Non-Deterministic Polling Systems

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Abstract

A non–deterministic polling system is considered in which a single server serves a number of stations. The service discipline at each station is, consistently, either non–exhaustive, semi–exhaustive, gated, or exhaustive. If the server polls a station i which uses either the non–exhaustive or the semi–exhaustive service discipline, then the next station polled is station j with probability p_{ij} if there was service at station i. The service time at station i is a random variable which may depend on the station polled next. If no service is performed at station i, then the next station polled is station j with probability e_{ij}. The time to switch between stations i and j is a random variable which may depend on whether service was performed at station i or not.

If the server polls a station i that follows either the exhaustive service discipline or the gated service discipline, then the next station polled is station j with probability p_{ij} regardless of whether there was service at station i or not.

Cycle times and stability conditions are derived for this system, and Conservation Laws are obtained which express a weighted sum of the mean waiting times in terms of known data parameters. For systems with a mix of exhaustive and gated service stations, we show how the individual mean waiting times can be obtained.

1. Introduction

The single–server polling system is a queueing system in which a single server attends to M stations. Each station receives requests for service according to independent arrival processes, and the server polls these stations in some order, servicing the requests present there. The number of requests served at a station depends on the service discipline at that station. The service disciplines that are typically modeled and studied are: i) *non–exhaustive* service: wherein the server serves exactly one request if the station is non–empty at the polling instant; ii) *exhaustive* service: wherein the server continues to serve requests at the station until it is empty; iii) *gated* service: wherein the server serves exactly those requests present at the station at the polling instant, and iv) *semi–exhaustive* service: wherein the server continues to serve requests until the number of requests waiting for service is one less than the number found by the server at the time the station was polled. Polling systems have been extensively studied for the case where the order in which the stations are visited is deterministically specified in advance. We refer to these systems as "deterministic polling systems."

In this paper, we introduce the "non–deterministic polling system" where the order in which the stations are polled is not deterministically specified. We consider a system with a mix of non–exhaustive, semi–exhaustive, exhaustive, and gated service stations. The movement of the server between the stations in the system is specified as follows.

Suppose that the server has just polled a station i which follows either the non–exhaustive or the semi–exhaustive service discipline. If one or more requests are present at the polling instant, then following the service at station i, the server next polls station j with probability p_{ij} after a random amount of time, L_{ij}. The service time for a request at station i is a random variable, B_{ij}, which may depend on the station j that is polled next. If there are no requests present at station i when it is polled, then the server next polls station j with probability e_{ij} after a random amount of time, S_{ij}.

If station i follows either the exhaustive service discipline or the gated service discipline, then the server next polls station j with probability p_{ij} regardless of whether service was provided at station i or not. The service time for a request at station i a random variable B_{ij}, if the station j is polled next. The switchover time from station i to station j is a random variable S_{ij}.

This work has been motivated by a specific application in material handling systems studied by Bozer and Srinivasan (1988). In this application, the material handling system is a unit–load Automated Guided Vehicle (AGV) system in which a single vehicle serves a manufacturing "cell" moving loads from one machining center in the cell to another. When the AGV delivers a load to a center, it inspects (polls) the output buffer of that center to determine if there are any loads waiting to be transported. If at least one load is present in the output buffer, then the AGV takes a certain amount of time to pick up a load from this output buffer, and a certain amount of time to transport the load and deliver it at its destination. Following this, the AGV polls the output buffer of the center which receives the load. If there are no loads waiting at the center that is polled by the AGV, it immediately switches to poll another center with some predetermined probability. The time taken by the vehicle to travel loaded from i to j (L_{ij}) could be quite different from the time it takes to travel empty from i to j (S_{ij}). (For instance the empty vehicle may follow path different from the loaded vehicle.) Note that in this application, all the stations use the non–exhaustive service discipline.

As another application for the non–deterministic polling system, consider a "messenger boy" who serves a number of centers, picking up messages from one center and delivering them to another. The messages that the messenger picks up from a center will dictate which center he next visits. Upon arrival at, say, center i, if the messenger finds no waiting messages, then one policy he may adopt could be to remain at this center periodically checking for messages to be delivered from center i, until one arrives (this is a case with $e_{ii} = 1$, $S_{ii} > 0$); an alternate policy he may adopt could be to immediately switch to poll another center j with some predetermined probability e_{ij}.

An application in computer communication networks is presented in Kleinrock and Levy (1988), where a random polling system is used to predict the expected delay in a Slotted ALOHA system. In the random polling system proposed by Kleinrock and Levy, the server, after polling a station i, next polls station j with probability $\hat{p}_j$, whether or not a service was performed at station i.

In this paper, we obtain an expression for the cycle time which represents the expected time between two successive polls at a specified station, and present the stability conditions for the general system with an arbitrary number of stations. Following this we obtain a conservation law which relates a weighted sum of the mean waiting time at each station in terms of the data parameters for two special cases: i) the case where $p_{ij} = e_{ij}$ for all i,j and ii) the system with two stations and arbitrary p_{ij}'s and e_{ij}'s at the non–exhaustive and semi–exhaustive service stations. Finally, for systems with a mix of exhaustive and gated service stations, we obtain the individual mean waiting times by solving $O(M^3)$ linear equations.

2. Previous Work

There is considerable literature on deterministic polling models. An excellent survey of work on polling systems is presented by Takagi (1988). Some of the earliest work on polling systems with an arbitrary number of stations can be attributed to Cooper and Murray (1969), and Cooper (1970). Following this work, a number of papers have appeared on exact analysis of these systems with non–zero switchover times for both the continuous–time and the discrete–time case (Eisenberg 1972, Ferguson and Aminetzah 1985, Konheim and Meister 1974, Swartz 1980).

The analysis of the non–exhaustive and semi–exhaustive service systems presents considerable difficulties. Even obtaining the mean waiting times appears to be a challenging task in general. Although mean waiting times for symmetric non–exhaustive service systems can be obtained exactly (Fuhrmann 1985, Takagi 1985), such analysis for asymmetric systems is available only for the case of M=2 (Boxma and Groenendijk 1987b). Indeed, while mean waiting times for gated and exhaustive service systems can be computed with only the first two moments of service and switchover times, this does not appear to be the case for non–exhaustive service systems in general (Takagi 1988).

The development of conservation laws (Watson 1984, Boxma and Groenendijk 1987a, Srinivasan and Lee 1987) for these systems has thus proved to be very useful in providing insight on their behavior. These laws express a weighted sum of the mean waiting times at the stations in terms of data parameters, and require only the first and second moments of the service and the switchover times. In particular, Boxma and Groenendijk present an elegant analysis based on a work decomposition approach, which also provides some intuition as to why these conservation laws hold. For symmetric systems, the conservation laws obtain an expression for the mean waiting times in a closed form. These laws also enable the development of approximate solutions for asymmetric systems (Boxma and Meister 1986, Srinivasan 1988, Fuhrmann and Wang 1988). Most of the work on deterministic polling systems is based on cyclic polling systems, although there is some work on analyzing systems with a more general order of service (Baker and Rubin 1987, Mapp and Manfield 1986, Boxma et al. 1988).

All of the above work relates to deterministic polling systems in which the order of visits to the stations by the server is fixed deterministically. A system in which the server visits the stations in a probabilistic order (according to a "Bernoulli schedule") is considered by Keilson and Servi (1986) wherein the server, on completion of a service at station i, next switches to station i+1 with probability p_i or polls station i with probability $1–p_i$. The cases where $p_i = 1$ and $p_i = 0$ reduce to the deterministic non–exhaustive and the deterministic exhaustive service system respectively.

Kleinrock and Levy (1988) analyze a discrete–time random polling system, where the server next polls station j with probability $\hat{p}_j$. They obtain the mean waiting times in exhaustive and gated service systems through the solution of $O(M^3)$ linear equations. For completely symmetric random polling systems (in which all the stations adopt exactly one of three service disciplines, namely, the exhaustive, gated, or the non–exhaustive discipline), they obtain the mean waiting times explicitly.

Boxma and Weststrate (1989) have, independent of this paper, considered a polling system with "Markovian server routing", and obtained the conservation law for a system with a mix of non–exhaustive, exhaustive, and gated service stations in an elegant manner using the work decomposition approach. Their analysis requires the identification of a time–reversed Markov chain in order to obtain the conservation law. The system analyzed by Boxma and Weststrate is the special case considered in our paper where $p_{ij} = e_{ij}$ for all stations including the non–exhaustive service stations.

It may be observed that many other well–known polling systems, including the random polling system presented by Kleinrock and Levy, are special cases of the non–deterministic polling system.

3. The Model

A system with M stations is considered. Requests for service arrive at station i according to an independent Poisson process at rate λ_i. It is assumed that there is an infinite capacity to hold these requests until they can be served. Denote by N, S, E, and G, the set of non–exhaustive, semi–exhaustive, exhaustive, and gated service stations respectively.

