AIAA-92-2277-CP

Delamination Buckling Instability Near a Circular Hole in Laminated Composite Plates

Khaled W. Shahwan* and Anthony M. Waas †
Dept. of Aerospace Engineering, Univ. of Michigan, Ann Arbor, MI 48109-2140

Abstract

The motivation for the present study stems from the importance of delamination buckling as a viable mode of failure in laminated composite plates. This part of the study focuses on the prediction of the buckling loads and modes of unilaterally constrained rectangular plates. The laminated plates were modeled along the lines of classical lamination plate theory. Due to the out-of-plane thickness ratio of the delaminated near surface plies to that of the sublaminate (parent), the sublaminate was modeled as an infinitely rigid foundation which constrained the plate's out-of-plane response to be of one sign. The foundation was modeled as extensional springs exhibiting a nonlinear force-displacement relationship such that its stiffness as well as the plate's buckling mode sign can be controlled in a continuous fashion allowing the simulation of a rigid and tensionless foundation. Preliminary investigations of the buckling loads and modes of rectangular plates attached to such foundations and subjected to a uniform inplane stress field showed the validity of this model for the cases investigated and compared to previous exact results reported in the literature.

Introduction

The use of composite materials is becoming more frequent in today's modern structures. Such widely-used materials can be seen in many different applications especially in the aerospace industry. The desire to use composite materials over non-composite materials (e.g., metals) can be attributed mainly to the fact that composite materials exhibit a higher stiffness/weight ratio over their counterparts. The most commonly used type of composites, particularly in the aerospace industry, is laminated fibrous composites. In laminated composite plates the problem of delamination, which is the separation of plies, is of great importance since it resembles a viable mode of failure. The term "delamination failure" has two distinct implications. The first is when a failure occurs due to ply delamination which can be caused by high interlaminar stresses, or by fiber microbuckling in the case of fibrous laminated composites. The second is when the presence of a delamination prior to loading causes the composite structure to respond to the applied stresses in a certain way that will eventually lead to failure. It is the latter situation that is the focus of a current investigation being conducted by the authors.

The presence of a delaminated region between the surface plies in a laminated composite plate and the remaining portion of the laminate introduces the type of response that is observed when analyzing a thin elastic plate resting unattached on an elastic foundation. In a compressive load environment, near surface plies that are initially delaminated will undergo buckling prior to their growth [1]. When this occurs, the buckled displacements are of one sign, because the supporting sublaminate (parent) essentially acts like a rigid foundation. The models of delamination buckling and growth [2,3] display an interesting variety of behaviors depending upon the dimensions of the delamination, the load at which it is introduced as well as its history, the elastic properties, and the fracture energy. The growth of the delamination may be stable, unstable, or an unstable growth followed by a stable growth. At present, a global theory allowing the precise prediction of failure by delamination growth in real structures does not exist.

Ye [4] pointed out that the characteristics of two-dimensional delamination growth near a curved edge may differ significantly from those of one dimensional delamination near a straight edge in composite laminates. The author also pointed out that the role of out-of-plane delamination opening displacement on delamination growth needs further detailed identification.

In the investigation at hand we are interested in

*Graduate Research Assistant, Student Member AIAA
†Asst. Professor of Aerospace Engineering, Member AIAA.

studying the stability and growth of delaminated near surface plies that are formed adjacent to a circular cutout in compressively stressed laminated plates. Such a situation was observed in the results of experimental work reported in [5]. The study is conducted in two parts. In part I, the analysis is restricted to delaminations of a rectangular planform and which are subjected to a uniaxial compression load. In part II, the more general problem of delaminations conforming to planform of a curved annular sector (as reported in [5], to be formed adjacent to circular cutouts) in a non-homogeneous stress field (that which corresponds to the linear elastic solution of a remotely compressed anisotropic plate containing a circular cutout) is considered. Since a new approach to handling delamination buckling problems conforming to constraints imposed by the physical situation is introduced here, a two part study as described above was warranted. The present paper is limited to issues pertaining to part I.

