SOLID-STATE PLASMA ELECTROKINETIC POWER AND ENERGY RELATIONS

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FOREWORD

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ABSTRACT

A general theory is developed for the electrokinetic power and energy properties associated with the basic carrier modes present in plasma media. Both hydrodynamic and kinetic theoretical models are obtained for the media in the presence of applied static electric and magnetic fields.

In the hydrodynamic theory the effects of carrier collisions and thermal diffusion are properly accounted for and explained by developing a second-order quasi-linear analysis. In this manner it is shown that the negative kinetic power property is directly related to dc slowing of the active carrier. The distinction between absolute and convective instabilities leads to the formulation of a space-averaged temporal-energy basis for determining the existence of absolute instabilities as compared to a time-averaged spatial-power basis for convective instabilities. The analysis shows that it is possible to relate the causality criteria for instabilities developed by Briggs to the conservation of power and energy in the medium. Thus useful general information is obtained on the behavior of the root trajectories in complex-k space as the imaginary part of the frequency is varied.

The quasi-linear theory, as a by-product, allows the analysis of the second-order Hall effect and related phenomena in solids. In addition, a study of the physical meaning of the quasi-linear theory shows that this is a useful analytical tool for studying potential energy effects caused by the reaction of the growing RF fields on the carrier charges. This also enables the accuracy of the linear dispersion equation to be assessed.

The power and energy theorems applied to the kinetic theory determine the effects of nonlocality, anisotropic carrier temperatures, and carrier heating. Whenever possible the results obtained are rigorously compared with those of the hydrodynamics theory. By obtaining the respective dispersion equations, computer results for the hybrid-hybrid electron-hole interaction are related to published experimental work on the phenomenon of microwave emission from indium antimonide.

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LIST OF SYMBOLS

a	Constant used in Eq. 5.103.
a_s	Plasma variable given in Eq. 6.54.
$\underline{B}_o$	Static magnetic flux density vector.
$\underline{B}_1$	RF magnetic flux density vector.
$\underline{B}_{1\pm}$	Circularly polarized components of RF magnetic flux density.
$\underline{B}_2$	Second-order change in $\underline{B}_o$ due to the growing RF fields.
c	Velocity of light in media, $c = (\mu_o \epsilon)^{-1/2}$, m/s.
$\tilde{D}(\omega, k)$	Matrix introduced in Eq. 2.109.
$\underline{E}_o$	Static applied electric field vector, V/m.
$\underline{E}_1$	RF electric field vector, V/m.
$E_{1\pm}$	Circularly polarized RF electric field components introduced in Eq. 2.48.
$\underline{E}_2$	Second-order change in $\underline{E}_o$ due to the presence of growing RF fields.
e	Subscript denoting electrons.
$F_o(v_x)$	Variable introduced in Eq. 5.48.
f	Variable introduced in Eq. 2.96.
$f_o = f_{oL} + f_{1L}$	Carrier distribution function in the presence of the applied fields $(\underline{E}_o, \underline{B}_o)$.
f_{oL}	Equilibrium distribution function in the absence of any applied fields.
$f_{1\pm}$	Components of the reduced distribution function given in Eq. 5.43.
f_{1L}	Perturbation from f_{oL} caused by the applied fields $(\underline{E}_o, \underline{B}_o)$.
f_y, f_z	Reduced distribution functions given in Eq. 5.42.

$G(z)$	The plasma dispersion function introduced in Eq. 5.29.
$G_0(v_x), G_1(v_x)$	Variables introduced in Eq. 5.49 and related by Eq. 5.51.
$H_{1\pm}$	Circularly polarized components of the RF magnetic field intensity, A/m.
h	Subscript denoting holes.
$I_\ell(b)$	Modified Bessel function of order ℓ.
i	Subscript denoting ions.
$\underline{J}_0$	Static current density vector, A/m^2.
$\underline{J}_1$	RF current density vector, A/m^2.
$J_{1\pm}$	Circularly polarized RF current density components introduced in Eq. 2.48.
$\underline{J}_2$	Second-order change in $\underline{J}_0$ due to the presence of growing RF fields.
J_{2-}	Second-order change in J_0 due to the presence of growing RF fields for the left-hand circularly polarized mode.
$J_m(b)$	Bessel function of the first kind of order m.
$\underline{k}$	Wave vector, $k = k_r + jk_i$, m^{-1}.
L_1, L_2	Variables introduced in Eqs. 6.155 and 6.156.
$M(a,b,c)$	The confluent hypergeometric function.
m^*	Carrier effective mass.
N_0	Static carrier number density.
N_1	RF carrier number density.
N_0'	Effective carrier number density given in Eq. 5.112.
$P(\omega,k) = P_r + jP_i$	Polarization factor defined in Eq. 2.110.
$\underline{P}_{e\ell\pm}$	Circularly polarized components of the electromagnetic power vector introduced in Eq. 2.54, W/m^2.
P_k	Electrokinetic power, W/m^2.

P_{k-}	Electrokinetic power of the left-hand circularly polarized mode, W/m^2.
q	Carrier charge, including sign, C.
s	Subscript denoting the sth carrier species.
T	Carrier temperature, °K.
u	Velocity variable.
$\underline{v}$	General velocity variable, m/s.
$\underline{v}_0$	Carrier drift velocity, m/s.
$\underline{v}_1$	RF carrier velocity, m/s.
$v_{1\pm}$	Circularly polarized components of RF carrier velocity introduced in Eq. 2.48.
$\underline{v}_2$	Second-order change in $\underline{v}_0$ due to the presence of growing RF fields.
v_{2-}	Second-order change in $\underline{v}_0$ due to RF field growth for the left-hand circularly polarized mode.
v_F	Fermi speed, m/s.
$v_{F\parallel}$	Fermi speed parallel to $\underline{B}_0$, m/s.
v_H	Constant velocity introduced in Eq. 6.33.
v_{rs}	Velocity variable introduced in Eq. 6.87.
v_T	Carrier thermal speed, $v_T = \sqrt{\kappa T/m^*}$, m/s.
$v_{T\parallel}$	Component of thermal velocity parallel to $\underline{k}$ for the hydrodynamic distribution function.
$v_{T\perp}$	Component of thermal velocity perpendicular to $\underline{k}$ for the hydrodynamic distribution function.
$v_\parallel$	Component of thermal velocity parallel to $\underline{k}$ for the Maxwellian distribution function.
$v_\perp$	Component of thermal velocity perpendicular to $\underline{k}$ for the Maxwellian distribution function.
$W_{e\ell}$	Sum of electrostatic and electromagnetic energy densities, J/m^2.

$W_{e\ell-}$	Electromagnetic energy density of the left-hand circularly polarized mode, J/m^2.
W_k	Electrokinetic energy density, J/m^2.
W_{k-}	Electrokinetic energy density of the left-hand circularly polarized mode.
w	Velocity variable introduced in Eq. 6.19.
$Z(z)$	Tabulated plasma dispersion function introduced in Eq. 5.31.
z	Normalized plasma variable used in Eq. 5.16 and 5.33.
z_c	Normalized plasma variable introduced in Eqs. 5.64 and 5.94.
α	Constant used in Table 2.1.
β_s	Variable introduced in Eq. 2.99.
$\Gamma(n)$	Gamma function, $\Gamma(n) = (n - 1)!$.
γ_s	Variable introduced in Eq. 2.98.
$\delta(x)$	Dirac delta function, introduced in Eq. 5.3 and used thereafter. $\int_a^b f(x)\delta(x - x_o)dx = f(x_o)$ if $b \geq x_o \geq a$, otherwise the integral equals zero.
ϵ	Lattice permittivity, $\epsilon = \epsilon_r\epsilon_o$, F/m.
$\overleftrightarrow{\epsilon}$	Effective dielectric constant introduced in Eq. 6.49.
ϵ_o	Permittivity of free space, F/m.
ϵ_r	Lattice relative dielectric constant.
η	Carrier charge-to-mass ratio.
θ, θ', Θ	Angle variables.
κ	Boltzmann's constant.
λ	Wavelength, m.
λ_c	Variable introduced in Eq. 6.170.
λ_s	Variable used in Eq. 6.105 and thereafter.

μ	Normalized variable introduced in Eq. 5.17.
μ_o	Permeability of free space.
ν	Carrier collision frequency.
ν'	Effective carrier collision frequency.
ρ_o	Static charge density, $\rho_o = qN_o$.
ρ_1	RF charge density.
ρ_2	Second-order change in ρ_o due to the presence of growing RF fields.
ρ_{2-}	Second-order change in ρ_o for the left-hand circularly polarized mode.
σ	A constant used in Eq. 5.34 and again in a different context in Eq. 5.65.
$\overleftrightarrow{\sigma}$	RF conductivity tensor introduced in Eq. 6.49.
σ_-	RF conductivity of the left-hand circularly polarized mode introduced in Eq. 5.61.
Φ_o, Φ_3, Φ_4	Variablesintroduced in Eqs. 5.12 through 5.15.
Φ_1, Φ_2	Variables introduced in Eqs. 6.85 and 6.86.
φ, φ'	Angle variables.
Ψ_1	Variable introduced in Eq. 5.13.
$\omega = \omega_r + j\omega_i$	Angular frequency, rad/s.
ω_c	Carrier cyclotron frequency, $\omega_c = \eta B_o$, rad/s.
ω_p	Carrier plasma frequency, $\omega_p = \sqrt{\eta\rho_o/\epsilon}$, rad/s.
ω_p'	Effective plasma frequency introduced in Eq. 6.207.

CHAPTER I. INTRODUCTION

1.1 Streaming Instabilities in Solid-State Plasmas

1.1.1 Introduction. There has been considerable research in recent years directed toward the generation and amplification of microwave radiation utilizing solid-state plasmas. The encouraging properties offered by these media[1-10] are (1) densities of charge carriers far in excess of any that can be reasonably achieved in vacuum or in gas discharges and (2) the existence of two types of mobile charge carriers, electrons and holes, which have different effective masses; these masses are much less than the free electron mass. The high carrier densities afford large coupling strengths in any carrier interaction mechanism; the low effective masses provide both large drift velocities and very high cyclotron frequencies at moderate applied fields. On the other hand, the large Fermi (thermal) velocities of the carriers and their large collision frequencies (lattice interaction) limit the degree of spatial and temporal coherence available in any wave-carrier interaction. While the collisions in gaseous plasmas are often insignificant, in solid-state plasmas they are in many cases of vast importance. Under some conditions the presence of collisions actually induces new instabilities by permitting more general sets of carrier motions by which the system can reach lower energy states. In addition, as opposed to gaseous plasmas wherein instabilities naturally arise from the methods of plasma production, solid-state plasmas are in thermal equilibrium, and in most cases very high current densities are required to generate instabilities. Thus the solid-state plasma experimentalist is often operating at the limit of sample power dissipation and the theoretician will, as a rule, have self-magnetic field effects and carrier heating effects to consider. Since the present study is concerned primarily with velocity-driven instabilities, the relevant characteristics of those waves associated with the drifting carriers will be examined.

A moving medium of charged carriers can support two classes of modes.[11] One class carries positive kinetic energy so that energy must be supplied to the system to excite these modes. The other class of modes carries negative kinetic energy and, correspondingly, energy must be removed from the system to excite these modes. Useful instabilities arise from the interaction of a positive-kinetic-energy-carrying mode with a negative-kinetic-energy-carrying mode. If the group velocities of the interacting modes are in the same direction the instability is convective in nature; if the mode group velocities are oppositely directed, the instability is absolute (nonconvective). Regardless of the interaction under study, several criteria are available to investigate the instability characteristics of the system.[12-15]

Of the possible instabilities, those which have received the most attention in the literature are of the two-stream type. These can be divided into two major subsets via their distinctive wave properties. In one subset, termed the transverse instability, the related modes are essentially electromagnetic waves modified by the presence of both a static magnetic field in the direction of wave propagation and the charge carriers. In the second subset related to longitudinal instabilities, the natural oscillations of the charge carriers give rise to electrostatic-type modes (plasma waves), which in general are not affected by any applied magnetic field. In addition, these basic modes can be coupled together by applying the magnetic field at an angle to the direction of wave propagation (hybrid instability). These classifications will be reviewed in turn, followed by a description of the objective of the present study.

1.1.2 Transverse Two-Stream Instabilities. The possibility of transverse electromagnetic wave propagation in conducting solids employing a static magnetic field parallel to the direction of wave propagation was first discussed by Konstantinov and Perel.[16] Aigrain[17] termed the waves in uncompensated plasmas helicons on the basis of their circular polarization and proposed their possible amplification by drifting the charge carriers. Helicons propagate with small damping in undrifted plasmas if the wave frequency is less than the majority carrier cyclotron frequency and if the latter is large compared to the phenomenological majority carrier collision frequency. In compensated media, the helicon modes go over to Alfven waves propagating with small damping for large applied magnetic fields at wave frequencies above the mean carrier collision frequency.[18] Helicons may still propagate in compensated media if the mobility of one species is much larger than that of the compensating species.

Bok and Nozieres,[19] on the basis of an instability analysis of the dispersion equation describing the drifted two-component system, determined that amplification could occur if the drift velocity of the more mobile carriers is greater than the phase velocity of the wave.

Also, since the contribution to the effective dielectric constant of the more mobile carriers is small near synchronism, the presence of a second carrier species is necessary to maintain the high dielectric constant. The gain mechanism cannot be described as inverse Landau damping since there is no axial RF electric field or carrier bunching. However, if the applied electric or magnetic field is inclined slightly from being parallel to the propagation vector, Misawa[20] has found that a convective instability can be attributed to this mechanism. Rodriguez and Antoniewisz[21] have also studied the helicon-longitudinal plasma interaction. Indeed, Baynham and Braddock[22] have demonstrated gain for radio-frequency helicon waves propagating off-axis.

Hasegawa[23] has investigated the system of Bok and Nozieres[19] under conditions of mass anisotropy wherein the electron mobility is much greater than that of the holes. In this case, the two-stream instability degenerates to a one-stream resistive instability; i.e., the hole drift can be neglected and the holes may be regarded as a resistive medium which absorbs the electromagnetic energy which the electron stream carries. As a consequence, the amplitude of the negative-kinetic-energy mode on the electron stream increases. Similarly, Akai[24] has interpreted the two-stream instability of Bok and Nozieres in terms of the positive-kinetic-energy cyclotron wave supported by the drifting holes behaving as the passive wave which dissipates the energy. It was further shown via computer analysis by Bers and McWhorter[25] for intrinsic InSb at liquid nitrogen temperature that above a threshold of applied magnetic field the convective instability is overridden by the occurrence of an absolute instability. Thus a necessary condition for a convective instability is that no absolute instability occurs simultaneously. In a recent analysis, Bartelink[26] has analyzed the modes of propagation and associated instabilities of transverse disturbances in solid-state plasmas of varying degrees of compensation and mobility anisotropy with drift exactly parallel to the applied magnetic field.

The salient feature introduced by partial compensation was that the negative-kinetic-energy carrier helicon mode could exist with the mean carrier drift velocity less than the wave phase velocity. Thus the proper criterion for gain is that the negative-kinetic-energy mode must see a net positive resistance. In this regard, negative resistance can occur in p-type media in which the electrons cause greater collisional interaction with the wave than the holes either by having a larger drift mobility or smaller cyclotron mobility than the holes, or both. The negative resistance can lead to a second mode of amplification corresponding to a positive energy wave, as has been shown in bismuth.[27] In a one-component and infinite plasma it is still necessary that the carrier drift velocity exceed the wave phase velocity so as to obtain a negative kinetic energy wave.[28] The most promising scheme to attain amplification in one-component plasmas is the utilization of a multi-layered structure in which some of the layers would support the wave while the carriers in adjacent layers would be given a drift velocity. In one form of this Baraff and Buchsbaum,[29,30] Wallace and Baraff,[31] and Saunders and Baraff[32] employ the surface waves which must exist at the layer interfaces[33] to match the boundary conditions. The interaction between the surface wave and the bulk helicon wave can lead to gain even for carrier velocities much less than the wave phase velocity. McWhorter[34] has explained the instability as due to electron collisions at the interface between the drifting plasma layers. No experimental evidence of this effect has yet been obtained. In addition, Baraff[35] has studied the layered device structure for bulk interaction wherein the surface wave plays a secondary role. A threshold for instability is found when the carrier drift velocity in one layer is approximately twice the phase velocity in an adjacent layer. The growth rate of the instability can be understood in terms of the balance of energy between power generated in the drifted layers, collisional losses in the undrifted layers and the losses or gains at the interfaces. The instability arises from the reversal of the collisional losses in the bulk of the current-carrying layer. Nanney et al.[36] in an experimental study of this bulk phenomenon in n-PbTe have observed, from the propagation of MHz signals, a transmission increase of the order of 15 dB for the drifted vs. the undrifted case. They were not able to obtain drift velocities greater than the calculated wave phase velocity due to a limitation of the pulse current supply, so net spatial gain did not occur. For larger values of magnetic field the aforementioned surface effects will predominate whereas the bulk effects become more important as the magnetic field is decreased.

One of the major problems associated with the helicon mode instabilities, in the frequency range which has been studied to date, is that the corresponding wavelengths are typically of the order of a millimeter so that the specimen will introduce a strong boundary effect on the wave propagation. The boundary effect has been observed in the helicon wave propagation in a semiconductor,[36] and Grow[37] was unable to find evidence of the instability of Bok and Nozieres from an analysis of the finite-dimension sample case. In addition the size effect can result in a significantly increased wave phase velocity over the infinite medium case.[38] There is also the nonlocal effect of doppler-shifted cyclotron resonance which limits the threshold to which the applied magnetic field can be reduced before severe damping occurs.[39] This latter problem will be especially significant with regard to the bulk instabilities because the upper frequency limit of the instability is effectively limited. Since large currents are generally required the self-magnetic field generated can cause a nonnegligible radially inward force to be applied to the plasma resulting in pinching.[40-42] The effect of an external magnetic field is such that, when it is comparable to or larger in magnitude than the self-magnetic field, the pinch does not form.[43] Glicksman[1] states that this is not because of effects caused directly by the applied field, but rather by the rapid rise of helical instabilities which tend to prevent formation of the pinch.

At higher frequencies the helicon mode characteristic changes to that of a slow cyclotron wave. Absolute instabilities associated with the coupling between this slow wave and the fast hole cyclotron mode have been studied by Höfflinger,[44] Grow[37] and Vural and Steele.[11] Experimental verification of this instability has not yet been achieved.

<u>1.1.3 Longitudinal Two-Stream Instabilities</u>. In addition to the transverse polarization case, in which the waves are electromagnetic in character (their phase velocity being dependent upon the velocity of light in the medium), the plasma can also support waves arising from the collective modes of oscillation of the mobile carriers. They were first

studied by Tonks and Langmuir[45] in a gaseous plasma and by Pines[46] in a solid. Their electrostatic nature implies that the waves (termed plasma waves) are of purely longitudinal polarization (in the absence of an applied magnetic field). The number of distinct plasma waves is equal to the number of distinct carrier species in the plasma. Thus, corresponding to the electron-hole plasma in a semiconductor or semimetal, two collective modes of oscillation exist. One mode consists of a high-frequency oscillation in which the electrons and holes oscillate out of phase at a frequency which in the long wavelength limit is the mean plasma frequency. The other mode, corresponding to in-phase electron-hole oscillation, is typified by phase velocities of the order of the Fermi velocities (or, in a classical plasma, the thermal velocities).

By analogy to the vibration spectrum of polar crystals the former high-frequency mode is termed the optical mode of plasma oscillation, the latter mode at lower frequencies, the acoustic mode of plasma oscillation. Very little experimental work has been done on the optical branch because of the high frequencies involved (optical or ultraviolet), concurrent with their longitudinal character; however, there is considerable literature related to the characteristic energy losses due to excitation of the optical branch of the plasma wave by passage of fast particles through thin metal foils.[47] The acoustic branch is heavily Landau damped unless the Fermi or thermal velocities of the two carriers are widely disparate.

The possibility of observing a two-stream instability in a high-mobility semiconductor such as InSb was first investigated by Pines and Schrieffer[48] who demonstrated that an absolute instability should occur for the acoustic branch if the electron drift velocity relative to the hole velocity is comparable to the electron-thermal velocity and if the growth exceeds the collisional and thermal damping. Similar criteria were found by Harrison[49] in his study of degenerate plasmas and Vural and Bloom[50] in their study of guided plasmas. The general conclusion was that it appeared marginal whether or not the instability could be observed in practice with currently available materials. The major obstacle is the thermal condition since Glicksman and Hicinbothem[51] have determined that the application of large electric fields to InSb at liquid nitrogen temperature results in hole temperatures which are of the same order as the electron temperature. In addition, since the oscillations are longitudinal in field polarization, they would have to be coupled out through some type of gradient mechanism. Efforts to experimentally observe the instability in bulk materials such as InSb, pyrolytic graphite and bismuth have not been conclusive.[52] Recently, Robinson and Swartz[53] and Robinson and Vural[54] have analyzed layered structures of p- and n-InSb, with which the temperature condition can be more easily satisfied, with the result that the surface plasma waves can grow at rates only slightly less than those of the bulk plasma waves of the corresponding penetrating stream system.

Interpreting the longitudinal plasma waves in a coupled-mode manner shows theoretically[50] that a co-streaming two-component plasma can give rise to a convective instability provided that the collisional loss is not too severe. No suitable material has been suggested in the literature to observe this instability.

1.1.4 Hybrid-Mode Instabilities. When the magnetic field is inclined from being parallel to the wave vector, a mode propagates with characteristics of both the longitudinal plasma-wave and helicon or cyclotron modes. In drifted plasmas, a transverse component of magnetic field also gives rise to a Hall electric field which for high mobility materials can be comparable to the applied electric field. The hybrid system has been studied theoretically[55-57] in an attempt to explain one type of microwave emission from InSb.[58-59] An analysis by Vural[60] of finite solid-state plasmas in the presence of an axial magnetic field has shown that coupling between plasma waves and cyclotron excitations occurs for this case which indicates an absolute instability associated with the cyclotron space-charge wave interactions. The hybrid-wave interaction is unusual in that Landau damping is absent.[61] Hasegawa[62] has shown that the hybrid-hybrid interaction leads to an instability in the limit that the holes are collision dominated. Swartz and Robinson[63] consider this interaction to be responsible for coherent oscillations observed in InSb.

1.2 Outline of the Present Study

The Chu kinetic power theorem[64,65] for longitudinal space-charge waves is a well known general method for studying those power properties of the wave associated with a group of streaming charge carriers which enable wave growth, either spatial or temporal, to occur in electron-beam devices as well as in gaseous and solid-state plasmas. In Chapter II a kinetic-electromagnetic power theorem is derived for the basic carrier waves present in a static magnetic field. Past studies in this area[65,66] have been inadequate in that the RF fields (which determine the kinetic power properties) are neglected in the analysis so that the results thus obtained only apply to the carrier wave in a region of no interest. In addition, because of their importance in the solid-state area, the effects of collisions and thermal diffusion on the carrier mode kinetic power are examined.

In all past developments of power-energy theorems there has been no differentiation made between convective and absolute instabilities. Thus it is tacitly assumed that a carrier mode which is active for a convective instability (e.g., slow space-charge wave) can, under proper circuit configuration, be active for an absolute instability with the relevant carrier parameters (e.g., plasma frequency) playing the same role in both cases. In Chapter III, a kinetic-electromagnetic energy theorem is derived for both space-charge waves and the basic carrier modes present in a static magnetic field which shows that, in general, separate criteria are involved. The theorem also demonstrates the physical mechanisms whereby an absolute instability arises. Whereas in Chapter II the study of convective instabilities was formulated in a spatial-power framework, this study is undertaken in a temporal-energy framework. In this way the complete dual of the convectively unstable system is found and related to the time rate of change of the carrier kinetic energy at any point in the interaction region.

In Chapter IV the power and energy theorems are examined with relation to the potential energy effects which arise due to the reaction of the growing RF fields on the charge carriers. The quasi-linear theory employed is found to be a useful analytical technique for studying the evolvement of an instability from the point and time of its initiation.

The general formulation adopted for the carrier mode electrokinetic power flow and energy density enables these concepts to be extended to the kinetic theory in Chapter V. Thus the power and energy effects of Landau damping, cyclotron resonance, and temperature anisotropy can be examined and the results compared with those of hydrodynamic theory.

Chapter VI is concerned with a detailed analysis of the hybrid wave system and its application to microwave emission phenomena in solid-state materials. After the examination of the zeroth-order distribution function for a system in a transverse magnetic field the dispersion relations describing the wave propagation according to kinetic theory are obtained. Unstable cyclotron-harmonic behavior theoretically derived is compared with similar experimental results. Furthermore, from a rigorous study of carrier heating, it is found that an entirely new mode can appear. This mode is significant in that it exhibits synchronous behavior and has a small damping decrement (similar to the helicon mode).

Finally, a discussion of results and conclusions is given in Chapter VII together with suggestions for further study.

CHAPTER II. EFFECTS OF A MAGNETIC FIELD ON THE KINETIC POWER PROPERTIES OF CARRIER WAVES: CONVECTIVE INSTABILITY

2.1 Introduction

There are three basic modes associated with a stream of drifting charge carriers in a solid which will be of interest; namely, the longitudinal space-charge or plasma wave, the left- or right-hand circularly polarized modes which propagate parallel to a static magnetic field (e.g., helicon or cyclotron modes), and the hybrid mode propagating perpendicular to a static magnetic field. For any of these modes to be useful as a source of power in a convective instability (i.e., be active) it must be such that the carrier mode transports less total energy than the dc stream alone in the direction of wave propagation. The total energy flow or total power of a carrier mode is the sum of its associated electromagnetic power and its electrokinetic power. When this sum is less than the kinetic power of the dc stream alone it is designated a <u>negative power</u> mode.

In the specific case of the slow space-charge wave of an electron beam the electromagnetic power is of the order of $(\omega_p/\omega)^2$ times the kinetic power in magnitude[65] with the practical result that it can be neglected and in this case the designation of negative kinetic power mode suffices. When a negative power mode interacts properly with a passive circuit, exponentially growing RF fields occur without violation of conservation of power since the carrier mode power growth is negative. In general, the circuit can correspond to a helix in electron-beam devices, to a second carrier species in a plasma or to a mode of lattice vibration of a piezoelectric or polar solid, etc.

Of particular interest in the present work is the analysis of the manner in which the RF fields may extract dc power from the relevant carrier mode. This is especially important for the cyclotron modes since no bunching processes are present and it is not immediately clear how these modes can supply power. By use of a quasi-linear analysis it is shown that the negative kinetic power property can be directly related to dc slowing of the carrier motion for all cases. In addition this method indicates the conditions under which the plane-wave method of analysis is valid.

Because of its importance in solid-state plasmas special attention is directed to the effects of collisions, carrier diffusion, and a static magnetic field upon the kinetic power carried by the carrier mode. For small growth rates ($k_i \ll k_r$) it is shown that the collisions play a dominant role and can assist the instability process by contributing to the negative kinetic power property when $(\omega - k_r v_o) < 0$. In the study of convective instabilities in the present chapter the angular frequency of the fundamental field ω is assumed pure real.

2.2 Kinetic Power Characteristics of Space-Charge Waves

Chu's kinetic power theorem[64] is a well known general method for studying those power properties of the longitudinal space-charge wave associated with a group of streaming charge carriers which enable wave growth to occur in electron-beam devices as well as in gaseous and solid-state plasmas. A brief review of the longitudinal theorem as applied to the traveling-wave amplification process is now given.

Assume an electron beam in a drift region with a dc velocity $v_o\hat{x}$. RF perturbations are assumed to have a space and time dependence of $e^{j(\omega t-kx)}$. Expressions for the electromagnetic and kinetic power flow associated with the medium can be derived from a base of Maxwell's equations and the linearized force equation,

$$\nabla \times \underline{E}_1 = -j\omega \underline{B}_1 \ , \tag{2.1}$$

$$\nabla \times \underline{B}_1 = \mu_o \underline{J}_1 + j\omega\mu_o \epsilon \underline{E}_1 \tag{2.2}$$

and

$$j\omega \underline{v}_1 + (\underline{v}_o \cdot \nabla)\underline{v}_1 + \nu \underline{v}_1 + \frac{v_T^2}{\rho_o} \nabla \rho_1 = \eta \underline{E}_1 + \eta \underline{v}_o \times \underline{B}_1 \ , \tag{2.3}$$

where η = the charge-to-mass ratio for electrons,

v_T = the carrier thermal velocity and

ν = the collision frequency.

From Eqs. 2.1 and 2.2 the small-signal Poynting theorem can be found as

$$\frac{1}{\mu_o} \nabla \cdot (\underline{E}_1 \times \underline{B}_1^*) + j\omega\mu_o \underline{H}_1 \cdot \underline{H}_1^* + j\omega\epsilon \underline{E}_1 \cdot E_1^* + \underline{E}_1 \cdot \underline{J}_1^* = 0 \ , \tag{2.4}$$

so that

$$\mathrm{Re}\left[\frac{1}{\mu_o} \nabla \cdot (\underline{E}_1 \times \underline{B}_1^*)\right] = -\mathrm{Re}(\underline{E}_1 \cdot \underline{J}_1^*) \ , \tag{2.5}$$

and the electromagnetic power is not conserved but rather is balanced by $\mathrm{Re}(\underline{E}_1 \cdot \underline{J}_1^*)$.

From the $\hat{y}$ component of Eq. 2.3,

$$j(\omega - kv_o - j\nu)v_{1y} = \eta\left(1 - \frac{kv_o}{\omega}\right)E_{1y} \quad , \qquad (2.6)$$

and since $J_{1y} = \rho_o v_{1y}$, multiplying both sides of Eq. 2.6 by E_{1y}^* yields, after rearranging,

$$E_{1y}^* J_{1y} = -\frac{j\eta\rho_o(\omega - kv_o)}{\omega(\omega - kv_o - j\nu)}|E_{1y}|^2 \quad . \qquad (2.7)$$

Equation 2.7 is used to show that

$$\mathrm{Re}(E_{1y}J_{1y}^*) = \frac{\omega_p^2\epsilon\nu(\omega - k_r v_o)|E_{1y}|^2}{\omega[(\omega - k_r v_o)^2 + (k_i v_o + \nu)^2]} \quad . \qquad (2.8)$$

Similarly, the $\hat{z}$ component of Eq. 2.3 leads to

$$\mathrm{Re}(E_{1z}J_{1z}^*) = \frac{\omega_p^2\epsilon\nu(\omega - k_r v_o)|E_{1z}|^2}{\omega[(\omega - k_r v_o)^2 + (k_i v_o + \nu)^2]} \quad . \qquad (2.9)$$

Equations 2.8 and 2.9 show the important result that the transverse motion of the carriers provides a source mode for electromagnetic power growth because of the presence of collisions if $(\omega - k_r v_o) < 0$. Since there is no transverse bunching it is not immediately obvious how this can occur. This phenomenon will be explained later, however, in terms of the related second-order carrier dynamics wherein the equivalence is made between second-order dc beam slowing and the source properties of Eqs. 2.8 and 2.9.

From the $\hat{x}$ component of Eq. 2.3,

$$j(\omega - kv_o - j\nu)v_{1x} - j\frac{kv_T^2}{\rho_o}\rho_1 = \eta E_{1x} \quad . \tag{2.10}$$

Since $J_{1x} = \rho_o v_{1x} + \rho_1 v_o$, the equation of continuity

$$\nabla \cdot J_1 + \frac{\partial \rho_1}{\partial t} = 0 \tag{2.11}$$

gives

$$\rho_1 = \frac{k\rho_o v_{1x}}{\omega - kv_o} \quad . \tag{2.12}$$

Equation 2.12 permits writing Eq. 2.10 as

$$j[(\omega - kv_o)(\omega - kv_o - j\nu) - k^2v_T^2]\, J_{1x} = \eta\rho_o\omega E_{1x} \quad , \tag{2.13}$$

from which the following may be found:

$$\mathrm{Re}(E_{1x}J_{1x}^*) = \frac{\omega_p^2\epsilon\omega\left[(\omega - k_rv_o)(\nu + 2k_iv_o) + 2k_ik_rv_T^2\right]|E_{1x}|^2}{\left[\left\{\left[(\omega - k_rv_o)^2 - k_iv_o(k_iv_o + \nu) + v_T^2(k_r^2 - k_i^2)\right]^2 + \left[(2k_iv_o + \nu)(\omega - k_rv_o) + 2k_ik_rv_T^2\right]^2\right\}\right]} \quad . \tag{2.14}$$

Since $k_i > 0$ is the case of concern, corresponding to exponentially growing fields, Eq. 2.14 shows that for $Re(E_{1x}J_{1x}^*)$ to be a source for the field growth it is necessary that $(\omega - k_r v_o) < 0$. Note that when this is the case the collisions assist the source power. Indeed, by inspection it can be seen that if $\nu \to 0$ it is then necessary that $v_o > v_T$ in order for the longitudinal carrier mode to be a source of power. This explains why in solid-state plasmas, where $v_o > v_T$ is not in general possible, negative power modes are possible because of the collisional assistance provided. This will be further elaborated upon in Chapter III in the study of electrokinetic energy densities in solids.

The real part of the electromagnetic power flow associated with the carrier stream is now examined. Since it is in general difficult to take into account explicitly the effects of the external circuit on the wave dispersion the results obtained for this flow are approximate and only apply to weakly coupled systems. From the assumed space and time dependence,

$$\mathrm{Re}\left[\frac{1}{\mu_o}\nabla\cdot(\underline{E}_1 \times \underline{B}_1^*)\right] = \frac{2k_i}{\mu_o}\mathrm{Re}(\underline{E}_1 \times \underline{B}_1^*) \quad . \tag{2.15}$$

To determine this quantity, from Maxwell's equation it can be found that

$$\underline{E}_1 \times \underline{B}_1^* = \frac{1}{j\omega}\left(E_{1y}\frac{\partial E_{1y}^*}{\partial x} + E_{1z}\frac{\partial E_{1z}^*}{\partial x}\right)\hat{x} \quad , \tag{2.16}$$

so that from Eqs. 2.15 and 2.16

$$\mathrm{Re}\left[\frac{1}{\mu_o}\nabla\cdot(\underline{E}_1 \times \underline{B}_1^*)\right] = \frac{2k_ik_r}{\mu_o\omega}\left(|E_{1y}|^2 + |E_{1z}|^2\right) \quad , \tag{2.17}$$

which is of course positive for the case of interest ($k_i > 0$).

The conservation theorem is now obtained by integration of Eq. 2.4 over the volume containing both circuit and beam, which gives when the real part is taken

$$\int \left\{\mathrm{Re}\left[\frac{1}{\mu_o}\nabla\cdot(\underline{E}_1 \times \underline{B}_1^*)\right]_{\mathrm{circuit}} + \mathrm{Re}\left[\frac{1}{\mu_o}\nabla\cdot(\underline{E}_1 \times \underline{B}_1^*)\right]_{\mathrm{beam}}\right\} dV$$

$$= - \int \left[\mathrm{Re}(\underline{E}_1 \cdot \underline{J}_1^*)_{\mathrm{circuit}} + \mathrm{Re}(\underline{E}_1 \cdot \underline{J}_1^*)_{\mathrm{beam}}\right] dV \quad . \tag{2.18}$$

The function $\mathrm{Re}(\underline{E}_1 \cdot \underline{J}_1^*)_{\mathrm{circuit}}$ corresponds to the wall losses at the circuit and is usually negligible. In addition, since at the beam the electromagnetic power flow corresponding to Eq. 2.17 is small compared to the beam kinetic power flow (i.e., $|E_{1x}| \gg |E_{1y}|, |E_{1z}|$), Eq. 2.18 reduces to

$$\mathrm{Re}\int\left\{\left[\frac{1}{\mu_o}\nabla\cdot(\underline{E}_1 \times \underline{B}_1^*)\right]_{\mathrm{circuit}} + (\underline{E}_1 \cdot \underline{J}_1^*)_{\mathrm{beam}}\right\} dV \cong 0 \quad . \tag{2.19}$$

Make the definition

$$\underline{P}_{\mathrm{circuit}} = \frac{1}{\mu_o}\mathrm{Re}(\underline{E}_1 \times \underline{B}_1^*)\,\hat{x} \tag{2.20}$$

correspond to the circuit power and

$$\underline{P}_k \triangleq \frac{1}{2k_i} \mathrm{Re}(\underline{E}_1 \cdot \underline{J}_1^*)_{\mathrm{beam}} \hat{x} \quad , \quad (k_i \neq 0) \qquad (2.21)$$

correspond to the beam kinetic power. If Gauss' theorem is used in Eq. 2.19, the Chu kinetic power theorem is then obtained as

$$\mathrm{Re} \oint (\underline{P}_{\mathrm{circuit}} + \underline{P}_k) \cdot d\underline{S} = 0 \quad , \qquad (2.22)$$

which is a surface integral enclosing the volume of interest. From Eqs. 2.8, 2.9, 2.14 and 2.21 the most general form of the beam kinetic power should be given by

$$P_k = \frac{\omega_p^2 \epsilon \nu (\omega - k_r v_o)(|E_{1y}|^2 + |E_{1z}|^2)}{2k_i \omega [(\omega - k_r v_o)^2 + (k_i v_o + \nu)^2]}$$

$$+ \frac{\omega_p^2 \epsilon \omega v_o \left[(\omega - k_r v_o)\left(1 + \frac{\nu}{2k_i v_o}\right) + k_r \frac{v_T^2}{v_o} \right] |E_{1x}|^2}{\left[\begin{array}{l} \{[(\omega - k_r v_o)^2 - k_i v_o (k_i v_o + \nu) - v_T^2 (k_r^2 - k_i^2)]^2 \\ + [(2k_i v_o + \nu)(\omega - k_r v_o) + 2k_i k_r v_T^2]^2\} \end{array} \right]} \quad . \qquad (2.23)$$

In particular, when $\nu = 0 = v_T$, the well known form is obtained:

$$P_k = \frac{\omega_p^2 \epsilon \omega v_o (\omega - k_r v_o) |E_{1x}|^2}{[(\omega - k_r v_o)^2 + k_i^2 v_o^2]^2} = \mathrm{Re} \left[\left(\frac{\underline{v}_o \cdot \underline{v}_1}{\eta} \right) J_{1x}^* \right] \quad . \qquad (2.24)$$

The latter expression has commonly been used as the definition of beam kinetic power in the presence of collisions and thermal diffusion by

Vural and Bloom.[67] This is inappropriate however since this definition has no direct significance in the balance of power given in Eq. 2.19. Thus in the previous work[67] the real kinetic power as defined by the right-hand side of Eq. 2.24 is determined to be negative whenever $\omega - k_r v_o < 0$. Inspection of Eq. 2.23 shows that this is not in general sufficient. The fact that the definition of kinetic power of Eq. 2.24 is misleading has recently been pointed out.[68] To utilize the concept of kinetic power properly in the presence of collisions and thermal diffusion the definition given in Eq. 2.21 is required.

To proceed further with the study of the space-charge-wave case, from Eq. 2.22 the total real circuit power (e.g., in watts), $P^T_{circuit}$ is

$$P^T_{circuit} = \text{Re} \oint \underline{P}_{circuit} \cdot d\underline{S} = - \text{Re} \oint \underline{P}_k \cdot d\underline{S} \quad . \qquad (2.25)$$

From the assumed spatial dependence, the following definitions can be made:

$$E_{1x} = E_{10} \, e^{-jkx}$$

and

$$E^*_{1x} = E^*_{10} \, e^{jk^* x} \quad , \qquad (2.26)$$

where the multipliers E_{10}, E^*_{10} are independent of x. In the case where collisions and thermal diffusion are ignored, use of Eqs. 2.24 and 2.26 in Eq. 2.25 gives

$$P^T_{circuit} = - \frac{\omega_p^2 \epsilon \omega v_o (\omega - k_r v_o) |E_{10}|^2}{(\omega - k_r v_o)^2 + k_i^2 v_o^2} \oint e^{2k_i x} \, \hat{x} \cdot d\underline{S} \quad , \qquad (2.27)$$

where only the mode of interest has been retained. Equation 2.27 indicates that for amplification to occur two requirements must be met; the nature of the exponential term requires that $k_i > 0$ and at the same time it must be that $v_o > (\omega/k_r)$ in order that the circuit power be positive. If both these criteria are met the circuit power for a device of length L is given in terms of the input amplitude of the longitudinal RF electric field by

$$P^T_{circuit} = \frac{\omega_p^2 \epsilon \omega v_o (k_r v_o - \omega)}{[(\omega - k_r v_o)^2 + k_i^2 v_o^2]} |E_{10}|^2 \left(e^{2k_i L} - 1\right) . \qquad (2.28)$$

Hence an active or amplifying carrier mode is obtained if it has the property $\omega/k_r < v_o$. From the dispersion equation describing the uncoupled modes on the beam this then indicates that for weak coupling at least the slow space-charge wave $(\omega - kv_o = -|\omega_p|)$ and the fast space-charge wave $(\omega - kv_o = |\omega_p|)$ are active and passive, respectively.

It is now shown that the carrier mode power properties can be related to the beam RF conductivity. For $\nu = 0 = v_T$ the longitudinal force equation gives

$$\eta E_{1x} = j(\omega - kv_o) v_{1x} \quad ; \qquad (2.29)$$

so employing

$$J_{1x} = \rho_o v_{1x} + \rho_1 v_o \triangleq \sigma_{longtl} E_{1x} \qquad (2.30)$$

and the continuity equation, it can be found that

$$Re(\sigma_{longtl}) = \frac{\omega_p^2 \epsilon \omega 2k_i v_o (\omega - k_r v_o)}{[(\omega - k_r v_o)^2 + k_i^2 v_o^2]^2} . \qquad (2.31)$$

Hence under conditions of amplification the real part of the beam RF conductivity is negative and as a result the property of negative kinetic power flow is equivalent to that of negative RF beam conductivity. This equivalence will be true in general if the kinetic power flow is determined only by the first-order fields. This has been shown to be true for the space-charge waves by retaining variables to second order in the analysis.[69]

The commonly held interpretation of the negative kinetic power property of the slow space-charge wave is that an excess particle density, $N = N_o + |N_1|$, occurs spatially localized with a decreased beam velocity, $v = v_o - |v_1|$, and vice versa, so that less net kinetic energy is transported across a surface enclosing the beam than under strictly dc conditions. It is shown in Appendix A however that an alternate viewpoint can be adopted and the negative kinetic power property is simply due to second-order dc beam slowing. The two viewpoints give mathematically identical results.

2.3 Nature of the Transverse-Field Contributions to the Kinetic Power Flow

It is wondered by what physical means the transverse contributions to the kinetic power given in Eqs. 2.8 and 2.9 can be accounted for since, in the hydrodynamic model for the purely transverse modes, there is no carrier bunching present; the latter was found to be directly related to the kinetic power characteristic of the longitudinal space-charge wave. To see this a quasi-linear theory is used in which second-order effects on the carrier stream are expressed in terms of the fundamental RF fields. Write the force equation to second order to give

$$\frac{\partial \underline{v}_1}{\partial t} + (\underline{v}_o \cdot \nabla)\underline{v}_1 + (\underline{v}_1 \cdot \nabla)\underline{v}_o + \nu \underline{v}_1 + \frac{v_T^2}{\rho_o} \nabla \rho_1 + \frac{\partial \underline{v}_2}{\partial t} + (\underline{v}_o \cdot \nabla)\underline{v}_2 + \underline{v}_1 \cdot \frac{\partial \underline{v}_1}{\partial x}$$

$$+ (\underline{v}_2 \cdot \nabla)\underline{v}_o + \nu \underline{v}_2 + \frac{v_T^2}{\rho_o} \nabla \rho_2 = \eta(\underline{E}_1 + \underline{E}_2 + \underline{v}_1 \times \underline{B}_1 + \underline{v}_o \times \underline{B}_2) \; , \quad (2.32)$$

and from this take the time-average real part, giving for the longitudinal or $\hat{x}$ direction

$$v_o \frac{\partial v_2}{\partial x} + \nu v_2 + \frac{v_T^2}{\rho_o} \frac{\partial \rho_2}{\partial x} + \frac{1}{2} \mathrm{Re}\left(v_{1x} \frac{\partial v_{1x}^*}{\partial x} \right) = \eta \left(E_2 + \frac{1}{2} \mathrm{Re}(\underline{v}_1 \times \underline{B}_1^*) \right) \; , \quad (2.33)$$

where v_2, ρ_2 and E_2 are then the second-order changes in the dc state of the system caused by the presence of the RF fields. From Eq. 2.7 and the assumed spatial dependence, Eq. 2.33 may be written as

$$(2k_i v_o + \nu)v_2 = -\frac{v_T^2}{\rho_o} \frac{\partial \rho_2}{\partial x} - \frac{k_i}{2} |v_{1x}|^2 + \eta E_2$$

$$+ \frac{\eta^2\{k_r \nu(\omega - k_r v_o) - k_i[(\omega - k_r v_o)^2 + k_i v_o(k_i v_o + \nu)]\}}{2\omega^2[(\omega - k_r v_o)^2 + (k_i v_o + \nu)^2]} (|E_{1y}|^2 + |E_{1z}|^2) \; . \quad (2.34)$$

Inspection of this result indicates that when the source functions given in Eqs. 2.8 and 2.9 are negative this leads to a dc slowing of the beam velocity since a negative contribution is given to v_2. Hence any carrier mode with transverse field components which are such that $(\omega - k_r v_o) < 0$ acts through collisions to reduce the dc beam velocity so that the beam kinetic power is less when such modes are excited than in their absence.

In the case where the fundamental fields are purely transverse (TEM wave) the above results are still applicable with $\rho_1 = |v_{1x}| = 0$

since $E_{1x} = 0$. It is assumed that the mobile charge carriers drift in the direction of wave propagation. The general dispersion equation may be written as[50]

$$k^2c^2 - \omega^2 + \frac{\omega_p^2(\omega - kv_o)}{\omega - kv_o - j\nu} = 0 \quad , \tag{2.35}$$

where $c = 1/\sqrt{\mu_o \epsilon}$ is less than the free-space velocity of light. Equation 2.35 can be written in the form

$$(\omega - kv_o)(k^2c^2 - \omega^2 + \omega_p^2) = j\nu(k^2c^2 - \omega^2) \quad . \tag{2.36}$$

Thus in the absence of collisions ($\nu = 0$) two stable uncoupled types of modes are obtained. In the presence of collisions the forward traveling waves are coupled and the solution[50] for this case shows that an instability of the resistive type is obtained if $v_o > \omega/k_r \simeq c$. This is then in full agreement with the fact that the beam kinetic power flow as given by Eq. 2.23 (with $E_{1x} = 0$) is negative. Since the electromagnetic power flow is positive when the instability occurs it is clear on physical grounds that it must be that $v_2 < 0$, corresponding to beam slowing, since this is the only source of power available. A problem which arises in this connection is now examined. As no RF bunching is present, the second-order dc current is given by

$$J_2 = \rho_o v_2 + \rho_2 v_o \quad , \tag{2.37}$$

which by a consideration of continuity must be zero. Hence,

$$\rho_2 = -\rho_o \frac{v_2}{v_o} , \tag{2.38}$$

and, in the steady state, any alteration in the dc beam velocity must be accompanied by a dc bunching given by ρ_2. From Poisson's equation the function ρ_2 gives rise to a second-order dc electric field given by

$$\epsilon \nabla \cdot \underline{E}_2 = 2k_i \epsilon E_2 = \rho_2 . \tag{2.39}$$

Use of Eqs. 2.38 and 2.39 in Eq. 2.34 then provides

$$v_2 = \frac{\left[\begin{array}{l} \eta^2 \{ k_r \nu (\omega - k_r v_o) - k_i [(\omega - k_r v_o)^2 \\ + k_i v_o (k_i v_o + \nu)]\} \, (|E_{1y}|^2 + |E_{1z}|^2) \end{array} \right]}{\left[\begin{array}{l} 2\omega^2 \left[(\omega - k_r v_o)^2 + (k_i v_o + \nu)^2 \right] \\ \cdot \left[2k_i v_o \left(1 - \frac{v_T^2}{v_o^2} \right) + \nu + \frac{\omega_p^2}{2k_i v_o} \right] \end{array} \right]} . \tag{2.40}$$

First note that the dispersion relation alone, Eq. 2.36, is independent of v_T. On the other hand the alteration in the dc beam velocity given by Eq. 2.40 is dependent upon v_T, particularly when the growth rate k_i predicted by the dispersion relation is large. Thus, if causality is based on the requirement $v_2 < 0$ in this case, it must be concluded that the dispersion relation alone does not in general give sufficient information to determine causality. As an example, if $v_T \gg v_o$ in Eq. 2.40, then $v_2 > 0$ and the convective instability predicted by the solution of the dispersion relation (together with any causality criteria applied thereto) does not exist.

These results indicate that a basic inconsistency can be found when the dispersion relation alone is used to study the system. This inconsistency is in part removed when it is seen that if the effects of $\dot{\rho}_2$ are significant upon the steady-state system (as predicted by the quasi-linear plane-wave analysis) it is necessary a priori to take into account the self-consistent possibility of a dc density gradient by solving the system starting from an assumed spatial dependence of the fundamental fields, ξ_1, as $\xi_{1o}(x)\ e^{-jkx}$. In addition, the time-averaged or dc quantities become functions of x which are dependent upon the fundamental fields themselves. The problem is then highly nonlinear and cannot be solved by straightforward analytical means alone.

These results indicate that essential nonlinearities can exist in the determination of the stability of a system in the steady state. In general, to satisfy conservation laws, the presence of a convective instability in a system requires that the fundamental fields extract power from the carrier mode. To solve for the stability of a system it is common to assume that no significant dc gradient of density is present so that the plane-wave type of analysis can be performed for the steady-state case. When this is done, however, the plane-wave solution to second order may show that to alter the dc beam velocity to provide source power for the instability a dc density gradient must also exist, thus violating an initial assumption made in the analysis.

A balance of power which can also be obtained is found as follows. From Eqs. 2.38 and 2.39,

$$E_2 = -\frac{\rho_o v_2}{2k_i \epsilon v_o} , \qquad (2.41)$$

so that if amplification occurs ($k_i > 0$), and $v_2 < 0$, therefore

$$\mathrm{Re}(E_2 J_o) = -\frac{\rho_o^2 v_2}{2k_i \epsilon} > 0 . \qquad (2.42)$$

Thus, as in dc Poynting vector calculations related to Joule heating,[70] the viewpoint may be adopted that the external source (e.g., battery) is supplying the power for the amplification in addition to the usual heating effects when collisions are present, the latter being given by $E_o J_o$.

The plane-wave analysis is fully justified if it is assumed that recombination processes are present in the system so that no second-order density gradient is established and hence $\rho_2 = E_2 = 0$. In this case, $J_2 = \rho_o v_2 \neq 0$, and $\mathrm{Re}(\underline{E} \cdot \underline{J})$ provides the term $E_o J_2$ to account for the source of the amplification.

2.4 Power Characteristics of Purely Transverse Modes in a Static Magnetic Field

In this section the case of the carrier modes associated with transverse electromagnetic waves propagating parallel to a dc magnetic field is considered. If the plasma frequency is much less than the angular wave frequency, as is common with electron beams, the associated carrier waves are the slow and fast cyclotron modes. In solid-state plasmas these are the left- and right-hand circularly polarized modes.

2.4.1 Derivation of Electromagnetic Power for Purely Transverse Waves.

Assume a collisionless convected medium (e.g., an electron beam) in a static magnetic field $B_o\hat{x}$ with drift velocity $v_o\hat{x}$. Transverse $(\hat{y} - \hat{z})$ RF perturbations are assumed with space and time dependence $e^{j(\omega t-kx)}$. An expression is derived for the electromagnetic power flow associated with the convected medium from a base of Maxwell's equations and the linearized force equation. With the assumed spatial dependence temporarily suppressed, the following equations are then obtained:

$$\nabla \times \underline{E}_1 = -j\omega \underline{B}_1 \quad , \tag{2.43}$$

$$\nabla \times \underline{B}_1 = \mu_o \underline{J}_1 + j\omega\mu_o \epsilon \underline{E}_1 \tag{2.44}$$

and

$$j\omega \underline{v}_1 + (\underline{v}_o \cdot \nabla)\underline{v}_1 = \eta \underline{E}_1 + \eta \underline{v}_o \times \underline{B}_1 + \eta \underline{v}_1 \times \underline{B}_o \quad , \tag{2.45}$$

where $\nabla = \hat{x}(\partial/\partial x)$ and

η = the charge-to-mass ratio of the single species medium.

From Eqs. 2.1 and 2.3,

$$j\omega v_{1y} + v_o \frac{\partial v_{1y}}{\partial x} = \eta E_{1y} + \omega_c v_{1z} + \frac{\eta v_o}{j\omega} \frac{\partial E_{1y}}{\partial x} \tag{2.46}$$

and

$$j\omega v_{1z} + v_o \frac{\partial v_{1z}}{\partial x} = \eta E_{1z} - \omega_c v_{1y} + \frac{\eta v_o}{j\omega} \frac{\partial E_{1z}}{\partial x} \quad , \tag{2.47}$$

where $\omega_c = \eta B_o$. Introduce rotating coordinates with the following definitions:

$$v_{1+} = v_{1y} + jv_{1z} \;, \qquad v_{1-} = v_{1y} - jv_{1z} \;,$$

$$E_{1+} = E_{1y} + jE_{1z} \;, \qquad E_{1-} = E_{1y} - jE_{1z} \;,$$

$$J_{1+} = \rho_o v_{1+} = J_{1y} + jJ_{1z} \;, \qquad J_{1-} = \rho_o v_{1-} = J_{1y} - jJ_{1z} \;, \quad (2.48)$$

where ρ_o is the charge density of the medium. If Eqs. 2.48 are then applied, Eqs. 2.46 and 2.47 become

$$j(\omega \pm \omega_c)v_{1\pm} + v_o \frac{\partial v_{1\pm}}{\partial x} = \eta E_{1\pm} + \frac{\eta v_o}{j\omega} \frac{\partial E_{1\pm}}{\partial x} \;. \quad (2.49)$$

For the transverse fields of interest, the wave equation is readily obtained from Eqs. 2.43 and 2.44:

$$\frac{\partial^2 \underline{E}_1}{\partial x^2} + \frac{\omega^2}{c^2} \underline{E}_1 = j\omega\mu_o \underline{J}_1 \;. \quad (2.50)$$

From this, employing the definitions of Eq. 2.48, the independent left- and right-hand circularly polarized modes in the transformed system are shown to be

$$\frac{\partial^2 E_{1\pm}}{\partial x^2} + \frac{\omega^2}{c^2} E_{1\pm} = j\omega\mu_o J_{1\pm} \;. \quad (2.51)$$

The expression $\underline{E}_1 \times \underline{B}_1^*$ which is related to the real part of the electromagnetic power flow is now examined. The equation

$$\frac{1}{\mu_o} \underline{E}_1 \times \underline{B}_1^* = \frac{1}{j\omega\mu_o}\left(E_{1y}\frac{\partial E_{1y}^*}{\partial x} + E_{1z}\frac{\partial E_{1z}^*}{\partial x}\right)\hat{x} \tag{2.52}$$

transforms according to Eq. 2.48 as

$$\frac{1}{\mu_o} \underline{E}_1 \times \underline{B}_1^* = \frac{1}{2j\omega\mu_o}\left(E_{1+}\frac{\partial E_{1+}^*}{\partial x} + E_{1-}\frac{\partial E_{1-}^*}{\partial x}\right)\hat{x} \; . \tag{2.53}$$

The power flows are now defined by the vectors

$$\underline{P}_{e\ell\pm} = \frac{1}{2j\omega\mu_o} E_{1\pm}\frac{\partial E_{1\pm}^*}{\partial x}\hat{x} \; . \tag{2.54}$$

To obtain expressions for these power flows in terms of the RF carrier velocities, the following procedure is followed:

a. Apply the operator $\partial/\partial x$ to Eq. 2.49 to obtain

$$j(\omega \pm \omega_c)\frac{\partial v_{1\pm}}{\partial x} + v_o\frac{\partial^2 v_{1\pm}}{\partial x^2} = \eta\frac{\partial E_{1\pm}}{\partial x} + \frac{\eta v_o}{j\omega}\frac{\partial^2 E_{1\pm}}{\partial x^2} \; . \tag{2.55}$$

b. Use Eq. 2.51 now to obtain

$$j(\omega \pm \omega_c)\frac{\partial v_{1\pm}}{\partial x} + v_o\frac{\partial^2 v_{1\pm}}{\partial x^2} = \eta\frac{\partial E_{1\pm}}{\partial x} + \frac{\eta v_o}{j\omega}\left(j\omega\mu_o J_{1\pm} - \frac{\omega^2}{c^2}E_{1\pm}\right) \; . \tag{2.56}$$

c. The first term on the right-hand side of Eq. 2.56 is replaced using Eq. 2.49, and rearranging gives

$$E_{1\pm}\,\eta\left(1-\frac{v_o^2}{c^2}\right) = j\,\frac{v_o}{\omega}\left(\frac{\omega(\omega \pm \omega_c)}{v_o} - \frac{\omega_p^2 v_o}{c^2}\right) v_{1\pm} \mp \frac{\omega_c v_o}{\omega}\,\frac{\partial v_{1\pm}}{\partial x} + j\,\frac{v_o^2}{\omega}\,\frac{\partial^2 v_{1\pm}}{\partial x^2}\,, \tag{2.57}$$

where

$$c^2 = \frac{1}{\mu_o \epsilon} \tag{2.58}$$

and

$$\omega_p^2 = \frac{\eta \rho_o}{\epsilon}\,. \tag{2.59}$$

d. Applying the operator $\partial/\partial x$ to the complex conjugate of Eq. 2.57 and using the resulting expressions, together with Eq. 2.57, gives for the power flows of Eq. 2.54,

$$\underline{P}_{e\ell\pm} = \frac{\hat{x}}{2j\omega\mu_o\eta^2\left(1-\dfrac{v_o^2}{c^2}\right)^2}\left[j\left(\omega \pm \omega_c - \frac{\omega_p^2 v_o^2}{\omega c^2}\right) v_{1\pm} \mp \frac{\omega_c v_o}{\omega}\,\frac{\partial v_{1\pm}}{\partial x} + j\,\frac{v_o^2}{\omega}\,\frac{\partial^2 v_{1\pm}}{\partial x^2}\right]\left[-j\left(\omega \pm \omega_c - \frac{\omega_p^2 v_o^2}{\omega c^2}\right)\frac{\partial v^*_{1\pm}}{\partial x} \mp \frac{\omega_c v_o}{\omega}\,\frac{\partial^2 v^*_{1\pm}}{\partial x^2} - j\,\frac{v_o^2}{\omega}\,\frac{\partial^3 v^*_{1\pm}}{\partial x^3}\right]. \tag{2.60}$$

The spatial dependence assumed earlier is now invoked so that for example $\partial v_{1+}/\partial x = -jkv_{1+}$, $\partial v^*_{1+}/\partial x = jkv^*_{1+}$, etc. In this manner, Eq. 2.60 becomes, where $k = k_r + jk_i$,

$$\underline{P}_{e\ell\pm} = \hat{x}\,\frac{k|v_{1\pm}|^2}{2\omega\mu_o\eta^2\left(1-\dfrac{v_o^2}{c^2}\right)^2}\left|\omega \pm \omega_c - \frac{\omega_p^2 v_o^2}{\omega c^2} \pm \frac{k v_o \omega_c}{\omega} - \frac{k^2 v_o^2}{\omega}\right|^2. \tag{2.61}$$

Equation 2.61 gives the electromagnetic power associated with the carrier stream in the absence of interaction. It will be found that the presence of the interaction affects this result to a negligible degree. Since examination of Eq. 2.61 indicates that the electromagnetic power is positive, to obtain a negative power mode useful in carrier wave-circuit interactions the kinetic power must be negative and exceed the electromagnetic power in magnitude.

2.4.2 Kinetic Power of the Purely Transverse Waves. In the Poynting theorem given by Eq. 2.4, if the definitions of Eq. 2.48 are applied, the theorem is transformed to the following:

$$\nabla \cdot (\underline{P}_{e\ell+} + \underline{P}_{e\ell-}) + \frac{1}{2} j\omega\mu_o (H_{1+}H^*_{1+} + H_{1-}H^*_{1-}) + \frac{1}{2} j\omega\epsilon (E_{1+}E^*_{1+} + E_{1-}E^*_{1-})$$

$$+ \frac{1}{2} (E_{1+}J^*_{1+} + E_{1-}J^*_{1-}) = 0 \quad , \qquad (2.62)$$

where $\underline{P}_{e\ell\pm}$ are given by Eq. 2.54 and

$$H_{1\pm} \triangleq \frac{1}{j\omega\mu_o} \frac{\partial E_{1\pm}}{\partial x} \quad . \qquad (2.63)$$

Also, because the first-order fields were assumed purely transverse, any longitudinal terms, such as $E_{1x}J^*_{1x}$, are zero. Since solution of the dispersion equation indicates that the (+) and (-) modes are uncoupled, for our purposes assume that the (+) mode amplitude is zero, i.e., $|v_{1+}| = 0$, and hence only the (-) mode is present. It will be seen that from the results obtained from the (-) mode the (+) mode results can be derived by the replacement of ω_c by $-\omega_c$. Inspection of Eq. 2.49 provides the following table within the (-) mode framework, if electrons are assumed as the carriers and $\underline{B}_o = \alpha|B_o|\,\hat{x}$, where $\alpha = \pm 1$.

Table 2.1

Carrier-Mode Sign Convention

Sign of ω_c	Solid-State Plasma	Electron Beam ($\omega_p \ll \omega$)
< 0 ($\alpha = +1$)	Helicon-cyclotron mode	Slow-cyclotron mode
> 0 ($\alpha = -1$)	Right-hand polarized mode (RHP)	Fast-cyclotron mode

To proceed with the derivation of the kinetic power the source function $E_{1-}J^*_{1-}$ is derived in Appendix B and verified as a proper physical concept. From Eq. 2.21 and B.10, the carrier-mode kinetic power is given by

$$\underline{P}_{k-} = \frac{1}{2k_i}\,\mathrm{Re}\left(\frac{1}{2}E_{1-}J^*_{1-}\right)\hat{x} = \frac{\alpha\omega\omega_c J_o\,|v_{1-}|^2}{4\eta[(\omega - k_r v_o)^2 + k_i^2 v_o^2]}\,\hat{x}\ , \qquad (2.64)$$

which shows that the modes with $\omega_c < 0$ in Table 2.1 have the negative kinetic power property.

Compare this result with that obtained using coupled-mode theory.[71] The latter technique applied to the pure cyclotron modes by ignoring the RF fields and requiring that the power represented by the square of the cyclotron normal mode amplitudes be conserved gives for the power carried in these cyclotron normal modes,[71]

$$P_{\pm} = \pm\frac{\omega J_o\,|v_{1\pm}|^2}{8\eta\,|\omega_c|}\ . \qquad (2.65)$$

Except for a factor of two related to the normalization used in Poynting's theorem, this is the result obtained from Eq. 2.64 when the cyclotron dispersion ($\omega - kv_o - \omega_c = 0$) is inserted. Thus Eq. 2.64 is the more general result which reduces properly to the simple normal mode result.

As was done in the derivation of the longitudinal space-charge-wave case, Eq. 2.22, both sides of Eq. 2.62 are integrated over the volume of interest, Gauss' theorem is employed, and the real part is taken to obtain the conservation law,

$$\mathrm{Re} \oint (\underline{P}_{e\ell-} + \underline{P}_{k-})_{\mathrm{beam\ +\ circuit}} \cdot d\underline{S} = 0 \ , \tag{2.66}$$

as a surface integral over the volume of interest containing both carrier mode and circuit. As a simple example, assume the circuit losses given by $\mathrm{Re}(P_{k-})_{\mathrm{circuit}}$ are negligible, so that Eq. 2.66 becomes

$$\mathrm{Re} \oint (P_{e\ell-})_{\mathrm{circuit}} \cdot d\underline{S} + \oint \underline{P}_{T-} \cdot d\underline{S} = 0 \ , \tag{2.67}$$

where

$$P_{T-} = \mathrm{Re}(P_{e\ell-} + P_{k-})_{\mathrm{beam}} \ , \tag{2.68}$$

or from Eqs. 2.61 and 2.64,

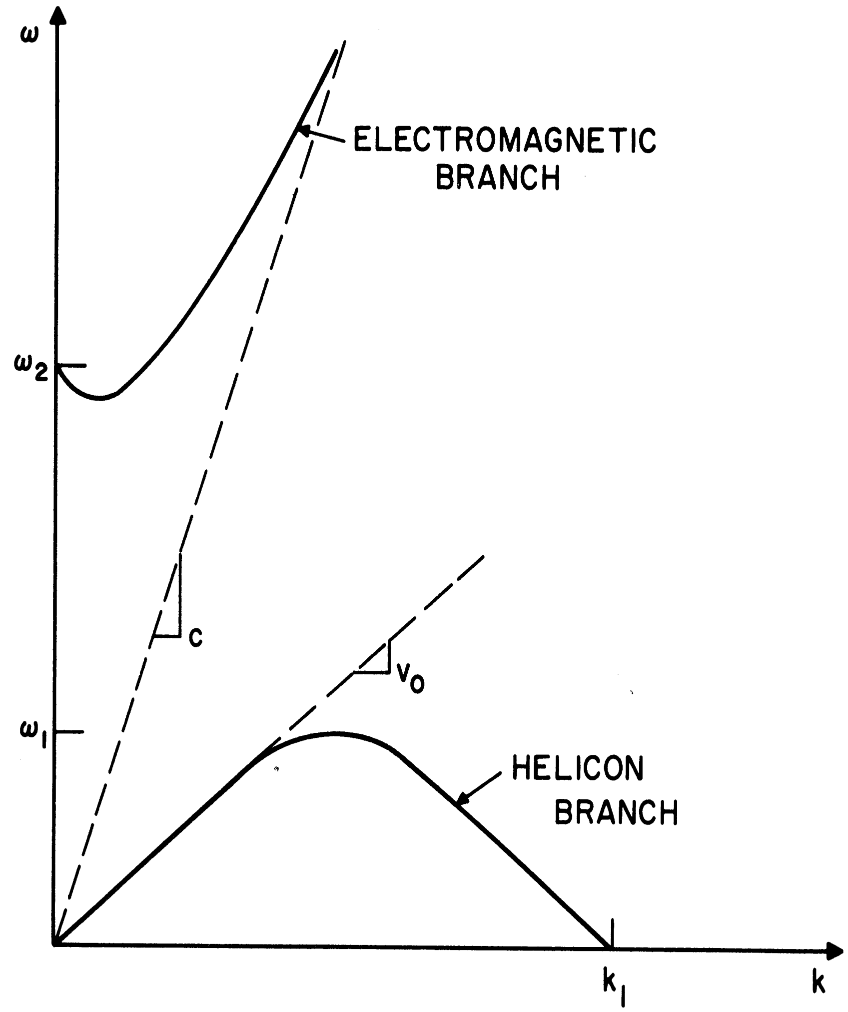
$$P_{T-} = \left[\frac{k_r \left| \omega - \omega_c - \dfrac{\omega_p^2 v_o^2}{\omega_c^2} - \dfrac{k v_o \omega_c}{\omega} - \dfrac{k^2 v_o^2}{\omega} \right|^2}{2\omega\mu_o \eta^2 \left(1 - \dfrac{v_o^2}{c^2}\right)^2} + \frac{\omega\omega_c J_o}{4\eta[(\omega - k_r v_o)^2 + k_i^2 v_o^2]} \right] |v_{1-}|^2 \ . \tag{2.69}$$

Equation 2.67 indicates that for the carrier mode to supply power to the circuit enabling wave amplification to occur it is necessary that $P_{T-} < 0$ corresponding to a negative power mode. For the particular case of a slow-cyclotron mode interacting weakly with a circuit, if the cyclotron mode dispersion ($\omega - kv_o - \omega_c \simeq 0$) is used in Eq. 2.61 it is found that $Re(P_{e\ell -}) \simeq 0$. Thus the slow-cyclotron normal mode has the attribute for instability processes that it carries no electromagnetic power of significance. It is important to note, however, that the entire spectrum in ω-k space of the left-hand polarized mode has the negative kinetic power property, as can be seen in the form of Eq. 2.64.

