THE UNIVERSITY OF MICHIGAN

INDUSTRY PROGRAM OF THE COLLEGE OF ENGINEERING

STATISTICAL ANALYSIS OF FATIGUE LIMITS USING THE LOGISTIC FUNCTION

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INTRODUCTION

Statistical approaches which assume a life distribution pertaining to a given alternating stress amplitude have limited value when employed to estimate fatigue strengths associated with lives of 10^6 cycles or greater. For such lives, scatter of datum points is large and consequently numerical estimates are relatively weak. However, this weakness can be overcome by dealing with an alternating stress-percentage failed function. Moreover, this approach relates directly to the engineering concept of reliability.

Development of the alternating stress-percentage failed function on the basis of fatigue data is not possible at present because insufficient data exist. However, it will be shown that the fatigue problem is analogous to problems successfully treated in the field of bio-statistics and that analysis of the limited fatigue data by bio-statistical techniques gives encouraging results.

Analogy Between Certain Biological and Fatigue Problems

Suppose that in a biological study it is desired to estimate the effectiveness of a particular drug. Tests may be conducted in which N animals are given a dose, d_1, and n_1 animals die, while $N-n_1$ survive. Then, another group of N animals are given a dose, d_2, and n_2

die, while $N\text{-}n_2$ survive. These tests may be continued until many different groups have been subjected to some particular dosage. In this biological problem, the statistics recorded are the dosages and, at the end of a given period of time, the number dead and the number surviving. Analogously, in the fatigue problem, the statistics recorded are the alternating stress levels and, for a predetermined fatigue life, the number of specimens failed and the number that run-out. In other words, the alternating stress level in fatigue corresponds directly to the dosage in the biological problem.

The logistic function has been used with success in the analysis of bio-assay problems such as the drug effectiveness analogy discussed above.(1-4) Consequently, because of the basic statistical similarities of these biological and fatigue problems, use of the logistic function is extended to analysis of fatigue limits as outlined below.

Logistic Function

This analysis consists of fitting fatigue data to the logistic function, which is

$$P = 1 - Q = \frac{1}{1 + e^{-(\alpha + \beta s)}} \qquad (1)$$

In Equation (1),

P = the "true" percentage failed

Q = the "true" percentage that run-out

s = the alternating stress level

α,β = population (not sample) parameters

The response is said to be quantal when it is measured not in terms of a continuous variable, but in terms of the observed number failed out of the total number tested at the given alternating stress level. In this analysis, it is assumed that the observed percentage failed, p, (based on a quantal observation) at the given alternating stress level, s, can be considered a random variable binomially distributed around the population P at s with a variance $\sigma_p^2 = PQ/N$.

Estimation of Population Parameters

The logistic function is completely defined when both α and β are known; hence, the basic problem is to estimate these population parameters from experimental results. While several different techniques exist for estimating these parameters, the non-iterative Berkson(1) method simplifies the computational procedure and gives maximum likelihood estimates.

Equation (1) can be transformed to give

$$\ln(P/Q) = \alpha + \beta s \tag{2}$$

Substituting ℓ for ln(P/Q), Equation (2) becomes

$$\ell = \alpha + \beta s \tag{3}$$

This linear transformation of Equation (1) is called the Berkson logit equation. The population parameters are estimated by a minimum χ^2 approach. Defining

$$\chi^2 = \sum_{i=1}^{k} N_i p_i q_i \,(\ell_i - \hat{\ell}_i)^2 \tag{4}$$

where

k = the number of alternating stress levels

$\hat{\ell}$ = the estimated $\ell[\ell = \ln(\hat{p}/\hat{q}) = \hat{\alpha} + \hat{\beta}s]$

p = the observed percentage failed

q = the observed percentage that run-out

N = the number of specimens tested at the i-th stress level,

the logit χ^2 is minimized with respect to the population parameters when

$$\sum Npq(\ell - \hat{\ell}) = 0 \tag{5}$$

$$\sum Npqs(\ell - \hat{\ell}) = 0 \; . \tag{6}$$

Simultaneous solution of Equations (5) and (6) yields

$$\hat{\beta} = \frac{\sum w \sum w\ell s - \sum w\ell \sum ws}{\sum w \sum ws^2 - (\sum ws)^2} \tag{7}$$

$$\hat{\alpha} = \frac{\sum w\ell - \hat{\beta} \sum ws}{\sum w} \tag{8}$$

where

$w = Npq$.