Previous work relevent to that reported here was undertaken by Seide [6], who considered isotropic, simply supported, infinite plates. Similar issues relating to problems of constraint plate deformation have been considered by other researchers in different contexts. In [7], Weitsman presented an approximate solution for the problem of contact between an elastic plate and a semi infinite elastic half space. In [8,9], the same author presented several examples of beam/plate type structures resting on an elastic half space and subjected to transverse distributed and concentrated loads. In these investigations the extent of the contact area that develops between the beam/plate and the half space was determined by an approximate technique. The case of a beam that is subjected to a single concentrated transverse force and resting on a rigid foundation was addressed by Civelek and Erdogan [10]. Gladwell [11] considered a variety of plane, frictionless, unbonded contact problems and provided approximate solutions in terms of Chebyshev polynomials. Some corrections to the earlier work by Weitsman [9] was provided in this article. More recently, Celep [12], addressed the behavior of transversely loaded rectangular elastic plates resting on a tensionless Winkler foundation. In this investigation Galerkin's method [13] was used to obtain results for the areas of contact as well as plate displacement distributions. A finite element solution for the unilateral contact problem of the heavy elastica was presented by Kooi [14]. Soong and Choi [15] addressed the problem of an elastica that involves continuous and multiple discrete contact with a boundary. These investigations presented examples in which the elastica curvature assumes values which are less than the curvature of the restraining boundary, thus resulting in line contact. Roorda [16], in a recent article, presented many aspects of unilateral buckling problems.

None of the models that have been presented to date include the physical constraint condition on the buckled displacement being of one sign. This constraint can play a significant role in the quantitative prediction of delamination buckling as was shown by Shahwan and Waas [17]. In that study, investigations on the buckling behavior of an isotropic, infinitely long, simply supported plate, showed an increase of 33% in the buckling load when the plate was unilaterally constrained. Further, using the exact solution of the governing differential equations, the authors [17] presented values for the % increase in the buckling load for unilaterally constrained specially orthotropic composite plates. Furthermore, it is reasonable to postulate that the growth behavior would also be significantly affected if this constraint is accounted for. In addition, delamination buckling and growth in the presence of non-uniform prebuckled stress state has not been investigated. Such a situation is of great practical value. Delamination formation at free edges and around cutouts are situations that fall into this category.

Problem formulation

Due to the fact that the delaminated surface plies must undergo an out-of-plane buckling prior to their growth, it is important to investigate first the buckling behavior of such plies. This can be done along the lines of classical laminated plate theory (CLT). The surface plies are modeled as a laminated composite plate with a number of plies that is equal to the number of delaminated surface plies. Since the current investigation is concentrating on the case of a laminated plate with a circular hole, with the delamination being near the edge of the hole, the geometry of the modeled plate will be assumed to have a shape of an annular sector. This geometry is an approximation to the actual situation. The plate will have clamped boundary conditions around all of its edges except at the free edge (the edge that is shared by the delaminated region and the hole). The entire structure, i.e., the laminate that includes the hole as well as the delamination, will be subjected to a uniaxial and uniform compressive far field stress. In order to include in this model the physical constraint imposed on the plate's buckled displacements, a nonlinear elastic foundation model was implemented which has the special property of being tensionless. Although a completely tensionless springs model was desirable, it was not numerically stable; for the cases where the exact results are known [17], it resulted in values for the buckling load that were lower than these exact values. As a result, the ratio of the tension to the compression stiffnesses was taken to be a small number (e.g., for a bilinear foundation model, this ratio was taken to be $\frac{1}{100}$).

An exact closed-form solution for this problem cannot be obtained. As such, an approximate method of solution must be followed to formulate the equations governing the plate's response. From the expression of the total potential energy (1), one can operate either on its functional form directly (e.g., Rayleigh-Ritz method), or on its first variation form (e.g., Galerkin method).

$$\begin{aligned}\hat{\Pi} = \frac{1}{2}\int_0^b\int_0^a [\ & D_{11}\hat{w}_{,\hat{x}\hat{x}}^2 + 2\ D_{12}\hat{w}_{,\hat{x}\hat{x}}\hat{w}_{,\hat{y}\hat{y}} \\ & + D_{22}\hat{w}_{,\hat{y}\hat{y}}^2 + 4\ (D_{16}\hat{w}_{,\hat{x}\hat{x}} + D_{26}\hat{w}_{,\hat{y}\hat{y}})\hat{w}_{,\hat{x}\hat{y}} \\ & + 4\ D_{66}\hat{w}_{,\hat{x}\hat{y}}^2\ ]\ d\hat{x}d\hat{y} + \int_0^b\int_0^a \mathbf{W}_f\ d\hat{x}d\hat{y} \\ & - \int_0^b\int_0^a [\ \frac{1}{2}N_{11}\hat{w}_{,\hat{x}}^2 + q(\hat{x},\hat{y})\ \hat{w}\ ]\ d\hat{x}d\hat{y}.\end{aligned} \tag{1}$$

where $\hat{\Pi}$ = total potential energy,
a = plate's dimension in the x-direction,
b = plate's dimension in the y-direction,
D_{ij} = plate bending stiffnesses,
$\hat{w}$ = plate out-of-plane displacement,
$\hat{x}$, $\hat{y}$ = spatial coordinates,
N_{11} = pre-buckling inplane compressive load,
$\mathbf{W}_f$ = strain energy density of the foundation,
$q(\hat{x},\hat{y})$ = distributed transverse load.