By inspection of Eq. 2.69 then, any region of carrier-mode presence near $k_r \simeq 0$ is an active mode. For example, with reference to Fig. 2.1 wherein the left-hand polarized mode in the uncoupled state is shown, the region near $\omega \simeq \omega_2$ corresponds to an active mode even though this is part of the electromagnetic branch. This important aspect of the power characteristic will be verified later in this chapter when the two-stream instability is studied.

2.5 Nature of Magnetic Field Effects on the Kinetic Power Flow

As in the collisionally induced source function contribution studied in Section 2.3, since in the hydrodynamic model there is no carrier bunching in the transverse plane, it is asked what physical means account for the transverse contributions given in Eq. 2.64. To see this it is necessary to study the second-order longitudinal carrier dynamics in the steady state. The presence of the transverse fundamental field components gives rise to a nonlinear real Lorentz force given by



$$\omega_2 = \frac{1}{2}\left(\sqrt{\omega_c^2 + \omega_p^2} - |\omega_c|\right) ,$$

$$\omega_1 = \frac{\omega_p^2 v_o^2}{4\ |\omega_c|\ c^2} ,$$

$$k_1 = \frac{4\omega_1}{v_o} ,$$

$$k^2c^2 - \omega^2 + \frac{\omega_p^2(\omega - kv_o)}{\omega - kv_o + |\omega_c|} = 0 .$$

FIG. 2.1 NATURE OF THE LEFT-HAND POLARIZED MODE DISPERSION.

$$F_L = q\,\frac{1}{2}\,\mathrm{Re}(\underline{v}_1 \times \underline{B}_1^* + \underline{v}_1 \times \underline{B}_1)\,\hat{x} \;, \qquad (2.70)$$

where q is the carrier charge. The second part of this force which varies as $(\underline{v}_1 \times \underline{B}_1)$ gives rise to second-order variables which have the form $e^{2j(\omega t - kx)}$. Such variables give no time-average contribution to the major power transfer (which is second order) and can be ignored. Analyzing the remaining time-averaged force in terms of Eqs. 2.48 provides

$$\mathrm{Re}(\underline{v}_1 \times \underline{B}_1^*) = \frac{1}{2}\,\mathrm{Re}\left(\frac{1}{j\omega}\,v_{1-}\,\frac{\partial E_{1-}^*}{\partial x} + \frac{1}{j\omega}\,v_{1+}\,\frac{\partial E_{1+}^*}{\partial x}\right) \;, \qquad (2.71)$$

showing that independent contributions arise from the left- and right-hand polarized modes. The second-order longitudinal force equation, since the fundamental fields are purely transverse and collisions are assumed absent, is given in the steady state by

$$v_o\,\frac{\partial v_2}{\partial x} = \eta E_2 + \eta\left(\mathrm{Re}(\underline{v}_1) \times \mathrm{Re}(\underline{B}_1)\right)_{\hat{x}} \;, \qquad (2.72)$$

which from Eq. 2.71 provides the following separate equation:

$$v_o\,\frac{\partial v_{2\pm}}{\partial x} = \eta E_{2\pm} + \eta\,\mathrm{Re}\left(\frac{1}{2j\omega}\,v_{1\pm}\,\frac{\partial E_{1\pm}^*}{\partial x}\right) \;, \qquad (2.73)$$

where

$$E_2 = \frac{1}{2}\,(E_{2+} + E_{2-}) \qquad (2.74)$$

and

$$v_2 = \frac{1}{2}\,(v_{2+} + v_{2-}) \;. \qquad (2.75)$$

Using Eq. 2.47 for the last term in Eq. 2.73 gives

$$\frac{1}{2j\omega} v_{1-} \frac{\partial E^*_{1-}}{\partial x} = \frac{1}{2J_o} E^*_{1-} J_{1-} + \frac{j(\omega - \omega_c)}{2\eta v_o} |v_{1-}|^2 - \frac{1}{2\eta} v_{1-} \frac{\partial v^*_{1-}}{\partial x} , \qquad (2.76)$$

which from the assumed spatial dependence and Eq. B.10 gives in Eq. 2.73 for the (-) mode

$$2k_i v_o v_{2-} = \eta E_{2-} - \frac{k_i}{2} |v_{1-}|^2 + \frac{k_i \omega \omega_c |v_{1-}|^2}{2[(\omega - k_r v_o)^2 + k_i^2 v_o^2]} . \qquad (2.77)$$

Inspection of this result shows that when the kinetic power given by Eq. 2.64 is negative this leads to a dc slowing of the beam velocity since a negative contribution is given to v_{2-}. Hence the circularly polarized carrier mode with $\omega_c < 0$ acts through the applied magnetic field to reduce the dc beam velocity so that the beam kinetic power is less when such a mode is excited than in its absence.

As no RF bunching is present, the second-order dc current is given by

$$J_{2-} = \rho_o v_{2-} + \rho_{2-} v_o , \qquad (2.78)$$

which from continuity considerations must be zero to conserve particles. Hence,

$$\rho_{2-} = -\rho_o \frac{v_{2-}}{v_o} \qquad (2.79)$$

and the dc beam-velocity alteration is accompanied by second-order dc bunching. A comparison of the fundamental mechanisms for effecting power transfer of the longitudinal vs. the transverse modes can then be made in Fig. 2.2.

For the one-carrier case, the field E_{2-} is given through Poisson's equation by

$$\epsilon\nabla \cdot \underline{E}_{2-} = 2k_i \epsilon E_{2-} = \rho_{2-} \quad . \tag{2.80}$$

Use of Eqs. 2.79 and 2.80 in Eq. 2.77 then provides

$$v_{2-} = \frac{\{\omega\omega_c - [(\omega - k_r v_o)^2 + k_i^2 v_o^2]\}\, |v_{1-}|^2}{4v_o \left(1 + \dfrac{\omega_p^2}{4k_i^2 v_o^2}\right) [(\omega - k_r v_o)^2 + k_i^2 v_o^2]} \quad . \tag{2.81}$$

The relationship of the total kinetic power flow to the result given in Eq. 2.64 is now derived and a fundamental problem discussed. From Tonk's theorem,[72] which is applicable to the present case since collisions are absent, the following may be found:

$$\mathrm{Re} \oint \left(\underline{E} \times \underline{B} + \frac{\underline{v} \cdot \underline{v}}{2\eta} \underline{J}\right) \cdot d\underline{S} = 0 \quad , \tag{2.82}$$

which relates the electromagnetic power flow to the kinetic power flow. In the present case, since $\underline{v} = (v_o + v_{2-})\,\hat{x} + v_{1y}\,\hat{y} + v_{1z}\,\hat{z}$,

$$\mathrm{Re}\left(\frac{\underline{v} \cdot \underline{v}}{2\eta} \underline{J}\right) = \frac{\underline{v} \cdot \underline{v}^*}{2\eta} \underline{J}_o$$

$$= \frac{v_o^2}{2\eta} J_o + \frac{2v_o v_{2-} + \dfrac{|v_{1+}|^2}{2} + \dfrac{|v_{1-}|^2}{2}}{2\eta} J_o \quad . \tag{2.83}$$

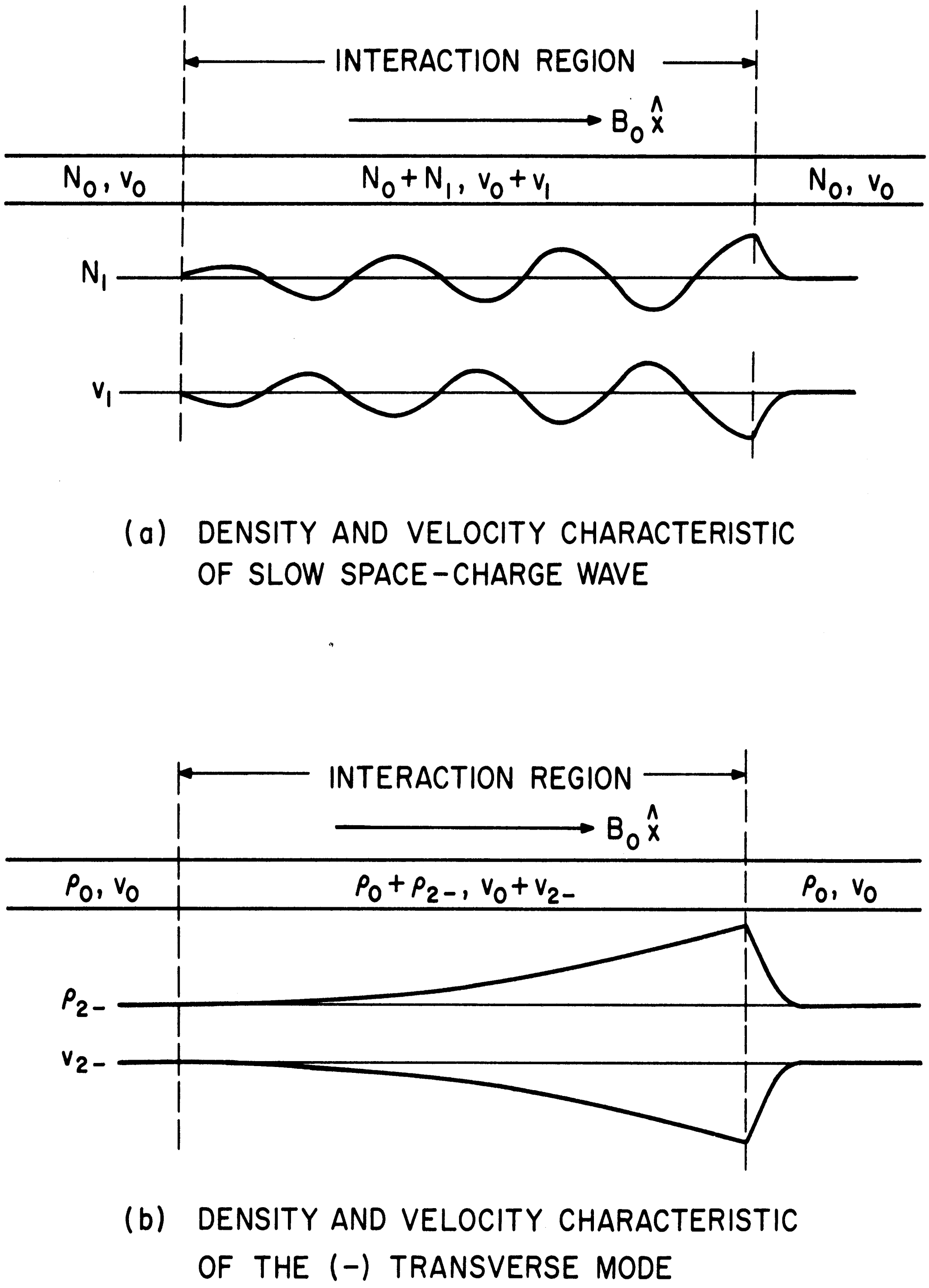


FIG. 2.2 COMPARISON OF LONGITUDINAL AND TRANSVERSE MODES UNDER CONVECTIVE GROWTH.

Equation 2.83 leads to defining the kinetic power as

$$P_{k-}^{T} = \frac{2v_o v_{2-} + \frac{|v_{1-}|^2}{2}}{2\eta} J_o \quad . \tag{2.84}$$

But from Eqs. 2.64 and 2.77 it can be found that

$$P_{k-}^{T} = \frac{1}{2k_i} \operatorname{Re}(E_{2-} J_o + E_{1-} J_{1-}^{*}) \quad . \tag{2.85}$$

Thus the kinetic power expression obtained from Tonk's or Chu's type of formalism is not in agreement with that obtained in Eq. 2.64. The difference lies in the fact that the field E_{2-} satisfies a separate conservation law which from Eq. 2.82 may be found as

$$\operatorname{Re} \oint (\underline{E}_{2-} \times \underline{B}_{os} + \underline{E}_{2-} \cdot \underline{J}_o) \cdot d\underline{S} = 0 \quad , \tag{2.86}$$

where $\underline{B}_{os}$ is the self-magnetic field due to $\underline{J}_o$. The kinetic power form given in Eq. 2.64 must be used for self-consistency with the dispersion equation obtained for the system.

Note that the field E_{2-} is in essence the reaction presented against the beam slowing. In the action of slowing, it must be that dc bunching occurs and via the field E_{2-} setup prevents the slowing. The field E_{2-} can have the aspect of an essential nonlinearity for some systems. For example, when thermal diffusion is introduced into the present model as in Eq. 2.34, Eq. 2.81 then becomes

$$v_{2-} = \frac{\{\omega\omega_c - [(\omega - k_r v_o)^2 + k_i^2 v_o^2]\} \, |v_{1-}|^2}{4v_o\left(1 + \dfrac{\omega_p^2}{4k_i^2 v_o^2} - \dfrac{v_T^2}{v_o^2}\right)[(\omega - k_r v_o)^2 + k_i^2 v_o^2]} \, . \qquad (2.87)$$

Assume that the linear dispersion equation for the system (which is independent of v_T in the hydrodynamic model) is solved for and a growing wave ($k_i > 0$) is obtained. Clearly $v_{2-} < 0$ is necessary for this solution to be permissible since this is the only source of power available to balance the positive electromagnetic power growth. However, if the solution is such that $v_T > v_o[1 + (\omega_p^2)/(4k_i^2 v_o^2)]^{1/2}$, Eq. 2.87 shows that $v_{2-} > 0$. Thus for those parameters used in the system the solution obtained is not self-consistent since $v_{2-} > 0$ indicates that in the steady state there is no beam slowing. The salient point is that in such occurrences the linear dispersion relation with homogeneous dc charge densities breaks down from the viewpoint of conservation of power and a more elaborate nonlinear analysis is required.

Previous investigators have stated that the power properties of the transverse modes in a longitudinal magnetic field depend upon the second-order RF electric field[50] or upon a transverse gradient of the fundamental RF electric field.[66] The former of these explanations is invalid since the second-order RF electric field has zero time average and hence does not contribute to the conservation of power to second order. The latter explanation, based upon gradients which are not in general present and are not included in the dispersion relation for the system, was invoked as a necessity since it is known that the $(\underline{v}_1 \times \underline{B}_1)$ force cannot alter the total carrier kinetic energy but only transfer energy between longitudinal and transverse forms. The important point however is that the $(\underline{v}_1 \times \underline{B}_1)$ force can alter the flow rate of the total carrier kinetic energy out of the surface of interest. It is because of this alteration of the flow rate from v_o to $(v_o + v_{2-})$ caused by the second-order time-averaged Lorentz force that enables the fundamental RF fields to extract power from the dc carrier motion without affecting its total kinetic energy. This becomes particularly clear in the following case.

Assume the system is such that, in the steady state, recombination processes are present which prevent any second-order density gradient ρ_{2-} from forming so that $E_{2-} = 0$. In this case, which is the only case which can be accurately evaluated based on the plane-wave analysis, the kinetic power forms given in Eqs. 2.64 and 2.85 are identical and if Eq. 2.83 is used the kinetic power may be written, since $\rho_1 = \rho_{2-} = 0$, as

$$P_{k-} = \operatorname{Re}\left(N_o \frac{\underline{v} \cdot \underline{v}}{2m} \underline{v}\right) , \tag{2.88}$$

where m is the carrier effective mass. By definition the total carrier kinetic energy is

$$E_k \triangleq N_o \frac{\underline{v} \cdot \underline{v}}{2m} , \tag{2.89}$$

so that the net kinetic power flow out of the surface of interest is

$$\oint \underline{P}_{k-} \cdot d\underline{S} = \operatorname{Re} \oint E_k \underline{v} \cdot d\underline{S} . \tag{2.90}$$

By selecting a cylinder encompassing the interaction region whose axis is longitudinal, the nature of the carrier dynamics shows that on the cylindrical surface $\underline{v} \cdot d\underline{S} = 0$ (i.e., no particles escape in the transverse direction), whereas at the end sections a net contribution arises since there $\underline{v} \cdot d\underline{S} = [v_o + v_{2^-}(x)]\, \hat{x} \cdot d\underline{S}$. Thus even if the function E_k is constant, if $v_{2^-} < 0$ the integral given in Eq. 2.90 is negative, indicating a power source for the instability.

In the alternate limit, where no recombination processes are present, the field E_{2^-} which must then be present can alter the carrier kinetic energy and thus account for the power transfer. Again, in this case, since a density gradient ρ_{2^-} is present, an assumption used in the steady-state plane-wave analysis is violated so that strictly speaking a more complete nonlinear analysis should be performed. For such systems, although $\rho_{2^-} \ll \rho_o$, the quasi-linear theory shows that the effect of ρ_{2^-} is significant in general in the power conservation process. It is reasonable to assume, this being the case, that any study of such systems <u>in the steady state</u> must include the dc density gradient in the analysis of instability phenomena.

The nature of the second-order field E_{2^-} shows that this property of the wave propagation applied to solid-state materials may be used to advantage in the measurement of carrier effective mass or in the detection of electromagnetic power. This aspect is presented in Appendix C.

<u>2.6 The Two-Stream Transverse Instability in a Longitudinal Magnetic Field</u>

It is desired to verify some of the results obtained in connection with the power characteristics of the circularly polarized waves. A useful case which will be solved explicitly under small-signal conditions is that of the two-stream interaction. In this system a second carrier, with superscript (2), is used as the circuit and the primary, active carrier is assigned superscript (1).

The quasi-linear theory is first used to study the second-order effects when no recombination processes are present. In this case, from continuity on each stream, Eq. 2.79 gives

$$\rho_{2-}^{(s)} = -\frac{\rho_o^{(s)} v_{2-}^{(s)}}{v_o^{(s)}} \quad ; \quad s = 1,2 \ . \tag{2.91}$$

From the assumed spatial dependence and Eq. 2.91, Poisson's equation for the second-order dc field, $\epsilon\nabla \cdot \underline{E}_{2-} = \sum_s \rho_{2-}^{(s)}$, becomes

$$2k_i \epsilon E_{2-} = -\left(\frac{\rho_{o1}}{v_{o1}}\right) v_{2-}^{(1)} - \left(\frac{\rho_{o2}}{v_{o2}}\right) v_{2-}^{(2)} \ , \tag{2.92}$$

where in general subscripts are used to delineate the carrier species in the case of zeroth-order quantities. Superscripts are used for reasons of clarity in the case of second-order quantities.

Employing Eq. 2.77 for each carrier gives

$$2k_i v_{os} v_{2-}^{(s)} = \eta_s E_{2-} + \frac{k_i \left|v_{1-}^{(s)}\right|^2 \left[\omega\omega_{cs} - (\omega - k_r v_{os})^2 - k_i^2 v_{os}^2\right]}{4v_{os}\left[(\omega - k_r v_{os})^2 + k_i^2 v_{os}^2\right]} \quad ; \quad s = 1,2 \ . \tag{2.93}$$

A further relation is provided by the first-order force equation, viz.,

$$E_{1-} = \frac{j(\omega - kv_{o1} - \omega_{c1})\omega v_{1-}^{(1)}}{\eta_1(\omega - kv_{o1})} = \frac{j(\omega - kv_{o2} - \omega_{c2})\omega v_{1-}^{(2)}}{\eta_2(\omega - kv_{o2})} \ , \tag{2.94}$$

from which it may be found that

$$|v_{1-}^{(2)}|^2 = ff^* |v_{1-}^{(1)}|^2 \quad , \tag{2.95}$$

where

$$f \triangleq \frac{\eta_2(\omega - kv_{o2})(\omega - kv_{o1} - \omega_{c1})}{\eta_1(\omega - kv_{o1})(\omega - kv_{o2} - \omega_{c2})} \quad . \tag{2.96}$$

Substituting Eqs. 2.92 and 2.95 into Eq. 2.93 gives the following result:

$$v_{2-}^{(1)} = \frac{|v_{1-}^{(1)}|^2 \; v_{o1}\left[(4k_i^2 v_{o2}^2 + \omega_{p2}^2)\gamma_1\beta_2 - \omega_{p2}^2\left(\frac{\eta_2}{\eta_1}\right)\gamma_2\beta_1\right]}{8\beta_2[(\omega - k_r v_{o1})^2 + k_i^2 v_{o1}^2][v_{o1}^2(4k_i^2 v_{o2}^2 + \omega_{p2}^2) + \omega_{p1}^2 v_{o2}^2]} \quad , \tag{2.97}$$

where

$$\gamma_s = \omega\omega_{cs} - (\omega - k_r v_{os})^2 - k_i^2 v_{os}^2 \tag{2.98}$$

and

$$\beta_s = (\omega - k_r v_{os} - \omega_{cs})^2 + k_i^2 v_{os}^2 \quad ; \qquad s = 1,2 \quad . \tag{2.99}$$

By a symmetry argument, the function $v_{2-}^{(2)}$ is readily obtained by the replacement $\binom{1 \to 2}{2 \to 1}$ throughout Eq. 2.97. Inspection of Eq. 2.97 shows that, in the absence of recombination processes, in the steady state the quasi-linear theory predicts a strong influence on the second-order carrier dynamics of the primary carrier by the secondary carriers and vice versa. As an example, consider the system with $\omega_{c1} < 0$, $\omega_{c2} > 0$, $(v_{o1}, v_{o2}) > 0$ and $\eta_1\eta_2 < 0$ so that based on the purely linear theory of Eq. 2.64 the mode of carrier (1) is active and that of carrier (2), passive. Inspection of Eq. 2.97 shows that if the interaction occurs with the right-hand polarized mode of carrier (2) near $(\omega - k_r v_{o2}) \simeq 0$ the situation can easily occur wherein $v_{2-}^{(1)} > 0$. In this case it is clear, since $v_{2-}^{(1)} < 0$ must occur to supply power for the interaction, that in this region any convective instability predicted by the linear dispersion relation alone has dubious validity since these second-order effects which affect the power conservation are ignored.

The quasi-linear theory is now used to study the two-stream transverse interaction for systems which are a priori known to possess recombination and generation mechanisms such that in the steady state no dc density gradients are permitted.

For the two-carrier case, since the circuit is well defined, the electromagnetic power flow for the system can be obtained. Equation 2.56 becomes, in the (-) mode formulation, for carrier (1),

$$j(\omega - \omega_{c1}) \frac{\partial v_{1-}^{(1)}}{\partial x} + v_{o1} \frac{\partial^2 v_{1-}^{(1)}}{\partial x^2} = \eta_1 \frac{\partial E_{1-}}{\partial x} - \eta_1 \frac{\omega v_{o1}}{jc^2} E_{1-} + \eta_1 v_{o1} \mu_o \cdot \sum_{s=1,2} \rho_{os} v_{1-}^{(s)} \quad . \tag{2.100}$$

Use Eq. 2.49 to replace the first term on the right-hand side and rearrange to give

$$\eta_1 \left(1 - \frac{v_{o1}^2}{c^2}\right) E_{1-} = j\left(\omega - \omega_{c1} - \frac{\omega_{p1}^2 v_{o1}^2}{\omega c^2}\right) v_{1-}^{(1)} + \frac{\omega_{c1} v_{o1}}{\omega} \frac{\partial v_{1-}^{(1)}}{\partial x} + j \frac{v_{o1}^2}{\omega} \frac{\partial^2 v_{1-}^{(1)}}{\partial x^2} - j \frac{\eta_1 \omega_{p2}^2 v_{o1}^2}{\eta_2 \omega c^2} v_{1-}^{(2)} \quad . \tag{2.101}$$

Apply the operator $\partial/\partial x$ to the complex conjugate of Eq. 2.101 and use the result obtained in Eq. 2.54 to provide, together with the definitions of Eqs. 2.95 and 2.96,

$$P_{\ell -} = \frac{k^* \left|v_{1-}^{(1)}\right|^2}{2\omega\mu_o\eta_1^2\left(1 - \frac{v_{o1}^2}{c^2}\right)^2} \left| \omega - \omega_{c1} - \frac{\omega_{p1}^2 v_{o1}^2}{\omega c^2} - \frac{kv_{o1}\omega_{c1}}{\omega} - \frac{k^2 v_{o1}^2}{\omega} - \frac{\omega_{p2}^2 v_{o1}^2 (\omega - kv_{o2})(\omega - kv_{o1} - \omega_{c1})}{\omega c^2 (\omega - kv_{o1})(\omega - kv_{o2} - \omega_{c2})} \right|^2 . \quad (2.102)$$

The dispersion relation for the two-carrier system can be readily obtained as

$$k^2c^2 - \omega^2 + \frac{\omega_{p1}^2(\omega - kv_{o1})}{\omega - kv_{o1} - \omega_{c1}} + \frac{\omega_{p2}^2(\omega - kv_{o2})}{\omega - kv_{o2} - \omega_{c2}} = 0 . \quad (2.103)$$

Use of this relation to replace ω_{p2}^2 in Eq. 2.102 leads to the form

$$\mathrm{Re}(P_{\ell -}) = \frac{k_r\omega \left|v_{1-}^{(1)}\right|^2}{2\mu_o\eta_1^2} \left| \frac{\omega - kv_{o1} - \omega_{c1}}{\omega - kv_{o1}} \right|^2 . \quad (2.104)$$

By a symmetry argument this result is unchanged if the replacement $(1 \rightarrow 2)$ is made in the carrier designation on the right-hand side. Equation 2.104 indicates that the cyclotron normal modes $(\omega - kv_{os} - \omega_{cs} \simeq 0)$ in the two-carrier system carry power almost entirely in electrokinetic form. Equations 2.64 and 2.104 used in the definition of total power flow given in Eq. 2.68 show that the mode associated with carrier (1) is active if

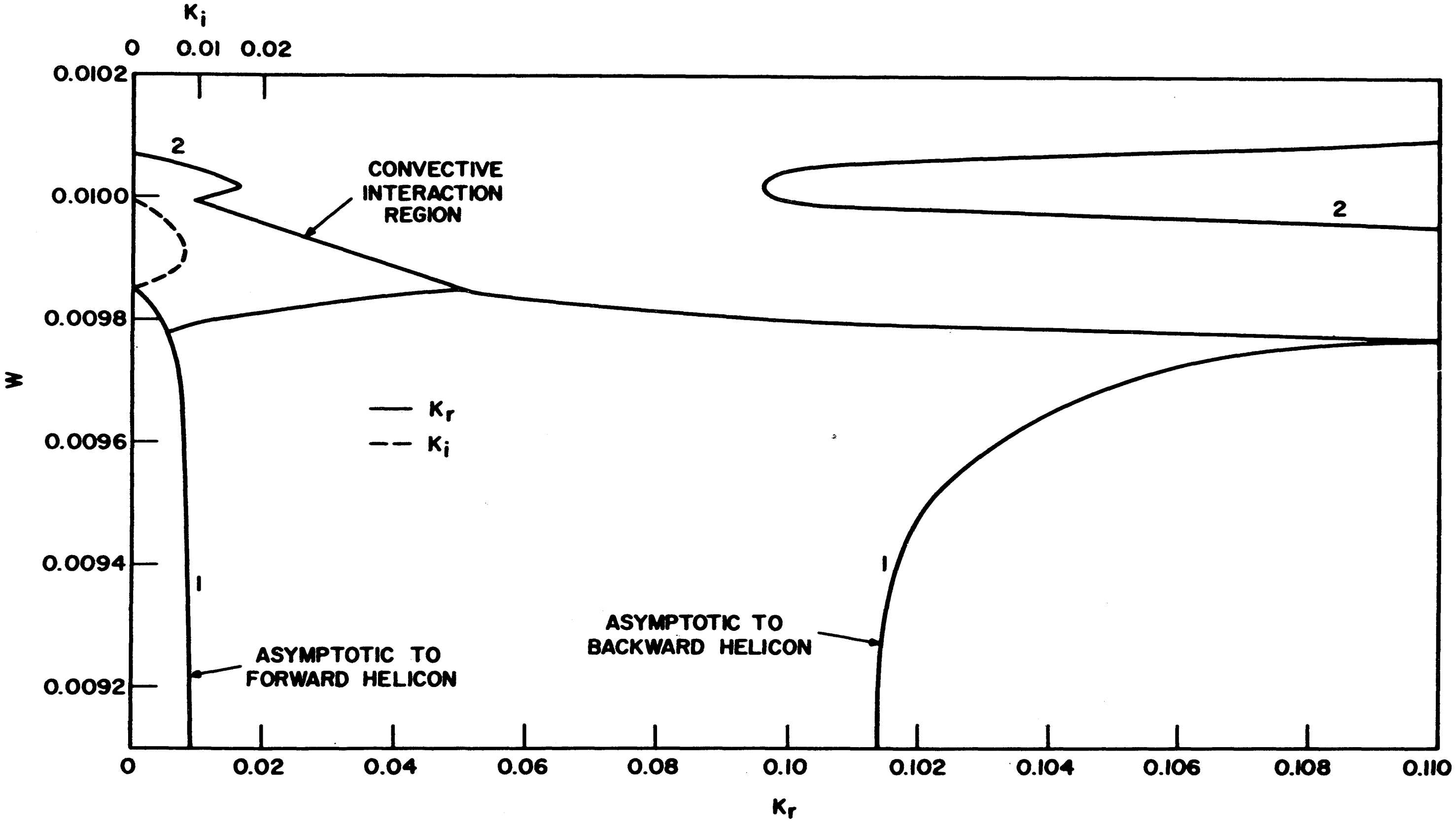
$$k_r < \frac{v_{o1}\,|\omega_{c1}|\,\omega_{p1}^2}{2c^2\,|\omega - kv_{o1} - \omega_{c1}|^2} , \quad (2.105)$$

where $\omega_{c1} < 0$ is assumed.

To verify that these power results are in agreement with the instability characteristics of the two-stream system, a computer solution of the dispersion relation, Eq. 2.103, is undertaken in sample regions. In Fig. 2.3 the interaction of a passive fast-cyclotron mode on carrier (2) with the forward and backward helicon branches of the primary carrier (1) is shown. A strong convective instability occurs in this case which severely distorts the uncoupled mode dispersion. A similar case in which coupling occurs with the backward helicon branch alone is shown in Fig. 2.4. The interaction of the normal cyclotron modes is shown in Fig. 2.5. In this region the coupling between the modes becomes vanishingly small[73] (p. 109) for any reasonable carrier densities so that no instability results. Equation 2.105 indicates that the negative power property is not necessarily limited to the region $k_r v_{o1} > \omega$. In particular, from the dispersion diagram of the uncoupled left-hand polarized mode for a single carrier, Fig. 2.1, the electromagnetic branch near $k_r \simeq 0$ may be examined utilizing the two-stream interaction. As the example of Fig. 2.6 indicates convective growth indeed occurs, although by the nature of the dispersion in this region ($k_r < \omega/c$) the interaction is limited to large wavelengths which in turn invokes the necessity of including boundary conditions and self-magnetic field effects for any finite system.

2.7 Effects of Collisions on the Mode Kinetic Power Properties in a Longitudinal Magnetic Field

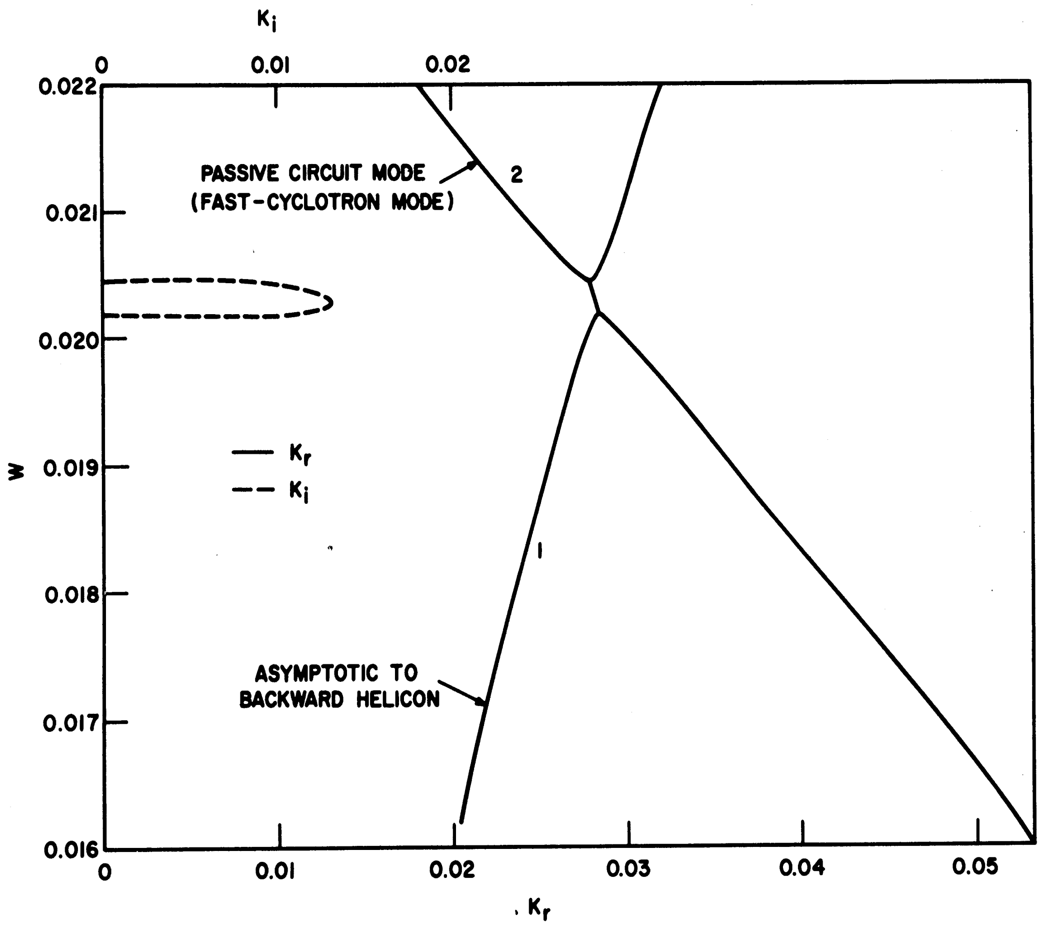
Collisions can be taken into account via the replacement $\omega \rightarrow \omega - j\nu$ in the force equation, Eq. 2.45. When this is done Eq. 2.64 for the carrier kinetic power in the (-) mode formulation is given by



$\omega_{p1} = 10\,|\omega_{c1}|$, $\omega_{p2} = 0.01\,\omega_{p1}$, $\omega_{c1} < 0$, $\omega_{c2} = -0.01\,\omega_{c1}$, $v_{o1} > 0$,

$v_{o2} = -0.001\,v_{o1}$, $\left(\frac{v_{o1}}{c}\right)^2 = 0.001$, $K_r = \frac{k_r v_{o1}}{|\omega_{c1}|}$, $W = \frac{\omega}{|\omega_{c1}|}$, $K_i = \frac{k_i v_{o1}}{|\omega_{c1}|}$.

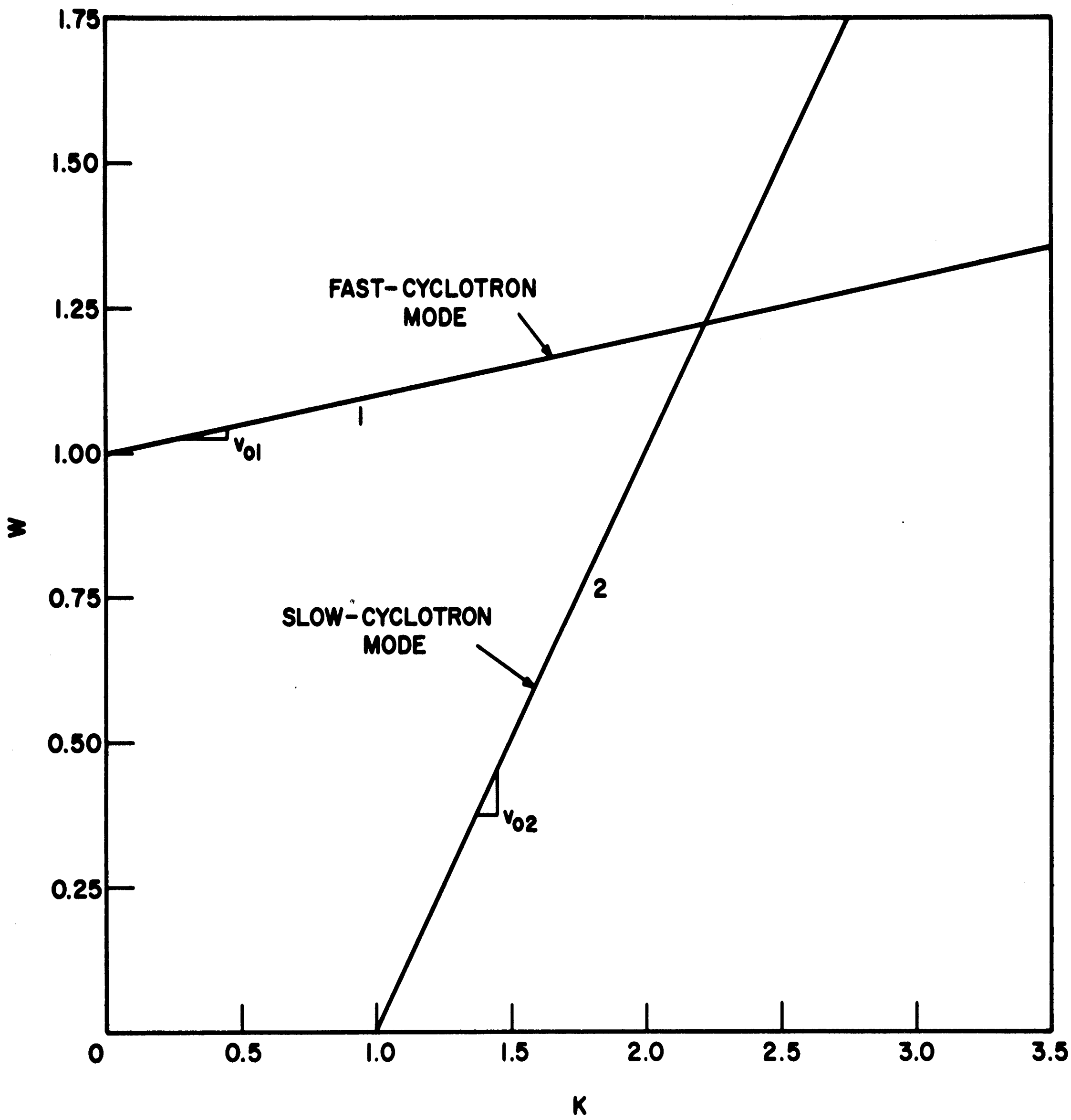
FIG. 2.3 UNSTABLE CONVECTIVE INTERACTION ASSOCIATED WITH HELICON REGION.



$\omega_{p1} = 10\,|\omega_{c1}|$, $\omega_{p2} = 0.001\,\omega_{p1}$, $v_{01} > 0$, $v_{02} = -0.167\,v_{01}$, $\omega_{c1} < 0$,

$\omega_{c2} = -0.025\,\omega_{c1}$, $\left(\frac{v_{01}}{c}\right)^2 = 0.001$, $W = \frac{\omega}{|\omega_{c1}|}$, $K = \frac{kv_{01}}{|\omega_{c1}|}$.

FIG. 2.4 CONVECTIVE INTERACTION WITH BACKWARD HELICON.

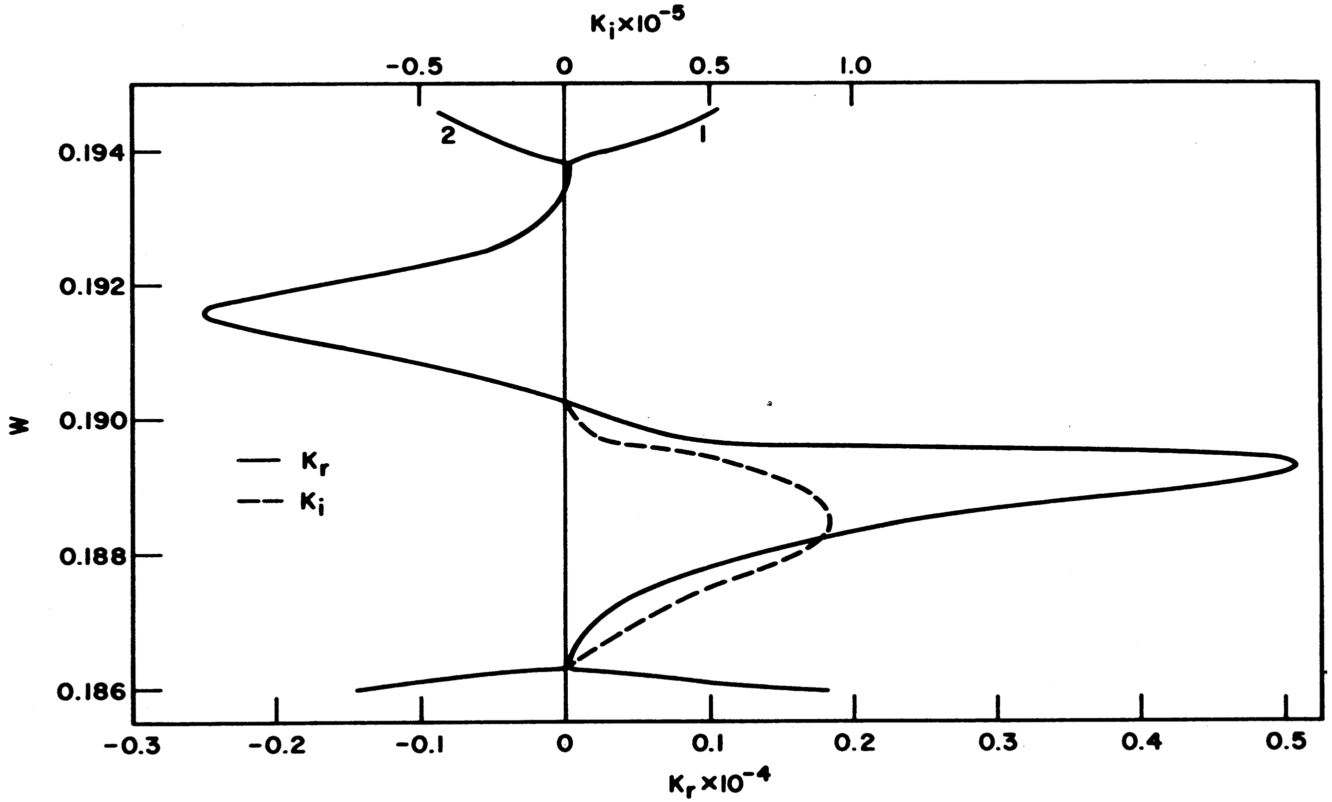


$$\omega_{p1} = |\omega_{c1}| = |\omega_{c2}| = \omega_{p2} , \quad v_{o1} = 0.1\, v_{o2} ,$$

$$\left(\frac{v_{o2}}{c}\right)^2 = 0.16 \times 10^{-4} , \quad K = \frac{kv_{o2}}{|\omega_{c2}|} , \quad W = \frac{\omega}{|\omega_{c2}|} ,$$

$$\omega_{c2} < 0 , \quad \omega_{c1} > 0 .$$

FIG. 2.5 STABLE INTERACTION OF SLOW- AND FAST-CYCLOTRON COSTREAMING MODES.



$\omega_{p1} = 0.5\,|\omega_{c1}|$, $\omega_{p2} = 0.01\,\omega_{p2}$, $\left(\frac{v_{o1}}{c}\right)^2 = 10^{-3}$, $v_{o2} = -0.01$, $v_{o1} > 0$,

$\omega_{c1} < 0$, $\omega_{c2} = -0.192\,|\omega_{c1}|$, $K = \frac{kv_{o1}}{|\omega_{c1}|}$, $W = \frac{\omega}{|\omega_{c1}|}$.

FIG. 2.6 TWO-STREAM TRANSVERSE INTERACTION WITH THE ELECTROMAGNETIC BRANCH OF THE HELICON SPECTRUM.

$$P_{k-}(\nu) = \frac{1}{2k_i} \operatorname{Re}\left(\frac{1}{2} E_{1-} J_{1-}^{*}\right) = \frac{\omega J_o |v_{1-}|^2 \left(\omega_c + \frac{\nu}{k_i v_o}(\omega - k_r v_o)\right)}{4\eta[(\omega - k_r v_o)^2 + k_i^2 v_o^2]} .$$

(2.106)

For the case where generation and recombination processes prevent the formation of any dc density gradients in the steady state, $\rho_{2-} = E_{2-} = 0$, so that in the presence of collisions Eq. 2.73 becomes

$$v_o \frac{\partial v_{2-}}{\partial x} + \nu v_{2-} = \eta \operatorname{Re}\left(\frac{1}{2j\omega} v_{1-} \frac{\partial E_{1-}^{*}}{\partial x}\right) , \tag{2.107}$$

which leads to

$$v_{2-} = \frac{\{\nu[k_r(\omega - k_r v_o) - k_i^2 v_o] + k_i \omega\omega_c - k_i[(\omega - k_r v_o)^2 + k_i^2 v_o^2]\} |v_{1-}|^2}{2(2k_i v_o + \nu)[(\omega - k_r v_o)^2 + k_i^2 v_o^2]} .$$

(2.108)

Comparison of Eqs. 2.40 and 2.108 shows that the general effect of collisions is to provide a power source for a convective instability if $(\omega - k_r v_o) < 0$ by inducing a component of dc beam slowing.

2.8 Kinetic Power Properties of the Hybrid Mode

A hybrid mode, which has characteristics of both the longitudinal space-charge wave and the circularly polarized electromagnetic wave, is obtained when propagation occurs at right angles to the direction of an applied static magnetic field. With the applied magnetic field in the $\hat{z}$ direction and wave vector $\underline{k}$ in the $\hat{x}$ direction, the hybrid extraordinary wave has the field structure (E_{1x}, E_{1y}). The remaining uncoupled ordinary wave with polarization E_{1z} has a dispersion relation independent of the magnetic field and is thus equivalent to the purely transverse mode studied in Section 2.3.

For the hybrid mode, since no simple separation into uncoupled modes occurs as in the longitudinal magnetic field case, in general it will be found that explicit knowledge of the dispersion equation for the interacting system is required. Assume it is known then that the dispersion equation describing the system is given by

$$\tilde{D}(\omega, k) \cdot \begin{pmatrix} E_{1x} \\ E_{1y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} , \qquad (2.109)$$

where $\tilde{D}(\omega,k)$ is a two by two matrix. From Eq. 2.109 the following may be found:

$$\frac{E_{1y}}{E_{1x}} = - \frac{D_{11}}{D_{12}} = - \frac{D_{21}}{D_{22}} \triangleq P(\omega, k) \quad , \qquad (2.110)$$

where the D_{ij}'s are the elements of $\tilde{D}(\omega,k)$.

The cold-plasma hydrodynamic force equations, with drift velocity assumed in the direction of wave propagation only, are given for the sth carrier species from Eq. 2.45 by

$$j(\omega - kv_{os})v_{1x}^{(s)} - \omega_{cs}v_{1y}^{(s)} = \eta_s E_{1x} \qquad (2.111)$$

and

$$j(\omega - kv_{os})v_{1y}^{(s)} + \omega_{cs}v_{1x}^{(s)} = \eta_s \left(1 - \frac{kv_{os}}{\omega}\right) E_{1y} \quad . \qquad (2.112)$$

Use of Eq. 2.110 in Eqs. 2.111 and 2.112 provides the relations

$$v_{1x}^{(s)} = \frac{\eta_s E_{1x} \left[j(\omega - kv_{os}) + \omega_{cs} \left(1 - \frac{kv_{os}}{\omega}\right) P(\omega, k) \right]}{\omega_{cs}^2 - (\omega - kv_{os})^2} \qquad (2.113)$$

and

$$v_{1y}^{(s)} = \frac{\eta_s E_{1x}\left(-\omega_{cs} + j\frac{1}{\omega}(\omega - kv_{os})^2 P(\omega,k)\right)}{\omega_{cs}^2 - (\omega - kv_{os})^2} . \tag{2.114}$$

The kinetic power flow of the sth carrier hybrid mode is given by

$$P_k^{(s)} = \frac{1}{2k_i} \mathrm{Re}\left(E_{1x}J_{1x}^{(s)*} + E_{1y}J_{1y}^{(s)*}\right) , \tag{2.115}$$

where $J_{1y}^{(s)} = \rho_{os}v_{1y}^{(s)}$ and from Eqs. 2.11 and 2.12

$$J_{1x}^{(s)*} = \frac{\omega\rho_{os}v_{1x}^{(s)*}}{\omega - k^* v_{os}} . \tag{2.116}$$

Equations 2.110 through 2.116 enable Eq. 2.115 to be written as

$$P_k^{(s)} = \frac{\omega_{ps}^2 \epsilon \, |E_{1x}|^2 \, v_{os}(\omega - k_r v_{os}) \left(\omega + 2P_i\omega_{cs} + \frac{|P|^2}{\omega}\omega_{cs}^2\right)}{[\omega_{cs}^2 - (\omega - k_r v_{os})^2 + k_i^2 v_{os}^2]^2 + 4k_i^2 v_{os}^2(\omega - k_r v_{os})^2} , \tag{2.117}$$

where $P(\omega,k) = P_r + jP_i$ has been used. Note that Eq. 2.117 reduces properly to the result found in Eq. 2.24 when $\omega_{cs} = 0$.

As the simplest example of the use of Eq. 2.117 consider a single carrier stream interacting weakly with an external circuit. In this case, for the present purposes, the dispersion equation can be approximated by that for the isolated single stream alone. It can readily be shown using Eqs. 2.111 and 2.112 together with Maxwell's equations that the elements of this dispersion relation are

$$D_{11} = -\frac{\omega^2}{c^2}\left(1 + \frac{\omega_p^2}{\omega_c^2 - (\omega - kv_o)^2}\right) , \tag{2.118}$$

$$D_{12} = -D_{21} = \frac{j\omega\omega_c\omega_p^2}{c^2[\omega_c^2 - (\omega - kv_o)^2]} \tag{2.119}$$

and

$$D_{22} = k^2 - \frac{\omega^2}{c^2} - \frac{\omega_p^2(\omega - kv_o)^2}{c^2[\omega_c^2 - (\omega - kv_o)^2]} \, . \tag{2.120}$$

Using Eqs. 2.118 and 2.119 in Eq. 2.110 provides

$$P(\omega,k) = \frac{\omega}{\omega_c\omega_p^2}\left(2k_iv_o(\omega - k_rv_o) + j[(\omega - k_rv_o)^2 - (\omega_c^2 + \omega_p^2) - k_i^2v_o^2]\right) . \tag{2.121}$$

This result when used in Eq. 2.117 then gives

$$P_k = \frac{\left[\omega_p^2\epsilon\,|E_{1x}|^2\,\omega v_o(\omega - k_rv_o)\left[\left(1 + \frac{[(\omega - k_rv_o)^2 - (\omega_c^2 + \omega_p^2) - k_i^2v_o^2]}{\omega_p^2}\right)^2 + \frac{4k_i^2v_o^2}{\omega_p^4}(\omega - k_rv_o)^2\right]\right]}{[\omega_c^2 - (\omega - k_rv_o)^2 + k_i^2v_o^2]^2 + 4k_i^2v_o^2(\omega - k_rv_o)^2} \, . \tag{2.122}$$

Inspection of this result shows clearly that $(\omega - k_rv_o) < 0$ is required to obtain the negative kinetic power property for this case. That this result is in general true in the absence of collisions and thermal diffusion can be seen when Eq. 2.117 is written in the form

$$P_k^{(s)} = \frac{\omega_{ps}^2\epsilon\,|E_{1x}|^2\,v_{os}(\omega - k_rv_{os})[(\omega + P_i\omega_{cs})^2 + P_r^2\omega_{cs}^2]}{\omega\left([\omega_{cs}^2 - (\omega - k_rv_{os})^2 + k_i^2v_{os}^2]^2 + 4k_i^2v_{os}^2(\omega - k_rv_{os})^2\right)} \, , \tag{2.123}$$

which shows directly that $(\omega - k_rv_{os}) < 0$ is required for $P_k^{(s)} < 0$.

As in the case of the purely transverse modes studied, the electromagnetic power flow associated with the hybrid mode cannot be neglected. In the present model, this power flow can be approximated as

$$P_{e\ell} = \frac{1}{2}\,\mathrm{Re}(E_{1y}B_{1z}^{*}) = \frac{k_r}{2\omega}\,|P|^2\,|E_{1x}|^2 \quad . \qquad (2.124)$$

Since the total power flow, $P_{e\ell} + P_k^{(s)}$, must be negative for the sth carrier hybrid mode to be active, Eqs. 2.123 and 2.124 then provide the limitation

$$k_r < \frac{2\omega_{ps}^2\,\epsilon v_{os}(k_r v_{os} - \omega)[(\omega + P_i\omega_{cs})^2 + P_r^2\omega_{cs}^2]}{|P|^2\left([\omega_{cs}^2 - (\omega - k_r v_{os})^2 + k_i^2 v_{os}^2]^2 + 4k_i^2 v_{os}^2(\omega - k_r v_{os})^2\right)} \qquad (2.125)$$

for the mode to be active, where $v_{os} > \omega/k_r$ is assumed.

2.9 Effects of Collisions and Thermal Diffusion on the Kinetic Power Properties of the Hybrid Mode

For completeness in the study of the basic carrier modes the effects of collisions and thermal diffusion on the hybrid mode kinetic power properties are studied, although in this case the results are particularly lengthy.

Collisions and thermal diffusion are introduced into Eqs. 2.111 and 2.112 in the standard manner, whereby Eqs. 2.113 and 2.114 then become

$$v_{1x}^{(s)} = \frac{\eta_s E_{1x}\left(j(\omega - kv_{os} - jv_s) + \frac{\omega_{cs}}{\omega}(\omega - kv_{os})P(\omega,k)\right)}{\omega_{cs}^2 - (\omega - kv_{os} - jv_s)^2 + \frac{k^2 v_{Ts}^2(\omega - kv_{os} - jv_s)}{\omega - kv_{os}}} \qquad (2.126)$$

and

$$v_{1y}^{(s)} = \frac{\eta_s E_{1x}\left(-\omega_{cs} + j[(\omega - kv_{os})(\omega - kv_{os} - jv_s) - k^2 v_{Ts}^2]\frac{P(\omega,k)}{\omega}\right)}{\omega_{cs}^2 - (\omega - kv_{os} - jv_s)^2 + \dfrac{k^2 v_{Ts}^2(\omega - kv_{os} - jv_s)}{\omega - kv_{os}}} , \tag{2.127}$$

from which Eq. 2.115 corresponding to the kinetic power flow becomes

$$P_k^{(s)} = \frac{\omega_{ps}^2 \epsilon\, |E_{1x}|^2}{2k_i\, |D_s|^2}\left(p_1^{(s)} q_1^{(s)} + p_2^{(s)} q_2^{(s)}\right) , \tag{2.128}$$

where

$$p_1^{(s)} = [(\omega - k_r v_{os})^2 + k_i^2 v_{os}^2][\omega_{cs}^2 - (\omega - k_r v_{os})^2 + (\nu_s + k_i v_{os})^2$$

$$+ (k_r^2 - k_i^2)v_{Ts}^2] + \nu_s v_{Ts}^2[k_i v_{os}(k_r^2 - k_i^2) + 2k_i k_r(\omega - k_r v_{os})] , \tag{2.129}$$

$$q_1^{(s)} = \frac{\omega\nu_s(\omega - k_r v_{os})}{(\omega - k_r v_{os})^2 + k_i^2 v_{os}^2} + \frac{|P|^2}{\omega}[2k_i k_r v_{Ts}^2 + (\nu_s + 2k_i v_{os})(\omega - k_r v_{os})] , \tag{2.130}$$

$$p_2^{(s)} = 2[(\omega - k_r v_{os})^2 + k_i^2 v_{os}^2][(\omega - k_r v_{os})(\nu_s + k_i v_{os}) + k_i k_r v_{Ts}^2]$$

$$+ \nu_s v_{Ts}^2[k_r v_{os}(k_r^2 + k_i^2) + \omega(k_r^2 - k_i^2)] , \tag{2.131}$$

$$q_2^{(s)} = 2P_i\omega_{cs} + \omega + \frac{\omega\nu_s k_i v_{os}}{(\omega - k_r v_{os})^2 + k_i^2 v_{os}^2} + \frac{|P|^2}{\omega}[(\omega - k_r v_{os})^2$$

$$- k_i v_{os}(\nu_s + k_i v_{os}) - (k_r^2 - k_i^2)v_{Ts}^2] \tag{2.132}$$

and

$$D_s = p_3^{(s)} + jq_3^{(s)} , \qquad (2.133)$$

where

$$p_3^{(s)} = (\omega - k_r v_{os})[\omega_{cs}^2 - (\omega - k_r v_{os})^2 + (\nu_s + k_i v_{os})^2] + 2k_i v_{os}(\nu_s + k_i v_{os})$$

$$\cdot (\omega - k_r v_{os}) + v_{Ts}^2[(k_r^2 - k_i^2)(\omega - k_r v_{os}) + 2k_i k_r(\nu_s + k_i v_{os})] \qquad (2.134)$$

and

$$q_3^{(s)} = k_i v_{os}[(\omega - k_r v_{os})^2 - \omega_{cs}^2 - (\nu_s + k_i v_{os})^2] + 2(\nu_s + k_i v_{os})(\omega - k_r v_{os})^2$$

$$+ v_{Ts}^2[2k_i k_r(\omega - k_r v_{os}) - (\nu_s + k_i v_{os})(k_r^2 - k_i^2)] \quad . \qquad (2.135)$$

As this result is difficult to analyze directly, the separate effects of diffusion and collisions are examined.

First consider $\nu_s = 0$ and the resultant effects of thermal diffusion. In this case, if Eqs. 2.128 through 2.132 are used, it can be shown that the kinetic power flow reduces to

$$P_k^{(s)}(\nu_s = 0) = \frac{\omega_{ps}^2 \epsilon \, |E_{1x}|^2 \, [(\omega - k_r v_{os})^2 + k_i^2 v_{os}^2][(\omega + P_i\omega_{cs})^2 + P_r^2\omega_{cs}^2]}{|D_s(\nu_s = 0)|^2}$$

$$\cdot [(\omega - k_r v_{os})v_{os} + k_r v_{Ts}^2] \quad . \qquad (2.136)$$

Inspection of this result shows that due to the presence of thermal diffusion $P_k^{(s)}(\nu_s = 0) < 0$ only if

$$v_{os} > \left(\frac{\omega}{k_r}\right) + \frac{v_{Ts}^2}{v_{os}} \quad , \qquad (2.137)$$

so that at the very least $v_{os} > v_{Ts}$ is required. Thus <u>regardless</u> of the dispersion relation for the (isotropic) system [i.e., regardless of the polarization factor $P(\omega,k)$, number of carriers, nature of the interacting circuit, etc.], if for all carrier species s it is known a priori that $v_{os} < v_{Ts}$, then <u>only</u> collisionally assisted convective instabilities are possible in the isotropic hydrodynamic model.

In a similar manner, let $v_{Ts} = 0$ in Eqs. 2.128 through 2.132 and obtain the kinetic power flow in the presence of collisions as

$$P_k^{(s)}(v_{Ts}=0) = \frac{\omega_{ps}^2 \epsilon \, |E_{1x}|^2}{2k_i \, |D_s(v_{Ts}=0)|^2} (\omega - k_r v_{os}) Q(\omega, k_r, k_i) \quad , \qquad (2.138)$$

where

$$Q(\omega,k_r,k_i) = \frac{|P|^2}{\omega} [(\omega - k_r v_{os})^2 + k_i^2 v_{os}^2]\{(\nu_s + 2k_i v_{os})\omega_{cs}^2 + \nu_s[(\omega - k_r v_{os})^2$$

$$+ (\nu_s + k_i v_{os})^2]\} + \omega\nu_s[\omega_{cs}^2 + (\omega - k_r v_{os})^2 + (\nu_s + k_i v_{os})^2 + 2k_i v_{os}(\nu_s + 2k_i v_{os})]$$

$$+ [(\omega - k_r v_{os})^2 + k_i^2 v_{os}^2][2k_i v_{os}\omega + 4P_i \omega_{cs}(\nu_s + k_i v_{os})] \quad . \qquad (2.139)$$

A close inspection of Eqs. 2.138 and 2.139 assuming $k_i > 0$ shows that again $(\omega - k_r v_{os}) < 0$ is required to obtain $P_k^{(s)} < 0$ and that when this is the case the collisions assist the source power.

These separate results would tend to indicate that in the general case, with both collisions and thermal diffusion present, the factor

$(\omega - k_r v_{os})$ must be negative to obtain a negative kinetic power mode, although even when this factor is negative it is still possible for the thermal diffusion to prevent this.

As with the previous carrier modes studied, the hybrid mode kinetic power flow can be related in part to the steady-state second-order dc carrier dynamics. As discussed previously, the only case which can be accurately studied with the plane-wave analysis used corresponds to the dc field $E_2 = 0$ or, equivalently, the medium is homogeneous in the steady state and $\rho_2^{(s)} = 0$. In this case it can easily be shown that the second-order longitudinal time-averaged force equation in the steady state is given by

$$(2k_i v_{os} + \nu_s) v_{2s} + \frac{k_i}{2} |v_{1x}|^2 = \frac{\eta_s}{2} \text{Re}(v_{1y} B_{1z}^*) \quad . \tag{2.140}$$

The right-hand side of this equation may be written as

$$\frac{\eta_s}{2} \text{Re}(v_{1y} B_{1z}^*) = \text{Re}\left(\frac{\eta_s k^*}{2\omega\rho_{os}} E_{1y}^* J_{1y} \right) , \tag{2.141}$$

so that when $k_r \text{Re}(E_{1y}^* J_{1y}) < 0$ a negative contribution to v_{2s} is obtained corresponding to beam slowing.

2.10 Utility of the Kinetic-Electromagnetic Power Theorem

Regardless of the carrier mode used, the present study has shown that the salient function of interest from the viewpoint of convective instabilities is the kinetic power function given by

$$P_k \triangleq \frac{1}{2k_i} \text{Re}(\underline{E}_1 \cdot \underline{J}_1^*) \quad . \tag{2.142}$$

The power relations, such as Eq. 2.22, show that an instability cannot occur unless $P_k < 0$ if $k_i > 0$. In a sense then, the power theorem provides a form of causality on the system indicating whether or not the root with $k_i > 0$ indeed corresponds to the amplification of RF electromagnetic power. Since the functional dependence of $P_k(\omega, k_r, k_i)$ can in general be ascertained by analytical methods, the primary utility of the power theorem is in determining, before any solutions of the dispersion equation are attempted, which regions of (ω, k) space and which carrier parameters are causal in nature. Note that this immediately delimits the investigation of the first-order dispersion relation to those carrier modes and/or those regions of (ω, k) space which can support a convective instability.

The present study also raises important questions related to the form of the assumed steady state of the system in convective instability analyses. If it is known at the outset that by its nature the system can support a steady-state dc density gradient of charge, such as is common in solid-state media (e.g., the Hall effect), then solving for the instability in such systems assuming a homogeneous medium in the steady state (i.e., using a plane-wave analysis) is in general inaccurate. Although the self-consistent dc density gradient is only second order in magnitude the power theorem indicates that it can still play an important role in the power exchange processes and hence that it cannot be neglected. This statement is particularly important for multiple carrier species interactions as evidenced by the discussion following and related to Eq. 2.99.

In the opposite case, in which it is known that the steady state of the system cannot support equilibrium dc carrier density imbalances through generation and recombination processes, the plane-wave dispersion equation is accurate and regardless of the carrier mode studied the kinetic power flows are in agreement with the second-order dc carrier dynamics. These considerations lend importance to the surface boundaries of the system (e.g., in solid-state devices the contact area) since these play a part in determining the steady state of the system.

These considerations lead to the speculation that for those systems in which the dc gradient of charge is permitted in the steady state, the possibility of nonlinearly generated instabilities exists wherein part of the power represented in the function $E_2 J_o$ is fed to the RF fields through the self-consistent charge density inhomogeneity. In such cases it may also be that the power represented in the plane-wave analysis of the system by $\mathrm{Re}(E_1 J_1^*)$ need not be negative. Verification of these hypotheses requires a full nonlinear analysis which is not attempted in this tract.

