Mathematical Knowledge for Teaching Teachers:
The Mathematical Work of and Knowledge Entailed by Teacher Education

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Education)
in The University of Michigan
2010

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Dedication

To my husband, Michael G. Zopf

And my children, Michael, Katherine, and David

Acknowledgements

Many people have contributed to the completion of this work. Beginning with my family, I would like to thank my parents, Joseph and Edna Rolecki for teaching me the value of hard work, discipline, and perseverance. I would like to thank my husband, Mike for encouraging me to pursue this dream. He has managed much of our family responsibilities as I spent long hours working on this document. For his love and support, I am most grateful. I would like to thank my children, Michael, Katie, and David for all the time they take from their lives to spend with Mike and me and the joy they, their spouses and children bring to our lives.

I would like to thank my colleagues at Henry Ford Community College and the University of Michigan – and at other institutions for their encouragement as I worked through this process.

This work would not have come to fruition without the guidance of my dissertation committee. I am grateful to Magdalene Lampert who pressed me to think beyond mathematical knowledge for teaching research and consider a more generic form of mathematical knowledge that might be the goal of teacher education. I appreciate Professor Lampert's careful reading of my late draft. Her comments helped me see minute errors and areas of repetitive text as well as prompted me to formulate my claims about MKTT.

My study is heavily dependent on the mathematical work of two teacher educators. Hyman Bass's thoughtful reading and commentary on the mathematics supported precise representation of the mathematical work presented in this dissertation. In addition, Professor Bass discussed the mathematical work with me opening up many perspectives on this work. His recommendations allowed me to be reassured that the mathematical content of this dissertation is correct. For this I am grateful.

The qualitative nature of my study required both broad, global examination of the mathematical work and a fine-grained analysis of the elements of this work. I am grateful

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Glossary

Common Content Knowledge (CCK) – Knowledge of mathematics that teachers use in teaching that may be used in other fields. As an example, teachers need to know that 2 x 4 = 8. Although knowledge of this fact is used in teaching, it is not specific to teaching. It is used in everyday life and in other types of work.

Knowledge of Content and Students (KCS) – Knowledge of mathematics and students is an amalgam of mathematical knowledge and knowledge of how students think about mathematics, the language they use easily or confound their conceptions and misconceptions.

Knowledge of Content and Teaching (KCT) – Knowledge of mathematics and teaching is the melding of mathematical knowledge and instruction about mathematics. This includes knowledge of how topics should be sequenced, the affordances of particular models, choice of language and representations, and the benefits of pursuing a line of reasoning or moving away from it.

Learning Mathematics for Teaching Project (LMT) – Research project directed by Heather C. Hill, Deborah Loewenberg Ball, and Hyman Bass which was designed to develop measures of Mathematical Knowledge for Teaching (MKT).

Mathematical Knowledge for Teaching (MKT) – The mathematical knowledge teachers use in the work of teaching. This construct is an evolving one. For this dissertation, four sub-domains will be considered: Common Content Knowledge, Specialized Content Knowledge, Knowledge of Content and Students, and Knowledge of Content and Teaching.

Mathematical Knowledge for Teaching Teachers (MKTT) – Mathematical Knowledge for Teaching Teachers is the mathematical knowledge used by mathematics teacher educators in the work of teaching mathematics to teachers.

Mathematics Methods Planning Group (MMPG) - Professional community consisting of mathematics teacher educators, a mathematician, and graduate students that plan, teach, debrief, and research the mathematics methods for elementary teachers course at a major university.

Specialized Content Knowledge (SCK) - Specialized Content Knowledge is the mathematical knowledge used by teachers for the work of teaching. It includes the mathematical knowledge used in choosing and modifying tasks, talking with and responding to students, using students' ideas to further conversation, assessing student work.

ABSTRACT

Mathematical Knowledge for Teaching Teachers:
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Deborah Ann Zopf

Chair: Deborah Loewenberg Ball

This dissertation investigates the mathematical work and knowledge demands of this work to teach mathematics to teachers. The last 20 years have seen progress in the study of the specialized knowledge of mathematics needed for the work of teaching, as well as much discussion about the importance of strengthening teachers' mathematical preparation. However, less attention has been paid to the mathematical work and the knowledge demands of the work for those who must teach the courses, write textbooks, or develop programs to help teachers learn mathematics. This study investigated the work of teaching teachers and the mathematical knowledge entailed by that work.

The overarching question of the study is: What is the work of teaching teachers mathematics and what are the mathematical knowledge demands entailed by this work? Two sub-questions are:

- What are some of the key tasks of teaching mathematical knowledge for teaching that are involved in teaching teachers mathematics?

- What are the mathematical knowledge demands entailed by teaching teachers mathematics?

This study examines the teaching of two mathematics teacher educators, educators significantly different in professional training. Their students are different. One teaches in-service teachers. The other teaches student teachers. The contrasting sites provided data to probe recurrent tasks and demands of the work of teaching mathematics to teachers.

First, I identified several task domains within the work of both teacher educators. Three appeared to be central: selecting interpretations and representations, selecting examples, and managing mathematical tasks. Four cases of teaching, two from each teacher educator were analyzed for elements of these task domains. Based on my cross case analysis I differentiate elements that seemed to be consistent and idiosyncratic across the cases. I proposed a framework for the study of the work of teacher education.

Second, I examined the mathematical work of these cases for the mathematical knowledge demands. I proposed a domain of mathematical knowledge, *mathematical knowledge for teaching teachers* (MKTT), discussed distinctive qualities of MKTT that appeared to characterize the ways MKTT is held and used for the work of teacher education.

Chapter 1

Formulating My Question

During the fall of 2006, I observed a university mathematics methods course. A team of instructors consisting of a mathematician, a mathematics educator (Deborah Ball), a graduate student with many years of elementary school teaching experience, an educational psychologist, and a practicing elementary teacher who also taught and conducted research at the university planned and taught several sections of the course. They met each week to design that week's lesson. Later in the same week, the team would observe the lead teachers teach the lesson and debrief the class immediately after it was taught. During the debriefing sessions, the mathematics teacher educators discussed many details about the mathematical issues involved in the classes. They discussed the examples they would use, tasks the teachers would complete, and assignments to reinforce or extend the work of the class. They thought about the mathematics for these tasks in very specific ways that attended to the needs of prospective teachers *and* maintained the integrity of the mathematics taught in elementary schools. The team anticipated what the teachers would find challenging, errors they might make, and ways to develop the mathematical ideas important for these teachers to know. As someone who teaches mathematics content course for pre-service elementary teachers, I was struck by the intensity of the team's work. Listening to their conversations about the mathematics a particular task might make visible, or what it might not, made me think about the mathematics to which they were attending in this course and how they planned to elicit

this mathematics through the course work. It appeared to me that this setting was particularly rich for an in-depth study of teaching mathematics to teachers. I began to wonder about the kinds of work in which the team engaged. I observed the mathematics methods planning group (referred to as MMPG), their planning and debriefing meetings, and Ball's teaching of the course. The mathematical work of this team of mathematics teacher educators was intriguing. It captured my attention and caused me to formulate some questions about the work of mathematics teacher education and the mathematical knowledge demands entailed by it.

This dissertation investigates the mathematical work and knowledge demands of the work of teaching mathematics to teachers. Although the last 20 years have seen much discussion about the importance of strengthening teachers' mathematical preparation, little attention has been paid to the mathematical work and the knowledge demands of that work for those who teach the courses, write textbooks, or develop programs to help teachers learn mathematics. One big discovery about the work on mathematical knowledge is that the mathematical knowledge used to teach children requires substantial mathematical skill not identical to the mathematics used in other contexts. My emerging question is similar. Could it be that the mathematical work to teach teachers might require a quality or type of mathematical knowledge beyond that needed to teach children?

In order to leverage some work on this question, I set out to study the instructional practice of mathematics teacher educators and to analyze and formulate conjectures about the mathematical work and the mathematical knowledge required by this work. The methods of this study parallel those used to make headway into the mathematical

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knowledge used by teachers to teach children. There too researchers studied the work and traced backwards analytically to the mathematical demands.

This study is not about the mathematics teacher educators whose courses I documented and analyzed. It is not about my personal theories about the tasks to which teachers should attend or what they should know. Neither is it about my personal theories about the mathematical tasks to which teacher educators should or need to attend or what mathematics teacher educators know or need to know. Rather, this study focuses closely on the tasks of teaching mathematics to teachers. It makes claims about these tasks and uses these as an empirical base to analyze the mathematical knowledge entailments of that work.

Research on teachers' mathematical knowledge has begun to reveal that the mathematical content used for teaching is a distinctive knowledge – mathematical knowledge for teaching. I hypothesize that this is the mathematical knowledge taught in courses for teachers. That is, the mathematical knowledge taught in mathematics teacher education is the mathematical knowledge that teachers use to teach children. It is special. It includes fundamental knowledge of mathematical concepts, procedures, and practices, and detailed unpacking of that content in ways germane to teaching it to others. Likewise, there are good reasons to think that the knowledge used to teach mathematics that is targeted for teachers as learners rather than children might present some special mathematical demands just as the work to teach children presents mathematical demands beyond that mathematical content.

It should be noted that this program of studying the mathematical knowledge for teaching teachers (MKTT) is part of a larger framework introduced by Hyman Bass and

Deborah Ball – the mathematical knowledge for mathematics education (MKME) – of which teacher education is one particular task domain.[1] This dissertation can be considered as a first empirical investigation of an important component, MKTT, of MKME.

In this introduction to my dissertation, I discuss what motivated my interest in this study and the work involved in developing the question. This work included preliminary observations that produced early forms of the dissertation questions and evolved into a pilot study that provided a first order test of the hypothesis and informed the study's design.

Beginnings of the Study

Decisions on How to Approach the Study

The work of teaching mathematics to teachers and its mathematical entailments might be investigated in a variety of ways. As a community college instructor who has developed and taught mathematics content courses for pre-service elementary teachers, a study of my practice and the mathematical knowledge demands of that work were feasible. A second approach might be to study the texts used for teaching mathematics to teachers and analyze the mathematical knowledge demands of the tasks and other work detailed for the instructor. A third approach would be to examine the enactment of actual courses taught by other mathematics teacher educators to learn about the mathematical knowledge entailed. Because of the opportunity to collect data on the work of the MMPG researchers, a study of mathematics teacher education practice seemed to be a productive way to approach this study. This approach builds on the work of mathematics teacher

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[1] Professor Bass discussed the broader domain of Mathematical Knowledge of Mathematics Educators at the Arthur F. Cox Lecture, Ann Arbor, Michigan, October 5, 2007.

educators who have examined their own practice (Tzur, 2001; Zaslavsky, Chapman, & Leikin, 2003). Further, it augments the work of researchers who have attended to pedagogical knowledge for teaching teachers (Geddis & Wood, 1997; John, 2002; Simon, 1994; Tzur, 2001)) or who have described mathematical knowledge for teaching teachers as evolving through the work of teaching teachers (Zaslavsky & Leikin, 2004). Tzur (2001) and Zaslavsky et al. (2003) claim that their knowledge of mathematics grew as a result of teaching children, pre-service and in-service teachers, and novice mathematics teacher educators.

This investigation is different from those which claim that mathematical knowledge grows or develops as a result of teaching teachers. Rather, it is focused on the *work* to teach mathematical knowledge for teaching to teachers. I hypothesize that this work is distinctive. And, I attempt to investigate both whether that seems to be the case as well as what the key distinctions might be. As someone who has taught mathematics to teachers, this question is important to me in identifying key aspects of this work and the mathematical entailments of the work of teacher education.

It might occur to the reader that the careful work already done on MKT might likely apply to this. So let's pause for a minute and consider what might lead one to surmise that the tasks and demands be different. First, the mathematical content is different. Children are taught mathematics; teachers are taught mathematical knowledge for teaching. Second, the learners are different. Children come to school with some mathematical understandings. The goal of school mathematics is mathematical proficiency. As an example, children learn the multi-digit multiplication algorithm and seek to become proficient at performing this algorithm. Juxtapose this with teachers'

mathematical knowledge when they enter teacher education. Teachers possess mathematical knowledge learned during their school experiences. They know the multi-digit multiplication algorithm. However, it is the goal of teacher education to develop this mathematical knowledge into mathematical knowledge for teaching. For the multi-digit multiplication algorithm, teachers learn interpretations and representations that develop the concept of multiplication. Interpretations and representations are used to explain algorithms. Connections are made between geometric and algebraic representations. Third, the purposes of instruction are different. Children learn mathematics for their own use; teachers learn mathematical knowledge for teaching to teach mathematics to students. Learning mathematical knowledge for teaching includes learning mathematical content used for unpacking mathematical ideas, ways of doing mathematical work that includes developing mathematical habits of mind, ways of reasoning, ways of talking and writing about mathematics, and reflecting on mathematical work. It is on the basis of this set of assumptions that this study rests.

Initial Observations: Brief Summary of the Work to Teach Place Value

During the observation of the MMPG research team, I observed the planning and teaching of a specific sequence within the elementary mathematics methods course. The mathematics teacher educators' work to teach place value to teachers involved several well-orchestrated tasks designed to unpack the fundamental understanding of place value used in teaching children of this concept. In planning and debriefing meetings, the MMPG researchers discussed tasks they used and the mathematics entailed by the tasks. They selected manipulatives appropriate to various grade levels. They discussed the language and representations appropriate for each set of manipulatives. They enacted

carefully planned lessons built around these tasks. As part of the lesson, they paused their

teaching and discussed the mathematics they were eliciting. They connected language

and representations to each set of manipulatives, making clear the importance of their

choices for each mathematical task. Further, they looked across the progression of

mathematical tasks and made explicit their various choices of manipulatives, and the

corresponding language and representations. Although I saw clear relationships between

the course work and the work that elementary teachers do with their pupils, this work by

the teacher educators appeared to be different. For example, the mathematics teacher

educators selected tasks that had the teachers "work backwards" into the mathematics,

because their students, student teachers, already knew place value, at least in some ways.

Primary grade teachers cannot approach place value in the same way with their pupils.

Primary grade teachers would not work across two or three grade levels in one lesson, so

the compression and organization of the content was different from what one might see in

first or second grade classes. Insights like this fueled my curiosity and added to my

conviction that I was pursuing an important question.

This brief summary of the planning for the work to teach place value

demonstrates the detailed work to plan and implement tasks for teaching mathematics to

teachers. In the next section, I provide the first vignette, that of the task to multiply 15 x

32. I present this vignette to demonstrate the initial set of data, my examination of the

data. My inquiry caused me to examine the teaching and pose a series of questions

catalyzed by the study of the teaching episode. This method of thinking carefully about

teaching, generating a plethora of questions, and reflecting on the teaching in light of

these questions allowed me through a recursive process to formulate the dissertation question. I turn now to the first vignette, that of the teaching of multi-digit multiplication.

An Introductory Vignette: 15 x 32

In order to make my question clearer, I drop into the middle of a class which the teacher educator, Ball, was working with student teachers on multiplication of multi-digit numbers. My purpose is not to begin the analysis of the study but to highlight the focus of the question and set up the study. Ball[2] asked a student teacher, Lana[3], to model the multiplication of 15 times 32, using a rectangular grid representation. She reminded the other teachers to watch and take notes on Lana's work so that they would be able to critique it. Lana agreed to present her work but stated that she was uncertain about it. Lana wrote out a series of steps to calculate the product. However, her work did not correspond to the rectangular grid representation. She used a set of partial products in a random order that did not align with the model. Her written work was unorganized. Her explanation lacked precision. Ball probed Lana's work.

Ball: Where is 15 x 32?
      Where is 5 x 2? … We need to show each part.

Janet, a second student responded.

Janet: We need to make explicit that $15 = 10 + 5$ and $32 = 30 + 2$. I would write it out.

Ball pointed out the order of the multiplication and recommended that the student teachers maintain consistency between the grid model, the symbolic representation, and the language they used when discussing the problem. She called on a second student, Amanda to demonstrate how she represented and explained the multiplication of 15 times

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[2] Two mathematics teacher educators enact the tasks that are the subject of this dissertation. I will refer to them as Ball and Suh.
[3] All student teachers' and teachers' names are pseudonyms.

32. Amanda pointed to the area that represented 10 times 30. She expanded 30 to 10 + 10 + 10 and produced the partial products 100 + 100 + 100.

Ball intervened. She reminded the student teachers that they needed to write out steps that corresponded to the representation in order to guide children's learning. Also, she noted that the teachers needed to unpack the mathematics that might be compressed within the algorithmic work of multiplication. She added that the teachers' demonstrations needed to make clear the multiplication of each digit in the multiplicand by each digit in the multiplier. Figure 1.1 depicts the representation with which the teachers worked.
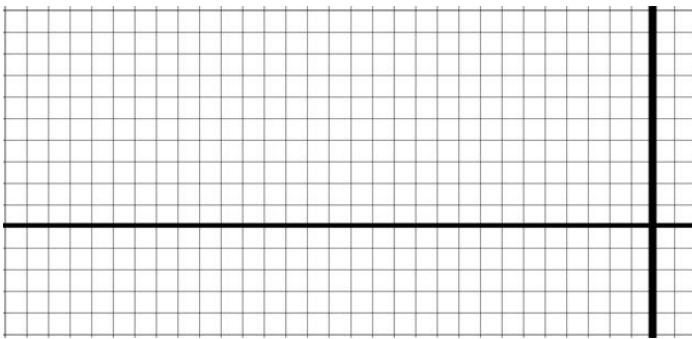
Figure 1.1. Representation of 15 x 32.

Ball analyzed the teachers' demonstrations of 15 x 32. In the moment, she assessed what each teacher produced and discussed the positive aspects of each teacher's work. In her critique, she attended to mathematics *and* she used pedagogical techniques suited for teaching teachers. She noted that the teachers were examining the mathematics closely which had been a goal of the course. She complemented the teachers noting that their attention to detail was developing as compared to the work they had done at the beginning of the semester. In her talk, Ball praised the teachers' for their attention to detail however, she reminded them that they needed to think carefully about *which* details

demanded attention. They needed to make visible the mathematics that children need to learn multi-digit multiplication. The following transcription provides evidence of the mathematical demands of the work to foster teachers' emerging mathematical knowledge for teaching.

Ball: I want to talk about several things that I have been hearing and pull some different pieces out. One of the things that is extremely good about what you are doing is all the things you are noticing. Do we want to think about this in the simplest way to add like the way Amanda is thinking? All these things are very, very subtle compared to the kind of things you would have done when the term began. It is the kind of careful use of representation that we hope to develop. As we are talking tonight we are using several things at the same time. One thing that is useful is to keep in mind is what your purposes are in your modeling. So one of your purposes might have been to understand what is the meaning of 15 times 32 and knowing that it is composed of separate pieces –that you can see in the area and also when you multiply and mapping it against the written form. And then Amanda's method when she is helping people see all the parts is important. In addition to becoming more careful about what you can see and use in the picture, you want to think carefully about what you want to show in a given moment. And then I think Sharon said it – be careful not to overwhelm the students with everything you have available to you at the same time. So it depends on what you want to show. (Ball transcript, 103106, p. 4).

Before Ball demonstrated the procedure explicating details important for teaching children multi-digit multiplication, she highlighted the mathematical work on which the teachers were improving. That is, she recognized their attention to details. However, she noted that the teachers needed to think about and make visible details that help children learn this procedure, the mathematical details used for teaching. She noted that although Amanda expanded 300 to $100 + 100 + 100$ which made the computation easier, this expansion did not identify the partial products in the rectangular array. Ball reminded the teachers that they needed to craft their explanations around the mathematical work of the lesson.

After this discussion, Ball multiplied 15 x 32; she demonstrated how the

algorithm connected to each step to the area representation. She made clear that the 1 in

15 represented 10 and the 3 in the 32 represented 3 tens, or 30. She computed the partial

products, pointed to the areas that corresponded to the partial products, and connected the

partial products to the area model in a clear demonstration of the multiplication

procedure.

Ball: We want to show 15 times 32. Can you see that? We have 15 on this side length [traces with pencil]. Ten plus five on this side. And 32 on this side, that is 30 plus 2. So we have a rectangle that is 15 on the side and 32 across the top. And in order to calculate this, the 5 x 2 which is 10. And the five times two is here as well [points to the 5 x 2 rectangle]. We have five [Ball traces length of five] times two [she traces width of two]. We have five times two. [Ball repeats the tracing of the 15 and the 32]. (Ball transcript, 103106, p. 5)

Ball continued to work through the multiplication with great detail. She attended

to the order of the multiplication and the dimensions of the rectangle. She cautioned the

teachers about steps that might appear correct but contain minute errors. She noted that:

So far I have only done that part of the multiplication. I have done five, which is this part here, times two. This is the first piece of what I have to do to get the answer. The second thing and *now I have to be careful* because I might say 10 times five.   This would be a little less conventional, because typically when we write it we do five times two and 10 times two. Just be consistent right now (Ball transcript, 103106, p. 5).

The demonstration continued for several minutes. Ball completed each step and

pointed out potential errors, such as confusing the order of the factors. She paused

intermittently to ensure that the teachers understood her work. After she had multiplied

each pair of digits, she stated,

Now I have multiplied all the numbers in the problem. And I have also multiplied all the pieces of the area. So we have 300. [Ball writes '300' in the green rectangle identifying the partial product with appropriate area.] Ten times 30 equals 300. And now there are two ways that I can find the answer. One way is

that I can add up all the areas inside the big rectangle. So I have 10 plus 20 plus 150 plus 300. I have the same thing written down – 10 plus 20 plus 150 plus 300. And altogether those equal 480. And I can see by adding down this way. One plus two plus five is eight and one hundred and three hundred is four hundred. Or I can also add up the areas" (Ball transcript, 103106, p. 6).

The data indicate that the mathematical work was detailed and worthy of study. I am curious to know about the mathematics made visible during the modeling of the multiplication of 15 times 32. What mathematics was unpacked for the student teachers through the use of the rectangular region? What were the mathematical ideas in this lesson important for student teachers' work but not be part of work with children? What were the mathematical demands of orchestrating a conversation among student teachers, of attending to the errors Lana and Amanda made, and of making judgments about what to take up and what to let go?

These initial explorations generated questions about the work to teach teachers and the mathematical entailments of that work. These questions and others raised my awareness of several tasks of teaching that appeared to be part of this work. The initial vignette and the questions it provoked pushed me to move forward with a pilot study.

Before discussing the pilot, I briefly discuss the purposes of mathematics teacher education and pose the questions which guided my study.

