For a basic aircraft weight of 14,000 lbs., Mr. Rutowski's derivation gives the numerical results

$$\Delta h/\Delta W = -(20{,}940/W) = -(20{,}940/14{,}000) = -1.5 \text{ ft. per lb.}$$
$$\Delta V/\Delta W = 0$$

for altitudes above 35,000 ft. and all speeds. His formula was derived on the assumption that Eq. (1) can be rebalanced after a change in weight by an appropriate change in ρ. He neglected the fact that any changes in ρ will act to unbalance Eq. (2) and that both Eqs. (1) and (2) must be satisfied for equilibrium during flight.

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[1] Rutowski, Edward S., *Energy Approach to the General Aircraft Performance Problem*, Journal of the Aeronautical Sciences, Vol. 21, No. 3, pp. 187–195, March, 1954.

[2] *Specification No. N-1603-C*, Pratt & Whitney Aircraft Division, United Aircraft Corporation, East Hartford, Conn.

Author's Reply

Edward S. Rutowski
Douglas Aircraft Company, Inc.
October 29, 1954

Mrs. Birman claims that the subject paper[1] neglects the thrust-drag equilibrium requirement in deriving a formula relating the change in gross weight of a jet-powered aircraft to the change in altitude during the cruise portion of the flight. Her claim, however, is not justified in that the thrust-drag equilibrium requirement is implicitly satisfied for a turbojet-powered aircraft flying at a constant attitude and at a constant Mach Number if the thrust is assumed to be directly proportional to the atmospheric density. This frequently made assumption, which was implicit in the derivation in the subject paper, is reasonably good for a turbojet at altitudes above 35,000 ft. in the isothermal layer since the turbojet is essentially a temperature-limited device. The engine characteristics used by Mrs. Birman apparently do not fit this assumption well, and hence her results differ from those in the subject paper.

If Mrs. Birman had chosen an engine with the characteristic of having a thrust directly proportional to the atmospheric density at a constant Mach Number above 35,000 ft., her results and those in the subject paper would have coincided. This may be seen by evaluating Mrs. Birman's Eqs. (8) and (9) for the case when $T = K\rho$.

Then, since

$$\partial T/\partial \rho = T/\rho$$

Eq. (8) reduces to

$$\Delta \rho/\Delta W = \rho/W$$

and Eq. (9) reduces to

$$\Delta V/\Delta W = 0$$

These equations now are equivalent to those in the subject paper and are therefore not the result of neglecting the thrust-drag equilibrium requirement as claimed by Mrs. Birman but rather the result of an implicit assumption for the thrust variation of a turbojet above 35,000 ft. Furthermore, despite Mrs. Birman's having demonstrated one example where it is not strictly valid, this approximation still has justification in my mind for the purposes of the analysis in the paper.

Reference

[1] Rutowski, Edward S., *Energy Approach to the General Aircraft Performance Problem*, Journal of the Aeronautical Sciences, Vol. 21, No. 3, pp. 187–195, March, 1954.

Forced Convection Through a Laminar Boundary Layer over an Arbitrary Surface with an Arbitrary Temperature Variation*

Myron Tribus† and John Klein‡
University of California, Los Angeles, and University of Michigan, Ann Arbor, Respectively
August 11, 1954

In a previous report the authors considered the general problem of forced convection in several systems when the wall temperature is nonuniform.[1] Table 1 summarizes the results of that report and is reproduced here because several new entries have been made in the table and a number of typographical errors have been discovered in reference 1. In all cases, if the wall temperature is given, the heat flux is to be calculated from

$$q(x) = \int_{\xi=0}^{x} h(\xi, x) dTw(\xi) \tag{1}$$

where

$q(x)$ = heat flux from the wall, B.t.u./hr. ft.^{2}
$h(\xi, x)$ = an integrating kernel (see Table 1), B.t.u./hr. ft.^{2} °F.
$T_w(x)$ = wall temperature, °F.
x = distance along the wall, ft.
ξ = dummy variable, ft.

and if the heat flux is given, the wall temperature is calculated from

$$T_w(x) - T_0 = \int_{\xi=0}^{x} g(\xi, x) q(\xi)\, d\xi \tag{2}$$

where

T_0 = reference temperature of fluid, °F.
$g(\xi, x)$ = integrating kernel (see Table 1), °F./(B.t.u./hr. ft.^{2}) ft.