At a station i which uses either the non–exhaustive or the semi–exhaustive service discipline, the switchover time L_{ij} (which arises when there is service at station i) has mean and second moment γ_{ij} and $\gamma_{ij}^{(2)}$, respectively, and the switchover time S_{ij} (which arises when there is no service at station i) has mean and second moment σ_{ij} and $\sigma_{ij}^{(2)}$, respectively. We assume that $\gamma_{ij} \geq \sigma_{ij}$ for all i,j. At stations which adopt the exhaustive or the gated service disciplines the switchover time to station j is S_{ij} (regardless of whether service is performed at station i or not), with mean and second moment σ_{ij} and $\sigma_{ij}^{(2)}$, respectively.

For ease of exposition we assume that for $i \in \{S, E, G\}$, the service time is B_i, independent of the station that is polled next. (The analysis extends directly to the case where the service time could depend on the station next visited.) The mean and second moment of B_i are denoted by τ_i and $\tau_i^{(2)}$ respectively. For $i \in N$, the service time B_{ij} has mean and second moment β_{ij} and $\beta_{ij}^{(2)}$, respectively.

To analyze the system, we consider the system state at each polling instant. Let $(i; n_1, \ldots, n_M)$ denote the state of the system at an arbitrary polling instant, where i denotes the station that is currently being polled by the server, and n_m, $m = 1, \ldots, M$, denotes the number of requests present at station m at this instant. In the following discussion, unless otherwise specified, the index for summation is implicitly assumed to range over $1, \ldots, M$. (It is also implicit that station M+1 is just station 1.) Since the arrivals at the stations follow independent Poisson processes, it is easily seen that the state of the system at successive polling instants follows a Markov chain. We only consider systems in which for each station k, $\lambda_k > 0$, and there exists at least one station $i \neq k$ for which either $p_{ik} > 0$, or $e_{ik} > 0$. Then, if the system is stable, it can be seen that the set of all the possible states observed at polling instants is irreducible.

Purely for ease of exposition, we first consider a system in which all the stations adopt the non–exhaustive service discipline, and obtain the generating function of the number of customers present at each station when the server polls station i. This will facilitate the subsequent derivation of the corresponding generating function for the system with a mix of non–exhaustive, semi–exhaustive, exhaustive, and gated service stations.

When all stations use the non–exhaustive service discipline, the transition probabilities for the Markov chain are obtained as follows. Let $Q_m(k;t)$ denote the probability of k arrivals at station m during an interval of length t, and for any random variable U, with distribution function $F_U(.)$, we denote by $A\{a_1,\ldots, a_M; U\} \triangleq \int_0^\infty [\prod_{m=1}^M Q_m(a_m; t)]\, dF_U(t)$, the probability that there are a_i arrivals at station i; $i = 1,\ldots,M$, during a time U. Consider a transition from the state $(k; x_1,\ldots, x_M)$ to the state $(i; n_1, \ldots, n_M)$, with $n_m \geq x_m$, for $m \neq k$, and $n_k \geq x_k - 1$. When $x_k > 0$, the probability of this transition is $p_{ki}\, A\{n_1-x_1,\ldots,n_k-x_k+1,\ldots, n_M-x_M; B_{ki} + L_{ki}\}$, and when $x_k = 0$, the transition probability is $e_{ki}\, A\{n_1-x_1,\ldots, n_k,\ldots, n_M-x_M; S_{ki}\}$.

Hence, if we let $\pi(i; n_1, n_2, \ldots, n_M)$ denote the equilibrium probability that the server polls station i and the state is $(i; n_1, \ldots, n_M)$ at the polling instant, then this probability is obtained as

$$\pi(i; n_1,\ldots,n_M) = \sum_{k=1}^{M} \Big[\sum_{X:x_k>0} \pi(k; x_1,\ldots,x_k,\ldots, x_M)\, p_{ki}\, A\{n_1-x_1,\ldots,n_k-x_k+1,\ldots,n_M-x_M; B_{ki}+ L_{ki}\}$$

$$+ \sum_{X:x_k=0} \pi(k; x_1,\ldots,x_k,\ldots,x_M)\, e_{ki}\, A\{n_1-x_1, \ldots, n_k, \ldots, n_M-x_M; S_{ki}\}\Big], \quad (3.1)$$

where the above summations are taken over all feasible values of $X=(x_1, \ldots, x_M)$.

Let $P_i(n_1, n_2, \ldots, n_M)$ denote the equilibrium probability that there are n_m customers at station m, for $m = 1, \ldots, M$, when (i.e., given that) station i is polled, and let π_i denote the unconditional probability that station i is polled. Note that

$$\pi(i; n_1, \ldots, n_M) = P_i(n_1, \ldots, n_M)\, \pi_i. \quad (3.2)$$

Let $\bar{z} = (z_1,\ldots,z_M)$, and consider the generating function

$$F_i(\bar{z}) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_M=0}^{\infty} P_i(n_1, \ldots, n_M)\, z_1^{n_1} \cdots z_M^{n_M}. \tag{3.3}$$

Substituting in equation (3.1), the expression for $\pi(i; n_1, \ldots, n_M)$ given by equation (3.2), and using equation (3.3), we obtain:

$$\pi_i F_i(\bar{z}) = \sum_k \Big[\pi_k \frac{F_k(\bar{z}) - F_k(0)}{z_k} p_{ki} B^*_{ki}(\sum_m (\lambda_m - \lambda_m z_m)) L^*_{ki}(\sum_m (\lambda_m - \lambda_m z_m)) + \pi_k F_k(0)\, e_{ki}\, S^*_{ki}(\sum_m (\lambda_m - \lambda_m z_m))\Big],$$

where $F_k(0) = F_k(z_1,\ldots,z_{k-1},0,z_{k+1},\ldots,z_M)$, and $B^*_{ki}(\cdot)$, $L^*_{ki}(\cdot)$ and $S^*_{ki}(\cdot)$ denote the Laplace Stieltjes transforms (LST) of the service / switchover times, B_{ki}, L_{ki}, and S_{ki}, respectively. Note that $F_k(0)|_{\bar{z}=\bar{1}}$ is the probability that station k is empty at the instant it is polled. Let

$$q_k = F_k(0)|_{\bar{z}=\bar{1}}, \quad \text{and} \quad \bar{q}_k = 1 - q_k. \tag{3.4}$$

For ease of expression, we denote the state $(i; n_1,\ldots,n_M)$ by $(i; \bar{n})$. If the system is stable then, for at least one station, there will be a non–zero probability that the system is empty at the instant that the server polls this station. Without loss of generality, let station 1 be one such station. We now consider the regeneration epochs at which the server polls station 1 and finds the system empty. We shall refer to the time between two such epochs as a regeneration period and let $C(1; \bar{0})$ denote its expected value. Note that $C(1; \bar{0}) < \infty$ if the system is stable.

Between two visits to state $(1; \bar{0})$, the expected number of visits to state $(i; \bar{n})$ is $\pi(i; \bar{n}) / \pi(1; \bar{0})$. Since $\pi_i = \sum \pi(i; \bar{n})$, (the summation is taken over all possible values of $\bar{n}$) the expected number of visits to station i during a regeneration period is $\pi_i / \pi(1; \bar{0})$. The expected number of requests arriving at station i during a regeneration period is $\lambda_i C(1; \bar{0})$, and clearly all these requests are served during this same period of time. Since the server serves *exactly one request* per visit to station i if it is non–empty at the polling instant, the probability of a service per visit, $\bar{q}_i$, is thus

$$\bar{q}_i = \frac{\lambda_i C(1; \bar{0})\, \pi(1; \bar{0})}{\pi_i}. \tag{3.5}$$

Now consider systems in which all four types of service stations can be present. Let $H^*_k(\cdot)$, for $k \in$ *{S, E}*, denote the LST of a busy period at station k, that is initiated by a single request. For notational ease, let

$B_{ki}^* \triangleq B_{ki}^*(\sum_m (\lambda_m - \lambda_m z_m)),\quad L_{ki}^* \triangleq L_{ki}^*(\sum_m (\lambda_m - \lambda_m z_m)),\quad$ and $\quad S_{ki}^* \triangleq S_{ki}^*(\sum_m (\lambda_m - \lambda_m z_m)).$

Adopting the same approach as we used earlier for the system with only non–exhaustive service stations, we obtain the following equation for $\pi_i F_i(\overline{z})$:

$$\begin{aligned}
\pi_i F_i(\overline{z}) \quad = \quad & \sum_{k\in N} [\ \pi_k \frac{F_k(\overline{z})-F_k(0)}{z_k} p_{ki} B_{ki}^* L_{ki}^* \ + \pi_k F_k(0)\, e_{ki} S_{ki}^*] \\
+ \quad & \sum_{k\in S} [\ \pi_k \frac{F_k(\overline{z})-F_k(0)}{z_k} p_{ki} H_k^*(\sum_{m\neq k} (\lambda_m - \lambda_m z_m))\, L_{ki}^* \ + \pi_k F_k(0)\, e_{ki} S_{ki}^*] \\
+ \quad & \sum_{k\in E} \pi_k\, p_{ki}\, F_k(z_1,\ldots,z_{k-1},H_k^*(\sum_{m\neq k} (\lambda_m - \lambda_m z_m)),z_{k+1},\ldots,z_M)\, S_{ki}^* \\
+ \quad & \sum_{k\in G} \pi_k\, p_{ki}\, F_k(z_1,\ldots,z_{k-1},B_k^*(\sum_m (\lambda_m - \lambda_m z_m)),z_{k+1},\ldots,z_M)\, S_{ki}^*. \qquad (3.6)
\end{aligned}$$

Setting $z_1 = \ldots = z_M = 1$ in equation (3.6), we get

$$\pi_i \quad = \quad \sum_{k\in N,S} \pi_k\, (\overline{q}_k\, p_{ki} \ + q_k\, e_{ki}) \ + \qquad \sum_{k\in E,G} \pi_k\, p_{ki}, \qquad (3.7)$$

In the above equation, $\overline{q}_k$, for $k\in N$, is defined by equation (3.5).