A further point raised by the power theorem deals with the distinction between a negative kinetic power mode and a negative power mode. For example, neglecting collisions and thermal diffusion, the left-hand circularly polarized mode ($\omega_c < 0$) was found in Eq. 2.64 to be a negative kinetic power mode throughout (ω, k) space, although the total mode power flow for this case, given by Eq. 2.69, is negative only in restricted regions of (ω, k) space. The problem in obtaining the total power flow of a carrier mode is that a localization of the RF electromagnetic power flow is required. From the viewpoint of coupled-mode theory this may be permissible for weakly coupled modes, but in the actual interacting system this cannot be accomplished with certainty. It would then appear that for weakly interacting modes a negative power mode is required for instability, whereas for strong interactions a negative kinetic power mode may be sufficient. A negative kinetic power mode, although both necessary and sufficient to satisfy conservation of power requirements, may not be sufficient to guarantee that instability will occur when this mode interacts with a passive circuit.

The techniques used in the kinetic power theorem can also be applied to gain insight into decaying mode interactions ($k_i < 0$). This aspect is presented in Appendix D.

2.11 Summary

The kinetic power theorem has now been formulated for the basic carrier modes possible. Any new mode (e.g., static magnetic field applied at an arbitrary angle) should only be a superposition of these basic modes and hence provide no new physical phenomena.

CHAPTER III. KINETIC ENERGY PROPERTIES OF CARRIER WAVES: ABSOLUTE INSTABILITY

3.1 Introduction

It is generally held that the consideration of energy or power can only provide information on convective instabilities or that a mode which has negative kinetic power can be active for an absolute instability if the proper interacting circuit mode is chosen. It will be shown, however, that a separate but related conservation principle for absolute instabilities can be obtained. A kinetic-electromagnetic energy theorem will be derived for the three basic carrier modes studied in Chapter II. The theorem also demonstrates the physical mechanisms through which absolute instabilities can arise.

Whereas the convective instability theory of the previous chapter used a time-averaged spatial-power framework, the present study employs a space-averaged temporal-energy basis. For example, in the convective instability process the source arises from the active carrier mode having negative power so that exponentially growing waves in space are possible. In the absolute instability process, however, it will be determined that if the carrier mode has a negative energy property, exponentially growing waves evolving in time are possible. In the latter case, in general, at each point in the interaction region, the total carrier kinetic energy increases exponentially as time progresses, but in the negative direction. For the present analysis the condition for steady-state oscillations will not be dealt with since this state is ultimately determined by saturation processes which cannot in general be taken into account. This in no way diminishes the value of the theory for predicting which carrier modes and which regions of ω-k space are potentially absolutely unstable. Because of their importance in the case of solid-state plasmas, the effects of collisions and thermal diffusion are included.

It is shown through a study of the electrokinetic energy density functions that the dynamics of the k_r-k_i plots according to Briggs' criteria can be elaborated upon and the meaning of the root behavior explained as a function of time for both absolute and convective instabilities.

The theorem is also used to show that the characteristics of the electromagnetic branch of the helicon spectrum are such that unstable wave behavior is permissible without violation of the conservation laws. This region of ω-k space is usually rejected as a possible source of instabilities.

Whenever possible the quasi-linear theory is included to help in understanding the carrier dynamics. It is found that the second-order effects predicted by the quasi-linear theory are intimately related to the boundary conditions of the system. In particular it is shown in which cases the linear dispersion relation is accurate and also a discussion is given of how the boundary effects can enhance or quench the instability.

3.2 Kinetic Energy Characteristics of Space-Charge Waves

The development in many ways parallels that used in the convective instability study. Thus assume that fundamental variables vary as $\exp[j(\omega t - kx)]$, where now $\omega = \omega_r + j\omega_i$ and $k = k_r + jk_i$. The Poynting theorem is written in the form

$$\nabla \cdot (\underline{E} \times \underline{B}^*) + \underline{B}^* \cdot \frac{\partial \underline{B}}{\partial t} + \mu_o \epsilon \underline{E} \cdot \frac{\partial \underline{E}^*}{\partial t} + \mu_o \underline{E} \cdot \underline{J}^* = 0 \quad , \tag{3.1}$$

and the real part is taken, viz.,

$$\frac{1}{\mu_o} \operatorname{Re} [\nabla \cdot (\underline{E} \times \underline{B}^*)] + \frac{\partial}{\partial t} (W_{e\ell} + W_k) = 0 \quad , \tag{3.2}$$

where

$$W_{e\ell} \triangleq \frac{1}{2\mu_o} |B|^2 + \frac{\epsilon}{2} |E|^2 \tag{3.3}$$

and

$$W_k \triangleq - \frac{1}{2\omega_i} \operatorname{Re} (\underline{E} \cdot \underline{J}^*) \quad ; (\omega_i \neq 0) \quad . \tag{3.4}$$

The function $W_{e\ell}$ is the sum of the electromagnetic and electrostatic energy densities and W_k is the electrokinetic energy density.

It will first be verified that the slow space-charge wave is the source mode for the backward-wave oscillator type of interaction in which a single carrier stream interacts with an external circuit wave mode. Consider the simplest case in which collisions and carrier diffusion are neglected and it is assumed a priori that k is purely real. It is shown in Appendix E that the electrokinetic energy density of the longitudinal space-charge wave to second order can be expressed in terms of the fundamental fields alone in this case as

$$W_k = \frac{k \rho_o v_o (\omega_r - k v_o) e^{-2\omega_i t} |v_{1x}(0)|^2}{2\eta [(\omega_r - k v_o)^2 + \omega_i^2]} \quad , \tag{3.5}$$

where $|v_{1x}(0)|$ is the longitudinal RF velocity amplitude of the carrier at time t = 0. Since in the present case k is purely real, if $\omega_i \neq 0$, Eq. 3.2 gives

$$W_{e\ell} + W_k = 0 \quad . \tag{3.6}$$

Inspection of Eq. 3.3 indicates that $W_{e\ell}$ is positive and hence for conservation of energy to be satisfied it is necessary that $W_k < 0$, or equivalently from Eq. 3.5 that $v_o > \omega_r/k$. Thus the slow space-charge wave must be used for the oscillation to grow positively in time. Note that part of the function $W_{e\ell}$ in Eq. 3.3 is the electrostatic energy density associated with the carrier mode. Hence in general it is not sufficient that $W_k < 0$ for growth to occur, but rather that the function W_T defined by

$$W_T \triangleq W_{es} + W_k \tag{3.7}$$

be negative, where W_{es} represents the electrostatic energy density of the carriers. In order to evaluate W_T, the longitudinal force equation, Eq. 2.10, gives for the present case

$$|E_{1x}(o)|^2 = \frac{(\omega_r - kv_o)^2 + \omega_i^2}{\eta^2} |v_{1x}(o)|^2 \quad , \tag{3.8}$$

where $E_{1x}(0)$ is the longitudinal RF electric field at time t = 0, so that, if Eqs. 3.3 and 3.5 are used,

$$W_T = \left(\frac{\epsilon[(\omega_r - kv_o)^2 + \omega_i^2]}{\eta} + \frac{k\rho_o v_o(\omega_r - kv_o)}{(\omega_r - kv_o)^2 + \omega_i^2} \right) \frac{|v_{1x}(0)|^2}{2\eta} e^{-2\omega_i t} \quad . \tag{3.9}$$

The functional dependence of W_T on ω_i shows directly that ω_i must be limited in magnitude and moreover indicates the important limiting parameters involved.

The general form of the electrokinetic energy density of the space-charge waves employing the definition of Eq. 3.4 can be obtained directly from Eqs. 2.6 and 2.13 as

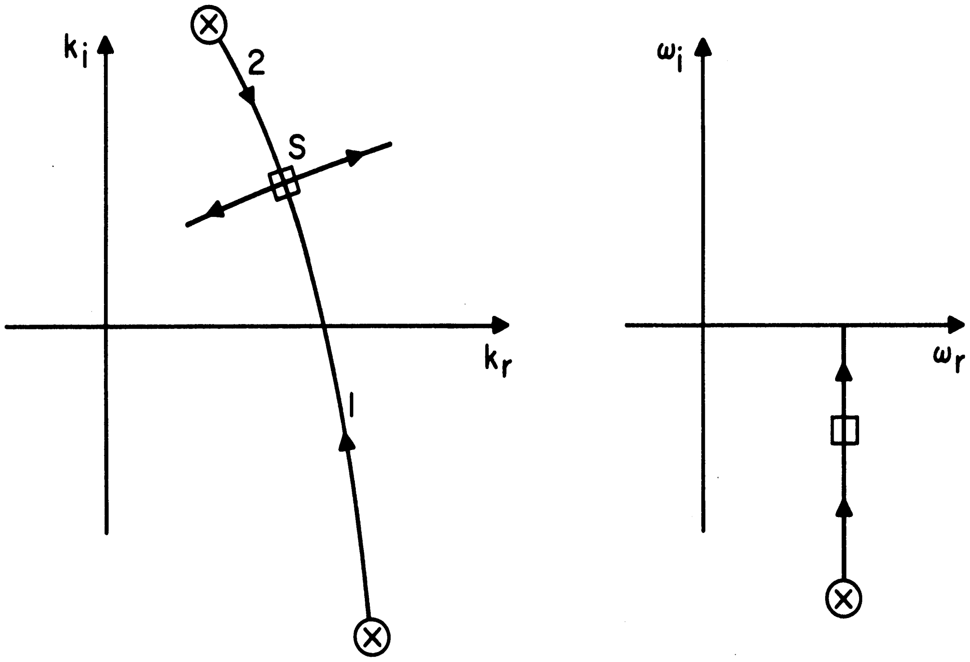
$$W_k = \frac{\omega_p^2\epsilon}{2} \left[\frac{|E_{1x}|^2 \left[\begin{array}{l} (\omega_r-k_rv_o)(\omega_r+k_rv_o)+(\omega_i-k_iv_o)(\omega_i-k_iv_o+\nu)+(k_r^2-k_i^2)v_T^2 \\ -\left(\frac{\omega_r}{\omega_i}\right) [2k_ik_rv_T^2+(2k_iv_o+\nu)(\omega_r-k_rv_o)] \end{array} \right]}{|(\omega-kv_o)(\omega-kv_o-j\nu)-k^2v_T^2|^2} \right.$$
$$\left. + \frac{(|E_{1y}|^2+|E_{1z}|^2)\left[(\omega_r-k_rv_o)^2+(\omega_i-k_iv_o)(\omega_i-k_iv_o-\nu) - \left(\frac{\omega_r}{\omega_i}\right)(\omega_r-k_rv_o)\nu\right]}{|\omega-kv_o-j\nu|^2 \; (\omega_r^2+\omega_i^2)} \right] \quad . \tag{3.10}$$

It will now be shown that it is possible to relate the energy characteristics of the system to the causality principle based on Briggs' criteria. For clarity assume the waves of interest are purely electrostatic and neglect the thermal diffusion ($v_T = 0$) so that Eq. 3.2 becomes when Eq. 3.10 is used

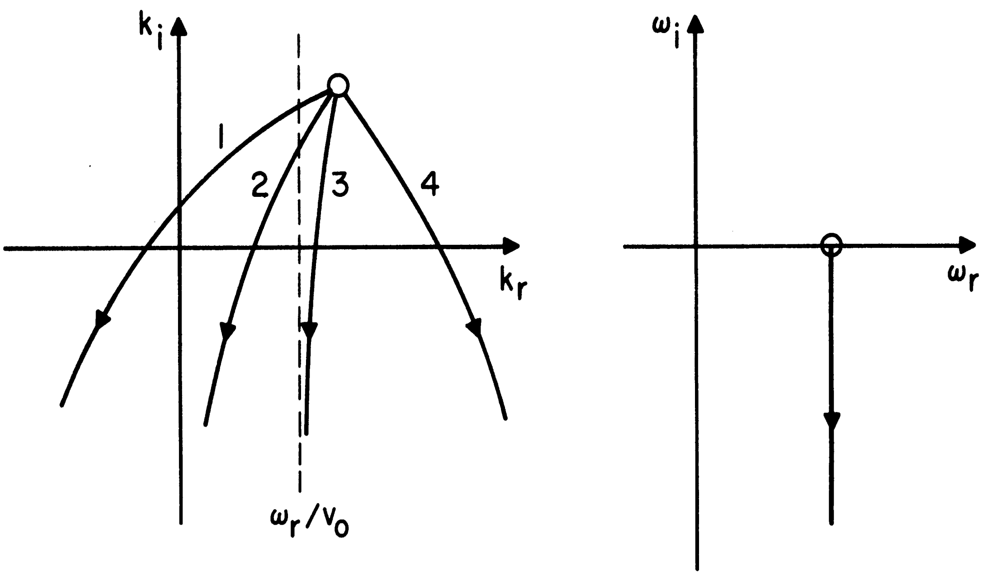
$$-\omega_i \epsilon \, |E_{1x}|^2 \left[1 + \frac{\omega_p^2 \left[(\omega_r - k_r v_o)(\omega_r + k_r v_o) + (\omega_i - k_i v_o)(\omega_i - k_i v_o - \nu) - \left(\frac{\omega_r}{\omega_i} \right) (2k_i v_o + \nu)(\omega_r - k_r v_o) \right]}{[(\omega_r - k_r v_o)^2 - (\omega_i - k_i v_o)(\omega_i - k_i v_o - \nu)]^2 + (\omega_r - k_r v_o)^2 [2(\omega_i - k_i v_o) - \nu]^2} \right] = 0 \; . \tag{3.11}$$

For an actual system a sum over the carrier species present can be made without altering the present relationships. If Briggs' method is applied to Eq. 3.11 by permitting $\omega_i \to -\infty$, inspection shows that for the electro-kinetic energy density to remain finite and satisfy Eq. 3.11 it is necessary that $k_i \to \omega_i / v_o \to -\infty$. Hence for either convective or absolute instabilities it is seen that Briggs' requirement on the amplifying mode, to satisfy causality, namely $k_i \to -\infty$ as $\omega_i \to -\infty$, is directly related to conservation of energy in the system.

Note that in a very real sense the state of the system defined by the operation of letting $\omega_i \to -\infty$ is an actual <u>potential</u> physical state of the system according to the linear analysis since it is a solution of the dispersion equation. Thus if the wave blows up very quickly in time $(\omega_i \to -\infty)$ it must be that it simultaneously decays spatially very quickly $(k_i \to -\infty)$ or otherwise there is too much energy in the system (conservation of energy is not satisfied). This nonequilibrium behavior can be related to the dynamics of the Briggs' plot for an absolute instability as shown in Fig. 3.1a. At some fixed time the system can have the solutions indicated by $\otimes$, wherein Mode 1 is the forward amplifying wave and Mode 2 is the backward passive wave. Mode 1 is then essentially in the state



(a) BRIGGS' MAPPING FOR ABSOLUTE INSTABILITY



(b) BRIGGS' MAPPING FOR CONVECTIVE INSTABILITY SHOWING POSSIBLE ROOT TRAJECTORIES

FIG. 3.1 TRAJECTORY STUDY OF BRIGGS' MAPPINGS.

discussed above. As time proceeds reflections (and possibly considerations of entropy) drive the system to the resonant double root at point S from which the eventual steady state is reached.

It is also now possible to relate the dynamics of the Briggs' plot for convective instabilities as shown in Fig. 3.1b to the dictum of conservation of energy. In particular, consider the possible forms the convectively unstable root may take in crossing the k_r axis ($k_i = 0$) as ω_i is varied, typified in the figure by potential trajectories 1 through 4. Even for convective instabilities the system must satisfy the conservation of energy equation, Eq. 3.2, which for $k_i = 0$ indicates that W_k is necessarily negative and nonzero. Inspection of W_k as given in Eq. 3.10 shows that for $k_i = 0$ it must be that $(\omega_r - k_r v_o) < 0$ in order that $W_k < 0$. Thus with reference to Fig. 3.1b trajectories such as 1 or 2 cannot occur, whereas trajectories 3 or 4 are permissible. Although this has been shown only for space-charge waves, it will be determined that this is a general result for the basic carrier modes in the hydrodynamic analysis.

Similar to the result found for the kinetic power function, Eq. 3.10 shows that the electrokinetic energy density can be negative through the action of the collisions for purely transverse fields alone. To understand this, consider the case $k_i = 0$ and take the dc part of Eq. 2.32 to obtain

$$\frac{\partial v_2}{\partial t} + \nu v_2 = \eta \left(E_2 + \frac{1}{2} \mathrm{Re}(\underline{v}_1 \times \underline{B}_1^*)_{\hat{x}} \right) . \qquad (3.12)$$

Then from the assumed time dependence, it is found that

$$v_2 = \frac{\eta E_2}{\nu - 2\omega_i} + \frac{\eta^2 k_r \nu(\omega_r - k_r v_o)(|E_{1y}|^2 + |E_{1z}|^2)}{2(\nu - 2\omega_i)(\omega_r^2 + \omega_i^2)\ |\omega - kv_o - j\nu|^2} , \qquad (3.13)$$

so that when $(\omega_r - k_r v_o) < 0$ the collisions act through the second-order Lorentz force to extract energy (at all points in the interaction region) from the carriers by slowing. Note that this result indicates that collective Cerenkov radiation may be strongly influenced by the presence of collisions.

In addition, as inspection of Eq. 3.10 indicates, under small-signal conditions, with $\omega_i \ll \omega_r$ and $k_i \ll k_r$, it is the presence of collisions which permits the electrokinetic energy density to be negative in the case $v_o < v_T$.

3.3 Electrokinetic Energy Density of Purely Transverse Waves in a Static Magnetic Field

Assume that the fundamental variables vary as $\exp[j(\omega t - kx)]$ where both ω and k are in general complex and $\underline{B}_o = B_o\hat{x}$ with $\underline{v}_o = v_o\hat{x}$ as before. The coordinate definitions of Eqs. 2.6 are used and again, since the transverse modes are uncoupled, only the (-) mode need be analyzed. Thus Eq. 2.7 is written in the form

$$\frac{\partial v_{1-}}{\partial t} - j(\omega_c - kv_o)v_{1-} = \eta E_{1-} + \frac{\eta v_o}{j\omega} \frac{\partial E_{1-}}{\partial x} . \qquad (3.14)$$

To examine the second-order longitudinal carrier dynamics in detail, consider the Lorentz force term

$$\underline{F}_\ell = q\ \mathrm{Re}(\underline{v}_1) \times \mathrm{Re}(\underline{B}_1) = \frac{1}{2} q\ \mathrm{Re}(\underline{v}_1 \times \underline{B}_1^* + \underline{v}_1 \times \underline{B}_1)_x \qquad (3.15).$$

which shows that two types of motion exist for the second-order variables, so that v_2 for example may be written as

$$\mathrm{Re}(v_2) = V_{21}\, e^{-2\omega_i t}\, e^{2k_i x} + V_{22}\, e^{-2\omega_i t}\, e^{2k_i x} \cos 2(\omega_r t - kx) \quad , \qquad (3.16)$$

where V_{21} and V_{22} are independent of x and t. The sinusoidally varying part of this dependence is related to the importance of device length for the time growth of the oscillation. In the limit $k_i \to 0$, a space integration over the device length shows that this contribution is zero if the device length, L, is an integral number of wavelengths. Also, if $k_i \to 0$, this contribution should be negligible if $L \gg \lambda$. In general $k_i \neq 0$ and it is necessary to take into account the total interaction so that at oscillation the interference between the waves present cancels the sinusoidal contribution and the expression for the ideal device length becomes dependent upon k_i, the wave parameters, etc. In order to determine the kinetic energy properties of a single mode then this sinusoidal contribution is neglected. This problem does not arise for convective instabilities since $\omega_i = 0$ and Eq. 3.5 is time averaged. In addition, using Eq. 2.6, if it can be rigorously assumed that $v_{1+} = 0$, the contribution of this sinusoidal part to the second-order motion of the (-) mode is identically zero.

From Eq. 2.62 the small-signal Poynting theorem can be written in the general form for the (-) mode:

$$\frac{1}{\mu_0} \nabla \cdot (\underline{E}_1 \times \underline{B}_1^*)_{(-)} + \frac{1}{2\mu_0} B_{1-}^* \frac{\partial B_{1-}}{\partial t} + \frac{\epsilon}{2} E_{1-} \frac{\partial E_{1-}^*}{\partial t} + \frac{1}{2} \sum_s E_{1-} J_{1-}^{(s)*} = 0 \quad ,$$

(3.17)

where the sum is over the carrier species present. Make the definitions, where the symbols have the usual meanings, as follows:

$$W_{k-} = -\frac{1}{4\omega_i} \operatorname{Re}(E_{1-} J^*_{1-}) \quad , \tag{3.18}$$

$$W_{e\ell-} = \frac{1}{4\mu_o} |B_{1-}|^2 + \frac{\epsilon}{4} |E_{1-}|^2 \tag{3.19}$$

and

$$P_{e\ell-} = \frac{k_r\omega_r - k_i\omega_i}{2\mu_o(\omega_r^2 + \omega_i^2)} |E_{1-}|^2 \quad , \tag{3.20}$$

so that, corresponding to the real part of Eq. 3.17, the following can be written:

$$2k_i P_{e\ell-} + \frac{\partial W_{e\ell-}}{\partial t} + \sum_s \frac{\partial W_{k-}^{(s)}}{\partial t} = 0 \quad . \tag{3.21}$$

The function W_{k-} is derived explicitly in Appendix F. Now from Briggs' criteria it is clear that for an absolute instability at least one root is required with the property that $\omega_i < 0$ for $k_i = 0$. For $k_i = 0$ and $\omega_i \neq 0$, Eq. 3.21 takes the form

$$W_{e\ell-} + \sum_s W_{k-}^{(s)} = 0 \quad . \tag{3.22}$$

Inspection of Eq. 3.19 shows that $W_{e\ell-}$ is necessarily positive (the case $|E_{1-}| = 0$ is of no interest) so that at least one carrier species must possess $W_{k-} < 0$, i.e., a negative electrokinetic energy density. Inspection of Eq. F.4 for this case thus indicates that for $W_{k-} < 0$ then $\omega_c < 0$ corresponding to the slow-cyclotron mode, helicon mode, etc., and that

$$|\omega_c| \, k_r v_o \; > \; (\omega_r - k_r v_o)^2 + \omega_i^2 \quad . \tag{3.23}$$

Equation 3.23 is easily satisfied by the helicon branch since $(\omega_r - k_r v_o) \simeq 0$ and also by the slow-cyclotron branch since $k_r > |\omega_c|/v_o$ for positive frequencies. In addition, however, this equation can still be satisfied in certain regions of the electromagnetic branch referred to in Fig. 2.1. Hence this branch which is usually neglected in instability studies can satisfy both the criteria of conservation of power and conservation of energy and hence be a source mode for convective or absolute instabilities. It must be immediately pointed out that the presence of collisions or consideration of finite size effects (since k_r is small in this region) can remove this branch from the negative kinetic power or energy property.

As an aid in understanding the absolute instability process the quasi-linear theory is now studied. The continuity equation for each carrier species is

$$\nabla \cdot \underline{J}_{2-} + \frac{\partial \rho_{2-}}{\partial t} = 0 \quad . \tag{3.24}$$

From the assumed time dependence together with physical considerations it must be that $\rho_{2-} = 0$. Equating current density contributions from outside and within the interaction region provides

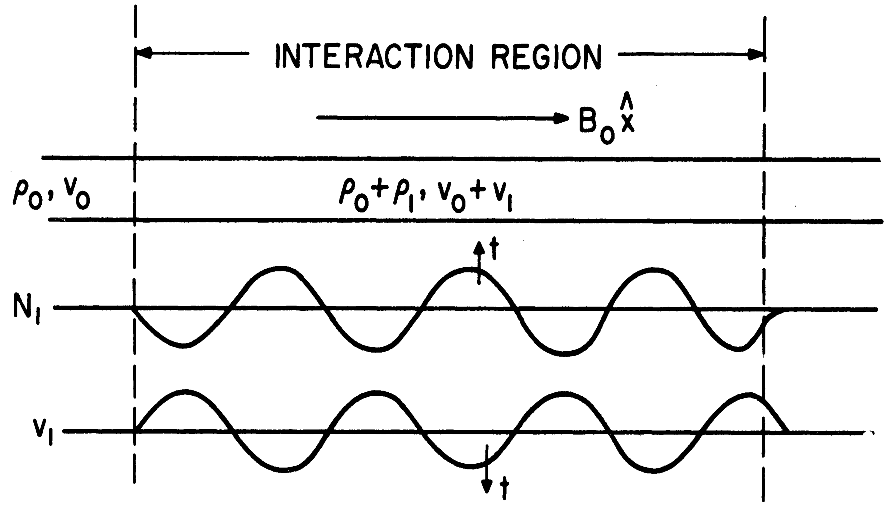
$$\sum_s \rho_{os} v_{os} = \sum_s \left(\rho_{os} v_{os} + \rho_{os} v_{2-}^{(s)} \right) + \epsilon \frac{\partial E_{2-}}{\partial t} \quad , \tag{3.25}$$

so that the displacement current balances the total current density in the interaction region. In particular, for the case of a single species of carrier interacting with an external circuit, Eq. 3.25 applies if the interaction is weak

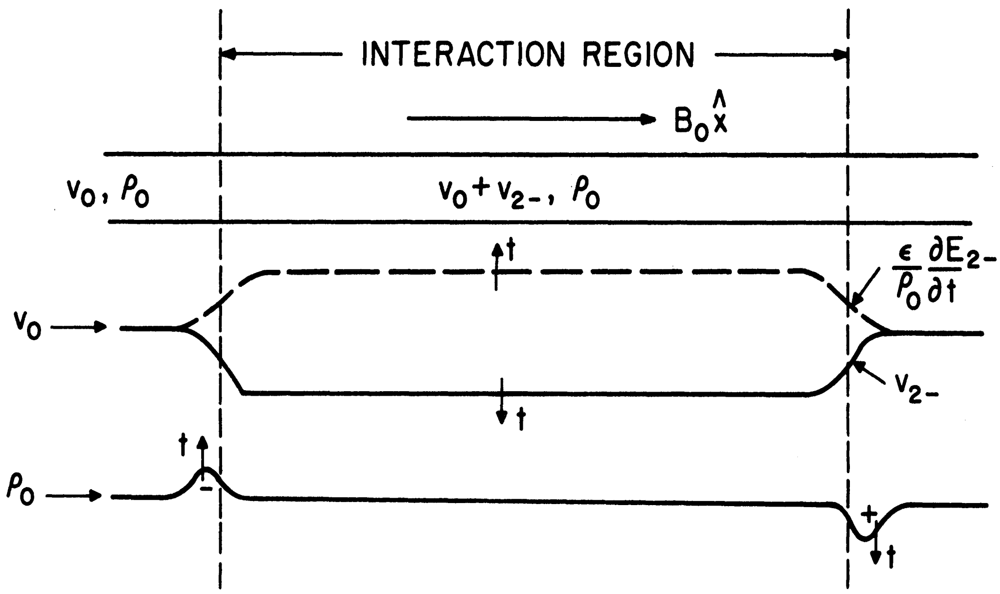
$$v_{2^-} = -\frac{\epsilon}{\rho_o} \frac{\partial E_{2^-}}{\partial t} . \qquad (3.26)$$

Thus in the case of absolute instabilities the slowing down of the dc beam in time is compensated by the buildup of displacement current whose magnitude increases with time until the eventual steady state is attained. It is in this manner that continuity is achieved. In an actual device this necessitates charge buildup at the extremities $x = 0$ and $x = L$ where $\rho_{2^-}(t) \neq 0$. The device analog is a leaky capacitor which "heals" itself as time progresses. The functions v_{2^-} and E_{2^-} are spatially uniform in the interaction region. The pictures obtained of the absolute instability process are thus quite different for the space-charge waves vs. the present transverse modes. The space-charge wave system evolves as shown in Fig. 3.2a, while for comparison the nature of the transverse mode is depicted in Fig. 3.2b for the simplest case, wherein $k_i = 0$. It is noted that in the most general case, $\omega_i \neq 0$ and $k_i \neq 0$, all the processes occurring in Fig. 2.3a and b and Fig. 3.2a and b may occur.

A similar problem to that connected with the suitability of the linear dispersion in the convective instability process occurs here also. The field E_{2^-} is the self-consistent reaction of the system to the dc beam slowing which must be present to conserve particles properly. This field



(a) DENSITY AND VELOCITY CHARACTERISTIC OF SLOW SPACE-CHARGE WAVE



(b) DENSITY AND VELOCITY CHARACTERISTIC OF THE (-) TRANSVERSE MODE

FIG. 3.2 COMPARISON OF LONGITUDINAL AND TRANSVERSE MODES UNDER ABSOLUTE INSTABILITY.

can quickly extinguish the instability so that in fact it never occurs or in some cases it can assist the instability. The quasi-linear theory can be used to show how this can occur. The second-order force equation for the active carrier can be obtained from Eq. 2.25 with collisions neglected as

$$\frac{\partial v_{2^-}}{\partial t} + v_o \frac{\partial v_{2^-}}{\partial x} = \eta E_{2^-} + \eta \operatorname{Re}\left(\frac{1}{2j\omega} v_{1^-} \frac{\partial E^*_{1^-}}{\partial x} \right) , \qquad (3.27)$$

where from Eq. 3.25 the field E_{2^-} satisfies the following:

$$\sum_s \rho_{os} v^{(s)}_{2^-} + \epsilon \frac{\partial E_{2^-}}{\partial t} = 0 \ . \qquad (3.28)$$

Clearly for the instability to proceed it must be that v_{2^-} of the active carrier must be negative since this is the only form of energy loss available for purely transverse waves. Equation 3.28 indicates, however, that the field E_{2^-} is in part determined by the properties of the passive carrier. Indeed by solving Eqs. 3.27 and 3.28 simultaneously it can be found that for the active carrier, for a two-carrier system,

$$v_{2^-} = - \frac{|v_{1^-}|^2}{4v_o} - \frac{\eta}{2\omega_i J_o} \operatorname{Re}(E_{1^-} J^*_{1^-}) + \frac{\eta \sum\limits_{s=1,2} \left(\frac{\rho_{os}}{v_{os}} |v^{(s)}_{1^-}|^2 + \frac{\eta_s}{\omega_i v_{os}} \operatorname{Re}(E_{1^-} J^{(s)*}_{1^-}) \right)}{16\, \omega_i^2 \epsilon \left(1 + \frac{\omega_{p1}^2 + \omega_{p2}^2}{4\omega_i^2} \right)} , \qquad (3.29)$$

where all unscripted variables refer to the active carrier, and the last term on the right-hand side is due to the field E_{2^-}. The linear dispersion relation would indicate that the sign of v_{2^-} is completely determined by the $\mathrm{Re}(E_{1-} J^{*}_{1-})$; however, the quasi-linear theory of Eq. 3.29 indicates that this is not in general true. The secondary or passive carriers can play a crucial role in determining the sign of v_{2-}. The physical reason for this is that as the active carriers are slowed an imbalance occurs in the continuity of the system which must be balanced by the secondary carriers. Unless a series of generation and recombination processes occur near the ends of the device the field of the displacement current E_{2-} builds up and affects the further evolvement of the active carrier dc motion. Thus the quasi-linear theory offers a method of taking into account the boundary effects of the system--something that is completely ignored in setting up the linear dispersion relationship.

In particular, for solid-state devices, one set of contacts for a particular device may be such that its injection characteristics never permit any charge discontinuities such as charge buildup near the contacts and the absolute instability can proceed. For a second set of contacts with opposite injection characteristics it may be that the charge buildup at the extremities near the contacts cannot be halted and the instability is quickly quenched by the effects of the field E_{2-} on the system. It is only in the former case that the results predicted by the linear dispersion equation will be valid. It would at first appear that an active mode could be obtained even if $W_{k-} < 0$ by appropriate variation of the system parameters to alter E_{2-} and hence obtain $v_{2-} < 0$. The difficulty with this approach is that in absolute instability studies the dynamic phenomena are nonequilibrium in nature and not a steady state of the system. Thus questions of which direction the system proceeds in time leads to the conclusion that the field E_{2-} (which is in a sense a measure of the effects of the system boundaries) can extinguish or enhance an instability already initiated, but cannot by itself play a role in the initiation itself.

3.4 Electrokinetic Energy Density of the Hybrid Mode

The field directions and conventions adopted in Section 2.8 for the hybrid mode are retained. The general form for the electrokinetic energy density of the hybrid mode is

$$W_k = -\frac{1}{2\omega_i}\,\mathrm{Re}(E_{1x}J_{1x}^* + E_{1y}J_{1y}^*) \quad , \tag{3.30}$$

with both ω and k complex. Equations 2.113, 2.114, and 2.116 can be substituted into Eq. 3.30 for the present case to obtain the following neglecting collisions and diffusion:

$$W_k = \frac{\left[\begin{array}{l}\omega_p^2\epsilon\,|E_{1x}|^2\left\{2(\omega_r - k_r v_o)(\omega_i - k_i v_o)\left(\omega_r + 2P_i\omega_c + \omega_r\omega_c^2\dfrac{|P|^2}{|\omega|^2}\right)\right.\\ \left.-\,\omega_i\left[[(\omega_r - k_r v_o)^2 - (\omega_i - k_i v_o)^2]\left(\dfrac{|P|^2}{|\omega|^2}\omega_c^2 + 1\right) - \omega_c^2\right]\right\}\end{array}\right]}{2\omega_i\,|\omega_c^2 - (\omega - kv_o)^2|^2} \quad , \tag{3.31}$$

where $P(\omega,k)$ is the polarization factor defined in Eq. 2.110. Inspection shows that even for the case $k_i \rightarrow 0$ the electrokinetic energy density is strongly influenced in magnitude and sign by the polarization factor $P(\omega,k)$. This is unlike the corresponding kinetic power flow of the hybrid mode given by Eq. 2.117 which indicated that $(\omega - k_r v_o) < 0$ is always required to obtain $P_k < 0$. Thus, in general, it is necessary to study the total system, thereby obtaining explicit information on the function

$P(\omega,k)$ before any specific statements can be made regarding the electrokinetic energy density. An exception to this occurs if $\omega_p > |\omega_c|$ so that $|E_{1x}| \gg |E_{1y}|$ and $|P| \simeq 0$ in which case inspection of Eq. 3.31 shows that the function W_k approaches in value that given for the slow space-charge wave.

3.5 Summary and Discussion

The electrokinetic energy density functions have been obtained for the basic carrier modes present in a static magnetic field. Since these basic carrier modes encompass all physical phenomena for isotropic media any general mode such as that obtained by applying the magnetic field at an arbitrary angle to the direction of wave propagation should simply be a superposition of the effects studied.

The potentially most valuable use of the present energy studies is that it offers a method for taking into account the fact that the finite length of the system introduces end boundaries to the interaction region which can play a significant role in determining the stability of the system in the steady state. Thus if the (linear) dispersion relation indicates that an absolute instability should develop in the system, application of the quasi-linear theory to second order, together with information about the end boundaries and the carrier generation-recombination characteristics, will determine whether or not these end effects enhance or quench the further evolvement of the instability. In the latter case no instability will exist in the steady state.

The energy theorems also provide useful correlations with the causality conditions of Briggs' criteria. Thus the behavior of the roots in k_r-k_i space as ω_i is varied is related to the fact that the electrokinetic energy density must remain finite for arbitrary perturbations. As in Appendix D, where it was hypothesized that even decaying convective roots must satisfy a causality condition similar to that for growing waves, this can also be extended to include causality criteria for waves which are predicted to decay in time. Note that these considerations raise the question of the definition of an instability. It is for practical reasons that an instability is defined to occur when the RF fields grow in time or space. It is possible however to define an instability more generally as any interaction in which a continuous flow of power and/or energy occurs between waves in the system. It is for this reason that causality criteria should be invoked even for decaying RF fields.

A central problem in the analysis of systems is that it is never clear in which state the system resides as the time t becomes larger. This ambiguity is related to the fact that one can arbitrarily assume ω complex, k complex, or both ω and k complex, and obtain a variety of solutions to the dispersion relation which all satisfy causality criteria. Thus, depending on the initial perturbation, these are all potential solutions at some time in the state of the evolvement of the system. These considerations immediately invoke the question of which direction the system proceeds after the time of the initial perturbation and for what reasons it does so. Although no attempt is made herein, it is suggested that consideration of the entropy of the system will provide knowledge of the system evolvement as a function of time for arbitrary perturbations.

CHAPTER IV. THE EVOLUTION OF PLASMA INSTABILITIES BASED ON QUASI-LINEAR THEORY

4.1 Introduction

In general the assessment of instability characteristics for a particular system involves the development of a dispersion relation based upon the linear wave equation. Solutions of this dispersion relation determine whether or not a small perturbation at some frequency of interest will at the perturbation onset grow spatially away from this point (convective instability) and/or build up in time from the perturbation onset (absolute instability).

The dispersion relation based on the linear theory cannot be used, however, to determine how the instability evolves in space and time away from the onset (e.g., $x = 0$, $t = 0$) point. The fact that the dispersion equation possesses a causal unstable root is then a necessary but not sufficient condition for the instability to persist as either the time or the spatial direction becomes finite. This is because the reaction of the carrier motion (e.g., carrier slowing) to the growing RF fields is not present in the linear equations.

The quasi-linear theory is first applied to aid in developing an understanding of how convective instabilities evolve from their points of initiation. This theory is a direct extension of the linear equations in that they are retained and used as a basis for formulating the equations of carrier motion to second order. This is permissible only if the small-signal approximation still holds and, more importantly, if the wave number predicted by the linear equations can be assumed to be a constant of the system for the root and frequency of interest. More generally it would be necessary to include the possibility that the wave number can be spatially dependent [e.g., $k \rightarrow k(x)$]. This latter approach is analytically non-tractable, however, and so the quasi-linear theory is used to obtain at least some indication of the system behavior due to the reactive forces of the growing RF fields on the carrier motion.

Even in the quasi-linear theory only the very simplest of convectively unstable systems is readily analyzed and effects such as the thermal diffusion, collisions, and the spatial dependence of the wave number are only qualitatively discussed. Some useful results relating the second-order effects to the potential energy of the charge carrier system are obtained and discussed for both gaseous and solid-state plasmas.

These methods are then applied to absolutely unstable systems. In this case the terminal boundaries of the system in the $\underline{k}$ direction can play an essential role in the development of the instability as is discussed.

4.2 Application of Quasi-Linear Theory to Convectively Unstable Systems

Various examples of convectively unstable systems will be examined to determine the reaction of the growing RF fields on the particle motions in the quasi-linear theory. Except when assessing causality according to Briggs' criteria the frequency is assumed purely real. In all cases the wave vector $\underline{k}$ is assumed to be parallel to $\hat{x}$.

4.2.1 Electron Stream-Plasma Electrostatic Interaction.

The dispersion relation for the purely longitudinal interaction of a cold electron beam with a cold stationary plasma is well known as

$$1 - \frac{\omega_{pp}^2}{\omega^2} - \frac{\omega_{pb}^2}{(\omega - kv_{ob})^2} = 0 \quad , \tag{4.1}$$

where the subscript p denotes plasma variables and b denotes the beam variables. This equation is readily solved as

$$kv_{ob} = \omega \pm \omega_{pb}\left(1 - \frac{\omega_{pp}^2}{\omega^2}\right)^{-1/2} \tag{4.2}$$

so that $k_i > 0$ can exist if $\omega \lesseqqgtr \omega_{pp}$ and moreover this root is causal since $k_i \to -\infty$ as $\omega_i \to -\infty$. In particular it is noted that if $\omega \to \omega_{pp}$, then $k_i \to \infty$ and the growth rate is infinite.

To proceed with the quasi-linear analysis it will be of interest to first study the case where ω_{pp} corresponds to a single charge carrier species alone denoted by subscript q so that $\omega_{pp} = \omega_{pq}$. The longitudinal force equation for a cold, collisionless carrier to second order is

$$\frac{\partial v_1}{\partial t} + \frac{\partial v_2}{\partial t} + v_o \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_1}{\partial x} + v_o \frac{\partial v_2}{\partial x} = \eta(E_1 + E_2) \quad . \tag{4.3}$$

The time-averaged real part of this gives

$$\frac{1}{2} \text{Re}\left(v_1 \frac{\partial v_1^*}{\partial x}\right) + v_o \text{ Re}\left(\frac{\partial v_2}{\partial x}\right) = \eta \text{ Re } (E_2) \quad , \tag{4.4}$$

where $v_2, E_2 \sim e^{2k_i x}$ as discussed in Chapter II since the second harmonic contributions give a zero time average. If Eq. 4.4 is applied to the q plasma carriers, the following is obtained since v_{oq} is assumed to be zero:

$$\frac{1}{2}\,\mathrm{Re}\left(v_{1q}\,\frac{\partial v_{1q}^{*}}{\partial x}\right) = \eta_q\,\mathrm{Re}\,(E_2)\quad . \tag{4.5}$$

From Eq. 4.3, by extracting the first-order equation, which is consistent with the dispersion equation of Eq. 4.1, the following is true:

$$v_{1q} = \frac{\eta_q}{j\omega}\,E_1\quad ; \tag{4.6}$$

therefore, from Eqs. 4.5 and 4.6 it can be found that

$$\mathrm{Re}\,(E_2) = \frac{\eta_q k_i}{2\omega^2}\,|E_1|^2\quad . \tag{4.7}$$

In a similar fashion Eq. 4.4 can be applied to the beam to obtain

$$j(\omega - kv_{ob})v_{1b} = \eta_b E_1 \tag{4.8}$$

so that Eq. 4.7 can be written in the form

$$\mathrm{Re}\,(E_2) = \frac{\eta_q k_i}{2\omega^2\eta_b^2}\,[(\omega - k_r v_{ob})^2 + k_i^2 v_{ob}^2]\,|v_{1b}|^2\quad . \tag{4.9}$$

Use of Eq. 4.9 in Eq. 4.4 leads to

$$v_{2b} = \frac{|v_{1b}|^2}{4v_{ob}}\left\{\left(\frac{\eta_q}{\eta_b}\right)\left[\left(1 - \frac{k_r v_{ob}}{\omega}\right)^2 + \frac{k_i^2 v_{ob}^2}{\omega^2}\right] - 1\right\}\quad . \tag{4.10}$$

Some remarks regarding the nature of the evolvement of the instability in space and time are now needed. The dispersion equation, Eq. 4.1, and its solution, Eq. 4.2, indicates that a perturbation at any frequency $\omega \leq \omega_{pq}$ at any point in the system will initially be unstable at that point. Since the present system is purely electrostatic the Poynting vector is zero so that no power is being fed into the RF field E_1. However, consideration of the RF electrostatic field energy density, $W_{e\ell}(x) = (\epsilon/2)|E_1(x)|^2$, shows clearly that by the definition of convective instability this quantity must be increasing with distance so that, corresponding to $k_r > 0$ and $k_i > 0$, $W_{e\ell}(x_2) > W_{e\ell}(x_1)$ if $x_2 > x_1$. Assume then that the instability is triggered by a continuous source at $x = 0$, $t = 0$ at a frequency $\omega_t < \omega_{pq}$ and examine the system at a later (elapsed) time $t = \delta$. Assume further that δ is much shorter than the characteristic times of, and the operating frequency ω_t is much greater than the characteristic frequencies of, any possible recombination or generation mechanisms present.

Since questions of recombination and generation are then inconsequential, continuity of the time-averaged charge then necessitates that

$$\mathrm{Re}\left(\nabla \cdot \underline{J}_2 + \frac{\partial \rho_2}{\partial t}\right) = 0 \quad . \tag{4.11}$$

Since ω is real, $\mathrm{Re}(\partial\rho_2/\partial t) = 0$, and hence Eq. 4.11 leads to

$$\rho_o v_2 + \frac{1}{2}\,\mathrm{Re}(\rho_1 v_1^*) + \rho_2 v_o = 0 \quad ; \quad k_i \neq 0 \quad . \tag{4.12}$$

For any carrier the time-averaged carrier-mode kinetic energy is defined by

$$\langle \mathrm{K.E.} \rangle = \frac{1}{2\eta} \langle \mathrm{Re}(\rho\underline{v} \cdot \underline{v}) \rangle \quad , \tag{4.13}$$

where < > denotes a time average, e.g., $\langle \mathrm{Re}(\rho_1 v_1) \rangle = (1/2)\mathrm{Re}(\rho_1 v_1^*)$. When expanded to second order Eq. 4.13 gives

$$\langle K.E. \rangle = \frac{1}{2\eta}\left(2v_o v_2 \rho_o + \langle \mathrm{Re}\ \rho_o(\underline{v}_1 \cdot \underline{v}_1) \rangle + 2v_o \langle \mathrm{Re}\ (\rho_1 v_1) \rangle + \rho_2 v_o^2 + \rho_o v_o^2\right) . \tag{4.14}$$

If Eq. 4.12 is used to replace ρ_2 in Eq. 4.14, the following results:

$$\langle K.E. \rangle = \frac{\rho_o}{2\eta} v_o^2 + \frac{\rho_o}{2\eta} v_o v_2 + \rho_o \frac{|v_1|^2}{4\eta} + \frac{v_o}{4\eta} \mathrm{Re}(\rho_1 v_1^*) . \tag{4.15}$$

Apply this now to the electron beam-q plasma interaction. Continuity of the RF beam charge gives

$$\nabla \cdot \underline{J}_{1b} + \frac{\partial \rho_{1b}}{\partial t} = 0 , \tag{4.16}$$

from which it can be determined that

$$\mathrm{Re}(\rho_{1b} v_{1b}^*) = \frac{\rho_{ob} |v_{1b}|^2 [k_r(\omega - k_r v_{ob}) - k_i^2 v_{ob}]}{[(\omega - k_r v_{ob})^2 + k_i^2 v_{ob}^2]} . \tag{4.17}$$

The solution with $k_i > 0$ from Eq. 4.2 when it is inserted into Eq. 4.17 gives

$$\mathrm{Re}(\rho_{1b} v_{1b}^*) = -\frac{\rho_{ob} |v_{1b}|^2}{v_{ob}} . \tag{4.18}$$

Finally, Eq. 4.18 used in Eq. 4.15 provides the following for the beam:

$$\langle K.E. \rangle_{beam} = \frac{\rho_{ob}}{2\eta_b}\left(v_{ob}^2 + v_{ob} v_{2b}\right) . \tag{4.19}$$

Recall now that the RF electrostatic field energy $W_{e\ell}(x)$ is increasing in the positive x-direction away from $x = 0$. A simple consideration of conservation of energy shows that it must then be that the kinetic energy of the beam is decreasing in this direction, or from Eq. 4.19 since $v_{ob} > 0$ if $\omega_t > 0$, $v_{2b} < 0$.

Examine now the function v_{2b} given in Eq. 4.10. Equation 4.2 shows that Eq. 4.10 can be written in the form

$$v_{2b} = \frac{|v_{1b}|^2}{4v_{ob}} \left[\frac{\eta_q}{\eta_b} \left(\frac{k_i^2 v_{ob}^2}{\omega^2} \right) - 1 \right] \quad . \qquad (4.20)$$

Until now no assumptions about the q-plasma have been made except that it corresponds to a single species of carrier charge. Assume now that the q-plasma is an electron-ion (gaseous) plasma in which the ions have infinite mass. The ion plasma frequency, ω_{pi}, is then zero and the plasma is effectively one component with $\omega_{pp} = \omega_{pq} = \omega_{pe}$. Moreover $(\eta_q/\eta_b) = 1$ and hence $v_{2b} < 0$ in Eq. 4.20 only if $k_i < \omega/v_{ob} = k_r$. From Eq. 4.2, since $\omega_t \leqq \omega_{pp}$,

$$k_i v_{ob} = \omega_{pb} \left(\frac{\omega_{pp}^2}{\omega_t^2} - 1 \right)^{-1/2} , \qquad (4.21)$$

and hence $v_{2b} < 0$ only if $\omega_t > (\omega_{pb}^2 + \omega_{pq}^2)^{1/2}$.

The physical reasoning behind this result can be seen if the reaction of the beam electrons to the growing RF field is examined after the time of initiation. For all practical purposes no dc bunching has yet occurred at or very near to $t = 0$ so that $E_2(x) = \rho_2(x) = 0$ and from

Eq. 4.4, $v_{2b} = -(1/4v_{ob})|v_{1b}|^2$. This result is in essence independent of the magnitude of k_i and is really as far in time as the linear dispersion relation, Eq. 4.1, can go with full accuracy. (In general, the linear dispersion relation is still of vast importance since its solution must necessarily be unstable for the instability to be initiated in the first place.) At a slightly later time, however, the reactive effects of the growing RF field E_1 on the beam charges lead to the spatial effect that in the region of interaction (i.e., in the region of the presence of E_1) the field E_2 is generated. In particular, the quasi-linear theory shows that for the infinite-mass ion case if $\omega_t \leq (\omega_{pb}^2 + \omega_{pq}^2)^{1/2}$ the field E_2 is sufficiently strong to quench the further evolvement of the instability.

Clearly the linear theory is in some ways extended too far by the quasi-linear theory. If the field E_2 is large, ρ_{2b} is then large, and hence $\rho'_{ob} \triangleq \rho_{ob} + \rho_{2b} = \rho_{ob}(x)$, and the time-averaged carrier density is dependent on x. Self-consistently then the dispersion equation cannot be formulated assuming a dependence as $\exp[j(\omega t - kx)]$ for the RF variables but rather as $\exp\{j[\omega t - \int^x k(x')dx']\}$, so that

$$\nabla \rightarrow -jk(x) \quad . \qquad (4.22)$$

In the quasi-linear approach k was held fixed so that the linear equations could be extended. This was useful in determining the direction in which the system moves from the point of initiation by studying the reaction of the charges to the growing RF fields. It is this charge reaction which itself causes the time-averaged charge density to become a function of x necessitating $k \rightarrow k(x)$. The quasi-linear theory then suggests the following method for incorporating nonlinear effects directly into the dispersion relation. The general carrier drift velocity is written as

$$v_o(x) = v_o(0) + v_2(0)\, e^{2\int^x k_i(x')\, dx'} , \qquad (4.23)$$

where now $k_i = k_i(x)$. The force equation becomes

$$\frac{\partial v_1}{\partial t} + v_o(x) \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_o(x)}{\partial x} = \eta E_1 \qquad (4.24)$$

so that when the dependence of Eq. 4.22 is used, wherein the $e^{j\omega t}$ dependence is retained since ω is assumed real, it is found that

$$j[\omega - kv_o(x)]v_1 + 2k_i v_2(0)\, v_1\, e^{2\int^x k_i(x')\, dx'} = \eta E_1 . \qquad (4.25)$$

On the basis of Eq. 4.23 the following may also be written:

$$\rho_o(x) = \rho_o(0) + \rho_2(0)\, e^{2\int^x k_i(x')\, dx'} . \qquad (4.26)$$

Equation 4.26 coupled with the form of Eq. 4.22 shows that the continuity equation takes the form

$$\nabla \cdot \underline{J}_1 + \frac{\partial \rho_1}{\partial t} = -jk(\rho_o v_1 + \rho_1 v_o) + 2k_i[\rho_2(0)v_1 + v_2(0)\rho_1]\, e^{2\int^x k_i(x')\, dx'}$$

$$+ j\omega\rho_1 = 0 . \qquad (4.27)$$

From Poisson's equation, $\epsilon\nabla \cdot \underline{E}_1 = \sum_s \rho_1^{(s)}$, and if the dependence of Eq. 4.22 is used as well as Eqs. 4.25 and 4.27, a form of the dispersion relation is obtained as follows:

$$1 = \sum_s \frac{\eta_s \left(-k\rho_{os}(x) - 2jk_i \, e^{2\int^x k_i(x')\,dx'} \rho_{2s}(0) \right)}{\epsilon k \left(j(\omega - kv_{os}) + 2k_i v_{2s}(0) \, e^{2\int^x k_i(x')\,dx'} \right)^2} \quad . \tag{4.28}$$

If $k_i \to 0$ this reduces to the usual dispersion equation, Eq. 4.1. Equation 4.28 can then be regarded as a dispersion relation which incorporates the nonlinear reactive effects. Note that v_{2s} and E_2 still satisfy Eq. 4.4 where now these functions are assumed to vary as $\exp 2 \int^x k_i(x')\, dx'$. To solve this dispersion relation it is necessary to obtain the sets $[k_r(x), k_i(x)]$ which at some real frequency ω of interest satisfy Eq. 4.28, bearing in mind that the functions $v_{2s}(0)$ and $\rho_{2s}(0)$ are themselves in general dependent upon $k_i(x = 0)$, (e.g., Eq. 4.20), in such a fashion as to maintain continuity of the time-averaged charge density. This problem is then too complex to solve by any analytical means.

A case in which at least the form of the solution for k can be found is if $k_i \ll k_r$ so that the terms containing $v_{2s}(0)$ and $\rho_{2s}(0)$ in Eq. 4.28 can be neglected and, in addition, it is assumed that the q-plasma is such that $|\eta_q/\eta_b| \ll 1$ so that $v_{oq}(x) \simeq 0$ and $\rho_{2q}(x) \simeq 0$. In this case Eq. 4.28 is approximated by

$$1 \approx \frac{\omega_{pb}^2(x)}{\left(\omega - kv_{ob}(x)\right)^2} + \frac{\omega_{pq}^2}{\omega^2} , \qquad (4.29)$$

which is readily solved as

$$k(x) = \frac{\left[\omega \pm \omega_{pb}(x)\left(1 - \frac{\omega_{pq}^2}{\omega^2}\right)^{-1/2}\right]}{v_{ob}(x)} . \qquad (4.30)$$

In particular when $v_{ob}(x)$ and $\omega_{pb}(x)$ are taken as constants the result of Eq. 4.2 is retrieved. Even in this simple case Eq. 4.30 is difficult to solve since $\omega_{pb}(x)$ and $v_{ob}(x)$ are actually dependent upon $k_i(x)$.

To proceed with the physical reasoning behind the quasi-linear theory it will be instructive to consider the case where the q-plasma corresponds to ions alone (e.g., positive ion sheath) so that in its interaction with the electron stream $\eta_q/\eta_b < 0$ and from Eq. 4.10 it is seen that there is now an additional component of beam slowing due to E_2. Recall now that the linear theory provides exponentially growing fields, $E_1 \sim e^{k_i x}$, so that a beam electron downstream has lost more energy than one at a smaller value of x. This energy loss cannot come from RF bunching alone since there would then be a net time-averaged, spatially dependent, second-order dc current $J_2 = (1/2)\mathrm{Re}(\rho_1 v_1^*)$ and this would violate conservation of charge. Hence beam slowing and dc bunching also occur and moreover these quantities are spatially dependent (increasing in magnitude with x) varying as $e^{2k_i x}$ to match and annul the time-averaged RF bunching current. This dc bunching in general then gives rise to the field E_2.

In extending the linear theory it was assumed that the k value (independent of x) could be retained. Hence the quasi-linear theory imposes the constraint upon the system that it provide the same field growth as the linear theory even when second-order effects are included. In the ion q-plasma case this gave the result that the beam was losing even more kinetic energy (e.g., the additional component of beam slowing provides in Eq. 4.19 that $|\langle K.E.\rangle|$ has increased) than that required of the linear theory. The reason for this is that the electron beam now must not only slow by an amount to feed a growth rate k_i to the field E_1 but it must slow further (lose further kinetic energy) in order to account for the fact that the potential energy of the electron beam-q plasma system has increased. Stated alternatively, for the electron beam to slow in the presence of the ion-q plasma the quasi-linear theory states that the beam electrons must go uphill (do additional work and lose more energy), whereas the linear theory has the beam electrons on a flat plane.

Intuitively it is clear that in this case the quasi-linear theory indicates that it is now more difficult for the RF field E_1 to extract energy from the beam since it competes for this energy with the potential energy of the charge carrier system. Since the interaction strength is determined for all practical purposes by the parameters v_{ob}, ω_{pb}, and ω_{pq} (and these are effectively constant for small signals), in actuality it is expected that the growth rate of the system will be less than that predicted by the linear theory. Indeed because the interaction strength is independent of E_2-field effects the quasi-linear theory strongly suggests that when the magnitude of the potential energy part of v_{2b} exceeds that of the RF field part, viz.,

$$\left| \frac{\eta_q}{\eta_b} \right| k_i^2 v_{ob}^2 > \omega^2 \quad , \tag{4.31}$$

where Eq. 4.20 has been used, the instability does not occur in fact.

In the opposite case $\eta_q/\eta_b > 0$, corresponding to the electron dominated or infinite ion mass q-plasma case, Eq. 4.20 indicates a component of beam speeding results from E_2. In this case, to retain the growth rate k_i predicted by the linear theory, the beam electrons must slow less than the amount needed to feed the RF field growth in order to account for the decreased potential energy of the system. Instability occurs more readily in this system since the charge separation effects tend to keep the beam electrons and the wave field in synchronism. A limiting action (on k_i) due to E_2 is also suggested for this case, since if Eq. 4.31 is satisfied $v_{2b} > 0$, and the unacceptable situation results that the beam energy and/or the field energy is increasing at the expense of the carrier charge potential energy. Rather, the instability does not ensue in this case and the primary effect of E_2 is to alter the saturation characteristics and growth rate predicted by the linear theory.

The quasi-linear theory thus provides some important differences in the description of the evolvement of convective instabilities in this system as compared with the linear result. On the basis of the linear dispersion relation alone it is concluded that the sign of the charges of the interacting carriers is of no importance (in the dispersion relation the charge always enters as a squared quantity so that the sign is lost). The quasi-linear theory on the other hand stresses the importance of the charge sign.

4.2.2 Two-Stream Longitudinal Amplification. For the two-stream convective instability the dispersion relation neglecting collisions and carrier diffusion is well known to be

$$1 = \frac{\omega_{p1}^2}{(\omega - kv_{01})^2} + \frac{\omega_{p2}^2}{(\omega - kv_{02})^2} \quad . \tag{4.32}$$

To facilitate the analysis consider only the case where

$$\frac{\omega_{p1}^2}{v_{01}^2} = \frac{\omega_{p2}^2}{v_{02}^2} \triangleq k_0^2 \quad \text{and} \quad v_{01}, v_{02} > 0 \quad . \tag{4.33}$$

Introduce the following definitions:

$$k_s \triangleq \frac{\omega}{v_{os}} \quad ; \quad s = 1,2 \quad , \tag{4.34}$$

and

$$k_{\pm} \triangleq \frac{1}{2}(k_1 \pm k_2) \quad . \tag{4.35}$$

The dispersion equation, Eq. 4.32, is solved for ω assumed purely real as

$$(k - k_+)^2 = k_0^2 + k_-^2 \pm (k_0^2 + 4k_-^2)^{1/2} \quad . \tag{4.36}$$

It can be shown by letting $\omega_i \to -\infty$ that $k \to \omega/v_{01}$, ω/v_{02} so that all four roots are causal for $k_i > 0$. Inspection of Eq. 4.36 shows that for complex k to exist for real ω

$$k_0^2 + k_-^2 < +k_0(k_0^2 + 4k_-^2)^{1/2} \quad , \tag{4.37}$$

wherein the (-) sign must be chosen in Eq. 4.36 indicating that two of the roots are dropped from consideration. Equation 4.37 can be simplified to

$$\omega \left| \frac{1}{v_{01}} - \frac{1}{v_{02}} \right| < \sqrt{2}\, k_o \quad ; \quad v_{01} \neq v_{02} \ , \tag{4.38}$$

and the only root of interest is given by

$$k = \frac{1}{2}\left(\frac{\omega}{v_{01}} + \frac{\omega}{v_{02}} \right) + j|\gamma| \ , \tag{4.39}$$

where

$$|\gamma| = \left(k_o \sqrt{k_o^2 + 4k_-^2} - (k_o^2 + k_-^2)\right)^{1/2} . \tag{4.40}$$

As discussed in the previous example assume that questions of recombination and generation do not arise in the time and frequency scales considered. For the quasi-linear theory Eqs. 4.4 and 4.12 are applicable to the present case for each of the charge carriers (s = 1,2). These relations, together with Poisson's equation in the form $\epsilon\nabla \cdot \underline{E}_2 = \sum_s \rho_{2s}$, yield

$$\mathrm{Re}(E_2) = \frac{k_i k_o^2}{4(k_o^2 + 2k_i^2)} \sum_{s=1,2} \frac{|v_1^{(s)}|^2}{\eta_s} \frac{(\omega - k_r v_{os})(\omega - 3k_r v_{os}) + 3k_i^2 v_{os}^2}{(\omega - k_r v_{os})^2 + k_i^2 v_{os}^2} . \tag{4.41}$$

From the first-order equation,

$$j(\omega - kv_{os})v_1^{(s)} = \eta_s E_1 \quad ; \quad s = 1,2 \tag{4.42}$$

it can be determined that

$$|v_1^{(2)}|^2 = \left(\frac{\eta_2}{\eta_1}\right)^2 \frac{(\omega - k_r v_{o1})^2 + k_i^2 v_{o1}^2}{(\omega - k_r v_{o2})^2 + k_i^2 v_{o2}^2} |v_1^{(1)}|^2 \qquad (4.43)$$

Use of Eqs. 4.41 and 4.43 in Eq. 4.4 for s = 1 leads to

$$v_2^{(1)} = \frac{|v_1^{(1)}|^2}{4v_{o1}} \left\{ \frac{2k_o^2}{k_o^2 + 2k_i^2} \left[\frac{(\omega - k_r v_{o1})(\omega - 3k_r v_{o1}) + 3k_i^2 v_{o1}^2}{(\omega - k_r v_{o1})^2 + k_i^2 v_{o1}^2} + \left(\frac{\eta_2}{\eta_1}\right) \right.\right.$$

$$\left.\left. \cdot \frac{[(\omega - k_r v_{o2})(\omega - 3k_r v_{o2}) + 3k_i^2 v_{o2}^2][(\omega - k_r v_{o1})^2 + k_i^2 v_{o1}^2]}{[(\omega - k_r v_{o2})^2 + k_i^2 v_{o2}^2]^2} \right] - 1 \right\} . \qquad (4.44)$$

By symmetry considerations $v_2^{(2)}$ is obtained by the replacement $\binom{1 \to 2}{2 \to 1}$ throughout Eq. 4.44. The present case cannot be reduced to the system studied in Section 4.2.1 because of the assumption made in Eq. 4.33. By comparison with Eq. 4.10,however, it is seen that Eq. 4.44 introduces an additional term in $v_2^{(1)}$ due to the fact that the carrier with which it interacts is now drifting.

The solution given in Eqs. 4.39 and 4.40 is too lengthy to analytically deal with the general behavior of $v_2^{(s)}$. It is clear however that the introduction of drift on the second carrier has permitted an extra degree of control over the second-order charge dynamics to be established and as a result this leads to an additional control of the maximum growth rate k_i and the saturation length (e.g., the length L such that $v_2(0)\, e^{2k_i L} \simeq v_o$) of the system.

4.2.3 Two-Stream Transverse Amplification. It will be assumed that one of the charge carrier species is stationary, e.g., hole or ion, and its interaction with a drifted electron stream will be studied. The dispersion relation for the purely transverse interaction is given by

$$\omega^2 - k^2c^2 - \frac{\omega\omega_{pi}^2}{\omega - \omega_{ci}} - \frac{\omega_{pe}^2(\omega - kv_o)}{\omega - kv_o - \omega_{ce}} = 0 \quad , \tag{4.45}$$

where the (-) mode framework is used and $\omega_{cs} = \eta_s B_o$; s = e,i. For the undrifted species Eq. 2.26 is used to give

$$E_{2-} = -\mathrm{Re}\left(\frac{1}{2j\omega} v_{1-}^{(i)} \frac{\partial E_{1-}^*}{\partial x} \right) , \tag{4.46}$$

where

$$v_{1-}^{(i)} = \frac{\eta_i E_{1-}}{j(\omega - \omega_{ci})} . \tag{4.47}$$

In addition it can readily be found from the first-order equations of Chapter II that

$$|v_{1-}^{(i)}|^2 = \left(\frac{\eta_i}{\eta_e}\right)^2 \frac{\omega^2[(\omega - k_r v_o - \omega_{ce})^2 + k_i^2 v_o^2]}{[(\omega - k_r v_o)^2 + k_i^2 v_o^2](\omega - \omega_{ci})^2} |v_{1-}^{(e)}|^2 . \tag{4.48}$$

Equations 4.46 and 4.48 then give

$$E_{2-} = \frac{k_i \eta_i \omega^2[(\omega - k_r v_o - \omega_{ce})^2 + k_i^2 v_o^2]}{2\omega\eta_e^2(\omega - \omega_{ci})[(\omega - k_r v_o)^2 + k_i^2 v_o^2]} |v_{1-}^{(e)}|^2 . \tag{4.49}$$

From the time-averaged second-order force equation for the electron stream, viz.,

$$2k_i v_o v_{2-}^{(e)} = \eta_e E_{2-} + \eta_e \,\mathrm{Re}\left(\frac{1}{2j\omega} v_{1-}^{(e)} \frac{\partial E_{1-}^*}{\partial x} \right) , \tag{4.50}$$

it can then be found when Eq. 4.49 is used that

$$v_{2^-}^{(e)} = \frac{|v_{1^-}^{(e)}|^2}{4v_o[(\omega - k_r v_o)^2 + k_i^2 v_o^2]} \left[\left(\frac{\eta_i}{\eta_e} \right) \frac{\omega}{\omega - \omega_{ci}} [(\omega - k_r v_o - \omega_{ce})^2 + k_i^2 v_o^2] - [(\omega - k_r v_o)^2 + k_i^2 v_o^2 - \omega\omega_{ce}] \right] , \quad (4.51)$$

where, from Eq. 4.15,

$$v_{2^-}^{(e)} < 0 \quad \text{and} \quad |v_{2^-}^{(e)}| > \frac{|v_{1^-}^{(e)}|^2}{v_o} \quad \text{for } \langle K.E. \rangle < \frac{\rho_o^{(e)} v_o^2}{2\eta_e} .$$

Now inspection of the k-cubic dispersion equation shows that an unstable convective root ($k_i > 0$) is expected if $\omega_{ce} < 0$ and

$$\left(\frac{v_o}{c} \right)^2 \gtrsim \frac{(\omega + |\omega_{ce}|)^2 (\omega - |\omega_{ci}|)}{3[\omega_{pe}^2 (\omega - |\omega_{ci}|) + \omega\omega_{pi}^2]} > 0 , \quad (4.52)$$

and this is easily satisfied near the resonance $\omega \simeq |\omega_{ci}| + |\delta|$. Since $(\eta_i/\eta_e) < 0$, Eq. 4.51 then shows that the reactive effects attempt to drive the charge carrier system to a higher level of potential energy. As discussed for the electron beam-q plasma interaction in Section 4.2.1, in this case instability is more difficult to achieve and the system in actuality has a growth rate diminished from the value predicted by the linear theory.