Integral 1 is interpreted in the Stieltjes sense.

In this note we give explicit directions for using the fourth entry of Table 1 to compute convection from an arbitrary surface. The fluid properties are constant.** Aerodynamic heating is zero.

The method consists of first solving the momentum equation by using the velocity distributions of Hartree[2] and the "patching" technique of Eckert.[3] The energy equation is then solved using the method of Lighthill.[4] The shear stresses obtained from the momentum equation are modified according to the method of Tifford[5] before being used in the Lighthill integrating kernel (line 4, Table 1). The results of two typical calculations are shown in Figs. 1 and 2, where the experimental results of Giedt[6] and Drake[7] are given. We refer to the method as the H.E.L.T. method (Hartree, Eckert, Lighthill, Tifford). A study of five references[1–5] will provide the background for following these instructions for making the calculations such as are demonstrated in Figs. 1 and 2. For a given surface it is presumed the pressure distribution is known and either the heat flux or wall temperature are prescribed along the surface, with the unprescribed quantity to be determined.

(1) From the pressure distribution a calculation is made and a graph prepared showing $u(x)$ and du/dx, where $u(x)$ = velocity just outside the boundary layer in feet per second, and x = distance along the surface in feet.

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† Associate Professor of Engineering.

‡ Research Assistant.

** See reference 18 for extension to air with variable properties.

FEBRUARY, 1954

TABLE I
INTEGRATING KERNELS FOR NONISOTHERMAL CONVECTION

$$q(x)=\int_{\xi=0}^{x} h(\xi,x)\,dT_w(\xi) \qquad T_w(x)-T_\infty=\int_{\xi=0}^{x} g(\xi,x)\,q(\xi)\,d\xi$$