After the proper coordinate transformation (1) can be modified and used for circular plates. The strain energy density of the elastic foundation $\mathbf{W}_f$ is defined as,

$$\mathbf{W}_f = \int k\ \hat{f}(\hat{w})\ d\hat{w}, \tag{2}$$

where k = stiffness parameter,
$\hat{f}(\hat{w})$ = force-displacement shape function.

For the case of a linear elastic foundation, $\hat{f}(\hat{w}) = \hat{w}$, and hence $\mathbf{W}_f = \frac{1}{2}k\hat{w}^2$, where k being the linear stiffness. Nondimensionalizing (1) will result in the following expression for the total potential energy:

$$\begin{aligned}\Pi = \frac{1}{2}\int_0^1\int_0^\xi [\ & w_{,xx}^2 + 2\frac{D_{12}}{D_{11}}w_{,xx}w_{,yy} + \frac{D_{22}}{D_{11}}w_{,yy}^2 \\ & + 4(\frac{D_{16}}{D_{11}}w_{,xx} + \frac{D_{26}}{D_{11}}w_{,yy})w_{,xy} + 4\frac{D_{66}}{D_{11}}w_{,xy}^2\ ]\ dxdy \\ & + \int_0^1\int_0^\xi \alpha\ \pi^4\ (\ \int f(w)\ dw\ )\ dxdy \\ & - \int_0^1\int_0^\xi [\ \frac{1}{2}\lambda\pi^2 w_{,x}^2 + Q(x,y)w\ ]\ dxdy.\end{aligned} \tag{3}$$

where $\Pi = \frac{\hat{\Pi}\ b^2}{D_{11}\ h^2}$,
h = plate thickness,
$x = \frac{\hat{x}}{b}$,
$y = \frac{\hat{y}}{b}$,
$\xi = \frac{a}{b}$ (plate's aspect ratio),
$w = \frac{\hat{w}}{h}$,
$\lambda = \frac{N_{11}\ b^2}{\pi^2\ D_{11}}$,
$\alpha = \frac{k\ b^4}{\pi^4\ D_{11}}$,
$f(w)$ = non-dimensional form of $\hat{f}(\hat{w})$,
$Q(x,y) = \frac{q(x,y)\ b^4}{D_{11}\ h}$,

Calculating the first variation of Π and applying the divergence theorem yields the following equation:

$$\begin{aligned}\delta\Pi = \int_0^1\int_0^\xi [\ & w_{,xxxx} + 2\ (\frac{D_{12}}{D_{11}} + \frac{D_{66}}{D_{11}})\ w_{,xxyy} \\ & + \frac{D_{22}}{D_{11}}w_{,yyyy} + 4\ (\frac{D_{16}}{D_{11}}w_{,xxxy} + \frac{D_{26}}{D_{11}}w_{,xyyy}) \\ & + \lambda\pi^2 w_{,xx} + \alpha\ \pi^4\ f(w) - Q(x,y)\ ]\ \delta w\ dxdy \\ & - \int_0^1 (M_{xx}\ \delta w_{,x} + M_{xy}\ \delta w_{,y} - V_x\ \delta w)_{x=0}^{x=\xi}\ dy \\ & - \int_0^\xi (M_{xy}\ \delta w_{,x} + M_{yy}\ \delta w_{,y} - V_y\ \delta w)_{y=0}^{y=1}\ dx.\end{aligned} \tag{4}$$

where M_{xx}, M_{yy}, M_{xy}, V_x, and V_y are the nondimensional moments and shear forces at the plate's boundary and are given in the following equations:

