transfer, electromagnetic or nuclear transport). Less known is the fact that the dissipation may have a diffusive or hysteretic origin. However, except for a few cases (such as the strain hardening and the magnetic saturation) the majority of dissipative processes, including the dissipation of radiation, is of diffusive nature. A recent study by Arpaci(1) shows, in terms of the radiative stress obtained from the specular (kinetic) moments of the transfer equation, the diffusive nature of radiation for any optical thickness. Accordingly, the expression to be developed for entropy production is in terms of this stress, and includes also the dissipation resulting from conduction of heat and viscous friction.

The study consists of four sections: following this introduction, Section 2 deals with a brief review on the radiative stress, Section 3 develops an expression for entropy production in terms of this stress, and Section 4 concludes the study.

2. Radiative Stress

This section is devoted to a brief review on the radiative stress. The review is in terms of spectrally averaged radiation because of its simplicity. A monochromatic review, which maybe needed for a quantitative study, is not essential here because of the conceptual nature of the intended study.

As pointed out by Felske & Tien(2) there are a variety of practical situations in which scattering is not important. For these situations, consider the spectrally averaged transfer equation,

$$\ell_i \frac{\partial I}{\partial x_i} = \kappa\ (I_o - I), \tag{1}$$

where I denotes the intensity, I_o its equilibrium, κ the absorption coefficient, ℓ_i the direction of optical path, and x_i the Cartesian coordinate. The usual definitions of the radiative internal energy, heat flux and stress in terms of the intensity are

$$u^R = \frac{1}{c}\int_\Omega I d\Omega = \frac{1}{c}\ J, \tag{2}$$

$$q_i^R = \int_\Omega I\ell_i d\Omega \quad , \tag{3}$$

$$\tau_{ij}^R = \frac{1}{c}\int_\Omega I\ell_i\ell_j d\Omega = \frac{1}{c}\Pi_{ij}, \tag{4}$$

where the J-scalar and the Π_{ij}-tensor are introduced for notational convenience, c is the velocity of light, and Ω is the solid angle. In terms of these definitions, the first three specular moments of the transfer equation are

$$\frac{\partial q_i^R}{\partial x_i} = \kappa_p(B-J), \tag{5}$$

$$\frac{\partial \Pi_{ij}}{\partial x_j} = -\kappa_R q_i^R, \tag{6}$$

$$\Pi_{ij} = \frac{1}{3}B\delta_{ij} - \frac{1}{\kappa_M}\frac{\partial}{\partial x_k}\int_\Omega I\ell_i\ell_j\ell_k d\Omega \ , \tag{7}$$

where $B = 4E_b$, $E_b = \sigma T^4$ being the Stefan-Boltzmann law for the black body emissive power, κ_p and κ_R are the Planck and Rosseland means of the absorption coefficient, respectively, and $\kappa_M = (\kappa_p\kappa_R)^{\frac{1}{2}}$ is the geometric mean of these coefficients. The incorporation of κ_p and κ_R into the foregoing equations is discussed by Traugott[3], Cogley, Vincenti & Gilles[4], Lord & Arpaci[5], Arpaci & Gözüm[6], Arpaci & Bayazitoğlu[7], Phillips & Arpaci[8], Arpaci & Tabaczynski[9], Arpaci[10] and Arpaci & Tabaczynski[11]. Clearly, equation (5) denotes the thermal balance, equation (6) the momentum balance associated with radiation, and equation (7) gives the definition of the Π_{ij}-tensor. Note that the radiative heat flux given by equation (6), rearranged as

$$q_i^R = -\frac{1}{\kappa_R}\frac{\partial \Pi_{ij}}{\partial x_j} \tag{8}$$

can be interpreted as a generalized diffusion process associated with this tensor.

For the evaluation of the Π_{ij}-tensor, rearrange the transfer equation in terms of κ_M as:

$$I = I_o - \frac{\ell_i}{\kappa_M}\frac{\partial I}{\partial x_i} \ . \tag{9}$$

Express the gradient in equation (9) by equation (9) itself so that

$$I = I_o - \frac{\ell_i}{\kappa_M}\frac{\partial}{\partial x_i}\left(I_o - \frac{\ell_j}{\kappa_M}\frac{\partial I}{\partial x_j}\right) \ . \tag{10}$$

Repeat the process to obtain

$$I = I_o + \sum_{n=1}^{\infty}\frac{(-1)^n}{\kappa_M^n}\left(\ell_p\ell_q\cdots\frac{\partial}{\partial x_p}\frac{\partial}{\partial x_q}\cdots\right)I_o + 0\left(\frac{1}{\kappa_M^{n+1}}\right) \tag{11}$$