Whenever $p = 0$ or $p = 1$ is observed; $p = 0$ should be replaced by $p = 1/2N$, and $p = 1$ should be replaced by $p = (2N - 1)/2N$. These "corrections" are required for a minimum variance of the estimates.[5]

Example of Estimating Population Parameters

The only data found in the literature which are amenable to analysis is that by Stulen.[6] Table I lists these data for a 4330 steel with an ultimate tensile strength of 130 ksi and a yield strength of 110 ksi.

The analysis of these data is presented in Table II. The logits used in the computational procedure are given in Table III.

Having computed the estimates $\hat{\alpha}$ and $\hat{\beta}$, $\hat{p}$ can be determined by using

$$\hat{\ell} = \hat{\alpha} + \hat{\beta}s \tag{9}$$

$$\hat{\ell} = \ln(\hat{p}/\hat{q}) \tag{10}$$

and the antilogits of Table IV. The last two columns in Table II list $\hat{\ell}$ and $\hat{p}$, respectively.

Figure 1 shows the results of this analysis. It can be seen that this straight line estimate is somewhat conservative at each tail. It is clear that p is a function of the number of specimens tested and that p becomes critical at the tails when the sample size is relatively small. Unless "correction" factors are applied for p equal zero and p equal unity, these observations bias the estimates of the population parameters.

Standard Error of Estimates and Confidence Limit

The formulae for the variances of the estimates of the population parameters are

$$S^2_{\hat{\alpha}} = \frac{1}{\sum \hat{w}} + \frac{\hat{\bar{s}}^2}{\sum \hat{w}(s - \hat{\bar{s}})^2} \tag{11}$$

$$S^2_{\hat{\beta}} = \frac{1}{\sum \hat{w}(s - \hat{\bar{s}})^2} \tag{12}$$

where

$$\hat{\bar{s}} = \sum \hat{w}s / \sum \hat{w} \ .$$

The scatter band for a desired level of confidence in the estimated percentage failed, $\hat{p}$, is given by

$$\hat{\ell} \pm t_{k-2} \sqrt{\left(\frac{\chi^2_{residual}}{k-2}\right) \left[\frac{1}{\sum \hat{w}} + \frac{(s - \hat{s})^2}{\sum \hat{w}(s - \hat{s})^2}\right]} \qquad (13)$$

where

$$\chi^2_{residual} = \sum N\hat{p}(1-\hat{p})(\ell-\hat{\ell})^2$$

t = Student's t corresponding to the desired confidence level, with degrees of freedom $= k - 2$. For values of Student's t, refer to: R.A. Fisher and F. Yates, Statistical Tables for Biological, Argicultural and Medical Research, Fourth Edition, Oliver and Boyd, London, 1953.

The 95 per cent confidence band appears in Figure 1. See Table V.

Discussion

It is very difficult to draw definite conclusions regarding the actual theoretical distribution from a single series of tests. While a visual comparison of plotted data and conceivable distributions may help in deciding which distribution is to be regarded as the most probable, it is generally not possible to exclude these other distributions on a theoretical basis.(7) In other words, in order that a certain distribution may be selected as the most probable and other conceivable distributions may be rejected by means of appropriate numerical analysis, it is necessary to have very extensive test data. Such data do not exist at present.

As previously mentioned, the logistic function has found wide use in the field of bio-statistics. One important reason for its wide use is that the population parameters can be estimated without becoming involved in iterative procedures. The computational procedures involved are adaptable to tabulation and are no more difficult than common least-squares curve fitting techniques. Yet, this method gives the maximum likelihood estimates of the population parameters.