$$\begin{aligned}M_{xx} &= -(w_{,xx} + \frac{D_{12}}{D_{11}}w_{,yy} + 2\frac{D_{16}}{D_{11}}w_{,xy}) \\ M_{yy} &= -(\frac{D_{12}}{D_{11}}w_{,xx} + \frac{D_{22}}{D_{11}}w_{,yy} + 2\frac{D_{26}}{D_{11}}w_{,xy}) \\ M_{xy} &= -(\frac{D_{16}}{D_{11}}w_{,xx} + \frac{D_{26}}{D_{11}}w_{,yy} + 2\frac{D_{66}}{D_{11}}w_{,xy}) \\ V_x &= -(w_{,xxx} + 3\frac{D_{16}}{D_{11}}w_{,xxy} + \frac{D_{12}+2D_{66}}{D_{11}}w_{,xyy} \\ &\qquad + \frac{D_{26}}{D_{11}}w_{,yyy} + \lambda\pi^2\ w_{,x}) \\ V_y &= -(\frac{D_{16}}{D_{11}}w_{,xxx} + \frac{D_{12}+2D_{66}}{D_{11}}w_{,xxy} \\ &\qquad + 3\frac{D_{26}}{D_{11}}w_{,xyy} + \frac{D_{22}}{D_{11}}w_{,yyy})\end{aligned} \tag{5}$$

Investigation of the stability of any equilibrium state requires the necessary and sufficient condition on the total potential energy Π be stationary. From the theorem of minimum total potential energy,

$$\delta\Pi\ =\ 0. \tag{6}$$

To determine the values of the buckling load and the corresponding mode, one can assume the following separable form for $w(x,y)$:

$$w(x,y) \;=\; \sum_{i=1}^{M}\sum_{j=1}^{N} A_{ij}\phi_{ij}(x,y), \qquad (7)$$

where A_{ij} are unknown constant coefficients, and $\phi_{ij}(x,y)$ are the assumed deformation shape functions. $\phi_{ij}(x,y)$ must satisfy the plate's kinematic boundary conditions. In the Galerkin method formulation, the weight function that must be multiplied by the governing equation and integrated over their domain in order to minimize the error in approximating the plate's response, are taken to be the shape functions $\phi_{ij}(x,y)$. Since $\phi_{ij}(x,y)$ are known functions, the first variation of $w(x,y)$ is

$$\delta w(x,y) \;=\; \sum_{i=1}^{M}\sum_{j=1}^{N} \delta A_{ij}\;\phi_{ij}(x,y). \qquad (8)$$

Substituting the above expression for $\delta w(x,y)$ into the expression for $\delta\Pi$ in conjunction with the statement of minimum total potential energy theorem one arrives at the following set of MxN algebraic equations;

$$\begin{aligned}
&\int_0^1\int_0^\xi \Big[\, w_{,xxxx} + 2\,\Big(\frac{D_{12}}{D_{11}} + \frac{D_{66}}{D_{11}}\Big)\, w_{,xxyy} \\
&+\frac{D_{22}}{D_{11}} w_{,yyyy} + 4\,\Big(\frac{D_{16}}{D_{11}} w_{,xxxy} + \frac{D_{26}}{D_{11}} w_{,xyyy}\Big) \\
&+\lambda\pi^2 w_{,xx} + \alpha\,\pi^4\, f(w) - Q(x,y)\,\Big]\,\phi(x,y)\,dxdy \qquad (9) \\
&-\int_0^1 \big(M_{xx}\,\phi_{ij,x} + M_{xy}\,\phi_{ij,y} - V_x\,\phi_{ij}\big)_{x=0}^{x=\xi}\,dy \\
&-\int_0^\xi \big(M_{xy}\,\phi_{ij,x} + M_{yy}\,\phi_{ij,y} - V_y\,\phi_{ij}\big)_{y=0}^{y=1}\,dx \;=\; 0.
\end{aligned}$$

Knowing the material properties D_{ij}, the transverse load parameter $Q(x,y)$ and the function describing the foundation type $f(w)$ the buckling load parameter λ as well as the unknown coefficients A_{ij} can be determined using an incremental load technique. Since the values of A_{ij} cannot be controlled individually, the algorithm increments the load parameter λ and monitors the determinant of the incremental stiffness matrix. The buckling load and its corresponding mode were obtained when the incremental stiffness matrix became singular.

The main distinction between this study and most of the previous ones lies in the elastic foundation model. The type of models needed to incorporate the physics of unilaterally constrained plates must exhibit not only a nonlinear force-displacement relationship but must also be deformation-sign dependent. A typical model that pertains to this category is the bilinear model.

$$F = \begin{cases} \alpha_1 w & \text{if } w > 0 \\ 0 & \text{if } w = 0 \\ \alpha_2 w & \text{if } w < 0 \end{cases} \qquad (10)$$

where F = spring force,
$\alpha_1 = \frac{k_1\, b^4}{\pi^4\, D_{11}}$,
$\alpha_2 = \frac{k_2\, b^4}{\pi^4\, D_{11}}$,
w = spring's (also plate's) displacement,
k_1 and k_2 are the linear stiffness coefficients.

