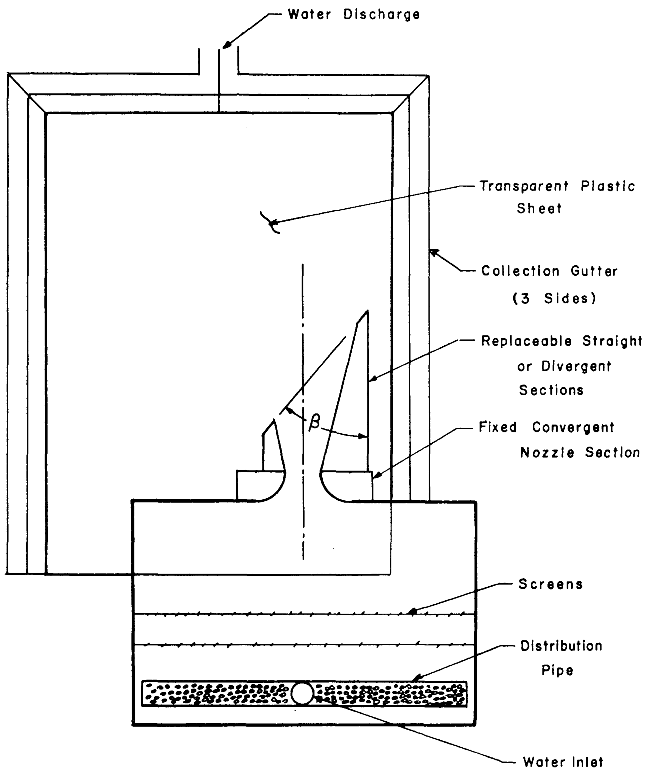

FIG. 1. Sketch of jet-deflection hydraulic-analogy test setup.

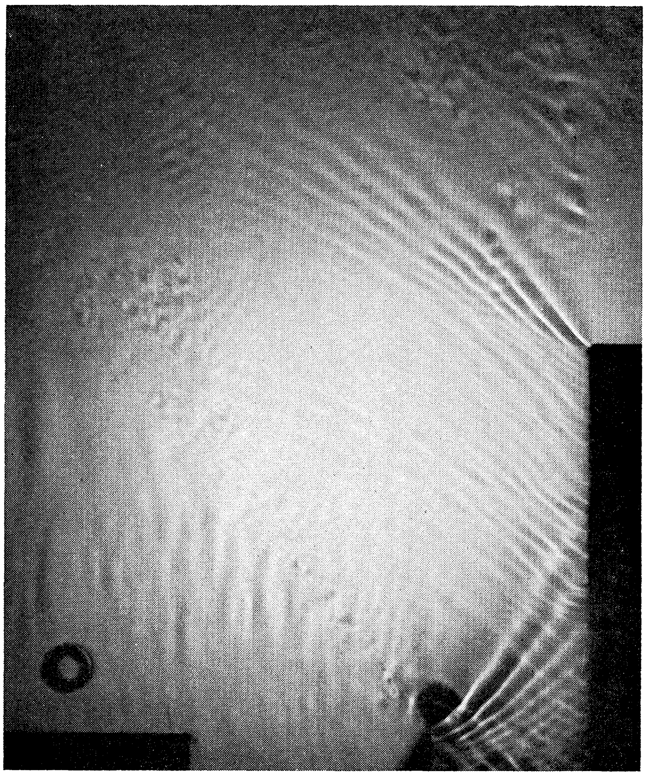

FIG. 2. Shadowgraph from convergent-straight nozzle by the hydraulic analogy.

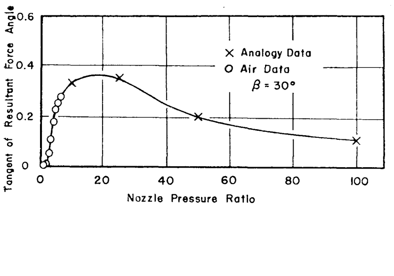

FIG. 3. Typical convergent-straight test result for $\beta = 30°$.

angle was shown to be a statistically significant function of both cut-off angle and simulated pressure ratio. Fig. 3 shows a typical curve of tangent of resultant-force angle on the nozzle as a function of pressure ratio, for a given cut-off angle, β. The deflection varied almost linearly with cut-off angle for given pressure ratios, and hence an empirical equation could be deduced which describes the test data for convergent-straight nozzles. The air-test data of Carter and Vick[2] are also plotted with the hydraulic-analogy results in Fig. 3.