4.3 Effects of Collisions and Carrier Diffusion

Up to this point to obtain useful analytical models for the quasi-linear theory the presence of collisions and carrier diffusion has been totally ignored. It was then determined that in multiple carrier interactions the presence of the growing RF fields alters the carrier dc characteristics to satisfy the condition of charge continuity. This in turn led to an alteration in the potential energy of the charge carrier system.

To obtain some understanding of the effects of collisions and thermal diffusion examine the time-averaged second-order equation given by

$$(\nu + 2k_i v_o) v_2 = -\frac{k_i}{2} |v_1|^2 + \eta \operatorname{Re}(E_2) - 2k_i \frac{v_T^2}{\rho_o} \rho_2 \qquad (4.53)$$

which corresponds to the purely longitudinal interaction when collisions and diffusion are retained. In addition Poisson's equation, $\epsilon \nabla \cdot \underline{E}_2 = \sum_s \rho_{2s}$, can be used to replace E_2 in Eq. 4.53 giving

$$(\nu + 2k_i v_o) v_2 = -\frac{k_i}{2} |v_1|^2 + \frac{\eta}{2k_i \epsilon} \left[\left(1 - \frac{4k_i^2 v_T^2}{\omega_p^2} \right) \rho_2 + \rho_2' \right], \qquad (4.54)$$

where ρ_2' is appropriate to the carrier being interacted with (e.g., the undrifted carrier). Also Eqs. 4.12 and 4.17 are still applicable to the present case and from these it can be found that

$$\rho_2 = -\frac{\rho_o v_2}{v_o} - \frac{\rho_o |v_1|^2}{2v_o} \frac{k_r(\omega - k_r v_o) - k_i^2 v_o}{(\omega - k_r v_o)^2 + k_i^2 v_o^2} . \qquad (4.55)$$

Finally, if Eq. 4.55 is used in Eq. 4.54 the following results:

$$\left[\nu + 2k_i v_o + \frac{\omega_p^2}{2k_i v_o} \left(1 - \frac{4k_i^2 v_T^2}{\omega_p^2} \right) \right] v_2 = -\frac{k_i}{2} |v_1|^2 + \frac{\eta \rho_2'}{2k_i \epsilon}$$

$$- \frac{\omega_p^2}{4k_i v_o} |v_1|^2 \frac{k_r(\omega - k_r v_o) - k_i^2 v_o}{(\omega - k_r v_o)^2 + k_i^2 v_o^2} \left(1 - \frac{4k_i^2 v_T^2}{\omega_p^2} \right) . \qquad (4.56)$$

Intuitively it would be expected that the carrier thermal diffusion would tend to prevent the formation of ρ_2 (and hence E_2) and as a result diminish the second-order potential energy effects. Indeed, from Eqs. 4.15 and 4.17, in the present case the general result

$$\langle K.E.\rangle_{beam} = \frac{\rho_o}{2\eta} v_o^2 + \frac{\rho_o}{2\eta} v_o v_2 + \frac{\rho_o |v_1|^2}{4\eta} \frac{\omega(\omega - k_r v_o)}{(\omega - k_r v_o)^2 + k_i^2 v_o^2} \qquad (4.57)$$

shows that the beam need not be slowed to obtain

$$\left(- \frac{\rho_o}{2\eta} v_o^2 + \langle K.E.\rangle_{beam}\right) < 0$$

if $(\omega - k_r v_o) < 0$.

The complexity of the problem of second-order effects is now described. It would appear justifiable to assume that if the charges have large thermal velocities concurrent with a low collision frequency (so that the diffusion coefficients are large) then $v_2, \rho_2, E_2 \to 0$. In this case Eq. 4.57 would suggest that the RF bunching is solely responsible for the energy lost by the beam. However in this same case the time-averaged charge density ρ_o is constant in the x-direction so that k is a constant corresponding to the medium being homogeneous. This then implies that Eq. 4.12 must be satisfied bringing in the contradiction that $v_2, \rho_2 \neq 0$ to satisfy charge continuity. On the other hand, if the collision frequency is sufficiently large, the thermal diffusion effects should be greatly reduced and the importance of potential energy considerations becomes reestablished.

The quasi-linear theory then at least helps to provide some insight into the difficult problem of the nature of the evolvement of a convective instability from its point of initiation. The field E_2 would appear to be necessarily important in the purely transverse interaction since there is no RF carrier bunching in the hydrodynamic theory so that carrier dc slowing must occur to provide the energy to drive the instability. On the other hand for space-charge-wave interactions the RF bunching process can be the dominant energy extraction process. In this latter case, if the beam slowing is small, the saturation length L (where L is such that $v_o \approx v_2(0)\, e^{2k_i L}$) is much larger than for the purely transverse waves. Note that since $v_2, \rho_2, E_2 \sim e^{2k_i x}$, whereas $v_1, E_1 \sim e^{k_i x}$, it would appear that the second-order dc effects, at least in some cases, will dominate the nonlinear behavior since the condition $v_2(0)\, e^{2k_i x} \approx v_o$ can arise in space prior to $|v_1(0)|\, e^{k_i x} \approx v_o$.

4.4 Application of Quasi-Linear Theory to Absolutely Unstable Systems

In general the absolute instability process is much more difficult to analyze than the convective instability case since both ω and k can be complex. For this reason only the case where k is assumed purely real is considered. This still retains the essential physics of absolute instabilities since the effect of complex k is primarily to permit time-averaged power to be extracted from one of the device terminals. Since Briggs' criteria shows that for any absolute instability the system must possess at least one root with $\omega_i < 0$ for real k, the present analysis is then meaningful in all cases.

In general to study the instability characteristics of a system a linear wave equation is used to determine whether a small perturbation of the RF field will grow from the onset point (in space or time) of the initial perturbation. Although the system boundaries transverse to the wave vector $\underline{k}$ can be taken into account in this theory (e.g., this usually introduces some additional mode structure into the solutions), this is not true of the terminal boundaries parallel to $\underline{k}$ (the end boundaries). For convective instabilities this is not a significant problem in general because the interaction proceeds spatially so that the condition of charge continuity is satisfied locally (questions concerning reflections are outside the scope of the present work).

In the absolute instability process the end boundaries can play a fundamentally important role however. To understand this assume that a system of purely transverse waves is absolutely unstable ($\omega_i < 0$) with k real. Since the waves are purely transverse, no RF bunching is present so that it must be that the active charge carrier is slowing as a function of time (at all points in the interaction region) as this is the only method available to feed energy to the RF field growth. This immediately raises questions about the end boundaries. For example, assume that this system corresponds to an electron beam as the active source entering and exiting a finite plasma at $x = 0$ and $x = L$. Assume further that the RF fields are zero for $x < 0$ and $x > L$ (e.g., in a gaseous plasma, coupling devices are at $x = 0$ and $x = L$ and in a solid-state plasma these are the contact points). In any case there is no interaction for $x < 0$ and $x > L$ so that in these regions the beam is unaffected. On the other hand at all points x in the plasma the beam is slowing exponentially with time. Clearly this raises questions of charge continuity at $x = 0$ and $x = L$.

If the growth rate ω_i is extremely slow it would appear feasible that generation and recombination mechanisms at $x = 0$ and $x = L$ could maintain the continuity. If this cannot happen the charge must build up at these terminal points and create a time-dependent field E_2 whose displacement current provides the continuity. If the former is the case the linear dispersion relation gives an accurate description of the system until such time as the small-signal approximation breaks down. To study the effects of the field E_2 it will be assumed that the time scales of the system are such that charge recombination and generation can be ignored.

4.5 Two-Stream Electrostatic Oscillation

With collisions and thermal diffusion neglected, the two-stream longitudinal interaction can be described by

$$1 = \frac{\omega_{p1}^2}{(\omega - kv_{01})^2} + \frac{\omega_{p2}^2}{(\omega + kv_{02})^2} \quad . \tag{4.58}$$

Consider the simplest case, $\omega_{p1} = \omega_{p2} \triangleq \omega_p$ and $v_{01} = v_{02} \triangleq v_0$, which then yields the solution

$$(kv_0)^2 = \omega^2 + \omega_p^2 \pm \omega_p(4\omega^2 + \omega_p^2)^{1/2} \quad . \tag{4.59}$$

For an absolute instability a double root of k is required with $\omega_i < 0$. Inspection of Eq. 4.59 shows that this only occurs at the purely imaginary frequency $\omega = \omega_0 = -j(\omega_p/2)$, with the corresponding real wave number $k = (\sqrt{3}/2)(\omega_p/v_0)$. As discussed in Chapter III assume the interaction length $L \gg \lambda$ or $L = n\lambda$ with n an integer. In that case the second-order variables of interest vary as $\exp(-2\omega_i t)$ alone.

From Eq. 4.4 therefore the following may be obtained for the terms which vary as $\exp(-2\omega_i t)$:

$$\mathrm{Re}\left(\frac{\partial v_2^{(s)}}{\partial t}\right) = \eta_s \ \mathrm{Re}\,(E_2) \quad ; \qquad s = 1,2 \ , \tag{4.60}$$

where all other possible contributions are zero since k is purely real. The first-order equations provide the following:

$$j(\omega - kv_o)v_1^{(1)} = \eta_1 E_1$$

and

$$j(\omega + kv_o)v_1^{(2)} = \eta_2 E_1 \ , \tag{4.61}$$

so that

$$|v_1^{(1)}|^2 = \left(\frac{\eta_1}{\eta_2}\right)^2 \frac{(\omega_r + kv_o)^2 + \omega_i^2}{(\omega_r - kv_o)^2 + \omega_i^2} |v_1^{(2)}|^2 \ . \tag{4.62}$$

As discussed in Chapter III, from physical considerations, in the interaction region,

$$\rho_2^{(s)}(t) = 0 \quad ; \qquad s = 1,2 \ , \tag{4.63}$$

so that

$$J_2^{(s)}(t) = \rho_{os} v_2^{(s)}(t) + \frac{1}{2}\,\mathrm{Re}\,(\rho_{1s}^{*} v_1^{(s)}) \quad ; \qquad s = 1,2 \ , \tag{4.64}$$

and the proper form of continuity in the absence of any generation or recombination processes is

$$\sum_{s=1,2} J_2^{(s)}(t) + \epsilon \frac{\partial E_2}{\partial t} = 0 \quad . \tag{4.65}$$

Use of Eqs. 4.60, 4.64, and 4.65 provides

$$\mathrm{Re}(E_2) = \frac{k\omega_i\omega_p^2}{2(\omega_p^2 + \omega_i^2)} \left(\frac{(\omega_r - kv_o)\ |v_1^{(1)}|^2}{\eta_1[(\omega_r - kv_o)^2 + \omega_i^2]} + \frac{(\omega_r + kv_o)\ |v_1^{(2)}|^2}{\eta_2[(\omega_r + kv_o)^2 + \omega_i^2]} \right) , \tag{4.66}$$

so at the unstable root $\omega = \omega_o$, from Eq. 4.63, $\eta_2^2\ |v_1^{(1)}|^2 = \eta_1^2\ |v_1^{(2)}|^2$ and as a result

$$\mathrm{Re}(E_2) = \frac{\omega_i\omega_p^2k^2v_o\ |v_1^{(1)}|^2}{2\eta_2(\omega_p^2 + 2\omega_i^2)(k^2v_o^2 + \omega_i^2)} \left(1 - \frac{\eta_1}{\eta_2}\right) . \tag{4.67}$$

Inspection of this result shows that if $\eta_1 = \eta_2$, (i.e., two equivalent beams in contra-flow) then $\mathrm{Re}(E_2) = 0$ and, consequently, from Eq. 4.60, $v_2(t) = 0$ for each stream. Consider the kinetic energy balance which results when $\eta_1 = \eta_2$. Apply Eq. 4.13, where now $\rho_2 = v_2 = 0$, to obtain (where carrier 1 is understood)

$$\langle\langle K.E.\rangle\rangle = \frac{1}{2\eta}\left(\rho_o v_o^2 + 2v_o\rho_o \langle\langle v_1\rangle\rangle + \rho_o\langle\langle v_1^2\rangle\rangle + \langle\langle\rho_1\rangle\rangle v_o^2 + 2v_o\langle\langle v_1\rho_1\rangle\rangle\right) , \tag{4.68}$$

wherein since $v_1 \sim v_1(0)\ e^{-\omega_i t}$ the first-order terms must be retained so that $\langle\langle\ \rangle\rangle$ now represents a space average and the kinetic energy (as expected) is always a function of the time and contains variation both as $\exp(-\omega_i t)$ and $\exp(-2\omega_i t)$. It is clear, however, that if the device length $L = \eta\lambda$, then $\langle\langle v_1\rangle\rangle = \langle\langle\rho_1\rangle\rangle = 0$. If from the continuity equation, Eq. 4.16, the result

$$\rho_1 = \frac{k\rho_o v_1}{\omega - kv_o} \tag{4.69}$$

is used in Eq. 4.68, assuming $L = n\lambda$, the following is true:

$$\langle\langle K.E.\rangle\rangle = \frac{\rho_o}{2\eta} v_o^2 + \frac{\rho_o}{2\eta} \frac{(\omega_r - kv_o)(\omega_r + kv_o) + \omega_i^2}{(\omega_r - kv_o)^2 + \omega_i^2} \langle\langle v_1^2 \rangle\rangle \ . \tag{4.70}$$

It can be seen by inspection that the $\langle\langle K.E.\rangle\rangle$ functions of streams 1 and 2 are identical at $\omega = \omega_o$ and moreover $(\partial/\partial t) \langle\langle K.E.\rangle\rangle < 0$. Indeed, on substituting the solution at $\omega = \omega_o$, for each stream,

$$\langle\langle K.E.\rangle\rangle = \frac{\rho_o}{2\eta}\left(v_o^2 - \frac{\langle\langle v_1^2\rangle\rangle}{2}\right) \quad ; \quad \frac{\rho_o}{\eta} = \frac{\epsilon\omega_p^2}{\eta_1^2} \ . \tag{4.71}$$

When $\eta_1 \neq \eta_2$, however, the field E_2 arises and can affect the evolution in time of the absolute instability. Note that now potential energy changes in the overall system will occur spatially at or near the end boundaries as opposed to the convectively unstable system in which these occurred at all points in the interaction region. The more general form of the stream kinetic energy when $E_2 \neq 0$ is given from Eqs. 4.15 and 4.68 as

$$\langle\langle K.E.\rangle\rangle = \frac{\rho_o}{2\eta} 2v_o v_2 + \frac{\rho_2}{2\eta} v_o^2 + \langle\langle K.E.\ (\text{Eq. } 4.68)\rangle\rangle \ , \tag{4.72}$$

where the last term is the $\langle\langle K.E.\rangle\rangle$ function of Eq. 4.68 which at the unstable root is given by Eq. 4.71 for either carrier.

Recall that in the quasi-linear theory the system is constrained to provide the same solution (e.g., growth rate) as the linear theory even when second-order effects are taken into account. This alters the magnitude of the rate of loss of kinetic energy of the streams from the linear theory (Eq. 4.71) to account for potential energy changes caused by the instability. From Eq. 4.72 these effects become important at least when

$$\left| \frac{\rho_o}{2\eta} 2v_o v_2 + \frac{\rho_2}{2\eta} v_o^2 \right| \approx \langle\langle \text{K.E. (Eq. 4.68)} \rangle\rangle - \frac{\rho_o v_o^2}{2\eta} , \qquad (4.73)$$

or, since in the interaction region $\rho_2(t) = 0$,

$$\left| \frac{\rho_o}{2\eta} 2v_o v_2 \right| \approx \frac{\rho_o}{4\eta} \langle\langle v_1^2 \rangle\rangle . \qquad (4.74)$$

To solve for $v_2^{(1)}$ in the quasi-linear theory, Eq. 4.65 when expanded using Eq. 4.64 gives

$$\rho_{o1} v_2^{(1)} + \frac{k\rho_o \left|v_1^{(1)}\right|^2 (-kv_o)}{2(k^2v_o^2 + \omega_i^2)} + \frac{k\rho_o \left|v_1^{(2)}\right|^2 (kv_o)}{2(k^2v_o^2 + \omega_i^2)} + \rho_{o2} v_2^{(2)} = 2\omega_i \epsilon E_2 . \qquad (4.75)$$

Equation 4.60 can be used to replace $v_2^{(2)}$ which, together with Eqs. 4.62 and 4.67, yields the following at $\omega = \omega_o$:

$$v_2^{(1)} = \frac{k^2 v_o \left|v_1^{(1)}\right|^2}{2(k^2v_o^2 + \omega_i^2)} \left[1 - \left(\frac{\eta_2}{\eta_1}\right)^2 + \frac{2\omega_i^2 \left(\frac{\eta_1}{\eta_2}\right)\left(1 - \frac{\eta_1}{\eta_2}\right)}{(\omega_p^2 + 2\omega_i^2)} \left(1 + \frac{\omega_p^2}{4\omega_i^2}\right) \right] , \qquad (4.76)$$

and since $\omega_i = -\omega_p/2$ the last term can be simplified to the following eventual result:

$$v_2^{(1)} = \frac{3\,|v_1^{(1)}|^2}{8v_o}\left[1 - \left(\frac{\eta_2}{\eta_1}\right)^2 + \frac{2}{3}\,\frac{\eta_1}{\eta_2}\left(1 - \frac{\eta_1}{\eta_2}\right)\right] . \qquad (4.77)$$

When used in Eq. 4.74, since $\langle\langle v_1^2 \rangle\rangle = |v_1^{(1)}|^2$ if $L = n\lambda$, the condition under which the effects of the field E_2 become important (corresponding to the alteration of the system potential energy becoming comparable to the alteration of stream 1 kinetic energy in the absence of E_2) is

$$f(\eta_1,\eta_2) \triangleq \left| \frac{\eta_1}{\eta_2}\left(1 - \frac{\eta_1}{\eta_2}\right) + \frac{3}{2}\left(1 - \frac{\eta_2^2}{\eta_1^2}\right) \right| \simeq 1 . \qquad (4.78)$$

When $f(\eta_1,\eta_2) > 1$ it is expected that the instability is quickly quenched or else driven to a much smaller growth rate. Inspection of the form $f(\eta_1,\eta_2)$ shows that if $\eta_1 \neq \eta_2$ it is easily possible for $f \gtrsim 1$.

The method used herein for the two-stream electrostatic oscillation can readily be applied to any system described by the linear dispersion equation to be absolutely unstable. Note that since in general $\rho_2(t) = 0$ in the interaction region the medium remains homogeneous so that it is not expected that $k \rightarrow k(x)$ due to the RF field growth as in convectively unstable processes. Rather, the linear dispersion equation remains valid (e.g., $\eta_1 = \eta_2$ in the preceding example) until the small-signal approximation breaks down or else the quasi-linear theory is a good approximation to take into account the end effects on the wave growth. Indeed for systems which are absolutely unstable with either k real or $\omega_i \gg k_i v_o$ it is expected that the quasi-linear theory is the best practical method to account for the end boundaries.

When an absolute instability occurs at a double root with $k_i \gtrsim \omega_i/v_o$ all the second-order effects studied in the convective instability process in Section 4.2 become important, e.g., $k \rightarrow k(x)$.

The effects of collisions and thermal diffusion are expected to play a similar role to that discussed in Section 4.3. A much more significant role may be played by the nature of the end boundaries with regard to the buildup or depletion of the carrier charge density near these regions.

4.6 Summary and Discussion

The utility of the quasi-linear theory in aiding in the understanding of the evolvement of plasma instabilities has been shown. It permits studying the instability in the intermediate regime after the initiation of the instability but prior to any large-signal effects. In addition, the quasi-linear theory offers a method for taking into account the effects of the end boundaries of the system on the instability evolvement--something which can be quite important in absolutely unstable systems.

In general, the quasi-linear theory shows that potential energy effects cannot be ignored in the analysis of plasma instabilities even when the thermal diffusion coefficients are large. Since some control can be effected over the parameters determining the second-order effects the quasi-linear theory can thus be used to suggest methods to help quench or diminish undesirable instabilities or enhance desirable ones by altering the growth rate and saturation characteristics.

CHAPTER V. THE POWER THEOREM ACCORDING TO KINETIC THEORY WITH APPLICATIONS

5.1 Introduction

The general definition of the kinetic power flow given by

$$P_k = \frac{1}{2k_i} \mathrm{Re}(\underline{E}_1 \cdot \underline{J}_1^*) \tag{5.1}$$

is readily adopted within the framework of kinetic theory for any carrier velocity distribution function. In the past the power theorem has been restricted to hydrodynamic theory by virtue of the fact that a sufficiently general formulation was not constructed.

The major distribution functions examined are the hydrodynamic or square distribution, the Maxwellian, and the degenerate distribution functions. These are applied in turn to obtain the kinetic power expressions for the basic carrier modes--the space-charge, cyclotron, and hybrid modes. Whenever applicable comparison with the hydrodynamic results of Chapter II is made.

Since a fundamental part of the present work is concerned with solid-state plasmas, special attention is devoted to the role of collisions in the kinetic power functions and the consequent wave behavior. It is shown that the magnetic field and the collisions play a crucially important part in active wave phenomena near the cyclotron resonance point of the carrier cyclotron modes. Applied to the phenomenon of microwave emission from indium antimonide (InSb) it is shown that even a small density of holes can lead to large wave slowing and consequent amplification by the electrons.

Comparison with the hydrodynamic theory of Chapter II shows that in many cases the hydrodynamic theory has been improperly used and the limitations on its applicability are shown. In those cases where the hydrodynamic theory is not valid, the results from the kinetic theory show that resonance and/or anisotropic temperatures can readily lead to instability.

A discussion of causality within the kinetic theory is discussed and suggestions made for its implementation.

5.2 Power Theorem for Longitudinal Space-Charge Waves

The Boltzmann equation is written for this case in the form

$$j(\omega - kv_x)f_1 + \eta E_{1x} \frac{\partial f_o}{\partial v_x} = \left(\frac{\partial f}{\partial t} \right)_{coll} , \tag{5.2}$$

where it has been assumed that the fundamental variables $\sim \exp[j(\omega t - kx)]$ and where various forms for the collision term on the right-hand side will be studied.

5.2.1 The Hydrodynamic Distribution Function. Consider first the drifted-square distribution defined by

$$f_o = \frac{N_o}{2v_T} \delta(v_y)\delta(v_z) \quad \text{for} \quad v_T + v_o \geqq v_x \geqq v_o - v_T ,$$

$$f_o = 0 \quad \text{otherwise} , \tag{5.3}$$

so that

$$\frac{\partial f_o}{\partial v_x} = \frac{N_o}{2v_T} \delta(v_y)\delta(v_z) \Big(\delta(v_x - v_o + v_T) - \delta(v_x - v_o - v_T) \Big) . \tag{5.4}$$

The RF current is in general given by

$$J_{1x} = q \iiint_{-\infty}^{\infty} v_x f_1 \, dv_x \, dv_y \, dv_z \qquad (5.5)$$

for each charge species present.

In the collisionless case, $(\partial f/\partial t)_{coll} = 0$, Eqs. 5.1 through 5.5 can be readily solved to obtain

$$P_k(\nu = 0) = \frac{\omega_p^2 \epsilon \omega v_o \left((\omega - k_r v_o) + k_r \frac{v_T^2}{v_o} \right) |E_{1x}|^2}{\left((\omega - k_r v_o)^2 - k_i^2 v_o^2 - v_T^2 (k_r^2 - k_i^2) \right)^2 + 4k_i^2 \left(v_o (\omega - k_r v_o) + k_r v_T^2 \right)^2} \; . \qquad (5.6)$$

Note that this result is identical to that obtained in the hydrodynamic analysis, Eq. 2.23, as is expected for the square distribution. If one proceeds in a similar fashion for the commonly used collision term defined by

$$\left(\frac{\partial f}{\partial t} \right)_{coll} = -\nu f_1 \; , \qquad (5.7)$$

it can be shown that

$$P_k = \frac{\left[\omega_p^2 \epsilon \omega v_o \, |E_{1x}|^2 \left[(\omega - k_r v_o) \left(1 + \frac{\nu}{k_i v_o} \right) + k_r \frac{v_T^2}{v_o} + \frac{\nu}{2 k_i v_o \omega} \left((\nu + k_i v_o)^2 - (\omega - k_r v_o)^2 + (k_r^2 - k_i^2) v_T^2 \right) \right] \right]}{\left[\left((\omega - k_r v_o)^2 - (\nu + k_i v_o)^2 - (k_r^2 - k_i^2) v_T^2 \right)^2 + 4 \left((\nu + k_i v_o)(\omega - k_r v_o) + k_i k_r v_T^2 \right)^2 \right]} \; . \qquad (5.8)$$

Inspection of this result shows that the collisionally assisted form of the negative kinetic power flow obtained in the hydrodynamic analysis, Eq. 2.23, when $(\omega - k_r v_o) < 0$ is not in general present. To some extent this result is to be expected since it is known that the collision term given by Eq. 5.7 does not conserve particles properly. Although it has been asserted that this collision term is valid if $\omega \gg \nu$,[74] Eq. 5.8 shows that even in this case Eq. 5.7 may not be an accurate representation if $\omega|\omega - k_r v_o| < k_r^2 v_T^2$.

The collision term which conserves particles properly is

$$\left(\frac{\partial f}{\partial t}\right)_{coll} = -\nu\left(f_1 - \frac{N_1}{N_o} f_o\right) , \tag{5.9}$$

where N_1 is the perturbed number density given by

$$N_1 = \iiint_{-\infty}^{\infty} f_1 \, dv_x \, dv_y \, dv_z \; . \tag{5.10}$$

If the resulting Eq. 5.2 is integrated over velocity space and Eq. 5.10 is used, the perturbed distribution function can be written as

$$f_1 = \frac{j\eta E_{1x}}{\omega - kv_x - j\nu}\left[\left(\frac{\partial f_o}{\partial v_x}\right) + \frac{\Phi_o}{\Psi_1} f_o\right] , \tag{5.11}$$

where

$$\Phi_o = \nu \iiint_{-\infty}^{\infty} \frac{\frac{\partial f_o}{\partial v_x}}{\omega - kv_x - j\nu} \, dv_x \, dv_y \, dv_z \tag{5.12}$$

and

$$\Psi_1 = jN_o - \nu \iiint_{-\infty}^{\infty} \frac{f_o}{\omega - kv_x - j\nu} dv_x\, dv_y\, dv_z \quad . \qquad (5.13)$$

For the distribution function of Eq. 5.3,

$$\Phi_o = - \frac{\nu k N_o}{(\omega - kv_o - j\nu)^2 - k^2 v_T^2} \quad . \qquad (5.14)$$

To compute the kinetic power flow it is found from a transformation of variables that the following integrals in the complex μ-plane are required:

$$\Phi_3 = \int_{-1}^{1} \frac{d\mu}{z - \mu} \quad \text{and} \quad \Phi_4 = \int_{-1}^{1} \frac{\mu d\mu}{z - \mu} \, , \qquad (5.15)$$

where

$$z = \frac{\omega - kv_o - j\nu}{kv_T} \qquad (5.16)$$

and

$$\mu = \frac{v_x - v_o}{v_T} \quad . \qquad (5.17)$$

If it is assumed that $k_i \ll k_r$, it is seen from Eq. 5.16 that Im $(z) < 0$ if $k_i > 0$ and $\nu > k_i v_o$. Since the time-dependent Fourier transforms $\sim \exp[j(\omega t - kx)]$ it is necessary to integrate just above the singularity by choosing the contour in the complex μ-plane with corners $[-1, -1 - j|\gamma|, 1 - j|\gamma|, 1]$ and $|\gamma| = -\text{Im}\,(z)$, leading to

$$\Phi_3 = \log_e \left(\frac{z + 1}{z - 1} \right) - j\pi\sigma \qquad (5.18)$$

and

$$\Phi_4 = -2 + z\Phi_3 \, , \qquad (5.19)$$

where

$$\sigma = 1 \ , \quad \text{if} \quad |\text{Re}\ (z)| < 1 \ ; \quad \text{Im}\ (z) > 0 \ ,$$
$$= 0 \qquad \text{otherwise} \ . \tag{5.20}$$

Use of these results in Eqs. 5.5 and 5.11 leads to the kinetic power flow of Eq. 5.1 being given as

$$P_k = \frac{\omega_p^2 \epsilon |E_{1x}|^2}{|k|^2 v_T |(\omega - kv_o - j\nu)^2 - k^2 v_T^2|^2} \cdot \text{Re}\left(-jk^* \ [(\omega - kv_o - j\nu)^2 - k^2 v_T^2]^* \ [\omega - j\nu F(\Phi_3)]\right) \ , \tag{5.21}$$

where

$$F(\Phi_3) = \frac{2k^* v_T(\omega - j\nu)\Phi_3 + 2j\nu k v_T \Phi_3^* - j\nu\omega\Phi_3\Phi_3^*}{4|k|^2 v_T^2 + \nu^2 \Phi_3 \Phi_3^*} \ . \tag{5.22}$$

The function $F(\Phi_3)$ is the term taking conservation of particles into account. The resultant form, Eq. 5.21, can be studied in various limits.

<u>Case a</u>:

$$|z| \gg 1 \ , \quad |\text{Re}\ (z)| > 1 \ .$$

For this case $\sigma = 0$, $\log_e[(z + 1)/(z - 1)] \simeq j \tan^{-1}(kv_T/\nu) \simeq 0$, and hence $F(\Phi_3) \simeq 0$ so that under the assumption $k_i \ll k_r$

$$P_k \approx \frac{\omega_p^2 \epsilon \omega v_o \ |E_{1x}|^2 \left[k_r \left(1 + \frac{\nu}{k_i v_o}\right) (\omega - k_r v_o)\right]}{|k|^2 v_T \ |(\omega - kv_o - j\nu)^2 - k^2 v_T^2|^2} \ , \tag{5.23}$$

which by comparison with Eq. 2.23 shows that the present conditions are similar in power flow characteristics to the hydrodynamic case.

Case b:

$$|z| \gg 1 \ , \qquad |\mathrm{Re}\ (z)| < 1 \ .$$

From Eq. 5.16 it can be found that in general

$$\mathrm{Im}\ (z) = - \frac{k_i \omega - k_r \nu}{|k|^2\ v_T} \ , \tag{5.24}$$

so that from Eq. 5.20 the residue only contributes if $k_i < 0$ corresponding to damped waves with $|k_i| > k_r \nu/\omega$. Thus the residue cannot play any role in instability (growing wave) processes. For this case $\sigma = 1$ and $\Phi_3 \approx - j\pi$ leading to

$$P_k \approx \frac{\omega_p^2 \epsilon\ |E_{1x}|^2\ 4k_r^2[(\omega - k_r v_o)^2 + \nu^2]}{2k_i\ |k|^2\ v_T^2\ |(\omega - kv_o - j\nu)^2 - k^2 v_T^2|^2} \ , \tag{5.25}$$

so that the self-consistent result obtained is that the interaction can only proceed if $k_i < 0$ in which case $P_k < 0$ to balance the positive electromagnetic power flow. When it is assumed that $|k_i| < k_r \nu/\omega$ the result of Eq. 5.23 is again applicable. For any particular system it is necessary to either solve the difficult dispersion equation exactly or else assume initial constraints such as $|k_i| < k_r \nu/\omega$ to analytically solve the dispersion equation, in which case the results obtained must be in agreement with the initial assumptions for self-consistency. This is a general problem associated with the kinetic theory.

Case c:

$$|z| \ll 1 \ , \qquad |\mathrm{Re}\ (z)| < 1 \ .$$

Again for the case $|k_i| > k_r\nu/\omega$ with $k_i < 0$ the residue contributes so that $\sigma = 1$. In addition $\log_e[(z+1)/(z-1)] \simeq j\pi$ leading to $F(\Phi_s) \approx 0$ and

$$P_k \approx \frac{\omega_p^2 \epsilon\omega\, |E_{1x}|^2\, v_o \left[k_r\left(1 + \frac{\nu}{k_i v_o}\right)(\omega - k_r v_o) + \frac{3}{2} k_r^2 \frac{v_T^2}{v_o}\right]}{|k|^2\, v_T\, |(\omega - kv_o - j\nu)^2 - k^2 v_T^2|^2} \quad . \qquad (5.26)$$

This result is of value in indicating which regions of $(\omega\text{-}k)$ space will potentially provide a self-consistent solution. For example in the present case if $|k_i| \stackrel{\sim}{>} k_r\nu/\omega$ it is required that $|\omega - k_r v_o|^2 > (3/2)k_r^2 v_T^2$ which is not self-consistent with $|z| \ll 1$ and hence no damped solutions are possible. If $|k_i| \gg k_r\nu/\omega$, however, a potential solution is possible $(P_k < 0)$ only if $(\omega - k_r v_o) < 0$ and $v_o > (3/2)k_r v_T^2/(k_r v_o - \omega)$.

For the unstable case $(k_i > 0)$ or $|k_i| < k_r\nu/\omega$, $k_i < 0$, the residue does not contribute and it can be found that, if $k_i \ll k_r$,

$$P_k \approx \frac{4\omega_p^2 \epsilon\, |E_{1x}|^2\, \nu k_r^2 \left(2\omega k_r v_T(\omega - k_r v_o) + \nu\pi[\nu^2 + k_r^2 v_T^2 - (\omega - k_r v_o)^2]\right)}{|k|^2\, |(\omega - kv_o - j\nu)^2 - k^2 v_T^2|^2\, (4\, |k|^2\, v_T^2 + \pi^2 \nu^2)} \quad . \qquad (5.27)$$

Inspection shows that since $|z| \ll 1$ the $k_r^2 v_T^2$ term above dominates and $P_k > 0$ so that an instability is not possible.

These results demonstrate the difference between the hydrodynamic theory of Chapter II and the kinetic theory with the square hydrodynamic distribution function. For unstable waves with $|z| \gg 1$ the two theories are in agreement provided $k_i > 0$ or $|k_i| < k_r\nu/\omega$. However with $|z| \ll 1$ there exist opposing results. Thus, for example, if $v_o < v_T$ it is only if

$\nu > |kv_T|$ that unstable solutions (such as acoustic-phonon amplification) predicted by the hydrodynamic theory are in accord with the kinetic theory.

5.2.2 The Maxwellian Distribution Function. Equations 5.11 through 5.13 can be applied to the isotropic drifted Maxwellian distribution function given by

$$f_o = \frac{N_o}{(2\pi v_T^2)^{3/2}} \exp\left(-\frac{(v_x - v_o)^2}{2v_T^2}\right) \exp\left(-\frac{(v_y^2 + v_z^2)}{2v_T^2}\right) . \qquad (5.28)$$

Define the plasma dispersion function as

$$G(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\xi^2}}{z - \xi} d\xi , \qquad (5.29)$$

from which it can be shown by an integration by parts that

$$\int_{-\infty}^{\infty} \frac{\xi e^{-\xi^2} d\xi}{z - \xi} = \sqrt{\pi} [zG(z) - 1]$$

and

$$\int_{-\infty}^{\infty} \frac{\xi^2 e^{-\xi^2} d\xi}{z - \xi} = \sqrt{\pi} z[zG(z) - 1] . \qquad (5.30)$$

The function G(z) is related to the tabulated function[75] Z by

$$G(z) = -Z(z^*) , \qquad (5.31)$$

where the complex conjugation is due to the tabulated functions being derived assuming variations as $\exp[j(kx - \omega t)]$.

With these definitions and the use of Eqs. 5.5 and 5.11 in Eq. 5.1, wherein only the collision term which conserves particles properly (Eq. 5.9) is studied, the power is found as

$$P_k = \frac{\omega_p^2 \epsilon \omega \, |E_{1x}|^2}{2\sqrt{2}\, k_i \, |k|^2 \, v_T^2 (2 \, |k|^2 \, v_T^2 + \nu^2 G G^*)} \, \mathrm{Re} \left(-jk^* (zG - 1)(\sqrt{2}\, k^* v_T - j\nu G^*) \right) \tag{5.32}$$

where $G = G(z)$ and

$$z \triangleq \frac{\omega - kv_o - j\nu}{\sqrt{2}\, kv_T} \, . \tag{5.33}$$

A central problem exists in the nature of the tabulated function Z (or the function G) as has been pointed out by Montgomery and Tidman;[76] namely, the function so obtained is only valid in the limit $t \to \infty$ and hence cannot be applied to unstable systems. When the buildup occurs in time ($\omega_i < 0$) this may be the case; however, for a convective instability, the linear steady state is well defined and this problem should not occur. In addition this function is commonly used for application to unstable systems so that the kinetic power results using the G function will be obtained in various limits.

<u>Case a</u>:

$$|z| \gg 1 \, .$$

For this case,

$$G(z) \approx -j\sqrt{\pi}\, \sigma \exp(-z^2) + \frac{1}{z}\left(1 + \frac{1}{2z^2} + \frac{3}{4z^4} + \cdots \right) , \tag{5.34}$$

where

$$\sigma = \begin{cases} 0 & \mathrm{Im}\,(z) < 0 \\ 1 & \mathrm{Im}\,(z) = 0 \, , \\ 2 & \mathrm{Im}\,(z) > 0 \end{cases}$$

and from Eq. 5.32

$$\text{Im}\ (z) = -\frac{k_r(\nu + k_i v_o) + k_i(\omega - k_r v_o)}{\sqrt{2}\, v_T(k_r^2 + k_i^2)} . \tag{5.35}$$

For growing waves ($k_i > 0$) inspection of Eq. 5.35 indicates that necessarily Im $(z) < 0$ so that the residue term in Eq. 5.34 does not contribute ($\sigma = 0$) and Eq. 5.32 becomes for this case, retaining the first two terms in the expansion,

$$P_k \approx \frac{\omega_p^2 \epsilon\omega\ |E_{1x}|^2\ (\omega - k_r v_o)(\nu + 2k_i v_o)\Big[|k|^4\ v_T^2 - k_r^2[3(\omega - k_r v_o)^2 - \nu^2]\Big]}{2k_i\ |k|^2\ v_T(2\ |k|^2\ v_T^2 + \nu^2 GG^*)\ |(\omega - kv_o - j\nu)^2|^2} . \tag{5.36}$$

This result shows that $P_k < 0$ provided $(\omega - k_r v_o) < 0$ unless the drift velocity is sufficiently nonsynchronous with the wave phase velocity that $3(\omega - k_r v_o)^2 > k_r^2 \nu^2 + |k|^4\ v_T^2$.

Case b:

$$|z| \ll 1 .$$

For this case, $G(z) \approx 2z[1 - (2z^2/3) + (4z^4/15) - \cdots]$, so that retaining only the first term

$$P_k \approx \frac{\omega_p^2 \epsilon\ |E_{1x}|^2\Big[\nu\{k_r[k_r(\omega - k_r v_o) - k_i(2\nu + k_i v_o)] - k_i^2\omega\} + 2k_i k_r(k_i^2 + k_r^2)v_T^2\Big]}{2k_i\ |k|^2\ (2\ |k|^2\ v_T^2 + \nu^2 GG^*)(k_r^2 + k_i^2)v_T^3} . \tag{5.37}$$

Inspection of this result shows that for the hot plasma case it is again necessary that $(\omega - k_r v_o) > 0$ and in addition that the growth rate be limited to

$$k_i < \frac{\nu \left|\omega - k_r v_o\right|}{2k_r v_T^2} \approx |z| \frac{\nu}{v_T} . \tag{5.38}$$

5.2.3 The Degenerate Distribution Function. The degenerate distribution function is defined by

$$f_o = \frac{N_o}{2\pi v_{T_\perp}^2 v_{T_\parallel}} \quad \text{for} \begin{cases} \sqrt{v_y^2 + v_z^2} \leqq v_{T_\perp} \\ v_o + v_{T_\parallel} \geqq v_x \geqq v_o - v_{T_\parallel} \end{cases} ,$$

$$= 0 \qquad \text{otherwise} , \tag{5.39}$$

so that

$$\frac{\partial f_o}{\partial v_x} = \frac{N_o}{2\pi v_{T_\perp}^2 v_{T_\parallel}} \left[\delta\left(v_x - v_o + v_{T_\parallel}\right) - \delta\left(v_x - v_o - v_{T_\parallel}\right) \right] . \tag{5.40}$$

Note that this distribution is an improvement over the hydrodynamic distribution function of Eq. 5.3 since the latter function ignores the transverse thermal motion. The collision term which conserves particles properly is chosen so that Eq. 5.11 is applicable. Inspection of the pertinent integrals involved in forming the kinetic power function shows that in all cases the transverse thermal distribution has no effect and hence the results obtained for the hydrodynamic distribution function, Eqs. 5.14 through 5.26, are strictly applicable. This is to be expected since the space-charge wave system is one dimensional so that the random transverse thermal motion cannot alter the longitudinal RF current. This is, of course, alleviated by the presence of a transverse static magnetic field.

5.3 Power Theorem for Purely Transverse Waves in a Static Magnetic Field

The kinetic power functions for purely transverse waves with $\underline{B}_o \parallel \underline{k} \parallel \hat{x}$ are now developed. The Boltzmann equation for this case, assuming that the fundamental frequency variables vary as $\exp[j(\omega t - kx)]$, is in linearized form

$$j(\omega - kv_x)f_1 + \omega_c\left(v_z \frac{\partial f_1}{\partial v_y} - v_y \frac{\partial f_1}{\partial v_z}\right) + \eta E_{1y}\frac{\partial f_o}{\partial v_y} + \eta E_{1z}\frac{\partial f_o}{\partial v_z} + \eta(\underline{v} \times \underline{B}_1) \cdot \frac{\partial f_o}{\partial \underline{v}} = \left(\frac{\partial f}{\partial t}\right)_{coll} , \quad (5.41)$$

where $\omega_c = \eta B_o$. Define the reduced distribution functions

$$f_y = \iint_{-\infty}^{\infty} v_y f_1 \, dv_y \, dv_z$$

and

$$f_z = \iint_{-\infty}^{\infty} v_z f_1 \, dv_y \, dv_z , \quad (5.42)$$

together with a transformation to rotating coordinates, viz.,

$$E_{1\pm} = E_{1y} \pm jE_{1z} ,$$

$$f_{1\pm} = f_y \pm jf_z$$

and

$$B_{1\pm} = B_{1y} \pm jB_{1z} . \quad (5.43)$$

Equation 5.41 is multiplied successively by v_y and v_z and integrated over the transverse velocity space (v_y, v_z) to obtain

$$j(\omega - kv_x)f_y - \omega_c f_z - \eta E_{1y} \iint_{-\infty}^{\infty} f_o \, dv_y \, dv_z + \eta \frac{d}{dv_x}$$

$$\cdot \left[\iint_{-\infty}^{\infty} (B_{1z}v_y^2 - v_y v_z B_{1y}) f_o \, dv_y \, dv_z \right] + \eta v_x B_{1z} \iint_{-\infty}^{\infty} f_o \, dv_y \, dv_z$$

$$= \iint_{-\infty}^{\infty} v_y \left(\frac{\partial f}{\partial t} \right)_{coll} dv_y \, dv_z \qquad (5.44)$$

and

$$j(\omega - kv_x)f_z + \omega_c f_y - \eta E_{1z} \iint_{-\infty}^{\infty} f_o \, dv_y \, dv_z + \eta \frac{d}{dv_x}$$

$$\cdot \left[\iint_{-\infty}^{\infty} (v_y v_z B_{1z} - v_z^2 B_{1y}) f_o \, dv_y \, dv_z \right] - \eta v_x B_{1y} \iint_{-\infty}^{\infty} f_o \, dv_y \, dv_z$$

$$= \iint_{-\infty}^{\infty} v_z \left(\frac{\partial f}{\partial t} \right)_{coll} dv_y \, dv_z \, , \qquad (5.45)$$

and where, if i,k subscripts represent y or z it has been assumed that f_o, $f_1 \rightarrow 0$ as $v_i \rightarrow \pm\infty$ so that in obtaining Eqs. 5.44 and 5.45 use has been made of

$$\int_{-\infty}^{\infty} \frac{\partial f_{o,1}}{\partial v_i} dv_i = 0 \ ; \qquad \text{e.g.,} \iint_{-\infty}^{\infty} v_k \frac{\partial f_1}{\partial v_i} dv_i \, dv_k = 0 \ ,$$

$$\iint_{-\infty}^{\infty} v_i v_k \frac{\partial f_1}{\partial v_i} dv_i \, dv_k = -f_k$$

and

$$\iint_{-\infty}^{\infty} v_i \frac{\partial f_{o,1}}{\partial v_i} dv_i\, dv_k = - \iint_{-\infty}^{\infty} f_{o,1}\, dv_i\, dv_k \quad , \tag{5.46}$$

which can easily be shown via integration by parts. Perform the operation [(Eq. 5.44) - j(Eq. 5.45)] and use the definition of Eq. 5.43 to obtain

$$j(\omega - kv_x - \omega_c)f_{1-} - \eta F_o(v_x)E_{1-} + j\eta v_x F_o(v_x)B_{1-} + \eta \frac{d}{dv_x} G_1(v_x)$$
$$\doteq \iint_{-\infty}^{\infty} \left(\frac{\partial f}{\partial t}\right)_{coll} (v_y - jv_z)\, dv_y\, dv_z \quad , \tag{5.47}$$

where

$$F_o(v_x) = \iint_{-\infty}^{\infty} f_o\, dv_y\, dv_z \tag{5.48}$$

and

$$G_1(v_x) = \frac{j}{2} \iint_{-\infty}^{\infty} \left[B_{1-}(v_y^2 + v_z^2) + B_{1+}(v_z^2 - v_y^2) + 2jv_y v_z B_{1+}\right] f_o\, dv_y\, dv_z \quad . \tag{5.49}$$

Now, in the presence of the constant external field B_o, f_o is necessarily isotropic about the direction of this field so that $f_o = f_o(v_x, v_y^2 + v_z^2)$. Thus if the B_{1+} part of the integral in Eq. 5.49 is considered and written as

$$v_y = u \sin\theta$$

and

$$v_z = u \cos\theta \quad ,$$

the result is

$$\frac{j}{2} B_{1+} \iint_{-\infty}^{\infty} (v_z^2 - v_y^2 + 2jv_y v_z)\, f_o \, dv_y \, dv_z$$

$$= \frac{j}{2} B_{1+} \int_0^{2\pi}\int_0^{\infty} u^3 f_o(u^2) e^{2j\theta} \, d\theta \, du = 0 \; . \qquad (5.50)$$

Thus, in general,

$$G_1(v_x) = jB_{1-}G_o(v_x) = \frac{j}{2} B_{1-} \iint_{-\infty}^{\infty} (v_y^2 + v_z^2) f_o \, dv_y \, dv_z \; . \qquad (5.51)$$

For the collision integral defined by Eq. 5.7 the right-hand side of Eq. 5.47 simplifies using Eq. 5.43 and Eq. 5.47 becomes

$$j(\omega - kv_x - \omega_c) f_{1-} - \eta F_o(v_x) E_{1-} + j\eta B_{1-} \left[v_x F_o(v_x) + \frac{dG_o}{dv_x}(v_x) \right] = -\nu f_{1-} \; . \qquad (5.52)$$

From the equation $\nabla \times \underline{E}_1 = -j\omega \underline{B}_1$ and Eqs. 5.43 it can be found that

$$B_{1\pm} = \pm j \frac{k}{\omega} E_{1\pm} \; , \qquad (5.53)$$

so that from Eq. 5.52

$$f_{1-} = - \frac{j\eta E_{1-} \left[F_o(v_x) \left(1 - \frac{kv_x}{\omega} \right) - \frac{k}{\omega} \frac{dG_o}{dv_x} \right]}{\omega - kv_x - \omega_c - j\nu} \; . \qquad (5.54)$$

Equation 5.54 shows that the circularly polarized modes separate. It can be shown by proceeding in a similar fashion that

$$f_{1+} = - \frac{j\eta E_{1+}\left[F_o(v_x)\left(1 - \frac{kv_x}{\omega}\right) - \frac{k}{\omega}\frac{dG_o}{dv_x}\right]}{\omega - kv_x + \omega_c - j\nu} , \qquad (5.55)$$

so that the results for the (+) mode can be obtained from those of the (-) mode by the simple replacement $\omega_c \to -\omega_c$, and only the (-) mode need be examined further. Recall that in the (-) mode formulation $\omega_c < 0$ corresponds to the helicon or slow-cyclotron mode, etc.

Because reduced distribution functions have been used the RF current density is

$$J_{1-} = q \int_{-\infty}^{\infty} f_{1-}\, dv_x , \qquad (5.56)$$

where q is the species charge.

As a simple example of the utilization of this result consider the cold plasma distribution function,

$$f_o = N_o \delta(v_y)\delta(v_z)\delta(v_x - v_o) , \qquad (5.57)$$

and use Eqs. 5.54 and 5.56 together with the definition of the kinetic power flow, Eq. 2.64, to obtain

$$P_{k-} = \frac{1}{2k_i} \mathrm{Re}(E_{1-} J_{1-}^{*}) = \frac{v_o \omega_p^2 \epsilon \, |E_{1-}|^2 \left[\omega_c + \frac{\nu}{k_i v_o}(\omega - k_r v_o)\right]}{2\omega[(\omega - k_r v_o - \omega_c)^2 + (k_i v_o + \nu)^2]} . \qquad (5.58)$$

By using Eq. 2.94 to replace $|E_{1-}|^2$ by the appropriate $|v_{1-}|^2$ term it can be shown that Eq. 5.58 is completely equivalent to the hydrodynamic result, Eq. 2.106.

In addition, integration of Eq. 5.54 can be performed to verify that $N_1 = 0$ so that only the collision term of Eq. 5.7 need be examined.

Prior to analyzing the various distribution functions it will be of value to examine the general requirements for instability of the purely transverse waves. For a multiple stream interaction it is readily found from Maxwell's equations, where $J_{1-} = J_{r-} + jJ_{i-}$, that for real ω

$$(k_r^2 - k_i^2)c^2 - \omega^2 - \frac{\omega\mu_o c^2}{E_{1-}} \sum_s J_{i-}^{(s)} = 0 \qquad (5.59)$$

and

$$2k_i k_r c^2 + \frac{\omega\mu_o c^2}{E_{1-}} \sum_s J_{r-}^{(s)} = 0 \ . \qquad (5.60)$$

The RF conductivity of a carrier is given by $\sigma_- = \sigma_{r-} + j\sigma_{i-}$, where

$$\sigma_{r-} = \left(\frac{J_{r-}}{E_{1-}}\right) \quad \text{and} \quad \sigma_{i-} = \left(\frac{J_{i-}}{E_{1-}}\right) \ . \qquad (5.61)$$

Thus under conditions of amplification ($k_i k_r > 0$) Eq. 5.60 shows that it is necessary that the net real RF conductivity, $\sum_s \sigma_{r-}^{(s)}$, must be negative. Also, if $k_i \ll k_r$, Eq. 5.59 indicates that for slow waves to be present (which is a necessary requirement for the interaction to occur) the net reactive RF conductivity, $\sum_s \sigma_{i-}^{(s)}$, must be large and positive. The latter condition indicates that relatively large RF reactive currents are necessary which in turn suggest that resonance processes may be important. Note that the kinetic theory can account for resonance processes whereas the hydrodynamic theory cannot.

<u>5.3.1</u> <u>The Hydrodynamic Distribution Function</u>. For the hydrodynamic distribution function defined in Eq. 5.3, Eqs. 5.54 and 5.56 can be used to obtain

$$P_{k-} = -\frac{\omega_p^2 \epsilon \, |E_{1-}|^2}{4k_i v_T \omega} \, \mathrm{Re} \int_{v_o - v_T}^{v_o + v_T} \frac{j(\omega - kv_x)}{\omega - kv_x - j\nu - \omega_c} \, dv_x \quad . \tag{5.62}$$

The pole present in the integrand can lead to a component of wave damping (cyclotron damping) or growth similar to the Landau damping of the purely longitudinal waves. By the transformation of variables $\mu = (v_x - v_o)/v_T$ it is seen that the integrals Φ_3 and Φ_4 defined in Eq. 5.15 are introduced and can be used to obtain the result

$$P_{k-} = \frac{\omega_p^2 \epsilon \, |E_{1-}|^2}{4k_i v_T \omega \, |k|^2} \, \mathrm{Re} \left\{ (-jk_r - k_i)(\omega_c + j\nu) \left[\log_e \left(\frac{z_c + 1}{z_c - 1} \right) - j\pi\sigma \right] \right\} \, , \tag{5.63}$$

where

$$z_c = \frac{\omega - kv_o - j\nu - \omega_c}{kv_T} \tag{5.64}$$

and

$$\sigma = 1 \text{ if } \quad |\mathrm{Re}\,(z_c)| < 1 \text{ and } \quad \mathrm{Im}\,(z_c) > 0 \, ,$$
$$= 0 \qquad \text{otherwise} \quad . \tag{5.65}$$

<u>Case a</u>:

$$|z_c| \gg 1 \; .$$

Assume that either $|\mathrm{Re}\,(z_c)| > 1$ or $\mathrm{Im}\,(z_c) < 0$ so that the residue does not contribute. In this case $\log_e[(z_c + 1)/(z_c - 1)] \approx 2/z_c$ and hence from Eq. 5.63

$$P_{k-} = \frac{\omega_p^2 \epsilon \, |E_{1-}|^2 \, v_o \left[\omega_c + \frac{\nu}{k_i v_o} (\omega - k_r v_o) \right]}{2\omega[(\omega - k_r v_o - \omega_c)^2 + (k_i v_o + \nu)^2]} \, , \tag{5.66}$$

which is a recovery of the hydrodynamic results of Chapter II with the carrier thermal velocity playing no role. Indeed it will now be shown that for this case the hydrodynamic results can be retrieved exactly (e.g., helicon mode). To see the effects of the transverse thermal motion this will actually be done for the degenerate distribution function defined in Eq. 5.39 and by letting $v_{T_\perp} \to 0$ the actual hydrodynamic distribution function result is obtained. Define the multiplier $\rfloor\overline{v_x}\lfloor$ to mean that, when this symbol appears as part of the numerator of a function, the function is multiplied by unity if $(v_o + v_{T_{||}} \geqq v_x \geqq v_o - v_{T_{||}})$ and by zero otherwise. Thus, when the integration is performed in Eq. 5.51 for the degenerate distribution function,

$$G_o(v_x) = \frac{N_o v_{T_\perp}^2}{8 v_{T_{||}}} \left(\rfloor\overline{v_x}\lfloor \right) , \tag{5.67}$$

so that

$$\frac{dG_o(v_x)}{dv_x} = G_o \left[\delta(v_x - v_o + v_{T_{||}}) - \delta(v_x - v_o - v_{T_{||}}) \right] . \tag{5.68}$$

Similarly from Eq. 5.48,

$$F_o(v_x) = \frac{N_o}{2 v_{T_{||}}} \left(\rfloor\overline{v_x}\lfloor \right) , \tag{5.69}$$

so that in Eq. 5.54,

$$f_{1-} = - \frac{j\eta E_{1-} N_o}{2\omega v_{T_{||}} (\omega - k v_x - j\nu - \omega_c)} \left\{ (\omega - k v_x)\left(\rfloor\overline{v_x}\lfloor \right) - \frac{k^2 v_{T_\perp}^2}{4} \cdot \left[\delta(v_x - v_o + v_{T_{||}}) - \delta(v_x - v_o - v_{T_{||}}) \right] \right\} , \tag{5.70}$$

and from Eq. 5.56,

$$J_{1-} = -\frac{j\omega_p^2 \epsilon E_{1-}}{2\omega v_{T_{\parallel}}} \left[\frac{k^2 v_{T_\perp}^2 v_{T_{\parallel}}}{2[(\omega - kv_o - j\nu - \omega_c)^2 - k^2 v_{T_{\parallel}}^2]} + \int_{v_o - v_{T_{\parallel}}}^{v_o + v_{T_{\parallel}}} \frac{(\omega - kv_x)dv_x}{\omega - kv_x - j\nu - \omega_c} \right] . \quad (5.71)$$

The integral in this equation is simply related to the functions Φ_3 and Φ_4 of Eqs. 5.18 and 5.19 (with $v_T \rightarrow v_{T_{\parallel}}$) so that in particular under the conditions of Case a, wherein $|z_c| \gg 1$ and either $|Re\ (z_c)| > 1$ or $Im\ (z_c) < 0$, it is found that

$$J_{1-} = -\frac{j\omega_p^2 \epsilon E_{1-}}{\omega} \left[\frac{\omega - kv_o}{\omega - kv_o - j\nu - \omega_c} + \frac{k^2 v_{T_\perp}^2}{4[(\omega - kv_o - j\nu - \omega_c)^2 - k^2 v_{T_{\parallel}}^2]} \right] . \quad (5.72)$$

Inspection shows that if $v_{T_\perp} = 0$ this result is in exact agreement with the hydrodynamic theory of Chapter II and hence gives the same dispersion relation. The second term in brackets in Eq. 5.72 is the correction term due to the finite transverse temperature. Note that $|z_c| \gg 1$ gives the following when the substitutions are made:

$$\left| \frac{\omega - kv_o - j\nu - \omega_c}{kv_{T_{\parallel}}} \right| \gg 1 , \quad (5.73)$$

which is independent of $v_{T_\perp}$. Thus if $v_{T_\perp} \gg v_{T_{\parallel}}$ the correction term can be large, whereas if $v_{T_\perp} \approx v_{T_{\parallel}}$ it can be ignored. In general the dispersion equation is derived from the Maxwell equations as

$$k^2c^2 - \omega^2 + j\,\frac{\omega}{\epsilon E_{1^-}} \sum_s J_{1^-}^{(s)} = 0 \quad , \tag{5.74}$$

so that using Eq. 5.72 in Eq. 5.74 the following result is found, for the case of a single carrier:

$$k^2c^2 - \omega^2 + \frac{\omega_p^2(\omega - kv_o)}{\omega - kv_o - j\nu - \omega_c} + \frac{\omega_p^2 k^2 v_{T_\perp}^2}{4[(\omega - kv_o - j\nu - \omega_c)^2 - k^2 v_{T_\parallel}^2]} = 0 \quad , \tag{5.75}$$

where any solution obtained therefrom must satisfy Eq. 5.73 and either have $|\mathrm{Re}\,(z_c)| > 1$ or $\mathrm{Im}\,(z_c) < 0$. Inspection shows that this equation is now fifth order in k so that two additional waves have been introduced due to the finite transverse temperature. In particular, although not attempted herein, this equation can be used to study the effects of finite $v_{T_\perp}$ on the drifted helicon dispersion relation with significant departures from the hydrodynamic theory occurring if $v_{T_\perp} \gg v_{T_\parallel}$.

Now from Eq. 5.64 it can easily be found that

$$\mathrm{Re}\,(z_c) = \frac{k_r(\omega - k_r v_o - \omega_c) - k_i(\nu + k_i v_o)}{|k|^2\, v_T} \tag{5.76}$$

and

$$\mathrm{Im}\,(z_c) = -\,\frac{k_r \nu + k_i(\omega - \omega_c)}{|k|^2\, v_T} \quad . \tag{5.77}$$

The latter equation shows that for $k_i > 0$ it is always true that $\mathrm{Im}\,(z_c) < 0$ if $\omega_c < 0$ corresponding to the helicon or slow-cyclotron mode, etc. Thus under growing-wave conditions the RF current density expressions never contain a residue contribution for carriers with $\omega_c < 0$.

Case b:

$$|z_c| \ll 1 \; .$$

This is often referred to as the case where nonlocal conditions apply. It is first demonstrated how a major nonlocal effect, namely cyclotron resonance absorption, occurs. To make this clear assume that $v_o = 0$ and the collisions are absent, i.e., $\nu = 0$. Assume that $\omega \to \omega_c$, i.e., $\omega_c > 0$, so that, by inspection of Eq. 5.77, Im $(z_c) \to 0$ and hence the residue is absent, $\sigma = 0$. In this case the kinetic power flow from Eq. 5.63 is

$$P_{k-} = \frac{\omega_p^2 \epsilon \; |E_{1-}|^2 \; \pi k_r \omega_c}{4 k_i v_T \omega \; |k|^2} \; , \tag{5.78}$$

where $\log_e[(z_c + 1)/(z_c - 1)] \approx j\pi$ has been used since $|z_c| \ll 1$. Thus $P_{k-} < 0$ when $k_i < 0$ and the resonant damping occurs. The carrier mode with $\omega_c > 0$ is as a result often termed the cyclotron resonance active mode.

The RF current density is now considered for the case $|z_c| \ll 1$ to compare results with the hydrodynamic theory of Chapter II. This will again be done retaining the transverse thermal motion so that Eq. 5.70 is applicable. The general result can be written, using Eqs. 5.18 and 5.19 in Eq. 5.71, as

$$J_{1-} = - \frac{j\omega_p^2 \epsilon E_{1-}}{2\omega} \left[2 + \frac{\omega_c + j\nu}{k v_{T_{\parallel}}} \Phi_3 + \frac{k^2 v_{T_{\perp}}^2}{2[(\omega - k v_o - j\nu - \omega_c)^2 - k^2 v_{T_{\parallel}}^2]} \right] , \tag{5.79}$$

where Φ_3 is given in Eq. 5.18. Since $|z_c| \ll 1$ the asymptotic expansion gives $\log_e[(z_c + 1)/(z_c - 1)] \approx j\pi + 2z_c$ in Φ_3 leading to

$$J_{1^-} = -\frac{j\omega_p^2 \epsilon E_{1^-}}{\omega}\left[1 + \frac{\omega_c + j\nu}{kv_{T_{\|}}}\left(j\frac{\pi}{2}(1-\sigma) + \frac{\omega - kv_o - j\nu - \omega_c}{kv_{T_{\|}}}\right) + \frac{k^2 v_{T_\perp}^2}{4[(\omega - kv_o - j\nu - \omega_c)^2 - k^2 v_{T_{\|}}^2]}\right] \quad . \quad (5.80)$$

The important point now is that irrespective of whether or not the residue is present the RF current density given in Eq. 5.80 is drastically altered from the conventional result of hydrodynamic theory, viz.,

$$J_{1^-} = -\frac{j\omega_p^2 \epsilon E_{1^-}(\omega - kv_o)}{\omega(\omega - kv_o - j\nu - \omega_c)} \quad , \quad (5.81)$$

so that if $|z_c| \ll 1$, i.e., $|\omega - kv_o - j\nu - \omega_c| \ll kv_{T_{\|}}$, the hydrodynamic theory should be considered invalid. Note, however, that in many investigations of solid-state plasmas (such as the helicon-phonon interaction or the electron-hole interaction[73]) the results obtained from the hydrodynamic dispersion equation satisfy $|z_c| \ll 1$ for any reasonable carrier densities and hence these results are invalid since a nonlocal theory should be used. The hydrodynamic theory thus breaks down at the higher frequencies, although at the lower frequencies (typically less than 1 GHz) the equality $|z_c| \gg 1$ is satisfied and the hydrodynamic theory is acceptable (unless $v_{T_\perp} \gg v_{T_{\|}}$).

From Eq. 5.80, since $|z_c| \ll 1$, it can be seen very roughly that the RF current density can be approximated by

$$J_{1-} \simeq -\frac{j\omega_p^2 \epsilon E_{1-}}{\omega}\left[1 + \frac{k^2 v_{T_\perp}^2}{4[(\omega - kv_o - j\nu - \omega_c)^2 - k^2 v_{T_\parallel}^2]}\right] . \qquad (5.82)$$

A particularly simple result is found if it is assumed that only one species of mobile charge is present with the anisotropy $v_{T_\perp} \ll v_{T_\parallel}$. In this case the second term in the brackets is negligible and use of Eq. 5.82 in Eq. 5.74 provides the dispersion relation,

$$k^2c^2 - \omega^2 + \omega_p^2 = 0 , \qquad (5.83)$$

and undamped waves are possible if $\omega > \omega_p$. For example, if $\omega \gg \omega_p$ so that $k \approx \omega/c$, the condition that Eq. 5.83 be valid is, since $|z_c| \ll 1$ is required,

$$|kc - kv_o - j\nu - \omega_c| \ll |kv_{T_\parallel}| , \qquad (5.84)$$

which can be achieved if $\omega_c > 0$, $\omega_c \simeq kc$, and the collision frequency satisfies $\nu \ll \omega v_{T_\parallel}/c$. These conditions can be approximated by choosing a doped-degenerate material with significant mass anisotropy. For the isotropic case the first and the third terms in Eq. 5.80 cancel and the cyclotron resonance condition is reestablished in this case.

Opposite to the limit $v_{T_\perp} \to 0$, consider $v_{T_\perp} \gg v_{T_\parallel}$. Equation 5.82 under the assumption $k_i \ll k_r$ and of course $|z_c| \ll 1$ provides the following:

$$J_{1-} \approx \frac{j\omega_p^2 \epsilon}{4\omega}\left(\frac{v_{T_\perp}}{v_{T_\parallel}}\right)^2 E_{1-} , \qquad (5.85)$$

so that $\sigma_{r-} \approx 0$ and

$$\sigma_{i-} \approx \frac{\omega_p^2 \epsilon}{4\omega}\left(\frac{v_{T_\perp}}{v_{T_\parallel}}\right)^2 . \tag{5.86}$$

Recall now from Eqs. 5.59 through 5.61 that Eq. 5.86 is just the requirement for slow waves with $k_i \ll k_r$ provided σ_{i-} is large. The part of the dispersion equation given by Eq. 5.59 becomes in the present case

$$(k_r^2 - k_i^2)c^2 - \omega^2 - \omega_p^2\left(\frac{v_{T_\perp}}{2v_{T_\parallel}}\right)^2 + \sum_r (-\omega\mu_o c^2)\sigma_{i-}^{(r)} = 0 , \tag{5.87}$$

where $\sum_r$ represents a sum over any other carrier species present. If the latter are absent, since in Eq. 5.86 $\sigma_{r-} \approx 0$, the dispersion equation is

$$k^2c^2 - \omega^2 = \omega_p^2\left(\frac{v_{T_\perp}}{2v_{T_\parallel}}\right)^2 , \tag{5.88}$$

which shows that the wave is significantly slowed if the plasma frequency is large and there is a large anisotropy. Since the distribution function used actually corresponds to a degenerate material this in turn is interpreted as being a large anisotropy of the Fermi surface between the transverse and the longitudinal directions. It is then readily conceivable that with a mobile secondary carrier present the slow wave predicted by Eq. 5.88 can be amplified. For self-consistency the solution must satisfy $|z_c| \ll 1$ which from Eq. 5.88 becomes in the present case

$$|\omega - kv_o - \omega_c - j\nu| \ll \left|\frac{\omega_p v_{T_\perp}}{2c}\right| , \tag{5.89}$$

from which it is seen that the collisions will be the basic limiting factor since large ratios of (ω_p/ν) are difficult to achieve in solid-state plasmas.