Although this model is continuous, it is not differentiable and has an undefined stiffness at the origin ($w = 0$). Another model that is continuous as well differentiable is

$$F \;=\; \alpha\; w\; \left[\frac{1}{2}\,(1 - Tanh\,(\beta w))\right], \qquad (11)$$

where F = spring force,
$\alpha = \frac{k\; b^4}{\pi^4\; D_{11}}$,
k is the linear stiffness coefficient,
β = spring (foundation) attachment coefficient.

In the above model, the foundation attachment can be controlled by changing the value of the parameter β. Large values of β implies less attachment. Theoretically, $\beta = \infty$ implies the foundation is fully unattached, while $\beta = 0$ implies the foundation is fully attached (note that if $\beta = 0$ the $\frac{1}{2}$ factor in (11) should be replaced by 1 in order to recover the linear case where $F = \alpha\; w$). Furthermore, the foundation (springs) stiffness can be increased to larger values by changing the value of α to simulate a stiffer foundation.

Results and Discussion

Example results are presented for isotropic rectangular plates. Although the results given are for simply supported plates, the model as well as the approach are completely independent of the plate's boundary conditions. For large aspect ratio ξ, the buckling load parameter λ as well as its corresponding deformation are independent of the boundary conditions at $x = 0$ and $x = \xi$ (except in the neighborhood of $x = 0$ and $x = \xi$ where boundary effects are very significant).

In the case of isotropic plates the value of λ has an exact lower bound value, which is obtained when the elastic foundation is linear (as opposed to bilinear). Although this exact lower bound value can be obtained for the bilinear foundation model (10), it cannot as easily be obtained for the $Tanh(\beta w)$ model. Assuming an approximate deformation function w of the form

$$w(x,y) = \sum_{i=1}^{M}\sum_{j=1}^{N} A_{ij} \, sin\left(i\pi\frac{x}{\xi}\right) \, sin\left(j\pi y\right), \quad (12)$$

the expression for $\lambda_{lowerbound}$ can be found to be,

$$\lambda_{lowerbound} = 2 + \left(\frac{i}{\xi}\right)^2 + (1+\alpha_1)\left(\frac{\xi}{i}\right)^2 \quad (13)$$

The fact that the effect of the end boundary conditions (at $x = 0, x = \xi$) on the buckling load and mode diminishes with increasing plate aspect ratio ξ can attributed to the fact that, when the plate gets longer, its response is governed mainly by the inner region (region away from $x = 0$ and $x = \xi$), and in turn this region does not "feel" the boundary effects.

The elastic foundations models investigated in this study yielded results that are consistent with each other (bilinear model vs. the $Tanh(\beta w)$ model). Ideally the foundation model should have zero tensile stiffness (α_1) and an infinite compressive stiffness (α_2). In the bilinear foundation model (10) this was accomplished by assigning to the ratio of the tension stiffness α_1 to that of compression α_2 a value of $\frac{1}{100}$. This $\frac{\alpha_1}{\alpha_2}$ ratio was found to give physically as well as mathematically consistent results. In the $Tanh(\beta w)$ model (11), the value of the attachment coefficient β was taken to be of the order of 10^3 to simulate a rigid foundation. It is worthwhile mentioning that the parameter β in the $Tanh(\beta w)$ model has a similar meaning to the ratio $\frac{\alpha_1}{\alpha_2}$ in the bilinear model.

As mentioned earlier, in this analysis $Q(x,y)$ was not taken to be zero. However, it was established that λ was independent of Q_{max} and $Q(x,y)$. For small aspect ratio plates, different $Q(x,y)$ resulted in slightly different buckling modes (due to 'end' effects) but this diminished with increasing plate aspect ratio ξ. A completely negative $Q(x,y)$ (i.e., compressing the plate towards the foundation) will result in buckling loads that are extremely high. In [18], Allan shown that, a strip that is resting on a rigid foundation and subjected to a negative transverse pressure as well a compressive in-plane load will have an infinite buckling load. In this study, different transverse load distributions were investigated, and it was observed that, although there is no change (for the plates considered) in the buckling load and the periodic portion of the response, the 'end effects' (x-boundary conditions) can be minimized if $Q(x,y)$ has negative values near the plate's boundary.