For a station k which uses the semi–exhaustive service discipline, $\overline{q}_k$ is obtained in an analogous manner as follows. Consider a standard M/G/1 queueing system in which the server is always available to serve customers, and with the same arrival rate and service time distribution as station k in the non–deterministic polling system. Then the busy period initiated by a single arrival in the M/G/1 system has a mean $\tau_k/(1-\lambda_k\tau_k)$. Now, in the non–deterministic polling system, suppose the server finds station k non–empty at the instant it is polled. Since this station adopts the semi–exhaustive service discipline, when the server subsequently leaves the station, the number of requests present there will be exactly one less than the number present at the polling instant. Hence it is easily seen that the expected number of requests served at station k is equal to 1 plus the expected number of requests served during a busy period in the corresponding standard M/G/1 queueing system, i.e., equal to $1 + \lambda_k\tau_k/(1-\lambda_k\tau_k)$. Since the expected number of visits to station k during a regeneration period is $\pi_k / \pi(1;\overline{0})$, the expected number of requests served at station k during a regeneration period is $\overline{q}_k \left(\frac{1}{1-\lambda_k\tau_k}\right) \frac{\pi_k}{\pi(1;\overline{0})}$. This is just equal to $\lambda_k\, C(1;\overline{0})$, which is the expected number of arrivals at station k during the regeneration period. Hence we get

$$\bar{q}_k = \frac{\lambda_k\, C(1;\bar{0})\, \pi(1;\bar{0})}{\pi_k}(1-\lambda_k\tau_k), \qquad k \in S. \tag{3.8}$$

As in the non–exhaustive service case, $q_k = 1 - \bar{q}_k$, for $k \in S$. Let

$$\tau_i = \sum_k \beta_{ik}\, p_{ik}, \quad i \in N, \qquad \varphi_i = \sum_k \gamma_{ik}\, p_{ik}, \quad i \in \{N, S\}, \tag{3.9.a}$$

and

$$\theta_i = \sum_k \sigma_{ik}\, e_{ik}, \qquad \theta_i^{(2)} = \sum_k \sigma_{ik}^{(2)}\, e_{ik}; \quad i = 1,\ldots,M. \tag{3.9.b}$$

Let T_i denote the expected time from the instant that the server polls station i until the instant when the next station is polled. For $i \in N$, the server finds one or more requests waiting at station i with probability $\bar{q}_i$ and in this case the expected time until the next polling instant is given by $\tau_i + \varphi_i$. With probability q_i, the server finds no requests waiting at station i at the polling instant, in which case the expected time until the next polling instant is given by θ_i. Hence,

$$T_i = q_i\,\theta_i + \bar{q}_i\,(\tau_i + \varphi_i); \quad i \in N. \tag{3.10.a}$$

For $i \in S$, the expected number of requests served per visit is $1/(1-\lambda_i\tau_i)$ if station i is non–empty at the polling instant. Hence the expected time until the next polling instant is $\tau_i/(1-\lambda_i\tau_i) + \varphi_i$ with probability $\bar{q}_i$, and is otherwise θ_i with probability q_i. So,

$$T_i = q_i\,\theta_i + \bar{q}_i\,[\frac{\tau_i}{1-\lambda_i\tau_i} + \varphi_i]; \quad i \in S. \tag{3.10.b}$$

For $i \in \{E, G\}$, the expected number of requests arriving at station i during a regeneration period is $\lambda_i\, C(1;\bar{0})$, and the expected number of visits to i during this period is $\pi_i / \pi(1;\bar{0})$. Hence it follows that the expected amount of time the server spends at i per visit is just $\tau_i\,\lambda_i\, C(1;\bar{0})\, \pi(1;\bar{0}) / \pi_i$. So,

$$T_i = \lambda_i\,\tau_i \frac{C(1;\bar{0})\, \pi(1;\bar{0})}{\pi_i} + \theta_i. \quad i \in \{E, G\}. \tag{3.10.c}$$

Since the expected number of visits to *any* station i during a regeneration period is $\pi_i / \pi(1;\bar{0})$,

$$C(1;\bar{0}) = \sum_i \frac{\pi_i}{\pi(1;\bar{0})} T_i. \tag{3.11}$$

Substituting in equation (3.11) the expression for T_i given by equations (3.10.a) through (3.10.c), setting $q_i = 1 - \bar{q}_i$ and using equations (3.5) and (3.8) to express $\bar{q}_i$ for $i \in \{N, S\}$,

$$
\begin{aligned}
C(1;\bar{0}) \quad = \quad & \sum_{i \in N} [\lambda_i \tau_i C(1;\bar{0}) + \frac{\pi_i}{\pi(1;\bar{0})} \theta_i + \lambda_i C(1;\bar{0}) (\varphi_i - \theta_i)] \\
+ \quad & \sum_{i \in S} [\lambda_i \tau_i C(1;\bar{0}) + \frac{\pi_i}{\pi(1;\bar{0})} \theta_i + \lambda_i C(1;\bar{0}) (1 - \lambda_i \tau_i) (\varphi_i - \theta_i)] \\
+ \quad & \sum_{i \in \{E,G\}} [\lambda_i \tau_i C(1;\bar{0}) + \frac{\pi_i}{\pi(1;\bar{0})} \theta_i].
\end{aligned}
$$

Collecting terms involving $C(1;\bar{0})$, and setting

$$\vartheta_i = \lambda_i \tau_i + \lambda_i (\varphi_i - \theta_i), \qquad i \in N, \tag{3.12.a}$$

$$= \lambda_i \tau_i + (1 - \lambda_i \tau_i) \lambda_i (\varphi_i - \theta_i), \qquad i \in S, \tag{3.12.b}$$

$$= \lambda_i \tau_i, \qquad i \in \{E, G\}, \tag{3.12.c}$$

and

$$\vartheta = \sum_i \vartheta_i, \tag{3.12.d}$$

we obtain

$$\pi(1;\bar{0})\, C(1;\bar{0}) = \frac{\sum_i \pi_i \theta_i}{1 - \vartheta}. \tag{3.13}$$

From equation (3.13) we observe that for $C(1;\bar{0})$ to be non–negative and bounded, we need $\vartheta <$ 1. (In addition, we obviously require that $0 \le \bar{q}_k < 1$, $k \in \{N, S\}$.) Let

$$C_i = \pi(1;\bar{0})\, C(1;\bar{0}) / \pi_i, \tag{3.14}$$

and set

$$v_i = \pi_i / \pi_1. \tag{3.15}$$

The term C_i denotes the expected time between two successive visits by the server to station i. We shall refer to C_1 as the "cycle time", and the v_i's will be referred to as the "visit ratios". Note that by definition, $v_1 = 1$. From equations (3.4), (3.5), (3.8), (3.14) and (3.15), we obtain alternate expressions for q_k, $k \in \{N, S\}$, stated as

Proposition 3.1:

$$q_k = 1 - \lambda_k C_k = 1 - \lambda_k C_1 / v_k, \quad k \in N, \quad (3.16.a)$$

$$= 1 - \lambda_k C_k (1-\lambda_k\tau_k) = 1 - \lambda_k C_1 (1-\lambda_k\tau_k)/ v_k, \quad k \in S, \quad (3.16.b)$$

■

From equations (3.13) through (3.15), an expression for C_1 is obtained from

Proposition 3.2:

$$C_1 = \frac{\sum_i v_i\theta_i}{1-\vartheta}. \quad (3.17)$$

■

The conditions for the stability of this system are obtained from the above development as:

$$\vartheta < 1, \quad (3.18.a)$$

$$\lambda_k C_k < 1, \quad k \in N, \quad (3.18.b)$$

$$\lambda_k C_k(1-\lambda_k\tau_k) < 1, \quad k \in S, \quad (3.18.c)$$

By suitably choosing the parameters p_{ij} and e_{ij}, we can consider several polling systems of interest. For notational convenience, in the following we set

$$\Theta = \sum_i \theta_i. \quad (3.19)$$

The deterministic service system

The corresponding deterministic service system is obtained by setting $p_{i,i+1} = e_{i,i+1} = 1$ for all i. For this system the switchover time from station i (to station i+1) is the same regardless of whether the server performed a service at station i or not. In other words, for i ∈ *{N, S}*, $\sigma_{i,i+1} = \gamma_{i,i+1}$, and so $\theta_i = \varphi_i$. From equations (3.7) and (3.15), we have $v_i = v_{i-1}$. Hence the visit ratio is the same for all stations and so from equation (3.17) the cycle time is the same for all stations. Since $v_1 = 1$ by definition, the resulting cycle time is the well known expression

$$C = \frac{\Theta}{1 - \sum_i \lambda_i \tau_i}. \quad (3.20)$$

The random polling system of Kleinrock and Levy

Consider the dispatching rule $p_{ij} = e_{ij} = \hat{p}_j$. This rule has been studied by Kleinrock and Levy (1988) and specifies that the station next polled by the server is station j with probability $\hat{p}_j$

regardless of the station the server is currently at, and whether a service is performed there or not. For this system, equations (3.7) and (3.15) give

$$v_i \quad = \quad \sum_j v_j \hat{p}_i,$$

and so

$$v_i / \hat{p}_i \quad = \quad v_1 / \hat{p}_1 \quad = \quad \sum_j v_j, \qquad \text{for all i.}$$