Simplicity of analysis and conservatism in analyzing the tails of the observed distribution are of prime importance in fatigue because of the many uncertainties involved in extrapolating these results to design applications. Although other approaches to analysis of fatigue limits have been suggested,(8) this method has definite advantages and should not be overlooked in fatigue analysis.

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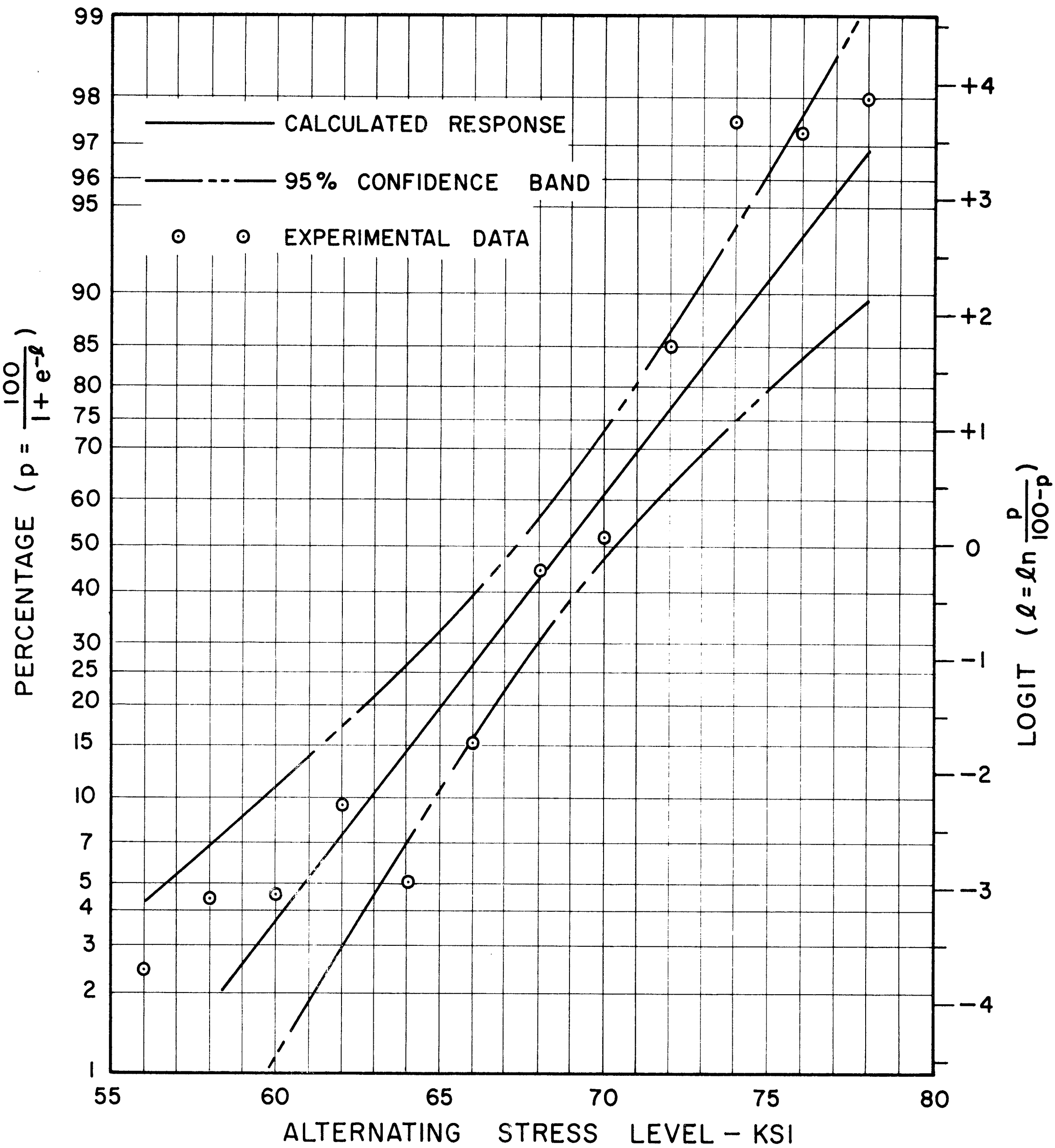

Figure 1. Results of Analysis of Rotating Bending Fatigue Tests. See Table I and Table V for analysis and Equation (13) for confidence band.