Figures 2 and 4 show plots of the 'response curves' for two different plates studied. The compressive load λ is plotted against the normalized amplitude ratio $\frac{\|A\|}{\|A_0\|}$. The corresponding 'slope' of these plots is shown in Figures 3 and 5. $\|A_0\|$ is the amplitude of the unknown Galerkin coefficient's vector for $\lambda = 0$. Thus, the local slopes are normalized with respect to the initial slope. This normalized slope is a measure of the determinant of the incremental stiffness matrix. At buckling the incremental stiffness matrix becomes singular and hence the slope is zero. In Fig.(3), one can notice a sudden increase in the slope near the point of buckling. This is due to the plate "touching" the foundation and experiencing high stiffness. On the other hand, in Fig. (5) where the transverse load has negative as well as positive regions, this is not strongly present. Thus, the transverse load affects the plate response, but not the buckling load nor the mode. As can be seen from Figures 2-4, the results obtained from the methodology presented here, are in very good agreement with previously reported results [6].

Concluding Remarks

A methodology is developed to obtain the buckling loads of unilaterally constrained plates. The problem arose in connection with the calculation of buckling loads of delaminated surface plies in laminated plates. The developed methodology is used to calculate example results for isotropic rectangular plates. The results are shown to be in good agreement with 'exact' results available in the literature [6]. The methodology used in obtaining the 'exact' results was limited to special plate configurations and it was the purpose of the first part of this study to develop a more general method to study the problem of plates of arbitrary shape applicable to surface delaminations in laminated plates. Results for orthotropic rectangular plates as well as for plates of different shapes (annular sectors) will be reported in the near future.

Acknowledgements

The authors are appreciative of the financial support received through AFOSR Initiation Grant RI-B-91-06 awarded through the Engineering Foundation.

References

1. Chai, H., Babcock, C., and Knauss, W., "One Dimensional Modelling of Failure in Laminated

Plates by Delamination Buckling," *International Journal of Solids and Structures*, Vol. 17, No. 11, 1981, pp. 1069-1083.

2. Evans, A. G., and Hutchinson, J. W., "On the Mechanics of Delamination and Spalling in Compressed Films," *International Journal of Solids and Structures*, Vol. 20, No. 5, 1984, pp. 455-466.

3. Chai, H., and Babcock, C., "Two Dimensional Modelling of Compressive Failure in Delaminated Laminates," *Journal of Composite Materials*, Vol. 19, 1985, pp. 67-98.

4. Ye, L., "Characteristic of Delamination Growth in a Notched Graphite/Epoxy Composite Laminate," *Journal of Reinforced Plastics and Composites*, Vol. 8, January 1989, pp. 79-90.

5. Waas, A., Babcock, C.D. and Knauss, W.G., "Compression Failure of Laminates in the Presence of Cutouts," *Int. J. Solids and Structures*, Vol. 26, No. 9/10, 1990, pp. 1071-1098.

6. Seide, P., "Compressive Buckling of a Long Simply Supported Plates on an Elastic Foundation," *Journal of Aeronautical Sciences*, June 1958, pp. 382-392.

7. Weitsman, Y., "On the Unbonded Contact Between Plates and an Elastic Half Space," *Journal of Applied Mechanics*, Vol. 36, No. 2, ASME Trans., Vol. 91, Series E, June 1969, pp. 198-202.

8. Weitsman, Y., "On Foundations That React in Compression Only," *Journal of Applied Mechanics*, ASME Trans., December 1970, pp. 1019-1030.

9. Weitsman, Y., "A Tensionless Contact Between Beam and an Elastic Half-Space," *International Journal of Engineering Science*, 1972, Vol. 10, pp. 73-81.

10. Civelek, M. B., and Erdogan, F., "Interface Separation in a Frictionless Contact Problem for an Elastic Layer," *Journal of Applied Mechanics*, ASME Trans., March 1976, pp. 175-177.

11. Gladwell, G. M. L., "On Some Unbonded Contact Problems in Plane Elasticity Theory," *Journal of Applied Mechanics*, ASME Trans., June 1976, pp. 263-267.

12. Celep, Z., "Rectangular Plates Resting on Tensionless Elastic Foundations," *Journal of Engineering Mechanics*, Vol. 114, No. 12, Dece
2083-2092.