Equation (7) yields, in terms of equation (11),

$$\Pi_{ij} = \frac{1}{3}B\delta_{ij} + \sum_{n=1}^{\infty}\frac{1}{\kappa_M^{2n}}\left(M_{ijpq}\cdots\frac{\partial}{\partial x_p}\frac{\partial}{\partial x_q}\right)B \ , \tag{12}$$

where

$$M_{ijpq\cdots} = \frac{1}{4\Pi}\int_\Omega (\ell_i\ell_j\ell_p\ell_q\cdots)d\Omega. \tag{13}$$

A procedure for the evaluation of equation (13) in terms of the Wallis Integrals is described in Unno & Spiegel (12). This procedure leads to

$$\Pi_{ij} = \sum_{n=o}^{\infty}\frac{\nabla^{2n-2}(2n\partial_i\partial_j + \nabla^2\delta_{ij})B}{\kappa_M^{2n}(2n+1)(2n+3)} \ , \tag{14}$$

where $\partial_i \equiv \partial/\partial x_i$ and $\partial_j \equiv \partial/\partial x_j$ are used for notational convenience. The same result may be found also in earlier works (see, for example, Milne[13]). The formal similarity of equation (14) to the Hookean constitution for elastic solids should be noted.

An alternate form for this stress may be given in terms of the isotropic radiative pressure. First, invoking the assumption of isotropy, equations (2) and (4) are related as

$$\tau_{ij}^R = \frac{1}{3}u^R\delta_{ij}, \tag{15}$$

or, equivalently, as

$$\Pi_{ij} = \frac{1}{3}\Pi_{kk}\delta_{ij} = \frac{1}{3}J\delta_{ij}, \tag{16}$$

where

$$\frac{1}{3}\Pi_{kk} = -p \tag{17}$$

is the (isotropic) pressure of radiation. Then, from the trace of Π_{ij}, noting that $\ell_k\ell_k = 1$,

$$\Pi_{kk} = \sum_{n=o}^{\infty}\left(\frac{\nabla^2}{\kappa_M^2}\right)^n\frac{B}{(2n+1)}. \tag{18}$$

Now, in a manner similar to the inclusion of the isotropic pressure to the development of viscous stress from elastic stress, (see, for example, Arpaci & Larsen (14)), adding the identity

$$\frac{1}{3}J\delta_{ij} - \frac{1}{3}\Pi_{kk}\delta_{ij} = 0 \tag{19}$$

to equation (14), the Π_{ij}-tensor may be rearranged in terms of the radiation pressure,

$$\Pi_{ij} = \frac{1}{3}J\delta_{ij} + \sum_{n=o}^{\infty}\frac{2n\nabla^{2n-2}(\partial_i\partial_j - \frac{1}{3}\nabla^2\delta_{ij})B}{\kappa_M^{2n}(2n+1)(2n+3)} \tag{20}$$

The operational similarity of equation (20) to the viscous (Stokesean) stress should be noted. The use of the first term of equation (20) in place of equation (14) is the well-known Eddington approximation. The next section develops an expression for the radiative entropy production in terms of Π_{ij} given by equations (14) and (20).

3. Radiative Entropy Production

As is well-known, the major contribution of thermal radiation to thermomechanics is the radiation heat flux, q_i^R (see, for example, Sparrow & Cess (15), Ozisik (16), Howell & Siegel (17), Vincenti & Kruger[18], and Sampson

(19). Accordingly, with the addition of this flux, the thermal energy becomes, in terms of the usual notation,

$$\rho \frac{Du}{Dt} + p \left(\frac{\partial v_i}{\partial x_i}\right) = - \frac{\partial}{\partial x_i} (q_i^K + q_i^R) + \tau_{ij} s_{ij}, \quad (21)$$

and the balance of entropy becomes

$$\rho \frac{Ds}{Dt} = - \frac{\partial}{\partial x_i} \left(\frac{q_i^R + q_i^R}{T}\right) + s''' \quad (22)$$

Now, consider the Gibbs (thermodynamic) relation

$$\rho \frac{Du}{Dt} = \rho T \frac{Ds}{Dt} + \frac{p}{\rho} \frac{D\rho}{Dt} , \quad (23)$$

and rearrange it in terms of the conservation of mass to get

$$\rho \frac{Du}{Dt} = \rho T \frac{Ds}{Dt} - p\left(\frac{\partial v_i}{\partial x_i}\right). \quad (24)$$