TABLE I

RESULTS OF ROTATING BENDING FATIGUE TESTS ON SAE 4330 STEEL
(Data by F. B. Stulen,[6] pertain to a fatigue life of 10^7 cycles)

Test Series	Stress Level ksi	Number Tested	Number Failed	Percentage Failed
1	56	20	0	0.0
2	58	23	1	4.3
3	60	22	1	4.5
4	62	21	2	9.5
5	64	20	1	5.0
6	66	20	3	15.0
7	68	27	12	44.4
8	70	27	14	51.9
9	72	20	17	85.0
10	74	20	20	100.0
11	76	18	18	100.0
12	78	24	24	100.0

TABLE II

ANALYSIS OF ROTATING BENDING FATIGUE DATA

1	2	3	4	5	6	7	8	9	10	11
Test Series	s_i Stress Level	N_i Number Tested	Number Failed	p_i Proportion Failed	w_i $N_i p_i(1-p_i)$	ℓ_i	$s_i \ell_i$	s_i^2	$\hat{\ell}_i$	$\hat{p}_i$
1	56.0	20	0	0.025*	0.4875	-3.6636	-205.1616	3136.0	-4.7416	.009
2	58.0	23	1	0.043	0.9465	-3.1026	-179.9508	3364.0	-4.0004	.018
3	60.0	22	1	0.045	0.9455	-3.0551	-183.3060	3600.0	-3.2592	0.037
4	62.0	21	2	0.095	1.8055	-2.2541	-139.7542	3844.0	-2.5180	0.074
5	64.0	20	1	0.05	0.9500	-2.9444	-188.4416	4096.0	-1.7768	0.144
6	66.0	20	3	0.150	2.550	-1.7346	-114.4836	4356.0	-1.0356	0.261
7	68.0	27	12	0.444	6.6653	-0.2249	- 15.2932	4624.0	-0.2944	0.426
8	70.0	27	14	0.519	6.7403	0.0760	5.3200	4900.0	0.4468	0.611
9	72.0	20	17	0.850	2.550	1.7346	124.8912	5184.0	1.1880	0.767
10	74.0	20	20	0.975*	0.4875	3.6636	271.1064	5476.0	1.9292	0.873
11	76.0	18	18	0.972*	0.4899	3.5472	269.5872	5776.0	2.6704	0.935
12	78.0	24	24	0.979*	0.4934	3.8420	299.6760	6084.0	3.4116	0.968

$\Sigma\, ws = 1700.4220$ $\quad \Sigma\, w = 25.1114$ $\quad \Sigma\, w\ell = -10.0455$

$\Sigma\, ws^2 = 115624.0432$ $\quad \Sigma\, w\ell s = -502.4420$

$$\hat{\beta} = \frac{\Sigma w \,\Sigma w\ell s - \Sigma w\ell \,\Sigma ws}{\Sigma w\, \Sigma ws^2 - (\Sigma ws)^2} = \frac{177.7904}{479.7271} = 0.3706$$

$$\hat{\alpha} = \frac{\Sigma w\ell}{\Sigma w} - \hat{\beta}\frac{\Sigma ws}{\Sigma w} = -0.4000 - 25.0952 = -25.4952$$

$$\hat{\ell} = \hat{\alpha} + \hat{\beta} s$$

* Necessary corrections made for p = 0 and p = 1.