With convergent-divergent nozzles, a Latin square experimental design using divergence half-angle, cut-off angle, and simulated pressure ratio as the variables failed to yield statistical significance for jet deflection as a function of any of the experimental factors. However, the general trend of all the data when grouped together as a function of simulated pressure ratio was a band of similar shape to that of Fig. 3.

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On a Distribution Function Satisfying the Local H-Theorem†

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The statistical description of a gaseous system utilizes a probability-distribution function in order to describe the behavior of the particles in the six-dimensional phase space. However, each distribution function represents an extremely large number of actual arrangements of the various particles. This fact is used within the Boltzmann H-Theorem to restrict the choice of the distribution function itself. Boltzmann's classical theorem states that a system has the greatest probability of having the distribution function which exhibits the largest number of molecular arrangements. This is stated mathematically as follows:

$$H = \int_{\mathbf{x}} \int_{\xi} f \log f d\mathbf{x} d\xi = \text{minimum}$$

In the case of a homogeneous system the solution has been shown to be the Maxwell-Boltzmann distribution law, where ρ_0 = density, β = (most probable speed)$^{-2}$, and c = random velocity—viz.,

$$f_0 = \rho_0(\beta/\pi)^{3/2} e^{-\beta c^2}$$

(For a complete description of the H-Theorem and subsequent developments see Ref. 1.)

One may go farther than this and postulate that the H-Theorem is satisfied locally in the physical space as well as throughout the phase space. This hypothesis cannot be proved, nor should it be true in general. However, it has been used fruitfully by Epstein[2] and may lead to useful results if one remembers that it is only a hypothesis. The conclusions reached here should be interpreted critically for this reason, but as we shall show they are consistent with the results of other contemporary analysis.

In order to obtain meaningful results one must apply the local H-Theorem, and must also insist that the macroscopic densities, velocities, stresses, etc., are also satisfied. These last conditions

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may be imposed as integral constraints in the formal development of the variational statement of the local H-Theorem. One should note that it is these constraints or macroscopic moments which provide the variation from equilibrium and thus generate meaningful results.

In general the problem outlined above can be carried out with any number of moments, however here the number of moment constraints will be restricted to the first thirteen. This is consistent with the truncation used by Grad[3] and will be subject to the same criticism and justification as he mentions in his development. Thus if one is willing to tentatively accept these hypotheses including the local H-Theorem then the distribution function is obtained from the solution of an elementary variational problem with integral constraints:

$$H = \int f \log f d\xi = \text{minimum} \tag{1}$$

where this statement is subject to the following constraints:

$$\left.\begin{aligned} \rho &= \int f d\xi \qquad P_{ij} = p\delta_{ij} + p_{ij} = \int c_i c_j f d\xi \\ \rho \mathbf{u} &= \int \xi f d\xi \qquad q_i = \int c_i (c^2/2) f d\xi \end{aligned}\right\} \tag{2}$$

where

$$c_i = \xi_i - u_i = \text{random velocity}$$

This set may be restated using the methods of variational calculus.[4] Taking the variation of H and using the method of Lagrangian multipliers on the constraints yields

$$\delta \int f \{\log f + a_0 + b_i \xi_i + d_{ij} c_i c_j + e_i c_i (c^2/2)\} d\xi = 0 \tag{1a}$$

This contains thirteen Lagrangian functions required to satisfy the thirteen constraints. A solution to this equation is

$$f = \exp\{-(a_0 + b_i \xi_i + d_{ij} c_i c_j + e_i c_i c^2/2)\} \tag{3}$$

One now applies the condition of small variations from local equilibrium and linearizes the exponential. Thus,

$$f = f_0\{1 - b_i' c_i - d_{ij}' c_i c_j - e_i' c_i c^2/2\} \tag{3a}$$

The thirteen constraint equations are now evaluated yielding thirteen equations for the unknown Lagrangian functions:

$$\left.\begin{aligned} \rho &= \rho_0 \left(1 - \sum_1^3 \frac{d_{ii}'}{2\beta}\right) \\ b_i &+ (5/4)(e_i/\beta) = 0 \\ P_{ij} &= -d_{ij}' \rho_0 / 4\beta^2 \quad i \neq j \\ P_{ii} &= (\rho_0/2\beta)\{1 - (1/2\beta)(3d_{ii}' + d_{kk}' + d_{ll})'\} \\ q_i &= (5/8)(\rho_0/\beta^2)\{b_i' + (7/4)(e_i'/\beta)\} \end{aligned}\right\} \tag{4}$$