Hence, from equation (3.17),

$$C_1 \quad = \quad \frac{\sum_i \hat{p}_i \, \theta_i}{\hat{p}_1 \, (1 - \vartheta)}. \tag{3.21}$$

The First–Encountered–First–Served Rule

The First–Encountered–First–Served (FEFS) rule has been used by Stone and Fuller (1973) to model the retrieval of records from a rotating drum. This rule provides a natural application for the non–deterministic polling system: consider a system where *all* the stations use the non–exhaustive service discipline, and for all i, $e_{i,i+1} = 1$ and $p_{ii} = 0$, with the other p_{ij}'s taking on arbitrary values. Such a policy specifies that the server serves exactly one request at a station if the station is non–empty at the polling instant; otherwise, the server polls the next station in some logical order. The FEFS policy has been used in this manner by Bozer and Srinivasan (1988) to model the unit–load AGV system as a polling system. For this policy we have, from equations (3.7) and (3.15),

$$v_i \quad = \quad \sum_j v_j \, \bar{q}_j \, p_{ji} + v_{i-1} \, q_{i-1}, \quad i=1,..,M. \tag{3.22}$$

(Note that if we set $p_{i,i+1} = 1$, then this is just the deterministic non–exhaustive service system.)

Set $\Lambda_i = \sum_j \lambda_j \, p_{ji}$. Using Proposition 3.1, equation (3.22) is rewritten as

$$v_i \quad = \quad v_{i-1} + (\Lambda_i - \lambda_{i-1}) \, C_1. \tag{3.23.a}$$

Continuing to express v_i in terms of $v_{i-2}, \dots, v_1$,

$$v_i \quad = \quad \sum_{j=2}^{i} (\Lambda_j - \lambda_{j-1}) \, C_1 \; + \; v_1, \qquad i=2,\dots,M. \tag{3.23.b}$$

Hence, the cycle time C_1 is obtained from Proposition 3.2 and equations (3.22) and (3.23.b) as

$$C_1 \quad = \quad \frac{\Theta}{1 - \vartheta - \sum_{i=2}^{M} \sum_{j=2}^{i} (\Lambda_j - \lambda_{j-1}) \, \theta_i}. \tag{3.24}$$

4. Conservation laws and mean waiting times:

In this section, we obtain the conservation law for two special cases of the non–deterministic polling system: the case where $p_{ij} = e_{ij}$ for all $i \in \{N, S\}$, and the case with M=2 but $p_{ij} \neq e_{ij}$ for $i \in \{N, S\}$. Following this, we obtain the individual mean waiting times in a system with a mix of exhaustive and gated service stations, through the solution of $O(M^3)$ equations.

To obtain the conservation laws, we need the first and second partial derivatives of $F_i(\overline{z})$ with respect to j, j = 1,…,M, evaluated at $\overline{z}=\overline{1}$. Let

$$f_{ij} = \frac{\partial F_i(\overline{z})}{\partial z_j}\Big|_{\overline{z}=\overline{1}}, \qquad f^{(2)}_{ijm} = \frac{\partial^2 F_i(\overline{z})}{\partial z_j \partial z_m}\Big|_{\overline{z}=\overline{1}}, \quad \text{and}$$

$$g_{ij} = \frac{\partial F_i(0)}{\partial z_j}\Big|_{\overline{z}=\overline{1}}.$$

4.1. The conservation law when $p_{ij} = e_{ij}$ for all stations

We first obtain the conservation law for the case where $p_{ij} = e_{ij}$ for all $i \in \{N, S\}$. Purely for ease of exposition, for all $i \in \{N, S\}$, we set $L_{ij} = S_{ij}$, and for $i \in N$, we set $B_{ij} = B_i$, and $E[B_i] = \tau_i$. (Note that with this we have $\lambda_i \tau_i = \vartheta_i$ for all i.) The extension to the case where $L_{ij} \neq S_{ij}$, and where B_{ij} can depend on the next station visited, is straightforward.

In order to obtain the first and second partial derivatives of $F_k(\overline{z})$ for this system, we need the first and second partial derivatives of

$$F_k(H_k^*) \triangleq F_k(z_1,\ldots,z_{k-1},H_k^*(\sum_{m\neq k} (\lambda_m - \lambda_m z_m)),z_{k+1},\ldots,z_M); \quad k \in E,$$

and

$$F_k(B_k^*) \triangleq F_k(z_1,\ldots,z_{k-1},B_k^*(\sum_{m} (\lambda_m - \lambda_m z_m)),z_{k+1},\ldots,z_M); \quad k \in G,$$

evaluated at $\overline{z}=\overline{1}$. Let $\delta(m,j) = 1$, if m = j, and equal to 0 otherwise. Set $\overline{\delta}(m,j) = 1 - \delta(m,j)$. Then the first and second derivatives of $F_k(H_k^*)$ and $F_k(B_k^*)$, evaluated at $\overline{z}=\overline{1}$, are obtained as follows (also refer Takagi 1986). For $k \in E$,

$$\frac{\partial F_k(H_k^*)}{\partial z_j}\Big|_{\overline{z}=\overline{1}} = (f_{kj} + f_{kk}\lambda_j \frac{\tau_k}{1-\vartheta_k})\ \overline{\delta}(k,j),$$

$$\frac{\partial^2 F_k(H_k^*)}{\partial z_j \partial z_m}\Big|_{\overline{z}=\overline{1}} = \Big[f^{(2)}_{kjm} + (f^{(2)}_{kkj}\lambda_m + f^{(2)}_{kkm}\lambda_j)\frac{\tau_k}{1-\vartheta_k} + f_{kkk}\lambda_j\lambda_m \frac{\tau_k^{\,2}}{(1-\vartheta_k)^2}$$

$$+ f_{kk}\lambda_j\lambda_m \frac{\tau_k^{(2)}}{(1-\vartheta_k)^3}\Big]\ \overline{\delta}(k,j)\ \overline{\delta}(k,m),$$

and for $k \in G$,

$$\frac{\partial F_k(B_k^*)}{\partial z_j}\Big|_{\bar{z}=\bar{1}} = f_{kj}\,\bar{\delta}(k,j) + f_{kk}\,\lambda_j\,\tau_k,$$

$$\frac{\partial^2 F_k(B_k^*)}{\partial z_j \partial z_m}\Big|_{\bar{z}=\bar{1}} = f^{(2)}_{kjm}\,\bar{\delta}(k,j)\,\bar{\delta}(k,j) + (f^{(2)}_{kkm}\,\lambda_j\,\bar{\delta}(k,m) + f^{(2)}_{kkj}\,\lambda_m\,\bar{\delta}(k,j))\tau_k + f_{kkk}\,\lambda_j\,\lambda_m\,\tau_k^2 + f_{kk}\,\lambda_j\,\lambda_m\,\tau_k^{(2)}.$$

Let $T = \{N, S, E, G\}$ and define the indicator function

$$\Delta(j,T) = 1; \quad j \in T; \qquad = 0; \quad j \notin T.$$

Differentiating equation (3.6) once with respect to j at $\bar{z}=\bar{1}$, we get (note that $F_k(0)\,|_{\bar{z}=\bar{1}} = q_k$)

$$\begin{aligned} v_i f_{ij} = & \sum_{k\in N} v_k\,p_{ki}\,[f_{kj} + \bar{q}_k\,\lambda_j\,(\tau_k + \sigma_{ki}) + q_k\,\lambda_j\,\sigma_{ki}] - v_j\,\bar{q}_j\,p_{ji}\,\Delta(j,N) \\ & + \sum_{k\in S} v_k\,p_{ki}\,[f_{kj} + \bar{q}_k\,\lambda_j\,(\frac{\tau_k}{1-\vartheta_k} + \sigma_{ki}) + q_k\,\lambda_j\,\sigma_{ki}] - (1 + \frac{\vartheta_j}{1-\vartheta_j})\,v_j\,\bar{q}_j\,p_{ji}\,\Delta(j,S) \\ & + \sum_{k\in E} v_k\,p_{ki}\,[f_{kj} + f_{kk}\,\lambda_j\,\frac{\tau_k}{1-\vartheta_k} + \lambda_j\,\sigma_{ki}] - v_j\,p_{ji}\,\frac{f_{jj}}{1-\vartheta_j}\,\Delta(j,E) \\ & + \sum_{k\in G} v_k\,p_{ki}\,[f_{kj} + f_{kk}\,\lambda_j\,\tau_k + \lambda_j\,\sigma_{ki}] - v_j\,p_{ji}\,f_{jj}\,\Delta(j,G). \end{aligned} \tag{4.1}$$