AUTHOR	REF.	SYSTEM	METHOD OF SOLUTION	$h(\xi,x)$	$g(\xi,x)$
RUBESIN	8	Laminar flow over a flat plate. Zero pressure gradient. Fluid properties constant.	Velocity and temperature profiles postulated linear in y. Thermal boundary layer thickness proportional to momentum thickness. (Integral)	$\frac{0.304k}{x} Pr^{1/3} Re_x^{1/2}\left[1-(\frac{\xi}{x})^{3/4}\right]^{-1/3}$	$\frac{1}{6(1/3)!(2/3)!}\frac{1}{0.304k} Pr^{-1/3} Re_x^{-1/2}\left[1-(\frac{\xi}{x})^{3/4}\right]^{-2/3}$
ECKERT	9	Laminar flow over a flat plate. Zero pressure gradient. Fluid properties constant.	Velocity and temperature profiles postulated as a cubic in y. Integral method of solution.	$\frac{0.33k}{x} Pr^{1/3} Re_x^{1/2}\left[1-(\frac{\xi}{x})^{3/4}\right]^{-1/3}$	$\frac{1}{6(1/3)!(2/3)!}\frac{1}{0.33k} Pr^{-1/3} Re_x^{-1/2}\left[1-(\frac{\xi}{x})^{3/4}\right]^{-2/3}$
LEVEQUE	10	Laminar flow over a flat plate. Shear at the wall postulated constant. Fluid properties constant.	Velocity profile taken as u=cy, not dependent upon x. Term $v\partial T/\partial y$ thereby drops out of energy equation and resulting equation is solved.	$\frac{kPr^{1/3}}{3(1/3)!}(\rho/9\mu)^{1/3}\left[(du/dy)_{y=0}\right]^{1/3}(x-\xi)^{-1/3}$	$\frac{2}{3k(2/3)!} Pr^{-1/3}(\rho/9\mu)^{-1/3}\left[(du/dy)_{y=0}\right]^{-1/3}(x-\xi)^{-2/3}$
LIGHTHILL	4	Laminar flow over a surface with known variation in surface shear stress. Constant fluid properties.	Velocity profile taken as $u=\frac{\tau(x)}{\mu}y$ where $\tau(x)$= wall shear. Resulting differential equations solved.	$\frac{k}{(1/3)!}Pr^{1/3}(\rho/9\mu^2)^{1/3}\sqrt{\tau(x)}\left[\int_\xi^x\sqrt{\tau(z)}\,dz\right]^{-1/3}$	$\frac{2}{9(2/3)!k}\left(\frac{9\mu^2}{\rho Pr}\right)^{1/3}\left[\int_\xi^x\sqrt{\tau(z)}\,dz\right]^{-2/3}$
BOND	11	"Wedge-flows". Fluid properties constant. (Velocity over wedge given by $u=c_1x^m$)	Velocity near wall taken as linear in y. Resulting differential equation solved for two cases: (a) Step function in temperature. (b) Step function in heat flux.	$\left(\frac{1+m}{2}\right)^{1/2}\frac{k}{(1/3)!x} b^{1/3} Re_x^{1/2}\left[1-(\frac{\xi}{x})^{c}\right]^{-1/3}$ $b=\frac{Pr}{6}f''(0) \qquad c=\frac{3}{4}(1+m)$ $f''(0)$ is a dimensionless velocity gradient tabulated as a function of m in reference 2.	$\frac{2c}{9(2/3)!k}\left(\frac{2}{1+m}\right)^{1/2} b^{-1/3} Re_x^{-1/2}\left(\frac{x}{\xi}\right)^{\frac{m+1}{2}}\left[\frac{x^c-\xi^c}{\xi^c}\right]^{-2/3}$
MODIFIED LEVEQUE	1	Laminar flow over a surface. Fluid properties constant. $d\tau/dx$ small.	Velocity profile taken as $u=\frac{\tau(x)}{\mu}y$. $d\tau/dx$ small enough to make $v\partial T/\partial y$ negligible. $\tau(x)$ = wall shear.	$\frac{3k}{(1/3)!}Pr^{1/3}\left[9\rho/\mu^2\right]^{1/3}\left[\int_\xi^x\frac{dz}{\tau(z)}\right]^{-1/3}$	
RUBESIN	12	Turbulent flow over a flat plate. Fluid properties constant.	Velocity and temperature profiles taken as following 1/7 power law. Integral method.	$\frac{0.0288k}{x} Pr^{1/3} Re_x^{0.8}\left[1-(\frac{\xi}{x})^{39/40}\right]^{-7/39}$	$\frac{(28/195)\ Pr^{-1/3}}{(32/39)!(7/39)!0.0288k} Re_x^{-0.8} x^{0.8}(x^{\frac{39}{40}}-\xi^{\frac{39}{40}})^{-\frac{32}{39}}$
SEBAN	13	Turbulent flow over a flat plate. Constant fluid properties.	Velocity profile taken as 1/7 power of y. Temperature profile taken as linear in y near the wall, 1/7 power of y outside the laminar sub-layer.	$\frac{0.0289k}{x} Pr^{1/9} Re_x^{0.8}\left[1-(\frac{\xi}{x})^{9/10}\right]^{-1/9}$	$\frac{(8/90)\ Pr^{-1/9}}{(8/9)!(1/9)!0.0289k} Re_x^{-0.8} x^{0.8}(x^{\frac{9}{10}}-\xi^{\frac{9}{10}})^{-\frac{8}{9}}$
MAISEL AND SHERWOOD	14	Experimental measurements on a mass transfer apparatus.	Empirical equation "best-fit" to data. (Translated to air heat transfer system by present author.)	$\frac{0.035k}{x} Re_x^{0.8}\left[1-(\frac{\xi}{x})^{0.8}\right]^{-0.11}$	
GRAETZ	15	Laminar flow in a tube. Parabolic velocity distribution. *Constant fluid properties.*	Differential equation solved by separation of variables. Only *first three eigen values known.*	$\frac{4k}{d}\sum_{i=1,2,3}^{\infty} c_i e^{-a_i\beta(x-\xi)}$ $\beta=\frac{\pi k}{2wc_p}$ i: 1, 2, 3 a_i: 7.113, 44.49, 113.8 c_i: 0.748, 0.538, 0.459	$\frac{\beta d}{4k}\sum c'_n e^{-a'_n\beta(x-\xi)}$ n = 1, 2, 3 c'_n = 8.00, 8.81, 12.02 a'_n = 0.00, 25.64, 84.62
POPPENDIEK	16	Turbulent flow of a liquid metal in a tube. Velocity profile established. Constant fluid properties.	Thermal conductivity of metal postulated large enough to render eddy heat transport negligible. Velocity profile follows. $u=B(\frac{y}{s})^p$	$\frac{\frac{k}{s}\left[\frac{Re_s Pr}{2(P+1)}\right]^{1/2(P+1)}}{(\frac{P+3}{P+2})!}\sum_{n=1,2,3}^{\infty} c_n\beta_n^{\frac{1}{P+2}} e^{-\frac{\beta_n^2}{s}(x-\xi)}$ Constants c_n and β_n tabulated in reference 12.	
POPPENDIEK MODIFICATION OF LEVEQUE	17	Turbulent flow of a liquid metal in a tube. Velocity profile established. Constant fluid properties.	Thermal conductivity of metal postulated large enough to render eddy heat transport negligible. Velocity profile follows. $u=B(\frac{y}{s})^p$	$\frac{k}{(\frac{P+3}{P+2})!}\left[\frac{B}{(P+2)^2 s^p a}\right]^{\frac{1}{P+2}}(x-\xi)^{-\frac{1}{P+2}}$	$\frac{(P+3)(P+1)}{(P+2)^3 k(\frac{P+1}{P+2})!}\left[\frac{B}{(P+2)^2 s^p a}\right]^{-\frac{1}{P+2}}(x-\xi)^{-\frac{P+1}{P+2}}$