TABLE III

LOGITS

For *p* less than .50 on left, logit is negative. For *p* greater than .50 on right, logit is positive.

p	Thousandths, for *p* in left column											
	0	1	2	3	4	5	6	7	8	9		
.00	—	6.90675	6.21261	5.80614	5.51745	5.29330	5.10998	4.95482	4.82028	4.70149	4.59512	.99
.01	4.59512	4.49880	4.41078	4.32972	4.25460	4.18459	4.11904	4.05740	3.99922	3.94413	3.89182	.98
.02	3.89182	3.84201	3.79447	3.74899	3.70541	3.66356	3.62331	3.58455	3.54715	3.51103	3.47610	.97
.03	3.47610	3.44228	3.40950	3.37769	3.34680	3.31678	3.28757	3.25914	3.23143	3.20441	3.17805	.96
.04	3.17805	3.15232	3.12718	3.10260	3.07857	3.05505	3.03202	3.00947	2.98736	2.96569	2.94444	.95
.05	2.94444	2.92358	2.90311	2.88301	2.86326	2.84385	2.82477	2.80601	2.78756	2.76941	2.75154	.94
.06	2.75154	2.73394	2.71662	2.69955	2.68273	2.66616	2.64982	2.63371	2.61783	2.60215	2.58669	.93
.07	2.58669	2.57143	2.55637	2.54149	2.52681	2.51231	2.49798	2.48382	2.46984	2.45601	2.44235	.92
.08	2.44235	2.42884	2.41548	2.40227	2.38920	2.37627	2.36348	2.35083	2.33830	2.32591	2.31363	.91
.09	2.31363	2.30149	2.28946	2.27754	2.26574	2.25406	2.24248	2.23101	2.21965	2.20839	2.19722	.90
.10	2.19722	2.18616	2.17520	2.16433	2.15355	2.14286	2.13227	2.12176	2.11133	2.10100	2.09074	.89
.11	2.09074	2.08057	2.07047	2.06046	2.05052	2.04066	2.03087	2.02115	2.01151	2.00193	1.99243	.88
.12	1.99243	1.98299	1.97363	1.96432	1.95508	1.94591	1.93680	1.92775	1.91876	1.90983	1.90096	.87
.13	1.90096	1.89215	1.88339	1.87469	1.86605	1.85745	1.84892	1.84043	1.83200	1.82362	1.81529	.86
.14	1.81529	1.80701	1.79878	1.79059	1.78246	1.77437	1.76632	1.75833	1.75037	1.74247	1.73460	.85
.15	1.73460	1.72678	1.71900	1.71126	1.70357	1.69591	1.68830	1.68072	1.67318	1.66569	1.65823	.84
.16	1.65823	1.65081	1.64342	1.63607	1.62876	1.62149	1.61425	1.60704	1.59987	1.59273	1.58563	.83
.17	1.58563	1.57856	1.57152	1.56451	1.55754	1.55060	1.54369	1.53681	1.52996	1.52314	1.51635	.82
.18	1.51635	1.50959	1.50286	1.49615	1.48948	1.48283	1.47621	1.46962	1.46306	1.45652	1.45001	.81
.19	1.45001	1.44353	1.43707	1.43063	1.42423	1.41784	1.41148	1.40515	1.39884	1.39256	1.38269	.80
.20	1.38629	1.38006	1.37384	1.36765	1.36148	1.35533	1.34921	1.34310	1.33702	1.33096	1.32493	.79
.21	1.32493	1.31891	1.31291	1.30694	1.30098	1.29505	1.28913	1.28324	1.27736	1.27150	1.26567	.78
.22	1.26567	1.25985	1.25405	1.24827	1.24251	1.23676	1.23104	1.22533	1.21964	1.21397	1.20831	.77
.23	1.20831	1.20267	1.19705	1.19145	1.18586	1.18029	1.17474	1.16920	1.16368	1.15817	1.15268	.76