The term f_{jj} represents the expected number of requests present at station j when the server polls station j. This term is easily obtained for the exhaustive and gated service stations as (also refer Watson 1984 and Takagi 1986):

$$\frac{f_{jj}}{1-\vartheta_j} = \lambda_j\,C_j; \qquad j \in E, \tag{4.2.a}$$

$$f_{jj} = \lambda_j\,C_j; \qquad j \in G. \tag{4.2.b}$$

From equations (3.16), (4.1) and (4.2), noting that $\Delta(j,N) + \Delta(j,S) + \Delta(j,E) + \Delta(j,G) = 1$, we obtain the following expression for f_{ij} stated as

Proposition 4.1

$$v_i f_{ij} = \sum_k v_k\,p_{ki}\,f_{kj} + \lambda_j \Big(\sum_k (\lambda_k\,C_1\,\tau_k + v_k\,\sigma_{ki})\,p_{ki} - C_1\,p_{ji}\Big). \tag{4.3}$$

■

Observe that when $p_{ij} = e_{ij}$ for $i \in \{N, S\}$, then from equations (3.7) and (3.15),

$$v_i \quad = \quad \sum_k v_k\, p_{ki}, \qquad i = 1,\ldots,M. \tag{4.4}$$

It may also be observed, from equations (4.3) and (4.4), that we can express f_{ij} in terms of f_{jj} as:

$$f_{ij} \quad = \quad f_{jj} + \lambda_j\, y_{ij}, \tag{4.5}$$

where the y_{ij} s are constants which need to be determined. From equations (4.3) through (4.5),

$$\begin{aligned} v_i f_{ij} &= \sum_k v_k\,(f_{jj} + \lambda_j\, y_{kj})\, p_{ki} + \lambda_j \Big(\sum_k (\vartheta_k C_1 + v_k\, \sigma_{ki})\, p_{ki} - C_1\, p_{ji}\Big) \\ &= v_i\, f_{jj} + \lambda_j \Big(\sum_k v_k\, y_{kj}\, p_{ki} + \sum_k (\vartheta_k C_1 + v_k\, \sigma_{ki})\, p_{ki} - C_1\, p_{ji}\Big). \end{aligned} \tag{4.6}$$

Hence, from equations (4.5) and (4.6), we get

$$v_i\, y_{ij} \quad = \quad \sum_k v_k\, y_{kj}\, p_{ki} + \sum_k (\vartheta_k C_1 + v_k\, \sigma_{ki})\, p_{ki} - C_1\, p_{ji}, \quad i=1,\ldots,M, \tag{4.7.a}$$

with

$$v_j\, y_{jj} \quad = \quad 0. \tag{4.7.b}$$

For each j, equations (4.7.a) and (4.7.b) form a system of M linear equations in the M unknowns y_{ij}, $i = 1,\ldots,M$, which can be solved to obtain the y_{ij}'s. The resulting solution can now be used in equation (4.6) to obtain the f_{ij}'s in terms of f_{jj}. Note from equations (4.2.a) and (4.2.b) that for exhaustive and gated stations, the f_{jj} terms can be obtained explicitly in terms of the data parameters. This observation will be seen to be key in determining the individual mean waiting times for systems with a mix of exhaustive and gated service stations, in section 4.3.

We now differentiate equation (3.6) twice with respect to j and m, and obtain an expression for $f^{(2)}_{ijm}$ in terms of the unknowns $f^{(2)}_{kjm}$; $k,j,m = 1,\ldots,M$, and the unknowns f_{kj} and g_{kj}, $k \in \{N, S\}$ and $j = 1,\ldots,M$. Following this, we eliminate all unknowns except for f_{jj}, for $j \in \{N, S\}$, and $f^{(2)}_{jjj}$, for $j \in \{E,G\}$, to get one equation in M unknowns. The resulting expression is stated as Proposition 4.2. A detailed derivation of Proposition 4.2 in presented in the Appendix.

Proposition 4.2

$$\sum_{j\in N} v_j f_{jj}\tau_j \Big[\frac{1-\vartheta}{\lambda_j} - \sum_m \frac{v_m\theta_m}{v_j}\Big] + \sum_{j\in S} v_j f_{jj}\tau_j \Big[\frac{1-\vartheta}{\lambda_j(1-\vartheta_j)} - \sum_m \frac{v_m\theta_m}{v_j}\Big] + (1-\vartheta)\sum_{j\in E} v_j f^{(2)}_{jjj}\frac{\tau_j}{2\lambda_j(1-\vartheta_j)} + (1-\vartheta)\sum_{j\in G} v_j f^{(2)}_{jjj}\tau_j\frac{(1+\vartheta_j)}{2\lambda_j}$$

$$= \frac{\vartheta\, C_1}{2}\sum_j \lambda_j\, \tau_j^{(2)} + \frac{\vartheta}{2}\sum_j v_j\, \theta_j^{(2)} - (1-\vartheta)\sum_j \vartheta_j \theta_j\, C_1 + \sum_j \vartheta_j \sum_m v_m\, \theta_m\, y_{mj}$$

$$+ \sum_{j\in\{E,G\}} v_j\, f_{jj}\, \tau_j \sum_m \frac{v_m\theta_m}{v_j} + (1-\vartheta)\sum_{j\in\{N,S\}} \tau_j\, C_1 - (1-\vartheta)\sum_{j\in\{S,E\}} \frac{\lambda_j\, \tau_j^{(2)}}{2(1-\vartheta_j)}\vartheta_j\, C_1. \tag{4.8}$$

■

Denote, by W_j, the mean waiting time at station j. Lemma 4.3 expresses the unknowns $f_{jj} : j \in \{N,S\}$, and $f^{(2)}_{jjj} : j \in \{E, G\}$, in terms of the mean waiting times at the stations. The proof of Lemma 4.3 is straightforward and is omitted. (Refer Watson (1984) and Takagi (1986) for analogous expressions in the case of deterministic polling systems.)

Lemma 4.3

$$f_{jj} = \bar{q}_j (\lambda_j W_j + 1), \qquad j \in N, \tag{4.9.a}$$

$$f_{jj} = \bar{q}_j \Big(\lambda_j W_j + 1 - \frac{\lambda_j^2 \tau_j^{(2)}}{2(1-\vartheta_j)}\Big), \qquad j \in S, \tag{4.9.b}$$

$$\frac{f^{(2)}_{jjj}}{2f_{jj}} = \lambda_j W_j - \frac{\lambda_j^2 \tau_j^{(2)}}{2(1-\vartheta_j)^2}, \qquad j \in E, \tag{4.9.c}$$

$$\frac{f^{(2)}_{jjj}}{2f_{jj}} = \frac{\lambda_j W_j}{(1+\vartheta_j)}, \qquad j \in G. \tag{4.9.d}$$

■

Substituting the values given by Lemma 4.3 in equation (4.8), we obtain

Lemma 4.4

The conservation law for the non–deterministic polling system with non–exhaustive, semi–exhaustive, exhaustive, and gated service stations, where $p_{ij} = e_{ij}$ for all stations is:

$$\sum_{j\in N} \vartheta_j W_j \Big(1 - \vartheta - \lambda_j \sum_m \frac{v_m \theta_m}{v_j}\Big) + \sum_{j\in S} \vartheta_j W_j \Big(1 - \vartheta - \lambda_j (1-\vartheta_j) \sum_m \frac{v_m \theta_m}{v_j}\Big) + (1-\vartheta) \sum_{j\in\{E,G\}} \vartheta_j W_j$$

$$= \frac{\vartheta}{2} \sum_j \lambda_j \tau_j^{(2)} + \frac{\vartheta}{2C_1} \sum_j v_j \theta_j^{(2)} + \vartheta \sum_j \vartheta_j \theta_j + \sum_j \vartheta_j \Big(\sum_m \frac{v_m \theta_m}{v_j} - \theta_j \Big)$$

$$- \sum_{j\in\{S,E\}} \vartheta_j^2 \sum_m \frac{v_m \theta_m}{v_j} + \sum_{j\in S} \frac{\vartheta_j \lambda_j^2 \tau_j^{(2)}}{2} \sum_m \frac{v_m \theta_m}{v_j} + \sum_j \vartheta_j \sum_m \frac{v_m \theta_m y_{mj}}{C_1}. \tag{4.10}$$

■

Lemma 4.4 can be used to obtain, explicitly, the mean waiting times in symmetric systems. Implicitly, these are systems in which all the stations adopt only one of the four service disciplines considered in this paper. Note, however, that symmetry here does not imply that the probabilities p_{ij} need to be the same for all i,j, nor that β_{ij} be the same for all i,j. The only requirement here is that all stations have identical characteristics with respect to each other. This is illustrated in example 4.1 below.