A Purpose for Mathematics Teacher Education: *Developing Mathematical Knowledge for Teaching*

As student teachers develop into teachers, their knowledge about mathematics evolves from the way students know mathematics for their own use to the way teachers know mathematics for teaching children. Their abilities to know definitions, rules, and algorithms in order to apply these to problem situations are fundamental to but not sufficient for the teaching of mathematics. Although teachers' mathematical knowledge

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for teaching depends on the knowledge they learned as students, it is a more fundamental, connected, and extensive knowledge necessary for teaching.

The purpose of mathematics teacher education is developing mathematical knowledge for teaching. It is a key setting in which teachers have the opportunity to take a look at mathematics and adjust their knowledge of it from the perspective of a student to that of a teacher. Consequently, one might ask: What is involved in creating opportunities for teachers to learn? This I call the work of mathematics teacher education. And for that work, what are the mathematical skills or insights entailed by the teaching of mathematics to teachers, people whose use for the mathematics will be to teach it?
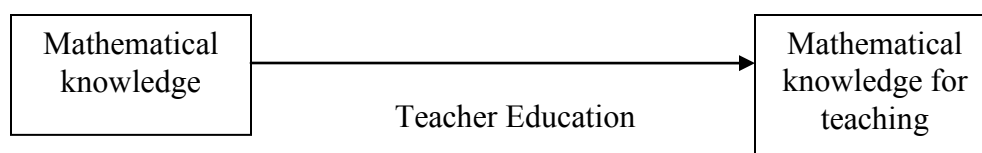
Figure 1.2. One component of the work of teacher education is to foster the development of mathematical knowledge for one's own use to mathematical knowledge for teaching.

Pre-service teachers' knowledge of mathematics begins to transform into mathematical knowledge for teaching during teacher preparation. Mathematics content and methods courses along with field work provide experiences that nurture pre-service teachers toward this goal. Courses that focus on mathematics, the teaching of mathematics, and reflection on the teaching of mathematics provide opportunities for pre-service teachers to learn mathematics for teaching. Course content that addresses mathematical language, concrete and abstract representation, and modeling includes specific experiences for pre-service teachers' learning of mathematics for teaching. The

two levels of learning opportunities, learning mathematics and learning to teach mathematics, are important for the development of mathematical knowledge for teaching. The central task of mathematics teacher education is to teach pre-service teachers *what* mathematics is needed for teaching and *how to think about the mathematics they use in teaching.* This professional education is intended to foster pre-service teachers' knowledge of mathematics for teaching.

Because teacher education is intended to foster pre-service teachers' shift from knowing about mathematics as students to knowing about mathematics as teachers, it is important to study factors that promote this shift. The focus of this dissertation is the work of teaching mathematics to teachers and the mathematical entailments of that work. The overarching question of this investigation is:

What is the work of teaching teachers mathematics and what are the mathematical knowledge demands entailed by this work?

To facilitate my investigation I focused on two supporting questions:

1. **What are some of the key tasks of teaching mathematical knowledge for teaching that are involved in teaching teachers mathematics?**

2. **What are the mathematical knowledge demands entailed by teaching teachers mathematics?**

For this dissertation, I investigate the mathematical work involved in teaching teachers mathematics for practice. I focus in particular on three tasks and the mathematical entailments of these tasks. I seek to identify sub-tasks or elements inherent to this work and use these to learn about the mathematical demands required by each element. My early exploration, the phase I call the pilot study, helped to develop and

sharpen my questions. The pilot study included the examination of two communities of mathematics teacher educators as they planned, taught, and debriefed mathematics methods lessons. It is my work at these sites and my analysis of these data that allowed the questions of this dissertation to crystallize and pushed me forward in my investigation. As I moved forward, I kept at the forefront the questions – what is the work of teaching mathematics to teachers and what are the mathematical demands entailed by this work?

Pilot Study

The focus of this investigation is mathematical work of teaching teachers and the knowledge entailments of this work. Through the analysis of mathematical problems, the work of teaching mathematics to teachers, and three tasks of teaching mathematics to teachers, I sought to identify the mathematical work and the associated knowledge demands. I assumed that this work is distinctive from the work of teaching mathematics to children. My assumptions, the learners are different, the mathematics is different, the purposes for instruction are different, set me off to learn about the nature of the work of mathematics teacher education. Because I was launching a study into uncharted territory, I conducted a pilot study that included two sites.

As described at the beginning of this chapter, the first site was a mathematics methods course for elementary teachers taught at a prominent state university. My selection of this site was based on the rich set of talented scholars who participated in the course preparation and teaching. This team had worked for several years to develop the course, individual lessons, and materials. In addition to the leadership team, there was a cohort of apprentice instructors. As previously discussed, this group met each week to

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plan and "walk through" the week's lesson. Also, the group observed the "lead section," co-taught by the mathematics teacher educator and graduate student. Finally, the group reviewed the lead section class and discussed recommendations for how each instructor might adjust the material for their needs. The work of the lead teacher educator, Professor Ball, is the object of this study.

My second site was a mathematics methods course for elementary teachers taught at a branch campus of the same state university. I selected this site because I sensed that I would learn much from the mathematics teacher educator, Professor Stein, whose practice I would be observing. Professor Stein has taught school mathematics and has been engaged in mathematics teacher education for several decades. Like the first site, there was a joint planning, teaching, and debriefing process for the mathematics methods course, but on a smaller scale. Two mathematics teacher educators met to plan their classes. They taught their classes separately and debriefed their experiences just prior to planning their next class.

I observed similarities between the sites. The teacher educators at both sites planned, taught, and debriefed lessons built around mathematical problems, examples, and assignments that fostered student teachers' understanding of the mathematics they teach, challenged misconceptions that students and teachers often hold, and nurtured teachers' mathematical knowledge of concrete and abstract models, representations, and language. This work presented many mathematical demands on the student teachers and on the mathematics teacher educators.

Analysis and Preliminary Results

I sifted through data from the two sites to learn about the teaching of mathematics to student teachers. I reviewed field notes recorded during planning meetings and class sessions as well as interview transcripts and noted instances of teaching mathematical knowledge for teaching in these data. I looked across the data from the two sites to see if there were common types of work. I studied the planning sessions to see what work was prominent. I examined the mathematical work in which the student teachers were engaged to extract pieces of work that looked similar – and different.

I identified eight tasks which require the teacher educator's use of mathematics: selecting the content, selecting tasks, modeling mathematics, selecting examples, managing discourse, responding to student teachers' questions, selecting materials, creating assignments, analyzing and responding to student work, and designing and grading assessments. These tasks appeared to be similar to those for teaching children. I wondered if they were different when teachers were the learners rather than children. Critical questions I might ask are: What might make this work distinctive for the work of teaching teachers? Does this distinctive work have special mathematical demands that transcend the mathematical demands of teaching of children? How are these mathematical demands different?

Distinguishing Characteristics of the Work of Mathematics Teacher Educators. My examination of data from both sites made me think about questions I had asked about the initial vignette and it raised new questions. It provided additional mathematical work that allowed me to sharpen the focus of my study.

At both sites, I observed the work entailed by selecting the tasks to be used for teaching student teachers. I observed the presentation of mathematics, then management

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of meta-level discussion on the mathematical work. This work at two levels appeared to demand an awareness of the mathematics being taught in the moment *and* the underlying mathematics be that substance or syntax, important to student teachers' knowledge of mathematics for teaching. These characteristics provided information about *what* mathematics is needed for teaching these courses, *how* the mathematics is used in teaching the courses, and *how the mathematics is made explicit* when teaching student teachers.

From this example, I questioned the tasks important for teaching mathematical knowledge for teaching and the mathematical entailments of these tasks. I wondered about the mathematical work entailed by planning, implementing, and assessing tasks used to foster student teachers' mathematical knowledge for teaching.

1. My pilot study results sharpened my questions and helped me focus on the teaching of teachers in the full study. Also, because of the richness of the data from one site, I chose to analyze these data in greater depth for the full study. Consequently, my investigation builds on my pilot study findings. I add to my data by studying the teaching of one additional mathematics teacher educator, one who is different in professional training and work. I wrote case studies of each subject's work to synthesize all sources of data. I analyzed these cases at several levels to learn more about the work of mathematics teacher education and the mathematical entailments of this work. In Chapter 3, I provide a detailed discussion of my data collection and analysis methods.

Overview of the Dissertation

In Chapter 2, I present findings in the literature on the purposes and work of teacher education. To examine the mathematical work of teacher educators more closely, I discuss scholars' perceptions of what mathematical knowledge *is* and what mathematical knowledge for teaching *is*. I focus on the concept of mathematical knowledge for teaching (MKT) developed by our research team led by Professor Ball to provide a vision of this domain of mathematical knowledge. Because her teaching is the object of study, one might assume that Ball is teaching student teachers with the goal of developing MKT. With a definition of mathematical knowledge for teaching established and a premise that a goal of mathematics teacher education is to foster student teachers' mathematical knowledge into mathematical knowledge for teaching, I examine teachers' opportunities to learn mathematical knowledge for teaching and discuss three projects designed to present student teachers opportunities to learn mathematical knowledge for teaching *and* examine student teachers' use of this knowledge in student teaching experiences. Finally, I examine literature on mathematics teacher educators' work to teach mathematics to student teachers. In this literature, I present work about mathematics teacher educators' study of their practice and several discussions on mathematics teacher educators' self-study of their development into their roles as mathematics teacher educators. A gap in this literature, empirical studies of the work of mathematics teacher education and the mathematical entailments of this work, is the focus of my dissertation.

In Chapter 3, I present my methods of data collection and analysis. My subjects, two mathematics teacher educators, provide a wealth of data from which I select episodes rich in mathematical tasks for analysis. My study consists of four phases: analysis of

observation and interview data to identify instances of the use of mathematical knowledge that appeared distinctive from mathematical knowledge for teaching, analysis of mathematics problems used to teach mathematical knowledge for teaching, analysis of three tasks for teaching mathematical knowledge for teaching for its purposes and levels, and analysis of three tasks of teaching mathematical knowledge for teaching for the sub-tasks or elements of work and the mathematical demands of that work. For each analytic phase, I study the emerging ideas and use those to inform the next stage of analysis.

In Chapter 4, I present the two cases of the work of Professor Suh, a research mathematician who designs and teaches professional development institutes for teachers. From these data, I selected Suh's teaching of the concept of fraction and fraction multiplication on which to focus. For each case, I begin with a vignette of Suh's teaching and I present a micro-analysis of his work of selecting interpretations and representations, selecting examples, and managing the enactment of mathematical tasks. I examine each task for sub-tasks or elements of that task that appear to be part of the work.

In Chapter 5, I examine the work of a second teacher educator, Professor Ball. Professor Ball comes to this work having taught elementary school for many years. I hoped that studying teaching of prospective teachers by a teacher educator who came to this work from the career of teaching might provide differences in the teaching of mathematics for teaching. I focus on the work to teach the concept of multi-digit whole number and decimal multiplication to student teachers. After the analysis of the work to examine two interpretations of whole number division, I present a vignette and analysis of the teaching of decimal multiplication.

In Chapter 6, I look across the four cases presented in Chapters 4 and 5 for similarities and distinctions in the work of teacher education. I present a cross-case analysis and hypothesize about elements of the work of selecting interpretations and representations, selecting examples, and managing the enactment of mathematical tasks. I make inferences about the mathematical knowledge demands of this work and claim that a distinctive domain of mathematical knowledge is used for the work of teaching mathematical knowledge for teaching. I discuss ways in which this knowledge supports the teaching of mathematical knowledge for teaching and characteristics of this mathematical knowledge that appear to be present in the teaching entailed by this work.

In this investigation, I analyzed the work entailed by teaching mathematical knowledge for teaching to teachers. I identified many elements of the work of selecting interpretations and representation, selecting examples, and managing the enactment of mathematical tasks for the purpose of fostering with teachers' incoming mathematical knowledge to a mathematical knowledge for teaching. Although my findings appear consistent with these dissertation data, they generate many questions, prompt additional research, and offer several trajectories for the study of the mathematical work and knowledge demands of teacher education. In Chapter 7, I anticipate and attempt to answer objections to this study and offer several ways to extend the study to confirm or disaffirm the claims of this investigation.

Chapter 2

Reviewing Prior Research

My purpose in writing this literature review is to situate my investigation within the mathematics education research that precedes it. Looking back at my research question, I ask:

What is the work of teaching teachers mathematics and what are the mathematical knowledge demands entailed by this work?

My investigation focused on two supporting questions:

1. **What are some of the key tasks of teaching mathematical knowledge for teaching that are involved in teaching teachers mathematics?**

2. **What are the mathematical knowledge demands entailed by teaching teachers mathematics?**

The key elements of my question are the mathematical content of teacher education, the mathematical work of teacher education, and the mathematical knowledge demands of this work. To begin my literature review, I discuss the research about mathematical knowledge and mathematical knowledge for teaching, research on the work about mathematics teacher education, and research about mathematics teacher educators.

It is important to ask: How do teachers develop from knowing mathematics for their personal use – success in a mathematics class, everyday life, or a career that uses mathematics to knowing mathematics for teaching?

Pre-service teachers come to teacher preparation after observing teaching throughout their K-12 experiences in what Lortie (1975) calls an "apprenticeship of observation" (p. 75). Thus, they have developed notions about what teaching entails. Feiman-Nemser (2001) claims that these notions influence what pre-service teachers can learn.

Feiman-Nemser (2008) conceptualizes the work of learning to teach as including four themes: "learning to *think* like a teacher, learning to *know* like a teacher, learning to *feel* like a teacher and learning to *act* like a teacher" (p. 698). Her first two themes, *thinking* and *knowing* like a teacher, are most closely linked to this dissertation study. Feiman-Nemser notes that learning to think like a teacher embraces the "intellectual work of teaching" (p. 698). This includes learning that teaching is more than the communication of information to students; the work of teaching and learning links teaching to student outcomes; and, teaching involves the ability to respond to classroom situations in the moment, to reflect on one's practice, and adjust one's practice to meet students' needs.

Further, the work of teaching requires teachers' deep and broad content knowledge. Their possession of this knowledge is foundational to their ability to "organize and 'hold' their knowledge." Teachers' *use* of this knowledge[4], "knowledge *for* teaching" and "knowledge *of* teaching" is the ultimate goal of teacher education (Feiman-Nemser, 2001, p. 699).

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[4]Feiman-Nemser (2001) defines "knowledge *for* teaching" as the knowledge teachers learn outside of their work. This includes knowledge of content and pedagogy. She defines "knowledge *of* teaching" as the knowledge teachers learn in their practice. This includes knowledge most closely tied to practice learned in practice (p. 699).

Feiman-Nemser's third and fourth themes for learning to be a teacher demonstrate the personal commitment teachers make to their profession. Teachers learn to feel like a teacher as they develop a deep personal commitment to their work and the success of their students. Finally, teachers learn to act like a teacher when they develop the "skills, strategies, and routines and the judgment to figure out when to use them" (p. 699).

Researchers have demonstrated that students come to teacher education with underdeveloped knowledge of mathematics (e.g., Ball, 1988). Further, they have found that novice teachers have difficulty applying the mathematics learned in courses designed for pre-service teachers and focused on mathematics for elementary teachers when they were in teaching situations, whether student teaching or novice teaching (Borko, Eisenhardt, Brown, Underhill, Jones, & Avard, 1992; Eisenhart, Borko, Underhill, Brown, Jones, & Agard, 1993; Lappan & Even, 1990).

Mathematical knowledge is a critical component of teaching for student achievement (Feiman-Nemser, 2001; Fennema & Franke, 1992; Hill & Ball, 2004; Hill, Ball, & Schilling, 2004; Hill, Sleep, Lewis, & Ball, 2007). Teachers need to know the content they teach. Specifically, they must know "three aspects of subject matter knowledge: (a) knowledge of central facts, concepts, theories, and procedures within a given field; (b) knowledge of explanatory frameworks that organize and connect ideas; and (c) knowledge of the rules of evidence and proof" (Shulman, 1986 as quoted in Feiman-Nemser). Shulman's identification of the subject matter teachers need to know expanded what teachers need to know to include the content that they teach and the "nature of knowledge and inquiry in different fields" (p. 1017). It is important for teachers to know how the nature of knowledge and inquiry are similar and different

between disciplines. Teacher's knowledge of mathematics and understanding of the syntax of the discipline appears to influence the types of tasks selected and the level of questions asked. If a teacher perceives mathematics as a set of memorized facts, that teacher will elicit responses to rote memorization tasks or descriptions of ideas. However, if a teacher sees mathematics as a discipline of inquiry, conjecture, and proof, the teacher is more likely to structure mathematics lessons that elicit these mathematical activities. In addition, a teacher's syntactical understanding may determine how and to what extent concepts are developed. (Ball, 1999; Feiman-Nemser, 2001; Shulman, 1986, 1987; Simon, 1994; Simon & Tzur, 1999).

Mathematical Content Knowledge for Mathematics Teacher Education

Boaler (2002) found that many countries define *mathematical knowledge* as students' success on standardized tests which measure knowledge of facts and procedures (p. 10). Boaler and other mathematics educators claim that this is an unsatisfactory definition (Boaler, 2002; Burton, 1999; Noss, Healy, & Hoyle, 1997).

Rather, mathematical knowledge is an integrated body of knowledge along with ways to reason and work to generate knowledge (Ball & Bass, 2003a; Ball and Bass, 2003b; Boaler, 2002; Burton, 1999; Noss et al., 1997). This includes the use of concrete and written representations, the ability to examine cases, make generalizations, and the ability to work between "formal and informal, analytic and perceptual, rigorous and intuitive" mathematical ideas (Noss et al., 1997, p. 209). Mathematical knowledge encompasses knowledge of procedures along with how and why they work, knowledge of concepts along with ways to represent and talk about those concepts, knowledge of how mathematics is structured, the construction and use of definitions, axioms, properties, and

theorems; knowledge of how mathematics is explored, and knowledge of how mathematics is thought about and how new mathematical ideas are generated (Boaler, 2002; Burton, 1999; Noss et al, 1997).

Given this broadened view of mathematical knowledge, there are implications for knowledge of mathematics for teaching. Ball and Bass (2003b) investigate an extended data set of one teacher's work to teach mathematics to third grade students for the mathematical entailments of this work. They enumerate four characteristics in this work that are consistent with the work of mathematicians. First, Ball and Bass emphasize the students' interactions among themselves, how they listen, respond, and question each other. Second, they call attention to the public work of recording and presenting mathematics. Third, Ball and Bass highlight the mathematical culture of creating mathematics within a third grade classroom. They make explicit the importance of teachers' knowledge of definitions and how to use definitions in the course of doing mathematical work with students. And fourth, they discuss the importance of mathematical language for teaching students. They claim that language "is central to constructing mathematical knowledge because it provides resources with which claims are developed, made, and justified" (p. 33). Mathematical language must be clear and precise to communicate concepts, state problems, and explain procedures.

Using this work analysis, Ball and Bass (2003b) identify four characteristics of knowing mathematics for teaching. First, they claim that knowledge of mathematics for teaching is *unpacked,* the details of mathematics must be explicit and the layers of the mathematics must be exposed. Second, knowledge of mathematics for teaching is connected across mathematical domains and across levels of school mathematics. Third,

knowledge of mathematics for teaching supports a teacher's communication of ideas that evolve over time so that the ideas are understandable at the level at which the teacher is teaching *and* maintains the integrity of the mathematics. And fourth, knowledge of mathematics for teaching includes knowledge of mathematical ideas and knowledge of "fundamental mathematical practices" (p. 12).

Just as our research group has studied the mathematical knowledge demands entailed by examining the work of teaching mathematics, I approach the study of the mathematical knowledge demands of teacher education by analyzing the mathematical work of teaching teachers. To frame this study, I present a key work on teachers' knowledge of mathematics for teaching -- Ball's work to create the construct of MKT and subsequent work that has supported further development of this construct.

Teachers' Knowledge of Mathematics for Teaching

Just as mathematical knowledge has been defined as the mathematical knowledge needed to score well on a standardized test, a review of the mathematics education literature reveals that teacher's mathematical knowledge has been defined in the same way. Some scholars define the mathematics needed for teaching as the mathematics learned in college level course work and use proxies such as numbers of completed college mathematics courses or standardized test scores as measures of teachers' knowledge of mathematics. However, studies demonstrate that teachers' knowledge of college mathematics does not influence student achievement (Begle, 1979; Eisenberg, 1977; Hanushek, 1996; Rivkin, Hanushek, & Kain, 2005). Wilson, Shulman, and Richert (1987) conclude that "research done in this [quantitative] tradition has failed to provide insight into the character of the knowledge held by students and teachers and the ways in

which that knowledge is developed, enriched, and used in classrooms" (p. 107). With some researchers' discontent with defining and measuring mathematical knowledge in a generic sense as a way to define and measure teachers' mathematical knowledge for teaching, educators sought new ways of thinking about, defining, and measuring this knowledge. This paved the road to developing a different domain of knowledge, mathematical knowledge for teaching (MKT).

Beyond Knowledge of Mathematics

Lee Shulman, in his 1986 American Education Research Association address, proposed that teachers have a disciplinary knowledge that is important to their work. He identifies this knowledge as *pedagogical content knowledge* (PCK), a special kind of knowledge that intertwines the knowledge of how students learn, how content might be presented, and what students find difficult with disciplinary knowledge (Ball, Lubienski, & Mewborn, 2001; Shulman, 1986; Wilson, Shulman, & Richert, 1987). Pedagogical content knowledge includes packages of knowledge that combine knowledge of content, students, and pedagogy. Since Shulman's address, mathematics educators have studied this "missing paradigm" or subject matter knowledge for mathematics. They claim that subject-matter knowledge for teaching goes beyond knowledge of facts, concepts, and procedures including "how the pieces fit together. It also includes knowledge about knowledge -- where it comes from, how it grows, how truth is established" (Feiman-Nemser, 1986, p. 221). Specifically, mathematics educators have studied the knowledge of mathematics important to the work of teaching. New theories about this work (Lampert, 1999, 2001; Shulman, 1986; Wilson, Shulman, & Richert, 1987) catalyzed research into *what* mathematics teachers need to know and *how* they need to know it

(Ball, 1988, 1990). This research trajectory shifted mathematics educators' work to study knowledge of mathematics to knowledge of mathematics for teaching and teachers' use of mathematical knowledge while teaching (Ball, Lubienski & Mewborn, 2001).

Our research group headed by Deborah Ball claims that mathematical knowledge for teaching (MKT) includes several sub-domains: *common content knowledge (CCK), specialized content knowledge (SCK),* and *knowledge of content and students (KCS),* and *knowledge of content and teaching (KCT),* and most recently *horizon content knowledge.* A teacher's *common content knowledge (CCK)* is her "mathematical knowledge and skill used in settings other than teaching. Teachers need to know the material they teach" (p. 339). A teacher's *specialized content knowledge (SCK)* is "knowledge and skill unique to teaching. …[It] is mathematical knowledge not typically needed for purposes other than teaching" (p. 400). This includes knowledge of concepts and procedures and multiple ways to represent and explain these.