5.3.2 The Maxwellian Distribution Function. The Maxwellian distribution function is altered to take into account the possibility of temperature anisotropy so that Eq. 5.28 becomes

$$f_o = \frac{N_o}{(2\pi)^{3/2}\, v_\perp^2 v_\parallel} \exp\left(-\frac{(v_x - v_o)^2}{2v_\parallel^2}\right) \exp\left(-\frac{v_y^2 + v_z^2}{2v_\perp^2}\right) . \qquad (5.90)$$

where $v_\parallel$ and $v_\perp$ are the components of the thermal velocity parallel and perpendicular to x, respectively. The integration in Eq. 5.48 is performed directly, viz.,

$$F_o(v_x) = \frac{N_o}{\sqrt{2\pi}\, v_\parallel} \exp\left(-\frac{(v_x - v_o)^2}{2v_\parallel^2}\right) , \qquad (5.91)$$

and similarly from Eqs. 5.51 it can be determined that

$$\frac{dG_o}{dv_x} = \frac{N_o v_\perp^2 (v_o - v_x)}{\sqrt{2\pi}\, v_\parallel^3} \exp\left(-\frac{(v_x - v_o)^2}{2v_\parallel^2}\right) . \qquad (5.92)$$

Use of Eqs. 5.91 and 5.92 in Eq. 5.54 provides the following:

$$f_{1-} = -\frac{j\eta E_{1-} N_o \left[\omega - kv_x\left(1 - \frac{v_\perp^2}{v_\parallel^2}\right) - kv_o \frac{v_\perp^2}{v_\parallel^2}\right]}{\sqrt{2\pi}\, \omega v_\parallel (\omega - kv_x - j\nu - \omega_c)} \exp\left(-\frac{(v_x - v_o)^2}{2v_\parallel^2}\right) .$$

(5.93)

Define for the Maxwellian case

$$z_c = \frac{\omega - kv_o - j\nu - \omega_c}{\sqrt{2}\, kv_\parallel} \qquad (5.94)$$

and compute the current density according to Eq. 5.56 using Eqs. 5.29 through 5.31 and the definiton of Eq. 5.94 to obtain

$$J_{1-} = -\frac{j\omega_p^2 \epsilon E_{1-}}{\sqrt{2\pi}\ \omega k v_{\|}} \int_{-\infty}^{\infty} \frac{\left[(\omega - kv_o) - \sqrt{2}\, kv_{\|}\left(1 - \frac{v_{\perp}^2}{v_{\|}^2}\right)\xi\right]}{z_c - \xi} e^{-\xi^2}\, d\xi \tag{5.95}$$

or

$$J_{1-} = -\frac{j\omega_p^2 \epsilon E_{1-}}{\sqrt{2}\ \omega k v_{\|}} \left[(\omega - kv_o)G(z_c) - \sqrt{2}\, kv_{\|}\left(1 - \frac{v_{\perp}^2}{v_{\|}^2}\right)\left(z_c G(z_c) - 1\right)\right] , \tag{5.96}$$

where again $G(z_c)$ is simply related to the tabulated Z function by Eq. 5.31.

<u>Case a</u>:

$$|z_c| \gg 1 .$$

For this case Eq. 5.34 is applicable with $z \rightarrow z_c$. If only the first term therein of the asymptotic expansion is retained, the following results:

$$G(z_c) \approx -j\sqrt{\pi}\,\sigma \exp(-z_c^2) + \frac{1}{z_c} . \tag{5.97}$$

Apply this to Eq. 5.96 assuming $\text{Im}(z_c) < 0$ so that $\sigma = 0$ to obtain

$$J_{1-} \approx -\frac{j\omega_p^2 \epsilon E_{1-}(\omega - kv_o)}{\omega(\omega - kv_o - j\nu - \omega_c)} , \tag{5.98}$$

which is exactly equivalent to the result of hydrodynamic theory. This is an unusual and important result which shows that if $|z_c| \gg 1$ and $\text{Im}(z_c) < 0$, regardless of the distribution of the thermal velocities, even if anisotropic, to a first approximation, the RF current density, P_{k-}, and hence the dispersion relation for Maxwellian plasmas is identical to that given by hydrodynamic theory. Including further terms from the asymptotic expansion of Eq. 5.34 will provide only a small correction.

From Eq. 5.94 it can be found that

$$\mathrm{Re}(z_c) = \frac{k_r(\omega - k_r v_o - \omega_c) - k_i(\nu + k_i v_o)}{\sqrt{2}\ |k|^2\ v_{\|}} \qquad (5.99)$$

and

$$\mathrm{Im}\ (z_c) = -\frac{k_r \nu + k_i(\omega - \omega_c)}{\sqrt{2}\ |k|^2\ v_{\|}} \ . \qquad (5.100)$$

In Eq. 5.34 it is seen that for the residue to contribute it is necessary that $\mathrm{Im}\ (z_c) \geqq 0$. Inspection of Eq. 5.100 shows that this can only occur under growth conditions ($k_i > 0$) if $\omega_c > 0$ and

$$k_i \omega_c > (k_r \nu + k_i \omega) \ ; \qquad (5.101)$$

or stated alternatively, the fast-cyclotron-type mode is required and if $k_i \ll k_r$ it is necessary that $(\omega_c - \omega) \gg \nu$. Recall that for the purely longitudinal waves Eq. 5.35 showed that for growing waves the residue term cannot contribute. The static magnetic field provides a method for potentially retrieving this residue. Assume then that Eq. 5.101 is satisfied with $k_i > 0$ so that from Eq. 5.34 $\sigma = 2$. Also neglect anisotropic effects or heating effects by setting $v_\perp = v_{\|}$ in Eq. 5.96. Use of Eq. 5.97 in Eq. 5.96 then gives the following:

$$J_{1^-} \approx \frac{\omega_p^2 \epsilon E_{1^-}}{\omega\ |k|^2\ v_{\|}} \left([k_i^2 v_o - k_r(\omega - k_r v_o)] + j k_i \omega \right) \left(\sqrt{2\pi}\ e^{-z_c^2} + j\,\frac{1}{\sqrt{2}\ z_c} \right) \ .$$

$$(5.102)$$

Let $a = \mathrm{Re}\ (z_c)$ and $b = \mathrm{Im}\ (z_c)$ so that the exponential term can be written as

$$\exp(-z_c^2) = \exp(b^2 - a^2) \cdot [\cos(2ab) - j \sin(2ab)] \ . \qquad (5.103)$$

Recall now from Eqs. 5.59 through 5.61 and the discussion thereafter that if $k_i \ll k_r$ it is necessary that the net resistive RF current be negative and the net RF reactive conductance be large and positive (to slow the wave). Inspection of Eqs. 5.102 and 5.103 shows that large RF conductivities are obtainable if $b > a$ because of the exponential function.

Under the condition $|z_c| \gg 1$, however, inspection of Eqs. 5.99 and 5.100 indicates that if $k_i \ll k_r$ then necessarily $a > b$ and the exponential term will be negligibly small, unless $(\omega - k_r v_o - \omega_c) \approx 0$ and $\mathrm{Im}\ (z_c) > 1$. In the latter case the exponential term is large and dominates the RF current density in Eq. 5.102, so that

$$J_{1-} \approx \frac{\sqrt{2}\ \omega_p^2 \epsilon E_{1-}\ \pi}{\omega\ |k|^2\ v_{||}} (-k_r\omega_c + jk_i\omega)\ \exp\left(\frac{[k_r\nu + k_i(\omega - \omega_c)]^2}{2\ |k|^4\ v_{||}^2}\right) \ , \qquad (5.104)$$

where it has been assumed that $a \approx 0$, and from previous assumptions $\omega_c > 0$ and $k_i(\omega_c - \omega) > k_r\nu$. Inspection of Eq. 5.104 shows that this current density is appropriate for unstable wave growth since $J_{r-} < 0$ and $J_{i-} > 0$ and their corresponding conductivities are large due to the exponential factor. In addition the kinetic power flow in the present case may be found from Eq. 5.104 as

$$P_{k-} = \frac{1}{2k_i} \mathrm{Re}(E^*_{1-} J_{1-}) = - \frac{\pi \omega_p^2 \epsilon \, |E_{1-}|^2 \, k_r \omega_c}{\sqrt{2} \, k_i \omega \, |k|^2 \, v_{\|}} \exp\left(\frac{[k_r \nu + k_i(\omega - \omega_c)]^2}{2 \, |k|^4 \, v_{\|}^2} \right) , \tag{5.105}$$

which shows that the kinetic power flow is negative for $k_i > 0$ and hence assists the instability. Although v_o is not explicitly present it has been taken into account by the assumption $a \approx 0$, i.e., $(\omega - k_r v_o - \omega_c) \approx 0$. Note that without the carrier drift this latter assumption cannot be made since it would violate the assumption $k_i(\omega_c - \omega) > k_r \nu$. On the other hand it is seen that v_o can be replaced by $-v_o$ without altering any of the conclusions reached. The assumptions again are

$$|z_c| \gg 1 \ , \quad \omega_c > 0 \ , \quad k_i(\omega_c - \omega) > k_r \nu \ , \quad k_i \ll k_r \ ,$$

$$v_{\|} = v_{\perp} \ \text{and} \ (\omega - k_r v_o - \omega_c) \approx 0 \ , \quad \text{or} \quad a \approx 0 \ . \tag{5.106}$$

The first of the assumptions above can be relaxed to $|z_c| > 1$ since the asymptotic expansion is no longer important when the residue is present with a large value $(b - a > 1)$. The condition $(\omega - k_r v_o - \omega_c) \approx 0$, together with $k_i(\omega_c - \omega) > k_r \nu$, implies that the condition $b > 1$ is equivalent to

$$|v_o| > \frac{\sqrt{2} \, (k_r^2 + k_i^2) v_{\|}}{k_i k_r} , \tag{5.107}$$

where $v_o < 0$ is required. Hence for $k_i < k_r$ it is necessary that $|v_o| > v_{\|}$. If the restriction $k_i < k_r$ is lifted, however, the condition that $a \approx 0$ and $b > 1$ from Eqs. 5.99 and 5.100 yields the following:

$$|v_o| > \sqrt{2}\left(\frac{k_r}{k_i}\right) v_{\parallel} + \left(\frac{\nu}{k_i}\right) \quad ; \quad v_o < 0 \ , \qquad (5.108)$$

so that it is potentially possible for unstable waves to exist with $k_i > k_r$ and $|v_o| < v_{\parallel}$. From Eq. 5.60 such large growth rates are feasible since σ_{r-} can be large.

Also under the present conditions the RF current density is altered if $v_{\parallel} \neq v_{\perp}$. In this case with $G(z_c)$ large corresponding to the presence of the residue, Eq. 5.96 gives

$$J_{1-} \simeq \frac{\sqrt{2}\,\omega_p^2 \epsilon E_{1-}\,\pi}{\omega\,|k|^2\,v_{\parallel}} \left[[k_i^2 v_o - k_r(\omega - k_r v_o)] + jk_i\omega - \left(1 - \frac{v_{\perp}^2}{v_{\parallel}^2}\right) \cdot [k_r\omega_c + k_i\nu + j(k_r\nu - k_i\omega_c)] \right] \cdot \exp(-z_c^2) \quad . \qquad (5.109)$$

As an example of the utilization of these results it will be shown that even a carrier species with a small number density can provide large slowing factors in solids under growth conditions. To see this assume $k_i < k_r$ so that Eq. 5.59 is approximated by

$$k_r^2 c^2 \approx \left(\frac{\omega}{\epsilon}\right) \sigma_{i-} \ , \qquad (5.110)$$

where $\omega \ll k_r c$ and the presence of any other carrier species in the equation is assumed to be negligible. Equation 5.107 is then applicable so that $|v_o| > v_{\parallel}$ corresponding to a large drift velocity either due to a large applied static electric field or to the high field regions near the contacts in any solid-state device. If the simpler case $v_{\parallel} = v_{\perp}$ is assumed it is found from Eq. 5.104 that

$$\sigma_{i-} = \frac{\sqrt{2}\,\pi\omega_p^2\epsilon k_i}{|k|^2\, v_{\|}} \exp\left(\frac{[k_r\nu + k_i(\omega - \omega_c)]^2}{2\,|k|^4\, v_{\|}^2}\right) , \qquad (5.111)$$

where Re $(z_c) \simeq 0$ is assumed. Equation 5.111 shows that the effect of the residue in the present case can be interpreted by defining an effective number density,

$$N_o' = N_o \exp\left(\frac{[k_r\nu + k_i(\omega - \omega_c)]^2}{2\,|k|^4\, v_{\|}^2}\right) . \qquad (5.112)$$

Now some typical values for a semiconductor could be chosen as $\nu \approx 3 \times 10^{11}\ s^{-1}$, $\omega_c \approx 10^{13}\ s^{-1}$, $\sqrt{2}\, v_{\|} \approx 3 \times 10^7$ cm/s, so that with $k_i \approx 0.1\ k_r$ Eq. 5.112 becomes if $\omega \ll \omega_c$

$$N_o' = N_o \exp\left(\frac{10^5}{3k_r}\right)^2 . \qquad (5.113)$$

Thus, for an example, if $k_r = 5 \times 10^3\ cm^{-1}$ corresponding to a wave phase velocity of 2×10^7 cm/s at $\omega = 10^{11}\ s^{-1}$, the effective number density is approximately a factor of 10^{19} times the actual number density. Clearly then, even if the actual number density is small, the resonant nature of the interaction provides that such carriers are still important to the interaction, whereas from the hydrodynamic theory such carriers may be a negligible factor in the dispersion relation. Self-consistently Eq. 5.113 can be solved to find that for the present example $N_o \approx 10^5\ cm^{-3}$. This serves to indicate that under wave growth conditions the presence of even a small density of a carrier species can be important.

It is also to be noted that these results are even somewhat pessimistic for the Maxwellian distribution. This is because a constant collision frequency ν has been assumed to simplify the analysis. In general, however, the collision frequency is dependent upon the carrier energy so that at any time some of the carriers in the distribution function will be colder than the mean so that their collision frequency will be less than the mean value ν. For such a subset of carriers at any particular time, then, the condition given in Eq. 5.101 is relaxed. In this case, for example, even if $\omega_c \approx \nu$ the residue can still be large. It is suggested that this may be the case for the holes in n-InSb, where $|\omega_{ch}| \approx \nu_h$ is typical and N_{oh} is small.

Case b:

$$|z_c| \ll 1 .$$

For this case even if the residue is present it is limited in magnitude since $\exp(-z_c^2) \simeq 1$ so that

$$G(z_c) \approx -j\pi\sigma + 2z_c\left(1 - \frac{2z_c^2}{3} + \frac{4z^4}{5} - \dots\right) . \tag{5.114}$$

For a comparison with hydrodynamic theory consider that the system satisfies $\sigma = 0$ and $v_{\|} = v_{\perp}$. For this case Eq. 5.96 gives, retaining only the first term in the expansion,

$$J_{1-} = -\frac{j\omega_p^2\epsilon E_{1-}(\omega - kv_o)(\omega - kv_o - j\nu - \omega_c)}{\omega k^2 v_{\|}^2} , \tag{5.115}$$

which introduces the nonlocal effect of the longitudinal carrier thermal velocity. Thus for finite temperatures the slow-cyclotron mode predicted by hydrodynamic theory does not exist since these solutions have $|\omega - kv_o - j\nu - \omega_c| \approx 0$. This consideration does not apply for the forward helicon branch since $|\omega - kv_o| \approx 0$ and $|\omega_c| \gg |kv_{||}|$ so that $|z_c| \gg 1$ applies.

In addition the backward helicon branch which has negative group velocity is predicted by the hydrodynamic theory to be restricted to $k_r < |\omega_c|/2v_o$ for positive frequencies so that if $|\omega_c| \gg |\sqrt{2}\, kv_{||}|$, i.e., $v_o > v_{||}$, this entire branch is present. If $v_o < v_{||}$, however, only part of the backward helicon is present and the single-carrier dispersion relation breaks away from this branch at some positive frequency near the helicon maximum frequency, $\omega_{max} = \omega_p^2 v_o^2/(4\ |\omega_c|\ c^2)$, and joins a root of the kinetic dispersion relation. Note in addition that, since Eq. 5.115 will lead to a fourth-order k and second-order ω dispersion equation as compared with the third order ω and k hydrodynamic dispersion equation, it is clear that the root structure in $(\omega, \mathbf{k})$ space is significantly modified. From Eq. 5.115 and Maxwell's equations the single carrier kinetic dispersion relation is found as

$$k^2c^2 - \omega^2 + \frac{\omega_p^2(\omega - kv_o)(\omega - kv_o - \omega_c - j\nu)}{k^2 v_{||}^2} = 0 \quad , \qquad (5.116)$$

where any solution must satisfy $|z_c| \ll 1$ and $\text{Im}\ (z_c) < 0$, and for the helicon mode $\omega_c < 0$. Although not explicitly analyzed, since $|z_c| \ll 1$ and J_{1-} varies proportionally to z_c, it is clear that the effective dielectric constant remains small and the solutions to Eq. 5.116 are characterized by heavily damped modes which have $k_i \gg k_r$.

The case of potential interest when $v_{||} = v_{\perp}$ corresponds to the residue being present $(\sigma \neq 0)$. The conditions for this are as discussed for the previous Case a. For this case the RF current density is given from Eqs. 5.34 and 5.96 as

$$J_{1^-} \approx -\frac{2\pi\omega_p^2\epsilon E_{1^-}}{\sqrt{2}\,\omega k v_{\|}}(\omega - k_r v_o) \quad ; \qquad \text{Im}\,(z_c) > 0 \ . \tag{5.117}$$

In addition the kinetic power flow associated with this carrier mode is

$$P_{k-} = \frac{1}{2k_i}\,\text{Re}(E_{1^-}^* J_{1^-}) = -\frac{\pi\omega_p^2\epsilon k_r\,|E_{1^-}|^2}{\sqrt{2}\,k_i\omega\,|k|^2\,v_{\|}}\,k_r(\omega - k_r v_o) \tag{5.118}$$

and the unusual result is noted that it is necessary that $v_o < \omega/k_r$ for this mode to be active for a convective instability. Since $|z_c| \ll 1$ this necessitates $\omega_c > 0$ which is in agreement with the conditions required for the presence of the residue when $k_i > 0$. This result is independent of the sign of v_o in that it plays no role in the sign of P_{k-}. As discussed in Case a, however, it is advantageous if $v_o < 0$ corresponding to a backward wave. When $P_{k-} < 0$ this indicates that the real part of the RF conductivity is negative and since the net real RF conductivity must be negative this implies that the instability is assisted.

The effect of significant temperature anistropy is now considered for the case $|z_c| \ll 1$. Assume that the conditions are such that the residue is absent, i.e., $\sigma = 0$, and the longitudinal and transverse thermal velocities are widely disparate so that Eqs. 5.96 and 5.101 indicate that

$$J_{1^-} \approx -\frac{j\omega_p^2\epsilon E_{1^-}}{\omega}\left(1 - \frac{v_\perp^2}{v_{\|}^2}\right) \ . \tag{5.119}$$

It can readily be seen from this result that $\text{Re}(E_{1^-}J_{1^-}^*) = 0$ and no power transfer is expected. Indeed the dispersion equation is

$$k^2c^2 - \omega^2 + \omega_p^2\left(1 - \frac{v_\perp^2}{v_\parallel^2}\right) = 0 \quad , \tag{5.120}$$

so that if $v_\perp > v_\parallel$ and $\omega_p^2[(v_\perp^2/v_\parallel^2) - 1] > \omega^2$, purely real solutions for k are obtained. Since an approximation has been used in deriving Eq. 5.119, namely $G(z_c) \approx 0$ since $|z_c| \ll 1$, this indicates that the system is actually at a point of marginal stability. This has been pointed out previously[77] where the present system was analyzed by a different approach. Note that if $k_i = 0$ then the residue is indeed necessarily absent since $\text{Im}\,(z_c) < 0$. If the correction term is included as the first term in the asymptotic expansion given in Eq. 5.114, Eq. 5.96 becomes

$$J_{1-} = -\frac{j\omega_p^2\epsilon E_{1-}}{\omega}\left[\left(1 - \frac{v_\perp^2}{v_\parallel^2}\right) + \frac{\omega - kv_o - j\nu - \omega_c}{k^2v_\parallel^2} \cdot \left(j\nu + \omega_c + \frac{v_\perp^2}{v_\parallel^2}(\omega - kv_o - j\nu - \omega_c)\right)\right] \quad , \tag{5.121}$$

and the kinetic power flow is found from this to be

$$P_{k-} = -\frac{\omega_p^2\epsilon\,|E_{1-}|^2\,\nu}{2k_i\omega\,|k|^2\,v_\parallel^2}\left[2\omega_c\left(1 - \frac{v_\perp^2}{v_\parallel^2}\right) + \left(\frac{2v_\perp^2}{v_\parallel^2} - 1\right)(\omega - k_rv_o)\right] \quad , \tag{5.122}$$

where $k_i \ll k_r$ has been assumed. Inspection shows that for $v_\perp > v_\parallel$ it is desirable to have $\omega_c < 0$ and $(\omega - k_rv_o) < 0$ to drive the instability.

At large magnetic fields the residue can appear when $\omega_c > 0$ under growth conditions. The RF current density is then given from Eqs. 5.96 and 5.114 as

$$J_{1^-} \approx -\frac{j\omega_p^2 \epsilon E_{1^-}}{\omega}\left(1 - \frac{v_\perp^2}{v_\parallel^2}\right)\left(\frac{2j\sqrt{\pi}\,(\omega - kv_o - j\nu - \omega_c)}{\sqrt{2}\,kv_\parallel} - 1\right) , \qquad (5.123)$$

so that the solutions are again near $k_i = 0$ and it is necessary that $\omega_c \gg \nu, \omega$ when $\sigma \neq 0$. Inspection shows directly that the sign of P_{k-} is directly dependent upon the sign of $[1 - (v_\perp^2/v_\parallel^2)](\omega - k_r v_o - \omega_c)$ and since $(\omega - k_r v_o - \omega_c) \approx 0$ it should be possible to obtain growing waves with either $v_\perp > v_\parallel$ or $v_\perp < v_\parallel$.

These results show that the Maxwellian distribution function for the purely transverse waves in a static magnetic field yields a wide variety of solutions according to the kinetic theory depending upon the assumptions made regarding their location in (ω, k) space and the carrier temperatures. For the kinetic theory, since the general solutions obtained involve complex functions whose values are themselves dependent upon the nature of the solutions, it is usually necessary for the purpose of analysis to assume a priori some properties of the solution in (ω, k) space. This leads to the formulation of a dispersion equation with a limited range of applicability so that solutions obtained therefrom must satisfy the initial assumptions made.

If a convective instability is predicted it should be verified that this convective root follows a trajectory in k_r-k_i space such that for some $\omega_i < 0$ the wave number k is purely real. The difficulty is clear, however, since the act of letting ω become complex starts to alter the nature of the variables which were initially delimited by a priori assumptions. The form of the dispersion equation itself can then change as ω_i is varied. This makes it difficult if not impossible to follow roots analytically as ω_i is varied. A necessary condition can be established by solving for the dispersion relation with k real and ω complex and verifying that solutions with $\omega_i < 0$ exist. Note that an assumption such as $|z_c| \gg 1$ for the complex ω dispersion relation does not necessarily correlate with the complex k dispersion relation with $|z_c| \gg 1$ and indeed it may join onto the complex k dispersion relation with $|z_c| \ll 1$.

The degenerate distribution function need not be analyzed further since it was incorporated into the hydrodynamic distribution function of Section 5.3.1.

5.4 Kinetic Power Theorem for Hybrid Waves

As was found in Chapter III the hybrid modes are difficult to analyze since both the longitudinal and transverse motions are coupled. Thus the hybrid modes in the present case will only be studied assuming the quasi-static assumption. In addition, only the isotropic Maxwellian distribution is examined since the hybrid mode for hydrodynamic and degenerate distribution functionsis studied rigorously in Chapter VI.

For the Maxwellian distribution function,

$$f_o = \frac{N_o}{(2\pi v_T^2)^{3/2}} \exp\left(-\frac{(v_x - v_o)^2}{2v_T^2}\right) \exp\left(-\frac{v_y^2 + v_z^2}{2v_T^2}\right) , \qquad (5.124)$$

where only the drift velocity in the direction of wave propagation is assumed significant. Assume that $\underline{B}_o = B_o \hat{z}$ and the fundamental field varies as $\exp[j(\omega t - kx)]$ so that if E_{ox} is the applied field and E_{oy} is the Hall field the Boltzmann equation for the present case is

$$\frac{\partial f_o}{\partial t} + v_x \frac{\partial f_1}{\partial x} + \eta[\underline{E}_o + (\underline{v} \times \underline{B}_o)] \frac{\partial f_1}{\partial \underline{v}} + \eta[\underline{E}_1 + (\underline{v} \times \underline{B}_1)] \cdot \frac{\partial f_o}{\partial \underline{v}} = \left(\frac{\partial f}{\partial t} \right)_{coll} \quad (5.125)$$

which becomes under the quasi-static assumption

$$j(\omega - kv_x)f_1 + \eta(E_{ox} + v_y B_o) \frac{\partial f_1}{\partial v_x} + \eta(E_{oy} - v_x B_o) \frac{\partial f_1}{\partial v_y} + \eta E_{1x} \frac{\partial f_o}{\partial v_x} = \left(\frac{\partial f}{\partial t} \right)_{coll} \quad . \quad (5.126)$$

Make the following definitions:

$$v_x = u \cos \theta + v_o$$

$$v_y = u \sin \theta \quad . \quad (5.127)$$

Assume that the carrier studied has high mobility and that the magnetic field is moderate so that $v_o \approx E_{oy}/B_o$ and $|E_{ox}| \ll |v_y B_o|$. Thus any carrier heating effects are neglected. In this case when the transformation of Eq. 5.127 is made, Eq. 5.126 takes a particularly simple form given by

$$\frac{\partial f_1}{\partial \theta} - j \frac{\omega - kv_o - ku \cos \theta}{\omega_c} = \eta E_{1x} \frac{\partial f_o}{\partial v_x} - \left(\frac{\partial f}{\partial t} \right)_{coll} \quad . \quad (5.128)$$

The differential equation is solved using an integrating factor to obtain

$$f_1 = \exp\left(j\,\frac{(\omega - kv_o)\theta - ku\sin\theta}{\omega_c}\right)\int_C^{\theta} \exp\left(-j\,\frac{(\omega - kv_o)\theta' - ku\sin\theta'}{\omega_c}\right)$$

$$\cdot\left[\eta E_{1x}\frac{\partial f_o}{\partial v_x} - \left(\frac{\partial f}{\partial t}\right)_{coll}\right] d\theta' \quad , \quad (5.129)$$

where C is a constant such that $f_1(\theta + 2\pi) = f_1(\theta)$. Equation 5.127 applied to Eq. 5.124 shows that

$$f_o = \frac{N_o}{(2\pi v_T^2)^{3/2}} \exp\left(-\frac{u^2}{2v_T^2}\right) \exp\left(-\frac{v_z^2}{2v_T^2}\right) \quad , \quad (5.130)$$

so that

$$\frac{\partial f_o}{\partial v_x} = \frac{\partial f_o}{\partial u}\left(\frac{\partial u}{\partial v_x}\right) = \cos\theta\,\frac{\partial f_o}{\partial u} \quad . \quad (5.131)$$

In addition the collision integral can be neglected to first study the collisionless case. Define

$$a_c = \frac{\omega - kv_o}{\omega_c} \quad \text{and} \quad b_c = \frac{ku}{\omega_c} \quad (5.132)$$

so that if Eqs. 5.131 and 5.132 are used in Eq. 5.129,

$$f_1 = \frac{\eta E_{1x}}{2} \exp[j(a_c\theta - b_c\sin\theta)]$$

$$\cdot\frac{\partial f_o}{\partial u}\int_C^{\theta} (e^{j\theta'} + e^{-j\theta'})\, e^{-ja_c\theta'}\, e^{jb_c\sin\theta'}\, d\theta' \quad , \quad (5.133)$$

where the replacement

$$\cos\theta' = \frac{e^{j\theta'} + e^{-j\theta'}}{2} \tag{5.134}$$

has been made. The Bessel function identities given by

$$e^{\pm jb_c \sin\theta'} = \sum_{n=-\infty}^{\infty} J_n(b_c)\, e^{\pm jn\theta'} \tag{5.135}$$

are used in Eq. 5.133 to obtain

$$f_1 = \frac{\eta E_{1x}}{2}\left(\frac{\partial f_o}{\partial u}\right) \sum_{\ell,m=-\infty}^{\infty} e^{ja_c\theta}\, e^{-j\ell\theta}\, J_\ell(b_c) J_m(b_c)$$

$$\cdot \int_C^{\theta} e^{jm\theta'}\, e^{-ja_c\theta'} \left(e^{j\theta'} + e^{-j\theta'}\right) d\theta' \quad , \tag{5.136}$$

which gives

$$f_1 = \frac{\eta E_{1x}}{2}\left(\frac{\partial f_o}{\partial u}\right) \sum_{\ell,m=-\infty}^{\infty} J_\ell(b_c) J_m(b_c)$$

$$\cdot \left(\frac{\exp[j(m-\ell+1)\theta]}{j(m-a_c+1)} + \frac{\exp(j(m-\ell-1)\theta]}{j(m-a_c-1)} \right) \quad . \tag{5.137}$$

Now the longitudinal RF current density is given by

$$J_{1x} = q \iiint\limits_{-\infty}^{\infty} v_x f_1 \, dv_x \, dv_y \, dv_z$$

$$= q \int_0^{\infty} \int_0^{2\pi} \int_{-\infty}^{\infty} (u \cos\theta + v_o) f_1 \, u \, du \, d\theta \, dv_z \, , \qquad (5.138)$$

where Eq. 5.127 has been used. Now from Eq. 5.124,

$$\frac{\partial f_o}{\partial u} = \frac{N_o}{(2\pi v_T^2)^{3/2}} \left(-\frac{u}{v_T^2}\right) \exp\left(-\frac{u^2}{2v_T^2}\right) \exp\left(-\frac{v_z^2}{2v_T^2}\right) \, , \qquad (5.139)$$

which is independent of θ, so that in Eq. 5.132

$$J_{1x} = \frac{N_o}{(2\pi v_T^2)^{3/2}} \frac{\eta E_{1x}}{4j} \int_0^{\infty} \sum_{\ell, m=-\infty}^{\infty} J_m(b_c) J_\ell(b_c) \left(-\frac{u}{v_T^2}\right) \exp\left(-\frac{u^2}{2v_T^2}\right) u \, du$$

$$\cdot \left(\int_{-\infty}^{\infty} \exp[-v_z^2/2v_T^2] \, dv_z \right) \left[\int_0^{2\pi} \left(\frac{\exp[j(m - \ell + 1)\theta]}{m - a_c + 1} + \frac{\exp[j(m - \ell - 1)\theta]}{m - a_c - 1} \right) \right.$$

$$\left. \cdot (u \cos\theta + v_o) d\theta \right] \, . \qquad (5.140)$$

The θ integral is given by

$$\left[\quad \right] = \frac{2\pi v_o}{\ell - a_c} [\delta(m - \ell + 1) + \delta(m - \ell - 1)]$$

$$+ \frac{u}{2} \int_0^{2\pi} \left(\frac{\exp[j(m - \ell + 2)] + \exp[j(m - \ell)\theta]}{m - a_c + 1} \right.$$

$$\left. + \frac{\exp[j(m - \ell - 2)\theta] + \exp[j(m - \ell)\theta]}{m - a_c - 1} \right) d\theta \, . \qquad (5.141)$$

If the Bessel function identities

$$J_{m-1}(b_c) - J_{m+1}(b_c) = 2 \frac{dJ_m(b_c)}{db_c}$$

and

$$J_{m-1}(b_c) + J_{m+1}(b_c) = \frac{2m}{b_c} J_m(b_c) \qquad (5.142)$$

are used in Eqs. 5.141 and 5.140, then

$$J_{1x} = \frac{jv_o\omega_p^2\epsilon E_{1x}}{v_T^4} \sum_{\ell=-\infty}^{\infty} \int_0^\infty \frac{2\ell\omega_c}{k(\ell - a)} uJ_\ell^2\left(\frac{ku}{\omega_c}\right) \exp\left(-\frac{u^2}{2v_T^2}\right) du + \frac{j\omega_p^2\epsilon E_{1x}}{v_T^4}$$

$$\cdot \sum_{\ell=-\infty}^{\infty} \frac{1}{(\ell - a_c)^2 - 1} \left\{ \int_0^\infty u \frac{\ell\omega_c^2}{k^2} [\ell(\ell - a_c) - 1] J_\ell^2\left(\frac{ku}{\omega_c}\right) \exp\left(-\frac{u^2}{2v_T^2}\right) du \right.$$

$$\left. + \int_0^\infty \left(\frac{a_c\omega_c}{2k}\right) \frac{d}{du}\left[J_\ell^2\left(\frac{ku}{\omega_c}\right)\right] u^3 \exp\left(-\frac{u^2}{2v_T^2}\right) du \right\} . \qquad (5.143)$$

Since

$$\int_0^\infty J_\ell^2(\mu) \exp\left(-\frac{\mu^2}{2\lambda}\right) \mu \, d\mu = \lambda e^{-\lambda} I_\ell(\lambda) , \qquad (5.144)$$

the following is true:

$$J_{1x} = \sum_{\ell=-\infty}^{\infty} \frac{j\omega_p^2\epsilon E_{1x}}{v_T^2} \left(\frac{\ell\omega_c v_o}{k(\ell - a_c)} + \frac{\ell\omega_c^2}{k^2} \frac{\ell(\ell - a_c) - 1}{(\ell - a_c)^2 - 1} \right) . \qquad (5.145)$$

From this result the kinetic power properties can be readily derived. This is reserved for application to instabilities in Chapter VI where a more complete case can be examined thoroughly.

5.5 Summary and Conclusions

The concept of the kinetic power function has been applied to the basic carrier modes within the framework of kinetic theory and the results obtained compared with those of the hydrodynamic theory of Chapter II.

Analysis shows that Maxwellian plasmas can exhibit resonance behavior leading to large effective dielectric constants even for small carrier densities. This effect leads to large wave slowing and consequent wave amplification if a second carrier species is present.

The possibility of temperature anisotropy is taken into account leading to several possible convective instabilities associated with the carrier modes.

Whenever possible corrections or improvements have been pointed out with regard to the utilization of the hydrodynamic theory, thus defining its realm of applicability and accuracy.

CHAPTER VI. KINETIC THEORY OF SOLID-STATE PLASMAS FOR PROPAGATION NORMAL TO THE STATIC MAGNETIC FIELD

6.1 Introduction

When the wave-vector $\underline{k}$ is taken perpendicular to the applied static magnetic field $\underline{B}_o$, in an isotropic medium with a single species of mobile charge, a mode termed the hybrid mode results with characteristics of both the longitudinal space-charge oscillations and the transverse (electromagnetic) helicon-cyclotron mode. This mode is of special interest in solid-state plasma interactions since Landau damping is absent and the Poynting vector is nonzero so that the coupling of electromagnetic radiation to this mode is possible.

A useful approximation which is commonly made when the slow waves are of interest is to assume that the RF magnetic field is negligible corresponding to the transverse component of the RF electric field being much less than the longitudinal component. In this case the resulting slow wave, termed the quasi-static hybrid mode, has a purely longitudinal RF current density which is dependent solely upon the longitudinal component of RF electric field. This quasi-static assumption will always be made and understood in this work. With two species of mobile charge present the interacting quasi-static hybrid modes of each carrier lead to the hybrid-hybrid instability.

For a rigorous analysis of the waves possible with $\underline{k} \perp \underline{B}_o$ it is first necessary to accurately ascertain the form of the carrier distribution function in the presence of applied static magnetic and electric fields.

This is done for equilibrium Maxwellian and degenerate distribution functions and it is shown under which conditions the often used approximation of a drifted form of the distribution function will be a reasonable solution. In addition, for the Maxwellian case, the dc electric field is more fully taken into account in the solution and the carrier distribution function which incorporates carrier heating effects has been obtained.

These results are then used to study the quasi-static hybrid mode. By introducing a modification to the degenerate distribution function, as an approximation, a particularly simple solution results which is readily analyzed. Computer results of the resulting hybrid-hybrid electron-hole instability obtained from this model are applied to the phenomenon of microwave emission from InSb and related to the harmonic structure observed in experimental studies. A discussion is also given of the transition from this plasma type of instability to acoustoelectric amplification at large magnetic fields. The dispersion relation is also obtained for the degenerate and Maxwellian distribution functions. In the latter case the full dispersion relation is solved by computer analysis and it is verified that the approximation of isolated resonances in the wave structure is quite acceptable.

In all cases a comparison is made of the results obtained from the present kinetic theory with those of the hydrodynamic theory. By incorporating a collision term which conserves particles properly it is found that, in essence, the hydrodynamic theory can be retrieved from the kinetic theory at large magnetic fields.

Since in these analyses the variation of the carrier collision frequency with carrier speed has been neglected by using a constant collision frequency, a study is made of the effect of including this variation in the analysis of the waves.

An examination is then conducted of the electrokinetic energy and power properties of the hybrid carrier mode, and the results applied to the RF bunching characteristics of the charge carriers. A comparison with the results of hydrodynamic theory is also made here to aid in understanding the mode behavior.

The effects of carrier heating on the hybrid mode for the Maxwellian distribution function are then studied. The concept of a complex temperature is critically examined by deriving the exact dispersion relation for this case. Significant departures are introduced by the dc electric field heating from the previous kinetic analyses including collisionless modes with a negative electrokinetic energy density at $k < \omega/v_o$.

Finally, a second mode present in the configuration with $\underline{k} \perp \underline{B}_o$, which is electromagnetic in character and termed the ordinary mode, is examined. Unlike the hydrodynamic theory for this wave, the kinetic theory exhibits resonance behavior due to the static magnetic field. Potential instabilities of this mode are then discussed including the effects of carrier heating on this mode.

6.2 Distribution Functions in Applied Static Electric and Magnetic Fields

It is desired to determine the distribution function for a drifting stream of charge carriers in a static magnetic field $\underline{B}_o = B_o\hat{z}$ and general electric field $\underline{E}_o$. This distribution function is given as

$$f_o = f_{oL} + f_{1L} \quad , \tag{6.1}$$

where f_{oL} is the known equilibrium distribution function in the absence of any external fields and f_{1L} is the perturbation due to the applied fields $(\underline{E}_o, \underline{B}_o)$. From Boltzmann's equation, written in the well known form

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{r}} + \frac{1}{m} \underline{F} \cdot \frac{\partial f}{\partial \underline{v}} = \left(\frac{\partial f}{\partial t} \right)_{coll} \quad , \tag{6.2}$$

where $\underline{F}$ is the external force, the function f_o satisfies the following in the steady state:

$$\eta \left(\underline{E}_o \cdot \frac{\partial f_{oL}}{\partial \underline{v}} + (\underline{v} \times \underline{B}_o) \cdot \frac{\partial f_{1L}}{\partial \underline{v}} \right) = -\nu f_{1L} \quad , \tag{6.3}$$

where

$$(\underline{v} \times \underline{B}_o) \cdot \frac{\partial f_{oL}}{\partial \underline{v}} = 0 \tag{6.4}$$

has been used corresponding to isotropic media and the term $\eta \underline{E}_o \cdot (\partial f_{1L}/\partial \underline{v})$ has been ignored. The latter term brings in carrier-heating effects which are negligible unless the carrier drift velocity is comparable to the carrier thermal velocity and this case is presently neglected. Since $\underline{B}_o = B_o \hat{z}$, Eq. 6.3 then becomes

$$B_o \left(v_y \frac{\partial f_{1L}}{\partial v_x} - v_x \frac{\partial f_{1L}}{\partial v_y} \right) + \frac{\nu}{\eta} f_{1L} = -\underline{E}_o \cdot \frac{\partial f_{oL}}{\partial \underline{v}} \quad , \tag{6.5}$$

which, employing the definitions

$$v_x = w \cos \varphi$$

$$v_y = w \sin \varphi \ , \tag{6.6}$$

becomes

$$\frac{\partial f_{1L}}{\partial \varphi} - \frac{\nu}{\eta B_o} f_{1L} = \frac{1}{B_o} \underline{E}_o \cdot \frac{\partial f_{oL}}{\partial \underline{v}} \ . \tag{6.7}$$

By utilizing an integrating factor, Eq. 6.7 is solved as

$$f_{1L} = \frac{\exp\left(\frac{\nu}{\omega_c}\varphi\right)}{B_o} \int_c^{\varphi} \exp\left(-\frac{\nu}{\omega_c}\varphi'\right) \underline{E}_o \cdot \frac{\partial f_{oL}}{\partial \underline{v}} \, d\varphi' \ , \tag{6.8}$$

where $\omega_c = \eta B_o$ and the constant $c \to \pm \infty$. The latter arises from the condition that f_{1L} be bounded and be periodic in φ with period 2π, with the choice of $\pm$ determined by the sign of ω_c. For the case where the equilibrium distribution function is Maxwellian, viz.,

$$f_{oL} = \frac{N_o}{(2\pi v_T^2)^{3/2}} \exp\left(-\frac{(v_x^2 + v_y^2 + v_z^2)}{2v_T^2}\right) \tag{6.9}$$

and

$$\underline{E}_o = E_{ox}\,\hat{x} + E_{oy}\,\hat{y} \ , \tag{6.10}$$

Eq. 6.8 becomes

$$f_{1L} = -\frac{wf_{oL}}{B_o v_T^2} \exp\left(\frac{\nu}{\omega_c}\varphi\right) \int_c^{\varphi} \exp\left(-\frac{\nu}{\omega_c}\varphi'\right) (E_{ox} \cos \varphi' + E_{oy} \sin \varphi') d\varphi' \ , \tag{6.11}$$

so that when the integration is carried out, Eq. 6.1 gives the following :

$$f_o = f_{oL}\left[1 - \frac{1}{B_o v_T^2\left(\frac{\nu^2}{\omega_c^2}+1\right)}\left[E_{ox}\left(-\frac{\nu}{\omega_c}v_x + v_y\right) + E_{oy}\left(-\frac{\nu}{\omega_c}v_y - v_x\right)\right]\right] . \tag{6.12}$$

Note that for f_{oL} Maxwellian, the expression

$$\iiint_{-\infty}^{\infty} f_{1L}\, dv_x\, dv_y\, dv_z = 0$$

is true, and this indicates that the solution conserves particles properly.

As an example of the application of this result consider n-InSb with the applied electric field E_{ox}. For moderate magnetic fields (typically 3 kG) and rectangular bar geometries, since the Hall electric field $E_{oy} \gg E_{ox}$ and $|\omega_{ce}| \gg \nu_e$, the distribution function becomes, from Eq. 6.12,

$$f_{oe} \approx f_{oL}\left(1 + \frac{v_x E_{oy}}{B_o v_{Te}^2}\right) . \tag{6.13}$$

The carrier drift velocity becomes, in the $\hat{x}$ direction,

$$v_{oe} = \frac{1}{N_o}\iiint_{-\infty}^{\infty} v_x f_{oe}\, dv_x\, dv_y\, dv_z \approx \frac{E_{oy}}{B_o} , \tag{6.14}$$

so that Eq. 6.13 can be written as

$$f_{oe} \simeq f_{oL}\left(1 + \frac{v_{oe} v_x}{v_{Te}^2}\right) . \tag{6.15}$$

Consider now the displaced Maxwellian f'_{oL} given by the Taylor expansion of $f_{oL}(v_x)$ as

$$f'_{oL} \triangleq f_{oL}(v_x \rightarrow v_x - v_{oe}) = f_{oL}(v_x) - v_{oe}\frac{\partial f_{oL}}{\partial v_x} + \frac{v_{oe}^2}{2!}\frac{\partial^2 f_{oL}}{\partial v_x^2} + \ldots , \tag{6.16}$$

which for $v_{oe} \ll v_{Te}$ gives

$$f'_{oL} \approx f_{oL}\left(1 + \frac{v_{oe}v_x}{v_{Te}^2}\right) . \tag{6.17}$$

Comparison of Eq. 6.15 with this result shows that for $v_{oe} \ll v_{Te}$

$$f_{oe} \approx \frac{N_{oe}}{(2\pi v_{Te}^2)^{3/2}} \exp\left(-\frac{(v_x - v_{oe})^2}{2v_{Te}^2}\right) \exp\left(-\frac{v_y^2 + v_z^2}{2v_{Te}^2}\right) , \tag{6.18}$$

thus illustrating that under the conditions assumed the drifted Maxwellian distribution function of Eq. 6.18 is an excellent approximation to Eq. 6.12.

In order to study the degenerate distribution function, the following transformation is needed:

$$v_x = w \cos\varphi = v_\rho \cos\varphi \sin\theta ,$$

$$v_y = w \sin\varphi = v_\rho \sin\varphi \sin\theta$$

and

$$v_z = v_\rho \cos\theta , \tag{6.19}$$

where

$$v_\rho = (v_x^2 + v_y^2 + v_z^2)^{1/2} = (w^2 + v_z^2)^{1/2} \tag{6.20}$$

and

$$\iiint_{-\infty}^{\infty} dv_x \, dv_y \, dv_z = \int_0^{v_F} \int_0^{2\pi} \int_0^{\pi} v_\rho^2 \sin\theta \, dv_\rho \, d\varphi \, d\theta \quad . \qquad (6.21)$$

The equilibrium degenerate distribution function by definition is

$$f_{oL} = \frac{3N_o}{4\pi v_F^3} \quad \text{if} \quad v_\rho \leqq v_F \ ,$$

$$= 0 \quad \text{otherwise} \ , \qquad (6.22)$$

where v_F is the isotropic Fermi velocity. It can readily be verified that

$$\iiint_{-\infty}^{\infty} f_{oL} \, dv_x \, dv_y \, dv_z = N_o \quad .$$

For the case $\underline{E}_o = E_{ox}\hat{x} + E_{oy}\hat{y}$, use of Eq. 6.22 in Eq. 6.8 provides that

$$f_{1L} = \frac{1}{v_\rho}\left(\frac{\partial f_{oL}}{\partial v_\rho}\right)\frac{1}{B_o\left(\frac{\nu^2}{\omega_c^2}+1\right)}\left[E_{ox}\left(-\frac{\nu}{\omega_c}v_x + v_y\right) - E_{oy}\left(v_x + \frac{\nu}{\omega_c}v_y\right)\right] \ , \qquad (6.23)$$

so that Eq. 6.1 gives, directly,

$$f_o = f_{oL} + \frac{1}{v_\rho}\left(\frac{\partial f_{oL}}{\partial v_\rho}\right)\frac{1}{B_o\left(\frac{\nu^2}{\omega_c^2}+1\right)}\left[E_{ox}\left(-\frac{\nu}{\omega_c}v_x + v_y\right) - E_{oy}\left(v_x + \frac{\nu}{\omega_c}v_y\right)\right] . \qquad (6.24)$$

It can be shown that the drift velocity in the $\hat{x}$ direction given by

$$v_o = \frac{1}{N_o}\iiint_{-\infty}^{\infty} v_x f_o \, dv_x \, dv_y \, dv_z \qquad (6.25)$$

becomes, when Eq. 6.24 is used,

$$v_o = \frac{1}{B_o\left(\frac{\nu^2}{\omega_c^2} + 1\right)}\left(E_{oy} + \frac{\nu}{\omega_c} E_{ox}\right) , \tag{6.26}$$

which is identical to that found for the Maxwellian case.

From Eq. 6.22 it can be found that

$$\frac{\partial f_{oL}}{\partial v_\rho} = f_{oL}\left(\delta(v_\rho) - \delta(v_\rho - v_F)\right) , \tag{6.27}$$

and from Eq. 6.19,

$$\frac{\partial f_{oL}}{\partial v_x} = \frac{\partial f_{oL}}{\partial v_\rho}\left(\frac{\partial v_x}{\partial v_\rho}\right) . \tag{6.28}$$

The drifted degenerate distribution function is defined by

$$f'_{oL} \triangleq f_{oL}(v_x \rightarrow v_x - v_o) = \frac{3N_o}{4\pi v_F^3} \text{ if } \left((v_x - v_o)^2 + v_y^2 + v_z^2\right)^{1/2} \leqq v_F ,$$

$$= 0 \quad \text{otherwise} . \tag{6.29}$$

A Taylor expansion similar to Eq. 6.16 shows that, if $v_o \ll v_F$,

$$f'_{oL} \approx f_{oL} - v_o \frac{\partial f_{oL}}{\partial v_x} , \tag{6.30}$$

so that from Eqs. 6.26 and 6.28

$$f'_{oL} \simeq f_{oL} + \frac{1}{v_\rho}\left(\frac{\partial f_{oL}}{\partial v_\rho}\right)\frac{1}{B_o\left(\frac{\nu^2}{\omega_c^2} + 1\right)}\left(-E_{oy}v_x - \frac{\nu}{\omega_c} E_{ox}v_x\right) . \tag{6.31}$$

A comparison of Eq. 6.31 with Eq. 6.24 shows that if $|E_{ox}| < |E_{oy}|$ and $|\omega_c| \gg \nu$ the drifted degenerate distribution function is an excellent approximation to the general result, Eq. 6.24.

6.3 Effects of High Electric Fields on the Carrier Distribution Functions

The case wherein the carrier drift velocity may be comparable to its thermal or Fermi velocity is now considered by including the term $\eta \underline{E}_o (\partial f_{1L}/\partial \underline{v})$ in the analysis. When this is done Eq. 6.5 becomes

$$B_o \left[\left(\frac{E_{ox}}{B_o} + v_y \right) \frac{\partial f_{1L}}{\partial v_x} + \left(\frac{E_{oy}}{B_o} - v_x \right) \frac{\partial f_{1L}}{\partial v_y} + \frac{E_{oz}}{B_o} \frac{\partial f_{1L}}{\partial v_z} \right] + \frac{\nu}{\eta} f_{1L}$$

$$= -\underline{E}_o \cdot \frac{\partial f_{oL}}{\partial \underline{v}} \quad .. \quad (6.32)$$

Consider first the case where E_{ox} and E_{oz} are negligible compared to E_{oy} and define the new transformation as

$$v_x = v_H + u \cos\theta \quad ; \quad v_H \triangleq E_{oy}/B_o$$

$$v_y = u \sin\theta \quad , \qquad (6.33)$$

so that Eq. 6.32 transforms as

$$\frac{\partial f_{1L}}{\partial \theta} - \frac{\nu}{\omega_c} f_{1L} = v_H \frac{\partial f_{oL}}{\partial v_y} \quad ; \qquad (6.34)$$

therefore,

$$f_{1L} = \exp\left(\frac{\nu}{\omega_c} \theta \right) \int_c^{\theta} \exp\left(- \frac{\nu}{\omega_c} \theta' \right) v_H \frac{\partial f_{oL}}{\partial v_y} d\theta' \quad . \qquad (6.35)$$

The equilibrium Maxwellian distribution, Eq. 6.9, provides that

$$\frac{\partial f_{oL}}{\partial v_y} = -\frac{v_y}{v_T^2} f_{oL} = -\frac{N_o}{(2\pi v_T^2)^{3/2}} \frac{u \sin\theta}{v_T^2} \exp\left(-\frac{v_H^2}{2v_T^2}\right) \exp\left(-\frac{u^2}{2v_T^2}\right)$$

$$\cdot \exp\left(-\frac{v_z^2}{2v_T^2}\right) \exp\left(-\frac{uv_H}{v_T^2}\cos\theta\right) \quad ; \quad (6.36)$$

thus, since

$$\exp\left(-\frac{uv_H}{v_T^2}\cos\theta\right) = \sum_{n=-\infty}^{\infty} I_n\left(-\frac{uv_H}{v_T^2}\right) e^{jn\theta} \quad , \qquad (6.37)$$

the following is true:

$$\frac{\partial f_{oL}}{\partial v_y} = g_o(u) \sin\theta \sum_{n=-\infty}^{\infty} I_n\left(-\frac{uv_H}{v_T^2}\right) e^{jn\theta} \quad , \qquad (6.38)$$

where

$$g_o(u) = -\frac{N_o}{(2\pi v_T^2)^{3/2}} \frac{u}{v_T^2} \exp\left(-\frac{v_H^2}{2v_T^2}\right) \exp\left(-\frac{u^2}{2v_T^2}\right) \exp\left(-\frac{v_z^2}{2v_T^2}\right) \, . \qquad (6.39)$$

Equation 6.35 is then readily solved as

$$f_{1L}(\theta) = v_H g_o(u) \sum_{n=-\infty}^{\infty} I_n\left(-\frac{uv_H}{v_T^2}\right) \frac{[(jn - \nu/\omega_c)\sin\theta - \cos\theta]}{1 + (jn - \nu/\omega_c)^2} e^{jn\theta} \quad , \qquad (6.40)$$

where the constant c has been taken into account via the requirement $f_{1L}(\theta) = f_{1L}(\theta + 2\pi)$.

The distribution function in the presence of the applied fields is then given from Eqs. 6.1, 6.9 and 6.40 as

$$f_o = \frac{N_o}{(2\pi v_T^2)^{3/2}} \exp\left(-\frac{u^2 + v_H^2 + v_z^2}{2v_T^2}\right) \sum_{n=-\infty}^{\infty} I_n\left(-\frac{uv_H}{v_T^2}\right) e^{jn\theta}$$

$$\cdot \left(1 - \frac{uv_H}{v_T^2} \frac{[(jn - \nu/\omega_c)\sin\theta - \cos\theta]}{1 + (jn - \nu/\omega_c)^2}\right) , \qquad (6.41)$$

which, if desired, can be transformed back to the original (v_x, v_y, v_z) reference frame from Eq. 6.33. Inspection of Eq. 6.41 shows that to conserve particles properly it is necessary that

$$0 = \iiint_{-\infty}^{\infty} f_{1L}\, dv_x\, dv_y\, dv_z = -\frac{N_o v_H}{2\pi v_T^4} \exp\left(-\frac{v_H^2}{2v_T^2}\right) \int_0^{2\pi}\int_0^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{1+(jn-\nu/\omega_c)^2}$$

$$u^2 \exp\left(-\frac{u^2}{2v_T^2}\right) I_n\left(-\frac{uv_H}{v_T^2}\right) [(jn - \nu/\omega_c)\sin\theta - \cos\theta] e^{jn\theta}\, du\, d\theta , \qquad (6.42)$$

and the integral on the right-hand side is found as

$$\sum_{n=-\infty}^{\infty} \frac{1}{1 + (jn - \nu/\omega_c)^2} \int_0^{2\pi}\int_0^{\infty} u^2 \exp\left(-\frac{u^2}{2v_T^2}\right) I_n\left(-\frac{uv_H}{v_T^2}\right)$$

$$\cdot \Big(\exp[j(n+1)\theta]\, [-j(jn - \nu/\omega_c) - 1] + \exp[j(n-1)\theta]\, [j(jn - \nu/\omega_c) - 1]\Big)\, du\, d\theta$$

$$= 2\pi \int_0^{\infty} (-j) \frac{\omega_c}{\nu} u^2 \exp\left(-\frac{u^2}{2v_T^2}\right) \left[I_1\left(-\frac{uv_H}{v_T^2}\right) - I_{-1}\left(-\frac{uv_H}{v_T^2}\right)\right] = 0 , \qquad (6.43)$$

wherein use was made of the following:

$$I_{n+1}(x) - I_{n-1}(x) = -\frac{2n}{x} I_n(x) \quad . \qquad (6.44)$$

Thus the solution given by Eq. 6.41 conserves particles properly. The drift velocity in the $\hat{x}$ direction is found as

$$v_o = \frac{1}{N_o} \iiint_{-\infty}^{\infty} v_x f_o \, dv_x \, dv_y \, dv_z = v_H + v_\theta \quad , \qquad (6.45)$$

where

$$v_\theta = -\frac{v_H}{2\pi v_T^4} \exp\left(-\frac{v_H^2}{2v_T^2}\right) \sum_{n=-\infty}^{\infty} \frac{1}{1 + (jn - \nu/\omega_c)^2} \int_0^{2\pi} \int_0^{\infty}$$

$$\cdot \, u^3 \exp\left(-\frac{u^2}{2v_T^2}\right) I_n\left(-\frac{uv_H}{v_T^2}\right) [(jn - \nu/\omega_c)\sin\theta \cos\theta - \cos^2\theta] e^{jn\theta} \, du \, d\theta,$$

$$= \frac{v_H}{2v_T^4} \exp\left(-\frac{v_H^2}{2v_T^2}\right) \int_0^{\infty} \frac{u^3}{1 + \nu^2/\omega_c^2} \exp\left(-\frac{u^2}{2v_T^2}\right)$$

$$\cdot \left[I_o\left(-\frac{uv_H}{v_T^2}\right) - I_2\left(-\frac{uv_H}{v_T^2}\right)\right] du \quad ,$$

$$= \frac{2v_H}{1 + \nu^2/\omega_c^2} \exp\left(-\frac{v_H^2}{2v_T^2}\right) M\left(2 \; ; 1 \; ; \frac{v_H^2}{2v_T^2}\right) \quad , \qquad (6.46)$$

where M is the confluent hypergeometric function and $v_H > 0$ has been assumed. For example, if

$$v_H = \frac{E_{oy}}{B_o} = \frac{1}{\sqrt{5}} v_T \quad , \qquad M(2 \; ; 1 \; ; 0.1) = 2.71828$$

and

$$v_o \approx v_H \left(\frac{2 + \nu^2/\omega_c^2}{1 + \nu^2/\omega_c^2} \right) . \qquad (6.47)$$

If $v_H \ll v_T$, then it is clear that the n = 0 term will dominate in Eq. 6.41 since for small arguments $|I_0| > |I_1|, |I_2|, |I_3|, \ldots$. Furthermore if $|\omega_c| \gg \nu$, Eq. 6.41 is approximated by

$$f_o \approx f_{oL} \left(1 + \frac{v_H v_x}{v_T^2} \right) , \qquad (6.48)$$

which is identical to the result found previously, Eq. 6.13, which led to the drifted Maxwellian form.

6.4 The Quasi-Static Hybrid Mode: General Solution

Assume that the quasi-static case is valid so that in the effective dielectric constant

$$\overleftrightarrow{\epsilon} = \overleftrightarrow{I} + \frac{1}{j\omega\epsilon} \sum_s \overleftrightarrow{\sigma}_s , \qquad (6.49)$$

where $\overleftrightarrow{I}$ is the unit matrix and $\overleftrightarrow{\sigma}_s$ is the conductivity tensor of the sth carrier species, the element ϵ_{xx} is much larger than any remaining ϵ_{ij} where a variation as $\exp[j(\omega t - kx)]$ is taken for the RF fields. The linearized Boltzmann equation, with $\underline{B}_o = B_o \hat{z}$, is then given for each carrier species s by

$$j(\omega - kv_{xs})f_{1s} + \eta_s B_o \left(v_{ys} \frac{\partial f_{1s}}{\partial v_{xs}} - v_{xs} \frac{\partial f_{1s}}{\partial v_{ys}} \right) + \eta_s \left(E_{ox} \frac{\partial f_{1s}}{\partial v_{xs}} + E_{oy} \frac{\partial f_{1s}}{\partial v_{ys}} \right)$$

$$+ \eta_s E_{1x} \frac{\partial f_{os}}{\partial v_{xs}} = -\nu_s \left(f_{1s} - \frac{N_{1s}}{N_{os}} f_{os} \right) , \qquad (6.50)$$

where $E_{oz} = 0$ has been assumed and the collision term which conserves particles properly has been taken on the right-hand side, where N_{1s} is the RF number density,

$$N_{1s} = \iiint_{-\infty}^{\infty} f_{1s}\, dv_{xs}\, dv_{ys}\, dv_{zs} \ . \tag{6.51}$$

To obtain a tractable result it is now assumed that $|\omega_{cs}| \gg \nu_s$, the Hall field E_{oy} is larger in magnitude than the applied field E_{ox} (or E_{oy} is the applied field and is larger in magnitude than E_{ox}), and the carrier drift velocity is sufficiently small compared to the carrier thermal velocity or Fermi velocity. Then, as discussed in Section 6.2 for either the Maxwellian or degenerate equilibrium distribution function, $v_{os} \approx E_{oy}/B_o$. The following transformation is then convenient:

$$v_{xs} = v_{os} + u_s \cos\theta$$

$$v_{ys} = u_s \sin\theta \ , \tag{6.52}$$

since Eq. 6.50 can now be written as

$$\frac{\partial f_{1s}}{\partial\theta} - j(a_s - b_s \cos\theta) f_{1s} = \frac{\eta_s E_{1x}}{\omega_{cs}} \left(\frac{\partial f_{os}}{\partial v_{xs}} \right) - \frac{\nu_s}{\omega_{cs}} \frac{N_{1s}}{N_{os}} f_{os} \ , \tag{6.53}$$

where

$$a_s = \frac{\omega - kv_{os} - j\nu_s}{\omega_{cs}} \quad \text{and} \quad b_s = \frac{ku_s}{\omega_{cs}} \ . \tag{6.54}$$

As discussed in Section 6.2, in the present case, the drifted distribution functions given by Eq. 6.18 for the Maxwellian case and Eq. 6.29 for the degenerate case can be used for f_{os}. In both instances, inspection of Eq. 6.52 indicates that f_{os} is independent of θ so that

$$\frac{\partial f_{os}}{\partial v_{xs}} = \frac{\partial f_{os}}{\partial u_s}\left(\frac{\partial u_s}{\partial v_{xs}}\right) = \cos\theta\left(\frac{\partial f_{os}}{\partial u_s}\right) . \qquad (6.55)$$

Equation 6.53 is solved employing an integrating factor together with Eq. 6.35 to obtain

$$f_{1s}(\theta)\exp[-j(a_s\theta - b_s\sin\theta)] = \frac{\eta_s}{\omega_{cs}} E_{1x}\left(\frac{\partial f_{os}}{\partial u_s}\right)\int_c^{\theta}\cos\theta'$$

$$\cdot \exp[-j(a_s\theta' - b_s\sin\theta')]d\theta' - \frac{\nu_s}{\omega_{cs}}\frac{N_{1s}}{N_{os}} f_{os}(u_s)\int_c^{\theta} .$$

$$\cdot \exp[-j(a_s\theta' - b_s\sin\theta')]d\theta' , \qquad (6.56)$$

where c is determined from the condition $f_{1s}(\theta + 2\pi) = f_{1s}(\theta)$. The Bessel function relations of Eq. 5.135 enable the solution of Eq. 6.56 to be written as follows:

$$f_{1s}(\theta) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} J_m(b_s)J_n(b_s)\left[\frac{\eta_s E_{1x}}{\omega_{cs}}\left(\frac{\partial f_{os}}{\partial u_s}\right)\frac{[-j(a_s - n)\cos\theta + \sin\theta]}{1 - (a_s - n)^2}\right.$$

$$\left. + j\frac{\nu_s N_{1s} f_{os}}{\omega_{cs}N_{os}(n - a_s)}\right]\exp[j(n - m)\theta] . \qquad (6.57)$$

The relations

$$dv_{xs}\, dv_{ys}\, dv_{zs} = u_s\, dv_s\, d\theta\, dv_{zs} \qquad (6.58)$$

and

$$J_{m-1}(b_s) + J_{m+1}(b_s) = \frac{2m}{b_s} J_m(b_s) \tag{6.59}$$

are now used in Eqs. 6.51 and 6.57 to obtain

$$N_{1s} = - \frac{j \dfrac{\eta_s E_{1x}}{k} \displaystyle\sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} \left(\frac{\partial f_{os}}{\partial u_s} \right) \frac{2\pi m J_m^2(b_s)}{m - a_s} \, du_s \, dv_{zs}}{1 - j \dfrac{2\pi \nu_s}{N_{os}\omega_{cs}} \displaystyle\sum_{m=-\infty}^{\infty} \frac{1}{m - a_s} \int_{-\infty}^{\infty} \int_{0}^{\infty} J_m^2(b_s) f_{os} u_s \, du_s \, dv_{zs}} . \tag{6.60}$$

At this point, prior to a study of the degenerate and Maxwellian distribution functions, a simple approximation to the degenerate distribution function is made. This will simplify the integrations involved while retaining the essential physics of the interaction.