Example 4.1 Consider a non–deterministic polling system with 4 stations, with all stations adopting the non–exhaustive service discipline. The data parameters for this system are as follows:

$$p_{ij} = e_{ij} = \begin{bmatrix} 0.10 & 0.40 & 0.30 & 0.20 \\ 0.20 & 0.10 & 0.40 & 0.30 \\ 0.30 & 0.20 & 0.10 & 0.40 \\ 0.40 & 0.30 & 0.20 & 0.10 \end{bmatrix},$$

$$\beta_{ij} = \begin{bmatrix} 0.10 & 0.15 & 0.20 & 0.30 \\ 0.30 & 0.10 & 0.15 & 0.20 \\ 0.20 & 0.30 & 0.10 & 0.15 \\ 0.15 & 0.20 & 0.30 & 0.10 \end{bmatrix}, \quad \text{All } B_{ij}\text{'s exponentially distributed,}$$

$$\gamma_{ij} = \sigma_{ij} = \begin{bmatrix} 0.10 & 0.15 & 0.20 & 0.30 \\ 0.30 & 0.10 & 0.15 & 0.20 \\ 0.20 & 0.30 & 0.10 & 0.15 \\ 0.15 & 0.20 & 0.30 & 0.10 \end{bmatrix}, \quad L_{ij} \text{ and } S_{ij} \text{ i.i.d., exponentially distributed,}$$

$$\lambda_i = [0.3125,\ 0.3125,\ 0.3125,\ 0.3125].$$

The derived data parameters for this system are $\tau_i = 0.45$, and $\theta_i = \varphi_i = 0.26$ for all i.

The visit ratio obtained from equation (4.4) is 1.0 for each station, and the cycle time, obtained from equation (3.17) is 1.3639 for each station. The mean waiting time at a station, obtained from equation (4.10) is 2.21124.

The random polling system

We now consider, as a special case of the conservation law, the random polling system introduced by Kleinrock and Levy (1988). For this system, $p_{ji} = \hat{p}_i$ for all j, and so the visit ratios are obtained from equation (4.4) as $v_i = \sum_j v_j \hat{p}_i$. In other words,

$$v_i / \hat{p}_i = \sum_j v_j = v_j / \hat{p}_j, \quad \text{for all i,j.} \tag{4.11}$$

Let

$$\kappa_i = \sum_k (\vartheta_k C_1 + v_k \sigma_{ki})\ \hat{p}_i - C_1\ \hat{p}_i\ . \tag{4.12}$$

So, from equations (4.7.a) and (4.7.b),

$$\begin{aligned} y_{ij} &= \sum_k v_k\, y_{kj}\ \hat{p}_i / v_i + \kappa_i / v_i \\ &= \sum_k \hat{p}_k\, y_{kj} + \kappa_i / v_i. \end{aligned} \tag{4.13}$$

Since $y_{jj} = 0$, we must have

$$\sum_k \hat{p}_k y_{kj} \quad = \quad -\kappa_j / v_j,$$

and so from equations (4.11) and (4.12),

$$y_{ij} \quad = \quad \kappa_i / v_i - \kappa_j / v_j \quad = \quad (\hat{p}_i / v_i) \sum_k v_k(\sigma_{ki} - \sigma_{kj}).$$

Thus, it may be observed that the conservation law for the random polling system does not require the solution of M sets of M linear equations in order to determine the y_{ij}'s.

Remark: Kleinrock and Levy obtain the mean waiting times for the discrete–time, symmetric system with constant service times. Note that the conservation law gives an explicit expression for the mean waiting times for a symmetric random polling system in the continuous–time case, which is very similar to the expression for the mean waiting times obtained by Kleinrock and Levy. ■

4.2. The Conservation Law when $p_{ij} \neq e_{ij}$

We present the conservation law for a system with two stations, for the case where $p_{ij} \neq e_{ij}$. The derivation of the conservation law for the system with more than two stations is a topic for further research. We consider a system where *both* stations adopt the *non–exhaustive* service discipline, since this appears to be one of the more complicated systems to analyze when there are only two stations. The conservation law in this case is more difficult to obtain since the g_{ij} terms now constitute an additional $M^2 - M$ unknowns. (For the case $p_{ij} = e_{ij}$ these terms vanished simultaneously resulting in equation 4.1.) From equation (3.6) with M=2, we have

$$v_i F_i(z_1,z_2) \quad = \quad \sum_{k=1}^{2} [\, v_k \frac{F_k(z_1,z_2) - F_k(0)}{z_k} p_{ki} B^*_{ki} L^*_{ki} \; + \; v_k F_k(0)\, e_{ki} S^*_{ki} \,], \quad i=1,2.$$

Let $\mathcal{V}^*_{ki} = B^*_{ki} L^*_{ki}$. In the above equations, setting z_1=z, and z_2=1, we have two equations in the three unknowns $F_1(z,1)$, $F_2(z,1)$, and $F_2(z,0)$ (note that $F_1(0,1) = q_1$). We can now eliminate $F_2(z,0)$ from these two equations, and express $F_2(z,1)$ in terms of $F_1(z,1)$ as follows:

$$v_2 F_2(z,1) \quad = \quad v_1 F_1(z,1) \frac{a(z)}{b(z)} - v_1 q_1 \frac{c(z)}{b(z)}, \qquad (4.14)$$

where

$$a(z) = z\,(p_{22}\mathcal{V}^*_{22} - e_{22}S^*_{22}) + p_{12}\mathcal{V}^*_{12}\, p_{21}\mathcal{V}^*_{21} - p_{11}\mathcal{V}^*_{11}\, p_{22}\mathcal{V}^*_{22} - p_{12}\mathcal{V}^*_{12}\, e_{21}S^*_{21} + p_{11}\mathcal{V}^*_{11}\, e_{22}S^*_{22},$$

$$b(z) = z\,(p_{21}\mathcal{V}^*_{21} - e_{21}S^*_{21} + p_{22}\mathcal{V}^*_{22}\, e_{21}S^*_{21} - p_{21}\mathcal{V}^*_{21}\, e_{22}S^*_{22}),$$

$$c(z) = a(z) + z(p_{22}\mathcal{V}^*_{22}\, e_{11}S^*_{11} - p_{21}\mathcal{V}^*_{21}\, e_{12}S^*_{12} + e_{12}S^*_{12}\, e_{21}S^*_{21} - e_{11}S^*_{11}\, e_{22}S^*_{22} - p_{22}\mathcal{V}^*_{22} + e_{22}S^*_{22}).$$

An expression for $v_1 F_1(1,z)$ in terms of $v_2 F_2(1,z)$ can be obtained in a similar manner. Differentiating equation (4.14) with respect to z, and evaluating it at z=1 gives an expression for f_{21} in terms of f_{11} and the data parameters. (This involves two applications of L'Hospital's rule.) In a similar manner, we can express f_{12} in terms of f_{22} and the data parameters. We now substitute these expressions for f_{21} and f_{12} in equation (4.13) and then express f_{ii} in terms of W_i, i=1,2, using Proposition 4.3. The resulting expression, obtained after considerable algebraic manipulation, is stated as

Lemma 4.5

The conservation law for the non–deterministic polling system with two non–exhaustive service stations and arbitrary p_{ij} and e_{ij} values is

$$\chi \sum_{i=1}^{2} \lambda_i W_i v_i q_i \eta_i = \left(\sum_{k=1}^{2} \frac{\lambda_k \eta_k e_{jk}}{2}\right) \sum_{i=1}^{2} (v_i \bar{q}_i \varpi_i^{(2)} + v_i q_i \theta_i^{(2)}) +$$
$$\sum_{k=1}^{2} \lambda_k \eta_k \theta_j \left[\sum_{i=1}^{2} (v_i \bar{q}_i p_{ij} \beta_{ij} + v_i q_i e_{ij} \sigma_{ij})\right], \quad j = k \bmod 2 + 1. \qquad (4.15)$$

■

In equation (4.15), we have set

$$\varpi_i^{(2)} = \sum_j [\beta_{ij}^{(2)} + \gamma_{ij}^{(2)} + 2\, \beta_{ij}\, \gamma_{ij}]\, p_{ij},$$

$$\chi = e_{21}\, \theta_1 + e_{12}\, \theta_2 ,$$

$$\eta_1 = e_{12}\, (\tau_1 + \varphi_1) - p_{12}\, \theta_1, \text{ and}$$

$$\eta_2 = e_{21}\, (\tau_2 + \varphi_2) - p_{21}\, \theta_2 .$$

Since there are only two stations, it is possible to express the cycle time C_1 given by equation (3.17) explicitly in terms of the data parameters as

$$C_1 = \frac{\chi}{e_{21} - \lambda_1[(\tau_1 - \theta_1)e_{21} + (e_{11} - p_{11})\theta_2] - \lambda_2 \eta_2}. \qquad (4.16)$$

The terms $v_i q_i = v_i - \lambda_i C_1$, i=1,2, are easily obtained using equations (3.7), (3.15) and (4.16).