We define a third sub-domain, *knowledge of content and students (KCS)* as the "knowledge that combines knowing about students and knowing about mathematics" (Ball et al., 2008, p. 401). This includes knowing what students might know from prior instruction, anticipating what they might find difficult and common misconceptions they might incur and knowledge of mathematics, students, and how students think about mathematics. We define a fourth sub-domain *knowledge of content and teaching (KCT)* as the knowledge that "combines knowing about teaching and knowing about mathematics" (Ball et al., 2008, p. 401). This includes knowledge of mathematical ideas important for the development of a lesson and pedagogical strategies and how these affect student learning.

Concurrent with the writing of this dissertation, our research group is investigating fifth sub-domain *horizon content knowledge.* We defined this knowledge as "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum (Ball et al., 2008, p. 403). As our study and discussion of this domain continues, we have broadened our definition to include four elements:

- A sense of the mathematical environment surrounding the current "location" in instruction,
- Major disciplinary ideas and structures,
- Key mathematical practices,
- Core mathematical values and sensibilities (Ball & Bass, 2009, pp. 4-5).

For the work of teaching school mathematics, we claim that MKT is the mathematical knowledge needed for this work (Ball, et al., p. 396). Thus, it is important to ask: How does mathematics teacher education develop pre-service teachers' mathematical knowledge into mathematical knowledge for teaching?

Teaching Mathematical Knowledge for Teaching to Teachers

Our claim is that the mathematical content of courses designed for pre-service teachers is different. We might ask if the pedagogy used for teacher education is different as well.

Approaches to Teachers' Learning

Simon (1994) developed a theoretical framework for mathematics teacher learning. Building on constructivist theories for mathematics learning and French didactical theory, Simon holds that "mathematics teacher educators must be guided by theories as to how mathematics teachers learn and how teacher education can effectively support these learning processes" (p. 88, as cited in Goldsmith & Schifter, 1995).

Goldsmith and Schifter (1995) claim that change in the recommended content and pedagogy for teaching children mathematics has motivated teacher educators to develop ways to help teachers shift their practice from teacher-centered to learner-centered approaches. The authors sort studies into teachers' learning into three categories: knowledge-based, sociocultural, and developmental approaches.

Carpenter and his colleagues (1996) conducted a knowledge-based study seeking to change teachers' practice by teaching them about how children reason about mathematical concepts and assisting the teachers as they learned to teach in a way that builds on children's understandings. The teachers worked in collaboration with other teachers in the project and with the project team. Learning about children's ways of mathematical reasoning, how they understand mathematics and applying this learning to teaching appeared to produce "generative" knowledge about mathematics in the teachers (Carpenter, Fennema, & Franke, 1996; Fennema, Carpenter, Franke, et al., 1996; Franke, Carpenter, Levi, & Fennema, 2001)

The sociocultural approach to teacher learning claims that teachers learn all of the time whether there are explicit tasks designed for learning or not. Stein and Brown (1995) investigated a professional community, its activities, and available resources. "[T]he practices of the community can be viewed as the curriculum; learning is conceived as changes over time in social participation patterns in the work practices of the community" (Stein & Brown, 1995, p. 162). This approach builds on Lave and Wenger's (1991) work on situated learning and the work of Cobb, Yackel, and Wood (1992) to study the sociocultural nature of children's learning.

The developmental approach examines the "experiences and individual processes [that contribute] to the reconstruction of teaching practice" (Goldsmith & Schifter, 1995, p. 21). Several professional development projects used this approach and studied ways that teachers reorganized and developed their knowledge of mathematics and their teaching practices. Using "transition mechanisms" (Goldsmith & Schifter, 1995, p. 41) appears to provide insights into how teachers' development progresses and offers ways to foster change from one level of understanding to another. These mechanisms are designed to help teachers make small or large adjustments.

Goldsmith & Schifter suggest several mechanisms. One mechanism helps teachers (and students) manage cognitive conflicts or bits of information that do not fit with the knower's established set of knowings. For more challenging adjustments, the Piagetian process of managing a perturbation, a disruption in what the knower knows, is one mechanism for managing change. When the knower meets such a perturbation, he makes an accommodation, and new learning occurs. For teacher learning, this may mean confronting and resolving new mathematical challenges in teachers' work. Examples might be examining an atypical piece of children's work, sorting through the mathematics, and coming to an understanding of the work and ways to move it along or adjust it (Goldsmith & Schifter, 1995; von Glaserfeld, 1995).

A second mechanism uses prior success as a basis for knowledge growth (Goldsmith & Schifter, 1995). Karmiloff-Smith recommends use of teachers' correct mathematics and making implicit understandings explicit. By doing this, she notes that "knowledge that is originally tied to acting in a narrow set of instances can be broadened and applied to a wider variety of situations" (p. 42). Along with making implicit

understandings explicit, Karmiloff-Smith suggests "borrowing across areas of understanding" (p. 42).

Study of Teachers' Learning at San Diego State University

Sowder (1998) applies the results of cognitive scientists' studies to advance the learning of pre-service and in-service teachers. She notes that for teachers to teach authentic mathematics lessons that involve mathematical problem solving, reasoning, and concepts that they may not have experienced, the teachers must be able to do these mathematical tasks. Sowder finds that many pre-service and in-service teachers have not had opportunities to be engaged in work that fostered problem solving or reasoning skills. For some, they may have developed poor ways of reasoning. For these teachers, "patterns of reasoning developed prior to formal instruction have effects on [teachers'] approaches to problems that outlast or intrude in powerful ways on the formally taught approaches" (Kaput & West, 1994, p. 237, as quoted in Sowder, 1998).

Characterizations of Teachers' Learning

Putman and Borko (2000) identify three characterizations of teachers' learning. First, they claim that teachers' learning is situated within teaching. When teachers are engaged in authentic activities that mirror the work of teaching, their knowledge about the mathematics used for their work is fostered. Second, teachers learn when they are engaged in discourse communities. Professional learning communities like those studied by McLaughlin and Talbert (2001) working within or across disciplines allows teachers to "draw upon and incorporate each others' expertise to create rich conversations and new insights into teaching and learning" (Putman & Borko, 2000, p. 8). For in-service teachers, such professional communities may be formed and facilitated within schools or

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across districts. However, pre-service teacher education programs may be challenged to create these communities that might require quality field placements and strong mentoring. Putnam and Borko (2000) claim that the theory that knowledge is socially constructed "an important part of learning to teach is becoming acculturated into the teaching community – learning to think, talk, and act as a teacher" (p. 9).[5] Finally, Putnam and Borko (2000) note the importance of *tools* in teacher' work. They discuss the importance of technology for professional productivity (e.g., use of computer technology for writing, record keeping, etc.), and pedagogical productivity (e.g., use of educational software to enhance lessons, internet websites as sources of content, etc.), as well as professional development (e.g., use of video case studies, internet websites for professional development activities).

Teachers' Opportunities to Learn Mathematical Knowledge for Teaching

      Subject matter knowledge. Along with fostering images of good teaching, Feiman-Nemser (2001) contends that teachers must know the content they teach. Specifically, they must know "three aspects of subject matter knowledge: (a) knowledge of central facts, concepts, theories, and procedures within a given field; (b) knowledge of explanatory frameworks that organize and connect ideas; and (c) knowledge of the rules of evidence and proof" (Shulman, 1986 as quoted in Feiman-Nemser). Feiman-Nemser claims that teachers must know the content of disciplines they teach *and* they must understand the "nature of knowledge and inquiry in different fields" (p. 1017). In addition, they must know how the nature of knowledge and inquiry are similar and different between disciplines. Teachers must know how description, explanation, and

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[5] Other scholars cite the importance of professional learning communities. As noted above, Goldsmith & Schifter, 1995, Cobb and his colleagues, 1992; Feiman-Nemser, 2001.

justification differ in mathematics. Further, they must recognize the differences between these mathematical tasks and similar tasks in other domains such as literature or history. A teacher's understanding of the syntax of a discipline influences the types of tasks selected and the level of questions asked. Further, a teacher's knowledge of what students' know and her ability to build upon students' existing conceptions, understand student difficulties with content and provide multiple explanations, representations, and models of the same concept are important to the type of mathematics lessons she teaches. A teachers' knowledge of concepts and applications within and across disciplines is important for instruction (Feiman-Nemser, 2001; Simon, 1994; Simon & Tzur, 1999).

Ball (1989) perceives teacher education as a site for significant "unlearning" of images of mathematics and mathematics class developed through K-12 experiences and learning new conceptions. She envisions mathematics courses for elementary teachers as opportunities for pre-service teachers to learn mathematics important for their teaching, experience mathematics as mathematicians do mathematics, and reconstruct their images of themselves as doers of mathematics.

Geddis and Wood (1997) address the importance of transforming teachers' content knowledge into content knowledge for teaching. In a study of Wood's work to prepare secondary mathematics teachers for student teaching, the authors developed a framework that depicts teachers' subject matter transforming from content knowledge to content knowledge for teaching. They hold that teachers' content knowledge for teaching includes learners' prior concepts, subject matter representations, instructional strategies, curriculum materials, and curricular saliency. Further, they hold that these categories are

useful to "provide some conceptual order to [classroom instruction that] is in practice non-linear, intuitive, and usually messy" (Geddis & Wood, 1997, p. 614).

Hiebert, Morris, and Glass (2003) claim that there are "two primary learning goals for prospective teachers" (p. 203). First, teachers must become mathematically proficient. Hiebert and his co-authors use the definition of mathematical proficiency from the NRC (2001) publication *Adding it up: Helping children learn mathematics.* Based on this definition, they hold that teachers must increase their mathematical proficiency in order to teach to the five interwoven strands of mathematical competencies proposed by NRC.

Second, teachers must "prepare to learn to teach for mathematical proficiency" (p. 204). This goal entails two components: "preparing to learn to teach, and preparing to learn to teach for mathematical proficiency" (p. 204). First, Hiebert et al. (2003) note that there are many tasks of teaching that are important for teachers to learn. The complexity of learning to teach along with the added demands of learning to teach for mathematical proficiency creates enormous expectations for teachers. Second, the authors claim that prospective teachers' learning "includes the importance of using mathematical proficiency as the standard for measuring one's own teaching effectiveness" (p. 206).

Pedagogical content knowledge. Along with fostering strong subject-matter knowledge, teacher educators help pre-service teachers develop a beginning repertoire of strategies that promote student learning of mathematics. These are strategies for designing lessons, assigning homework and projects, and assessing student learning. Pre-service teachers need to recognize that as teachers, they have two roles, teaching and learning about teaching. Teacher educators must foster dispositions, skills, and habits that encourage the continuous study of teaching. Abilities such as critical observation and

interpretation of children's work and interactions are important for generative knowledge for teaching. Analysis of student work, both written and oral, are critical for formative and summative assessment of student learning, rethinking of teaching strategies, and re-teaching to foster student achievement. Constructive evaluation and adaptation of curricular materials and colleagues' work allow for variety in teaching strategies and learning of content. Further, teacher educators nurture productive dispositions and habits toward work with students, parents, colleagues, and administrators. Given these demands on teachers' content knowledge, the mathematics teacher educator must have the knowledge to foster teachers' knowledge of and about mathematics for their teaching (Feiman-Nemser, 2001; Simon, 1994; Simon & Tzur, 1999).

Research on the Work of Mathematics Teacher Education

Just as teachers are important agents in children's learning, one might hypothesize that mathematics teacher educators' work to teach teachers may be important to teachers' learning of mathematical knowledge for teaching. I examine one last set of literature to frame my study, that of mathematics teacher educators. In this set of literature, I found two types of studies: studies of mathematics teacher educators' work with pre-service teachers and studies of mathematics teacher educators' development into this profession. I begin the next section with a review of three studies in which the mathematics teacher educators examine their practice.

Mathematics Teacher Educators' Study of Their Work to Teach Teachers

John (2002) studies the knowledge and understandings that teacher educators use in their work with student teachers. John's study produces several interesting insights into his subject's work. First, he discusses the subject's insight that she learns more about

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mathematics through her teaching. Second, he discusses her work to teach pre-service teachers. Specifically, he describes the subject's movement between teaching mathematics and engaging in meta-talk about mathematics and the teaching of mathematics.

Geddis and Wood (1997) discuss Wood's work to engage the pre-service secondary teachers in "a critical analysis of their own understandings" and in an exploration of ways to articulate and represent mathematics. Geddis and Wood find that Wood's own knowledge of the mathematics transformed into a pedagogical content knowledge for teaching mathematics to pre-service teachers. They claim that Wood's knowledge of pre-service teachers' knowledge of mathematics and the teachers' concerns about teaching mathematics are important for "*teacher educator pedagogical content knowledge"* (Geddis & Wood, 1997, p. 623).

Geddis and Wood (1997) summarized Wood's transformation of knowledge to a pedagogical content knowledge for teaching pre-service teachers.

> [Wood] has transformed his own understandings about the pedagogy of teaching integers into a repertoire of ways of representing the addition of integers, and also into a learning activity that has engaged his students in articulating and evaluating these representations. It is worth noting, that under our formulation of teaching as the transformation of subject matter, representations and teaching strategies are not just examples of pedagogical content knowledge (knowledge used by a teacher in constructing transformations, but that they are also (as in this case) the end products of the knowledge transformation itself (p. 624).

These studies reveal mathematics teacher educators' work to teach mathematics in ways that they perceive will open up mathematics and make it more usable for teaching children. This characterization of mathematics teacher educators' work is similar to the work of both subjects in this study. However, my study is different in several respects.

I continue to review studies in which mathematics teacher educators investigate their development into their roles as mathematics teacher educators.

Mathematics Teacher Educators' Evolution from Teachers to Mathematics Teacher Educators

Tzur (2001) presents a self-study of his evolution from mathematics student to mathematics teacher, then from mathematics teacher to mathematics teacher educator, and finally to mentor of mathematics teacher educators. Framed with concepts of reflection and interaction based on Dewey, Piaget, and Schön's work, Tzur examines his practice and his knowledge of mathematics at these various stages. He discusses three aspects of his early teaching: "[his] continuous learning of mathematics and how it differed from [his] students' learning, [his] instructors' expectation that pre-service teachers should conduct research, and [his] learning to teach as a result of reflecting on related activities of teaching, course work, and research" (p. 267). Tzur emphasizes his learning of mathematics while preparing for and teaching mathematics and he discusses an awareness of how his knowledge of mathematics is distinct from students' knowledge.

Tzur credits his "growing understanding of teaching-learning processes" (p. 271) as enabling him to transition from being a teacher to teacher educator and later to mentor of teacher educators. In particular, Simon and Tsur postulate a new construct, *conception-based perspective*, "to emphasize how a teacher thinks about the ways extant knowledge affords and constrains what learners can perceive, understand, and do in given situations" (p. 271).

Zaslavsky and Leikin (2004) study a group of mathematics teachers, mathematics teacher educators, and an educator of mathematics teacher educators for two purposes. First, they seek to learn about mathematics teacher educators' professional growth as

members of a community of mathematics educators. Second, they intend to test a theoretical model by applying it to their work. In this model, Zaslavsky and Leikin claim that designing mathematical tasks for mathematics teachers' professional development activities requires teacher educators to use mathematical knowledge, knowledge of teachers as learners, and use of "innovative approaches" to manage teachers' learning (p. 7). In addition, this work requires all participants to "maintain a reflective state of mind and collaborative disposition" (p. 30). Further, they find that teacher educators grow professionally through their work to develop professional development activities and reflect on their practice.

Tzur and Simon as well as Zaslavsky and Leikin's works focus on professional growth of mathematics teacher educators. While they address professional growth, which includes knowledge of mathematics, there is no explicit examination of knowledge *of* and *about* mathematics that mathematics teacher educators use for their work. Tzur employs reflection on his own practice to develop a framework for mathematics teacher educators' professional growth. Simon applies a recursive learning cycle that includes exploration, identification, and application to design a theory of learning of mathematics teacher educators. He concludes that "the learning demanded at each level (mathematics student, mathematics teachers, and mathematics teacher educators) increased exponentially" (p. 91). Further, Simon claims that people who are knowledgeable about mathematics must go beyond their knowledge of mathematics to become competent as mathematics educators. In making this claim, Simon acknowledges that mathematics teacher educators' knowledge goes beyond the knowledge of mathematics students, or mathematics teachers. However, he does not say how this knowledge is different from

students' or teachers' knowledge of mathematics. This dissertation builds on the work of

Tzur, Simon, Zaslavsky and Leikin, but is different. I analyze the work of two

mathematics teacher educators to learn about the work of teacher education and the

mathematical knowledge demands of this work.

Chapter 3

Designing My Study: Data Collection and Analysis

The goal of this dissertation is to investigate the mathematical knowledge demands of the work of teaching mathematics to teachers. I seek through my research to describe the work of teaching mathematics to teachers, to identify components of that work, to identify distinctions between the work to teach mathematics to teachers and the work to teach mathematics to children, to make inferences about the mathematical knowledge entailed by this work, and to hypothesize about a domain of mathematical knowledge used for this work.

My dissertation research evolved over four phases: the pilot study (phase 1), the analysis of tasks of teaching mathematical knowledge for teaching (phase 2), and the analysis of mathematics problems used for teaching mathematical knowledge for teaching (phase 3). After these phases, I paused my study and refocused on the tasks of teacher education analyzing these for the elements of this work and the mathematical knowledge demands. Each phase helped me articulate a more clearly defined image of this work and the mathematical knowledge entailed by it. The subsequent phases built on what was learned from the previous. Across the study, the hypothetical concept of the work of teaching mathematical knowledge for teaching and the knowledge demands became sharper; a more detailed description evolved as layers of the work were unpacked.

In this chapter, I discuss my data collection and analytical methods. As I transition between phases, I briefly discuss the results of each phase that informed the structure of the next. First, I discuss the methodology which I used to guide my work: case study methods and conceptual-analytic work.

<p style="text-align:center">General Analytic Strategy</p>

I began by generating descriptive cases of the teaching of mathematics to teachers. These cases, like those in the pilot study, needed to be distinctive.

Case narratives. The case narratives are one form of data. From these, I formulated analytic generalizations, I confirmed or disaffirmed these generalizations, and I proposed theoretical propositions about the work of mathematics teacher education (Yin, 2003). I used multiple case studies because "[a]nalytic conclusions independently arising from two [or more] cases… [are] more powerful than those coming from a single case" (Yin, 2003, p. 53). I used cases that have similar characteristics to demonstrate similar findings. More importantly, replication of findings from cases that have varying contexts provided strong evidence for claims. For cases where findings are different, these differences were explored to determine reasons for these differences. My in-depth study of these cases is intended to "immeasurably [expand] the external generalizability of [these] findings" (Yin, 2003, p. 53).

The case selection strategy for this study serves several purposes: to investigate the *nature of the work* to teach mathematical knowledge for teaching to teachers, to explore the *mathematical knowledge demands* of this work, to determine the extent to which this *knowledge is distinctive to the work* of mathematics teacher educators, and to generate *hypotheses about the existence and nature of mathematical knowledge* for

teaching teachers (MKTT). I selected cases based on variation in individuals' disciplinary and professional backgrounds. These differences bring different ways of knowing and using mathematics to the work of the mathematics teacher educators. Also, I selected mathematics teacher educators whose work in the field is considered exemplary. Hence, I have conducted a *reputational case selection.*

For the study of the work of teaching mathematics to teachers, I analyze four cases for the elements entailed by the work. I compare the elements across the cases. In doing so, the findings are strengthened as are "the precision, the validity, and the stability of the findings. … If a finding holds in one setting and, given its profile, also holds in a comparable setting but does not in a contrasting case, the finding is more robust. … [T]he multiple-case sampling gives us confidence that our emerging theory is generic, because we have seen it work out – and not work out – in predictable ways" (Miles & Huberman, 1994, p. 29). In particular, I use these cases as a resource. They provide records of data from which I learn about the mathematical tasks of the work to teach mathematics to teachers. In the third phase of my analysis, I identify and focus on three tasks of teaching mathematical knowledge for teaching and I examine the mathematical demands across these tasks. This is not to say that there are only three tasks for this work. Rather, I am focusing my analysis on three tasks of teaching that I find to be particularly rich sites for this analysis.

This study is not about the particular mathematics teacher educators whose work I studied. Rather, I use their teaching to learn about the work of teaching teachers and to test a hypothesis that there is a distinctive type of work for teaching mathematical knowledge for teaching. In addition, I use their work to reveal the nature of that work and

to unpack the mathematical demands of that work. At each phase of the analysis, I use the findings of the previous one to revisit the data, find a different approach to analyze the data and develop the description of the work of teaching mathematical knowledge for teaching and the mathematical demands of this work.

The purpose of this study is *not* to develop a theory that is generalizable. I am *not* seeking to state that in general, mathematics teacher educators engage in this particular work, attend to certain tasks of teaching mathematical knowledge for teaching, or use specific forms of mathematical knowledge when teaching teachers. Rather, I am attempting to take a hypothetical concept, the work of teaching mathematical knowledge for teaching and the mathematical knowledge demands of this work, and use data to explore its plausibility and usefulness, clarify the concept, to add detail to it and to make it usable for the work of studying mathematics teacher education. I summarize the analytical phases and initial findings in figure 3.1.

Phase	Findings
Phase 1: Pilot Study	• There appear to be several tasks of teaching mathematical knowledge for teaching. • Stein discusses use of more detailed yet broad mathematical knowledge for teaching teachers; evidence from Ball interview of her use of detailed yet broad mathematical knowledge for teaching teachers. • Stein and Ball discuss distinctions between teaching teachers and children.
Phase 2: Analysis of 3 tasks of teaching teachers and the knowledge entailed by the tasks	• Purposes of the tasks appear to have similarities and differences for teaching children and teachers. • Inferences on knowledge used for teaching MKT indicate that there is a domain of knowledge entailed by this work.
Phase 3: Analysis of mathematical problems	• Confirm tasks of teaching; three seem prominent. • Ball and Suh discuss use of mathematical knowledge for teaching teachers. Hypotheses about a domain of mathematical knowledge entailed by the work of teaching mathematical knowledge for teaching.

Findings from Phases 2 and 3
- Three tasks of teacher education identified: use of interpretations and representations, use of examples, managing discourse about MKT tasks.
- Purposes of these tasks appear to have similarities and differences for teaching children and teachers
- It appears that a domain of mathematical knowledge for teaching mathematical knowledge for teaching is used for this work.