6.4.1 The "Cylindrical" Degenerate Distribution Function. The "cylindrical" degenerate distribution function is now defined by

$$f_{os} = \frac{N_{os}}{2\pi v_{F\|} v_{Fs}^2} \quad \text{if} \quad u_s \leqq v_{Fs} \quad \text{and} \quad |v_{zs}| \leqq v_{F\|}$$

$$= 0 \quad \text{otherwise} , \tag{6.61}$$

where $v_{F\|}$ is the Fermi speed parallel to B_o. This is then similar to the degenerate distribution function studied previously which is given by Eqs. 6.29 and 6.52 as

$$f_{os} = \frac{3N_{os}}{4\pi v_{Fs}^3} \quad \text{if} \quad (u_s^2 + v_{zs}^2)^{1/2} \leqq v_{Fs} ,$$

$$= 0 \quad \text{otherwise} . \tag{6.62}$$

From the viewpoint of achieving instability it is expected that the function of Eq. 6.61 is actually more pessimistic than that of Eq. 6.62. The distribution function of Eq. 6.61 used in Eq. 6.60 yields

$$N_{1s} = \frac{-\dfrac{j2\eta_s N_{os} E_{1x}}{kv_{Fs}^2} \displaystyle\sum_{m=-\infty}^{\infty} \frac{m\omega_{cs} J_m^2\left(\dfrac{kv_{Fs}}{\omega_{cs}}\right)}{\omega - kv_{os} - j\nu_s - m\omega_{cs}}}{1 + j\nu_s \displaystyle\sum_{m=-\infty}^{\infty} \frac{\left(1 - \dfrac{m^2\omega_{cs}^2}{k^2 v_{Fs}^2}\right) J_m^2\left(\dfrac{kv_{Fs}}{\omega_{cs}}\right) + \left[J_m'\left(\dfrac{kv_{Fs}}{\omega_{cs}}\right)\right]^2}{\omega - kv_{os} - j\nu_s - m\omega_{cs}}} , \qquad (6.63)$$

where

$$J_m'\left(\frac{kv_{Fs}}{\omega_{cs}}\right) = \left.\frac{dJ_m\left(\dfrac{ku_s}{\omega_{cs}}\right)}{d\left(\dfrac{ku_s}{\omega_{cs}}\right)}\right|_{u_s = v_{Fs}} , \qquad (6.64)$$

and in obtaining Eq. 6.63 use has been made of

$$mJ_m^2(0) = 0 \quad \text{for } m \text{ integer} , \qquad (6.65)$$

and

$$\int J_m^2(kx)\, x\, dx = \frac{1}{2}\left(x^2 - \frac{m^2}{k^2}\right) J_m^2(kx) + \frac{1}{2} x^2 \left[J_m'(kx)\right]^2 . \qquad (6.66)$$

Note that in the present case the Fermi velocity parallel to B_o, $v_{F_{||}}$, plays no role in the quasi-static case. From Poisson's equation,

$$\nabla \cdot \underline{E}_1 = -jkE_{1x} = \sum_s \frac{q_s}{\epsilon} N_{1s} \quad , \tag{6.67}$$

and the dispersion relation is found from Eq. 6.63 as

$$1 = \sum_s \frac{\left(\dfrac{2\omega_{ps}^2}{k^2 v_{Fs}^2}\right) \displaystyle\sum_{m=-\infty}^{\infty} \frac{m\omega_{cs} J_m^2\left(\dfrac{kv_{Fs}}{\omega_{cs}}\right)}{\omega - kv_{os} - j\nu_s - m\omega_{cs}}}{1 + j\nu_s \displaystyle\sum_{m=-\infty}^{\infty} \frac{Q_m\left(\dfrac{kv_{Fs}}{\omega_{cs}}\right)}{\omega - kv_{os} - j\nu_s - m\omega_{cs}}} \quad , \tag{6.68}$$

where $Q_m(kv_{Fs}/\omega_{cs})$ is defined directly from Eq. 6.63. The term in Q_m is due directly to the conservation of particles aspect of the collision term defined in Eq. 6.50. Thus if the collision term $(\partial f/\partial t)_{coll} = -\nu f_{1s}$ had been used this would correspond to $Q_m = 0$. The physical effect of imposing particle conservation is seen if it is assumed that a single resonance at m = n gives the major contribution to the sums in Eq. 6.68 for some carrier "s" in which case the dispersion relation is

$$1 = \frac{2\omega_{ps}^2 n\omega_{cs} J_n^2\left(\dfrac{kv_{Fs}}{\omega_{cs}}\right)}{k^2 v_{Fs}^2 \left\{\omega - kv_{os} - n\omega_{cs} - j\nu_s\left[1 - Q_n\left(\dfrac{kv_{Fs}}{\omega_{cs}}\right)\right]\right\}} + \left(\begin{array}{c}\text{terms due to}\\ \text{other}\\ \text{carrier species}\end{array}\right) . \tag{6.69}$$

Thus the modification introduced by the conservation of particles can be taken into account by replacing the collision frequency ν_s by an effective collision frequency ν_s', where

$$\nu_s' = \nu_s \left[1 - \left\{ \left(1 - \frac{n^2\omega_{cs}^2}{k^2 v_{Fs}^2} \right) J_n^2\left(\frac{kv_{Fs}}{\omega_{cs}} \right) + \left[J_n'\left(\frac{kv_{Fs}}{\omega_{cs}} \right) \right]^2 \right\} \right] . \quad (6.70)$$

Moreover, from Eq. 6.60, the more general statement for the distribution function f_{os} is, when a single resonance is assumed as shown above,

$$\nu_s' = \nu_s \left[1 - \frac{2\pi}{N_{os}} \int_{-\infty}^{\infty} \int_0^{\infty} J_n^2\left(\frac{ku_s}{\omega_{cs}} \right) f_{os} u_s \, du_s \, dv_{zs} \right] , \qquad (6.71)$$

wherein it is assumed that f_{os} is independent of θ since Eq. 6.55 has been used to obtain Eq. 6.60. This latter condition is true for the drifted Maxwellian distribution, Eq. 6.18, and the drifted degenerate distribution, Eq. 6.62, and of course the cylindrical distribution, Eq. 6.61. Hence for $v_{os} < v_{Ts}$ for the Maxwellian case and $v_{os} < v_{Fs}$ for the degenerate case the respective distribution function is independent of θ. When k is real and f_{os} is independent of θ it can be shown that ν_s', the effective collision frequency, is necessarily positive or zero. For k real, it follows that $|J_o(kv_{Fs}/\omega_{cs})| \leq 1$ and $|J_n(kv_{Fs}/\omega_{cs})| \leq 1/\sqrt{2}$, so it must be that

$$I' \triangleq \int_{-\infty}^{\infty} \int_0^{\infty} J_n^2\left(\frac{kv_{Fs}}{\omega_{cs}} \right) f_{os} u_s \, du_s \, dv_{zs} \leq \int_{-\infty}^{\infty} \int_0^{\infty} f_{os} u_s \, du_s \, dv_{zs} .$$

Therefore, if f_{os} is independent of θ, since

$$N_{os} = \int_0^{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} f_{os} u_s \, du_s \, d\theta \, dv_{zs} ,$$

then the following is true:

$$I' \leq \frac{N_{os}}{2\pi} \quad . \tag{6.72}$$

Use of this result in Eq. 6.71 then provides that $\nu_s' \geq 0$. Note that only in the case n = 0 and $(k/\omega_{cs}) \to 0$ does $\nu_s' \to 0$.

To investigate the properties of the theoretical model, consideration is given to high-field instabilities in n-InSb since experimental data is available for this material. The carrier resonances are well defined if $|\omega_{cs}| \gg \nu_s$ which is readily satisfied by the electrons but not by the holes. Nevertheless for the purposes of analysis the resonances will be separated out even for the holes. In addition the effective collision frequency defined in Eq. 6.70 can now be used. The dispersion equation then is written from Eq. 6.69 as

$$1 = -\left(\frac{2\omega_{pe}^2}{k^2 v_{Fe}^2}\right) \frac{J_1^2\left(\frac{kv_{Fe}}{\omega_{ce}}\right)|\omega_{ce}|}{\omega - kv_{oe} - j\nu_e' + |\omega_{ce}|} + \left(\frac{2\omega_{ph}^2}{k^2 v_{Fh}^2}\right) \frac{J_m^2\left(\frac{kv_{Fh}}{\omega_{ch}}\right) m|\omega_{ch}|}{\omega - j\nu_h' - m|\omega_{ch}|} \quad , \tag{6.73}$$

which describes the interaction of the fundamental electron slow-cyclotron mode with the mth harmonic of the hole cyclotron wave. The hole drift velocity is assumed negligible and the effect of the Hall electric field on the hole dispersion properties is neglected. The solution for real k is

$$2\omega = j(\nu_e' + \nu_h') + m|\omega_{ch}|\,(1 + \theta_h) + kv_{oe} - |\omega_{ce}|\,(1 + \theta_e) \pm \sqrt{R} \quad , \tag{6.74}$$

where

$$R = \left(j(\nu_e' - \nu_h') + kv_{oe} - m|\omega_{ch}|\ (1 + \theta_h) - |\omega_{ce}|\ (1 + \theta_e)\right)^2$$

$$- 4m|\omega_{ch}|\ |\omega_{ce}|\ \theta_e\theta_h \qquad (6.75)$$

and

$$\theta_e = \frac{2\omega_{pe}^2}{k^2 v_{Fe}^2} J_1^2\left(\frac{kv_{Fe}}{\omega_{ce}}\right) , \qquad \theta_h = \frac{2\omega_{ph}^2}{k^2 v_{Fh}^2} J_m^2\left(\frac{kv_{Fh}}{\omega_{ch}}\right) \geqq 0 \ . \qquad (6.76)$$

For the possibly growing solutions ($\omega_i < 0$) of interest, where $\omega = \omega_r + j\omega_i$, Eq. 6.74 simplifies, if $\nu_e' \approx \nu_h'$, to the following:

$$2\omega_r = kv_{oe} + m|\omega_{ch}|\ (1 + \theta_h) - |\omega_{ce}|\ (1 + \theta_e) \ . \qquad (6.77)$$

and

$$\omega_i = \frac{\nu_e' + \nu_h'}{2} \pm \left[-\left(\omega_r - m|\omega_{ch}|\ (1 + \theta_h)\right)^2 + m|\omega_{ch}|\ |\omega_{ce}|\ \theta_e\theta_h\right]^{1/2} \ . \qquad (6.78)$$

Inspection of this result shows that growth should occur near the hole cyclotron harmonics in frequency provided that

$$\frac{4m\ |\omega_{ch}|\ |\omega_{ce}|\ \omega_{pe}^2\omega_{ph}^2\ J_1^2\left(\frac{kv_{Fe}}{\omega_{ce}}\right) J_m^2\left(\frac{kv_{Fh}}{\omega_{ch}}\right)}{k^4 v_{Fe}^2 v_{Fh}^2} > \frac{(\nu_e' + \nu_h')^2}{4} \ . \qquad (6.79)$$

However, because of the strong dependence of the growth factor on the wave number, a solution for k from Eq. 6.77 indicates that, for moderate magnetic field strengths ($B_o \approx 3$kG), larger growth rates may be attained for $2\omega_r \rightarrow m|\omega_{ch}|$, provided that $|\omega_{ce}| \gg |\omega_{ch}|$. Exact computer solutions of Eq. 6.73 have shown this to be the case in that the emissive peaks are shifted to magnetic field valueslarger than those corresponding to the hole cyclotron harmonics. The amount of the shift increases with a decrease in carrier density. Thus in Fig. 6.1 with equilibrium electron and hole densities $N = P = 4 \times 10^{15}$ cm^{-3} the shift from the fundamental resonance for a chosen frequency $\omega_r = 2\pi \times 9.4$ GHz (corresponding to the experimental value[78]) is approximately 600 G, whereas in Fig. 6.2 with $N = P = 10^{16}$ cm^{-3} the shift is only 100 G. Other parameters used in these computations are as follows:

$$m_e^* = 0.03\ m_o \ ,$$
$$m_h^* = 0.61\ m_o \ ,$$
$$v_{Fe} = 8 \times 10^7\ cm/s \ ,$$
$$v_{Fh} = 4 \times 10^7\ cm/s \ .$$

This carrier density range is selected in accordance with the current density values attained in the experimental work[78] under the stipulation that the drift velocity never exceeds the thermal velocity and the fact that the harmonic radiation occurs after the onset of impact ionization.[78] The theoretical results obtained in Fig. 6.1 are in general agreement with the experimental data reproduced in Fig. 6.3 with the following characteristics:

1. The quantitative position of the emissive peaks and their relation to the hole cyclotron harmonics.

2. The shift at low-current densities to higher values of magnetic field corresponding to a decrease in carrier density at low-field values.

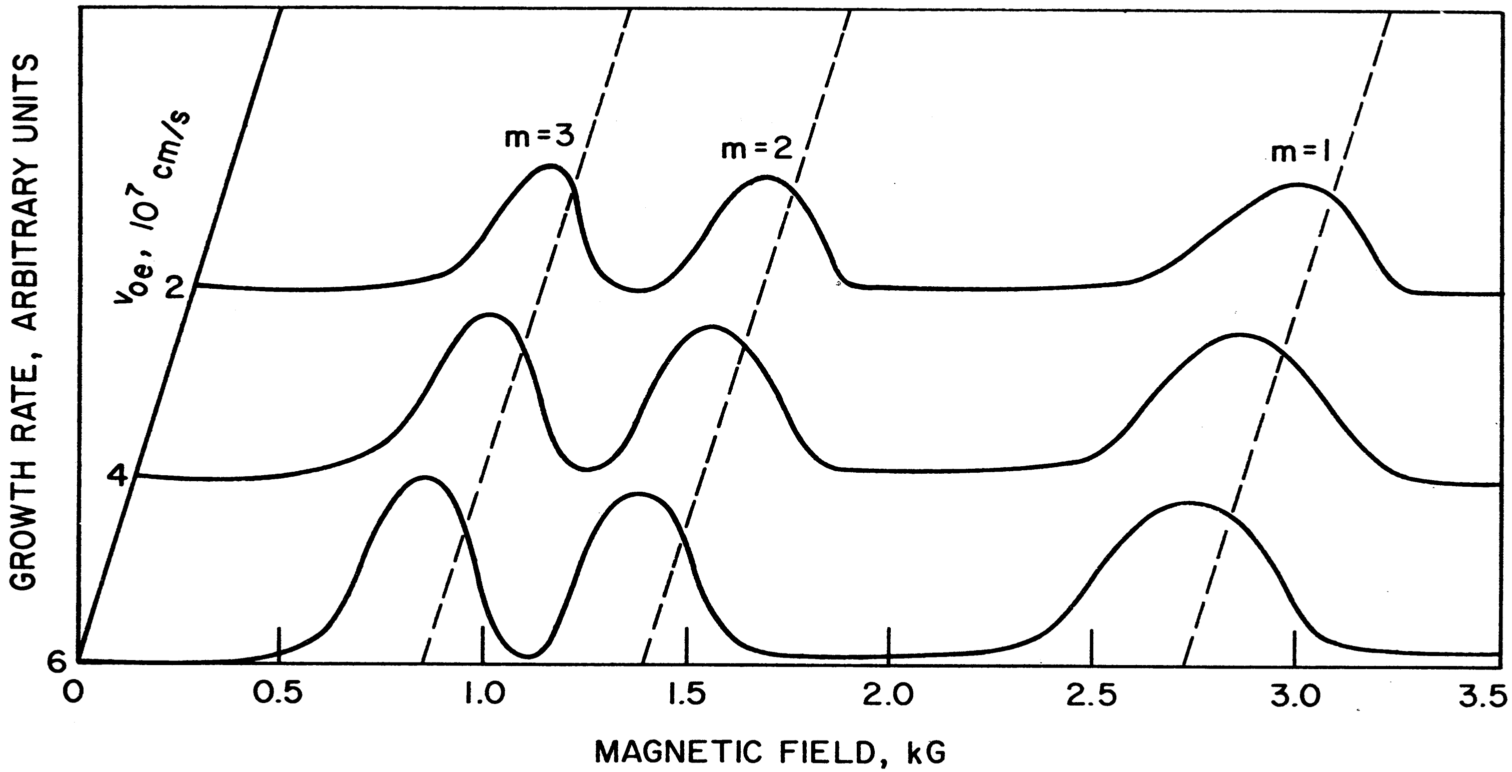


FIG. 6.1 GROWTH RATE AS A FUNCTION OF DRIFT VELOCITY AND APPLIED MAGNETIC FIELD.

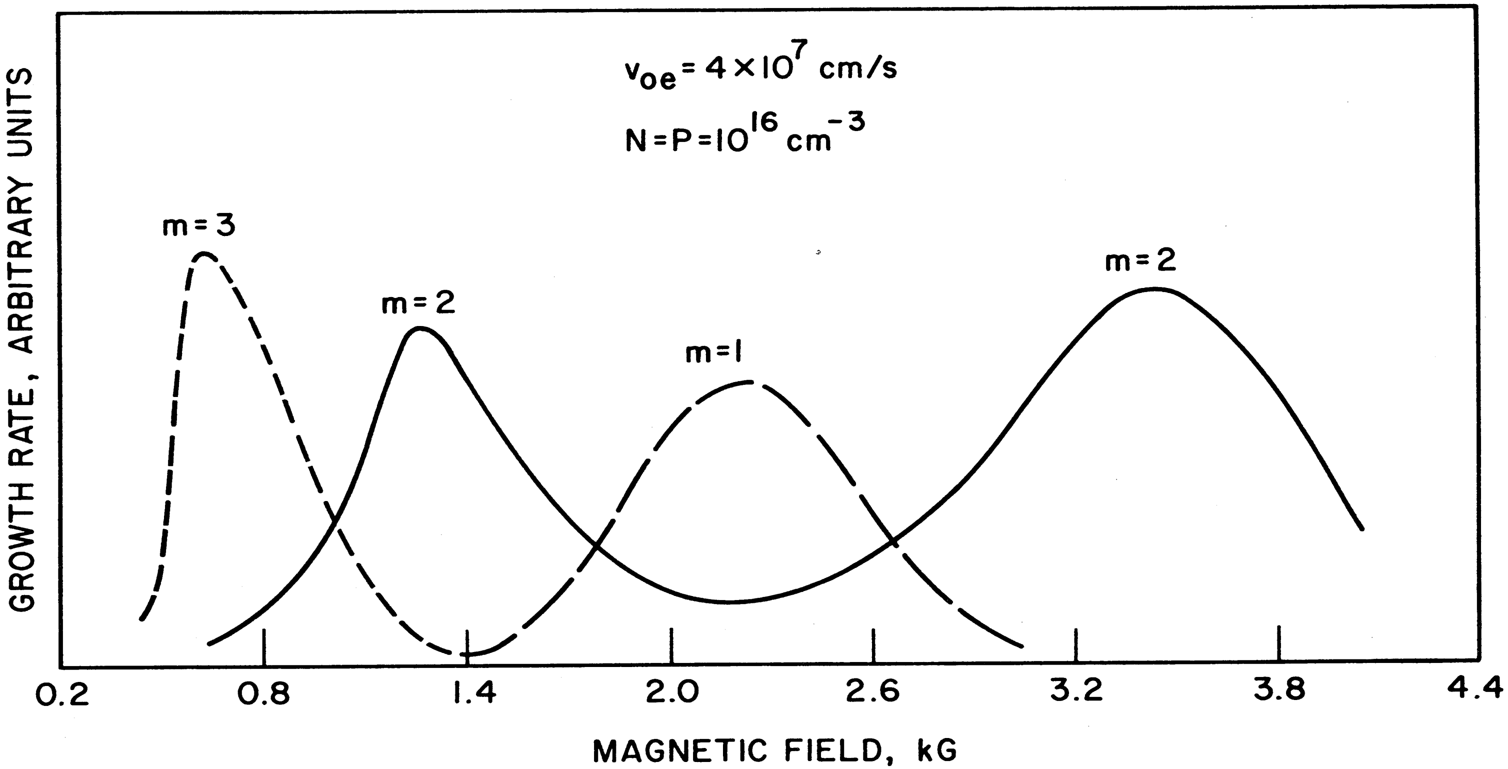


FIG. 6.2 GROWTH RATE AS A FUNCTION OF MAGNETIC FIELD.

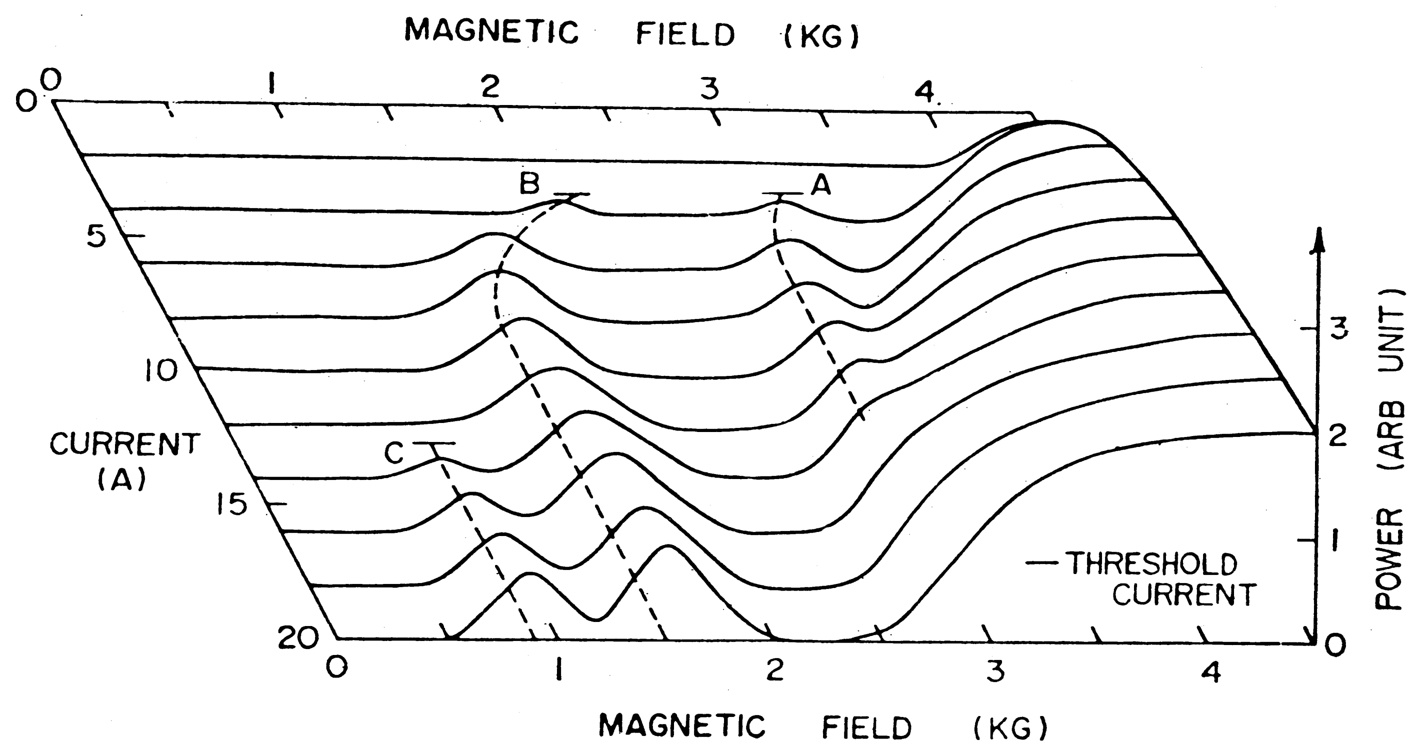


FIG. 6.3 MICROWAVE POWER AS A FUNCTION OF THE APPLIED CURRENT AND MAGNETIC FIELD. (MORISAKI AND INUISHI[78])

3. The independence of the position of the emission peaks on the drift velocity.

The fact that the linear theory presented here predicts higher growth rates at the m = 3 resonance than at the m = 2 or m = 1 resonances in contradiction to the experimental results may be due to any of several factors. The theory presented here does not take into account explicitly any carrier generation or recombination phenomena, whereas the experimental current densities correspond to the post-impact-ionization regime. In addition the m = 3 resonance corresponds to a larger wavelength than the m = 1 resonance and hence the experimental coupling of radiation will differ in these cases. Also in the theoretical work the effects of carrier heating have been neglected in developing the carrier distribution functions in Section 6.2. Of course there are also the possibilities of boundary effects and nonlinearities in any actual system.

Also from the experimental data of Fig. 6.3 it appears that a different and stronger form of instability sets in when the transverse magnetic field becomes sufficiently large ($B_o \gtrsim 3$ kG). There is ample evidence[73,79] to conclude that this region corresponds to (electron-phonon) acoustoelectric amplification enhanced by the transverse magnetic field. Theoretically,[73] a large transverse magnetic field is necessary in high-mobility semiconductors such as InSb in order to lower the electron drift velocity to a value which maximizes the gain. Experimentally,[79] transverse magnetic fields have been found to enhance greatly the acoustoelectric effect in the III-V semiconductors InSb, GaSb and GaAs. These experiments showed that the formation of high-resistance domains can be induced by applying a sufficiently large transverse B_o, and in addition, from a qualitative comparison of the results of different materials, that for a given B_o the enhancement is more pronounced as the mobility increases. Also whereas the resonance-type emission only occurred[78] above the breakdown field for impact avalanche ionization in agreement with the present kinetic theory (i.e., below breakdown in n-InSb the hole plasma frequency in Eq. 6.79 is negligible thus preventing instability), the emission with $B_o \gtrsim 3$ kG can occur below the impact ionization field in agreement with the theory of the acoustoelectric effect in high-mobility materials.[73]

6.4.2 Correlation of Kinetic and Hydrodynamic Theory for the "Cylindrical" Degenerate Distribution Function. It is of interest to determine if the present results can be correlated in any way with the corresponding hydrodynamic theory for the quasi-static hybrid mode. Inspection of the denominator of Eq. 6.63 shows that if $|\omega_{cs}| \gg kv_{Fs}$ the primary contribution is from the m = 0 term. In addition the summation in the numerator may be written as

$$\sum_{m=-\infty}^{\infty} \frac{m\omega_{cs} J_m^2\left(\frac{kv_{Fs}}{\omega_{cs}}\right)}{\omega - kv_{os} - j\nu_s - m\omega_{cs}} = \sum_{m=1}^{\infty} \frac{2m^2\omega_{cs}^2 J_m^2\left(\frac{kv_{Fs}}{\omega_{cs}}\right)}{(\omega - kv_{os} - j\nu_s)^2 - m^2\omega_{cs}^2} \quad . \qquad (6.80)$$

For $|\omega_{cs}| \gg kv_{Fs}$ then Eq. 6.63 may be written as follows:

$$N_{1s} = -\frac{j2\eta_s N_{os} E_{1x}}{kv_{Fs}^2} \sum_{m=1}^{\infty} \frac{2m^2\omega_{cs}^2 J_m^2\left(\frac{kv_{Fs}}{\omega_{cs}}\right)(\omega - kv_{os} - j\nu_s)}{[(\omega-kv_{os}-j\nu_s)^2 - m^2\omega_{cs}^2]\left\{\omega - kv_{os} - j\nu_s\left[1 - J_o^2\left(\frac{kv_{Fs}}{\omega_{cs}}\right)\right]\right\}} . \tag{6.81}$$

From the asymptotic expansion for small argument, viz.,

$$J_m(z) \approx \frac{\left(\frac{z}{2}\right)^m}{\Gamma(m+1)} = \left(\frac{z}{2}\right)^m\left(\frac{1}{m!}\right) , \tag{6.82}$$

it follows that $J_o^2 \approx 1$ and $J_1^2(kv_{Fs}/\omega_{cs}) \approx k^2v_{Fs}^2/4\omega_{cs}^2$. Use of these latter approximations in Eq. 6.81 together with Eq. 6.67 permits the dispersion equation to be found for the fundamental (m = 1) as

$$1 = \sum_s \frac{\omega_{ps}^2(\omega - kv_{os} - j\nu_s)}{(\omega - kv_{os})[(\omega - kv_{os} - j\nu_s)^2 - \omega_{cs}^2]} + R_s , \tag{6.83}$$

where R_s represents the contribution from any carrier species which do <u>not</u> satisfy $|\omega_{cs}| \gg kv_{Fs}$ and hence have a different form. Now the hybrid quasi-static dispersion equation is given by[73]

$$1 = \sum_s \frac{\omega_{ps}^2(\omega - kv_{os} - j\nu_s)}{(\omega - kv_{os})[(\omega - kv_{os} - j\nu_s)^2 - \omega_{cs}^2] - \frac{1}{2}k^2v_{Ts}^2(\omega - kv_{os} - j\nu_s)} , \tag{6.84}$$

where v_{Ts} is the thermal velocity. Comparison of these results shows good agreement and in fact becomes exact when $v_{Ts} \to 0$. Thus if $|\omega_{cs}| \gg kv_{Fs}$

it is quite clear that the hydrodynamic analysis (for that carrier) is accurate although the kinetic theory suggests the dropping of the thermal term in the denominator of Eq. 6.84 entirely. As a result the utilization of the hydrodynamic analysis in the study of the acoustoelectric interaction in high mobility media is justified for the case of a large transverse magnetic field such that $|\omega_c| \gg kv_T$. For smaller magnetic fields, however, the kinetic theory raises serious questions about the legitimacy of the hydrodynamic analysis.

In addition it is to be noted that in multiple carrier interactions, such as the electron-hole hybrid interaction, it can easily occur that the condition $|\omega_{cs}| \gg kv_{Ts}$ for the one carrier species (electrons) is well satisfied but not for the other (holes). In this case a hydrodynamic analysis with direct substitutions in Eq. 6.84 is of dubious validity, whereas the kinetic theory incorporates departures from the $|\omega_{cs}| \gg kv_{Ts}$ format in the term R_s in Eq. 6.83. Also, since the hydrodynamic and kinetic theories depart at smaller magnetic fields, it is reasonably certain that the resonance emission for $B_o < 3$ kG obtained in n-InSb from the kinetic theory will not be predicted by the hydrodynamic theory.

<u>6.4.3</u> <u>The Drifted Degenerate Distribution Function</u>. Reconsider now the drifted degenerate distribution function of Eq. 6.62 which was found to be a good approximation to the general form, Eq. 6.24, when $v_{os} < v_{Fs}$ and $|\omega_{cs}| \gg \nu_s$. Identify the integrals of interest in Eq. 6.42 by

$$\Phi_1 = \int_0^\infty \int_{-\infty}^\infty \left(\frac{\partial f_{os}}{\partial u_s}\right) J_m^2(b_s)\, du_s\, dv_{zs} \tag{6.85}$$

and

$$\Phi_2 = \int_0^\infty \int_{-\infty}^\infty J_m^2(b_s) f_{os} u_s\, du_s\, dv_{zs} \quad , \tag{6.86}$$

where it suffices to have m take any positive integer or zero value since the summation can be reduced as was done for example in Eq. 6.80. Define a new transformation,

$$u_s \cos\theta = v_{rs} \cos\Theta \sin\varphi \quad ,$$

$$u_s \sin\theta = v_{rs} \sin\Theta \sin\varphi \quad ,$$

$$v_{zs} = v_{rs} \cos\varphi \quad , \tag{6.87}$$

so that

$$u_s = v_{rs} \sin\varphi \quad , \qquad (\pi \geqq \varphi \geqq 0)$$

and

$$\int_0^\infty \int_0^{2\pi} \int_{-\infty}^\infty u_s\, du_s\, d\theta\, dv_{zs} = \int_0^\infty \int_0^{2\pi} \int_0^\pi v_{rs}^2 \sin\varphi\, dv_{rs}\, d\Theta\, d\varphi \quad . \tag{6.88}$$

It can be shown that the angles Θ and θ are equivalent. The distribution function, Eq. 6.62, transforms as

$$f_{os} = \frac{3N_{os}}{4\pi v_{Fs}^3} \quad \text{if} \quad v_{rs} \leqq v_{Fs}$$

$$= 0 \quad \text{otherwise} \quad , \tag{6.89}$$

so that

$$\frac{\partial f_{os}}{\partial u_s} = \frac{\partial f_{os}}{\partial v_{rs}}\left(\frac{\partial v_{rs}}{\partial u_s}\right) = \sin\varphi\left(\delta(v_{rs}) - \delta(v_{rs} - v_{Fs})\right)\frac{3N_{os}}{4\pi v_{Fs}^3} \quad . \tag{6.90}$$

Use of these relations in Eqs. 6.85 and 6.86 provides the following:

$$\Phi_1 = -\frac{3N_{os}}{2\pi v_{Fs}^2}\int_0^{\pi/2} \sin\varphi\, J_m^2\left(\frac{kv_{Fs}}{\omega_{cs}}\sin\varphi\right) d\varphi \tag{6.91}$$

and

$$\Phi_2 = \frac{3N_{os}}{2\pi v_{Fs}^3}\int_0^{\pi/2}\int_0^{v_{Fs}} v_{rs}^2 J_m^2\left(\frac{kv_{rs}}{\omega_{cs}}\sin\varphi\right)\sin\varphi\, d\varphi\, dv_{rs} \quad . \tag{6.92}$$

Corresponding directly to these integrals the cylindrical degenerate distribution function, Eq. 6.61, gave the values (e.g., from Eqs. 6.60, 6.63, 6.85 and 6.86)

$$\Phi_1' = -\frac{N_{os}}{\pi v_{Fs}^2} J_m^2\left(\frac{kv_{Fs}}{\omega_{cs}}\right) \tag{6.93}$$

and

$$\Phi_2' = \frac{N_{os}}{2\pi v_{Fs}^2}\left\{\left(v_{Fs}^2 - \frac{m^2\omega_{cs}^2}{k^2}\right) J_m^2\left(\frac{kv_{Fs}}{\omega_{cs}}\right) + v_{Fs}^2\left[J_m'\left(\frac{kv_{Fs}}{\omega_{cs}}\right)\right]^2\right\} , \tag{6.94}$$

where the cylindrical Φ functions will be primed. Although Eqs. 6.91 and 6.92 can be directly integrated using the relations

$$-J_m^2(0) + \int_0^{\pi/2} J_m^2(z \sin\theta) \sin\theta \, d\theta = \sum_{n=0}^{\infty} J_{2m+2n+1}(z) \quad , \qquad [\mathrm{Re}(m) > -1]$$

(6.95)

and

$$\int_0^{v_{Fs}} x^2 J_m(\alpha x) \, dx = \frac{v_{Fs}^2}{\alpha} \frac{\Gamma\left(\frac{m+3}{2}\right)}{\Gamma\left(\frac{m-1}{2}\right)} \cdot \sum_{\ell=0}^{\infty} \frac{(m + 2\ell + 1)\Gamma\left(\frac{m-1}{2} + \ell\right)}{\Gamma\left(\frac{m+5}{2} + \ell\right)} J_{m+2\ell+1}(\alpha v_{Fs}) \quad , \qquad (6.96)$$

where Γ is the gamma function, $\Gamma(n + 1) = n!$, the results are too lengthy to be detailed here. By inspection of Eq. 6.91, however, it can be seen in general that $\Phi_1(-m) = \Phi_1(m)$ and necessarily $\Phi_1 \leq 0$. Similarly, from Eq. 6.92, $\Phi_2(-m) = \Phi_2(m)$ and $\Phi_2 \geq 0$. For simplicity the Φ integrals will only be investigated for the fundamental interactions $|m| = 1$ and $m = 0$. From Eqs. 6.91 and 6.95, and the relation

$$\sin z = 2J_1(z) - 2J_3(z) + 2J_5(z) - 2J_7(z) + \ldots \quad , \qquad (6.97)$$

it can be found that

$$\Phi_1(|m| = 1) = \frac{3N_{os}}{2\pi v_{Fs}^2} \left[-\frac{1}{2} \sin\left(\frac{kv_{Fs}}{\omega_{cs}}\right) + J_1\left(\frac{kv_{Fs}}{\omega_{cs}}\right) - 2J_3\left(\frac{kv_{Fs}}{\omega_{cs}}\right) - 2J_7\left(\frac{kv_{Fs}}{\omega_{cs}}\right) - 2J_{11}\left(\frac{kv_{Fs}}{\omega_{cs}}\right) - \ldots \right] \quad . \qquad (6.98)$$

Similarly from Eqs. 6.92, 6.95, 6.96 and 6.97,

$$\Phi_2(m = 0) = \frac{3N_{os}}{2\pi v_{Fs}^3} \left\{ \frac{v_{Fs}^3}{3} + \frac{\omega_{cs}^3}{k^3} \left[\left(\frac{kv_{Fs}}{\omega_{cs}} \right) \sin \left(\frac{kv_{Fs}}{\omega_{cs}} \right) - \left(\frac{k^2 v_{Fs}^2}{\omega_{cs}^2} - 1 \right) \right.\right.$$

$$\left.\left. \cdot \cos \left(\frac{kv_{Fs}}{\omega_{cs}} \right) - 1 \right] - 2 \sum_{n,\ell=0}^{\infty} \frac{\omega_{cs} v_{Fs}^2}{k} \frac{\Gamma(2n + 2)\Gamma(2n + \ell)}{\Gamma(2n)\Gamma(2n + \ell + 3)} J_{4n+2\ell+2} \left(\frac{kv_{Fs}}{\omega_{cs}} \right) \right\}$$

(6.99)

and

$$\Phi_2(|m| = 1) = \frac{3N_{os}}{2\pi v_{Fs}^3} \left\{ 2 \sum_{n,\ell=0}^{\infty} \frac{\omega_{cs} v_{Fs}^2}{k} \frac{\Gamma(2n + 2)\Gamma(2n + \ell)}{\Gamma(2n)\Gamma(2n + 3 + \ell)} (4n + 2\ell + 2) \right.$$

$$\cdot J_{4n+2\ell+2} \left(\frac{kv_{Fs}}{\omega_{cs}} \right) - \frac{\omega_{cs}^3}{k^3} \left[\left(\frac{kv_{Fs}}{\omega_{cs}} \right) \sin \left(\frac{kv_{Fs}}{\omega_{cs}} \right) - \left(\frac{k^2 v_{Fs}^2}{\omega_{cs}^2} - 1 \right) \right.$$

$$\left.\left. \cdot \cos \left(\frac{kv_{Fs}}{\omega_{cs}} \right) - 1 \right] - \sum_{\ell=0}^{\infty} \frac{\omega_{cs} v_{Fs}^2}{k} 2(\ell + 1) \frac{\Gamma(\ell)}{\Gamma(3 + \ell)} J_{2\ell+2} \left(\frac{kv_{Fs}}{\omega_{cs}} \right) \right\} .$$

(6.100)

Corresponding to these, the cylindrical degenerate distribution function gives, from Eq. 6.93,

$$\Phi_1'(|m| = 1) = - \frac{N_{os}}{\pi v_{Fs}^2} J_1^2 \left(\frac{kv_{Fs}}{\omega_{cs}} \right) ,$$

and from Eq. 6.75,

$$\Phi_2'(m = 0) = \frac{N_{os}}{2\pi} \left[J_0^2 \left(\frac{kv_{Fs}}{\omega_{cs}} \right) + J_1^2 \left(\frac{kv_{Fs}}{\omega_{cs}} \right) \right] ,$$

$$\Phi_2'(|m| = 1) = \frac{N_{os}}{2\pi v_{Fs}^2} \left\{ \left(v_{Fs}^2 - \frac{\omega_{cs}^2}{k} \right) J_1^2 \left(\frac{kv_{Fs}}{\omega_{cs}} \right) + v_{Fs}^2 \left[J_1' \left(\frac{kv_{Fs}}{\omega_{cs}} \right) \right]^2 \right\} . \tag{6.101}$$

From the point of view of the dispersion relation, the only difference between the cylindrical and the drifted degenerate functions is contained in the respective Φ_1 and Φ_2 integrals.

By going to the limit $(kv_{Fs}/\omega_{cs}) \to 0$ it can be found that $\Phi_2(m = 0) = \Phi_2'(m = 0) \gg \Phi_1, \Phi_1'$. Thus the hydrodynamic form found for the cylindrical degenerate distribution function, Eq. 6.83, also exists for the drifted degenerate case. For the resonant interaction, if the replacements

$$\theta_s = \frac{2\omega_{ps}^2}{k^2 v_{Fs}^2} \left(- \frac{\pi v_s^2}{N_{os}} \right) \Phi_1(|m| = 1)$$

and

$$\nu_s' = \nu_s \left(1 - \frac{2\pi}{N_{os}} \Phi_2(|m| = 1) \right) , \tag{6.102}$$

corresponding to Eqs. 6.98 and 6.100 are made in Eqs. 6.74 through 6.78, the solution for the drifted degenerate interaction is obtained directly. The results for the drifted and cylindrical degenerate systems are then quite similar and differ only in the magnitude of their respective Φ functions. Since these functions involve infinite sums for the drifted degenerate case these are not dealt with directly. This fact points out the utility of the cylindrical function which is much more readily analyzed yet is similar in instability characteristics to the drifted degenerate function.

<u>6.4.4 The Hybrid Dispersion Relation for Maxwellian Carriers.</u> The quasi-static RF current density, J_{1x}, for the hybrid mode with a drifted Maxwellian carrier velocity distribution function was developed in Chapter V, Section 5.5. This will be used later in the discussion of the electrokinetic energy density and kinetic power flow of the carrier modes. To develop the dispersion equation, Eq. 6.52 used in Eq. 6.18 shows that the drifted Maxwellian distribution function if $|v_{os}| \ll v_{Ts}$ takes the form

$$f_{os} = \frac{N_{os}}{(2\pi v_{Ts}^2)^{3/2}} \exp\left(-\frac{u_s^2}{2v_{Ts}^2}\right) \exp\left(-\frac{v_{zs}^2}{2v_{Ts}^2}\right) , \tag{6.103}$$

so that Eqs. 6.85 and 6.86 provide the following:

$$\Phi_1 = \left(-\frac{1}{v_{Ts}^2}\right) \Phi_2 = -\frac{N_{os}}{2\pi k^2 v_T^4} \lambda_s e^{-\lambda_s} I_m(\lambda_s) , \tag{6.104}$$

where

$$\lambda_s = \left(\frac{kv_{Ts}}{\omega_{cs}}\right)^2 , \tag{6.105}$$

$I_m(\lambda_s)$ is the modified Bessel function of order m, and the standard integral

$$\int_0^\infty \mu \exp\left(-\frac{\mu^2}{2\lambda}\right) J_m^2(\mu)\, d\mu = \lambda e^{-\lambda} I_m(\lambda) \tag{6.106}$$

has been used. From Eqs. 6.60, 6.85 and 6.86 the following general result is found:

$$N_{1s} = - \frac{\dfrac{j\eta_s E_{1x}}{k} \displaystyle\sum_{m=-\infty}^{\infty} \frac{2\pi m}{m - a_s} \Phi_1}{1 - j \dfrac{2\pi \nu_s}{N_{os}\omega_{cs}} \displaystyle\sum_{m=-\infty}^{\infty} \frac{1}{m - a_s} \Phi_2} , \tag{6.107}$$

so that using Eq. 6.104 in Eq. 6.107 provides that

$$N_{1s} = \frac{\dfrac{j\eta_s N_{os} E_{1x}}{kv_{Ts}^2} \displaystyle\sum_{m=-\infty}^{\infty} \frac{m}{m - a_s} e^{-\lambda_s} I_m(\lambda_s)}{1 - j \dfrac{\nu_s}{\omega_{cs}} \displaystyle\sum_{m=-\infty}^{\infty} \frac{1}{m - a_s} e^{-\lambda_s} I_m(\lambda_s)} . \tag{6.108}$$

Use of this result in Eq. 6.67 gives the dispersion relation as

$$1 + \sum_s \frac{\left(\dfrac{\omega_{ps}^2}{k^2 v_{Ts}^2} \right) \displaystyle\sum_{m=-\infty}^{\infty} \frac{m}{m - a_s} e^{-\lambda_s} I_m(\lambda_s)}{1 - j \dfrac{\nu_s}{\omega_{cs}} \displaystyle\sum_{m=-\infty}^{\infty} \frac{1}{m - a_s} e^{-\lambda_s} I_m(\lambda_s)} = 0 . \tag{6.109}$$

In the limit that $(kv_{Ts}/\omega_{cs}) \to 0$ the behavior of the function $e^{-\lambda_s} I_m(\lambda_s)$ shows that m = 0 provides the dominant contribution in the denominator, so that when only the fundamental ($|m| = 1$) is retained in the numerator, Eq. 6.109 becomes

$$1 + \sum_s \left(\frac{\omega_{ps}^2}{k^2 v_{Ts}^2} \right) \frac{2\omega_{cs}^2 (\omega - kv_{os} - j\nu_s) e^{-\lambda_s} I_1(\lambda_s)}{[\omega_{cs}^2 - (\omega - kv_{os} - j\nu_s)^2](\omega - kv_{os})} = 0 , \tag{6.110}$$

where

$$\sum_{m=-\infty}^{\infty} \frac{m}{m - a_s} e^{-\lambda_s} I_m(\lambda_s) = \sum_{m=1}^{\infty} \frac{2m}{m - a_s^2} e^{-\lambda_s} I_m(\lambda_s) \qquad (6.111)$$

has been used with $|m| = 1$. To understand this result further, from the behavior of $e^{-\lambda_s} I_m(\lambda_s)$ for small λ_s,

$$e^{-\lambda_s} I_m(\lambda_s) \approx \frac{1}{m!}\left(\frac{\lambda}{2}\right)^m , \qquad (6.112)$$

$$e^{-\lambda_s} I_1(\lambda_s) \approx \left(\frac{\lambda_s}{2}\right) , \qquad (6.113)$$

leading to the following result in Eq. 6.104:

$$1 + \sum_s \frac{\omega_{ps}^2 (\omega - kv_{os} - j\nu_s)}{(\omega - kv_{os})[\omega_{cs}^2 - (\omega - kv_{os} - j\nu_s)^2]} = 0 . \qquad (6.114)$$

This result is identical to that found for the cylindrical degenerate distribution function under the same limit, Eq. 6.83, and is quite similar to the result of hydrodynamic theory, Eq. 6.84. Thus for the drifted Maxwellian the discussion following Eq. 6.84 comparing kinetic and hydrodynamic theory is also applicable. This hydrodynamic limit corresponds to retaining only the m = 0 term in the denominator of Eq. 6.109. If the $|m| = 1$ term is also retained, and having used the method of Eq. 6.111, the denominator becomes as follows (for small λ so that Eqs. 6.112 and 6.113 can be used):

$$D = 1 + j \frac{\nu_s}{\omega - kv_{os} - j\nu_s} - j \frac{\nu_s}{\omega_{cs}} \frac{k^2 v_{Ts}^2}{\omega_{cs}^2 - (\omega - kv_{os} - j\nu_s)^2} . \qquad (6.115)$$

When this improved form is used Eq. 6.126 becomes

$$1 + \sum_s \frac{\omega_{ps}^2(\omega - kv_{os} - j\nu_s)}{(\omega - kv_{os})[\omega_{cs}^2 - (\omega - kv_{os} - j\nu_s)^2] - j\dfrac{\nu_s}{\omega_{cs}} k^2 v_{Ts}^2(\omega - kv_{os} - j\nu_s)} = 0 \ . \qquad (6.116)$$

This result should be compared with that of the hydrodynamic theory, Eq. 6.84. This also suggests returning to the cylindrical distribution function of Chapter V, Section 5.3.1 and retaining the $|m| = 1$ term in the denominator under $|\omega_{cs}| \gg kv_{Fs}$. When this is done the following result is found, which is an improvement of Eq. 6.83:

$$1 = \sum_s \frac{\omega_{ps}^2(\omega - kv_{os} - j\nu_s)}{(\omega - kv_{os})[(\omega - kv_{os} - j\nu_s)^2 - \omega_{cs}^2] + j\dfrac{3\nu_s k^2 v_{Fs}^2}{4\omega_{cs}^2}(\omega - kv_{os} - j\nu_s)^2 - j\nu_s\dfrac{k^2 v_{Fs}^2}{4}} \ , \qquad (6.117)$$

which also indicates divergence from the hydrodynamic result.

The case is now considered where the resonance approximation can be made so that only the resonance terms contribute in the $\sum_m$. The dispersion relation then takes the following form for the nth resonance:

$$1 + \sum_s \frac{\omega_{cs}\omega_{ps}^2 n\, e^{-\lambda_s} I_n(\lambda_s)}{k^2 v_{Ts}^2[n\omega_{cs} - (\omega - kv_{os} - j\nu_s')]} = 0 \ , \qquad (6.118)$$

where ν_s' is the effective collision frequency given by

$$\nu_s' = \nu_s\left(1 - e^{-\lambda_s} I_n(\lambda_s)\right) . \qquad (6.119)$$

For example, for n = 1 the maximum effect occurs when $\lambda \approx 1.5$ in which case $\nu_s' \approx 0.8\ \nu_s$. Thus the general statement can be made that ν_s' is never altered appreciably from ν_s for n = 1 (20 percent) and for larger n values this is even more true; for n = 2 the maximum is ≈ 12 percent, for n = 3 the maximum is 8 percent, etc.

From Eq. 6.118 the dispersion equation for the electron fundamental interacting with the hole harmonic modes is

$$1 + \frac{|\omega_{ce}|\ \Gamma_e}{\omega - kv_{oe} - j\nu_e' + |\omega_{ce}|} - \frac{m|\omega_{ch}|\ \Gamma_h}{\omega - j\nu_h' - m|\omega_{ch}|} = 0 \ , \qquad (6.120)$$

where

$$\Gamma_e = \frac{\omega_{pe}^2}{k^2 v_{Te}^2} e^{-\lambda_e} I_1(\lambda_e) \qquad (6.121)$$

and

$$\Gamma_h = \frac{\omega_{ph}^2}{k^2 v_{Th}^2} e^{-\lambda_h} I_m(\lambda_h) \ , \qquad (6.122)$$

so that $\Gamma_e, \Gamma_h > 0$. The solution is identical to that given in Eqs. 6.74 through 6.76 with the replacement $\theta_s \rightarrow \Gamma_s$, s = e,h.

For self-consistency it should be verified that the resonance approximation is valid. First it is demonstrated that the term corresponding to m = 0 in the denominator of Eq. 6.109 has a negligible effect on the dispersion equation. The solution given by Eq. 6.74 for $\omega_i < 0$ indicates that $kv_{oe} \approx |\omega_{ce}|$ for reasonable carrier densities so that $\lambda_e > 1$ since $v_{oe} < v_{Te}$. For $\lambda_e > 1$, $I_o(\lambda_e) \approx I_1(\lambda_e)$ so that defining the following as

$$D_{s,m} = \frac{1}{m - a_s} e^{-\lambda_s} I_m(\lambda_s) \quad ; \qquad s = e,h \quad , \tag{6.123}$$

it follows that $|D_{e,1}| \gg |D_{e,o}|$, since

$$\left| \omega - kv_{oe} - j\nu_e + |\omega_{ce}| \right| \gg |\omega - kv_{oe} - j\nu_e| \tag{6.124}$$

corresponding to $|\omega_{ce}| \gg \nu_e$. Thus the m = 0 term should have a negligible effect in the electron term. For the holes

$$D_{h,o} + D_{h,m} = e^{-\lambda_h} \left(- \frac{\omega_{ch} I_o(\lambda_h)}{\omega - j\nu_h} + \frac{\omega_{ch} I_m(\lambda_h)}{m\omega_{ch} - \omega - j\nu_h} \right) , \tag{6.125}$$

and since $\lambda_e > 1$ implies that $\lambda_h > 1$, it is true that $I_o(\lambda_h) \approx I_1(\lambda_h)$. Thus for m = 1 and $\omega \ll \nu_h$ the right-hand side of Eq. 6.125 is approximately zero and hence, by inspection of Eq. 6.109, the resonance approximation is valid although ν_h' should be replaced by ν_h. The approximation $I_o(\lambda_h) \approx I_m(\lambda_h)$ becomes successively weaker for higher m values. However near $\omega = |\omega_{ch}|m$ inspection of Eq. 6.125 shows that the effects should be negligible and this is the region of interest.

It should also be verified that the harmonic terms can be isolated for analysis. Since $\nu_e' \approx \nu_e$ and $\nu_h' \approx \nu_h$ the effects due to the sum in the denominator of Eq. 6.121 can be neglected and the denominator set to unity. From Eq. 6.123 the dispersion equation, Eq. 6.121, becomes

$$1 = \sum_s \sum_{n=1}^{\infty} \left(\frac{2\omega_{ps}^2}{k^2 v_{Ts}^2} \right) \frac{e^{-\lambda_s} I_n(\lambda_s) n^2 \omega_{cs}^2}{(\omega - kv_{os} - j\nu_s)^2 - n^2\omega_{cs}^2} , \tag{6.126}$$

which can be written in the integrated form, viz.,

$$1 = - \sum_s \left(\frac{\omega_{ps}^2}{\omega_{cs}^2} \right) e^{-\lambda_s} \int_0^{\pi} \frac{\sin x \sin \left[x \left(\frac{\omega - kv_{os} - j\nu_s}{\omega_{cs}} \right) \right] e^{-\lambda_s \cos x}}{\sin \left[\pi \left(\frac{\omega - kv_{os} - j\nu_s}{\omega_{cs}} \right) \right]} dx \ . \tag{6.127}$$

In particular since this result is amenable to complex k, Eq. 6.127 will be solved for real ω and complex k. Thus, since $\omega_i < 0$ is not necessarily sufficient for instability, if amplifying roots are obtained it can be stated with good certainty that a causal instability exists. In addition, if the harmonic modes are clearly separated in the solution, the use of the resonance approximation as an analytically acceptable technique will be verified.

Equation 6.127 is solved via computer for the following parameter values:

$$\begin{aligned}
&N_{oe} = N_{oh} = 10^{15}\ \text{cm}^{-3} \ , \\
&\nu_e = \nu_h = 4 \times 10^{11}\ \text{s}^{-1} \ , \\
&m_e^* = 0.03\ m_o \ , \quad m_h^* = 0.6\ m_o \ , \\
&B_o = 2\ \text{kG} \ , \\
&v_{Te} = 6 \times 10^7\ \text{cm/s} \ , \quad v_{Th} = 4 \times 10^7\ \text{cm/s} \ , \\
&v_{oe} = 10^7\ \text{cm/s} \ , \quad v_{oh} = 10^4\ \text{cm/s} \ ,
\end{aligned} \tag{6.128}$$

which are again in the range of values corresponding to the reported experimental work in n-InSb of interest,[78] namely that the instability occurs in the post-impact ionization range. The solution shown in Fig. 6.4 indicates clearly separated harmonic modes except for small wave numbers which are not in the interaction region. A convective instability at the hole

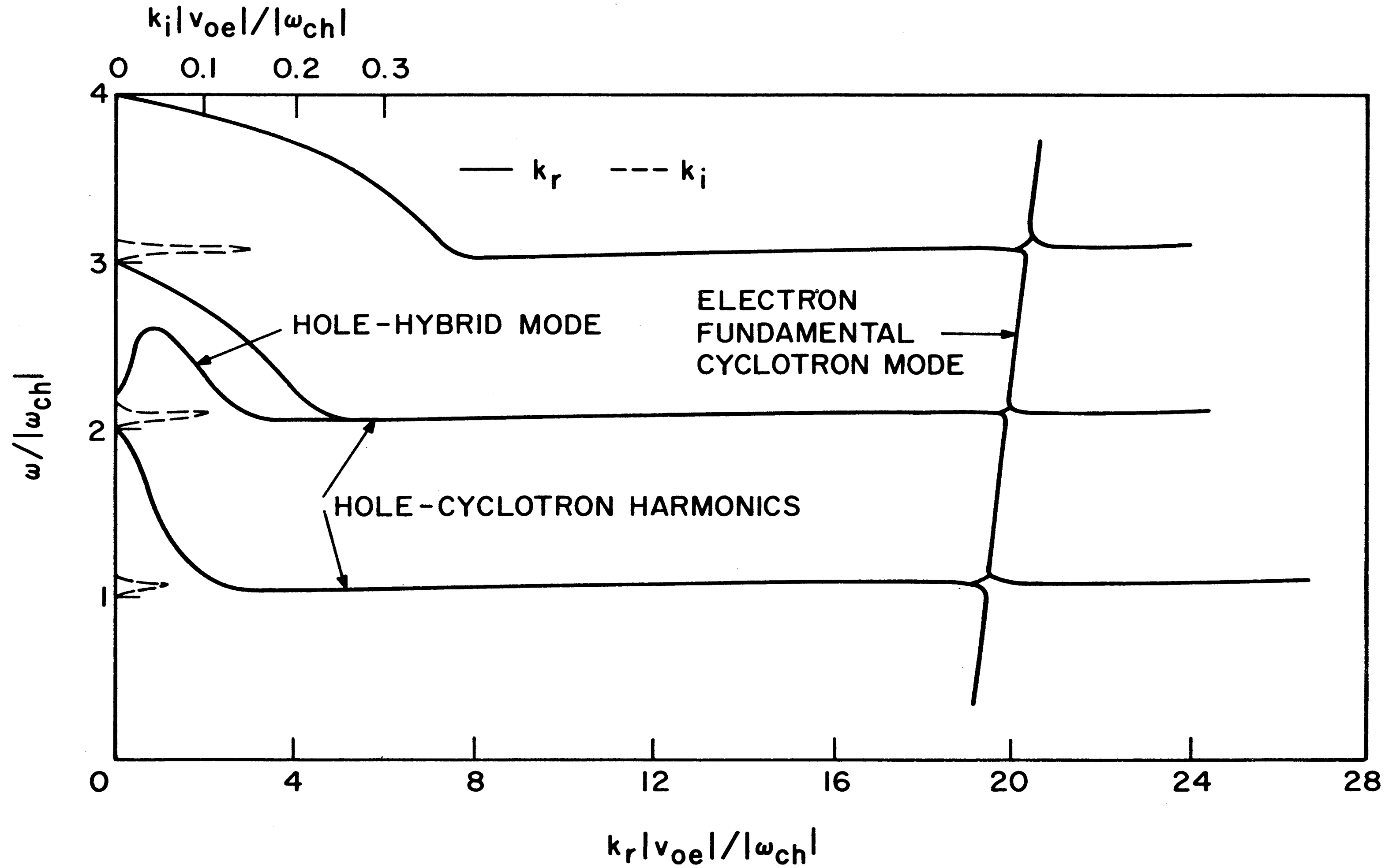


FIG. 6.4 LOW-FREQUENCY HYBRID-CYCLOTRON HARMONIC DISPERSION DIAGRAM.

cyclotron harmonic frequencies is also present, thus verifying that the $\omega_i < 0$ solution of the resonance approximation has a corresponding amplifying wave for real ω.

6.4.5 Effect of Collision Frequency Variation with Carrier Speed. In general, the collision frequency ν_s is a function of the carrier speed $v = (v_x^2 + v_y^2 + v_z^2)^{1/2}$, whereas up to this point it has been assumed constant. If $v_{os} < v_{Ts}$ or v_{Fs} it is apparent that ν_s is independent of θ as defined in Eq. 6.52. It then follows that Eq. 6.60 is still valid if the replacement $\nu_s \rightarrow \nu_s(v)$ is made so that the denominator of Eq. 6.60 should now be written, where $a_s(v) = [\omega - kv_{os} - j\nu_s(v)]/\omega_{cs}$, as

$$D_s = \left[1 - j\frac{2\pi}{N_{os}\omega_{cs}} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{u_s \nu_s(v) J_m^2(b_s) f_{os} \, du_s \, dv_{zs}}{m - a_s(v)}\right], \quad (6.129)$$

where $v = (u_s^2 + v_{zs}^2)^{1/2}$ when $v_o < v_{Ts}$ or v_{Fs}.

Inspection shows that near resonance at small growth rates, $m\omega_{cs} - (\omega - kv_{os}) \approx 0$, $\nu_s(v)/[m - a_s(v)] \approx$ constant so that the collisional dependence on carrier speed will have negligible effect on the value obtained for D_s if it is assumed that ν_s is a constant. For hole cyclotron resonant interactions the resonance assumption can be made independent of the dispersion relation since it can be assumed that $\omega_r \approx m|\omega_{ch}|$, $|v_{oh}| \approx 0$, and $k \approx |\omega_{ce}|/v_{oe}$, $\omega \ll |\omega_{ce}|$, so that in each case $m\omega_{cs} - (\omega - kv_{os}) \approx 0$. Even away from resonance it is not expected that the replacement $\nu_s \rightarrow \nu_s(v)$ will seriously alter D_s because of the nature of the integrand in Eq. 6.129. Thus it will be assumed that for any carrier velocity distribution function,

f_{os}, independent of θ the use of a constant collision frequency ν_s in the denominator D_s is justified for the collision term which conserves particles properly.

With this in mind the numerator can now be examined. For the drifted degenerate distribution function of Eq. 6.62, Eq. 6.90 applied to Eq. 6.60 shows directly that because of the delta function dependence only the value of the collision frequency at $v = v_{Fs}$ is involved. Thus the assumption of a constant collision frequency for this distribution function is well justified for resonant processes with $\nu_s = \nu_s(v = v_{Fs})$. For the drifted Maxwellian distribution function for resonant processes the function D_s in Eq. 6.129 can be set equal to unity since the denominator has little effect so that Eq. 6.60 becomes

$$N_{1s} = \frac{j\eta_s E_{1x}}{kv_{Ts}^2} \sum_{m=-\infty}^{\infty} 2\pi m\omega_{cs}$$

$$\cdot \int_{-\infty}^{\infty} \int_0^{\infty} J_m^2\left(\frac{ku_s}{\omega_{cs}}\right) \frac{u_s \exp\left(-\frac{u_s^2}{2v_{Ts}^2}\right) \exp\left(-\frac{v_{zs}^2}{2v_{Ts}^2}\right)}{m\omega_{cs} - [\omega - kv_{os} - j\nu_s(v)]}\, du_s\, dv_{zs} \quad . \tag{6.130}$$

Since the case of interest corresponds to $\omega_i < 0$ for k real no singularities can arise. The effects of the variation of collision frequency with carrier speed will only be examined qualitatively because of the difficulty associated with the integrals involved.

For scattering due to thermal vibrations in nonpiezoelectric materials the mean free path is a constant, ℓ_o, and $\nu_s = v\ell_o^{-1}$. Inspection of Eq. 6.130

shows that since $v \approx (u_s^2 + v_{zs}^2)^{1/2}$ the carriers with low carrier speed in the distribution will interact more strongly with the wave (i.e., the resonance is more clearly defined) than in the constant collision frequency case. The net effect of this is that the values of Γ_s in Eqs. 6.121 and 6.122 are increased thus easing the requirements for instability without altering the essential physics of the interaction. Alternatively, if the scattering is predominantly due to lattice thermal vibrations, the analysis employing a constant collision frequency should use an appropriate constant ν_s less than that appearing in the low field mobility.

In a similar fashion it can be determined that if impurity scattering predominates, since $\nu_s(v) \sim v^{-1}$, the resonances are more poorly defined at low carrier speeds (where the exponentials in Eq. 6.130 dominate) so that the analysis employing a constant collision frequency should use a larger value of collision frequency than that associated with the low field mobility.

In general, the inclusion of the variation of collision frequency with carrier speed is not expected to lead to any significant departures from the constant collision frequency theory.

6.5 Electrokinetic Energy and Power Properties of the Hybrid Mode

6.5.1 Carrier Distribution Function f_o Independent of θ. Since the quasi-static assumption has been made $\nabla \times \underline{B}_1 \approx 0$ so that it follows that

$$\sum_s J_{1x}^{(s)} + j\omega\epsilon E_{1x} = 0 \quad , \qquad (6.131)$$

from which it can readily be found that

$$\sum_s W_k^{(s)} + \frac{\epsilon}{2} |E_{1x}|^2 = 0 \quad , \tag{6.132}$$

where $W_k^{(s)}$ is the electrokinetic energy density of the sth carrier species, and is defined by

$$W_k^{(s)} = - \frac{1}{2\omega_i} \text{Re} \left(E_{1x}^* J_{1x}^{(s)} \right) \quad . \tag{6.133}$$

Equation 6.132 indicates that $\sum_s W_k^{(s)}$ must be negative for unstable ($\omega_i < 0$) interaction to occur. From Eq. 6.131 and Poisson's equation, Eq. 6.67, it can be found that

$$\sum_s J_{1x}^{(s)} = \frac{\omega}{k} \sum_s q_s N_{1s} \quad , \tag{6.134}$$

so that if the carrier species are assumed to be separately conserved in number, then $J_{1x}^{(s)} = (\omega/k) q_s N_{1s}$. Applying this result to the case of the cylindrical degenerate distribution function, Eq. 6.63, gives the following for the mth resonance:

$$J_{1x}^{(s)} = - \frac{j2\omega_{ps}^2 \epsilon E_{1x} \omega \, m \, \omega_{cs} J_m^2 \left(\dfrac{kv_{Fs}}{\omega_{cs}} \right)}{k^2 v_{Fs}^2 (\omega - kv_{os} - j\nu_s' - m\omega_{cs})} \quad , \tag{6.135}$$

where ν_s' is given by Eq. 6.70. If k is assumed purely real Eq. 6.133 used in Eq. 6.135 provides that

$$W_k^{(s)} = \frac{\omega_{ps}^2 \epsilon \, |E_{1x}|^2 \, m\omega_{cs} J_m^2\left(\frac{kv_{Fs}}{\omega_{cs}}\right)}{k^2 v_{Fs}^2 \, |\omega - kv_{os} - j\nu_s' - m\omega_{cs}|^2} \left[kv_{os} + m\omega_{cs} - \left(\frac{\omega_r}{\omega_i}\right)\nu_s'\right] . \tag{6.136}$$

For example if this result is applied to the electron-hole interaction studied in Section 6.5.1, then the following results:

$$W_k^{(e)} = - \frac{\omega_{pe}^2 \epsilon \, |E_{1x}|^2 \, |\omega_{ce}| \, J_1^2\left(\frac{kv_{Fe}}{|\omega_{ce}|}\right)}{k^2 v_{Fe}^2 \left|\omega - kv_{oe} - j\nu_e' + |\omega_{ce}|\right|^2} \left[kv_{oe} - |\omega_{ce}| - \left(\frac{\omega_r}{\omega_i}\right)\nu_e'\right] \tag{6.137}$$

and

$$W_k^{(h)} = \frac{\omega_{ph}^2 \epsilon \, |E_{1x}|^2 \, m|\omega_{ch}| \, J_m^2\left(\frac{kv_{Fe}}{|\omega_{ce}|}\right)}{k^2 v_{Fh}^2 \left|\omega - j\nu_h' - m|\omega_{ch}|\right|^2} \left[m|\omega_{ch}| - \left(\frac{\omega_r}{\omega_i}\right)\nu_h'\right] , \tag{6.138}$$

where $m = 1,2,3,\ldots$. For the unstable solution of Eqs. 6.74 through 6.78, it is true that $\omega_i < 0$ and $kv_{oe} > |\omega_{ce}|$ so that $W_k^{(e)} < 0$ and $W_k^{(h)} > 0$. It can also be found from Eqs. 6.134 and 6.135 applied to the electron-hole interaction that

$$\rho_1^{(e)} = - \frac{2\omega_{pe}^2 \epsilon \, |\omega_{ce}| \, J_1^2\left(\frac{kv_{Fe}}{|\omega_{ce}|}\right)}{kv_{Fe}^2 \left|\omega - kv_{oe} - j\nu_e' + |\omega_{ce}|\right|^2} [(\nu_e' - \omega_i) - j(\omega_r - kv_{oe} + |\omega_{ce}|)]E_{1x} \tag{6.139}$$

and

$$\rho_1^{(h)} = \frac{2\omega_{ph}^2 \epsilon m|\omega_{ch}| \; J_m^2\left(\frac{kv_{Fh}}{|\omega_{ch}|}\right)}{kv_{Fh}^2 \left|\omega - j\nu_h' - m|\omega_{ch}|\right|^2} [(\nu_h' - \omega_i) - j(\omega_r - m|\omega_{ch}|)]E_{1x} \quad . \tag{6.140}$$

A comparison of this result with Eqs. 6.137 and 6.138 shows that the sign of $W_k^{(s)}$ is equal to the sign of $Re(\rho_1^{(s)}/E_{1x})$. This is similar to the result of hydrodynamic theory for space-charge waves in cold plasma, viz.,

$$\rho_1 = \frac{\omega_p^2 \epsilon k\{(\nu - 2\omega_i)(\omega_r - kv_o) - j[(\omega_r - kv_o)^2 - \omega_i^2 + \omega_i \nu]\}}{|(\omega - kv_o)(\omega - kv_o - j\nu)|^2} E_{1x} \quad , \tag{6.141}$$

in which if $(\omega_r - kv_o) < 0$ then $Re(\rho_1/E_{1x}) < 0$. Thus carrier modes with negative electrokinetic energy density are characterized by the property of bunching charge carriers in regions where the passive modes (e.g., the mode with $v_o = 0$ in Eq. 6.141) have carrier depletion and vice versa. The same energy exchange process occurs for the hybrid mode as for the space-charge mode then. A basic difference, however, exists between the hydrodynamic space-charge result, Eq. 6.141, and the present kinetic case in that, although the former requires $v_o > \omega_r/k$, in the latter case the sign of the electrokinetic energy density is dominated by the function $m\omega_{cs}$ in Eq. 6.136.