.24	1.15268	1.14720	1.14175	1.13630	1.13087	1.12546	1.12006	1.11468	1.10931	1.10395	1.09861	.75
.25	1.09861	1.09329	1.08797	1.08268	1.07739	1.07212	1.06686	1.06162	1.05639	1.05117	1.04597	.74
.26	1.04597	1.04078	1.03566	1.03043	1.02528	1.02014	1.01501	1.00990	1.00479	0.99970	0.99462	.73
.27	0.99462	0.98955	0.98450	0.97945	0.97442	0.96940	0.96439	0.95939	0.95440	0.94943	0.94446	.72
.28	0.94446	0.93951	0.93456	0.92963	0.92471	0.91979	0.91489	0.91000	0.90512	0.90025	0.89538	.71
.29	0.89538	0.89053	0.88569	0.88086	0.87604	0.87122	0.86642	0.86162	0.85684	0.85206	0.84730	.70
.30	0.84730	0.84254	0.83779	0.83305	0.82832	0.82360	0.81889	0.81418	0.80949	0.80480	0.80012	.69
.31	0.80012	0.79545	0.79079	0.78613	0.78148	0.77685	0.77222	0.76759	0.76298	0.75837	0.75377	.68
.32	0.75377	0.74918	0.74460	0.74002	0.73545	0.73089	0.72633	0.72179	0.71724	0.71271	0.70819	.67
.33	0.70819	0.70367	0.69915	0.69465	0.69015	0.68566	0.68117	0.67669	0.67222	0.66775	0.66329	.66
.34	0.66329	0.65884	0.65439	0.64995	0.64552	0.64109	0.63667	0.63225	0.62784	0.62344	0.61904	.65
.35	0.61904	0.61465	0.61026	0.60588	0.60150	0.59713	0.59277	0.58841	0.58406	0.57971	0.57536	.64
.36	0.57536	0.57103	0.56669	0.56237	0.55804	0.55373	0.54942	0.54511	0.54081	0.53651	0.53222	.63
.37	0.53222	0.52793	0.52365	0.51937	0.51509	0.51083	0.50656	0.50230	0.49805	0.49379	0.48955	.62
.38	0.48955	0.48531	0.48107	0.47683	0.47260	0.46838	0.46416	0.45994	0.45573	0.45152	0.44731	.61
.39	0.44731	0.44311	0.43891	0.43472	0.43053	0.42634	0.42216	0.41798	0.41381	0.40963	0.40547	.60
.40	0.40547	0.40130	0.39714	0.39298	0.38883	0.38467	0.38053	0.37638	0.37224	0.36810	0.36397	.59
.41	0.36397	0.35983	0.35570	0.35158	0.34745	0.34333	0.33922	0.33510	0.33099	0.32688	0.32277	.58
.42	0.32277	0.31867	0.31457	0.31047	0.30637	0.30228	0.29819	0.29410	0.29002	0.28593	0.28185	.57
.43	0.28185	0.27777	0.27370	0.26962	0.26555	0.26148	0.25741	0.25335	0.24928	0.24522	0.24116	.56
.44	0.24116	0.23710	0.23305	0.22900	0.22494	0.22089	0.21685	0.21280	0.20875	0.20471	0.20067	.55
.45	0.20067	0.19663	0.19259	0.18856	0.18452	0.18049	0.17646	0.17243	0.16840	0.16437	0.16034	.54
.46	0.16034	0.15632	0.15229	0.14827	0.14425	0.14023	0.13621	0.13219	0.12818	0.12416	0.12014	.53
.47	0.12014	0.11613	0.11212	0.10811	0.10409	0.10008	0.09607	0.09206	0.08806	0.08405	0.08004	.52
.48	0.08004	0.07604	0.07203	0.06803	0.06402	0.06002	0.05601	0.05201	0.04801	0.04401	0.04001	.51
.49	0.04001	0.03600	0.03200	0.02800	0.02400	0.02000	0.01600	0.01200	0.00800	0.00400	0.00000	.50
		9	8	7	6	5	4	3	2	1	0	*p*
		Thousandths, for *p* in right column										