Pause the study and refocus
- Modification of hypotheses
  ▪ There are many tasks of teacher education. Three are:
    • Selecting interpretations and representations
    • Selecting examples
    • Managing the enactment of mathematical tasks
  ▪ There is a domain of mathematical knowledge used for teaching that is beyond mathematical knowledge used for other professions.
  ▪ There is a domain of mathematical knowledge for teaching mathematical knowledge for teaching.

Phase	Findings
Phase 4: Analysis of 3 tasks of teaching for elements of the work entailed and mathematical knowledge demands	• Elements of the tasks of teaching in these data identified. • Conjectures about two sub-domains of MKTT made. • Characteristics of MKTT emerge.

Figure 3.1. Phases of analysis used for this investigation and initial findings.

With this brief discussion of the case methods and conceptual-analytic work

complete, I present a detailed discussion of the specific case selection strategy,

participants, instruments, and analytic methods for the second, third, and fourth phases of the study.

<p style="text-align:center">Data Collection</p>

My work to collect, code, and analyze data simultaneously caused the operations to "blur and intertwine continually, from the beginning of investigation to the end" (Glaser & Strauss, 1967). Results from initial work shaped later data collection and analyses by allowing for the formulation of hypotheses about the work of mathematics teacher education. These hypotheses determined the lenses for later analyses. This recursive data collection and analysis fostered the generation of claims from within my data (Charmaz, 1983; Glaser & Strauss, 1967).

Participant Selection

I used three criteria to select participants. First, to maximize the variation in mathematical knowledge used by participants, I chose teacher educators with different disciplinary backgrounds. I anticipated that this variation might cause the teacher educators to choose varying types of tasks for their work and to approach their work with different expectations for the teachers' learning of mathematics. Second, I identified teacher educators who work with pre-service and in-service teachers. I expected that these subjects may have diverse emphases in their work with respect to research and teaching. Consequently, I anticipated that there would be variation in the mathematical tasks they selected, the emphases they placed on developing mathematical ideas, and the mathematical demands of their work. Third, I selected teacher educators who recognize that the mathematical knowledge needed to teach as different from mathematical knowledge used for other professions and that this distinctive domain of mathematical

knowledge was the focus of their teaching.[6] Subjects are experts in the field. Hence, I have completed a *reputational case selection* (Yin, 2003).

Other case selection strategies were possible, but those did not seem likely to fruitfully generate a robust description of the mathematical work of teaching mathematical knowledge for teaching and the mathematical knowledge demands of this work. Although the work of each participant is not the final unit of analysis, the descriptions of the participants and their teaching context provide insights into their work.

Professor Deborah Loewenberg Ball. The first site is a mathematics methods course for elementary teachers taught at a large public research university. I selected this site for four reasons. First, at this site, a group of faculty and graduate students regularly works together to prepare and teach the course. This group is formally called the Mathematics Methods Planning Group (MMPG) (Ball, Sleep, Boerst, & Bass, 2009). During the data collection semester, there were four sections of the mathematics methods course for elementary teachers taught by people with a variety of backgrounds. There was a leadership team consisting of a prominent mathematics educator, a research mathematician, a graduate student with many years of elementary teaching experience, and an elementary school teacher who worked part-time teaching fifth grade and part-time as a mathematics teacher educator and researcher. Second, this team worked collaboratively to develop, teach, and research their work to teach teachers. The expertise of each member – mathematics teacher educator, mathematician, elementary school

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[6] When using in phrase "mathematical knowledge for teaching," I am referring to knowledge of mathematics that the mathematics teacher identifies as important for teachers to know for the work of teaching. This allows for mathematics teacher educators to have images of what this mathematical knowledge might be. Whereas, when referring to Ball's concept of MKT, I use the acronym.

teachers, and educational psychologist – contributed to this work. Third, this team worked for several years to develop the course, individual lessons, and materials. The data include lessons that had been taught and revised across several years of work, benefitting from many teach – revise cycles. Fourth, the course included focused mathematical work. The lesson segments during which mathematical work was the focus comprise my data.

The work of this group included several phases. First, the MMPG met each week to plan and "walk through" the week's lesson. Second, the group observed as the mathematics teacher educator-graduate student team taught each lesson. Finally, the group reviewed the team-taught class and discussed recommendations for how each instructor might adjust the material for their needs. Consequently, I selected this site because the teacher educators are prominent in the field and their collaboration extended across several years.

The head researcher and catalyst for the MMPG group is the first mathematics teacher educator in my study, Ball. Ball came to the work of mathematics teacher education after teaching elementary school for many years. Ball's work to study pre-service teachers is discussed earlier in this dissertation.

Professor Jin Suh. The second source of data for my study was the work of Suh, a research mathematician and teacher educator at a large public research university. I chose this site because it provided a contrasting case to Ball's mathematics methods course and her work with MMPG.

Suh conducted a mathematics professional development institute at a large state university. He designed this institute to teach in-service teachers the mathematics that

underpins the school mathematics the teachers teach. The teachers met from 8:30 a.m. to 4:30 p.m. five days a week for three weeks. During this institute, the teachers studied mathematics beginning with fundamental definitions, properties, and theorems. They examined the mathematics foundational to the mathematics of whole numbers and rational numbers found in school curricula.

Suh first became interested in providing professional development for teachers when he reviewed reform-based curricula several years ago. His work in mathematics education is noteworthy because of his early stance on traditional mathematics content and pedagogy. Over the years, Suh has engaged in conversations with mathematics educators who maintain differing views about how and what mathematics should be taught in K-12 classrooms. Also, he has studied and written about his work with teachers. These documents provide insights into his work. As an example, Suh (1997) noted that "[a] future mathematics teacher may join the profession with only a very meager knowledge of mathematics even under the best of circumstances" (p.2). He does not fault pre-service or in-service teachers for this dilemma. Rather, he questions teacher preparation institutions for what he claims is the lack of attention to the kind of mathematics curriculum prospective teachers need to provide the foundational knowledge and understanding important for teaching mathematics. Suh (2002) noted that "[m]ost teachers cannot bridge the gap between what we teach them in the undergraduate curriculum and what they teach students in schools" (p. 1). He argues that mathematics as a discipline rests on fundamental characteristics and that teachers must have a firm understanding of these characteristics and must use these when teaching. Suh claims that definitions, logical reasoning, and connectedness are integral to mathematics. He asserts:

(1) that precise definitions form the basis of any mathematical explanation, and without explanations mathematics becomes difficult to learn,

(2) that logical reasoning is the lifeblood of mathematics, and one must always ask *why* as well as find out the answer, and finally,

(3) that concepts and facts in mathematics are tightly organized as part of a coherent whole so that the understanding of any fact or concept requires the understanding of its interconnections with other facts and concepts (Suh, 2002, p.2).

Along with his clear vision of the content and syntax of mathematics necessary for teachers' knowledge of mathematics, he communicates a sensibility for pedagogical techniques he uses to teach prospective teachers. First, he noted that "if we show teachers by example and not just by words that mathematics can be taught according to reason," (p. 40) then mathematics teacher educators may influence the teaching of mathematics to children by improving the mathematical content and pedagogy of teachers.

Suh communicated an understanding of the difficulties teachers face when they teach. He cited the inadequacy of mathematics preparation at the university level, weaknesses of school curricula, and lack of content based professional development. Most importantly, Suh (2002) noted that "*…mathematics of elementary school is not trivial*" (p. 41).

Unit of Analysis

My study is focused on the mathematical work of teacher education. During each analytical phase, I extract from the data repository different components of the data. For the analysis of pilot data, I examined the data on Ball and Stein's teaching to learn about this work, to determine whether my hypotheses might develop beyond that status. I attempted to parse their work to identify components that I might hypothesize as tasks for teaching mathematical knowledge. I analyzed the work to identify the mathematics and make inferences about the mathematical knowledge they drew upon to teach this

mathematics. For the second analytic phase, the unit of analysis was the vignette. These data were analyzed for the work to attend to the three tasks of teaching: selection of interpretations and representations, selection of examples, and management of enactment of mathematical tasks. For the third analytic phase, the unit of analysis shifted from a holistic examination of the data to a fine-grained analysis of the mathematical problems used by Ball and Suh. I extracted 31 problems and analyzed them for the mathematical demands I perceived necessary for their enactment. For the fourth analytic phase, the unit of analysis shifted back to the vignettes. However, the foci changed to an analysis of the elements that comprised the work of attending to the tasks. Thus, the data set, the narratives were repositories for the data used at each phase of analysis, the unit of analysis varying at each analytic phase.

Data Sets

    Ball: observations. The Ball-Teachers data set consists of several components. First, the data were collected from three consecutive mathematics methods class cycles. These cycles included planning meetings, the class session, and debriefing meetings. Data include records of field notes from the Mathematics Methods Planning Group (MMPG) planning sessions, class sessions, and debriefing meetings. The planning sessions met for two hours each week; the classes met three hours once a week; the debriefing sessions met for approximately one hour each week. In addition to field notes, there are lesson plans, course assignments, and video recordings for all classes.

    Ball: interviews. Ball participated in follow-up interviews within several days of each class. These interviews include her discussion about the mathematics to which she attended during that class, the teachers' work on mathematics, and thoughts she had

about each class. I transcribed the interviews to prepare them for analysis. The interview protocols are found in Appendix 3.1.

Suh: observations. The Suh-teachers data set consists of data from three consecutive days of a professional development institute. These data include video recordings of three days of the institute and field notes from two days. Suh conducted a lecture style class each morning. The afternoon classes included some lecture, group work, and small group homework sessions.

Suh: interview. Suh participated in a follow-up interview approximately one month after the class observations. This interview was delayed at Professor Suh's request. He found the demands of conducting the professional development such that he asked that the interview be conducted at a later date. This interview focused on Suh's work to teach the class, the mathematical tasks he selected for the class, and his perceptions about the teachers' learning. In addition, Suh provided monographs he had written for this class as a data source.

These data provided multiple sources from which I drew upon to write narratives and vignettes, sources of data which I analyzed for the work of teaching mathematical knowledge for teaching.

The pilot study having been discussed in Chapter 1, I discuss the subsequent analytical phases of this study. I present a summary of the findings from phases two and three and review threats to validity and shortcomings of these phases.

Data Analysis

Beginning with the writing of narratives, I discuss my data analysis in detail. In addition, the recursive nature of my work is predicated on the theory that each phase of

analysis informs the next and each successive phase adds greater detail to reveal the mathematical work of teaching teachers and the mathematical knowledge demands of that work (Charmaz, 2000). I continue this discussion by reviewing the work to write narratives.

Written Narratives

For the first phase of my analysis, I wrote narratives that record each instructor's teaching of mathematics to teachers. These narratives, through extensive use of transcript, detail the work of each teacher educator to orchestrate interactions among themselves, the teachers, and the mathematics, recurrent problems and tasks of the work, and the mathematical issues that arise in the course of the instruction. These narratives capture the classroom tasks and the mathematical demands of these tasks. Further, they reveal the work in which the teacher educator and teachers are engaged. The writing of the case studies synthesizes the data from each observation site and provides for an initial examination of the work of mathematics teacher education. The data in narrative form open up the evidence on which an analysis of the work of teacher education and the mathematical demands entailed by this work are created (Strauss & Corbin, 1998; Yin, 1994). For this analysis, the melding of the data from the field notes and the interviews created a picture of the work. It helped me "see" the teaching via the observation and video data. I could learn about the teaching from the teacher educators' discussion of their work. Together these components helped me learn about the work and the mathematical knowledge demands of the work. Individually they were helpful, but together they were more powerful. It was the writing of the narratives that helped open up the work of mathematics teacher education.

After writing narratives, my analysis entered three phases: (1) the identification of three tasks of teaching and a broad analysis of the work of teacher education with respect to these tasks and the mathematical demands entailed by this work, (2) the extraction of mathematical problems and the examination of mathematical demands entailed by these problems and, (3) a micro-analysis of the tasks and inferences about the mathematical demands at the micro-analytic level.

Analytical Phase 2: Analysis of Narratives and Interviews for evidence of MKTT

The case studies are intended to be repositories of data rather than content for my dissertation. Once written, I analyzed the narratives using a series of coding strategies (Strauss & Corbin, 1998). First, I open coded the cases line by line. Using the lines of text as the first unit of analysis allowed me to "to open up the text and expose" (p. 102) instances of mathematical work. During this phase, my detailed examination of the narratives allowed for "fine discrimination and differentiation among categories" (p. 102) of the mathematical work used to teach teachers. During this stage of the analysis, I was able to accomplish two objectives. By coding the Ball observation and interview data, I identified instances of mathematical demands of teacher educator's work to teach mathematical knowledge for teaching. Second, I was able to formulate conjectures about the mathematical knowledge demands of this work. These conjectures were a first step in hypothesizing about the tasks of teaching mathematical knowledge for teaching and the mathematical entailments of that work.

I conducted a second coding phase during which I used chunks of text from the narratives and the interviews rather than the line by line technique in my initial coding. I parsed text of both the case narratives and interviews by identifying instances of

mathematics instruction or identification by the teacher educator of the use of

mathematics for teaching. I made inferences about the mathematical knowledge used to

design and enact lessons for teaching mathematics to teachers. I attempted to identify the

mathematical knowledge as being mathematical knowledge used for teaching or

mathematical knowledge that might be different from mathematical knowledge for

teaching.

In analytical phases two and three, I attempted to examine the work of teacher

educators from a broad perspective. I formed claims about the purposes of the three

identified tasks of teaching and I modified the tasks. Also, I became more resolute that

there is a domain of mathematical knowledge entailed by the work of teaching

mathematics to teachers. I continue with a discussion of my findings about the work of

using interpretations and representations, using examples and managing discourse about

mathematics.
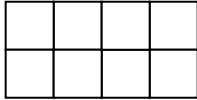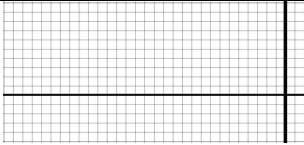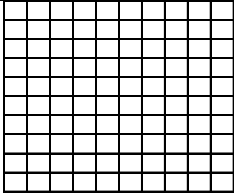
Use of interpretations and representations. One form of analysis was the

examination of the tasks for their purposes. I had identified two tasks as the *use* of

interpretations and representations and the *use* of examples. For the use of interpretations

and representations, I identified two purposes: developing teachers' mathematical

knowledge into mathematical knowledge for teaching and developing pedagogical

strategies that teachers might use to teach children. I cited Ball's use of the rectangular

array across the domain of multiplication, the work to modify the representation and to

use it to make visible increasingly more complicated mathematical ideas.[7] For the

purpose of developing mathematical knowledge for teaching, I found that representations

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[7] As the reader will see, this analysis could be used to demonstrate *threading* mathematical ideas across a
domain, an element of *managing the enactment of mathematical tasks* in analytic phase 4.

and interpretations were used for demonstrating mathematical ideas, challenging

mathematical knowledge, and unpacking mathematical ideas compressed within concepts

and operations. For Ball's teaching of multiplication, interpretations and representations

fostered knowledge of multiplication facts (e.g., 2 x 4 and 4 x 2) into conceptual

understanding using repeated addition and area interpretations of multiplication. Details

about the representations were made explicit (e.g., operators and operands are modeled

by the vertical and horizontal dimensions, respectively). For multi-digit multiplication,

the area interpretation and rectangular grid made visible the partial products of both the

standard and partial product algorithms. Finally, these interpretations and representations

fostered a justification for the decimal multiplication algorithm (e.g., why isn't 0.7 x 0.1

= 0.7?). In Ball's work, identification of one interpretation and representation, threading

of these throughout the teaching of multiplication with modifications to adapt

representations to address the developing complexity of the mathematics permitted

layered learning of underlying concepts of multiplication. My analyses of the use of

interpretations and representations identified purposes and ways in which interpretations

and representations were used for teacher education. These findings are in figure 3.2.

Uses of the Rectangular Region Across the Teaching of MKT for Teaching Multiplication		
Purpose of the Figure	Example of Figure	Mathematical Details
Representation of the area interpretation multiplication		For a x b, 'a' is the vertical dimension and 'b' is the horizontal dimension
Representation of the area interpretation of multi-digit multiplication		Since the multiplication of 15 is done by multiplying by 10 and 5, the vertical dimension has a dark line that allows for easy identification of these amounts along the vertical dimension. Also, the multiplication of 32 is done by multiplying by 30 and 2 so the horizontal dimension has a dark line that allows for easy identification of these amounts along the horizontal dimension.
Representation of area interpretation of decimal multiplication		To represent decimals, the size of the grid is restricted to a 10 x 10 that allows for 100 small squares each having side lengths of one one-tenth and areas of one one-hundredth.

Figure 3.2. Summary of characteristics of area interpretation and several representations used for the teaching of MKT for multiplication.

Use of examples. I analyzed the *use* of examples for teaching mathematical knowledge for teaching. I claimed that examples might have different purposes for different kinds of learners. Across the data, I found that pre-planned (e.g., sequences of examples in decimal multiplication and fraction multiplication cases) and learner generated examples (e.g., 38 ÷ 4 word problems) were used. I analyzed the use of

examples for the mathematical ideas examples elicited, the evolution of mathematical ideas across examples, and the dynamics of orchestrating lessons around these different kinds of examples. It appeared that examples were used for three purposes: to attempt to challenge the student teachers' mathematical knowledge (e.g., what is 0.7 x 0.1?), to unpack MKT concepts (e.g., what is $\frac{1}{3}$ of $\frac{6}{5}$?), and to develop the MKT student teachers may use when teaching children (e.g., model $12 \div 6$ using two interpretations). For both sets of analysis, I examined the use of examples in broad strokes.

A shift in analytical focus. Examining the *use* of interpretations, representations, and examples restricted the analysis to studying teaching during the enactment of the lessons. Shifting the tasks to *selecting* interpretations and representations and *selecting* examples broadened the analysis to an examination of preparatory work. In the fourth analytical phase, I analyzed these tasks for the work entailed by teaching. I identified elements of the tasks. These analyses provided greater insights into the work from which the mathematical knowledge demands became more visible and inferences about the mathematical knowledge entailed by this work were feasible.

Managing discourse. For the task of managing discourse about MKT, I analyzed the work for the use of pedagogical strategies and the purposes of discourse. I discussed the use of interactive strategies to elicit teachers' thinking about mathematics through their mathematical talk, the work of listening to, hearing, and responding to mathematical talk to develop mathematical knowledge for teaching. And, I examined three purposes of managing discourse: engaging teachers with mathematical ideas, interpreting others' talk about mathematics, and summarizing mathematical ideas. These claims appear to reflect the dynamics depicted in the instructional triangle developed for teacher education. My

analysis focused on identifying overarching characteristics of the work to manage discourse.
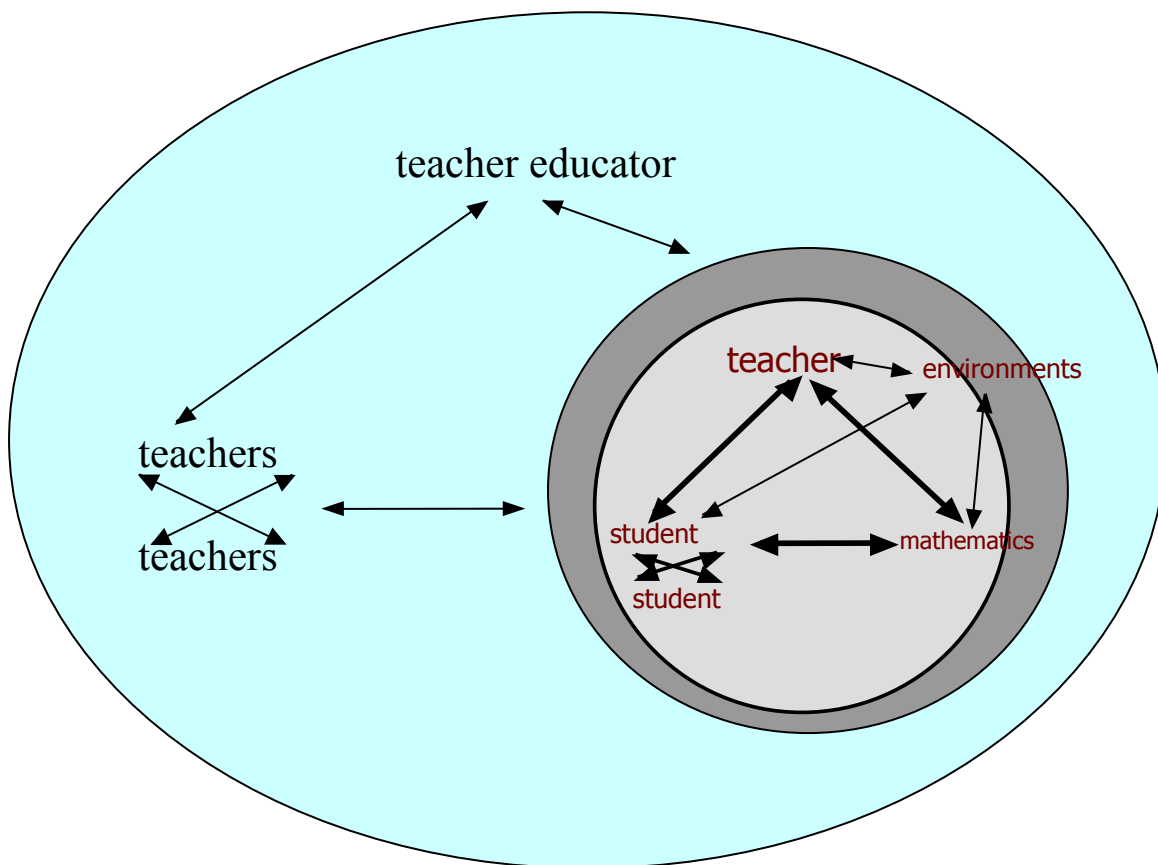
Figure 3.3. Instructional triangle for teacher education where the subject-matter is the work of teaching as depicted in the inner triangle, the instructional triangle for teaching students.[8]

Challenges to these analyses. The initial analytical phases offered insights into the mathematical work of teacher education; however they were broad and riveted to MKT as the content for mathematics teacher education courses. A closer look – a fine-grained analysis of the *work* of these tasks was in order. Also, I needed to step away from MKT research and broaden my conceptualization of the content taught in teacher education courses. Therefore, I paused my work and began fresh. I reformulated my hypotheses,

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[8] A version of this diagram is found in Ball, D. L., & Cohen, D. K. (1999). This diagram was used in a presentation by Ball, D. L. and members of our research team in Salt Lake City, UT, 2008.

refined the tasks of teacher education, refocused my analysis. I selected four vignettes from the data, two that introduced fundamental concepts (fraction and whole number division) and two that developed mathematical ideas within operations (fraction multiplication and decimal multiplication). I de-contextualized the three tasks of teacher education and made inferences about the parts of the work of each task. I went back to the vignettes to find evidence about these parts of the work, later named *elements*. I did this for each case independently then I conducted cross case analyses. Finally, I analyzed this work for the mathematical knowledge entailments. I claim that this work uses a special domain of mathematical knowledge. I attempt to characterize the knowledge based on these dissertation data.