These invert to yield the Lagrangian functions in terms of the thirteen moments when combined with the equation of state:

$$\left.\begin{aligned} \rho_0 &= \rho \qquad & e_i &= -(16\beta^3/5\rho) q_i \\ b_i &= +(4\beta^2/\rho) q_i \qquad & d_{ij} &= 4\beta^2/\rho \end{aligned}\right\} \tag{5}$$

These are combined with the equation of state and substituted into the distribution function to give the final distribution:

$$f = \rho(\beta/\pi)^{3/2} e^{-\beta c^2} \{1 + p_{ij}(c_i c_j / 2pRT) - (q_i c_i / pRT)[1 - (c^2/5RT)]\} \tag{6}$$

This result is identical with that used by Grad in his development of the thirteen-moment equations based on an expansion in Hermite polynomials.

Thus it has been shown that near equilibrium the local H-Theorem and the thirteen-moment distribution functions are consistent. This cannot be taken to be a justification for the use of either of these. However, it may be of some use in the criticism of both. That is to say, if one can show that one of these is inconsistent with physical reality in some region, then it follows that the other is also inconsistent. Unfortunately, at the present, very little can be said specifically about either and thus one should be very conservative about drawing broad conclusions from such circuitous reasoning.

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On the Postbuckling Behavior of Rectangular Plates

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THE OBJECT of this note is to make a comparison between the results of two approximate methods which have been applied to the postbuckling behavior of heated elastic rectangular plates. The problem to be considered is that of a rectangular plate with planform dimensions $2a$ and $2b$, which is heated symmetrically about the two centerlines of the plate. It is further assumed that the plate is simply supported for bending and that in-plane displacements normal to the edges are prevented while the plate is free to slide along its supports.

In Ref. 1, the form of the deflection function $w(x, y)$ is assumed as

$$w = \delta \sin \pi x/2a \sin \pi y/2b \tag{1}$$

where the coefficient δ, representing the center deflection, is to be

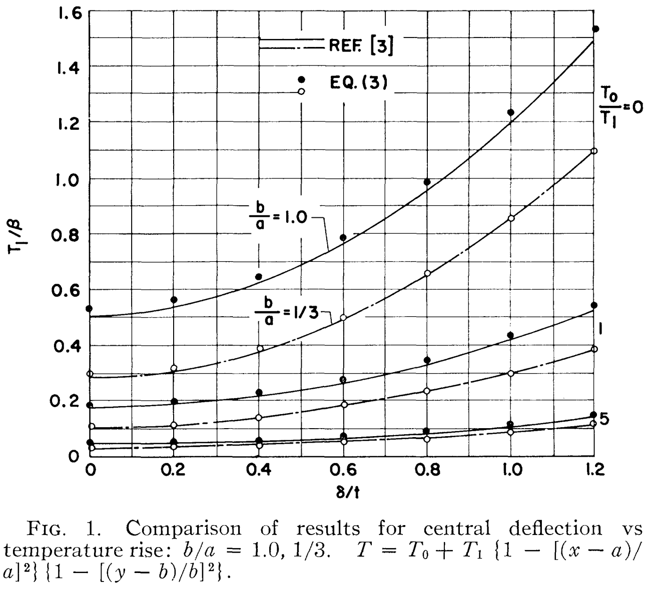

FIG. 1. Comparison of results for central deflection vs temperature rise: $b/a = 1.0, 1/3$. $T = T_0 + T_1 \{1 - [(x-a)/a]^2\}\{1 - [(y-b)/b]^2\}$.

determined. The compatibility equation is then employed to determine the stress function[2] from which the stresses are readily found, and finally a Galerkin procedure is utilized, where vertical equilibrium is satisfied approximately, to evaluate the unknown central deflection. For the particular case of a parabolic temperature variation,

$$T = T_0 + T_1 \left[1 - \left(\frac{x-a}{a}\right)^2\right]\left[1 - \left(\frac{y-b}{b}\right)^2\right] \tag{2}$$

the formula for determining the deflection amplitude is