13. Fung, Y. C., *Foundations of Solid Mechanics*, Prentice-Hall, Inc., Englewood Cliffs, NJ., 1965, pp. 337-338.

14. Kooi, B. W., "A Unilateral Contact Problem with the Heavy Elastica Solved by Use of Finite Elements," *Computers and Structures*, Vol. 21., No. 1/2, 1985, pp. 95-103.

15. Soong, T. C., and Choi, I., "An Elastica that Involves Continuous and Multiple Discrete Contacts with a Boundary," *International Journal of Mechanical Sciences*, Vol. 28, No. 1, 1986, pp. 1-10.

16. Roorda, J., "Buckles, Bulges and Blow-ups," *Applied Solid Mechanics*, Tooth, A. S. and Spence, J., eds., Elsevier, 1988, pp. 347-380.

17. Shahwan, K., and Waas, A.,"Elastic Buckling of Infinitely Long Specially Orthotropic Plates on Tensionless Foundations," *Journal of Engineering Materials and Technology*, ASME Trans., October 1991, pp. 396-403.

18. Allan, T., "One-Way Buckling of a Compressed Strip Under Lateral Loading," *Journal of Mechanical Engineering Science*, Vol. 10, No. 2, 1968, pp. 173-181.

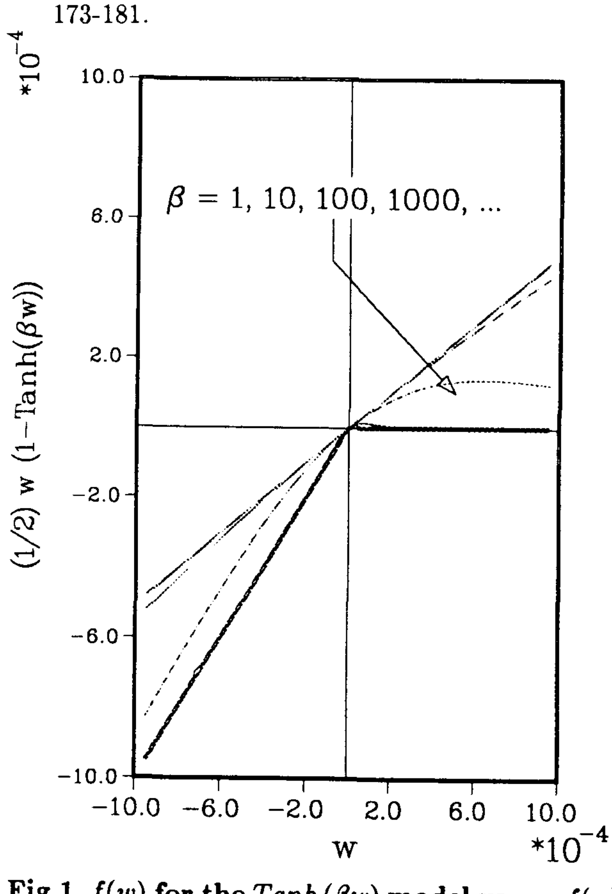

Fig.1 $f(w)$ **for the** $Tanh\,(\beta w)$ **model vs.** w, $f(w) = \frac{1}{2}\, w\, (1 - Tanh\,(\beta w))$