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See Reference 1.

TABLE IV

ANTILOGITS

Entries give value of p for specified positive value of logit l; if l is negative, p is 1 minus the tabled value.

l	0	1	2	3	4	5	6	7	8	9
0.0	.50000	.50250	.50500	.50750	.51000	.51250	.51500	.51749	.51999	.52248
0.1	.52498	.52747	.52996	.53245	.53494	.53743	.53991	.54240	.54488	.54736
0.2	.54983	.55231	.55478	.55725	.55971	.56218	.56464	.56709	.56955	.57200
0.3	.57444	.57689	.57932	.58176	.58419	.58662	.58904	.59146	.59387	.59628
0.4	.59869	.60109	.60348	.60587	.60826	.61064	.61301	.61538	.61775	.62011
0.5	.62246	.62481	.62715	.62948	.63181	.63414	.63645	.63876	.64107	.64337
0.6	.64566	.64794	.65022	.65249	.65475	.65701	.65926	.66150	.66374	.66597
0.7	.66819	.67040	.67261	.67481	.67700	.67918	.68135	.68352	.68568	.68783
0.8	.68997	.69211	.69424	.69635	.69847	.70057	.70266	.70475	.70682	.70889
0.9	.71095	.71300	.71504	.71708	.71910	.72112	.72312	.72512	.72711	.72909
1.0	.73106	.73302	.73497	.73692	.73885	.74077	.74269	.74460	.74649	.74838
1.1	.75026	.75213	.75399	.75584	.75768	.75951	.76133	.76315	.76495	.76674
1.2	.76852	.77030	.77206	.77382	.77556	.77730	.77903	.78074	.78245	.78415
1.3	.78583	.78751	.78918	.79084	.79249	.79413	.79576	.79738	.79899	.80059
1.4	.80218	.80377	.80534	.80690	.80845	.81000	.81153	.81306	.81457	.81608
1.5	.81757	.81906	.82054	.82201	.82346	.82491	.82635	.82778	.82920	.83062
1.6	.83202	.83341	.83480	.83617	.83753	.83889	.84024	.84158	.84290	.84422
1.7	.84553	.84684	.84813	.84941	.85069	.85195	.85321	.85446	.85570	.85693
1.8	.85815	.85936	.86057	.86176	.86295	.86413	.86530	.86646	.86761	.86876
1.9	.86989	.87102	.87214	.87325	.87435	.87545	.87653	.87761	.87868	.87974
2.0	.88080	.88184	.88288	.88391	.88493	.88595	.88695	.88795	.88894	.88993
2.1	.89090	.89187	.89283	.89379	.89473	.89567	.89660	.89752	.89844	.89935
2.2	.90025	.90114	.90203	.90291	.90378	.90465	.90551	.90636	.90721	.90805
2.3	.90888	.90970	.91052	.91133	.91214	.91293	.91373	.91451	.91529	.91606
2.4	.91683	.91759	.91834	.91909	.91983	.92056	.92129	.92201	.92273	.92344
2.5	.92414	.92484	.92553	.92622	.92690	.92757	.92824	.92891	.92956	.93022
2.6	.93086	.93150	.93214	.93277	.93339	.93401	.93462	.93523	.93584	.93643
2.7	.93703	.93761	.93820	.93877	.93935	.93991	.94048	.94103	.94159	.94213
2.8	.94268	.94321	.94375	.94428	.94480	.94532	.94583	.94634	.94685	.94735
2.9	.94785	.94834	.94883	.94931	.94979	.95026	.95073	.95120	.95166	.95212
3.0	.95257	.95302	.95347	.95391	.95435	.95478	.95521	.95564	.95606	.95648
3.1	.95689	.95730	.95771	.95811	.95851	.95891	.95930	.95969	.96007	.96046
3.2	.96083	.96121	.96158	.96195	.96231	.96267	.96303	.96339	.96374	.96408
3.3	.96443	.96477	.96511	.96544	.96578	.96610	.96643	.96675	.96707	.96739
3.4	.96770	.96802	.96832	.96863	.96893	.96923	.96953	.96982	.97011	.97040
3.5	.97069	.97097	.97125	.97153	.97180	.97208	.97235	.97262	.97288	.97314
3.6	.97340	.97366	.97392	.97417	.97442	.97467	.97491	.97516	.97540	.97564
3.7	.97587	.97611	.97634	.97657	.97680	.97702	.97725	.97747	.97769	.97790
3.8	.97812	.97833	.97854	.97875	.97896	.97916	.97937	.97957	.97977	.97996
3.9	.98016	.98035	.98054	.98073	.98092	.98111	.98129	.98148	.98166	.98184
4.0	.98201	.98219	.98236	.98254	.98271	.98288	.98304	.98321	.98337	.98354
4.1	.98370	.98386	.98402	.98417	.98433	.98448	.98463	.98478	.98493	.98508
4.2	.98523	.98537	.98551	.98566	.98580	.98594	.98607	.98621	.98635	.98648
4.3	.98661	.98674	.98687	.98700	.98713	.98726	.98738	.98751	.98763	.98775
4.4	.98787	.98799	.98811	.98823	.98834	.98846	.98857	.98868	.98879	.98890
4.5	.98901	.98912	.98923	.98933	.98944	.98954	.98965	.98975	.98985	.98995
4.6	.99005	.99015	.99024	.99034	.99043	.99053	.99062	.99071	.99081	.99090
4.7	.99099	.99108	.99116	.99125	.99134	.99142	.99151	.99159	.99167	.99176
4.8	.99184	.99192	.99200	.99208	.99215	.99223	.99231	.99239	.99246	.99253
4.9	.99261	.99268	.99275	.99283	.99290	.99297	.99304	.99310	.99317	.99324
l	0	1	2	3	4	5	6	7	8	9