Results of this form also follow when the kinetic power flow is examined. Only the case $k_i = 0+$ (i.e., $k_i > 0$ and $k_i \ll k_r$) is considered because of the difficulty associated with handling arbitrary complex arguments of the Bessel function. Since the quasi-static assumption has been made the Poynting vector is zero and the conservation of power takes the following form, with ω assumed real:

$$\sum_s P_k^{(s)} = 0 \quad ; \qquad P_k^{(s)} = \frac{1}{2k_i} \operatorname{Re}(E_{1x}^* J_{1x}^{(s)}) \quad . \tag{6.142}$$

For the cylindrical distribution function then, Eq. 6.135 used in Eq. 6.142 gives, at the mth resonance,

$$P_k^{(s)} \approx \frac{\omega_{ps}^2 \epsilon \, |E_{1x}|^2 \, \omega \, m \, \omega_{cs} J_m^2\left(\dfrac{kv_{Fs}}{\omega_{cs}}\right) \nu_s'}{k_i v_{Fs}^2 \, |k(\omega - kv_{os} - j\nu_s' - m\omega_{cs})|^2} \quad ; \qquad k_i \ll k_r \quad . \tag{6.143}$$

Comparison with Eq. 6.136 shows that $W_k^{(s)} < 0$ ($\omega_i < 0$) is in correspondence with $P_k^{(s)} < 0$ ($k_i > 0$). Hence similar statements with regard to the energy exchange can be made regarding the sink and source of the electrokinetic power.

Similar behavior is found when the drifted Maxwellian function is examined. From Eq. 5.120 in Chapter V, for the mth resonance, with the definition of ν_s' in Eq. 6.119 having been utilized, the following results:

$$J_{1x}^{(s)} = - \frac{j\omega_{ps}^2 \epsilon E_{1x} m\omega_{cs} \, e^{-\lambda_s} \, I_m(\lambda_s)}{k^2 v_{Ts}^2 (\omega - kv_{os} - j\nu_s' - m\omega_{cs})} \quad , \tag{6.144}$$

so that when this result is compared with Eq. 6.135 it can be seen that the $W_k^{(s)}$ and $P_k^{(s)}$ functions can be obtained from the results of the cylindrical distribution function by the replacement

$$J_m^2\left(\frac{kv_{Fs}}{\omega_{cs}}\right) \rightarrow \frac{1}{2} \exp\left[-\left(\frac{kv_{Fs}}{\omega_{cs}}\right)^2\right] I_m\left(\frac{kv_{Fs}}{\omega_{cs}}\right)^2 \quad . \tag{6.145}$$

In summation, for those distribution functions f_{os} which are independent of θ so that Eq. 6.60 may be used, the electrokinetic energy density and electrokinetic power flow are negative if $m\omega_{cs} < 0$ corresponding to the slow-cyclotron type of mode and hence are similar in general to hydrodynamic theory.

6.5.2 Carrier Distribution Function f_o Dependent on θ. As discussed in Sections 6.2 and 6.3, when the dc electric field is taken fully into account, the carrier distribution function f_o can become dependent on the angle θ in the x-y plane defined by Eq. 6.33. When this occurs Eq. 6.55 (and hence Eq. 6.60) is no longer valid and as a result interesting deviations from traditional requirements for instability (e.g., $v_o > \omega_r/k$) are potentially possible. Various cases in which the carrier velocity distribution function f_o is dependent on θ are now examined.

Up to this point the effect of the field E_{oy} has been ignored for the holes so that the general formulation could be used. This neglect is, strictly speaking, only permissible if $|E_{oy}/B_o| \ll v_{Th}$. The effect of the inclusion of this term is examined for a carrier (e.g., hole) whose drift velocity is negligible in any direction ($|v_o| \ll v_T$). In this case employ the transformation

$$v_x = \left(\frac{E_{oy}}{B_o}\right) + u \cos\theta$$

$$v_y = u \sin\theta \quad , \tag{6.146}$$

where the field E_{ox} is assumed negligible compared to E_{oy}. The solution given by Eqs. 6.53 and 6.54 is still valid where now, with Eqs. 6.9 and 6.146 having been used,

$$f_o = \frac{N_o}{(2\pi v_T^2)^{3/2}} \exp\left[-\left(\frac{E_{oy}}{\sqrt{2}\,B_o v_T}\right)^2\right] \exp\left(-\frac{u^2}{2v_T^2}\right) \exp\left[-\left(\frac{uE_{oy}}{B_o v_T^2}\right)\cos\theta\right]$$

$$\cdot \exp\left(-\frac{v_z^2}{2v_T^2}\right) \quad , \qquad (6.147)$$

so that $f_o = f_o(\theta)$. From Eqs. 6.9 and 6.146, the following results:

$$\frac{\partial f_o}{\partial v_x} = -\frac{v_x}{v_T^2} f_o = -\frac{1}{v_T^2}\left(\frac{E_{oy}}{B_o} + u\cos\theta\right) f_o(\theta) \quad . \qquad (6.148)$$

From the Bessel function identity,

$$\exp(z\cos\theta) = \sum_{\ell=-\infty}^{\infty} I_\ell(z)\exp(j\ell\theta) \quad , \qquad (6.149)$$

Eq. 6.147 becomes

$$f_o(\theta) = \frac{N_o}{(2\pi v_T^2)^{3/2}} \exp\left[-\left(\frac{E_{oy}}{\sqrt{2}\,B_o v_T}\right)^2\right] \exp\left(-\frac{u^2}{2v_T^2}\right) \exp\left(-\frac{v_z^2}{2v_T^2}\right)$$

$$\cdot \sum_{\ell=-\infty}^{\infty} I_\ell\left(-\frac{uE_{oy}}{B_o v_T^2}\right) e^{j\ell\theta} \quad , \qquad (6.150)$$

from which $g_o(u)$ is defined by

$$f_o(\theta) = g_o(u) \sum_{\ell=-\infty}^{\infty} I_\ell\left(-\frac{uE_{oy}}{B_o v_T^2}\right) e^{j\ell\theta} \quad . \qquad (6.151)$$

By proceeding in a manner similar to that used in obtaining Eq. 6.57, Eqs. 6.35, 6.148, 6.150 and 6.151 provide the following:

$$f_{1s}(\theta) = \sum_{\ell, m, n=-\infty}^{\infty} J_m(b) J_n(b) I_\ell \left(- \frac{uE_{oy}}{B_o v_T^2} \right) g_o(u)$$

$$\cdot \left[\beta_1 \frac{\exp[j(m+\ell-n)\theta]}{m+\ell-a} - \beta_2 u \left(\frac{\exp[j(m+\ell-n+1)\theta]}{m+\ell-a+1} + \frac{\exp[j(m+\ell-n-1)\theta]}{m+\ell-a-1} \right) \right] , \tag{6.152}$$

where

$$\beta_1 = - \frac{\eta E_{1x} E_{oy}}{j\omega_c v_T^2 B_o} - \frac{\nu N_1}{j\omega_c N_o} , \quad b = \frac{ku}{\omega_c} , \quad a = \frac{\omega - k\left(\dfrac{E_{oy}}{B_o} \right) - j\nu}{\omega_c}$$

and

$$\beta_2 = \frac{\eta E_{1x}}{j2\omega_c v_T^2} . \tag{6.153}$$

From Eq. 6.51 the RF number density is found as

$$N_1 = \frac{\displaystyle\sum_{m,n=-\infty}^{\infty} - \frac{j\eta E_{1x} N_o \exp\left[- \left(\dfrac{E_{oy}}{\sqrt{2}\, B_o v_T} \right)^2 \right]}{\omega_c v_T^4 (a - n)} \left[\left(\frac{E_{oy}}{B_o} \right) L_1 + L_2 \right]}{1 + j \dfrac{\nu \exp\left[- \left(\dfrac{E_{oy}}{\sqrt{2}\, B_o v_T} \right)^2 \right]}{\omega_c v_T^2} \displaystyle\sum_{m,n=-\infty}^{\infty} \frac{L_1}{a - n}} , \tag{6.154}$$

where

$$L_1 = \int_0^\infty J_m\left(\frac{ku}{\omega_c}\right) J_n\left(\frac{ku}{\omega_c}\right) I_{n-m}\left(-\frac{uE_{oy}}{B_o v_T^2}\right) \exp\left(-\frac{u^2}{2v_T^2}\right) u\, du \quad (6.155)$$

and

$$L_2 = \int_0^\infty J_m\left(\frac{ku}{\omega_c}\right) J_n\left(\frac{ku}{\omega_c}\right) I'_{n-m}\left(-\frac{uE_{oy}}{B_o v_T^2}\right) \exp\left(-\frac{u^2}{2v_T^2}\right) u^2\, du\ , \quad (6.156)$$

wherein use has been made of

$$I_{m-1}(x) + I_{m+1}(x) = 2\frac{d}{dx}\Big(I_m(x)\Big) = 2I'_m(x) \quad . \quad (6.157)$$

It can be verified that if $E_{oy} \to 0$, Eq. 6.154 reduces properly to the result of Eq. 6.108 since only the n = m term survives in Eq. 6.155 [$I_n(0) = 0$ except for $I_o(0) = 1$] and

$$L_2(E_{oy} \to 0) \to \frac{1}{2}\int_0^\infty J_m\left(\frac{ku}{\omega_{cs}}\right) J_n\left(\frac{ku}{\omega_{cs}}\right)\Big(I_{(n-m)-1}(0) + I_{(n-m)+1}(0)\Big)$$

$$\cdot \exp\left(-\frac{u^2}{2v_T^2}\right) u^2\, du = \frac{n\omega_{cs}}{k}\int_0^\infty J_n^2\left(\frac{ku}{\omega_{cs}}\right) \exp\left(-\frac{u^2}{2v_T^2}\right) u\, du \quad , \quad (6.158)$$

where $J_{n-1}(x) + J_{n+1}(x) = (2n/x)J_n(x)$ has been used.

Inspection of Eq. 6.154 indicates that the resonant form is still present in the factor (a - n) representing the nth resonance. In particular the n = 0 resonance can now appear, whereas this gave zero contribution for f_{os} independent of θ as can be seen in Eq. 6.60. However, for the n = 0 resonance in the hydrodynamic limit, $kv_{Ts}/\omega_{cs} \to 0$, Eqs. 6.154 through 6.156 give the following:

$$L_1 \approx \int_0^\infty J_o^2(0) I_o \left(- \frac{uE_{oy}}{B_o v_T^2} \right) \exp\left(- \frac{u^2}{2v_T^2} \right) u \, du \approx v_T^2 \exp\left(\frac{E_{oy}^2}{2B_o^2 v_T^2} \right) ,$$

$$L_2 \approx \int_0^\infty J_o^2(0) I_1 \left(- \frac{uE_{oy}}{B_o v_T^2} \right) u^2 \exp\left(- \frac{u^2}{2v_T^2} \right) du \approx - \frac{E_{oy} v_T^2}{B_o} \exp\left(\frac{E_{oy}^2}{2B_o^2 v_T^2} \right) \tag{6.159}$$

and thus $N_1 \approx 0$, wherein use was made of

$$I_\nu(z) = \exp\left(- \frac{1}{2} \nu\pi j \right) J_\nu(jz)$$

and

$$\int_0^\infty \exp(-a^2 t^2) t^{\nu+1} J_\nu(bt) = \frac{b^\nu}{(2a^2)^{\nu+1}} \exp\left(- \frac{b^2}{4a^2} \right) ;$$

$$[\mathrm{Re}(\nu) > -1, \ \mathrm{Re}(a^2) > 0] \quad . \tag{6.160}$$

In general the integrals in Eqs. 6.155 and 6.156 are not solvable by analytical means. Some information can be achieved by using Eqs. 6.134 and 6.154 in Eq. 6.133 to obtain the carrier-mode electrokinetic energy density for the nth resonance in the resonance approximation

$$W_k = \sum_{m=-\infty}^{\infty} \frac{\omega_p^2 \epsilon \, |E_{1x}|^2 \exp\left[- \left(\frac{E_{oy}}{\sqrt{2}\, B_o v_T} \right)^2 \right] \left[\left(\frac{E_{oy}}{B_o} \right) L_1 + L_2 \right]}{2kv_T^4 \left| \omega - k \left(\frac{E_{oy}}{B_o} \right) - j\nu' - n\omega_c \right|^2}$$

$$\cdot \left[- \frac{\nu' \omega_r}{\omega_i} + k \left(\frac{E_{oy}}{B_o} \right) + n\omega_c \right] , \tag{6.161}$$

where

$$\nu' = \nu \left\{ 1 - \frac{1}{v_T^2} \exp\left[-\left(\frac{E_{oy}}{\sqrt{2}\, B_o v_T} \right)^2 \right] \sum_{m=-\infty}^{\infty} L_1 \right\} , \qquad (6.162)$$

so that if $\omega_i \ll \omega_r$, $W_k < 0$ if $[(E_{oy}/B_o)L_1 + L_2] < 0$. By proper selection of $\underline{E}_o$, $\underline{B}_o$, etc., it is expected that this latter condition can be achieved. In this case for example the hole mode can act as the energy source to drive the instability.

An alternative method whereby the distribution function becomes θ dependent occurs when the field E_{ox} is taken more rigorously into account. The following transformation is now used to replace Eq. 6.146:

$$v_x = \left(\frac{E_{oy}}{B_o} \right) + u \cos \theta \quad ,$$

$$v_y = -\left(\frac{E_{ox}}{B_o} \right) + u \sin \theta \quad . \qquad (6.163)$$

Then, if the carrier drift velocity is negligible ($\nu \gg \omega_c$), Eqs. 6.9 and 6.163 give the distribution function as

$$f_o = \frac{N_o}{(2\pi v_T^2)^{3/2}} \exp\left(- \frac{E_{ox}^2 + E_{oy}^2}{2B_o^2 v_T^2} \right) \exp\left(- \frac{u^2}{2v_T^2} \right) \exp\left(- \frac{v_z^2}{2v_T^2} \right) \exp\left(\frac{uE_{ox}}{B_o v_T^2} \sin \theta \right)$$

$$\cdot \sum_{\ell=-\infty}^{\infty} I_\ell \left(- \frac{uE_{oy}}{B_o v_T^2} \right) e^{j\ell\theta} \quad . \qquad (6.164)$$

Inspection of Eqs. 6.53, 6.148 and 6.164 indicates that the term in θ introduced by the field E_{ox} can be taken into account by replacing the argument of $J_m(b)$ in Eq. 6.152 via

$$b = \frac{ku}{\omega_{cs}} \rightarrow \frac{ku}{\omega_{cs}} - j\frac{uE_{ox}}{B_o v_T^2} \quad . \tag{6.165}$$

The RF density given by Eq. 6.154 is still valid where now, however,

$$L_1 = \exp\left[-\left(\frac{E_{ox}}{\sqrt{2}\,B_o v_T}\right)^2\right]\int_0^\infty J_m\left(\frac{ku}{\omega_c} - j\frac{uE_{ox}}{B_o v_T^2}\right) J_n\left(\frac{ku}{\omega_c}\right) \cdot I_{n-m}\left(-\frac{uE_{oy}}{B_o v_T^2}\right)\exp\left(-\frac{u^2}{2v_T^2}\right) u\, du$$

and

$$L_2 = \exp\left[-\left(\frac{E_{ox}}{\sqrt{2}\,B_o v_T}\right)^2\right]\int_0^\infty J_m\left(\frac{ku}{\omega_c} - j\frac{uE_{ox}}{B_o v_T^2}\right) J_n\left(\frac{ku}{\omega_c}\right) \cdot I'_{n-m}\left(-\frac{uE_{oy}}{B_o v_T^2}\right)\exp\left(-\frac{u^2}{2v_T^2}\right) u^2\, du \quad . \tag{6.166}$$

To obtain a more tractable analysis assume that E_{oy} field effects are negligible by letting $E_{oy} \rightarrow 0$ (e.g., the applied field can be E_{oy} and the Hall field E_{ox}, with $E_{ox} \gg E_{oy}$, without loss of generality). From an analysis such as used in Eq. 6.158 it can then be found that

$$L_1 = \exp\left[-\left(\frac{E_{ox}}{\sqrt{2}\,B_o v_T}\right)^2\right] \int_0^\infty J_n\left(\frac{ku}{\omega_c} - j\,\frac{uE_{ox}}{B_o v_T^2}\right) J_n\left(\frac{ku}{\omega_c}\right) u \cdot \exp\left(-\frac{u^2}{2v_T^2}\right) du$$

and

$$L_2 = \frac{n}{\left(\frac{k}{\omega_c} - j\,\frac{E_{ox}}{B_o v_T^2}\right)} L_1 \quad . \tag{6.167}$$

The general result[80]

$$\int_0^\infty \exp(-p^2t^2) J_\nu(\alpha t) J_\nu(\beta t)\, t\, dt = \frac{1}{2p^2} \exp\left(-\frac{\alpha^2 + \beta^2}{4p^2}\right) I_\nu\left(\frac{\alpha\beta}{2p^2}\right) \quad ;$$

$$\mathrm{Re}(\nu) > -1 \quad , \quad |\arg p| < \frac{\pi}{4} \quad , \tag{6.168}$$

applied to Eqs. 6.167 and thence to Eq. 6.154 provides the following:

$$N_1 = \frac{\displaystyle\sum_{n=-\infty}^{\infty} - \frac{j\eta E_{1x} k N_o}{\omega - j\nu - n\omega_c}\, \frac{n\omega_c}{\omega_c^2 \lambda_c}\, e^{-\lambda_c} I_n(\lambda_c)}{\displaystyle 1 + j\nu \sum_{n=-\infty}^{\infty} \frac{e^{-\lambda_c} I_n(\lambda_c)}{\omega - j\nu - n\omega_c}} \quad ; \quad \mathrm{Re}(n) > -1 \quad , \tag{6.169}$$

where

$$\lambda_c = \frac{k^2 v_T^2}{\omega_c^2} - j\,\frac{kE_{ox}}{B_o \omega_c} \quad . \tag{6.170}$$

Since $v_T^2 = \kappa T/m^*$, where T is the carrier temperature, κ is Boltzmann's constant and m^* is the carrier effective mass, the view can be adopted that the effect of the field E_{ox} is to replace the temperature T by the complex temperature

$$T_c = T - j \frac{E_{ox} m^* \omega_c}{B_o k\kappa} . \qquad (6.171)$$

The resonance approximation is now made, corresponding to the omission of the n = 0 term in the denominator of Eq. 6.170, which is valid provided that $|\lambda_c| \gtrsim 1$ and is strictly invalid if $|\lambda_c| \ll 1$. Equation 6.169 then becomes the following for the nth resonance:

$$N_1 = - \frac{j\eta E_{1x} k N_o n\omega_c e^{-\lambda_c} I_n(\lambda_c)}{\omega_c^2 \lambda_c (\omega - j\nu' - n\omega_c)} \quad ; \quad |\lambda_c| \gtrsim 1 , \qquad (6.172)$$

where

$$\nu' = \nu \left(1 - e^{-\lambda_c} I_n(\lambda_c)\right) . \qquad (6.173)$$

A study of the function $e^{-\lambda_c} I_n(\lambda_c)$ indicates that $\mathrm{Re}(\nu') > 0$ must hold. Apply these results now to holes with $|\omega_{ch}| \ll \nu_h$ by comparing Eqs. 6.108 and 6.169 so that the solution given in Eq. 6.120 is still valid but wherein,

$$\Gamma_h \rightarrow \Gamma_h' = \frac{\omega_{ph}^2 e^{-\lambda_c} I_m(\lambda_c)}{\lambda_c \omega_{ch}^2} , \qquad (6.174)$$

where from Eq. 6.170,

$$\lambda_c = \frac{k^2 v_{Th}^2}{\omega_{ch}^2} - j\,\frac{kE_{ox}}{B_o\omega_{ch}} \quad . \tag{6.175}$$

Recalling that the field E_{oy} is negligible or zero (e.g., the applied field in E_{ox} and the Hall field E_{oy} is shorted) it should still be shown that the undrifted form of the carrier distribution function is permissible in the presence of E_{ox} when $\nu \gg |\omega_c|$. By following the system of Eqs. 6.32 through 6.41, but where now E_{ox} is retained and $E_{oy} = E_{oz} = 0$, so that Eq. 6.163 replaces Eq. 6.33, it can be found that

$$f_o = \frac{N_o}{(2\pi v_T^2)^{3/2}} \exp\left(-\frac{u^2}{2v_T^2}\right) \exp\left(-\frac{v_z^2}{2v_T^2}\right) \exp\left[-\left(\frac{E_{ox}}{\sqrt{2}\,B_o v_T}\right)^2\right] \cdot$$

$$\cdot \sum_{n=-\infty}^{\infty} J_n\left(-j\,\frac{uE_{ox}}{B_o v_T^2}\right) e^{jn\theta} \left[1 - \frac{uE_{ox}}{B_o v_T^2}\left(\frac{(jn - \nu/\omega_c)\cos\theta + \sin\theta}{1 + \left(jn - \frac{\nu}{\omega_c}\right)^2}\right)\right] . \tag{6.176}$$

Inspection of this result shows that for $\nu \gg |\omega_c|$ the last term should give a negligible contribution so that

$$f_o \approx \frac{N_o}{(2\pi v_T^2)^{3/2}} \exp\left(-\frac{u^2}{2v_T^2}\right) \exp\left(-\frac{v_z^2}{2v_T^2}\right) \exp\left[-\left(\frac{E_{ox}}{\sqrt{2}\,B_o v_T}\right)^2\right]$$

$$\cdot \sum_{n=-\infty}^{\infty} J_n\left(-j\,\frac{uE_{ox}}{B_o v_T^2}\right) e^{jn\theta} = \frac{N_o}{(2\pi v_T^2)^{3/2}} \exp\left(-\frac{(v_x^2 + v_y^2 + v_z^2)}{2v_T^2}\right) , \tag{6.177}$$

thus validating the assumption of a negligible drift velocity. The RF number density can also be found directly, without approximation, for the exact carrier distribution function f_o of Eq. 6.176 (although the result is not in closed form) by the following procedure. From the transformation of Eq. 6.163 with $E_{oy} = 0$ it can be determined that Eq. 6.176 can be written as

$$f_o = \frac{N_o}{(2\pi v_T^2)^{3/2}} \left[\exp\left(-\frac{v_x^2 + v_y^2 + v_z^2}{2v_T^2}\right) - \frac{E_{ox}}{B_o v_T^2} \exp\left(-\frac{v_z^2}{2v_T^2}\right) \right.$$

$$\cdot \exp\left(-\frac{(v_y + E_{ox}/B_o)^2}{2v_T^2}\right) \cdot \exp\left[-\left(\frac{E_{ox}}{\sqrt{2}\, B_o v_T}\right)^2\right] \exp\left(-\frac{v_x^2}{2v_T^2}\right) \cdot$$

$$\cdot \sum_{n=-\infty}^{\infty} J_n\left\{-j \frac{E_{ox}}{B_o v_T^2}\left[v_x^2 + \left(v_y + \frac{E_{ox}}{B_o}\right)\right]^2\right\} \frac{\left[\left(jn - \frac{\nu}{\omega_c}\right) v_x + \left(v_y + \frac{E_{ox}}{B_o}\right)\right]}{\left[1 + \left(jn - \frac{\nu}{\omega_c}\right)^2\right]}$$

$$\left. \cdot \exp\left[jn \tan^{-1}\left(\frac{v_y + E_{ox}/B_o}{v_x}\right)\right] \right] . \qquad (6.178)$$

From Eq. 6.50, with $E_{oy} = 0$, it follows that

$$f_1(\theta) = \sum_{m=-\infty}^{\infty} \exp(ja\theta) \exp(-jb \sin\theta) \left[\frac{\eta E_{1x}}{\omega_c} \int_c^\theta J_m(b) \exp[j(m - a)\theta'] \right.$$

$$\left. \cdot \frac{\partial f_o(\theta')}{\partial v_x} d\theta' - \frac{\nu N_1}{\omega_c N_o} \int_c^\theta J_m(b) \exp[j(m - a)\theta'] f_o(\theta')\, d\theta' \right] , \qquad (6.179)$$

where $a = (\omega - j\nu)/\omega_c$ and $b = ku/\omega_c$.

Use of Eq. 6.178 in Eq. 6.179 obtains $f_1(\theta)$ which applied to Eq. 6.51 provides the following:

$$N_1 = - \frac{\dfrac{j\eta E_{1x} N_o}{2\omega_c v_T^4} \exp\left[-\left(\dfrac{E_{ox}}{\sqrt{2}\, B_o v_T}\right)^2\right] \displaystyle\sum_{\ell,n=-\infty}^{\infty} \sum_{i=1}^{3} \frac{\mathcal{E}_i(\ell,n)}{a-\ell}}{1 + j\dfrac{\nu}{\omega_c v_T^2} \exp\left[-\left(\dfrac{E_{ox}}{\sqrt{2}\, B_o v_T}\right)^2\right] \displaystyle\sum_{\ell,n=-\infty}^{\infty} \frac{D(\ell,n)}{a-\ell}} , \qquad (6.180)$$

where

$$\mathcal{E}_1 = 2\ell \int_0^\infty J_\ell\left(\frac{ku}{\omega_c}\right) \exp\left(-\frac{u^2}{2v_T^2}\right) \frac{J_\ell\left(\dfrac{ku}{\omega_c} - j\dfrac{uE_{ox}}{B_o v_T^2}\right)}{\dfrac{k}{\omega_c} - j\dfrac{E_{ox}}{B_o v_T^2}}\, u\, du , \qquad (6.181)$$

$$\mathcal{E}_2 = \frac{(E_{ox}/B_o)}{1 + (jn - \nu/\omega_c)^2} \int_0^\infty J_\ell\left(\frac{ku}{\omega_c}\right) J_n\left(-j\frac{uE_{ox}}{B_o v_T^2}\right) \exp\left(-\frac{u^2}{2v_T^2}\right) uH(\ell,n)\, du , \qquad (6.182)$$

where

$$H(\ell,n) = \left[-\frac{u^2}{v_T^2}\left(jn - \frac{\nu}{\omega_c}\right) - jn + \left(jn - \frac{\nu}{\omega_c}\right)^2 \frac{2(\ell - n)}{b}\right] J_{\ell-n}\left(\frac{ku}{\omega_c}\right)$$

$$- 2j\left(jn - \frac{\nu}{\omega_c}\right) J'_{\ell-n}(b) + \left[-n\left(jn - \frac{\nu}{\omega_c}\right) + j\frac{u^2}{v_T^2} - \frac{u^2}{v_T^2}\left(jn - \frac{\nu}{\omega_c}\right) + jn\right]$$

$$\cdot J_{\ell-n-2}\left(\frac{ku}{\omega_c}\right) + \left[n\left(jn - \frac{\nu}{\omega_c}\right) - j\frac{u^2}{2v_T^2} - \frac{u^2}{v_T^2}\left(jn - \frac{\nu}{\omega_c}\right) + jn\right] J_{\ell-n+2}\left(\frac{ku}{\omega_c}\right) , \qquad (6.183)$$

$$\mathcal{E}_3 = - \frac{jE_{ox}}{B_o v_T^2 \left[1 + \left(jn - \frac{\nu}{\omega_c}\right)^2\right]} \int_0^\infty J_\ell\left(\frac{ku}{\omega_c}\right) J_n'\left(-j \frac{uE_{ox}}{B_o v_T^2}\right)$$

$$\cdot \exp\left(-\frac{u^2}{2v_T^2}\right) u^3 \, K(\ell,n) \, du \quad , \qquad (6.184)$$

where

$$K(\ell,n) = 2\left(jn - \frac{\nu}{\omega_c}\right) J_{\ell-n}\left(\frac{ku}{\omega_c}\right) + \left[\left(jn - \frac{\nu}{\omega_c}\right) - j\right]$$

$$\cdot J_{\ell-n-2}\left(\frac{ku}{\omega_c}\right) + \left[\left(jn - \frac{\nu}{\omega_c}\right) + j\right] J_{\ell-n+2}\left(\frac{ku}{\omega_c}\right) \quad , \qquad (6.185)$$

and

$$D(\ell,n) = \int_0^\infty J_\ell\left(\frac{ku}{\omega_c}\right) J_n\left(-j \frac{uE_{ox}}{B_o v_T^2}\right) \exp\left(-\frac{u^2}{2v_T^2}\right) L(\ell,n) u \, du \; , \qquad (6.186)$$

where

$$L(\ell,n) = J_{\ell-n}\left(\frac{ku}{\omega_c}\right) - \frac{uE_{ox}\left[\left(jn - \frac{\nu}{\omega_c}\right)\frac{2(\ell - n)}{b} J_{\ell-n}(b) - 2jJ_{\ell-n}'(b)\right]}{2B_o v_T^2[1 + (jn - \nu/\omega_c)^2]} \; . \qquad (6.187)$$

This is a general result valid for any charge carrier species. The integrals are not readily solved except for $\mathcal{E}_1$ which can be obtained using Eq. 6.168 and correspondsto the earlier result of Eq. 6.169. Inspection of Eqs. 6.180 through 6.187 under the condition $\nu \gg |\omega_c|$ shows that even though the drift velocity is negligible, e.g., Eq. 6.177, the full dependence of the distribution function f_o on θ can introduce additional nonnegligible terms in the RF number density beyond those of Eq. 6.169. Thus, in general, the dispersion relation obtained by first approximating the carrier distribution function f_o through $\nu \gg |\omega_c|$ can differ significantly from the result for the dispersion relation (for the exact function f_o) in the limit $\nu \gg |\omega_c|$. As a consequence, in the presence of a static magnetic field the use of a complex temperature, such as Eq. 6.171, to account for carrier heating effects is not in general justified. In addition there is the problem that when $\nu \gg |\omega_c|$ the resonances are not well isolated so that examining a particular resonance, i.e., selecting only one ℓ value, is a poor approximation. This explains some of the anomalous behavior which can arise for undrifted carriers when the resonance approximation is made. For example, if Eq. 6.172 is used in Eqs. 6.133 and 6.134, it can be found that the electrokinetic energy density of such carriers can be negative due to the field E_{ox}.

Conclusive statements regarding the effects of the dependence of the carrier distribution function f_o on θ require a rigorous solution of Eq. 6.180 which is not attempted in the present tract. This already cumbersome expression for the RF number density is further compounded by questions regarding the accuracy of a constant collision frequency assumed in the presence of carrier heating. Some qualitative statements regarding the effects of the field E_{ox} are still possible for the charge carrier with $|\omega_c| \gg \nu$. For example, inspection of Eqs. 6.180 through 6.187 shows that the $\ell = 0$ resonance now gives a contribution to the RF number density when $E_{ox} \neq 0$. In addition, the effective collision frequency concept is still valid where now at the ℓth resonance

$$\nu' = \nu \left[1 - \sum_{n=-\infty}^{\infty} \frac{\exp\left[- \left(\frac{E_{ox}}{\sqrt{2}\, B_o v_T} \right)^2 \right] D(\ell,n)}{v_T^2} \right] . \tag{6.188}$$

Inspection of Eqs. 6.186 and 6.187 shows that for $|\omega_c| \gg \nu$, the function $D(\ell,n)$ itself exhibits a resonance form at $n = \pm 1$; however, cancellations occur when the summation is made so that these terms have no special significance. An interesting case which can readily be analyzed is the small wave number limit $(k \to 0)$ when $\ell = 0$. If $|\omega_c| \gg \omega, \nu$ Eq. 6.180 is approximated by

$$N_1 \approx - \frac{j\eta E_{1x} N_o \left\{ \exp\left[- \left(\frac{E_{ox}}{\sqrt{2}\, B_o v_T} \right)^2 \right] \sum_{n=-\infty}^{\infty} [\mathcal{E}_2(0,n) + \mathcal{E}_3(0,n)] \right\}}{2\omega_c v_T^4 \left\{ a + j \frac{\nu}{\omega_c v_T^2} \exp\left[- \left(\frac{E_{ox}}{\sqrt{2}\, B_o v_T} \right)^2 \right] \sum_{n=-\infty}^{\infty} D(0,n) \right\}} \tag{6.189}$$

and by permitting $k \to 0$ in Eqs. 6.180 through 6.187 Eq. 6.189 becomes

$$N_1(k \to 0) \to - \frac{\sqrt{\pi}\, \eta E_{1x} N_o E_{ox}^2 \left\{ \exp\left[- \left(\frac{E_{ox}}{\sqrt{2}\, B_o v_T} \right)^2 \right] \right\}}{4\sqrt{2}\, \omega v_T^3 B_o^2} M\left(\frac{3}{2},\, 2,\, \frac{E_{ox}^2}{2B_o^2 v_T^2} \right) , \tag{6.190}$$

where again M is the confluent hypergeometric function. Note that implicit in Eq. 6.190 is the result, derived from Eq. 6.188, that the effective collision frequency $\nu' \to 0$ as $k \to 0$. This result is expected to be a good approximation, provided that $|\omega_c| \gg kv_T, \omega, \nu$, for finite k, and hence the field E_{ox} introduces an entirely new mode, corresponding to $\ell = 0$, in which the effect of the carrier collisions on the wave dispersion can be negligible. In addition for $E_{ox} \neq 0$ tables[81] provide that $M > 0$ and as a result if Eqs. 6.133 and 6.134 are used in Eq. 6.190 it can be shown that this mode has a negative electrokinetic energy density. Thus, since the carrier drift velocity plays no explicit role, the field E_{ox} introduces a negative kinetic energy wave without the traditional requirement of hydrodynamic theory that $v_o > \omega_r/k$ where $\underline{v}_o \parallel \underline{k}$. In the present case by using Eq. 6.176 in Eq. 6.25 it is found that

$$v_{ox} = \exp\left[-\left(\frac{E_{ox}}{\sqrt{2}\,B_o v_T}\right)^2\right]\left[\frac{\sqrt{\pi}}{\sqrt{2}}\,v_T\,M\left(\frac{3}{2},\,1,\,\frac{E_{ox}^2}{2B_o^2 v_T^2}\right) + \frac{2E_{ox}\nu}{B_o\omega_c\left(1+\frac{\nu^2}{\omega_c^2}\right)}\right.$$
$$\left.\cdot\,M\left(2,\,1,\,\frac{E_{ox}^2}{2B_o^2v_T^2}\right) - \frac{E_{ox}^3\nu\left(9+\frac{\nu^2}{\omega_c^2}\right)}{2B_o^3v_T^2\omega_c\left[\left(\frac{\nu^2}{\omega_c^2}-3\right)^2 + 16\,\frac{\nu^2}{\omega_c^2}\right]}\,M\left(3,\,3,\,\frac{E_{ox}^2}{2B_o^2v_T^2}\right)\right] \tag{6.191}$$

for the component of drift velocity $\parallel \underline{k}$.

It can be stated then that the dc electric field $\underline{E}_o \parallel \underline{k}$ when taken fully into account leads to significant new wave properties with energy characteristics quite different from the hydrodynamic theory. Similar phenomena also occur for the case $E_{ox} = E_{oz} = 0$ with $E_{oy} \neq 0$ ($\underline{E}_o \perp \underline{k}$) as discussed in Appendix G. No attempt is made herein to actually develop the dispersion relations resulting from the general number density, Eq. 6.180, due to its complexity, but Eq. 6.190 indicates that unstable wave behavior is to be expected when a carrier with $|\omega_c| \gg (\omega, \nu, kv_T)$ interacts with a secondary carrier with $|\omega_c| < \nu$. In addition, since $\nu'(k \to 0) \to 0$ in Eq. 6.190, it may be possible that for finite wave numbers the effective collision frequency actually becomes negative. The important point here is that when carrier heating occurs in the presence of a static magnetic field the effects of collisions on the wave propagation can be significantly reduced.

6.6 The Ordinary Mode in Solid-State Plasmas

When the geometry is retained with $\underline{B}_o \parallel \hat{z}$, $\underline{k} \parallel \hat{x}$, with the applied static electric field E_{ox} and the Hall field E_{oy}, the ordinary mode is defined as the electromagnetic mode with $\underline{E}_1 \parallel \hat{z}$ and $\underline{B}_1 = B_{1y}\hat{y}$. For isotropic media with negligible Hall drift velocities this mode is well defined by the component ϵ_{zz} of the effective dielectric constant, ($\tilde{\epsilon} = \tilde{I} + \tilde{\sigma}/j\omega\epsilon$). If the RF magnetic field B_{1y} is retained and $N_{1s} = 0$ is set, since there is no RF bunching, Eq. 6.32 becomes

$$j(\omega - kv_{xs})f_{1s} + \eta_s B_o\left(v_{ys}\frac{\partial f_{1s}}{\partial v_{xs}} - v_{xs}\frac{\partial f_{1s}}{\partial v_{ys}}\right) + \eta_s\left(E_{ox}\frac{\partial f_{1s}}{\partial v_{xs}} + E_{oy}\frac{\partial f_{1s}}{\partial v_{ys}}\right)$$

$$+ \eta_s E_{1z}\frac{\partial f_{os}}{\partial v_{zs}} + \eta_s v_{xs}B_{1y}\frac{\partial f_{os}}{\partial v_{zs}} - \eta_s v_{zs}B_{1y}\frac{\partial f_{os}}{\partial v_{xs}} = -\nu_s f_{1s} \quad . \qquad (6.192)$$

The following coordinate systems, previously used, will be needed:

$$v_{xs} - v_{os} = u_s \cos\theta = v_{rs}\cos\theta\sin\varphi$$

$$v_{ys} = u_s \sin\theta = v_{rs}\sin\theta\sin\varphi$$

$$v_{zs} = v_{rs}\cos\varphi \ , \quad u_s = v_{rs}\sin\varphi \ , \qquad (6.193)$$

from which the following can be found:

$$\iiint_{-\infty}^{\infty} dv_{xs}\, dv_{ys}\, dv_{zs} = \int_0^{\infty}\int_0^{2\pi}\int_{-\infty}^{\infty} u_s\, du_s\, d\theta\, dv_{zs}$$

$$= \int_0^{\infty}\int_0^{2\pi}\int_0^{\pi} v_{rs}^2 \sin\varphi\, dv_{rs}\, d\theta\, d\varphi \quad . \qquad (6.194)$$

Only the drifted Maxwellian, Eq. 6.18, and the drifted degenerate function, Eq. 6.70, are studied in the following with their respective distribution functions superscripted by M and D for clarity. The term E_{ox} is assumed to be negligible; therefore, the method of Eqs. 6.34 through 6.37 provides the following where now $a_s = (\omega - kv_{os} - j\nu_s)/\omega_{cs}$ and $b_s = ku_s/\omega_{cs}$:

$$\frac{\partial f_{1s}^M}{\partial\theta} - j(a_s - b_s \cos\theta)f_{1s}^M = -\frac{\eta_s E_{1z}}{\omega_{cs}}\left(1 - \frac{kv_{os}}{\omega}\right)\frac{v_{zs}}{v_{Ts}^2} f_{os}^M \qquad (6.195)$$

and

$$\frac{\partial f_{1s}^D}{\partial\theta} - j(a_s - b_s \cos\theta)f_{1s}^D = \frac{\eta_s E_{1z}}{\omega_{cs}}\left(1 - \frac{kv_{os}}{\omega}\right)\frac{v_{zs}}{v_{rs}}\left(\frac{\partial f_{os}^D}{\partial v_{rs}}\right) , \qquad (6.196)$$

wherein the following were used:

$$\frac{\partial f_{os}^M}{\partial v_{xs}} = -\frac{v_{xs} - v_{os}}{v_{Ts}^2} f_{os}^M , \quad \frac{\partial f_{os}^M}{\partial v_{ys}} = -\frac{v_{ys}}{v_{Ts}^2} f_{os}^M , \quad \frac{\partial f_{os}^M}{\partial v_{zs}} = -\frac{v_{zs}}{v_{Ts}^2} f_{os}^M ,$$

$$\frac{\partial f_{os}^D}{\partial v_{xs}} = (v_{xs} - v_{os})\frac{1}{v_{rs}}\left(\frac{\partial f_{os}^D}{\partial v_{rs}}\right) , \quad \frac{\partial f_{os}^D}{\partial v_{ys}} = \frac{v_{ys}}{v_{rs}}\left(\frac{\partial f_{os}^D}{\partial v_{rs}}\right) ,$$

$$\frac{\partial f_{os}^D}{\partial v_{zs}} = \frac{v_{zs}}{v_{rs}}\left(\frac{\partial f_{os}^D}{\partial v_{rs}}\right) \qquad (6.197)$$

and $B_{1y} = -(k/\omega)E_{1z}$ corresponding to variation as $\exp[j(\omega t - kx)]$. Let the right-hand side of Eqs. 6.195 and 6.196 be designated R^M and R^D, respectively. It can readily be demonstrated that $R^{M,D}$ is independent of θ. Thus the solutions for $f_{1s}^{M,D}$ are found in the same manner as that for the N_{1s} term in Eqs. 6.38 and 6.39, leading to

$$f_{1s}^M(\theta) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} J_m(b_s)J_n(b_s) \frac{j\eta_s E_{1z}\left(1 - \frac{kv_{os}}{\omega}\right)}{n - a_s} \frac{v_{zs}}{v_{Ts}^2} f_{os}^M \exp[j(n - m)\theta] \tag{6.198}$$

and

$$f_{1s}^D(\theta) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} J_m(b_s)J_n(b_s) \frac{(-j)\eta_s E_{1z}\left(1 - \frac{kv_{os}}{\omega}\right)}{\omega_{cs}(n - a_s)} \frac{v_{zs}}{v_{rs}} \cdot \left(\frac{\partial f_{os}^D}{\partial v_{rs}}\right) \exp[j(n - m)\theta] \; . \tag{6.199}$$

Since the ordinary mode is dependent solely on ϵ_{zz}, only the RF current density $J_{1z} \triangleq J_{1z}(\underline{E}_1 = E_{1z}\hat{z})$ is required where

$$J_{1z} = \sum_s q_s \iiint_{-\infty}^{\infty} v_{zs} f_{1s}(\theta) \, dv_{xs} \, dv_{ys} \, dv_{zs} = \sum_s J_{1z,s} \; . \tag{6.200}$$

Use of Eqs. 6.198 and 6.199 in Eq. 6.200 then gives the following:

$$J_{1z,s}^M = \sum_{m=-\infty}^{\infty} - \frac{j\epsilon\omega_{ps}^2 e^{-\lambda_s} I_m(\lambda_s)}{\omega - kv_{os} - j\nu_s - m\omega_{cs}} E_{1z} \frac{\omega - kv_{os}}{\omega} \quad ; \quad \lambda_s = \left(\frac{kv_{Ts}}{\omega_{cs}}\right)^2 \tag{6.201}$$

and

$$J^D_{1z,s} = \sum_{m=-\infty}^{\infty} - \frac{j2\pi q_s \eta_s E_{1z}\left(1 - \frac{kv_{os}}{\omega}\right)}{\omega_{cs}(m - a_s)} \int_0^\infty \int_0^\pi J_m^2\left(\frac{kv_{rs}}{\omega_{cs}} \sin\varphi\right)$$

$$\cdot \cos^2\varphi \sin\varphi \left(\frac{\partial f^D_{os}}{\partial v_{rs}}\right) v^3_{rs}\, dv_{rs}\, d\varphi \qquad (6.202)$$

and since, from Eq. 6.70,

$$\frac{\partial f^D_{os}}{\partial v_{rs}} = \frac{3N_{os}}{4\pi v^3_{Fs}} [\delta(v_{rs}) - \delta(v_{rs} - v_{Fs})] \quad , \qquad (6.203)$$

then

$$J^D_{1z,s} = \sum_{m=-\infty}^{\infty} + \frac{j\epsilon 3\omega^2_{ps} E_{1z}(\omega - kv_{os})}{2\omega\omega_{cs}(m - a_s)} \int_0^\pi \sin\varphi \cos^2\varphi\, J_m^2\left(\frac{kv_{Fs}}{\omega_{cs}} \sin\varphi\right) d\varphi \quad .$$

$$(6.204)$$

Employing Eq. 6.201 in the usual manner in Maxwell's equations leads to the dispersion relation for the Maxwellian distribution function,

$$k^2c^2 - \omega^2 + \sum_s \sum_{m=-\infty}^{\infty} \frac{\omega^2_{ps}(\omega - kv_{os})e^{-\lambda_s} I_m(\lambda_s)}{\omega - kv_{os} - j\nu_s - m\omega_{cs}} \quad . \qquad (6.205)$$

If now $(kv_{Ts}/\omega_{cs}) \to 0$, it is true that $I_m(\lambda_s) \to 0$ except for $I_o(\lambda_s) \to 1$ and Eq. 6.204 becomes

$$k^2c^2 - \omega^2 + \sum_s \frac{\omega^2_{ps}(\omega - kv_{os})}{\omega - kv_{os} - j\nu_s} = 0 \quad , \qquad (6.206)$$

which is identically the result found from hydrodynamic theory. Significant deviations from the hydrodynamic theory can then occur if $kv_{Ts} \gtrsim |\omega_{cs}|$, the most interesting of which is the possible resonant behavior $\omega - kv_{os} - m\omega_{cs} \approx 0$ leading to a slow-cyclotron-type mode. Similar statements can be made for the degenerate distribution function since if $kv_{Fs}/\omega_{cs} \rightarrow 0$ in Eq. 6.204 only the m = 0 term survives in the summation and consequently the hydrodynamic result, Eq. 6.206, is obtained exactly when the integration is then performed. In the general case this integral cannot be readily evaluated; however, this integral by inspection is always positive or zero and hence qualitatively behaves similar to the function $e^{-\lambda_s} I_m(\lambda_s)$.

It is important to note that unlike the purely transverse mode ($\underline{k} \parallel \underline{B}_o$) studied in Chapter V the ordinary mode does not present any problems associated with nonlocal effects which alter the resonance structure. To study the instability characteristics of the ordinary mode define the effective plasma frequency, ω_p', by the equation

$$(\omega_p')^2 = \omega_p^2 e^{-\lambda_s} I_m(\lambda_s) \tag{6.207}$$

for the Maxwellian distribution function, and by

$$(\omega_p')^2 = \frac{3}{2} \omega_p^2 \int_0^\pi \sin\varphi \cos^2\varphi \, J_m^2\left(\frac{kv_{Fs}}{\omega_{cs}} \sin\varphi\right) d\varphi \tag{6.208}$$

for the degenerate case. For fundamental resonant interactions $k \approx (\omega + \omega_{cs})/v_{os}$, so that if $\lambda_s \gtrsim 1$, the dispersion relation for either the Maxwellian or degenerate cases can be approximated for a two-carrier system from Eqs. 6.205 through 6.208, with m = 1 by

$$k^2c^2 - \omega^2 + \frac{(\omega'_{p1})^2(\omega - kv_{o1})}{\omega - kv_{o1} - j\nu_1 - \omega_{c1}} + \frac{(\omega'_{p2})^2(\omega - kv_{o2})}{\omega - kv_{o2} - j\nu_2 - \omega_{c2}} = 0 \quad . \qquad (6.209)$$

Simplify further by assuming that $v_{o2} \approx 0$ and $|v_{o1}| \ll c$ so that the coupled-mode form is obtained in which the coupling term appears on the right-hand side,

$$(\omega - kv_{o1} - j\nu_1 - \omega_{c1})(\omega - j\nu_2 - \omega_{c2}) = -\frac{(\omega'_{p1})^2}{k^2c^2}(\omega - kv_{o1})(\omega - j\nu_2 - \omega_{c2})$$

$$- \frac{(\omega'_{p2})^2}{k^2c^2}(\omega - kv_{o1} - j\nu_1 - \omega_{c1})\omega \quad , \qquad (6.210)$$

where $\omega \ll kc$ has been used. Inspection of this result shows that to achieve the coupling strength required for unstable wave behavior it is necessary that at least one of the carriers satisfy $(\omega'_p)^2/k^2c^2 \gtrsim 1$. Practically speaking, to avoid excessive Joule heating, this in turn requires a carrier species with large number density and negligible drift velocity in the direction of the applied electric field. For real k, Eq. 6.210 is solved as

$$\omega 2(1 + \theta_1 + \theta_2) = kv_{o1}\theta_1 + (kv_{o1} + \omega_{c1})(1 + \theta_2) + \omega_{c2}(1 + \theta_1) + j\nu_2(1 + \theta_1)$$

$$+ j\nu_1(1 + \theta_2) \pm \sqrt{R} \quad , \qquad (6.211)$$

where

$$R = [kv_{o1}\theta_1 + (kv_{o1} + j\nu_1 + \omega_{c1})(1 + \theta_2) - (\omega_{c2} + j\nu_2)(1 + \theta_1)]^2$$

$$+ 4\theta_1\theta_2(\omega_{c2} + j\nu_2)(\omega_{c1} + j\nu_1) \quad , \qquad (6.212)$$

and

$$\theta_s = \frac{(\omega'_{ps})^2}{k^2c^2} \quad ; \qquad s = 1,2 \quad . \tag{6.213}$$

Upon inspection of this result is is found that one of the carriers can have a very large collision frequency without diminishing the possibility for instability. For example, if $\nu_2 \gg \nu_1$, $\omega_{c1} < 0$, $\omega_{c2} > 0$, and $k \approx [|\omega_{c1}|(1 + \theta_2) + |\omega_{c2}|(1 + \theta_2)]/(1 + \theta_1 + \theta_2)$ instability occurs at $\omega_r = |\omega_{c2}|(1 + \theta_1)/(1 + \theta_1 + \theta_2)$ provided that

$$4\theta_1\theta_2 \, |\omega_{c1}| \, |\omega_{c2}| \gtrsim \nu_1^2(1 + \theta_2)^2 \quad . \tag{6.214}$$

In general instabilities of the ordinary mode are more difficult to achieve than the quasi-static hybrid mode since roughly speaking in the latter case the coupling constant is proportional to $(\omega_p/kv_T)^2$ or $(\omega_p/kv_F)^2$, whereas in the ordinary mode it is proportional to $(\omega_p/kc)^2$.

As in the hybrid mode study it is also of interest to study the effects of carrier heating on the cyclotron modes by taking the dc electric field fully into account in the carrier distribution function f_o. This aspect is presented and discussed in Appendix H.

6.7 Summary and Discussion

A rigorous analysis of the waves possible in the configuration with $\underline{k} \perp \underline{B}_o$ for various distribution functions has shown several important results brought about by the kinetic theory which have no correspondence in a hydrodynamic analysis. This has made it possible to relate experimental data to the cyclotron harmonic structure of the hybrid-hybrid kinetic dispersion relation. Also, the kinetic approach can take carrier heating effects into account explicitly. This has indicated that entirely new modes can appear in which the collisions play no role and unstable wave interaction is now possible in the fast wave regime ($v_o < \omega_r/k$). In this case the coupling of the longitudinal and transverse fields of the hybrid mode is increased, so that strictly speaking, the quasi-static assumption should be lifted and the full hybrid mode form examined.

A study of the variation of the carrier collision frequency with carrier speed has demonstrated that the use of a constant collision frequency is actually an excellent approximation since it is strictly accurate for the degenerate distribution function and it only introduces small quantitative changes for the Maxwellian distribution function.

An investigation of the ordinary mode in kinetic theory has revealed that cyclotron-resonant behavior is present although unless the charge carrier densities are larger and at least one of the carrier species has a small collision frequency ($\nu \ll \omega$) unstable interactions are difficult to achieve.

CHAPTER VII. SUMMARY, CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER STUDY

7.1 Summary and Conclusions

The primary purpose of this theoretical investigation has been to obtain an improved understanding of instability phenomena in plasma media. This is based on a study of the electrokinetic power and energy properties of the basic carrier modes present when the plasma medium is subject to applied static electric and magnetic fields. Particular emphasis has been placed on the investigation of solid-state plasmas in which case carrier collisions and thermal diffusion must be taken into account to obtain a meaningful analysis.

A discussion will be given of how the present work alleviates the problems associated with previous studies in this area. These previous studies have been incomplete and/or incorrect in the following respects:

1. The hydrodynamic expression for the electrokinetic power flow developed by Chu[64] for space-charge waves has been explicitly used as a meaningful expression when carrier collisions and thermal diffusion are included.[67, 82] This is incorrect since to balance the electromagnetic power flow it is not necessarily sufficient that the Chu electrokinetic power flow be negative.

2. The hydrodynamic development of the electrokinetic power for the cyclotron modes has been obtained on a normal mode basis.[71] This is incorrect since the coupled-mode technique ignores both the presence of the RF fields and the fact that the normal mode described by $\omega - kv_o - \omega_c \simeq 0$ is in a nonlocal regime for finite carrier temperatures so that cyclotron resonance phenomena should be included. In addition

past interpretations of the cyclotron mode electrokinetic power properties, since carrier bunching is absent, have been based upon a second-order RF electric field[50] or upon a transverse gradient of the fundamental RF electric field.[66] The former is incorrect because the second-order RF electric field has zero time average and hence cannot contribute to the second-order conservation of power. Even if this field is extended to include a time-average component this would be due to a charge separation effect related to the evolvement of the instability as opposed to any direct relationship with the linear dispersion relation. In the latter case[66] the explanation of the power properties based upon the RF electric field gradient is false simply because this gradient is not in general present.

3. No attempt has been made to ascertain the electrokinetic power properties of the hybrid mode which results when $\underline{k} \perp \underline{B}_o$ and since this is a basic carrier mode the previous analyses are incomplete.

4. In the application of the electrokinetic power concept no distinction has been made between convective and absolute instabilities, although it is usually implied that a mode which is active (i.e., a source mode) in a convective instability can also be active in an absolute instability.

5. There has been no published work relating the dispersion equation root structure in complex-k space based upon the mathematical causality criteria of Briggs[12] to the root behavior based upon the more physical approach of analysis of the carrier-mode power and energy properties.

6. In developing the expressions related to the conservation of power and energy, no satisfactory explanation has yet been provided for the

meaning and effects of the second-order time-averaged currents and fields which arise in a general second-order analysis.

7. Because of the manner in which the Chu definition of the electrokinetic power is obtained, no attempt has been made to extend the concept of electrokinetic power flow and energy to the kinetic theory. As a result, in addition to nonlocal effects, phenomena associated with the carrier distribution function, such as anisotropic carrier temperature and carrier heating, have not been explored on a power or energy basis.

The problems associated with items 1, 2, and 3 relating to convective instabilities were overcome in Chapter II by adopting a more general expression for the carrier-mode electrokinetic power flow based upon an examination of Poynting's theorem. In this manner the thermal kinetic power flow and the power flow associated with carrier collisions are incorporated directly into this carrier-mode electrokinetic power flow (P_k). Since the electromagnetic power flow must be positive for $k_r > 0$ this expression P_k must then be negative to conserve power properly, whereas the Chu electrokinetic power can be negative without necessarily satisfying conservation. For space-charge waves the carrier-mode electrokinetic power flow, P_k, reduces to that of the Chu theorem when collisions and thermal diffusion are absent.

When this more general approach is applied to the helicon-cyclotron modes discussed in item 2 it is found that the carrier waves must first be separated into their right- and left-hand circularly polarized modes. It is then self-consistently determined, by employing quasi-linear theory, that by proceeding to a full second-order analysis the source of power for the active mode is directly related to dc carrier slowing as opposed to the presence of any second-order RF electric field or gradient of the RF electric field. The general formulation for completeness is also readily applied to obtain the electrokinetic power flow for the hybrid carrier mode.

As a by-product, the quasi-linear theory applied to decaying electromagnetic waves is used in Appendix C to explain the experimental observation[83] of the induction of a dc voltage in the direction of the Poynting vector when helicons propagate through a solid-state plasma. This general phenomenon is termed the second-order Hall effect.

Associated with items 3 and 4, a kinetic-electromagnetic-electrostatic energy theorem is derived in Chapter III which shows that a general expression (W_k) for the carrier-mode electrokinetic energy density can be obtained from the requirement of conservation of energy. This plays a similar role for absolute instabilities, as does the carrier mode electrokinetic power flow (P_k) of Chapter II for convective instabilities. The energy theorem also explains the behavior of an unstable root in complex-k space as ω_i is varied to test for Briggs' causality criteria. The theorem demonstrates the fact that in this dynamic state as the wave blows up more quickly in time ($\omega_i \rightarrow -\infty$) it must correspondingly decrease its spatial growth rate ($k_i \rightarrow -\infty$); otherwise the energy is not conserved. Analysis of a particular dispersion relation can now be limited to those frequencies and wave numbers which provide negative electrokinetic power and energy modes.

In Chapter IV the quasi-linear theory is employed to determine the significance of the second-order products described in item 6 and shows that such products are directly related to the evolvement of the instability from the point and time of initiation. They are also generated by the distortion of the potential energy of the system due to the reactive effects of the growing RF fields upon the interacting carrier(s). The meaning of the quasi-linear theory is explained and it is shown that this is a valuable analytical technique for handling the difficult problem of nonlinear wave propagation in inhomogeneous media.

In connection with item 7 the electrokinetic power concept is extended to kinetic theory in Chapter V for the basic carrier modes and the major carrier distribution functions of interest. Comparison with the hydrodynamic theory of Chapter II shows that in many cases this hydrodynamic approach has been improperly used and the limitations on its applicability are pointed out. In particular, near the cyclotron resonance point, it is found that Maxwellian plasmas may exhibit a resonance-wave slowing in the presence of self-consistent spatially growing waves, even for small carrier number densities. This phenomenon is discussed in relation to the microwave emission observed from n-InSb in the configuration $\underline{k} \parallel \underline{B}_o$.

From a detailed analysis of wave propagation normal to the static magnetic field undertaken in Chapter VI, it is found that the harmonic nature of the microwave emission from n-InSb in the configuration where $\underline{k} \perp \underline{B}_o$ is in excellent agreement with computer results based on the electron-hole hybrid-hybrid interaction. In this regard, an examination of the effects of the variation of the carrier collision frequency with carrier speed indicates that the use of the constant collision frequency approximation is well justified. A rigorous study of carrier heating shows that an important quasi-static hybrid mode can arise for Maxwellian plasmas when carrier heating is included in the equilibrium dc state of the distribution function. This mode has a negligibly small effective collision frequency, exhibits synchronous behavior (e.g., $\omega \approx kv_o$ when $\underline{E}_o \perp \underline{k}$), and possesses a negative electrokinetic energy density independent of the sign of $(\omega - k_r v_o)$.

7.2 Recommendations for Further Study

From a practical viewpoint the present theoretical work is of most benefit in suggesting experimental studies related to the interaction of external radiation with matter as well as the generation of useful instabilities in solid-state plasmas.

In Appendix C it is illustrated that external electromagnetic radiation will produce a second-order dc electric field in the steady state in semiconductors due to the action of the Lorentz force. Thus it is expected that a pulse of radiation in passing through the material with exponential decay will generate a second-order unidirectional transient current.

In particular, assume now a material with some hysteresis in its dc I-V characteristic such that if the current exceeds some critical value, I_{crit}, the material switches to a second state which will persist when the current is removed. The point is now made that instead of supplying the current directly by attaching leads from the material to a current source, by the second-order Hall effect, this unidirectional current can be generated by external radiation. Indeed if the radiation has a sufficiently large power density the critical current, I_{crit}, can be attained. In addition, if the radiation is focused, the current (which is in the direction of wave propagation) will be localized if the pulse length is much less than the time characteristic of diffusion processes. Such localized switching action has been observed in thin-film amorphous semiconductors by employing focused pulsed laser radiation.[84]

In the work cited[84] the theory behind the observed effect is not clear and it is herein suggested that further study be given to the second-order Hall effect to determine if it is responsible. Such switching action is being studied for possible application to high-density memory storage systems.

The second-order Hall effect occurs in any medium with mobile charge carriers under the influence of radiation, with the strength of this effect proportional to the damping decrement k_i and the radiation power density. Thus it is suggested that with the present availability of high-power radiation sources (e.g., focused laser beam) important technologies can be derived wherein either the induced unidirectional current or unidirectional voltage (induced electrostatic effect) are used to advantage.

The study of Chapter IV utilizing the quasi-linear theory to study potential energy effects raises some important questions regarding the saturation length, maximum growth constant, etc., of an unstable system. As an example of this it can be asked by what means (e.g., doping) can these potential energy effects be used to enhance or quench the evolvement of the instability.

The correlation that was effected between the causality criteria of Briggs and the conservation of energy is worthy of further study. For example, to test a root with $k_r, k_i > 0$ (at $\omega_i = 0$) for convective instability Briggs' criteria[12] requires that $k_i \rightarrow -\infty$ as $\omega_i \rightarrow -\infty$, whereas that of Sturrock[13] requires only that $\omega_i < 0$ when $k_i = 0$. The conservation of energy approach, however, would appear to state that it is sufficient proof of causality if an increase in the total energy density is balanced by a decrease in the total power (e.g., if the stored energy in a closed volume increases, it must be balanced by less energy leaving that volume). This latter approach may lead to a simpler approach for assessing causality. This is particularly important with regard to the kinetic theory since the causality criteria of Briggs is often difficult to apply.

There are also fundamental problems raised between the concepts of causality and energy conservation. Thus, is there a one-to-one correspondence between all noncausal solutions and the violation of conservation of energy? Can it be shown that the advanced potential solutions of Maxwell's equations violate energy conservation?

Further study is also suggested to correctly analyze the residue associated with the cyclotron resonance when k is complex and ω is assumed real. The general problem of a residue is handled (even in collisionless plasmas) by introducing a small amount of damping so that the Landau pole, for example, is written as $\omega - kv_x \rightarrow \omega - kv_x - j\nu$. In this case for real k it is only with $\omega_i > 0$ (damped wave) that the pole can exist, as is well known. However, when the magnetic field is present and k is complex, the correct interpretation is not clear.

In Chapter VI it is felt that additional theoretical work is warranted in the study of carrier heating effects on the wave behavior. Although it has been commonly asserted that such effects are negligible if the drift velocity is small compared to the carrier thermal velocity, the present work indicates that this is not the case since an entirely new mode can appear corresponding to the extra degree of freedom in the system.

Finally, in the experimental study of instabilities in solid-state plasmas, the theoretical studies of Chapters V and VI suggest strongly that attempts be made to couple directly to the bulk of the material (as opposed to an antenna approach based on surface charge oscillations) by the use of high-dielectric constant end plates to match the waveguide field to the field in the bulk.

APPENDIX A. KINETIC POWER OF SPACE-CHARGE WAVES TO SECOND ORDER

It has been shown[69] that the kinetic power of the slow space-charge wave is dependent only on first-order variables. Since the presentation given therein is vague the theorem is rederived.