In the next section, I discuss the analytical work used to study the three selected tasks of teacher education. In this phase, I attempted to eliminate the situated nature of the first phases by beginning with the premise that the teacher educators were working to transition teachers' mathematical knowledge to a mathematical knowledge used for teaching. I began with a broad image of mathematical knowledge for teaching. That is, I allowed mathematical knowledge for teaching to be mathematical content intended to support or to be used for teaching mathematics. I analyzed tasks of teaching mathematical knowledge at a micro level. Then, I used the results of the analysis of the mathematical work to learn about the mathematical demands of this work. The recursive nature of my analyses is predicated on the theory that each phase of analysis informs the next and each successive phase adds greater detail to reveal the mathematical work of teaching teachers and the mathematical knowledge demands of that work (Charmaz, 2000).

Analytical Phase 4: Analysis of Work of Teacher Education

This phase of my data analysis focuses on the analysis of vignettes of teaching mathematical knowledge for teaching. From the narratives, I extracted four vignettes of the teaching of multiplication, division, and fractions. I examined each vignette for evidence of the teacher educators' work of selecting interpretations and representations, selecting examples, and managing the enactment of mathematical tasks. My analysis revealed several elements entailed by the tasks allowing for a more detailed picture of the mathematical work and knowledge demands entailed by teaching mathematical knowledge for teaching (Patton, 2002).

Analysis of tasks of teaching: review of findings from prior analyses. Looking across the initial analytical phases, the first phase produced two important hypotheses. First, it appeared that the teacher educators were teaching mathematical knowledge that they valued for teachers' work. Second, there were several mathematical tasks of teaching in which the teacher educators appeared to engage. Three appeared to be prominent across the cases: selecting interpretations and representations, selecting examples, and managing the enactment of mathematical tasks. These appeared to be central to the work in each case. In the second and third analytical phases, a domain of mathematical knowledge beyond the knowledge being taught appeared to be entailed by the work of teaching teachers. At this point, I re-focus my analysis to examine the three tasks of teacher education for the elements of this work. Then, I make inferences about the mathematical demands entailed by this work.

Conceptual-analytic study of the tasks of teacher education. In this phase of analysis, I use a *conceptual-analytic* approach to identify sub-tasks or elements of the tasks of teacher education. Strauss and Corbin (1998) define this as the process of

conceptualizing, a process of "abstracting in which data are broken down into discrete incidents, ideas, events, and acts and are then given a name that represents or stands for these" (p. 105). For this dissertation, I studied teaching by focusing on the tasks of teacher education and mining these for discrete elements of which they are comprised. This closer examination or microanalysis revealed many elements of the work of each task and provided the opportunity to make inferences about the mathematical knowledge entailed by these elements.

Each vignette was analyzed for the elements entailed by the work of the tasks of teacher education. A hypothetical list of elements was created and the data were examined for evidence of these elements. Moving back and forth between hypothetical elements and data allowed for some elements to emerge and others to go away. Lists of synonyms were subject to experimentation to determine which best fit the element. Cross case analyses were completed at the site level and finally across all four cases.

In the fourth analytic phase, I examined the teaching in the vignettes from a micro-analytic level. That is, I began with each task of teacher education and parsed the tasks of teacher education into finer-grained elements. This analysis entailed several layers of work. First, I considered components of work that might make up the task. Then, I examined data to find evidence that the stages or elements of work were present in the data. I deliberated about the words to use to identify the elements of work. After analyzing each vignette in this way, I conducted cross case analyses studying similarities and differences across the cases.

Analytic process for the task of selecting interpretations and representations. The analysis of the work of selecting interpretations and representations entailed several steps.

First, I listed the interpretations and representations used in a vignette. I summarized the mathematical ideas each could make visible and the instructional purposes for the lesson. Second, I examined all data sources to find evidence of the work entailed by selecting these. For Suh's work, the monographs written as texts for the professional development and interview data provided insights. It was evident from the appendix of one monograph that Suh had reviewed several publications focused on the teaching and learning of rational numbers (Suh monograph, June 21, 2007). Thus, for Suh, it appeared that selecting interpretations and representations began with reviewing literature and gathering interpretations and representations used by other mathematics teacher educators. For Ball's work, field notes from MMPG meetings and lesson plans offered information about these decisions. It appeared that the researchers' past work had informed their selections. Third, I listed several words that might capture this work: gathering, collecting, reviewing, scanning, surveying. Throughout my writing, I experimented with these words to see which captured the work precisely. I thought carefully about the definitions of these words. Of these words, I selected *surveying* because it captures the general study of items.[9] Once interpretations and representations had been surveyed for the mathematical details they might reveal, many details must be thought out carefully and choices must be made. Considering, deliberating, reflecting on, weighing were possible words to select.[10] I chose *deliberating* to capture the careful, reflective nature of the work of weighing many characteristics of the interpretations and representations and the work for which they were intended. Specifically, Suh sought to

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[9] Surveying: to view in detail, esp. to inspect, examine, or appraise formally or officially in order to ascertain condition, value, etc. Retrieved from the world wide web at: *dictionary.reference.com/browse/survey,* **January 7, 2010.**

[10] Deliberating: to weigh in the mind; consider: to think carefully or attentively; reflect. Retrieved from the world wide web at: *dictionary.reference.com/browse/survey,* **January 7, 2010.**

extend whole number operations to fraction operations. He threaded concepts of fractions throughout all fraction operations. Deliberating which interpretations and representations required attention to these details. Once details are weighed, interpretations and representations were chosen. From the words choosing or deciding, the words are synonymous. I selected *deciding*.[11] Fourth, I hypothesized that these elements were present in the teaching in the dissertation data. I examined the data identifying instances of the use of the elements.

 Analytic process for the task of selecting examples. To analyze the work of selecting examples, I continued with the same analytical process used for selecting interpretations and representations. First, I created a table of examples used in a vignette. I identified the mathematics elicited through the use of each example. I attempted to map the mathematics as it appeared to become visible through the examples. Second, I attempted to analyze the work from examination of instructional goals to the selection of examples. The elements appeared to be: reviewing instructional goals, checking criteria for use, fitting examples to interpretations and representations, identifying categories of examples, sequencing examples, developing repositories. Third, I examined each case for evidence of these elements of work. Fourth, I conducted a cross case analysis for similarities and differences in the work of selecting examples.

 Analytic process for the task of managing the enactment of mathematical tasks. The analysis of the work of managing the enactment of mathematical tasks was grounded in deep examination of vignettes and teacher educator interviews. First, I looked across each vignette and identified key components of a lesson. There may be the *launch,* the

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[11] Decide: to make a final choice or judgment. Retrieved from the world wide web at: *www.merriam-webster.com/dictionary/**deciding***

mathematical work, and the *closing.* I analyzed the vignettes for the management of mathematical tasks. I hypothesized that *launching* the lesson*, developing mathematical ideas, engaging teachers in mathematical talk, assessing teachers' knowledge* were components of this work. I examined the cases for evidence of this work. I found in each case a form of talk that was tangential to the lesson, yet important to teachers' learning. I referred to this as *digressing* throughout the writing of this dissertation. After the final cross case analysis, I recognized that the tangential conversations had a significant characteristic. These conversations focused on the mathematical task and addressed issues that examined mathematical practices or ways of doing mathematics and connected the mathematical task to teaching mathematics to children. The essence of these conversations - a closer examination of the mathematical task – warranted a label that communicated a deeper level of mathematical talk. I re-named this element *meta-talk.* As with the previous two tasks, I analyzed these data case by case. I made claims about each hypothesized element and found evidence of these elements. I conducted cross case analyses for similarities and distinctions among cases.

 Final review process. The analyses were done case by case across several months. Summarizing the elements for each task entailed the creation of a matrix that included the elements from each task and case. These were not identical. Consequently, I deliberated about each element across the cases. First, I asked if the work of the elements was similar. If so, I examined the labels I had selected. To demonstrate my work, I noticed that across the cases, there is an element of gathering and reviewing possible interpretations and representations. For some cases I had labeled this as *reviewing.* For other cases I used the label *surveying.* During the final drafts, I selected one word to

capture the meaning of the element. In the case of *reviewing* or *surveying,* I selected *surveying.* Second, I looked for elements that were distinctive. Some were evident. As an example, in the Interpretations of Whole Number Division case, *creating in-the-moment examples* was an element in the managing the enactment of mathematical task. This element was not visible in the Decimal Multiplication case where each example had been planned. I included those as different. My final review of the analyses resulted in a set of elements for each case.

Analysis to develop the theory of mathematical knowledge for teaching teachers. The work analysis identifying elements of the tasks of teacher education unpacked the work of selecting interpretations and representations, selecting examples and managing the enactment of mathematical tasks. In addition, this fine-grained analysis facilitated the analysis of the mathematical knowledge demands entailed by the work of teacher education. To analyze the mathematical knowledge demands of teacher education, I reviewed the work analysis for each case. As I read through the analyses, I asked, "What mathematical knowledge is entailed by this work?" I listed the types of mathematical knowledge and created a master list of these types found across all cases. Then, I attempted to categorize the types of knowledge. Initially, I attempted to map them to specialized content knowledge. I can make the case that each type of mathematical knowledge may be knowledge of mathematical knowledge for teaching that is beyond mathematical knowledge for teaching. I call this specialized knowledge of mathematical knowledge for teaching.

There is one caveat. That is, the evidence of the use of mathematical structures and ways of mathematical work were striking in the Concept of Fraction and Fraction

Multiplication cases. My initial conjecture was to suggest a sub-domain that includes a robust knowledge and ability to teach the epistemology of mathematics which I named foundational content knowledge. However, because evidence was different across all cases, I will delay this until I gather more data and find additional evidence of the use of the knowledge of mathematical epistemology.

Teaching children mathematics: Support for initial assumptions. I argue clearly in Chapter 1 why it is reasonable to assume that the work of teaching mathematical knowledge for teaching would look different from teaching mathematical knowledge to children. Therefore, it would follow that the knowledge entailed by this work is different. I did spend a phase of this study looking at the work of teaching children. However, the results are not reported in the findings section of the dissertation. At this point, I discuss data that substantiate prior assumptions and continue to refine my thinking.

An initial assumption of this study is that the learners, i.e., teachers are different. When teaching children, it is assumed that children do not know the mathematical content of a lesson. When teaching teachers, it is assumed that teachers possess some mathematical knowledge. The work of teacher education is to develop this mathematical knowledge into mathematical knowledge for teaching. The teaching of decimal multiplication provides evidence. Ball asks the student teachers to multiply 0.7 x 0.1. She challenges their knowledge of decimal multiplication and ways to represent the operation. Her challenge raises the student teachers' awareness to the complicated nature of this operation, ways of representing it, and ways of communicating about it. Provoking an error entails knowledge of these fragile understandings and the mathematical knowledge to make these visible. Because learners are different, mathematical tasks and the

mathematical knowledge entailed by developing and teaching these tasks appear to be different as evidenced by the work to teach decimal multiplication.

A second assumption is that the purposes of teacher education are different. When teaching children, mathematical knowledge is the goal. When teaching teachers, developing mathematical knowledge into mathematical knowledge for teaching is the goal. The teaching of fraction multiplication provides evidence. Suh begins the lesson "[i]n mathematics taking two thirds of a bag of rice means that you divide the bag of rice into three equal parts and you take two" (Suh transcript, 070309, p. 1). Suh parsed the multiplication by two thirds into two operations. A teacher asked, "Isn't this fraction multiplication?" (Suh transcript, 070309, p. 1), Suh explained that the work of this lesson was to develop an understanding about fraction multiplication. He justified this work by citing the importance of developing children's understandings of this operation. He proceeded through a detailed lesson that included three interpretations of fraction multiplication and work with a sequence of examples. This lesson focused on teaching multiple ways of reasoning about fraction multiplication for teachers' mathematical knowledge and for their mathematical knowledge for teaching. This evidence supports my assumption that the goals of teaching children and teachers are different and that these goals shape different learning experiences. Further, design and enactment of this lesson entailed the mathematical knowledge of fraction multiplication, interpretations of fraction multiplication and the mathematical ideas inherent in each, connections across several interpretations of this operation, and underlying mathematical knowledge about operations and composition of operations. The teaching of fraction multiplication

supported my assumption that the mathematical content of teacher education is beyond the mathematical content of teaching children.

Teaching children mathematics: distinctions in elements of tasks of teacher education. The analysis of the tasks of teacher education manifested several elements of these tasks that might not be present in the work of teaching children, and if present, might be different. I present three examples of elements that appear to be distinctive to teacher education as they appeared in these data: launching the mathematical task, engaging teachers in mathematical talk, and engaging in meta-talk. First, *launching the mathematical task* entailed contextualizing the task within teaching or challenging mathematical understandings and unpacking mathematical ideas to develop mathematical knowledge into mathematical knowledge for teaching. Second, *engaging teachers in mathematical talk* entailed an acute level of listening to, hearing, and responding to precision in teachers' mathematical talk beyond that done with children. Third, a prominent element in managing the enactment of mathematical tasks is *meta-talk.* Focused on making knowledge of and about mathematics explicit and connecting mathematical knowledge to the work of teaching children, *meta-talk* was used to reflect on mathematical work connecting it more closely to mathematical knowledge for teaching. These elements appeared to focus the tasks of teacher education more directly on the teaching mathematical knowledge used for teaching.

The four phases of analysis used for this investigation revealed, layer by layer findings about the work and mathematical knowledge demands entailed by teacher education. I conclude the preliminary discussion of this investigation and continue to

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present four cases from the data and analyses of these data for the mathematical work and knowledge demands entailed by the work of teacher education.

Chapter 4

Two Cases of the Teaching of Mathematics to Teachers

In this chapter, I provide a provisional framework for the work of teaching

mathematics to teachers. For my study, I examine four cases of teaching different aspects

of rational number arithmetic. I use these cases to identify and elaborate in more detail

the work of selecting interpretations and representations of mathematical ideas, choosing

examples to support these interpretations and representations, and managing the teaching

of this content to teachers. These three tasks of the work of teaching mathematics to

teachers appear to be each crucial in particular ways, and also interdependent. I begin this

chapter by offering a discussion of my focus on these particular tasks of teacher

education.

Selection of the Tasks of Teacher Education

Data-Driven Selection

My work to analyze the work of teaching mathematics to teachers mirrors Ball

and Bass's work to begin within the practice of teaching to identify tasks of that work

(Ball & Bass, 2003a, 2003b). My choice to analyze the work of selecting interpretations

and representations, selecting examples, and managing the enactment of a mathematical

task for teaching mathematical knowledge to teachers is rooted in these data. I examined

data collected at two sites to identify key mathematical tasks. My initial analyses led me

to identify nine tasks: creating the course, selecting tasks, modeling mathematics,

selecting examples, orchestrating conversation, responding to student teachers' questions, selecting materials, creating assignments, analyzing and responding to student work, and designing and grading assignments. I continued my field work gathering a third set of data. From these data, the work of selecting interpretations and representations of mathematical concepts stood out. The work to select examples was dominant as well. Finally, managing the engagement of teachers with mathematics attracted my attention. When considering the nine original tasks, these three seemed like the most likely to reveal the subtleties of mathematics educator work and the knowledge entailed by that work. They were observable. It seemed possible to extract examples of the work and make inferences into how this work was done. My second round of data collection and preliminary analysis encouraged me to focus my study on three tasks of teacher education that I saw in my data: selecting interpretations and representations, selecting examples, and managing the enactment of tasks for teaching mathematical knowledge for teaching. As a novice researcher, I sought out additional support for my decision.

Support for Selection

To justify these choices I rely on my own practice as a community college mathematics instructor, my work with the Learning Mathematics for Teaching (LMT) research project of which I was a research assistant throughout my doctoral work, and my study of research interests of other mathematics educators. As a community college mathematics instructor, I have designed and taught mathematics content courses for prospective teachers for almost two decades. This work has included informal study of interpretations and representations of mathematical concepts, selection of examples to manifest mathematical ideas, and ways of managing prospective teachers' engagement

with and learning of mathematics. These interests from a practitioner's perspective were strengthened as I engaged in research with the LMT research team. While we analyzed records from a set of elementary and middle school classrooms for insights into the mathematical knowledge used for teaching, we examined teachers' use of interpretations and representations. We found that teaching "requires understanding different interpretations of the operations in ways that students need not explicitly distinguish" (Ball, Thames, & Phelps, 2008, p. 400). We identified teachers' mathematical knowledge used "to choose, make, and use mathematical representations effectively (e.g., recognizing advantages and disadvantages of using rectangles or circles to compare fractions)" (Ball et al., 2008, p. 400) as a component of *specialized content knowledge*. Later, our research team concluded that

> Many of the mathematical tasks of teaching require a mathematical knowledge of the design of instruction. Teachers sequence particular content for instruction. They choose which examples to start with and which examples to use to take students deeper into the content. Teachers evaluate the instructional advantages and disadvantages of representations used to teach a specific idea and identify what different methods and procedures afford instructionally. Each of these tasks requires an interaction between specific mathematical understanding and an understanding of pedagogical issues that affect student learning (Ball et al., 2008, p. 401).

Because these tasks of teacher education targeted mathematical knowledge entailed by the work of teaching children, studying this work seemed valuable.

Looking beyond the LMT research, I found that there has been research on the use of interpretations and representations, use of examples, and work to manage engagement with mathematics. This literature provides insights into the importance of these tasks of teaching mathematics. Interpretations and representations are tasks of many levels of mathematical work (Cohen, 2005; Goldin, 2003; NCTM, 2000; Pape &

Tchoshanov, 2001). Likewise, examples are tools for mathematical work. For mathematics teacher education, both teacher developed and learner generated examples have been shown to be powerful pedagogical tools for helping students learn at a variety of levels (Mason & Watson, 2005; Zodik & Zaslavsky, 2008). Finally, the role of managing mathematical tasks entails many elements of teaching. Engaging students in conversation about mathematics is viewed as productive in fostering mathematical learning (Ball, 1988, 1991; Hiebert & Wearne, 1993; Lampert, 1990, 2001; Sfard, 2000; Sfard & Kieran, 2001). Given the importance of knowledge of interpretations, representations, and examples for teaching children mathematics as well as the importance of engaging students with mathematical tasks, I thought that investigating the work of selecting these for teaching teachers mathematics might reveal insights into this work. Consequently, my selection of these tasks of teacher education is rooted in my data. I can justify the value of this study by reflecting on the role these tasks play in my practice, the importance LMT placed on these as tasks of teaching children mathematics (i.e. the importance of teachers attending to these tasks), and the mathematics education community's investigation of these in other contexts.

Additional Thoughts about this Selection

I recognize that my analysis of this work is but a first attempt to identify key elements of the work of teaching mathematics to teachers. I assume that others might identify other elements of the three tasks on which I focus. I welcome and look forward to continued study and conversation necessary for the development of a more elaborated framework about the work of teaching mathematics to teachers that an extended investigation among many researchers might afford.

Further, my selection of three tasks of teacher education does not imply that these are the only tasks of this work or that these are the most important. I recognize that there are several other pieces of the work of teaching mathematics to teachers. The use of mathematical language is one example of an aspect of this work which I chose not to select. My decision not to include this component is not because I think it is less important for the work of teaching mathematics to teachers. On the contrary, I think the use of mathematical language during the work of teaching mathematics to teachers is essential and its study would be so intricate that it warrants exclusive treatment and elaboration in another work. Along with use of mathematical language, modeling mathematical work is an important component of teaching mathematics to teachers. Again, the breadth of this topic is so extensive that it might be a topic for another investigation.

The first two tasks selected for investigation are part of planning; they determine the instructional trajectory. The third component, managing the enactment of teaching teachers mathematics, is the *teaching* of the planned lesson. This work requires attention to two unique elements: the special nature of the learners, people who intend to teach children, and the special nature of the content, mathematical knowledge for teaching. It seemed that investigating the work to manage the enactment of mathematical tasks might expose aspects of this work that might be different from teaching mathematics to other types of learners. The special nature of the learners and the content might require characteristics of this work that might be distinctive to teaching teachers. Thus, the initial findings in my data prompted my interest in investigating the elements of the work entailed by these tasks of teacher education. My professional experiences as a

mathematics teacher educator, my research experiences as a doctoral student, and prior research by the broader mathematics education community led me to choose these three tasks, selecting interpretations and representations, selecting examples, and managing the enactment as important for the work of teaching mathematical knowledge to teachers and worthy of investigation.

I continue with the presentation of two cases of the teaching of mathematics to teachers. In the first case, the mathematics teacher educator introduces teachers to the concept of fraction, developing the point on the number line interpretation. In the second case, the mathematics teacher educator unpacks the meaning of fraction multiplication, using the definition of fraction and developing the concept of *taking part of.*

The Teaching of the Concept of Fraction

We drop into a class during the second week of a professional development institute for practicing teachers. At this point in the summer institute, Suh and the teachers have met for one week during which they completed a unit on whole number operations.
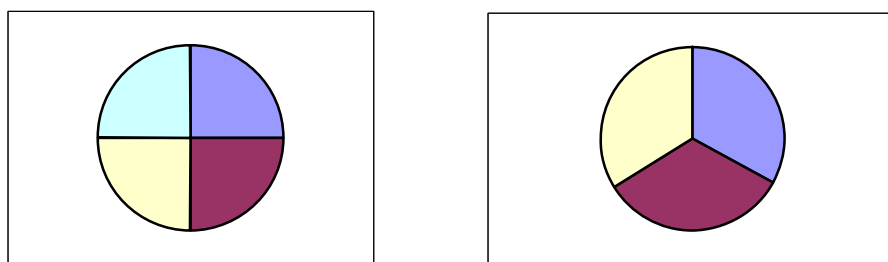
Figure 4.1. Suh's drawing of fourths and thirds for the teachers' inspection.