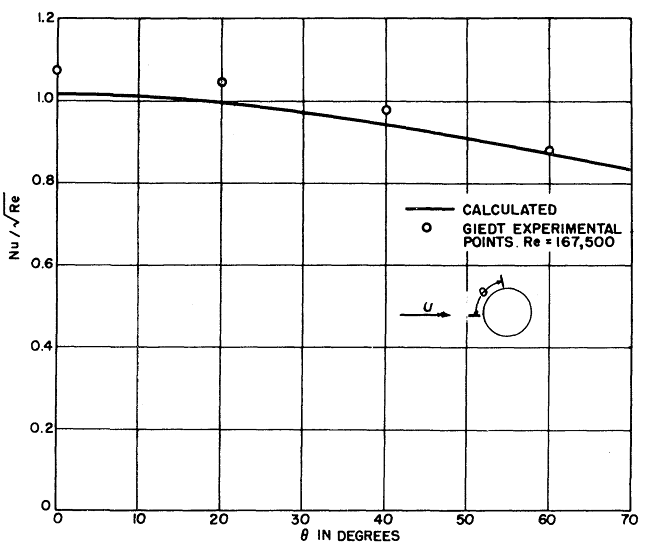

FIG. 1. Comparison of calculated and measured heat transfer from a uniformly heated circular cylinder in cross flow.[6]

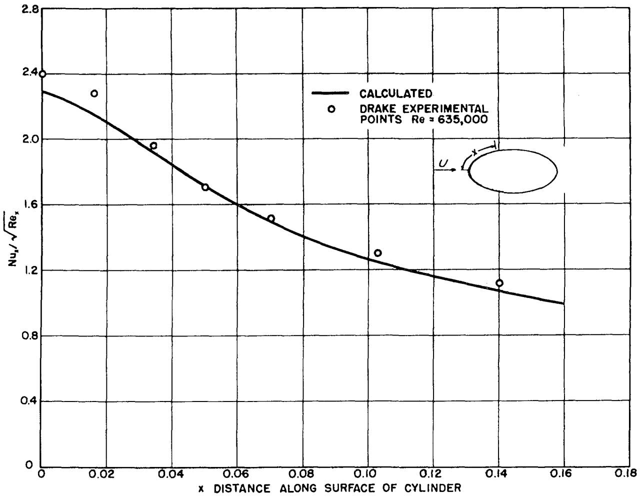

FIG. 2. Comparison of calculated and measured[7] heat transfer from a uniformly heated elliptical cylinder in cross flow.

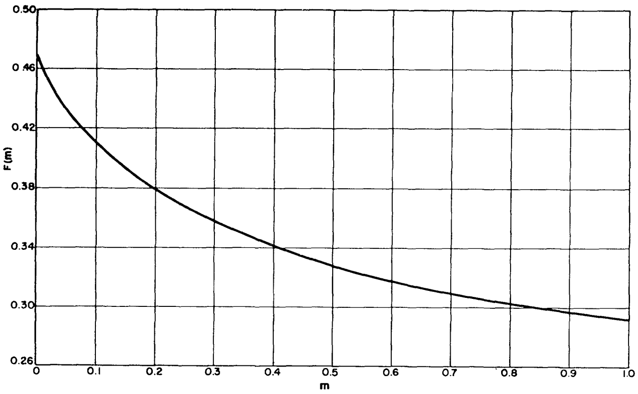

FIG. 3. Function to be used in calculating with Eq. (2).