As Tonks[72] has shown, the kinetic power may be written in the present case as

$$P_k = \frac{v^2}{2\eta} J \quad , \tag{A.1}$$

since all variables are one dimensional. To second order, the velocity may be written as

$$v = v_o + V_1 \cos(\omega t - k_r x) + v_2(x) + V_2 \cos^2(\omega t - k_r x) \quad , \tag{A.2}$$

where V_1, $v_2(x)$, and V_2 are time-independent. V_1 varies as $\exp(k_i x)$ and the functions $v_2(x)$ and V_2 vary as $\exp(2k_i x)$. Similarly, the current density to second order may be written as

$$J = J_o + J_1 \cos(\omega t - k_r x) + J_2(x) + J_2' \cos 2(\omega t - k_r x) \quad , \tag{A.3}$$

where J_1, $J_2(x)$, and J_2' are also time-independent, J_1 varies as $\exp(k_i x)$ and the functions $J_2(x)$ and J_2' vary as $\exp(2k_i x)$. From the time-averaged continuity equation, when the current density is equated from outside the interaction region (J_o) to that within $[J_o + J_2(x)]$, it is seen that

$$J_2(x) = 0 \quad . \tag{A.4}$$

If all contributions are retained to second order,

$$Jv^2 = J_o v_o^2 + J_o V_1^2 \cos^2(\omega t - k_r x) + 2J_o v_o V_1 \cos(\omega t - k_r x) + 2J_o v_o v_2(x)$$

$$+ 2J_o v_o v_2 \cos 2(\omega t - k_r x) + J_1 v_o^2 \cos(\omega t - k_r x) + 2J_1 v_o V_1 \cos^2(\omega t - k_r x)$$

$$+ v_o^2 J_2' \cos 2(\omega t - k_r x) \quad . \qquad (A.5)$$

When the time average of both sides of Eq. A.5 is considered, the following is obtained:

$$\text{Time average } (Jv^2) = J_o v_o^2 + \frac{1}{2} J_o V_1^2 + 2J_o v_o v_2(x) + v_o V_1 J_1 \quad . \qquad (A.6)$$

MacColl[85] has performed a mean-value analysis applicable to the present case which shows that

$$v_2(x) = -\frac{1}{4} \frac{V_1^2}{v_o} , \qquad (A.7)$$

so that the second-order variable contribution in Eq. A.6 is annulled and the time average of Eq. A.1 becomes

$$\text{Time average } (P_k) = \frac{1}{2\eta} (J_o v_o^2 + v_o V_1 J_1) , \qquad (A.8)$$

verifying that it is sufficient to know the fundamental fields to determine the carrier mode kinetic power. In addition, the second-order longitudinal force equation is

$$\frac{\partial}{\partial t}\,[V_2 \cos 2(\omega t - k_r x)] + v_o \frac{\partial v_2(x)}{\partial x} + v_o \frac{\partial}{\partial x}\,[V_2 \cos 2(\omega t - k_r x)]$$

$$+ V_1 \cos(\omega t - k_r x) \frac{\partial}{\partial x}\,[V_1 \cos(\omega t - k_r x)] = \eta E_2 \quad . \qquad (A.9)$$

The differentiations having been performed and the time-average real part having been taken, Eq. A.7 then shows, as expected, that

$$\text{Time average } [\mathrm{Re}(E_2)] = 0 \quad . \qquad (A.10)$$

Equation A.2 shows that

$$\text{Time average } (v) = v_o + v_2(x) \quad . \qquad (A.11)$$

On the basis of Eqs. A.7 and 2.54, the kinetic power flow may be expressed directly in terms of the beam slowing with distance,

$$\mathrm{Re}(P_k) = - \frac{4 v_o J_o v_2(x) \omega (\omega - k_r v_o)}{\eta [(\omega - k_r v_o)^2 + k_i^2 v_o^2]} \quad . \qquad (A.12)$$

Similarly, from Eq. 2.58, the circuit power may be written as

$$P^T_{circuit} = \frac{4 \epsilon \omega_p^2 \omega v_o^2}{\eta^2 (\omega - k_r v_o)} \oint v_2(x)\, \hat{x} \cdot d\underline{S} \quad , \qquad (A.13)$$

which gives the power (e.g., in watts) directly in terms of the beam slowing.

APPENDIX B. VERIFICATION OF THE CYCLOTRON-MODE KINETIC POWER

Since a complex transformation has been used (Eq. 2.6) it should be verified that the power expressions obtained are physically valid. From Eqs. 2.46 and 2.47, invoking the exponential dependence $\exp[j(\omega t - kx)]$,

$$j(\omega - kv_o)v_{1y} - \omega_c v_{1z} = \eta\left(1 - \frac{kv_o}{\omega}\right)E_{1y} \tag{B.1}$$

and

$$j(\omega - kv_o)v_{1z} + \omega_c v_{1y} = \eta\left(1 - \frac{kv_o}{\omega}\right)E_{1z} \quad . \tag{B.2}$$

These equations having been used, the contribution to the source function is

$$E_{1y}J_{1y}^{*} + E_{1z}J_{1z}^{*} = \frac{\omega\rho_o}{\eta(\omega - kv_o)}\Big(j(\omega - kv_o)v_{1y}v_{1y}^{*} - \omega_c v_{1z}v_{1y}^{*}$$

$$+ j(\omega - kv_o)v_{1z}v_{1z}^{*} + \omega_c v_{1z}^{*}v_{1y}\Big) \quad . \tag{B.3}$$

Since the circularly polarized modes are uncoupled, one of these modes is selected, e.g., (-) mode, so that the following definitions can be made (in isotropic media):

$$v_{1y} = V_1 \exp[j(\omega t - kx)]$$

and

$$v_{1z} = V_1 \exp\left[j\left(\omega t - kx + \frac{\pi}{2}\right)\right] \quad , \tag{B.4}$$

where V_1 is independent of x. The use of these definitions in Eq. B.3 provides the following:

$$\mathrm{Re}(E_{1y}J_{1y}^{*} + E_{1z}J_{1z}^{*}) = \frac{2\omega\rho_o \, |V_1|^2 \, e^{2k_i x}}{\eta} \, \mathrm{Re}\left(\frac{j(\omega - kv_o - \omega_c)}{\omega - kv_o} \right) . \quad \text{(B.5)}$$

Use of Eq. B.4 in Eq. 2.48 shows that

$$v_{1-} = 2V_1 \, \exp[j(\omega t - kx)] . \quad \text{(B.6)}$$

From Eq. 2.7 the following result is readily derived:

$$\mathrm{Re}(E_{1-}J_{1-}^{*}) = \mathrm{Re}\left(\frac{j\omega(\omega - kv_o - \omega_c)}{\eta(\omega - kv_o)} \, \rho_o v_{1-} v_{1-}^{*} \right) , \quad \text{(B.7)}$$

which from Eq. B.6 may be written as

$$\mathrm{Re}(E_{1-}J_{1-}^{*}) = \frac{4\omega \, |V_1|^2 \, \rho_o \, e^{2k_i x}}{\eta} \, \mathrm{Re}\left(\frac{j(\omega - kv_o - \omega_c)}{\omega - kv_o} \right) . \quad \text{(B.8)}$$

Comparison of Eqs. B.5 and B.8 shows that

$$\mathrm{Re}(E_{1y}J_{1y}^{*} + E_{1z}J_{1z}^{*}) = \frac{1}{2} \, \mathrm{Re}(E_{1-}J_{1-}^{*}) , \quad \text{(B.9)}$$

which verifies the physical significance of power terms in the transformed system and explains the factor of one half in the Poynting expression of Eq. 2.22. Taking the real part in Eq. B.8 gives

$$\mathrm{Re}(E_{1-}J_{1-}^{*}) = \frac{\omega\rho_o k_i v_o \omega_c \, |v_{1-}|^2}{\eta[(\omega - k_r v_o)^2 + k_i^2 v_o^2]} , \quad \text{(B.10)}$$

which is to be used in the kinetic power discussion.

An alternative proof which brings into play the dispersion relation can be shown. For isotropic systems, regardless of the number of carrier species present or the form of the interacting circuit, it is in general true that the dispersion relation for the purely transverse interaction may be written as

$$\begin{bmatrix} a & jb \\ -jb & a \end{bmatrix} \begin{bmatrix} E_{1y} \\ E_{1z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} , \tag{B.11}$$

from which the following can be found:

$$\frac{E_{1y}}{E_{1z}} = \mp j \quad . \tag{B.12}$$

Equations B.1 and B.2 are solved for the RF velocities of the sth carrier species in terms of the fields as

$$v_{1y}^{(s)} = \frac{\eta_s(\omega - kv_{os})E_{1y}}{\omega[\omega_{cs}^2 - (\omega - kv_{os})^2]} [j(\omega - kv_{os}) \pm j\omega_{cs}] \tag{B.13}$$

and

$$v_{1z}^{(s)} = \frac{\eta_s(\omega - kv_{os})E_{1y}}{\omega[\omega_{cs}^2 - (\omega - kv_{os})^2]} [-\omega_{cs} \mp (\omega - kv_{os})] \quad , \tag{B.14}$$

where Eq. B.12 has been used. Equations B.13 and B.14 are then sufficient to obtain the following:

$$\mathrm{Re}(E_{1y}^{*}J_{1y} + E_{1z}^{*}J_{1z}) = \frac{2\omega_{ps}^2 \epsilon \, |E_{1y}|^2 \, k_i v_{os}}{\omega[(\omega - k_r v_{os} - \omega_{cs})^2 + k_i^2 v_{os}^2]} (\pm\omega_{cs}) \quad ; \tag{B.15}$$

this provides the same information as that of Eqs. B.9 and B.10.

If one returns to the (-) mode formulation these results indicate clearly the dynamical reasons why the modes with $\omega_c < 0$ (slow-cyclotron mode, helicon mode, etc.) have the negative kinetic power property under growth conditions ($k_i > 0$). Since there are no collisions in the present analysis, taking the real part of Eq. B.3 shows directly that the only power exchange between the fields and the carriers arises due to the current J_{1z} from the field E_{1y} and the current J_{1y} from the field E_{1z}. When $\omega_c < 0$ and $k_i > 0$, the viewpoint can be adopted that the field $E_{1y}(E_{1z})$ acting through the static magnetic field produces a contribution to $J_{1z}(J_{1y})$ which is 180 degrees out of phase with $E_{1z}(E_{1y})$. Self-consistently then, when $k_i > 0$ and $\omega_c < 0$, the carriers supply net power to the fields. It will be shown in Chapter II that this power is ultimately derived from the dc carrier motion. Note the important result that this phenomenon is not explicitly dependent upon the sign of $(\omega - k_r v_o)$.

APPENDIX C. THE MEASUREMENT OF POWER UTILIZING THE SECOND-ORDER HALL EFFECT

The nature of the second-order time-averaged dc electric field E_{2-} is studied. With reference to Fig. C.1, a circularly polarized electromagnetic waveguide mode propagates through an undrifted n-type semiconductor sample (e.g., n-InSb). From Eq. 2.26, since $v_o = 0$ and for the closed system with one carrier species $J_{2-} = 0$, it must be that for the cold plasma model

$$\underline{E}_{2-} = -\mathrm{Re}\left(\frac{1}{2j\omega} v_{1-} \frac{\partial E^*_{1-}}{\partial x} \right) \hat{x} \tag{C.1}$$

within the sample. Equation 2.7 becomes, when collisions are introduced into the analysis,

$$\eta E_{1-} = j(\omega - j\nu - \omega_c)\, v_{1-} \quad . \tag{C.2}$$

From Eq. C.2, the result may be found for Eq. C.1:

$$E_{2-} = -\frac{(\omega_c - \omega)k_i - \nu k_r}{2\eta\omega} |v_{1-}|^2 \quad . \tag{C.3}$$

The second-order field E_{2-}, although varying with distance in the direction of wave propagation, can be considered a second-order Hall electric field since it is set up in the same manner as the well known zeroth-order phenomenon. Indeed, this field has been studied[86] using linear polarization, no static magnetic field, and germanium samples to construct Hall-effect wattmeters for the measurement of RF power.

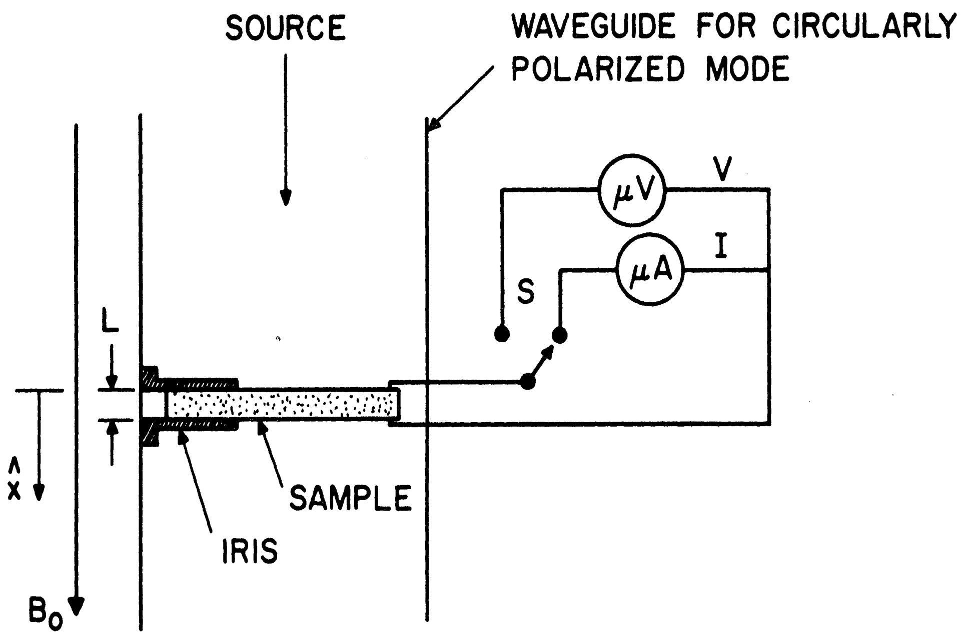


FIG. C.1 UTILIZATION OF THE SECOND-ORDER HALL EFFECT TO DETECT ELECTROMAGNETIC POWER OF CIRCULARLY POLARIZED WAVES.

Equation 2.19 in the presence of collisions with $v_o = 0$ gives

$$\mathrm{Re}(P_{e\ell-}) = \frac{k_r\,|v_{1-}|^2\,[(\omega_c - \omega)^2 + \nu^2]}{2\omega\mu_o\eta^2} \quad . \tag{C.4}$$

Using this result in Eq. C.3 gives

$$E_{2-}(x) = \frac{\eta\mu_o[(\omega_c - \omega)k_i - \nu k_r]}{k_r[(\omega_c - \omega)^2 + \nu^2]}\,\mathrm{Re}[P_{e\ell-}(x)] \quad . \tag{C.5}$$

Since

$$P_{e\ell-}(x) = P_{e\ell-}(0)\,e^{2k_i x} \quad , \tag{C.6}$$

where $P_{e\ell-}(0)$ is the electromagnetic power at the top surface of the sample and is the quantity to be measured (questions of reflected power from the front or back surfaces of the sample are ignored in this analysis), and $k_i < 0$ corresponding to a decaying wave. The voltage developed across the sample of length L (switch S in Fig. C.1 is set to the V branch) is

$$V = -\int_0^L E_{2-}\,dx = -\frac{\eta\mu_o[(\omega_c - \omega)k_i - k_r\nu]}{2k_i k_r[(\omega_c - \omega)^2 + \nu^2]}\left(1 - e^{2k_i L}\right)\mathrm{Re}[P_{e\ell-}(0)] \quad . \tag{C.7}$$

Alternatively, by moving the switch S to the I branch, the short-circuit current may be measured. Since $E_{2-} = 0$, when collisions are introduced into Eq. 2.26, the following results:

$$v_{2-} = \frac{1}{\nu}\,\eta\,\mathrm{Re}\left(\frac{1}{2j\omega}\,v_{1-}\,\frac{\partial E_{1-}^*}{\partial x}\right) \quad . \tag{C.8}$$

The approximate current drawn is

$$I_{2-} = \rho_o v_{2-}(0)A \quad , \tag{C.9}$$

where A is the device cross-sectional area. Use of Eqs. C.5 and C.8 in C.9 shows the following:

$$I_{2-} = \frac{\eta\omega_p^2 A[(\omega_c - \omega)k_i - \nu k_r]}{\nu c^2 k_r[(\omega_c - \omega)^2 + \nu^2]} \mathrm{Re}[P_{e\ell-}(0)] \quad . \tag{C.10}$$

As an example, exactly at resonance ($\omega = \omega_c$), of the right-hand polarized mode (B_o is then $-\hat{x}$ directed for n-type material), the current is

$$I_{2-} = -\frac{\eta\omega_p^2 A}{\nu^2 c^2} \mathrm{Re}[P_{e\ell-}(0)] \quad . \tag{C.11}$$

For example in indium antimonide at liquid nitrogen temperature when typical values of a moderately pure sample[87] are used, a responsivity may be found as

$$R \triangleq \frac{I_{2-}}{\mathrm{Re}[P_{e\ell-}(0)]} \cong 20 \left(\frac{\mathrm{mA}}{\mathrm{W/cm^2}} \right) \quad . \tag{C.12}$$

The absolute power density of the incident radiation may then be measured using this technique. In addition since the current should be a maximum at $\omega_c = \omega$, as the magnetic field is tuned, the frequency of the radiation is also measured. Because of the tuning properties of the magnetic field the device is inherently wideband in nature. Although only a local theory is presented here, it may be possible to determine carrier effective masses of materials of poor transmissivity (e.g., semimetals or metals) by utilization of this technique, wherein the unknown sample becomes the detector itself and the operating frequency ω is known.

It is to be noted that many processes are known to possibly occur which should be taken into account in a more detailed analysis. Thus the second-order Hall current or voltage may be obscured by thermoelectric effects at the metal-semiconductor junctions, generation and consequent diffusion of electron-hole pairs if the incident photon energy exceeds the bandgap energy, hot-electron effects, rectification at the contacts, etc.

For the warm plasma model a useful resonance is found which can enhance the voltage across the sample. The force equation in the one carrier case is given by

$$\frac{v_T^2}{\rho_o} \frac{\partial \rho_{2-}}{\partial x} = \eta E_{2-} + \eta \, \mathrm{Re}\left(\frac{1}{2j\omega} v_{1-} \frac{\partial E_{1-}^*}{\partial x} \right) , \qquad (C.13)$$

where

$$\epsilon \frac{\partial E_{2-}}{\partial x} = \rho_{2-} \quad . \qquad (C.14)$$

Since the second-order variables vary as $e^{2k_i x}$, Eqs. C.13 and C.14 can be solved similar to Eq. C.3 to obtain

$$E_{2-} = - \frac{(\omega_c - \omega)k_i - \nu k_r}{2\eta\omega\left(1 - \dfrac{4k_i^2 v_T^2}{\omega_p^2}\right)} |v_{1-}|^2 \quad . \qquad (C.15)$$

If Eqs. C.4 and C.6 are used, the open-circuit voltage is found as

$$V = -\frac{\eta\mu_o}{2k_i k_r} \frac{[(\omega_c - \omega)k_i - \nu k_r][1 - e^{2k_i L}]}{[(\omega_c - \omega)^2 + \nu^2]\left(1 - \frac{4k_i^2 v_T^2}{\omega_p^2}\right)} \mathrm{Re}[P_{e\ell-}(0)] \quad . \qquad (C.16)$$

Implicit in solving for E_{2-} in Eq. C.14 is the assumption that the surface recombination velocity of the top and bottom surfaces is zero. Inspection of Eq. C.16 shows that by operating with $k_i \simeq \omega_p/2v_T$ a resonance is possible leading to large voltages, with the peak possible voltage limited by the assumption $\rho_{2-}(0) \ll \rho_o$ inherent in the analysis. Within a restricted frequency range it should always be possible by variation of ω_c, ω_p, or ν to obtain $k_i \simeq -\omega_p/2v_T$. As an example of this consider the detection of the 10-μ line of the CO_2 laser, so that $\omega \simeq 2 \times 10^{14}$ rad/s. At such high frequencies it can be assumed that $\omega_c \ll \omega$, and consequently the dispersion relation is

$$k^2c^2 - \omega^2 + \frac{\omega_p^2\omega(\omega + j\nu)}{\omega^2 + \nu^2} = 0 \qquad (C.17)$$

or

$$k = \pm \frac{\omega}{c}\left(1 - \frac{\omega_p^2(\omega + j\nu)}{\omega(\omega^2 + \nu^2)}\right)^{1/2} \quad . \qquad (C.18)$$

From Eq. C.18, to satisfy the resonance condition $k_i = \omega_p/2v_T$, it is required that

$$\frac{\omega}{c}\left[\left(1 - \frac{\omega_p^2}{\omega^2 + \nu^2}\right)^2 + \frac{\nu^2\omega_p^4}{\omega^2(\omega^2 + \nu^2)^2}\right]^{1/4} \sin\left[\frac{1}{2}\tan^{-1}\left(-\frac{\nu\omega_p^2}{\omega(\omega^2 + \nu^2) - \omega\omega_p^2}\right)\right] = \frac{\omega_p}{2v_T} \quad , \qquad (C.19)$$

where $\omega^2 + \nu^2 > \omega_p^2$ has been assumed. This equation can be readily satisfied with materials with $\nu \simeq \omega$, corresponding to a low mobility of the free carriers. In addition, since $\omega \sim k_r c$, the condition of locality, i.e., $|\omega - j\nu - \omega_c| \gg |kv_T|$, is well satisfied so that the use of the dispersion equation in the form of Eq. C.18 is justified.

APPENDIX D. A STUDY OF DECAYING MODES BY KINETIC POWER CONCEPTS

For $k_i \neq 0$, it is in general possible to cast many carrier interactions in the form

$$\mathrm{Re} \oint (\underline{P}_{circuit} + \underline{P}_k) \cdot d\underline{s} = 0 \quad , \tag{D.1}$$

where $P_{circuit}$ is always positive ($k_r > 0$) or zero and P_k is the kinetic power flow. In general it can be assumed that the power flows are purely longitudinal for the purposes of analysis (e.g., $\hat{x}$-directed) so that Eq. D.1 may be written for real ω as

$$2k_i(P_{circuit} + P_k) = 0 \quad . \tag{D.2}$$

Hence even for decaying interactions ($k_i < 0$) the function $P_k \lesseqq 0$.

As an example, utilizing this, consider the kinetic power flow of the longitudinal space-charge wave, Eq. 2.23, with drift velocity $v_o = 0$,

$$P_k = \frac{\omega_p^2 \epsilon \omega \left[\dfrac{\omega \nu}{2k_i} + k_r v_T^2 \right]}{[\omega^2 - v_T^2(k_r^2 - k_i^2)]^2 + [\omega\nu + 2k_i k_r v_T^2]^2} \quad . \tag{D.3}$$

Inspection shows that the collisions provide $P_k < 0$ with $k_i < 0$ and that the thermal diffusion limits the possible damping to $|k_i| < \omega\nu/2k_r v_T^2$. For purely electrostatic interactions $P_{circuit}$ is zero and hence from Eq. D.2, $P_k = 0$ so that

$$k_i = -\frac{\omega\nu}{2k_r v_T^2} , \tag{D.4}$$

from which it is seen that the general effects of diffusion are to reduce the damping.

Hence, in the definition of kinetic power flow, if $k_i \neq 0$, the kinetic power flow is always negative for the interaction to proceed and in the presence of collisions knowledge of the sign of k_i is required to assess the meaning of the function P_k.

APPENDIX E. ELECTROKINETIC ENERGY DENSITY OF SPACE-CHARGE WAVES TO SECOND ORDER

The kinetic energy properties of longitudinal space-charge waves to second order can be derived in a similar manner to the kinetic power derivation in Appendix A.

Consider first the case in which it can be assumed that k is purely real. Following Tonks,[72] the carrier mode kinetic energy density may be defined as

$$W_k = \frac{1}{2\eta} \rho \underline{v} \cdot \underline{v} \quad . \tag{E.1}$$

Since all variables are one dimensional the velocity to second order can be written as

$$v = v_o + V_1' \exp j(\omega t - kx) + v_2(t) + V_2' \exp 2j(\omega t - kx) \quad , \tag{E.2}$$

where v_o, V_1', $v_2(t)$, and V_2' are independent of x. Similarly for the charge density,

$$\rho = \rho_o + \rho_1' \exp j(\omega t - kx) + \rho_2(t) + \rho_2' \exp 2j(\omega t - kx) \quad , \tag{E.3}$$

where ρ_o, ρ_1', $\rho_2(t)$ and ρ_2' are independent of x. Physical reasoning shows that for the charge to be conserved properly, in the interaction region

$$\rho_2(t) = 0 \quad . \tag{E.4}$$

If all products are retained to second order, Eqs. E.2, E.3, and E.4 provide in the interaction region,

$$\mathrm{Re}(\rho\underline{v} \cdot \underline{v}) = \rho_o v_o^2 + 2\rho_o v_o v_2(t) + 2\rho_o v_o V_1' \cos(\omega_r t - kx)\, e^{-\omega_i t}$$

$$+ 2\rho_o v_o V_2' \, e^{-2\omega_i t} \cos(2\omega_r t - 2kx) + \rho_o (V_1')^2 \, e^{-2\omega_i t} \cos^2(\omega_r t - kx)$$

$$+ \rho_1' v_o^2 \, e^{-\omega_i t} \cos(\omega_r t - kx) + 2v_o V_1' \rho_1' \, e^{-2\omega_i t} \cos^2(\omega_r t - kx)$$

$$+ \rho_2' v_o^2 \, e^{-2\omega_i t} \cos(2\omega_r t - 2kx) \ . \quad \text{(E.5)}$$

From Eq. E.5 the function

$$\langle W_k \rangle = \frac{1}{2\eta} \langle \rho\underline{v} \cdot \underline{v} \rangle = \frac{1}{2\eta L} \int_0^L \rho\underline{v} \cdot \underline{v} \, dx \qquad \text{(E.6)}$$

can be obtained, which is a space average of the total carrier-mode kinetic energy density over the device length L. If $L = N(\lambda/2)$, where N is any positive integer greater than zero,

$$2\eta \langle W_k \rangle = \rho_o v_o^2 + \frac{1}{2} \rho_o (V_1')^2 \, e^{-2\omega_i t} + 2\rho_o v_o v_2(t) + v_o \rho_1' V_1' \, e^{-2\omega_i t} \ . \quad \text{(E.7)}$$

Note that for $L \neq N(\lambda/2)$, Eq. E.7 is still an excellent approximation if $L \gg \lambda$. As in Appendix A, MacColl's results[69,85] when interpreted as a space average provide for the mean-square velocity in the interaction region,

$$\langle v^2 \rangle = v_o^2 \ . \qquad \text{(E.8)}$$

Use of this result, together with the squared and space-averaged value from Eq. E.2, gives

$$v_o v_2(t) = -\frac{1}{4} (V_1')^2 e^{-2\omega_i t} . \qquad (E.9)$$

Thus Eq. E.9 can be used in Eq. E.7 to obtain

$$\langle W_k \rangle = \frac{1}{2\eta} \rho_o v_o^2 + \frac{1}{2\eta} v_o \rho_1' V_1' e^{-2\omega_i t} , \qquad (E.10)$$

which shows that for the purely longitudinal space-charge modes it is sufficient to know the first-order fields alone to determine the kinetic energy density properties. Comparison of Eq. E.10 with Eq. A.8 shows that bunching plays the same role in absolute as in convective instabilities.

In the more general case with $k_i \neq 0$, the same space-averaging techniques may be used. Whereas the case $k_i = 0$ provided a minimum device length $L = \lambda/2$, when $k_i \neq 0$, this minimum length becomes a function of k_i. The spatially growing nature of the carrier mode does not appear to play any fundamental role in the oscillation, however.[65]

As in the developments of Eqs. A.11 and A.12 it is possible using Eq. E.9 to express the kinetic energy density function $\langle W_k \rangle$ directly in terms of the beam slowing.

APPENDIX F. ELECTROKINETIC ENERGY DENSITY OF CYCLOTRON MODES

From Eq. 3.14 the fundamental field source function may be written as

$$\mathrm{Re}(E_{1-}J^*_{1-}) = \mathrm{Re}\left(\frac{j\omega\rho_o(\omega-kv_o-\omega_c)v_{1-}v^*_{1-}}{\eta(\omega-kv_o)} \right) , \tag{F.1}$$

so that, in general,

$$\mathrm{Re}(E_{1-}J^*_{1-}) = \frac{\rho_o|v_{1-}|^2}{\eta} \; \frac{-\omega_i[(\omega_r-k_rv_o)^2+\omega_ck_rv_o+(\omega_i-k_iv_o)^2]+k_iv_o\omega_r\omega_c}{(\omega_r-k_rv_o)^2+(\omega_i-k_iv_o)^2} . \tag{F.2}$$

From Eq. 3.18 and with the dependence assumed,

$$W_k = \frac{\rho_o|v_{1-}|^2}{4\eta} \; \frac{[(\omega_r-k_rv_o)^2+\omega_ck_rv_o+(\omega_i-k_iv_o)^2] - \frac{k_iv_o}{\omega_i}\omega_r\omega_c}{(\omega_r-k_rv_o)^2+(\omega_i-k_iv_o)^2} . \tag{F.3}$$

Thus in the case $k_i = 0$,

$$W_k(k_i = 0) = \frac{\rho_o|v_{1-}|^2}{4\eta} \; \frac{(\omega_r-k_rv_o)^2+\omega_ck_rv_o+\omega_i^2}{(\omega_r-k_rv_o)^2+\omega_i^2} , \tag{F.4}$$

so that in this case, once again, only the polarization $\omega_c < 0$ corresponding to the slow-cyclotron mode, helicon mode, etc., gives a negative contribution to the carrier mode kinetic energy and synchronism is preferred to optimize the magnitude of the electrokinetic energy density.

APPENDIX G. EFFECT OF CARRIER HEATING TRANSVERSE TO $\underline{k}$ ON THE HYBRID MODE

For completeness in the study of the quasi-static hybrid mode the RF number density is found for a carrier which is part of an equilibrium Maxwellian distribution for the case $\underline{E}_o = E_{oy}\,\hat{y}$, $\underline{B}_o = B_o\,\hat{z}$, and $\underline{k} = k\,\hat{x}$. The carrier distribution function which includes carrier heating in the presence of the dc fields was found in Eq. 6.41 as

$$f_o = \frac{N_o}{(2\pi v_T^2)^{3/2}} \exp\left(-\frac{u^2 + v_H^2 + v_z^2}{2v_T^2}\right) \sum_{n=-\infty}^{\infty} I_n\left(-\frac{uv_H}{v_T^2}\right) e^{jn\theta}$$

$$\cdot \left[1 - \frac{uv_H}{v_T^2}\, \frac{(jn - \nu/\omega_c)\sin\theta - \cos\theta}{1 + (jn - \nu/\omega_c)^2}\right], \qquad (G.1)$$

where $v_H = E_{oy}/B_o$, $v_x = v_H + u\cos\theta$, and $v_y = u\sin\theta$. The method used then follows exactly that used for the case $\underline{E}_o = E_{ox}\,\hat{x}$ resulting in

$$N_1 = \frac{N}{D}, \qquad (G.2)$$

where

$$D = 1 + j\frac{\nu}{v_T^2} e^{-v_H^2/v_T^2} \sum_{\ell,n=-\infty}^{\infty} \int_0^{\infty} \frac{e^{-u^2/2v_T^2} J_\ell\left(\frac{ku}{\omega_c}\right) I_n\left(-\frac{uv_H}{v_T^2}\right) u}{\omega - kv_H - j\nu - \ell\omega_c}$$

$$\cdot \left[J_{\ell-n}\left(\frac{ku}{\omega_c}\right) + \frac{juv_H\left[2(jn - \nu/\omega_c)J'_{\ell-n}\left(\frac{ku}{\omega_c}\right) - j\,\frac{2(\ell - n)\omega_c}{ku}\,J_{\ell-n}\left(\frac{ku}{\omega_c}\right)\right]}{2v_T^2[1 + (jn - \nu/\omega_c)^2]}\right] du$$

(G.3)

and

$$N = - \frac{\eta E_{1x} N_o e^{-v_H^2/2v_T^2}}{v_T^2} \sum_{\ell,n=-\infty}^{\infty} \sum_{i=1}^{4} \int_0^{\infty} \frac{e^{-u^2/2v_T^2} J_\ell\left(\frac{ku}{\omega_c}\right) \mathcal{E}_i(\ell,n)}{\omega - kv_H - j\nu - \ell\omega_c} u \, du \quad , \tag{G.4}$$

where now, with the argument of I_n always understood as $(-uv_H/v_T^2)$,

$$\mathcal{E}_1(\ell,n) = \frac{1}{2j}\left[\frac{dI_n}{du} - \left(\frac{u}{v_T^2} + \frac{n}{u}\right) I_n\right] J_{\ell-n-1}\left(\frac{ku}{\omega_c}\right) ,$$

$$\mathcal{E}_2(\ell,n) = \frac{1}{2j}\left[\frac{dI_n}{du} - \left(\frac{u}{v_T^2} - \frac{n}{u}\right) I_n\right] J_{\ell-n+1}\left(\frac{ku}{\omega_c}\right) ,$$

$$\mathcal{E}_3(\ell,n) = \frac{v_H}{4v_T^2[1 + (jn - \nu/\omega_c)^2]} \left\{u \frac{dI_n}{du}\left(j(n-1) - \frac{\nu}{\omega_c}\right) + I_n\left[\frac{n\nu}{\omega_c}\right.\right.$$

$$\left.\left. - \left(jn - j - \frac{\nu}{\omega_c}\right)\frac{u^2}{v_T^2}\right]\right\} \cdot \left[J_{\ell-n-2}\left(\frac{ku}{\omega_c}\right) + J_{\ell-n+2}\left(\frac{ku}{\omega_c}\right)\right]$$

and

$$\mathcal{E}_4(\ell,n) = \frac{v_H}{v_T^2[1 + (jn - \nu/\omega_c)^2]} \left\{- j \frac{u}{2} \frac{dI_n}{du} + I_n\left[j\left(\frac{u^2}{2v_T^2} - 1\right)\right.\right.$$

$$\left.\left. + \frac{n}{2}\left(jn - \frac{\nu}{\omega_c}\right)\right]\right\} J_{\ell-n}\left(\frac{ku}{\omega_c}\right) . \tag{G.5}$$

The most important aspect of the carrier heating is that the interacting $\ell = 0$ mode can now appear. This mode is of interest because $\omega \approx kv_H$ and, since

$v_H \approx v_o$ in high mobility materials, interactions can occur at much smaller wave numbers than the resonant interactions previously studied (e.g., $k \approx |\omega_{ce}|/v_{oe}$). This carrier-heated mode has the additional attribute for instability that the effect of the collisions is negligible if $|\omega_c| \gg \nu$. To understand this, assume that $|\omega_c| \gg \nu$ so that the resonance approximation is reasonable and in particular for $\ell = 0$ the effective collision frequency is given from Eq. G.3 by

$$\nu' = \nu \Bigg\{ 1 - \frac{e^{-v_H^2/2v_T^2}}{v_T^2} \sum_{n=-\infty}^{\infty} \int_0^\infty e^{-u^2/2v_T^2} J_o\left(\frac{ku}{\omega_c}\right) I_n\left(-\frac{uv_H}{v_T^2}\right) u\,du$$

$$\cdot \left[J_{-n}\left(\frac{ku}{\omega_c}\right) + \frac{juv_H\left[2(jn - \nu/\omega_c)J'_{-n}\left(\frac{ku}{\omega_c}\right) + 2j\frac{n\omega_c}{ku} J_{-n}\left(\frac{ku}{\omega_c}\right)\right]}{2v_T^2[1 + (jn - \nu/\omega_c)^2]} \right] \Bigg\} . \tag{G.6}$$

Inspection of this result shows that the n = 0 and n = ±1 terms give the largest contributions in the sum resulting in

$$\mathrm{Re}(\nu') \approx \nu \left[1 - \frac{v_H\, e^{-v_H^2/2v_T^2}}{2v_T^4} \int_0^\infty e^{-u^2/2v_T^2} J_o^2\left(\frac{ku}{\omega_c}\right) I_1\left(\frac{uv_H}{v_T^2}\right) u^2\,du \right.$$

$$\left. - \frac{e^{-v_H^2/2v_T^2}}{v_T^2} \int_0^\infty e^{-u^2/2v_T^2} J_o^2\left(\frac{ku}{\omega_c}\right) I_o\left(-\frac{uv_H}{v_T^2}\right) u\,du \right] . \tag{G.7}$$

For a limiting case $v_H \to 0$ (which corresponds to the disappearance of the $\ell = 0$ term in N) it can be found from Eq. G.7 that $\mathrm{Re}(\nu') \to 0$. Hence for $v_H \lesssim v_T$ it is clear from the behavior of the Bessel functions involved that for this more general case $Re(\nu') \ll \nu$. The possibility that $Re(\nu') < 0$ for some range of (v_H/v_T) is also noted. In addition inspection of the $\mathcal{E}_i(\ell,n)$ function in Eq. G.5 shows that if $|\omega_c| \gg \nu$ the collision frequency ν should play an unimportant role in N. Thus for the $\ell = 0$ mode with $|\omega_c| \gg \nu$ the deleterious collisional effects are negligible and instability is to be expected. In the constant collision frequency approximation it is expected that these results are unaltered when an increased value of ν is selected, corresponding to the carrier heating, provided $|\omega_c| \gg \nu$ is maintained.

The physical reasoning for the appearance of the (interacting) $\ell = 0$ mode due to carrier heating is now discussed in relation to the electron-hole interaction. From Eq. 6.118, when the effects of carrier heating are not included in the analysis, the dispersion equation in the resonance approximation for the electron-hole interaction is

$$1 + \frac{\ell\omega_{ce}\omega_{pe}^2 \, e^{-\lambda_e} I_\ell(\lambda_e)}{k^2 v_{Te}^2[\ell\omega_{ce} - (\omega - kv_{oe} - j\nu_e')]} + \frac{m\omega_{ch}\omega_{ph}^2 \, e^{-\lambda_h} I_m(\lambda_h)}{k^2 v_{Th}^2[m\omega_c h - (\omega - kv_{oh} - j\nu_h')]} = 0 \; . \tag{G.8}$$

If this equation is solved for $\ell = 0$, it is readily seen that one root is given by $\omega - kv_{oe} - j\nu_e' = 0$. This is a damped noninteracting electron carrier wave given by Eq. 6.119 as

$$\omega_r = kv_{oe}$$

and

$$\omega_i = \nu_e' = \nu_e[1 - e^{-\lambda_e} I_o(\lambda_e)] \; , \tag{G.9}$$

so that as $\lambda_e \to 0$ (i.e., $B_o \to \infty$), $\omega_i \to 0$.

On the other hand the dispersion equation in the resonance approximation, for $\ell = 0$, with the effects of carrier heating included, can be obtained from Eq. G.2, together with Poisson's equation (Eq. 6.67), as

$$1 + \frac{j\omega_{pe}^2 \exp\left(-\frac{v_H^2}{2v_{Te}^2}\right)}{kv_{Te}^2(\omega - kv_H - j\nu_e')} \sum_{n=-\infty}^{\infty} \sum_{i=1}^{4} \int_0^\infty \exp\left(-\frac{u^2}{2v_{Te}^2}\right) J_o\left(\frac{ku}{\omega_{ce}}\right) \mathcal{E}_i(0,n)\, u\, du + [h] = 0 \quad , \qquad (G.10)$$

wherein [h] represents the hole term which is not of direct significance. The effect of the carrier heating then is to perturb the $\ell = 0$ electron carrier wave away from the solution given by Eq. G.9 since it is now an <u>interacting</u> mode. Since the solution of Eq. G.9 shows a mode approaching marginal instability as the magnetic field increases, this perturbation need not be large to drive this root into instability. Note also that the adopted viewpoint[48] in which the growth rate must exceed the hole collision frequency (when the dispersion equation is solved with $\nu_h = 0$) is erroneous when a static magnetic field is present.

The important effect of the carrier heating of the electrons then is that an extra degree of freedom of the system is permissible which enables the electron carrier wave to interact. From a quantum viewpoint, it is expected that this effect corresponds to the introduction of additional available energy states for the system enabling more general sets of motion by which the electrons can interact with the holes.

It is again pointed out that because of the strong dependence of the coupling constant on the wave number (e.g., Eq. 6.79) for this interaction in general, the carrier heated $\ell = 0$ mode interaction (with $k \simeq \omega_r/v_{oe}$) can readily dominate over the fundamental $\ell = 1$ mode interaction [with $k \simeq (\omega + |\omega_{ce}|)/v_{oe}$] when $\omega \ll |\omega_{ce}|$.

It is hypothesized that the carrier-heated electron carrier wave is potentially the most important carrier mode in solid-state plasmas. This is because not only does it possess a small damping decrement in the noninteracting state, but also, unlike the helicon mode, it is not severely frequency limited.

APPENDIX H. EFFECTS OF CARRIER HEATING ON THE CYCLOTRON MODES

The cyclotron modes ($\underline{k} \parallel \underline{B}_o \parallel \hat{x}$) were studied in Chapter V, Section 5.3 neglecting carrier heating. If $\underline{E}_o \perp \underline{k}$ it can be found by including $\underline{E}_o$ in Eqs. 5.41 through 5.44 that the right- and left-hand circularly polarized modes are now coupled. This case is difficult to analyze since there are then at least four RF current components, e.g., $J_{1y}(E_{1y})$, $J_{1y}(E_{1z})$, $J_{1z}(E_{1y})$, and $J_{1z}(E_{1z})$. Attention is therefore directed at the case $\underline{E}_o \parallel \underline{k}$. By including this field in Eqs. 5.41 through 5.52 the circularly polarized components still separate and now

$$\frac{\partial f_{1-}}{\partial v_x} + \frac{j(\omega - kv_x - \omega_c - j\nu)}{\eta E_{ox}} f_{1-} = \left(\frac{E_{1-}}{E_{ox}}\right)\left[F_o(v_x)\left(1 - \frac{kv_x}{\omega}\right) - \frac{k}{\omega}\frac{dG_o}{dv_x}\right] , \tag{H.1}$$

where again

$$F_o(v_x) = \iint_{-\infty}^{\infty} f_o \, dv_y \, dv_z \tag{H.2}$$

and

$$G_o(v_x) = \frac{1}{2} \iint_{-\infty}^{\infty} (v_y^2 + v_z^2) f_o \, dv_y \, dv_z \quad . \tag{H.3}$$

For self-consistency Eqs. H.2 and H.3 should be solved for the carrier distribution function f_o which includes any carrier effects due to E_{ox}. This is done as follows. Since $\underline{E}_o \parallel \underline{B}_o \parallel \hat{x}$, the dc equation,

$$\eta \underline{E}_o \cdot \left[\frac{\partial f_{oL}}{\partial \underline{v}} + \frac{\partial f_{1L}}{\partial \underline{v}} \right] + \eta(\underline{v} \times \underline{B}_o) \cdot \frac{\partial f_{1L}}{\partial \underline{v}} + \nu f_{1L} = 0 \ , \tag{H.4}$$

where $f_o = f_{oL} + f_{1L}$ and f_{oL} is the known equilibrium carrier distribution function in the absence of any applied fields, becomes

$$\frac{\partial f_{1L}}{\partial v_x} + \left(\frac{B_o}{E_{ox}} \right) \frac{\partial f_{1L}}{\partial \theta} = - \frac{\partial f_{oL}}{\partial v_x} - \frac{\nu}{\eta E_{ox}} f_{1L} \ , \tag{H.5}$$

where the definitions were made

$$v_y = u \sin \theta \ , \qquad v_z = u \cos \theta \ . \tag{H.6}$$

Then the functions $F_o(v_x)$ and $G_o(v_x)$ can be split up as

$$F_o(v_x) = F_{oL}(v_x) + F_{1L}(v_x) = \iint_{-\infty}^{\infty} f_{oL} \, dv_y \, dv_z + \iint_{-\infty}^{\infty} f_{1L} \, dv_y \, dv_z \tag{H.7}$$

and

$$G_o(v_x) = G_{oL}(v_x) + G_{1L}(v_x) = \frac{1}{2} \int_0^{\infty} \int_0^{2\pi} u^3 f_{oL} \, du \, d\theta + \frac{1}{2} \int_0^{\infty} \int_0^{2\pi} u^3 f_{1L} \, du \, d\theta \ . \tag{H.8}$$

By integrating Eq. H.5 over (v_y, v_z) space it can be found that

$$\frac{\partial F_{1L}}{\partial v_x} + \frac{\nu}{\eta E_{ox}} F_{1L} = - \frac{\partial F_{oL}}{\partial v_x} \ , \tag{H.9}$$

wherein use was made of $\int_0^{2\pi}(\partial f_{1L}/\partial\theta)d\theta = 0$. In a similar fashion premultiplying by u^2 and integrating over (v_y, v_z) space Eq. H.5 generates

$$\frac{\partial G_{1L}}{\partial v_x} + \frac{\nu}{\eta E_{ox}} G_{1L} = -\frac{\partial G_{oL}}{\partial v_x} \tag{H.10}$$

In principle then since f_{oL} is known (and hence F_{oL} and G_{oL}) Eqs. H.9 and H.10 can be solved to obtain $F_o(v_x)$ and $G_o(v_x)$. For example, if the function f_{oL} is selected as a Maxwellian,

$$f_{oL} = \frac{N_o}{(2\pi v_T^2)^{3/2}} \exp\left(-\frac{v_x^2 + v_y^2 + v_z^2}{2v_T^2}\right) , \tag{H.11}$$

it can be found by proceeding in the aforementioned manner that

$$F_o(v_x) = \frac{\nu N_o}{2\eta E_{ox}} \exp\left(-\frac{\nu v_x}{\eta E_{ox}}\right) \exp\left(\frac{\nu^2 v_T^2}{2\eta^2 E_{ox}^2}\right) \cdot \left[\text{erf}\left(\frac{v_x}{\sqrt{2}\, v_T} - \frac{\nu v_T}{\sqrt{2}\,\eta E_{ox}}\right) + C_1\right] , \tag{H.12}$$

where erf(x) is the error function of argument x and where C_1 is a constant which can be determined by the conservation of particles requirement $\int_{-\infty}^{\infty} F_o(v_x)dv_x = N_o$. The point is clear, however, that the use of a complex function such as Eq. H.12 in Eq. H.1 will lead to nontractable integrals in attempting to solve for f_{1-}. For the purpose of analysis assume then that the system is such that the drifted Maxwellian carrier distribution function,

$$f_o = \frac{N_o}{(2\pi v_T^2)^{3/2}} \exp\left[-\frac{(v_x - v_o)^2}{2v_T^2}\right] \exp\left[-\frac{u^2}{2v_T^2}\right] , \qquad (H.13)$$

is appropriate even if $E_{ox} > B_o v_T$. Equations 5.91 and 5.92 can then be used in Eq. H.1 to obtain for this case

$$\frac{\partial f_{1-}}{\partial v_x} + jaf_{1-} = \frac{E_{1-} N_o \left(1 - \frac{kv_o}{\omega}\right)}{E_{ox} \sqrt{2\pi}\, v_T} \exp\left[-\frac{(v_x - v_o)^2}{2v_T^2}\right] , \qquad (H.14)$$

where

$$a = \frac{\omega - kv_x - \omega_c - j\nu}{\eta E_{ox}} . \qquad (H.15)$$

When the integrating factor $\exp[\int a(v_x)\, dv_x]$ is noted, Eq. H.14 is solved as

$$f_{1-} = \frac{E_{1-} N_o \left(1 - \frac{kv_o}{\omega}\right)}{E_{ox} \sqrt{2\pi}\, v_T} \exp\left(-j(a_1 v_x + a_2 v_x^2)\right)$$

$$\cdot \int_{C_2}^{v_x} e^{ja_1 v_x'} e^{ja_2 (v_x')^2} \exp\left(-\frac{(v_x' - v_o)^2}{2v_T^2}\right) dv_x' , \qquad (H.16)$$

where

$$a_1 = \frac{\omega - \omega_c - j\nu}{\eta E_{ox}} \quad \text{and} \quad a_2 = -\frac{k}{2\eta E_{ox}} , \qquad (H.17)$$

and C_2 is a constant to be determined. Equation H.16 can be integrated directly to obtain

$$f_{1^-} = \frac{E_{1^-} N_o \left(1 - \frac{kv_o}{\omega}\right)}{E_{ox} \sqrt{2\pi}\, v_T} \exp\left(-j(a_1 v_x + a_2 v_x^2)\right) \left[\gamma_1 \operatorname{erf}\left[\gamma_2 v_x - \frac{\frac{v_o}{v_T^2} + ja_1}{\gamma_2} \right] + C_3 \right] , \tag{H.18}$$

where

$$\gamma_1 = \frac{\sqrt{\pi}}{2\gamma_2} \exp\left[\frac{\left(\frac{v_o}{v_T^2} + ja_1\right)^2 - \gamma_2^2 \frac{v_o^2}{2v_T^2}}{\gamma_2^2} \right] ,$$

$$\gamma_2 = \left(\frac{1}{2v_T^2} - ja_2 \right)^{1/2}$$

and C_3 is a new constant replacing C_2. Since it is necessary that $f_{1^-} \to 0$ as $v_x \to \pm\infty$ the function f_{1^-} exhibits a Stokes phenomenon characteristic of the asymptotic expansion of analytic functions, with the result that

$$f_{1^-} = f_{1^-}(v_x > 0) + f_{1^-}(v_x < 0) , \tag{H.19}$$

in which the constant C_3 has one value for $v_x > 0$ and another value for $v_x < 0$ such that in either half-space $f_{1^-} \to 0$ as $|v_x| \to \infty$. It is in this manner that the constant C_3 is determined in each half-space. The RF current density is then obtained directly from $J_{1^-} = q \int_{-\infty}^{\infty} f_{1^-}\, dv_x$. Even for the result for f_{1^-} of the drifted Maxwellian distribution function, Eq. H.18, (which is the simplest possible distribution function which retains thermal effects) this is a formidable task necessitating numerical computation and has not been undertaken.

LIST OF REFERENCES

1. Glicksman, M., "Instabilities and Growing Waves in Solid State Plasmas," Proc. 7th Int. Conf. on Phys. of Semiconductors, Dunod, Paris, pp. 149-164, 1965.

2. Vedenov, A. A., "Solid State Plasma," Soviet Phys.--Uspekhi, vol. 7, No. 6, pp. 809-822, May-June 1965.

3. Chynoweth, A. G. and Buchsbaum, S. J., "Solid-State Plasma," Phys. Today, vol. 18, No. 11, pp. 26-37, November 1965.

4. Bok, J., "Instabilities in Solid State Plasmas," J. Phys. Soc. Japan, vol. 21, Suppl., pp. 685-693, 1966.

5. Ancker-Johnson, B., "Plasma Effects in Semiconductors," Semiconductors and Semimetals, Physics of III-V Compounds, Willardson, R. K. and Beer, A. C. (Eds.), Academic Press, Inc., New York, vol. 1, Chap. 11, pp. 379-481, 1966.

6. Hoyaux, M. F., "Plasma Phenomena and the Solid State," Contemp. Phys., vol. 9, No. 2, pp. 165-196, March 1968.

7. Bowers, R. and Steele, M. C., "Plasma Effects in Solids," Proc. IEEE, vol. 52, No. 8, pp. 1105-1112, August 1964.

8. Veselago, V. G., Glushkov, M. V. and Prokhorov, A. M., "Microwave Properties of Solid State Plasma," Radio Eng. and Electronic Phys., vol. 12, No. 7, pp. 1134-1140, July 1967.

9. Morgan, D. P., "Helicon Waves in Solids," Phys. Stat. Sol., vol. 24, No. 1, pp. 9-36, November 1, 1967.

10. Kaner, E. A. and Skobov, V. G., "Electromagnetic Waves in Metals in a Magnetic Field," Soviet Phys.--Uspekhi, vol. 9, No. 4, pp. 480-503, January-February 1967.

11. Vural, B. and Steele, M. C., "Possible Two-Stream Instabilities of Drifted Electron-Hole Plasmas in Longitudinal Magnetic Fields," Phys. Rev., vol. 139, No. 1A, pp. A300-A304, 5 July 1965.

12. Briggs, R. J., Electron-Stream Interaction with Plasmas, Research Monograph No. 29, The M.I.T. Press, Cambridge, Mass., 1964.

13. Sturrock, P., "Kinematics of Growing Waves," Phys. Rev., vol. 112, No. 5, pp. 1488-1503, 1 December 1958.

14. Sudan, R. N., "Classification of Instabilities from Their Dispersion Relations," Phys. Fluids, vol. 8, No. 10, pp. 1899-1904, October 1965.

15. Polland, R., "Instabilities and Growing Waves," Phys. Rev., vol. 140, No. 38, pp. 776-787, 8 November 1965.

16. Konstantinov, O. V. and Perel, V. I., "Possible Transmission of Electromagnetic Waves Through a Metal in a Strong Magnetic Field," Soviet Phys.--JETP, vol. 11, No. 1, pp. 117-119, July 1960.

17. Aigrain, P., "Les 'Helicons' dan les Semiconducteurs," Proc. Int. Conf. on Phys. of Semiconductors, Academic Press, Inc., New York, pp. 224-226, 1961.

18. Buchsbaum, S. J. and Galt, J. K., "Alfven Waves in Solid State Plasmas," Phys. Fluids, vol. 4, No. 12, pp. 1514-1516, December 1961.

19. Bok, J. and Nozieres, P., "Instabilities of Transverse Waves in a Drifted Plasma," J. Phys. Chem. Solids, vol. 24, No. 6, pp. 709-714, June 1963.

20. Misawa, T., "Transverse Instabilities in Solid State Plasmas Under Steady Electric and Magnetic Fields," Japan. J. Appl. Phys., vol. 2, No. 8, pp. 500-515, August 1963.

21. Rodriguez, S. and Antoniewisz, P. R., "Interaction of Helicons with Longitudinal Plasma Waves in Solids," J. Phys. Chem. Solids, vol. 26, No. 4, pp. 747-750, April 1965.

22. Baynham, A. C. and Braddock, P. W., "Off Axis Helicon Amplification," Solid State Commun., vol. 4, No. 8, pp. 377-380, August 1966.

23. Hasegawa, A., "Resistive Instabilities in Semiconductor Plasmas," J. Phys. Soc. Japan, vol. 20, No. 6, pp. 1072-1079, June 1965.

24. Akai, S., "The Nature of Helicon-Wave Instabilities," J. Appl. Phys. Japan, vol. 5, No. 12, pp. 1227-1234, December 1966.

25. Bers, A. and McWhorter, A. L., "Absolute Instabilities with Drifted Helicons," Phys. Rev. Letters, vol. 15, No. 19, pp. 755-758, 8 November 1965.

26. Bartelink, D. J., "Propagation and Instability of Transverse Waves in Current-Carrying Electron-Hole Plasmas," Phys. Rev., vol. 158, No. 2, pp. 400-414, 10 June 1967.

27. Bartelink, D. J., "Amplification of Transverse Plasma Waves in Bismuth," Phys. Rev. Letters, vol. 16, No. 12, pp. 510-513, 21 March 1966.

28. Nanney, C., "Helicon-Drift-Current Interaction in a Solid-State Plasma Waveguide," Phys. Rev., vol. 138, No. 5A, pp. A1484-A1489, 31 May 1965.

29. Baraff, G. A. and Buchsbaum, S. J., "Surface-Wave Instability in Helicon Wave Propagation," Appl. Phys. Letters, vol. 6, No. 11, pp. 219-221, 1 June 1965.

30. Baraff, G. A. and Buchsbaum, S. J., "Surface-Wave Instability in Helicon Wave Propagation: Gain in Multilayered Structures," IEEE Trans. on Electron Devices, vol. ED-13, No. 1, pp. 203-205, January 1966.

31. Wallace, R. N. and Baraff, G. A., "Surface Wave Instability in Helicon Propagation II, Effect of Collisional Losses," J. Appl. Phys., vol. 37, No. 8, pp. 2937-2944, July 1966.

32. Saunders, L. M. and Baraff, G. A., "Surface Wave Instability in Helicon Wave Propagation III, Theory of Multilayered Structures," J. Appl. Phys., vol. 37, No. 12, pp. 4551-4558, November 1966.

33. Klozenberg, J. P., McNamara, B. and Thoremann, P. C., "The Dispersion and Attenuation of Helicon Waves in a Uniform Cylindrical Plasma," J. Fluid Mech., vol. 21, Part 3, pp. 545-563, July 1965.

34. McWhorter, A. L., Comment on paper by J. Bok, ref. No. 4, J. Phys. Soc. Japan, vol. 21, Suppl., p. 693, 1966.

35. Baraff, G. A., "Interaction of Helicon Waves with an Adjacent Drift Current," J. Phys. Chem. Solids, vol. 28, No. 6, pp. 1037-1053, June 1967.

36. Nanney, G. A., Libchaber, A. and Gains, J. P., "Helicon-Drift Interaction in a Layered Semiconductor Structure," Appl. Phys. Letters, vol. 9, No. 11, pp. 395-397, 1 December 1966.

37. Grow, R. W., "Electron Wave Interaction in Solids," Fifth Interim Report, Contract No. AF33(615)-3686, Microwave Device and Physical Electronic Laboratory, University of Utah, Salt Lake City, October 1967.

38. Wisseman, W. R. and Davies, E. J., "Investigation of Strong Guiding Effects in Helicon Propagation," J. Appl. Phys., vol. 38, No. 10, pp. 3940-3948, September 1967.

39. Buchsbaum, S. J., "Theory of Waves in Solid State Plasmas," Proc. 7th Int. Conf. on Phys. of Semiconductors, pp. 3-18, Dunod, Paris, 1965.

40. Steele, M. C. and Glicksman, M., "High Electric Field Effects in n-Indium Antimonide," J. Phys. Chem. Solids, vol. 8, pp. 242-244, January 1959.

41. Glicksman, M. and Steele, M. C., "Plasma Pinch Effects in Indium Antimonide," Phys. Rev. Letters, vol. 2, No. 11, pp. 461-463, 1 June 1959.

42. Chynoweth, A. G. and Murray, A. A., "Pinch Effect in Indium Antimonide," Phys. Rev., vol. 123, No. 2, pp. 515-520, 15 July 1961.

43. Toda, M., "Direct Observation of Self-Pinched Plasma Distribution in Indium Antimonide," Japan J. Appl. Phys., vol. 2, No. 8, pp. 467-470, August 1963.

44. Höfflinger, B., "A Method for Amplification of Plasma Waves in Solid-State Plasmas with Mass-Anisotropy," Solid State Electronics, vol. 8, No. 12, pp. 907-912, December 1965.

45. Tonks, L. and Langmuir, I., "Oscillations in Ionized Gases," Phys. Rev., vol. 33, No. 2, pp. 195-210, February 1929.

46. Pines, D., "Electron Interaction in Solids," Can. J. Phys., vol. 34, No. 12A, pp. 1379-1394, December 1956.

47. Nozieres, P. and Pines, D., "Electron Interaction in Solids. The Nature of the Elementary Excitations," Phys. Rev., vol. 109, No. 4, pp. 1062-1074, 15 February 1958.

48. Pines, D. and Schrieffer, J. R., "Collective Behavior in Solid-State Plasmas," Phys. Rev., vol. 124, No. 5, pp. 1387-1400, 1 December 1961.

49. Harrison, M. J., "Collective Excitation of Degenerate Plasmas in Solids," J. Phys. Chem. Solids, vol. 23, No. 7, pp. 1079-1086, August 1962.

50. Vural, B. and Bloom, S., "Streaming Instabilities and the Role of Collisions," IEEE Trans. on Electron Devices, vol. ED-13, No. 1, pp. 57-63, January 1966.

51. Glicksman, M. and Hicinbothem, W. A., "Hot Electrons in Indium Antimonide," Phys. Rev., vol. 129, No. 4, pp. 1572-1577, 15 February 1963.

52. "Study of Microwave Generation and Amplification Utilizing Double-Stream Instabilities in Anisotropic Media," Quar. Report Nos. 1-10, Contract No. DA36-039 AMC-03424(E), Microwave Electronics Corp., Palo Alto, Calif., 4 November 1963 to 30 June 1966.

53. Robinson, B. B. and Swartz, G. A., "Two-Stream Instability in Semiconductor Plasmas," J. Appl. Phys., vol. 38, No. 6, pp. 2461-2465, May 1967.

54. Robinson, B. B. and Vural, B., "Double-Stream Interaction in a Thin Semiconductor Layer," RCA Rev., vol. 29, No. 2, pp. 270-280, June 1968.

55. Suzuki, T., "Theory of Microwave Emission from Indium Antimonide," IEEE Trans. on Electron Devices, vol. ED-13, No. 1, pp. 202-203, January 1966.

56. Suzuki, T., "Instabilities of Semiconductor Plasmas in Crossed Electric and Magnetic Fields," J. Phys. Soc. Japan, vol. 21, No. 10, pp. 2000-2010, October 1966.

57. McNeill, P. R., "Instabilities in a Drifting Semiconductor Plasma," Proc. IEEE, vol. 54, No. 11, pp. 1596-1597, November 1966.

58. Suzuki, K., "The Generation of Microwave Radiation from InSb," Japan. J. Appl. Phys., vol. 4, No. 1, pp. 42-52, January 1965.

59. Suzuki, K., "The Spectrum of Microwave Radiation from InSb," IEEE Trans. on Electron Devices, vol. ED-13, No. 1, pp. 132-137, January 1966.

60. Vural, B., "Analysis of Double-Stream Interactions in the Presence of a Finite Axial Magnetic Field," RCA Rev., vol. 22, No. 4, pp. 753-779, December 1961.

61. Bernstein, I. B., "Waves in a Magnetic Field," Phys. Rev., vol. 109, No. 1, pp. 10-21, 1 January 1958.

62. Hasegawa, A., "Microinstabilities in Transversely Magnetized Semiconductor Plasmas," J. Appl. Phys., vol. 36, No. 11, pp. 3590-3595, November 1965.

63. Swartz, G. A. and Robinson, B. B., "Coherent Microwave Emission in n-Type Indium Antimonide," Appl. Phys. Letters, vol. 8, No. 7, pp. 183-185, 1 April 1966.

64. Chu, L. J., "A Kinetic Power Theorem," Paper Presented at AIEE-IRE Conference on Electron Tube Research, Durham, N. H., June 1951.

65. Chodorow, M. and Susskind, C., Fundamentals of Microwave Electronics, McGraw-Hill Book Co., Inc., New York, pp. 207-212, 1964.

66. Bobroff, D. L., Haus, H. A. and Kluver, J. W., "On the Small Signal Power Theorem of Electron Beams," J. Appl. Phys., vol. 33, No. 10, pp. 2932-2942, October 1962.

67. Vural, B. and Bloom, S., "Small-Signal Power Flow and Energy Density for Streaming Carriers in the Presence of Collisions," IEEE Trans. on Electron Devices, vol. ED-14, No. 7, pp. 345-349, July 1967.

68. Meyer, M. and Van Duzer, T., "Traveling-Wave Amplification and Power Flow in Conducting Solids," IEEE Trans. on Electron Devices, vol. ED-17, No. 3, pp. 193-199, March 1970.

69. Birdsall, C. K., "Electron-Stream Kinetic-Power Second-Order Terms as Obtained from MacColl's Mean Values," J. Appl. Phys., vol. 33, No. 1, pp. 235-236, January 1962.

70. Ramo, S., Whinnery, J. R. and Van Duzer, T., Fields and Waves in Communications Electronics, John Wiley and Sons, Inc., New York, pp. 243-244, 1965.

71. Louisell, W. H., Coupled Mode and Parametric Electronics, John Wiley and Sons, Inc., New York, pp. 193-194, 1960.

72. Tonks, L., "A New Form of the Electromagnetic Energy Equation When Free Charged Particles Are Present," Phys. Rev., vol. 54, p. 863, 15 November 1938.

73. Steele, M. C. and Vural, B., Wave Interactions in Solid State Plasmas, McGraw-Hill Book Co., Inc., New York, 1969.

74. Gueret, P., "Wave Propagation and Instabilities in Semiconductors," Microwave Laboratory Report No. 1533, Microwave Laboratory, W. W. Hansen Laboratories of Physics, Stanford University, Stanford, Calif., May 1967.

75. Fried, B. D. and Conte, S. D., The Plasma Dispersion Function, Academic Press, Inc., New York, 1961.

76. Montgomery, D. C. and Tidman, D. A., Plasma Kinetic Theory, McGraw-Hill Book Co., Inc., New York, 1964.

77. Schmidt, G., Physics of High Temperature Plasmas, Academic Press, Inc., New York, 1966.

78. Morisaki, H. and Inuishi, Y., "Harmonic Generation in Microwave Emission from InSb," Japan. J. Appl. Phys., vol. 5, No. 4, pp. 343-344, April 1966

79. Bray, R., Kumar, C. S., Ross, J. B. and Silva, P. O., "Acoustoelectric Domain Effects in III-V Semiconductors," J. Phys. Soc. Japan, vol. 21, Suppl., pp. 483-488, 1966.

80. Watson, G. N., The Theory of Bessel Functions, Cambridge University Press, London and New York, 1958.

81. Abramowitz, M. and Stegun, I. A., Handbook of Mathematical Functions, Applied Mathematics Series No. 55, National Bureau of Standards, Washington, D. C., Chaps. 6 and 7, 1965.

82. Freire, G. F., "Active Waves in Solid-State Plasmas," Int. J. Electronics, vol. 28, No. 1, pp. 1-18, January 1970.

83. Carter, D. L. and Libchaber, A., "Helicon-Induced dc Voltage in InSb," Bull. Am. Phys. Soc., vol. 9, No. 3, p. 264, 23 March 1964.

84. Feinleib, J., deNeufville, J., Moss, S. C. and Ovshinsky, S. R., "Rapid Reversible Light-Induced Crystallization of Amorphous Semiconductors," Appl. Phys. Letters, vol. 18, No. 6, pp. 254-257, 15 March 1971.

85. MacColl, L. A., "Certain Mean Values in the Theory of the Traveling-Wave Amplifier," Bell System Tech. J., vol. 39, No. 2, pp. 365-367, March 1960.

86. Barlow, H.E.M. and Kataoka, S., "The Hall Effect and Its Application to Measurement at 10 GHz," Proc. IEE, Part B, Paper No. 2450R, pp. 53-60, January 1958.

87. van Nieuwland, J. M. and Vlaardingerbroek, M. T., "Cyclotron Waves in InSb," IEEE Trans. on Electron Devices, vol. ED-14, No. 9, pp. 596-599, September 1967.

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13. ABSTRACT

A general theory is developed for the electrokinetic power and energy properties associated with the basic carrier modes present in plasma media. Both hydrodynamic and kinetic theoretical models are obtained for the media in the presence of applied static electric and magnetic fields.

In the hydrodynamic theory the effects of carrier collisions and thermal diffusion are properly accounted for and explained by developing a second-order quasi-linear analysis. In this manner it is shown that the negative kinetic power property is directly related to dc slowing of the active carrier. The distinction between absolute and convective instabilities leads to the formulation of a space-averaged temporal-energy basis for determining the existence of absolute instabilities as compared to a time-averaged spatial-power basis for convective instabilities. The analysis shows that it is possible to relate the causality criteria for instabilities developed by Briggs to the conservation of power and energy in the medium. Thus useful general information is obtained on the behavior of the root trajectories in the complex-k space as the imaginary part of the frequency is varied.

The quasi-linear theory, as a by-product, allows the analysis of the second-order Hall effect and related phenomena in solids. In addition, a study of the physical meaning of the quasi-linear theory shows that this is a useful analytical tool for studying potential energy effects caused by the reaction of the growing RF fields on the carrier charges. This also enables the accuracy of the linear dispersion equation to be assessed.

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ABSTRACT CONTINUED

The power and energy theorems applied to the kinetic theory determine the effects of nonlocality, anisotropic carrier temperatures, and carrier heating. Whenever possible the results obtained are rigorously compared with those of the hydrodynamics theory. By obtaining the respective dispersion equations, computer results for the hybrid-hybrid electron-hole interaction are related to published experimental work on the phenomenon of microwave emission from indium antimonide.

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