Suh began the session by asking the teachers to define "fraction." Their response was that a fraction is a *part-of-the-whole*. Then, Suh provided several problems to challenge the *part-of-the-whole* interpretation. Specifically, Suh asked the teachers, "I have $12\frac{1}{2}$ pounds of rice. I want $\frac{1}{3}$ of that. How much rice do I have? He asked, "How does [the] picture [of $\frac{1}{3}$] help? It does not." (Suh video notes, 35:15). Suh demonstrated that using pictures of fractions did not help solve problems involving weight because weight was not found in the picture. Next, he argued that pictures are not helpful with multiplication such as $\frac{1}{3} \times \frac{1}{4}$. He referred to pizzas cut into pieces measuring $\frac{1}{3}$ and $\frac{1}{4}$ and stated that the pictures provide images of what $\frac{1}{3}$ and $\frac{1}{4}$ of pizzas look like, but they do not help with the computation of $\frac{1}{3} \times \frac{1}{4}$. Suh concluded by stating that pictures provide an image of fractions but their usefulness "does not go much further" (Suh video notes, 062907, 36:50). He claimed that the *part-of-the-whole* interpretation is problematic when teaching children. He asked the teachers to make sense of the addition algorithm for unlike fractions. He asked, "How do you teach students to add $\frac{2}{27} + \frac{5}{36}$? What denominator would you use? Would you use 27 x 36? Would you use 27 + 36? "(Suh video notes, 062907, 20:30). Suh referred back to whole number addition. He noted that adding two whole numbers is *continued counting*. He stated that "If you add 4 + 5, you begin at 4 and add five to it by counting on – five, six, seven, eight, nine. So 4 + 5 equals 9" (Suh video notes, 062907, 20:46). Suh proposed that the teachers add fractions by

counting as they would count to add whole numbers. He suggested that the teachers use an algorithm to add $\frac{2}{27} + \frac{5}{36}$. He wrote:

$$\frac{2}{27} + \frac{5}{36} =$$
$$\frac{72 + 135}{27 \times 36} =$$
$$\frac{197}{27 \times 36}$$

Suh asked the teachers to make sense out of this algorithm. He asked:

> Do you sense that there is a problem? If you give me an idea about what something is about and you come to the computation and it is completely different, when that happens, how are your kids going to learn? What do you say? What do you say precisely? Are they going to trust you? How are they going to be able to make sense of what you are saying? (Suh transcript, 062907, 22:20).

After this stream of questions, Suh suggested that they examine work with fractions from a different stance. He challenged them to "make a new beginning" with respect to their knowledge of fractions. He sought to help the teachers develop an understanding that was "more precise, more workable" (Suh video notes, 062907, 34:20). In this introduction, Suh's goal was to convince the teachers that the part-of-the-whole interpretation is not an optimal interpretation for teaching.

The Teaching of the Concept of Fraction for Mathematical Knowledge for Teaching

Suh offered a new model of fraction to the teachers. He began with a *unit segment*. He divided the unit segment into thirds. He extended this by drawing a segment 3 units long and partitioned each unit into thirds (see figure 4.2).

$$\frac{0}{3} \qquad \frac{1}{3} \qquad \frac{2}{3} \qquad \frac{3}{3}$$

$$\frac{0}{3} \quad \frac{1}{3} \quad \frac{2}{3} \quad \frac{3}{3} \quad \frac{4}{3} \quad \frac{5}{3} \quad \frac{6}{3} \quad \frac{7}{3} \quad \frac{8}{3} \quad \frac{9}{3}$$
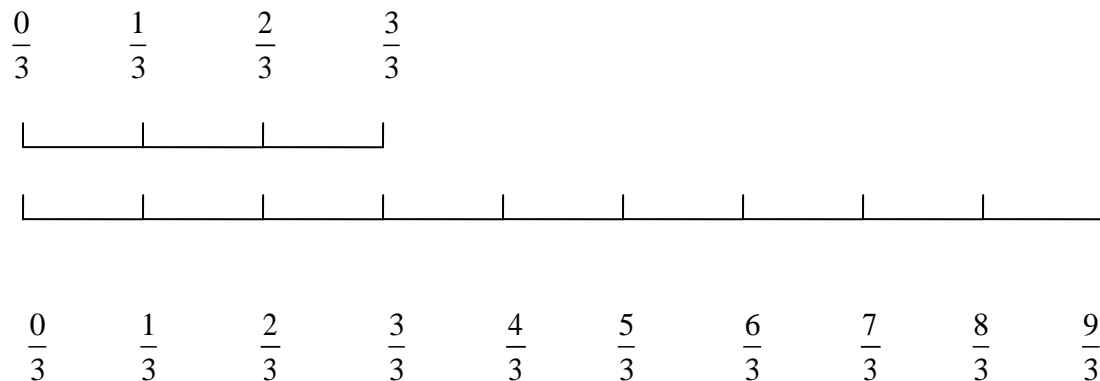
Figure 4.2. Suh's drawing of $\frac{3}{3}$ and $\frac{9}{3}$.

Suh elaborated. He noted that his pictures are not linear models of thirds. Rather, the number line allowed him to define $\frac{1}{3}$ as a *point* on the number line. Consequently, Suh's image of $\frac{1}{3}$ was not the *segment* from 0 to $\frac{1}{3}$, of length $\frac{1}{3}$. Given the unit segment, from 0 to 1, divided into three congruent segments, $\frac{1}{3}$ is the first leftmost endpoint $\frac{1}{3}$ of the distance from zero to one on the number line. Likewise, the number $\frac{2}{3}$ is the endpoint of the second leftmost segment from 0 to 1 on the number line, and so on. Suh suggested that the teachers work on this concept of fraction as a point on the number line, transition to this way of thinking, and adjust their teaching to having their students think about fractions as points on the number line.[12]

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[12] The underlying primitive notion here is *congruence* of intervals. So one would cut the unit interval from 0 to 1 into three equal subintervals, where "equal" is a geometric notion (congruent), not numerical equality. Then we define the number 1/3 to be the right endpoint of the first (left most) of these three intervals.

Suh provided more examples for the teachers. He began working with line segments. He attempted to work through the examples slowly and methodically. Suh asked the teachers to define several other fractions. He required precision in their language as they identified the fraction as a point on the number line. Then he transitioned his examples to non-unit fractions.

<p style="text-align:center">Analysis of the Tasks of Teaching the Concept of Fraction</p>

I step back from this presentation of Suh's teaching and move to the analysis of the mathematical work.

Work of Selecting Interpretations and Representations Used for Teaching Mathematical Knowledge for Teaching

To analyze the work of selecting interpretations and representations, I listed the interpretations and representations used in the vignette along with the mathematics each made visible and the purposes for the use of the interpretations and representations. After creating this visual summary, I examined all data sources to find evidence of the work entailed by selecting these. In the next section, I present three elements of the work of selecting interpretations and representations for the teaching of the concept of fraction: surveying, deliberating, and deciding interpretations and representations.

Surveying. Suh's work of selecting interpretations for his teaching of fractions was integral to designing lesson segments and involved several stages. First, Suh's monographs indicate that he gathered interpretations and representations from which he selected. Second, selecting interpretations and representations required reviewing the many interpretations of fraction. This in turn entailed analyzing these for the mathematics which each makes visible as well as the mathematics it does not. Lamon (2007) identified five major interpretations of rational numbers: "part/whole comparisons with unitizing, as

a measure, as a quotient, as an operator, and as a ratio" (p. 652). In Suh's work to select an interpretation, considering the interpretations of fractions required him to weigh the benefits each brought to the teaching of teachers. In this particular case, evidence suggested that Suh's criteria for selecting the interpretation included the precision which the interpretation afforded and the opportunity to have teachers reason about mathematics abstractly. Suh's consideration of interpretations is an example of the work to select these. This review of interpretations includes careful thinking about available interpretations, their mathematical details, and the mathematics they may reveal. Also, it involves taking a panoramic view of different interpretations to envision the ways these may be used to unpack the embedded mathematical ideas. And, it includes anticipation of adaptations and applications of interpretations across mathematical domains (Suh monograph, June 21, 2007).

Deliberating. The work of selecting requires considering the interpretations with which the teachers typically work when teaching early elementary grades, the part/whole interpretation and the interpretations used in grades five and up, measure, quotient, and ratio. This deliberation entailed thinking about how, if at all, these interpretations would be selected. In this particular case, rather than ignore the part/whole interpretation, Suh chose to use it strategically. He elicited it from the teachers and worked to demonstrate weaknesses with the interpretation.

Deciding. Suh selected an interpretation. In this case, he selected an interpretation which is a one dimensional model of the part/whole interpretation, the segment on the number line. This selection provided three important forms of scaffolding for the teachers. It used the part/whole interpretation, it provided a visual representation, *and* it

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transitioned the teachers to work on the number line which fostered their work with the point on the number line interpretation. The point on the number line interpretation supported several of Suh's instructional decisions. First, this interpretation allowed work with fractions to be extensions of whole number arithmetic allowing Suh to link work across these number systems. Second, as an abstract interpretation, it supported Suh's goal to foster teachers' abilities to reason abstractly. In the preview to his monograph on fractions, Suh (2007) noted that

> a fraction is very much like a whole number in that it is a point on the number line constructed in a well-determined manner, …, and once you accept this definition, you can use logical reasoning to explain *all* other meanings of the concept. Moreover, you also add, subtract, multiply, and divide fractions in the same way you do whole numbers. [However,] the concept of fraction … is an abstraction. [I]t is in the nature of an abstract concept that … its learning needs the support of precise descriptions and transparent reasoning. [He reiterated that] a fraction is a point on the number line and we use this fact as the basis of all explanations" (Suh, 2007, p. 3).

Making this selection cemented the direction of the lesson. It maintained the use of interpretation of number as point on the number line. And, it supported the decision to work with number as an abstraction and determined in a large part the mathematics to which the teachers would be exposed. A different decision would have yielded markedly different instruction and teacher opportunities to learn.

Discussion. Suh's work of selecting three interpretations (i. e., part-of-the-whole, segment-on-the-number-line, and point-on-the-number line) included his deliberation of several interpretations, his anticipation of the interpretations elementary teachers use and with which they are comfortable, and a decision to use an interpretation that melded an interpretation with which the teachers were familiar (part-of-the-whole) and a

representation that included his intended interpretation (point-on-the-number line).[13]

These three elements of selecting an interpretation, deliberating possible interpretations, anticipating teachers' responses, and deciding to select interpretations became visible through analyses of interview and video data as well as study of documents used by Suh and the teachers for this professional development. In this particular case, interview data suggest that Suh selected the point on the number line interpretation because it was mathematically precise, it provided the teachers an abstract interpretation with which to work, and it allowed the teachers to use their knowledge of whole numbers as a basis for their work with fractions. In addition, study of the course monograph reveals Suh's use of the segment on the number line interpretation as a transitional interpretation meant to be replaced by the point on the number line interpretation.[14] Suh writes:

> …the collection of segments, $\left[0, \frac{1}{3}\right], \left[0, \frac{2}{3}\right], \left[0, \frac{3}{3}\right], \left[0, \frac{4}{3}\right]$, etc., may be completely replaced by their right endpoints, $\frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}$, etc. In turn, the latter, instead of being described in terms of parts-of-a-whole, may be simply described as the division points when each segment between successive whole numbers, [0, 1], [1, 2], [2, 3],[3, 4], [4, 5], etc., is divided into equal thirds (in length), together with the whole numbers themselves (Suh, June 21, 2007, p. 10).

It was Suh's sense that teachers' mathematical knowledge needed to include mathematical knowledge at the abstract level. Consequently, his selection of the point-on-the-number line interpretation supported that learning goal.

Work of Selecting Examples Used for Teaching Mathematical Knowledge for Teaching

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[13] Interestingly, recent studies (Lamon, 2007) suggest that students who use the measurement interpretation of fraction across several grade levels of learning fractions and other multiplicative structures develop multiplicative reasoning skills and strong abilities to work with fractions (Lamon, 1999).

[14] Suh wrote text book like materials for the teachers' reference and produced them as monographs. These monographs included the content of his lectures, homework assignments, and appendices in which he discussed his work to produce the monographs. Suh revised these monographs after teaching an institute to make them more suitable for teaching teachers (Suh interview, 071507, p.6).

To analyze the work of selecting examples, I created a table of examples used in a vignette. I identified the mathematics that the teacher educator appeared to elicit through the use of the examples. I attempted to map the mathematics as it appeared to become visible through the examples. With this mapping of the mathematics complete, I turned to the work of selecting examples. I reviewed Suh's use of examples and analyzed the work to expose the elements of selecting examples. This analysis reveals that selection of examples requires attention to several aspects of this work and entailed several stages.

Reviewing. First, selecting examples required *reviewing instructional goals*. In this case, Suh wished to demonstrate shortcomings of the part/whole interpretation and support the point on the number line interpretation. Furthermore, he ultimately wanted to make connections between whole number operations and fraction operations, specifically the use of concatenation for addition with both sets of numbers. Also, he sought to develop the teachers' abilities to do mathematics at an abstract level. Suh's selection of point on the number line supported these instructional goals. Summoning these instructional goals to mind sets the stage for the selection of a sequence of examples that addresses each.

Checking. Selecting examples required that these examples satisfy some criteria for use. That is, there are standards by which examples are examined to determine whether they are usable for instruction. For the work of teaching fractions to teachers, there appear to be two criteria on which examples were judged. First, did the examples help to achieve instructional goals *and* were they usable for teaching? In this particular case, Suh compared examples to determine whether they would attend to the

aforementioned goals and would be simple enough to draw during the act of teaching. Furthermore, he selected examples that were diverse enough so that they did not generate misconceptions. Identifying thirds, fourths, and fifths as sequences of fractions on the number line attended to three important pieces of the lesson: support for the definition of sequences of fractions, order on the number line, and density. Suh's selection of the *set of examples* contributed to his broad instructional goals as well as those specific to this lesson segment. Selection of examples requires the negotiating between instructional goals and general criteria for instruction such as usability.

Sequencing. Suh developed the mathematics of the lesson by sequencing examples. This entailed taking a broad view of the lesson, considering the layers of mathematical knowledge that were packed within the lesson, and working backwards to see how to unpack these layers of knowledge. This required consideration of the mathematics and sequencing examples, building from simple tasks, to more complex tasks. It entailed introducing, reinforcing, and extending language, procedures, and concepts. In particular, Suh began with unit fractions, defining the *unit* and providing a precise process for identifying one third on the number line. He stated, "one third is the segment from zero to one third on the number line. This is the standard representation of one third. We can see what this chunk is. Pictorially, we see one third as the chunk. We then transition from the segment to the point" (Suh video notes, 062907, 53:55). Then, Suh selected the sequence of thirds to nine thirds. He repeated the pattern of defining the unit fraction (one fifth) and choosing a sequence (from one fifth to six fifths) for teachers' work. Later, he chose a variety of fractions, proper and improper, to support the learning of the definition of fraction and the procedure for locating fractions as he

prescribed. Finally, Suh extended his selection of examples to include fractions whose numerators were variables defined over intervals. This selection provided examples which included algebraic notation and opportunities to represent generalizations of key mathematical ideas. As this evidence suggests, sequencing examples entails examining instructional goals, the layers of mathematics, working backwards to construct a logical sequence of examples which unpacks this mathematics.

Developing repositories. The mathematical work entailed by using examples included developing and making ready repositories of examples to support teachers' learning, generating similar examples to provide more experiences and modifying examples to develop more complex ideas. Suh developed and had ready repositories of examples which he documented in monographs, used as examples in class and for assignments, and made available to teachers (Suh, June 15, 2007, June 21, 2007). For this professional development, the monograph was the result of several years' writing, testing with teachers, and revising. Suh noted that for this class he had approached some work in a new way and learned what was hard for teachers. He shared:

> What I discovered is that what I did in the past I thought they understood. But they didn't. And so I had to think about what did I have to do to get this over to them? All the way to the end of the third week, I was forced to think of ways to explain things that are closer to their knowledge base. So I think that was actually good. For example I have since then have written this into the chapter 2 (Suh interview, 071507, p. 6, lines 15-21).

Development of examples into repositories ready for use requires writing, testing with teachers, and revision. Creation of examples for use in the moment benefits from prior work to develop, test, and revise examples. Having a repository from which to work supports both planned examples and creation of additional examples.

The work of selecting examples entails thoughtful attention to instructional goals, the needs of the learners, and the unfolding of the lesson as it emerges.

Work of Managing the Enactment of Mathematical Tasks Used for Teaching Mathematical Knowledge for Teaching

The work of enacting tasks begins with the planning work of selecting interpretations and representations and selecting examples. Data from this case suggest that managing the enactment of mathematical tasks while teaching teachers may include four elements: launching, generating mathematical ideas, engaging teachers in conversation, and assessing.

Launching. A critical moment of the enactment of the task is the *launch* of the task. This might mean challenging teachers' incoming mathematical knowledge in an effort to generate teachers' questions about the mathematics, to open up the mathematics in a way that allows for knowledge of finer details. At other times, it might mean stating mathematical information for teachers to absorb either reviewing mathematics from past lessons or presenting ideas that are new to the teachers' work together. In this particular case, Suh had the teachers present the concept of fraction they use for teaching. The teachers stated that a fraction is "a part of a whole" (Suh video notes, 062907, 34:15). Suh challenged the usefulness of this concept by asking the teachers to find $\frac{1}{3}$ of $12\frac{1}{2}$ pounds of rice and multiplying $\frac{1}{3} \times \frac{1}{4}$ using pictures of parts of a whole. He launched this lesson by challenging the teachers' incoming concept of fraction and proposing that they work to make the concept "more precise" and "more workable" (Suh video notes, 062907, 34:20). The launch of the mathematical task stages the work of the lesson.

Developing mathematical ideas. A second element of managing engagement is generating mathematical ideas. This may be accomplished by presenting mathematics through a direct lecture or developing the ideas by engaging teachers in conversation. This vignette provides examples of both. First, Suh defined fractions as points on the number line using $\frac{1}{3}$ as an example. In this presentation, he provided several basic mathematical ideas. He defined *segment*. He noted that "one is not the segment from five to six, one is the segment from zero to one. … $\frac{1}{3}$ is the segment from zero to $\frac{1}{3}$" (Suh video notes, 062907, 53:55). He discussed $\frac{1}{3}$ as a "chunk." And he transitioned the teachers to think about $\frac{1}{3}$ as a point on the number line. Suh *presented* these mathematical ideas through a direct lecture. Later, he supported the teachers' learning of these concepts by engaging them in conversation. Suh's work to engage teachers took on several forms.

Engaging teachers. Suh's work to *engage* teachers in conversation around mathematics involved several strategies. He appeared to engage teachers, either in small or large group settings, to provide teachers opportunities to talk about mathematics, most importantly the mathematics they might use for teaching. Orchestrating conversation requires making decisions about the nature of the conversation. Is this a question / response session where teachers are asked to provide the "right answer?" Is this a demonstration session during which teachers presented mathematics as they might do when teaching? Is this an exploratory session during which teachers examined a variety of ways to work with a mathematical idea, skill, or procedure? Is this a discussion session during which teachers and teacher educators shared ideas about the mathematics and the

work of teaching? Each of these entails orchestration of mathematical ideas and response to teachers' mathematical conversation. They require knowing which ideas to pursue and which to put aside.

For this particular case, Suh chose to use three ways of engaging teachers in mathematical conversations. First, he conducted recitation sessions during which he posed a question and had teachers respond by reciting a precise definition or procedure. As an example, Suh asked the teachers to "[l]ocate seven ninths on the number line. (Suh video notes, 062908, p. 4). Second, he managed lesson segments during which the teachers worked with another teacher and rehearsed a mathematical procedure. As an example, Suh asked the teachers to "try to show that $2 \div 3 = \dfrac{2}{3}$. Explain to your neighbor. First remind yourself what $2 \div 3$ means" (Suh field notes, 063007, p. 3). Third, he designed lesson segments during which teachers worked together on their homework. These opportunities might be intended to foster teachers' exploration of mathematics, recitation of mathematical definitions, or practice with concrete materials, representations, procedures, and language. Engaging teachers in conversation about mathematics, specifically specialized content knowledge takes on many forms of conversation including recitation of precisely worded definitions and procedures, exploration of teachers' thinking, rehearsal of teaching scripts, and demonstrations of mathematical ideas.

Assessing. Suh's management of the mathematics of this lesson included a monitoring of the teachers' assimilation of the mathematics. Assessing teachers' mathematical work provides information on teachers' developing specialized content knowledge. The results of this process may guide the development of lessons and may be

an integral part of a teacher educator's management of tasks meant to foster specialized content knowledge. In this case, Suh assessed teachers' developing specialized content knowledge during large group recitation and small group practice. When conducting a recitation segment, Suh listened, evaluated, and reacted to the teachers' responses. For the location of fractions on the number line, Kara offered to locate $\frac{7}{9}$. She began "Divide [0,1] into nine equal parts. $\frac{7}{9}$ is the seventh point (Suh video notes, 062907, 42:45). Suh noted that her language was imprecise. He modeled the exact statement of the definition of fraction.

Suh: You have to say it more clearly. I want you to say it correctly because it will help you form the correct mental image. Begin with the segment [0, 1]. Divide it into 9 equal parts. The first division point is $\frac{1}{9}$. Count $\frac{1}{9}, \frac{2}{9}, \frac{3}{9}, \frac{4}{9}, \frac{5}{9}, \frac{6}{9}, \frac{7}{9}$. Now does anyone want to try $\frac{6}{11}$?

Suh asked Ray to define $\frac{6}{11}$. Ray used the language "*Break down* the segment into 11 equal parts." Suh corrected his statement, emphasizing a critical word for this definition, *divide.*

Suh: *Divided* into 11 equal parts. Now what?

Ray: Starting with 0, count 6 division points to the right.

Suh: Good. (Suh video notes, 062907, 43:00)

In this work, Suh listened carefully to the teachers' response and edited any part that did not comply with the agreed upon definition. When the teachers worked in pairs or small groups, Suh walked among them and monitored their work. He used the same technique

of correcting language that was inconsistent with the agreed upon definitions or procedures established during whole group work.

To provide more information on teachers' thinking, Suh often posed an additional problem for the teacher to complete. As in the case of Lynn, Suh suggested that he locate four tenths on the number line. He corrected Lynn's attempt. Lynn repeated Suh's corrected statement. Then Suh suggested, "Now do seventeen tenths" (Suh field notes, 063007, p. 12). In-the-moment assessment allows for thoughtful consideration of the lesson and formation of the next stages of the work. Deliberation of next steps might include modification of the lesson, development of a supplementary lesson to attend to pieces of the task teachers did not understand, or extension of the lesson to develop new mathematical ideas.

Engaging in meta-talk. Teaching mathematical knowledge for teaching appeared to include focused attention to detailed mathematics *and* crafting lessons to target mathematical ways of work and situating the mathematics within the work of teaching. In this case, Suh prefaced the lesson with a discussion of the importance of detail when working with mathematics. He noted that mathematics includes mathematical ideas and ways of working. Suh noted that

> it is about the *minute* details. Everything comes down to the details. If you limit yourself to a fixed amount of space, you telescope your thoughts into a small number of words and then you throw away the details. You are guaranteeing yourself lots of mistakes. … Concentrate on the substance and don't let the form limit you." (Suh fieldnotes, 070307, p. 3).