Eliminating the internal energy and the entropy among equations (21),(22) and (24) yields the expression for entropy production

$$s''' = \frac{1}{T}\left[- \frac{1}{T}(q_i^K + q_i^R)\left(\frac{\partial T}{\partial x_i}\right) + \tau_{ij} s_{ij}\right] \quad (25)$$

Inserting the usual conductive and viscous constitutions

$$q_i^K = -k \left(\frac{\partial T}{\partial x_i}\right) , \quad \tau_{ij} = 2\mu s_{ij}, \quad (26)$$

and the radiative constitution

$$q_i^R = - \frac{1}{\kappa_R} \left(\frac{\partial \Pi_{ij}}{\partial x_i}\right) \quad (8)$$

into this expression gives

$$s''' = \frac{1}{T}\left\{\frac{1}{T}\left[k\left(\frac{\partial T}{\partial x_i}\right) + \frac{1}{\kappa_R}\left(\frac{\partial \Pi_{ij}}{\partial x_j}\right)\right]\left(\frac{\partial T}{\partial x_i}\right) + 2\mu s_{ij} s_{ij}\right\} \quad (27)$$

Explicitly, in terms of equation (14),

$$s''' = \frac{1}{T}\left\{\frac{1}{T}\left[k\left(\frac{\partial T}{\partial x_i}\right) + \frac{4\sigma}{\kappa_R} \sum_{n=0}^{\infty} \left(\frac{\nabla^2}{\kappa_M^2}\right)^n \left(\frac{\partial T^4/\partial x_i}{(2n + 3)}\right)\right]\left(\frac{\partial T}{\partial x_i}\right) + 2\mu s_{ij} s_{ij}\right\}, \quad (28)$$

or, in terms of equation (20),

$$s''' = \frac{1}{T}\left\{\frac{1}{T}\left[k\left(\frac{\partial T}{\partial x_i}\right) + \frac{1}{3\kappa_R}\left(\frac{\partial J}{\partial x_i}\right) + \frac{4\sigma}{3\kappa_R} \sum_{n=0}^{\infty} \frac{4n}{(2n+1)(2n+3)} \left(\frac{\nabla^2}{\kappa_M^2}\right)^n \left(\frac{\partial T^4}{\partial x_i}\right)\right]\left(\frac{\partial T}{\partial x_i}\right) + 2\mu s_{ij} s_{ij}\right\} \quad (29)$$

The consideration only the first term of the summation involved with equation (28) gives

$$s''' = \frac{1}{T}\left\{\frac{1}{T}\left[k\left(\frac{\partial T}{\partial x_i}\right) + \frac{4\sigma}{3\kappa_R}\left(\frac{\partial T^4}{\partial x_i}\right)\right]\left(\frac{\partial T}{\partial x_i}\right) + 2\mu s_{ij} s_{ij}\right\} \quad (30)$$

which is valid only for large optical thicknesses (the thick gas approximation), while the first term of the summation involved with equation (30) gives

$$s''' = \frac{1}{T}\left\{\frac{1}{T}\left[k\left(\frac{\partial T}{\partial x_i}\right) + \frac{1}{3\kappa_R}\left(\frac{\partial J}{\partial x_i}\right)\right]\left(\frac{\partial T}{\partial x_i}\right) + 2\mu s_{ij} s_{ij}\right\} \quad (31)$$

which is valid for any optical thickness (the Eddington approximation). This approximation overestimates the radiative heat flux by about 30% in the neighborhood of $\tau = 1/\sqrt{3}$, and needs to be coupled with

$$\left(\frac{\partial^2}{\partial x_i \partial x_i} - 3\kappa_M^2\right) J = - 12\kappa_M^2 \sigma T^4 \quad (32)$$

(see, for example, Arpaci[1]).

4. Conclusions

The radiative stress obtained from the specular (kinetic) moments of the transfer equation is reviewed. In terms of an isotropic radiative pressure, this stress is given an alternate form by following a development similar to that of the viscous (Stokesean) stress from the elastic (Hookean) stress.

An expression for entropy production including the effect of radiation, as well as that of conduction and (viscous) friction, is developed. The radiative contribution to this production is expressed in terms of the radiative stress. Alternate forms of the entropy production are stated by considering the "elastic" and "viscous" equivalents of the radiative stress. These forms include also the usual contributions of conduction and (viscous) friction.

A study on the radiative entropy production associated with the classical aerodynamic heating problem is under progress and will be reported later. The objective of the study is to show the spatial distribution of entropy production resulting from separate effects of conduction, radiation and (viscous) friction. The extrema (minima) of this production gives the most efficient (least irreversible) operating conditions.

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