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TABLE V

ANALYSIS FOR THE CONFIDENCE BAND

1	2	3	4	5	6	7	8	9	10	11	12	13	14
N_i	$\hat{p}_i$	$1-\hat{p}_i$	$\hat{w}_i$ $N_i(1-\hat{p}_i)\hat{p}_i$	ℓ_i	$\hat{\ell}_i$	$(\ell_i-\hat{\ell}_i)^2$	s_i	$s_i-\hat{\bar{s}}$	$(s_i-\hat{\bar{s}})^2$	Formula 13 (-sign)	$\hat{p}_{i_{95}}$	Formula 13 (+sign)	$\hat{p}_{i_{95}}$
20	.009	.991	.1784	-3.6636	-4.7416	1.16208	56	-12.6022	158.8154	-6.35969	-*	-3.12351	.04229
23	.018	.982	.4065	-3.1026	-4.0004	0.80604	58	-10.6022	112.4066	-5.38923	-*	-2.61157	.06850
22	.037	.963	.7839	-3.0551	-3.2592	0.04166	60	- 8.6022	73.9978	-4.41832	.01189	-2.10008	.10910
21	.074	.926	1.4390	-2.2541	-2.5180	0.06964	62	- 6.6022	43.5890	-3.47016	.03018	-1.56584	.17365
20	.144	.856	2.4653	-2.9444	-1.7768	1.36329	64	- 4.6022	21.1802	-2.72896	.06123	-1.01934	.26503
20	.261	.739	3.8576	-1.7346	-1.0356	0.48860	66	- 2.6022	6.7746	-1.63540	.16247	- .43580	.39174
27	.426	.574	6.6021	-0.2249	-0.2944	0.00483	68	- 0.6022	0.3626	- .80871	.30789	.21991	.55478
27	.611	.389	6.4173	0.0760	0.4468	0.13749	70	1.3978	1.9538	- .09000	.47752	.98360	.72711
20	.767	.233	3.5742	1.7346	1.1880	0.29877	72	3.3978	11.5450	.53201	.62948	1.84399	.86296
20	.873	.127	2.2174	3.6636	1.9292	3.00814	74	5.3978	29.1362	1.09738	.73302	2.76102	.94048
18	.935	.065	1.0940	3.5472	2.6704	0.76878	76	7.3978	54.7274	1.63499	.83617	3.70581	.97611
24	.968	.032	0.7434	3.8420	3.4116	0.18524	78	9.3978	88.3186	2.15819	.89660	4.66501	.99071

$\sum \hat{w} = 29.7791$ $\qquad \hat{\bar{s}} = \frac{\sum \hat{w}s}{\sum \hat{w}} = 68.6022$ $\qquad \sum \hat{w}(s-\hat{\bar{s}})^2 = 519.4373$ $\qquad \chi^2_{residual} = 15.545$

* Close to zero; beyond range of Table IV.