In this meta-talk, Suh cautioned the teachers not to restrict their work. He emphasized that attending to "*minute*" details is important for mathematical work. For this particular

case, meta-talk informed teachers about an important aspect of mathematical work – attention to minute details.

The Teaching of Fraction Multiplication

Let's return to Suh's professional development institute on July 3, three class days after Suh introduced fractions. Suh devoted the morning session to completing work on addition of fractions. He ventured into addition of decimals before pausing for the morning break. We drop into Suh's class around mid-morning.

Introducing the Concept

The launch. Suh asked the teachers to find two thirds of a bag of rice. He noted that "[i]n mathematics taking two thirds of a bag of rice means that you divide the bag of rice into three equal parts and you take two" (Suh transcript, 070309, p. 1). A teacher asked, "Isn't this fraction multiplication?" (Suh transcript, 070309, p. 1). Suh responded that the answer can be computed using fraction multiplication however, he suggested that the teachers look closely at this problem and consider what it meant by *taking part of* a number. Suh discussed that for teaching children approaching the problem in this way appeals to children's understanding of two thirds. He justified his work stating that it is important for children to understand fraction multiplication. Further, if the teacher presents an algorithm for multiplying two thirds times six fifths, there is little substantiation for why multiplication is used. Consequently, Suh proposed a different approach to this problem. He suggested that the teachers think about the problem as defining *taking part of* a number. He presented the following definition for their work:

$\frac{m}{n}$ of a fraction $\frac{k}{l}$ means the length of m concatenated parts when the segment $[0,$ $\frac{k}{l}]$ is partitioned into n equal parts. In other words, $\frac{m}{n}$ could be a whole number.

'n' always tells you the number of parts it is divided into (Suh transcript, 070309, p. 1)

Suh continued to use the point on the number line interpretation for fraction. He posed the task:

> Take two-thirds of a bag of rice weighing 15 pounds. How would you do it? Two thirds of what? Size? Weight? …When you make up problems for your kids, you have to be careful. For the sake of your children, you would specify two thirds by weight. That is easier. If I say two thirds by weight, I put this on the number line where the unit is one pound. Then there is no ambiguity about units. Everything is a pound. When you are a teacher, you have to worry about these things (Suh transcription, 070307, p. 1).

Suh reiterated a process for finding the answer. He noted that the 15 pounds of rice could be divided into three equal parts, each part having 5 pounds. This amount could be multiplied by 2. The result would be 10 pounds. He wrote (15 ÷ 3) x 2 and restated the answer as 10 pounds of rice.

During his launch of the problem, Suh challenged the teachers in three ways. First, he asked them to use the number line to represent the problem. Second, he suggested that they refrain from thinking about the problem as a *multiplication* problem. Rather, he asked them to think about the steps they might take to find the solution without multiplying by two thirds. He asked them to *reason out* the solution. Third, he distinguished the mathematical work of teaching. He stressed attention to details and noted that *for teaching* details must be attended to without ambiguity.

Presentation of three interpretations of fraction multiplication. After posing this contextualized problem, Suh suggested that the teachers consider the problem as "pure mathematics." He de-contextualized the work of taking *part of* a number. He began with the problem $\frac{1}{3}$ of $\frac{6}{5}$. Applying his definition of fraction, Suh directed the teachers to

divide $[0, \frac{6}{5}]$ into 3 equal parts. A teacher, Ray suggested that Suh divide each unit on the

number line into 5 equal parts and count off 6 of those parts to find the point $\frac{6}{5}$ (see

figure 4.3 below).

0	1	2	3	4	5	6
$\frac{0}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{5}{5}$	$\frac{6}{5}$
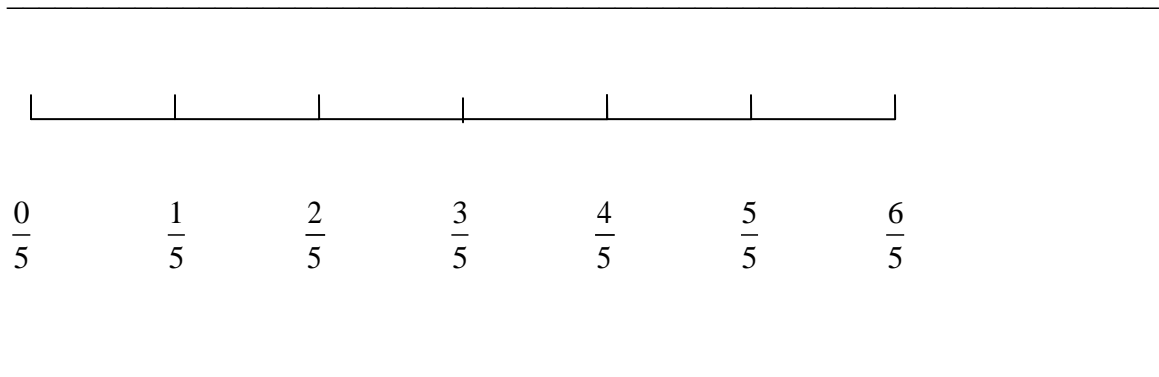
Figure 4.3. Suh's drawing of $\frac{6}{5}$.

Next, Suh applied the definition of fraction to find $\frac{1}{3}$ of $\frac{6}{5}$. He stated:

I divide $[0, \frac{6}{5}]$ into three equal parts. We want one third of this. Divide the

segment into three equal parts. One third is the first partition. The result is $\frac{2}{5}$

(Suh transcription, 070307, p. 1).

In this approach to find one third of six fifths, Suh directly applied the definition of

fraction. First, he identified six fifths as the *unit* and found one third of that unit. See

figure 4.4 for Suh's representation of this work.

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$$\begin{array}{ccccccc} \dfrac{0}{5} & \dfrac{1}{5} & \dfrac{2}{5} & \dfrac{3}{5} & \dfrac{4}{5} & \dfrac{5}{5} & \dfrac{6}{5} \end{array}$$
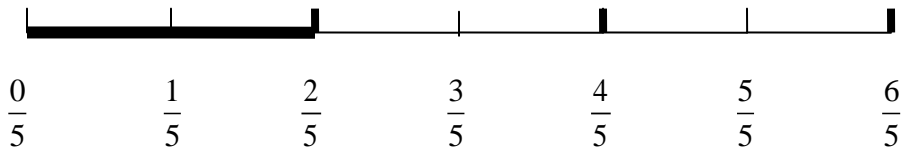
Figure 4.4. Suh's drawing of $\frac{1}{3}$ of $\frac{6}{5}$.

Immediately, Suh suggested a second approach to finding $\frac{1}{3}$ of $\frac{6}{5}$. He reverted to his claim that a non-unit fraction is composed of *copies of* the unit fraction. Consequently, $\frac{6}{5}$ is equal to 6 copies of $\frac{1}{5}$. To find one third of those copies, take $\frac{1}{3}$ of 6 which is 2. So $\frac{1}{3}$ of $\frac{6}{5}$ is $\frac{1}{3}$ of 6 copies of $\frac{1}{5}$ which is 2 copies of $\frac{1}{5}$ or $\frac{2}{5}$. Suh parsed the mathematics for this process. He noted:

> At the moment, I only deal with one operation. First, think of this as a number on the number line. Then, think of finding $\frac{1}{3}$ of the way from 0 to $\frac{6}{5}$. A second way to think of $\frac{6}{5}$ is as 6 copies of $\frac{1}{5}$. So if you want $\frac{1}{3}$, you have 2 copies. So, $\frac{1}{3}$ of $\frac{6}{5}$ is 2 copies of $\frac{1}{5}$ or $\frac{2}{5}$ " (Suh transcription, 070309, p. 2).

Finally, Suh added a third approach to this problem. He reminded the teachers that $\frac{6}{5}$ is the addition of $\frac{2}{5}+\frac{2}{5}+\frac{2}{5}$. He discussed the value in this approach.

> Why is this so good? What is addition? Concatenation. You see in front of your eyes – two things, two things, two things. I am drawing the picture for you. It is the virtue of knowing the definition. You can see right away that $\frac{1}{3}$ of $\frac{6}{5}$ is $\frac{2}{5}$ (Suh transcript, 070307, p. 2).

$$\frac{2}{5} \qquad + \qquad \frac{2}{5} \qquad + \qquad \frac{2}{5}$$

$$\frac{0}{5} \qquad \frac{1}{5} \qquad \frac{2}{5} \qquad \frac{3}{5} \qquad \frac{4}{5} \qquad \frac{5}{5} \qquad \frac{6}{5}$$
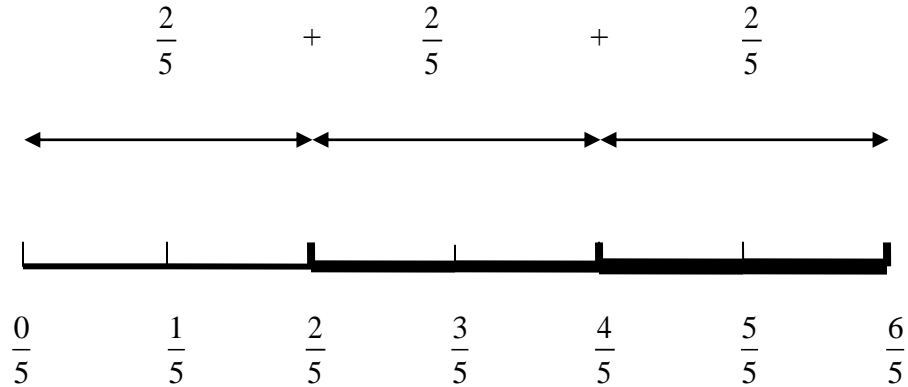
Figure 4.5. Suh's third interpretation of $\frac{1}{3}$ of $\frac{6}{5}$.

A second problem: What is three fourths of sixteen sevenths? Suh proposed a

second problem. He began to write the problem $\frac{3}{4}$ of $\frac{15}{7}$ but apologized and quickly

erased the 15. He changed the problem to $\frac{3}{4}$ of $\frac{16}{7}$. The teachers worked on this problem

individually and in pairs for approximately 20 minutes. They agreed that the answer

was $\frac{12}{7}$. Suh began a discussion about the multiplication.

Suh: Begin with the number line. That is a good place to begin. You cannot go wrong.

So you talk about sixteen sevenths. You have to begin by doing what?

Susan: Divide the segments.

Suh: You divide the segments between consecutive whole numbers into seven equal
parts. [Suh draws the number line from zero to three.] Okay, that is enough for
our purpose. And 16 over 7 will be 16 counts. I don't have to count. I can think
about 16 as being 2 x 7 + 2.

| 0 | $\frac{7}{7}$ | $\frac{14}{7}$ $\frac{16}{7}$ |
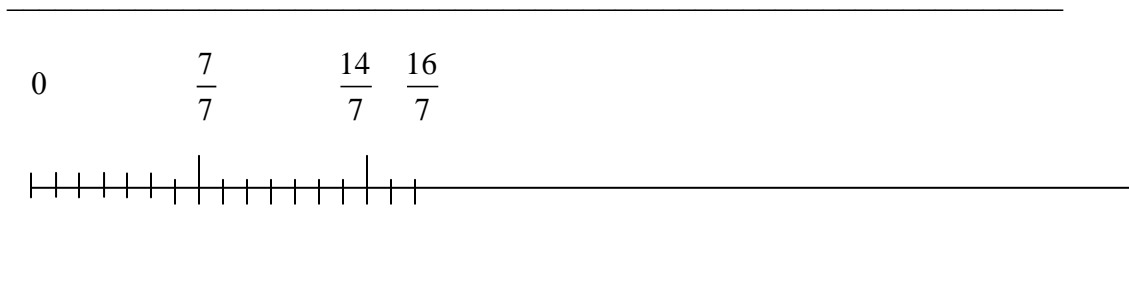
Figure 4.6. Suh's representation of $\frac{16}{7}$.

Suh reinforced the importance of beginning with the chosen interpretation of fraction, "Begin with the number line." He insisted that the teachers state the definition precisely and suggested that they use the "division with remainder theorem" to locate the point (Suh transcript, 070309, p. 2). At this point in the discussion, Suh deviated from the work of finding *part of* a number to discuss the use of mathematical definitions and theorems.

Suh repeated the work to provide a second interpretation of *taking part of* a fraction. He defined $\frac{16}{7}$ as 16 copies of $\frac{1}{7}$. He noted that $\frac{1}{4}$ of $\frac{16}{7}$ is 4 copies of $\frac{1}{7}$. To find $\frac{3}{4}$ of $\frac{16}{7}$, concatenate 4 copies of $\frac{1}{7}$ three times. That is (3 x 4) copies of $\frac{1}{7}$ or $\frac{12}{7}$.

Suh continued with the third interpretation. He asked, "So what about the last one? $\frac{16}{7}$ = $\frac{4}{7} + \frac{4}{7} + \frac{4}{7} + \frac{4}{7}$" (Suh transcript, 070307, p. 2). Suh identified $\frac{16}{7}$ as the repeated addition of $\frac{4}{7}$ and that this is

> the symbolic restatement of the picture, exactly! I am not saying more or less or roughly. It is the symbolic transcription of that picture. 4 parts of $\frac{1}{7}$, concatenate

with 4 parts of $\frac{1}{7}$, concatenate with 4 parts of $\frac{1}{7}$. The result is $\frac{3}{4}$ of $\frac{16}{7}$ which is $\frac{12}{7}$ (Suh transcription, 070307, p. 2).

Suh used three interpretations of *taking part of* to find $\frac{3}{4}$ of $\frac{16}{7}$.

A third example: Take $\frac{3}{4}$ of $\frac{15}{7}$. For the example take $\frac{3}{4}$ of $\frac{16}{7}$, Suh extended the

work of the first example (take $\frac{1}{4}$ of $\frac{16}{7}$) in which a unit fraction ($\frac{1}{3}$) was the operator

(the *part* of) to a non-unit fraction ($\frac{3}{4}$) as the operator in the second example. He

developed this work further using the example $\frac{3}{4}$ of $\frac{15}{7}$. Consistent with his work on

examples one and two, Suh continued to use three interpretations of fraction

multiplication.

Suh posed the problem: take $\frac{3}{4}$ of $\frac{15}{7}$. He asked, "What is the issue here? Last

time it worked and this time it doesn't. Sixteen is divisible by 4, 15 is not. So the key

issue is the 15" (Suh transcription, 070307, p. 3). Suh made the major change in the

problem explicit. He asked the teachers to think about the problem and paused while

teachers worked independently. Suh continued. He suggested redefining 15 as (3 x 4) + 3.

He wrote $\frac{15}{7} = \frac{3\cdot4}{7} + \frac{3}{7}$. Suh began by taking $\frac{1}{4}$ of $\frac{3\cdot4}{7}$. This is $\frac{3}{7}$. Then, he attempted to

take $\frac{1}{4}$ of the remaining $\frac{3}{7}$. He paused to note that only two numbers in the problem were

important. For $\frac{a}{b} \cdot \frac{c}{d}$ c had to be a multiple of b. For this particular problem, 15 needed to

be a multiple of 4. Suh noted that he wanted the numerator to be divisible by 4. So, he

multiplied the numerator and denominator by 4. The result was $\dfrac{15}{7} = \dfrac{15 \cdot 4}{7 \cdot 4}$. Then, he

took $\dfrac{1}{4}$ of $\dfrac{15 \cdot 4}{7 \cdot 4}$. The result was $\dfrac{15}{7 \cdot 4}$. Finally, Suh multiplied $\dfrac{3}{4}$ of $\dfrac{15 \cdot 4}{7 \cdot 4}$. He multiplied

15 by 3. His result was $\dfrac{3}{4}$ of $\dfrac{15}{7 \cdot 4} = \dfrac{45}{7 \cdot 4} = \dfrac{45}{28}$.

Suh paused the work of *taking part of* to examine the work of finding equivalent

fractions. At this point, he digressed and discussed the work of *explaining*. He posed a

rhetorical question, "What does it mean to explain something to someone?" (Suh

transcription, 070307, p. 3). Suh responded:

> When you say you want to explain something to somebody, you want to explain
> in terms of something that person already knows. Do you all agree? If I explain
> something to you using something you do not know about, do you know what I
> have explained? No. So now you are not explaining this to us, to yourself. You
> are explaining this to your young charges. I don't care how you do it, but you are
> only allowed to use what they already know (Suh transcription, 070307, p. 3).

Suh asked the teachers to imagine explaining the work of finding equivalent fractions to

students who did not know how to multiply fractions. He noted:

> I am going to use something that I know to be true and then I am going to make a
> logical deduction. You have to start with something you already know. That is the
> hierarchy in mathematics. It is layered learning. You have one layer – it is a solid
> fact you know already. On top of that you introduce another layer. The new layer
> rests on top of something solid. If the lower layer is full of holes, what it is resting
> on is thin ice. It is shaky. What is written there is correct, $\dfrac{15}{7} = \dfrac{15 \cdot 4}{7 \cdot 4}$. You might
> make it more compact by saying that I want to represent this so that the numerator
> is divisible by 4. *That* is the driving force. The other way is correct but it does not
> make mathematical sense. Why not $\dfrac{15 \cdot 25}{7 \cdot 25}$? Why not $\dfrac{15 \cdot 82}{7 \cdot 82}$? Why not all those
> things? What do you want? The numerator to be divisible by 4. All you want is
> that the numerator is divisible by 4 (Suh transcript, 070307, p. 4).

Suh discussed *how* mathematical work is done. He noted that for the work to take $\frac{3}{4}$ of $\frac{15}{7}$ the numerator of $\frac{15}{7}$ needs to be divisible by 4. Although fractions have multiple representations, Suh emphasized that $\frac{15 \cdot 4}{7 \cdot 4}$ was the representation needed.

Suh proceeded to demonstrate the difficulty in working with $\frac{3}{4}$ of $\frac{15}{7}$. He wrote $\frac{15}{7} = \frac{3 \times 4}{7} + \frac{3}{7}$. He noted $\frac{1}{4}$ of $\frac{3}{7}$ required the use of equivalent fractions. Suh noted that the mathematical work was cumbersome for the problem $\frac{3}{4} \times \frac{15}{7}$ and suggested using equivalent fractions. He wrote $\frac{3}{4}$ of $\frac{15}{7}$.

$$\frac{15}{7} = \frac{15 \times 4}{7 \times 4} \text{ by equivalent fractions.}$$

$$\frac{1}{4} \text{ of} \frac{15}{7} = \frac{15}{4 \times 7}.$$

$$\frac{15}{7} = \frac{15 \times 4}{7 \times 4} = \frac{60}{28}. \text{ Partition } \frac{60}{28} \text{ into 4 equal parts of } \frac{15}{28}. \text{ Concatenate three times}$$

and the result is $\frac{45}{28}$. And, $\frac{3}{4} \times \frac{15}{7} = \frac{45}{28}$. Suh completed this example and reinforced the mathematical work it revealed by completing one final example, take $\frac{5}{6}$ *of* $\frac{13}{5}$.

Analysis of the Tasks of Teaching Fraction Multiplication

At this point in the lesson, I step away from the vignette and analyze the work of developing the concept of *taking part of.*

Work of Selecting Interpretations and Representations Used for Teaching Mathematical Knowledge for Teaching

Surveying. Fraction multiplication entails two important concepts: fraction and multiplication. Consequently, the work to select interpretations and representations

required attention to both. At each step of the work to select interpretations and representations, an additional level of work – attention to how the interpretations and representations for fraction and multiplication *supported each other was required.*

For the teaching of the concept of fraction, Suh reviewed interpretations and representations for this concept. For this subsequent case, he continued to use the *point on the number line* interpretation for fraction. His integration of this interpretation throughout the fraction lessons reflects his stance that

> The whole point of having a definition is that, once accepted, *it has to be the starting point of all future discussions of fractions.* In your classroom, therefore, once you adopt this definition, *you as well as your students will be obligated to refer back to it for any explanation about fractions* (Suh, July 21, 2007, p. 13, *emphasis in original*).

An additional step for this lesson was the review of interpretations and representations for multiplication. He reviewed using repeated addition and area interpretations. For this lesson, Suh selected the repeated addition interpretation. As indicated in his monograph (Suh, June 21, 2007), his focused consideration of fraction multiplication included his study of several mathematics teacher educators' work to study the manner in which fractions are presented in school mathematics and mathematics educators' work to address teaching and learning of fractions (e.g., Hart, 2000; Huinker, 1998; Lamon, 1996). He described persistent problems he perceived with this work summarized as: (1) no clear definition of fraction; (2) inattention to the similarities between whole numbers and fractions; (3) conceptual development focused on the complexities of fractions rather than the "underlying mathematical simplicity" (p. 139); (4) fraction operations are presented as rules that "seem to be made up on an *ad hoc* basis" rather than as an outgrowth of whole number operations (p. 140); and (5) *mathematical* explanations are

not used to teach the many aspects of fractions (Suh July 21, 2007). These premises guided Suh's work with fractions. The work of deliberating interpretations and representations includes parsing the task for each mathematical component and considering available interpretations and representations for each and the confluence of the different components.

Deliberating. Deliberating interpretations involves considering interpretations of fraction, multiplication, and fraction multiplication and how these support each other. Suh's deliberations required thinking through the many possible interpretations and analyzing how these interpretations supported his instructional goals for fraction operations. He might have considered using the repeated addition and area interpretations for multiplication. In fact, he did that. For his work with fraction multiplication, he chose to break the instruction into two lessons. The first lesson addressed the concept of taking part of. He noted that "if [teachers] want to understand how to multiply two fractions, they have to understand what it means to multiply fractions" (Suh interview, 071607, p. 1, emphasis added). This is the goal of the lesson under examination.[15]

For this lesson, Suh used Lamon's (1996) work as a reference (Suh, June 21, 2007). He applied the interpretation of fractions as operators to unpack the meaning of taking part of. In this examination of fraction multiplication, using "$\frac{a}{b}$ of" as an operator "instructs you to multiply by [a] and divide the result by [b]" (Lamon, 1996, p. 94). Further, Lamon discusses working with fraction multiplication in a way that demonstrates the "effects of rational numbers acting as operators" (Lamon, 1996, p. 95). For the work

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[15] The second lesson which took place a few days after this one focused on the geometric interpretation of multiplication. The content of that lesson is in Suh's monograph.

of demonstrating "$\frac{a}{b}$ of", *a* copies of a quantity are divided by *b* or the quantity divided

by *b* taken *a* times. As an example "$\frac{3}{4}$ of" means the division by 4 of a quantity

multiplied by 3 or the multiplication by 3 of a quantity divided by 4. This can be written

"$\frac{3}{4}Q = \frac{3Q}{4} = 3\left(\frac{Q}{4}\right)$. Finally, Lamon discusses the use of "$\frac{a}{b}$ of" as the composition of

multiplication with division. "The individual operations that are composed are simply

multiplication and division by natural numbers" (Lamon, 1996, p.98).