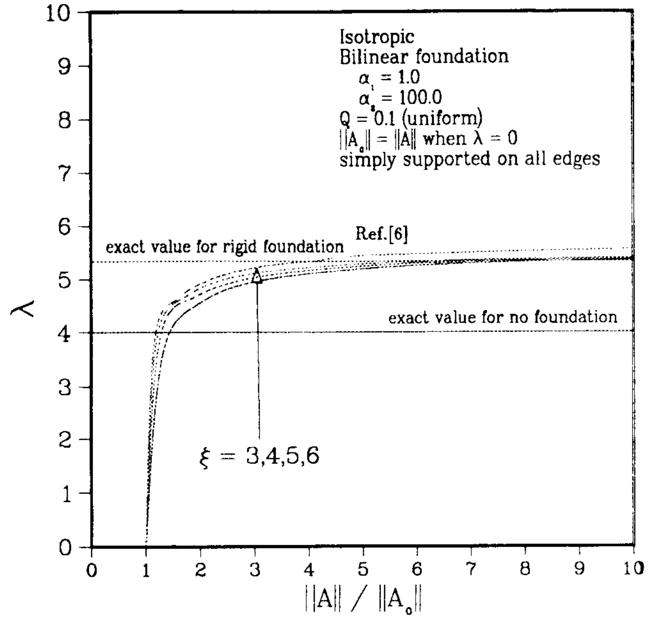

Fig.2 Buckling Load Parameter λ vs. Normalized $||A||$

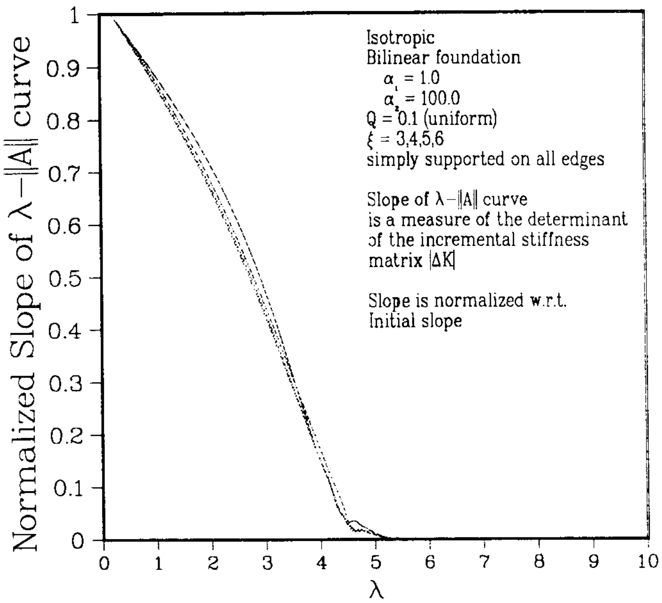

Fig.3 Normalized Slope of $\lambda - ||A||$ curve vs. λ

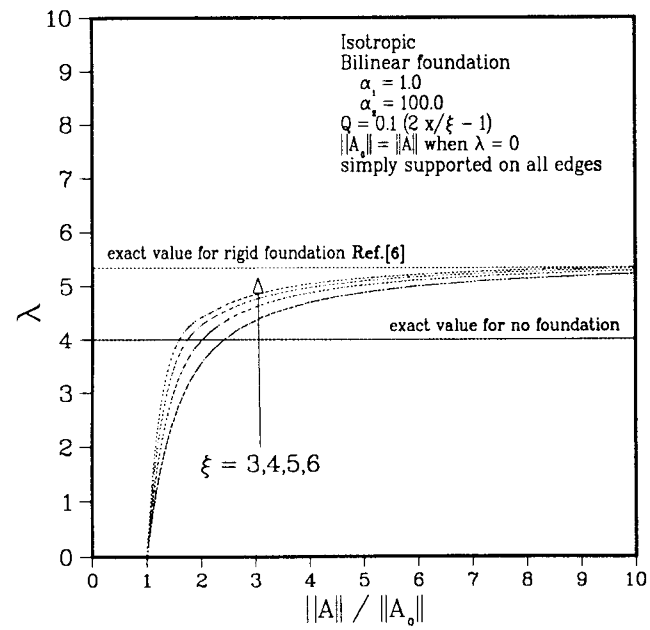

Fig.4 Buckling Load Parameter λ vs. Normalized $||A||$

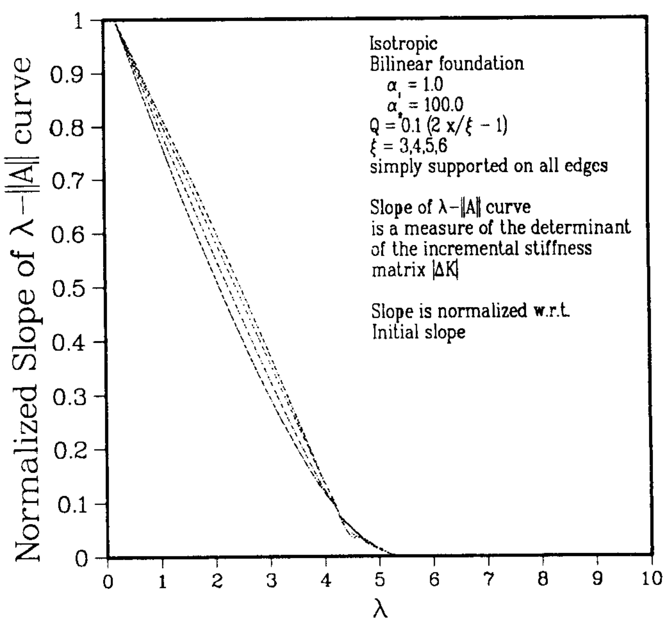

Fig.5 Normalized Slope of $\lambda - ||A||$ curve vs. λ