Although equation (4.15) is valid for only two stations, it is the most general form of the non–deterministic polling systems, and it may be verified that the conservation laws for the deterministic and random polling systems with non–exhaustive service can be obtained as special cases of equation (4.15), by a suitable choice of the data parameters. It is interesting to note that the conservation law for the deterministic and random polling systems with exhaustive service can also be obtained from equation (4.15). The conservation law for the deterministic exhaustive service system is obtained by setting $p_{ii} = e_{i,i+1} = 1$, $\beta_{ii} = \tau_i$, $\gamma_{ii} = 0$, $\sigma_{ij} = \theta_i$, $\beta^{(2)}_{ii} = \tau^{(2)}_{i}$, $\gamma^{(2)}_{ii} = 0$, and $\sigma^{(2)}_{ij} = \theta^{(2)}_{i}$ to get, after some elementary algebra, the well known expression:

$$\sum_{i=1}^{2} \vartheta_i W_i \;=\; \frac{\vartheta}{2(1-\vartheta)} \sum_{i=1}^{2} \lambda_i \tau^{(2)}_{i} + \frac{\vartheta}{2\Theta} \sum_{i=1}^{2} \mathrm{var}(\theta_i) + \frac{\Theta}{2(1-\vartheta)} \sum_{i=1}^{2} (\vartheta_i - \vartheta_i^2). \qquad (4.17)$$

The conservation law for the random polling system with exhaustive service can be obtained from equation (4.15) in a similar manner by setting $p_{ii} = 1$, $e_{ij} = \hat{p}_j$, $\beta_{ii} = \tau_i$, $\gamma_{ii} = 0$, $\sigma_{ij} = \theta_i$, $\beta^{(2)}_{ii} = \tau^{(2)}_{i}$, $\gamma^{(2)}_{ii} = 0$, and $\sigma^{(2)}_{ij} = \theta^{(2)}_{i}$.

4.3. Mean Waiting Times: systems with Exhaustive and Gated Service Stations

In this section we indicate how the individual mean waiting times may be obtained for non–deterministic polling systems consisting only of Exhaustive and Gated service stations. Differentiating equation (3.6) twice with respect to j and m, setting $\bar{z}=\bar{1}$, and using equations (4.2) and (4.5), we get

$$\begin{aligned}
v_i f^{(2)}_{ijm} = & \sum_{k\in E} v_k p_{ki} \Big\{ f^{(2)}_{kjm} + (f^{(2)}_{kkj}\lambda_m + f^{(2)}_{kkm}\lambda_j)\frac{\tau_k}{1-\vartheta_k} + f^{(2)}_{kkk}\lambda_j\lambda_m \frac{\tau_k^2}{(1-\vartheta_k)^2} \Big\} \\
& + \sum_{k\in G} v_k p_{ki} \Big\{ f^{(2)}_{kjm} + (f^{(2)}_{kkj}\lambda_m + f^{(2)}_{kkm}\lambda_j)\tau_k + f^{(2)}_{kkk}\lambda_j\lambda_m \tau_k^2 \Big\} \\
& - v_j p_{ji} \Delta(j,E) \Big(\frac{f^{(2)}_{jjm}}{1-\vartheta_j} + f^{(2)}_{jjj}\frac{\lambda_m\tau_j}{(1-\vartheta_j)^2}\Big) - v_m p_{mi} \Delta(m,E)\, \bar{\delta}(j,m) \Big(\frac{f^{(2)}_{mmj}}{1-\vartheta_m} + f^{(2)}_{mmm}\frac{\lambda_j\tau_m}{(1-\vartheta_m)^2}\Big) \\
& - v_j p_{ji} \Delta(j,G) (f^{(2)}_{jjm} + f^{(2)}_{jjj}\lambda_m\tau_j) - v_m p_{mi} \Delta(m,G) (\bar{\delta}(j,m) f^{(2)}_{mmj} + f^{(2)}_{mmm}\lambda_j\tau_m) + D_{ijm},
\end{aligned} \qquad (4.18)$$

where

$$\begin{aligned}
D_{ijm} \;=\; & \lambda_j\lambda_m C_1 \Big\{ \sum_{k} p_{ki} \Big[\lambda_k \tau^{(2)}_{k} + \frac{\sigma^{(2)}_{ki}}{C_1} + 2\vartheta_k\sigma_{ki} + \Big(\frac{v_k}{v_j} + \frac{v_k}{v_m}\Big)\sigma_{ki} + \frac{v_k (y_{kj} + y_{km})}{C_1}\sigma_{ki}\Big] \\
& + \sum_{k\in E} p_{ki}\Big[\lambda_k \tau^{(2)}_{k} \frac{2\vartheta_k - \vartheta_k^2}{(1-\vartheta_k)^2} - v_k\sigma_{ki}\Big(\frac{\vartheta_j}{v_j} + \frac{\vartheta_m}{v_m}\Big)\Big] - \sigma_{ji} p_{ji} - \sigma_{mi} p_{mi} \\
& - p_{ji}\Delta(j,E)\lambda_j \frac{\tau^{(2)}_{j}}{(1-\vartheta_j)^2} - p_{mi}\Delta(m,E)\bar{\delta}(j,m)\lambda_m \frac{\tau^{(2)}_{m}}{(1-\vartheta_m)^2} \Big\}.
\end{aligned} \qquad (4.19)$$

In equation (4.18), the indices i,j, and m range over 1,…,M. Since $f^{(2)}_{ijm} = f^{(2)}_{imj}$, it can be seen that equation (4.18) thus provides $M^2(M+1)/2$ equations in $M^2(M+1)/2$ unknowns, which can be solved, thereby obtaining the $f^{(2)}_{jjj}$ terms and hence the mean waiting times.

5. Conclusions

In this paper we introduced the non–deterministic polling system. We obtained the cycle times and the stability conditions for this system and developed conservation laws for two special cases: i) where $p_{ij} = e_{ij}$, and ii) for arbitrary p_{ij} and e_{ij} values (namely, $p_{ij} \neq e_{ij}$) in the case of a system with two stations, both using the non–exhaustive service discipline. The mean waiting times were obtained for systems with arbitrary number of stations, where the stations could adopt either the exhaustive or the gated service discipline.

The non–deterministic polling system finds application in many situations. It has been used in analyzing material handling systems, and in computer communication networks. Special cases of this system are the well–known deterministic exhaustive and non–exhaustive service systems, the Bernoulli schedule system considered by Servi, and the random polling system analyzed by Kleinrock and Levy.

There are several topics for future research which need to be addressed. The development of the conservation law for non–deterministic polling systems should foster interest in obtaining approximations for mean waiting times using these laws. The conservation law for arbitrary p_{ij} and e_{ij} values still remains an open issue for systems with more than two stations.[1]

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Appendix: Proof of Proposition 4.2

Note: Purely for clarity, we present the derivation for a system with non–exhaustive, exhaustive, and gated service stations. This will indicate how the derivation can, in fact, extend to consider several other service disciplines. For this system, we rewrite proposition 4.2 as follows:

$$\sum_{j\in \mathcal{N}} v_j f_{jj}\, \tau_j \Big[\frac{1-\vartheta}{\lambda_j} - \sum_m \frac{v_m\theta_m}{v_j}\Big] + (1-\vartheta)\sum_{j\in \mathcal{E}} v_j f^{(2)}_{jjj} \frac{\tau_j}{2\lambda_j(1-\vartheta_j)} + (1-\vartheta)\sum_{j\in \mathcal{G}} v_j f^{(2)}_{jjj}\, \tau_j \frac{(1+\vartheta_j)}{2\lambda_j}$$

$$= \frac{\vartheta\, C_1}{2}\sum_j \lambda_j \tau_j^{(2)} + \frac{\vartheta}{2}\sum_j v_j\, \theta_j^{(2)} + \vartheta\sum_j \vartheta_j\theta_j C_1 + \sum_j \vartheta_j \sum_m v_m\, \theta_m\, y_{mj} - \sum_j \vartheta_j\, \theta_j\, C_1$$

$$+ \sum_{j\in \mathcal{E},\mathcal{G}} v_j f_{jj}\, \tau_j \sum_m \frac{v_m\theta_m}{v_j} + (1-\vartheta)\sum_{j\in \mathcal{N}} \tau_j C_1 - (1-\vartheta)\sum_{j\in \mathcal{E}} \frac{\lambda_j\, \tau_j^{(2)}}{2(1-\vartheta_j)^2}\vartheta_j C_1. \quad \text{(A.1)}$$

Proof: Differentiating equation (3.6) twice with respect to j and m, and summing over all i, at $\bar{z}=\bar{1}$ we get

$$\sum_i v_i f^{(2)}_{ijm} = \sum_k v_k f^{(2)}_{kjm} + \lambda_m\, \Omega_j + \lambda_j\, \Omega_m + K_{jm} - v_j\, \Delta(j,\mathcal{N})\, (f_{jm} - g_{jm}) - v_m\, \Delta(m,\mathcal{N})\, (f_{mj} - g_{mj})$$

$$- v_j\, \Delta(j,\mathcal{E})\, \Big(\frac{f^{(2)}_{jjm}}{1-\vartheta_j} + f^{(2)}_{jjj}\frac{\lambda_m\tau_j}{(1-\vartheta_j)^2}\Big) - v_m\, \Delta(m,\mathcal{E})\, \bar{\delta}(j,m)\Big(\frac{f^{(2)}_{mmj}}{1-\vartheta_m} + f^{(2)}_{mmm}\frac{\lambda_j\tau_m}{(1-\vartheta_m)^2}\Big)$$

$$- v_j\, \Delta(j,\mathcal{G})\, (f^{(2)}_{jjm} + f^{(2)}_{jjj}\, \lambda_m\tau_j) - v_m\, \Delta(m,\mathcal{G})\, (f^{(2)}_{mmm}\, \lambda_j\, \tau_m + \bar{\delta}(j,m) f^{(2)}_{mmj}), \quad \text{(A.2)}$$