(2) Starting from the stagnation point, it will be found that for a certain distance the graph of $u(x)$ versus x is a straight line. In this region the momentum thickness ϑ, in feet, is given by: $\vartheta = 0.2921x\ (Re_x)^{-1/2}$ where the Reynolds modulus, Re_x, is formed from the local velocity, $u(x)$, and the distance x from the stagnation point. In this region the "wedge parameter," m = 1 (see reference 2).

(3) Downstream of this region it is necessary to solve the following equations (see reference 3):

$$\frac{d\vartheta^2}{dx} = \frac{1-m}{m}\frac{\vartheta^2}{u}\frac{du}{dx} \tag{1}$$

$$\frac{\vartheta^2(du/dx)}{v} = \frac{2m}{1+m}[F(m)]^2 \tag{2}$$

v = kinematic viscosity in feet per second. $F(m)$ = function obtained from references 2 and 3. Thus, when the velocity is no longer linear in x, we use the known values of ϑ^2, du/dx, to calculate $F(m)$ and from the graph, Fig. 3, we find m. From Eq. (1) then, a new value of ϑ^2 is computed for a position Δx downstream. In this way a tabular set of values of $\vartheta(x)$, $m(x)$, $F(m)$, etc., is prepared. Eckert[3] suggests the isocline method of solution.

(4) The wall shear stress is calculated from the equation

$$\tau_0 = \mu(du/dy)_{y=0} = \mu u(x)F(m)f''(0, m)/\vartheta(x)$$

Fig. 4 gives a graph of $f''(0, m)$.

(5) Before substituting this shear stress in the Lighthill integrating kernels (line 4, Table 1), the correction of Tifford is applied, which may be written:

$$\tau_{0\ (\text{effective})} = \tau_{0\ (\text{Hartree})}\left[1 - \frac{4}{3}\frac{2m}{1+m}\frac{(Fm)}{f''(0,m)}Pr^{-1/4}\right]$$

Pr = Prandtl modulus of the fluid, dimensionless.

This is a quasi-empirical correction to Lighthill's approximation for the more complicated velocity distribution in the boundary layer.

TABLE 2
The Functions $F(m)$ and $f''(0, m)$

m	$[F(m)]^2$	$f''(0, m)$
0.0	0.4696	0.470
0.1	0.410	0.674
0.2	0.379	0.805
0.3	0.357	0.900
0.4	0.340	0.977
0.5	0.327	1.040
0.6	0.3175	1.095
0.7	0.309	1.134
0.8	0.302	1.17
0.9	0.297	1.20
1.0	0.292	1.23

(6) Values of τ_0 (effective) are then substituted into the integrating kernels of Table 1, line 4, and the integrations performed graphically. Table 2 gives numerical values for $F(m)$, $f''(0, m)$.

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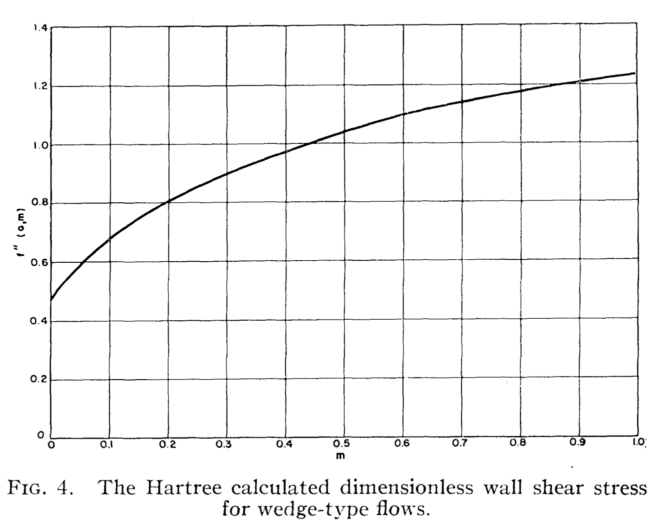

FIG. 4. The Hartree calculated dimensionless wall shear stress for wedge-type flows.