where

$$\Omega_j = \sum_{k\in \mathcal{N}} v_k(f_{kj}(\tau_k+\theta_k) - g_{kj}\, \tau_k) + \sum_{k\in \mathcal{E}} v_k\Big(f_{kj}\theta_k + \frac{\tau_k}{1-\vartheta_k} f^{(2)}_{kkj}\Big) + \sum_{k\in \mathcal{G}} v_k(f_{kj}\theta_k + \tau_k f^{(2)}_{kkj}), \quad \text{(A.3)}$$

and

$$K_{jm} = \lambda_j\, \lambda_m\, \Big[\, \Gamma - N_{jm} + \sum_{k\in \mathcal{E}} v_k\, f^{(2)}_{kkk} \frac{\tau_k^2}{(1-\vartheta_k)^2} + \sum_{k\in \mathcal{G}} v_k\, f^{(2)}_{kkk}\, \tau_k^2\, \Big] + 2\Delta(j,\mathcal{N})\, \delta(j,m)\, \lambda_j C_1, \quad \text{(A.4)}$$

with

$$\Gamma = \sum_k [\lambda_k C_1(\tau_k^{(2)} + 2\theta_k\tau_k) + v_k\, \theta_k^{(2)}\,] + \sum_{k\in \mathcal{E}} \lambda_k C_1 \tau_k^{(2)} \frac{2\vartheta_k - \vartheta_k^2}{(1-\vartheta_k)^2}, \quad \text{(A.5)}$$

and

$$N_{jm} = (\tau_j + \theta_j)\, C_1\, \Delta(j,\mathcal{N}) + (\tau_m + \theta_m)\, C_1\, \Delta(m,\mathcal{N}) + \Big(\frac{\tau_j^{(2)}}{(1-\vartheta_j)^2}\lambda_j + \theta_j\Big)\, C_1\, \Delta(j,\mathcal{E})$$

$$+ \Big(\bar{\delta}(j,m)\frac{\tau_m^{(2)}}{(1-\vartheta_m)^2}\lambda_m + \theta_m\Big)\, C_1\, \Delta(m,\mathcal{E}) + \Delta(j,\mathcal{G})\, \theta_j\, C_1 + \Delta(m,\mathcal{G})\, \theta_m\, C_1. \quad \text{(A.6)}$$

From equation (A.2) we get

$$0 = \lambda_m \Omega_j + \lambda_j \Omega_m + K_{jm} - v_j \Delta(j,\mathcal{N}) (f_{jm} - g_{jm}) - v_m \Delta(m,\mathcal{N}) (f_{mj} - g_{mj})$$

$$- v_j \Delta(j,E) \left(\frac{f^{(2)}{}_{jjm}}{1-\vartheta_j} + f^{(2)}{}_{jjj} \frac{\lambda_m \tau_j}{(1-\vartheta_j)^2} \right) - v_m \Delta(m,E)\, \bar{\delta}(j,m) \left(\frac{f^{(2)}{}_{mmj}}{1-\vartheta_m} + f^{(2)}{}_{mmm} \frac{\lambda_j \tau_m}{(1-\vartheta_m)^2} \right)$$

$$- v_j \Delta(j,G) \left(f^{(2)}{}_{jjm} + f^{(2)}{}_{jjj} \lambda_m \tau_j \right) - v_m \Delta(m,G) \left(f^{(2)}{}_{mmm} \lambda_j \tau_m + \bar{\delta}(j,m) f^{(2)}{}_{mmj} \right). \qquad \text{(A.7)}$$

Consider the case where m = j. Note that for $j \in \mathcal{N}$, $g_{jj} = 0$. So, from equation (A.7),

$$2 \lambda_j \Omega_j - \xi_j + K_{jj} = 0, \text{ where} \qquad \text{(A.8)}$$

$$\xi_j = 2 v_j f_{jj}; \qquad j \in \mathcal{N}, \qquad \text{(A.9.a)}$$

$$= \frac{v_j f^{(2)}{}_{jjj}}{(1-\vartheta_j)^2}; \qquad j \in E, \qquad \text{(A.9.b)}$$

$$= v_j f^{(2)}{}_{jjj} (2 \vartheta_j + 1); \qquad j \in G, \qquad \text{(A.9.c)}$$

We see that Ω_j can be expressed, from equation (A.8) as $\Omega_j = \xi_j / 2\lambda_j - K_{jj} / 2\lambda_j$. Set

$$a_j = \sum_{k \in \mathcal{N}} v_k f_{kj} (\tau_k + \theta_k) + \sum_{k \in E,G} v_k f_{kj} \theta_k - \frac{\xi_j}{2\lambda_j} + \frac{K_{jj}}{2\lambda_j}. \qquad \text{(A.10)}$$

So, from equations (A.3), (A.8) and (A.10), we can write

$$a_j = \sum_{k \in \mathcal{N}} v_k \tau_k g_{kj} - \sum_{k \in E} v_k f^{(2)}{}_{kkj} \frac{\tau_k}{1-\vartheta_k} - \sum_{k \in G} v_k f^{(2)}{}_{kkj} \tau_k. \qquad \text{(A.11)}$$

Again, from equation (A.7), setting

$$b_{jm} = v_j \Delta(j,\mathcal{N}) f_{jm} + v_m \Delta(m,\mathcal{N}) f_{mj} + v_j \Delta(j,G)\, f^{(2)}{}_{jjj} \lambda_m \tau_j + v_m \Delta(m,G) f^{(2)}{}_{mmm} \lambda_j \tau_m$$

$$+ v_j \Delta(j,E) f^{(2)}{}_{jjj} \frac{\lambda_m \tau_j}{(1-\vartheta_j)^2} + v_m \Delta(m,E)\, \bar{\delta}(j,m) f^{(2)}{}_{mmm} \frac{\lambda_j \tau_m}{(1-\vartheta_m)^2} - \lambda_m \Omega_j - \lambda_j \Omega_m - K_{jm},$$

we can write

$$b_{jm} = v_j \Delta(j,\mathcal{N}) g_{jm} + v_m \Delta(m,\mathcal{N}) g_{mj} - v_j \Delta(j,G)\, f^{(2)}{}_{jjm}$$

$$- v_m \Delta(m,G)\, \bar{\delta}(j,m) f^{(2)}{}_{mmj} - v_j \Delta(j,E) \frac{f^{(2)}{}_{jjm}}{1-\vartheta_j} - v_m \Delta(m,E)\, \bar{\delta}(j,m) \frac{f^{(2)}{}_{mmj}}{1-\vartheta_m}. \qquad \text{(A.12)}$$

From equations (A.11) and (A.12), we observe that

$$\sum_{j} \tau_j a_j \quad = \quad \sum_{m}\sum_{\substack{j \\ m>j}} \tau_j \tau_m b_{jm} - \sum_{j\in E} \tau_j^2 \frac{v_j f^{(2)}_{jjj}}{1-\vartheta_j} - \sum_{j\in G} \tau_j^2 v_j f^{(2)}_{jjj}. \qquad \text{(A.13)}$$

So, from equation (A.13), after some elementary algebraic manipulation, we get

$$\sum_{m}\sum_{j} v_m \theta_m \tau_j f_{mj} - (1-\vartheta)\sum_{j\in N}\frac{\tau_j v_j f_{jj}}{\lambda_j} - (1-\vartheta)\sum_{j\in E}\frac{\tau_j v_j f^{(2)}_{jjj}}{2\lambda_j (1-\vartheta_j)} - (1-\vartheta)\sum_{j\in G}\frac{\tau_j v_j f^{(2)}_{jjj}}{2\lambda_j}(1+\vartheta_j)$$

$$+\frac{\Gamma\vartheta}{2} + (1-\vartheta)\sum_{j\in N}\tau_j C_1 + \sum_{j\in N}\vartheta_j \tau_j C_1 - (1-\vartheta)\sum_{j}\frac{\vartheta_j N_{jj}}{2} - \sum_{m}\sum_{j}\frac{\vartheta_j \vartheta_m N_{jm}}{2} = 0. \qquad \text{(A.14)}$$

Using equation (A.6), we can express

$$\sum_{j}\frac{\vartheta_j N_{jj}}{2} \quad = \quad \Big\{\sum_{j}\vartheta_j \theta_j + \sum_{j\in N}\vartheta_j \tau_j + \sum_{j\in E}\frac{\lambda_j \tau^{(2)}_j}{2(1-\vartheta_j)^2}\vartheta_j\Big\} C_1, \qquad \text{(A.15)}$$

and

$$\sum_{j}\frac{\vartheta_j \vartheta_m N_{jm}}{2} \quad = \quad \Big\{\vartheta\Big[\sum_{j}\vartheta_j \theta_j + \sum_{j\in N}\vartheta_j \tau_j + \sum_{j\in E}\frac{\lambda_j \tau^{(2)}_j}{(1-\vartheta_j)^2}\vartheta_j\Big] - \sum_{j\in E}\frac{\lambda_j \tau^{(2)}_j}{2(1-\vartheta_j)^2}\vartheta_j^2\Big\} C_1. \qquad \text{(A.16)}$$

Substituting, in (A.14), the values for f_{mj} in terms of f_{jj}, as given by equation (4.5), and using equations (A.5), (A.15) and (A.16), we get the desired result.

■

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