Deliberating interpretations for fraction multiplication required surveying

interpretations for both fractions *and* multiplication. It entailed considering how these

interpretations supported each other. Further, it required thinking through the

mathematics that needed to be unpacked in order to reveal the *mathematics* within

fraction multiplication. It appears that Suh deliberated over how these interpretations

built on the definition of fraction which he selected for this work, how they developed the

concept of fraction multiplication as an operation that built upon whole number

multiplication, *and* how they allowed the concept of fraction multiplication to evolve

from *taking part of* to the standard algorithm. Further, it required considering how

teachers often teach fraction multiplication, the strengths and weaknesses of this work,

and how this lesson on fraction multiplication might unpack the mathematical

underpinnings of this operation. In Suh's examination of school mathematics texts and

mathematics educators' work, he found that "algorithms are justified through

'connections among real-world experiences, concrete models and diagrams, oral

language, and symbols' (Huinker, 1998, p. 181 as quoted in Suh, June 21, 2007, p. 140).

Suh rejected such connections and selected "*mathematical* explanations…of what the

usual algorithms mean and why they are reasonable – a simple task *if* one starts from a

precise definition of fraction" (Suh, June 21, 2007, p. 140, emphasis in original). Suh's

stance, that the mathematics within fraction multiplication be unpacked using

conceptualized, fundamental mathematics was the basis for his selections of

interpretations and representations.

Deciding. A final step in the selecting interpretations and representations is

decision making. Deciding the interpretations and representations for the lesson entails

several considerations. For this case, choosing those that supported the instructional goals

for the lesson and that were consistent with prior and future instruction. For the

observation period of this professional development institute, Suh worked entirely with

the point on the number line interpretation of fraction. As noted above, he decided to

thread this interpretation of fraction throughout the instruction. He stated "I believe if

teachers are more aware of it [point on the number line], they first of all emphasize it.

Secondly, I think it increases the capacity for learning if they see a thread running

through everything" (Suh interview, 071607, p. 4). For the concept of multiplication, he

selected the repeated addition interpretation. To work with the specific case of *fraction*

multiplication, evidence from Suh's teaching and his monograph indicated that he

decided to unpack details of this operation by using the interpretations presented in

Lamon (1996): application of the definition of fraction as point on the number line,

application of the concept of "copies of" or concatenation of the unit, and application of

repeated addition interpretation of multiplication. Suh's use of these interpretations

defined the trajectory of this lesson. It reinforced his commitment to work with fractions

as points on the number line. It was consistent with his philosophy to work as

mathematicians do to generate mathematics. That is, he began with a definition and developed the mathematics based on that definition. He generated the concept of fraction multiplication through the use of three interpretations of this operation.

Impact on learning opportunities. If Suh had chosen different interpretations, the mathematical task and the mathematics would have been very different. The references Suh used provided many other interpretations. To demonstrate that others may not have supported Suh's instructional goals, I present one from Lamon (1996, p. 99) that interprets the operation *take part of* as an exchange. For this interpretation, ' $\frac{3}{4}$ of' is interpreted as an exchange of three items for every four that are given. As an example, if there are 12 coins, for each set of four, three will be returned. This interpretation would not support Suh's work with fraction multiplication. First, it does not use the point on the number line interpretation of fraction. Second, it does not use the concept of *copies* and concatenation to connect whole number and fraction work. Third, it does not support work with repeated addition to connect fraction multiplication with whole number multiplication. See figure 4.7 for a diagram of this exchange.
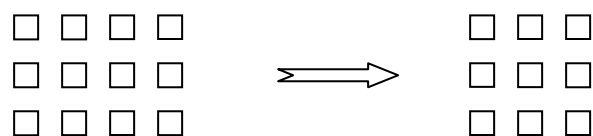
Figure 4.7. Lamon's depiction of take $\frac{3}{4}$ of 12 using the operator interpretation and the discrete interpretation of fraction.

Work of Selecting Examples Used for Teaching Mathematical Knowledge for Teaching

Suh used four examples for this lesson. These examples and the mathematics each elicited are summarized in figure 4.8.

Example	Mathematical characteristics	Mathematics elicited
$\frac{1}{3}$ of $\frac{6}{5}$	Unit fraction as operator Denominator of operator is factor of numerator of operand	Taking part of based on definition of fraction Copies of
$\frac{3}{4}$ of $\frac{16}{7}$	Non-unit fraction as operator Denominator of operator is factor of numerator of operand	Taking part of based on definition of fraction Copies of Concatenation
$\frac{3}{4}$ of $\frac{15}{7}$	Non-unit fraction as operator Denominator of operator is not a factor of numerator of operand	Taking part of based on definition of fraction Copies of Concatenation Need for equivalent fractions
$\frac{5}{6}$ of $\frac{13}{5}$	Non-unit fraction as operator Denominator of operator is not a factor of numerator of operand	Taking part of based on definition of fraction Copies of Concatenation Need for equivalent fractions

Figure 4.8. Examples used for fraction multiplication and mathematical characteristics.

Reviewing. An element of selecting examples is reviewing instructional goals. Consistent with prior instruction, it appears that Suh intended to make visible the

mathematical underpinnings of fraction multiplication. He sought to develop the concept

of fraction multiplication beginning with the definition of fraction. He noted:

> First, we have to give a precise meaning to a common expression, "two thirds of something," or more generally, "$\frac{k}{l}$ of something" … In general, *suppose a unit of measurement has been made clear,* then $\frac{m}{n}$ of something means the totality of *m* parts when that something is divided into *n* equal parts according to the chosen unit of measurement. Since the minute a unit has been chosen, we know what 1 means and we would have a number line, this definition can be put in more precise terms:
>
> Definition. $\frac{m}{n}$ of a fraction $\frac{k}{l}$ means the length of m concatenated parts when the segment $[0, \frac{k}{l}]$ is divided into n parts of equal length (Suh, June 21, 2007, p. 33-34).

Then, he presented three ways to interpret fraction multiplication and to demonstrate

links between these interpretations. Finally, he sought to develop a strategy that would be

applicable to any problem. Reviewing instructional goals made visible the purposes of

the instruction which determined the types of examples needed to address these goals.

Checking. Examples that support selected interpretations and representations

require that examples satisfy criteria for instruction. Because this lesson was built on

prior work, examples needed to foster instructional goals, connect with prior and future

lessons, and be usable for teaching. In this case, Suh selected examples which fostered

the unpacking of the mathematical underpinnings of fraction multiplication and

connected the current task to mathematics on which the teachers had worked together.

Specifically, Suh developed taking *part of* something using the definition of fraction,

working with multiple *copies of* unit fractions, and partitioning non-unit fractions into

equal parts for use with repeated addition. In the lesson on the concept of fraction, Suh

presented the teachers with a carefully developed definition of fraction. He applied this to

the work of *taking part of* for unit fractions, taking 3 copies of $\frac{1}{4}$ to extend the definition

of unit fraction, and applying repeated addition to foster the idea of equal parts when

parsing $\frac{6}{5}$ into $\frac{2}{5} + \frac{2}{5} + \frac{2}{5}$. Suh's application of the definition of fraction supported his

statement that "it increases the capacity for learning if [teachers] see a thread running

through everything" (Suh interview, 071607, p. 4). Further, Suh held that if teachers

"give students a framework, then they learn much better in comparison to when you just

give them a collection of things with no interconnection…. So I have a thread that runs

through it and they learn better" (Suh interview, 071607, p. 4). Using these

interpretations allowed Suh to begin with mathematical ideas which the teachers had

explored and develop them further into work that unpacked concepts primitive to fraction

multiplication.

Identifying categories. Examples support interpretations and representations and

make visible minute details of mathematical concepts and operations. Selecting and using

examples entails the identification of categories of examples that attend to specific

instructional goals. For this particular case, the categories included examples to

demonstrate: taking part of using unit fraction as operator / denominator with common

factor, non-unit fraction as operator / denominator with common factor, and non-unit

fraction as operator / denominator without common factor. Having generated these

categories, repositories of examples might be developed. Also, having defined categories

might permit for the generation of new examples during teaching. To model what this

might look like, consider this particular case. Suh explicated three levels of mathematics

in this lesson's examples: what it means by *taking part of* a unit fraction where the denominator of the operator is a factor of the numerator of the operand, what it means by *taking part of* a non-unit fraction where the denominator of the operator is a factor of the numerator of the operand, what it means by *taking part of* a non-unit fraction where the denominator of the operator is *not* a factor of the numerator of the operand. Generating these categories of examples facilitates the development of repositories of examples and the creation of examples within the categories during teaching.

   Sequencing. Thinking through the evolution of ideas, examples are selected and sequenced. For Suh's work with *taking part of,* he began with the example $\frac{1}{3}$ of $\frac{6}{5}$. This example had three critical characteristics. First, the operator is a unit fraction. This allowed for the use of the definition of fraction in a simple manner. That is, $\frac{6}{5}$ was divided into three equal parts of size $\frac{2}{5}$ and one of those parts was identified as $\frac{1}{3}$ of $\frac{6}{5}$. Second, the numerator of the operand was a multiple of the denominator of the operator. This allowed for the equal parts. Third, $\frac{6}{5}$ is improper. This is inconsistent with the typical approach of working with proper fractions. It is consistent with his attempt to help teachers work with fractions as points on the number line. Suh's second example began with a slip. Suh began with the problem: $\frac{3}{4}$ of $\frac{15}{7}$. He quickly erased this problem and wrote $\frac{3}{4}$ of $\frac{16}{7}$. This slip seems to demonstrate that Suh had a carefully planned sequence of examples. $\frac{3}{4}$ of $\frac{15}{7}$ was an example. However, it did not belong at this stage of the

lesson. At this stage of the lesson, Suh's example, $\frac{3}{4}$ of $\frac{16}{7}$, advanced the development of

fraction multiplication by one layer. That is, it required the concatenation of a unit

fraction. To take $\frac{3}{4}$ of $\frac{16}{7}$, Suh found $\frac{1}{4}$ of $\frac{16}{7}$ and concatenated that three times resulting

in $\frac{3}{4}$ of $\frac{16}{7}$. With the presentation of the concatenation of the unit fraction, Suh moved on

to a more complicated example, $\frac{3}{4}$ of $\frac{15}{7}$. This example involved a numerator of the

operand that was not a multiple of the denominator of the operator. This required the

introduction of equivalent fractions. Finally, Suh used the example $\frac{5}{6}$ of $\frac{13}{5}$ to reinforce

the use of equivalent fractions to take part of $\frac{13}{5}$. This step-by-step sequencing of four

examples allowed Suh to introduce nested bits of mathematics, the step-by-step levels of

mathematics as the task of *taking part of* develops from the simplest task to the more

complicated task requiring the use of equivalent fractions. The examples allowed Suh to

develop the concept of fraction as operator (Lamon, 2006), building from the simplest

relying entirely on the definition of fraction to the more complex requiring the use of

equivalent fractions to allow for the interaction between the operator and operand. In the

cases of Suh's last two examples, these were selected so that the fourth example could

reinforce the third. Generally, sequencing of examples is intended to build from simple to

more complex mathematical situations, revealing mathematical ideas as the examples

progress. Alternately, examples may be sequenced to reinforce mathematical ideas. An

important part of this work is Suh's intentionality. He had a purpose for each example.

He selected and sequenced these to unpack fraction multiplication in a detailed, specific

way. Interchanging examples would not have accomplished his goals. Making examples more varied ($\frac{3}{4}$ of $\frac{16}{7}$ then $\frac{1}{2}$ of $\frac{15}{7}$) might not have piqued the teachers' attention as $\frac{3}{4}$ of $\frac{16}{7}$ and $\frac{3}{4}$ of $\frac{15}{7}$. Changing one component of the problem raised a critical piece of the instruction. That is, it allowed Suh to address multiplication when the denominator of the operator is not a factor of the numerator of the operand. This sequencing of examples fostered the evolution of mathematical ideas and the unpacking of the work of fraction multiplication.

It appears that the work to select examples entails four elements: reviewing instructional goals, attending to criteria for instruction, generating general categories of examples, and sequencing examples.

Work of Managing the Enactment of the Mathematical Tasks Used for Teaching Mathematical Knowledge for Teaching

With the selection of interpretations and representations and selection of examples complete, the lesson is enacted. This begins with an opening segment or launch.

Launching. Launching the task is the conversation starter. In some way, it generates work on or conversation about a piece of mathematics. The launch may take the form of presentation of ideas central to the mathematics. For teachers, it may be a challenge to think more carefully about mathematics they may be accustomed to doing algorithmically, a common misconception, or a piece of children's mathematical work. In any case, the launch is intended to generate work on mathematics. In this particular case, Suh presented a contextual problem. He stated that "in mathematics taking two thirds of a bag of rice means that you divide the bag of rice into three equal parts and you take two"

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(Suh transcript, 070307, p. 1). In his solution, Suh parsed the work to *"take two thirds of"*

into two separate operations: division by 3 and multiplication by 2. It appears that Suh

challenged the teachers to reason through a problem that they might solve using fraction

multiplication. When a teacher asked, "Is this multiplication?" Suh framed his purpose

for the lesson. He noted, "It is multiplication. … You will see that the definition of

fraction is the basic element in everything we do with fractions. … For now, there is no

operation" (Suh transcript, 070307, p. 2). In doing this, Suh laid the groundwork for a

lesson intended to have teachers explore alternative ways to think about fraction

multiplication. The launch provides a starting point at which teachers begin to examine

underlying mathematical concepts for a given piece of mathematics as well as the

approach taken to the mathematics.

 Developing mathematical ideas. The work of the lesson is production of

mathematical ideas and may take several forms. In this case, Suh used the definition of

fraction, the work of *taking part of,* and repeated addition to unpack the mathematics

foundational to fraction multiplication. He presented this mathematics in a lecture format.

He began with the problem $\frac{1}{3}$ of $\frac{6}{5}$. He applied the agreed-upon definition of fraction by

dividing $[0, \frac{6}{5}]$ into 3 equal parts. He stated:

> I divide $[0, \frac{6}{5}]$ into three equal parts. We want one third of this. Divide the
>
> segment into three equal parts. One third is the first partition. The result is $\frac{2}{5}$
>
> (Suh transcription, 070307, p. 1).

Immediately, Suh suggested a second approach to finding $\frac{1}{3}$ of $\frac{6}{5}$ by examining six

copies of the unit fraction. Six fifths is equal to 6 copies of $\frac{1}{5}$. He found one third of those

copies, $\frac{2}{5}$. Suh detailed this process. He noted:

> At the moment, I only deal with one operation. First, think of this as a number on the number line. Then, think of finding one third of the way from zero to six fifths. A second way to think of six fifths is as six copies of one fifth. So if you want a third, you have two copies. So, one third of six fifths is two copies of one fifth or two fifths" (Suh transcription, 070309, p. 2).

Suh presented a third approach. He stated that $\frac{6}{5}$ is the addition of $\frac{2}{5} + \frac{2}{5} + \frac{2}{5}$. Suh

presented these approaches in quick lecture-style fashion.

Developing mathematical ideas includes several sub-elements; parsing, organizing, layering, threading, all done with intentionality. Further, the work of making mathematics visible to teachers includes the elements of explicitness, making mathematical sense and raising teachers' awareness to this mathematical sense -- when to go back to the definition, when to apply equivalent fractions -- and connecting to work of teaching mathematics to children.

Parsing, organizing, layering and threading. First, parsing the task entails examining the task for the mathematical details. For Suh's work with *taking part of*, he explored the operation through the use of the definition of fraction, the notions of *copies of* and concatenation, the repeated addition interpretation of multiplication. Parsing the task for the mathematics within the task allows for the opening up and examination of the mathematics.

Second, organizing these parsed mathematical details allows the mathematics to evolve. Suh referred to this as "layered learning" (Suh transcript, 070307, p. 4). He noted that "you have to start with something you already know. That is the hierarchy in

mathematics. You have one layer -- it is a solid layer. The new layer rests on top of something solid" (Suh transcript, 070307, p. 4). For fraction multiplication, Suh began with the definition of fraction and used unit fraction as a fraction that exemplified the equal partitioning of the whole and location of endpoints of the partitions. He applied the notion of unit fraction by *taking part of* that fraction. Then, Suh developed *taking part of* from work with a unit fraction to non-unit fractions by applying *copies of* to develop the work with non-unit fractions. He offered a supporting way to interpret non-unit fractions as operators by using repeated addition interpretation for multiplication. Organizing the parsed mathematical details provides for the unpacking of compressed mathematics and layering of those details.

Third, threading mathematical concepts fosters connections within a topic and across mathematical domains. Suh shared his contention that "it increases the capacity for learning if [teachers] see a thread running through everything. … [If] you give students the framework then they learn much better in comparison to when you just give them a collection of things with no interconnection" (Suh interview, 071607, p. 4). He acted on this contention by threading mathematical concepts within the topic of fractions by establishing the definition of fraction and reverting to it for work with *taking part of.* Also, Suh's work to thread concepts across mathematical domains is seen in his use of concatenation for addition of whole numbers and fractions. Threading mathematical concepts within and across mathematical domains fosters a coherent network of mathematical ideas.

Finally, parsing the mathematics, organizing it for layered learning, and threading mathematical concepts within and across mathematical domains appear to require

intentionality. This work does not appear to be left to chance. For this case, the work to develop the concept of fraction multiplication entailed studying other mathematics educators' research and work with children and teachers (e.g., Freudenthal, Hart, Huinker, Lamon, and others) and developing lessons that began with the teachers' incoming knowledge, generate the "idea that there is much more to what they know," (Suh interview, 071607, p. 1), and develop teachers' mathematical knowledge into the mathematical knowledge used for teaching mathematics to children.

This analysis examined the work to unpack mathematics to foster teachers' learning of mathematical knowledge for teaching during the enactment of the task. A second piece of this work is raising teachers' awareness of the mathematical knowledge used for teaching. From this case, the work of making mathematics visible to teachers includes explicitness, making mathematical sense and raising teachers' awareness to this mathematical sense -- when to go back to the definition, when to apply equivalent fractions -- and connecting to work of teaching mathematics to children.

Explicitness, making mathematical sense, raising teachers' awareness.

Making underlying mathematics explicit raises teachers' awareness to mathematical details embedded in mathematical work. In this case, there are three instances when Suh stated explicitly the mathematics to which he attended. First, he noted that when *taking part of* a number, for example, $\frac{a}{b} \cdot \frac{c}{d}$ the b and c were important. That is, c had to be a multiple of b. Second, the problem, take $\frac{3}{4}$ of $\frac{15}{7}$ challenged the mathematics used in the previous examples. For this example, use of equivalent fractions was introduced. Suh stated, "What is the issue here? Last time it worked and this time it

doesn't. Sixteen is divisible by four, 15 is not. So the key issue is the 15" (Suh

transcription, 070307, p. 3). Suh made the mathematical distinction explicit. Third, when

using equivalent fractions, he emphasized the "driving force" for teachers' selection of an

appropriate equivalent fraction. Suh stated

> What is written there is correct, $\frac{15}{7} = \frac{15 \cdot 4}{7 \cdot 4}$. You might make it more
> compact by saying that I want to represent this so that the numerator is
> divisible by 4. *That* is the driving force. The other way is correct but it
> does not make mathematical sense. Why not $\frac{15 \cdot 25}{7 \cdot 25}$? Why not $\frac{15 \cdot 82}{7 \cdot 82}$?
> Why not all those things? What do you want? …the numerator to be
> divisible by 4. All you want is that the numerator is divisible by 4 (Suh
> transcript, 070307, p. 4).

Suh's direct statements, "the key issue is" and "*that* is the driving force" alerted teachers

to the mathematical details that were important for the work of taking $\frac{3}{4}$ of $\frac{15}{7}$ and

finding an appropriate equivalent fraction for their work. Making the mathematics

explicit appears to be an element of the work of enacting the task.

Making mathematical sense requires thinking about the task and reasoning about

mathematics rather than applying algorithms mindlessly. In this particular case, Suh

asked the teachers to appeal to mathematical sense making on several occasions. First, he

asked them to put aside fraction multiplication and reason through *taking part of.* He

asked them to make sense of this process. Second, he asked them to make sense of their

work to use *copies* of and repeated addition for *taking* $\frac{3}{4}$ of $\frac{16}{7}$. Third, he pressed the

teachers to determine which equivalent fraction made sense for them to use for the work

to *take* $\frac{3}{4}$ of $\frac{15}{7}$. He asked, "Why not $\frac{15 \cdot 25}{7 \cdot 25}$? Why not $\frac{15 \cdot 82}{7 \cdot 82}$?" He pushed the teachers

to think about the reasonableness of their selection of an equivalent fraction to take

$\frac{3}{4}$ of $\frac{15}{7}$. Throughout the enactment phase of his work, Suh pressed teachers to make

sense of their mathematical work. Enacting the mathematical task entails working in a

way that engages mathematical reasoning and makes mathematical sense of that work.

Connecting the mathematical task to work of teaching mathematics to children

requires considering the mathematics taught to children, the underlying concepts

necessary to support instruction, and ways to work with this mathematics that maintains

mathematical integrity. For this particular case, Suh probed the teachers' understanding

of explanation. He asked, "What does it mean to explain something to someone?" (Suh

transcription, 070307, p. 3). He discussed explanation with respect to the teachers' work

together and then pressed further. Suh asked:

> So now you are not explaining this to us, to yourself. You are explaining this to
> your young charges. I don't care how you do it, but you are only allowed to use
> what they already know (Suh transcription, 070307, p. 3).

At this point of the lesson, Suh asked the teachers to think about their work with children

and to create explanations of equivalent fractions as they would for that work. During an

interview, Suh discussed the use of threading mathematical ideas not just for the teachers'

own learning, but for their use of this skill in their teaching. He noted,

> I think it wears on children that disconnected collection of things that they are
> supposed to get through. If teachers know it and they make the children know it,
> they see that there is a thread that runs through it…they learn it better" (Suh
> interview, 071607, p. 4).

Suh expressed the value of teachers' learning mathematics in a way that developed the

mathematical knowledge they possessed not only for their learning of mathematical

knowledge for teaching but for their use of this knowledge for teaching children.

Connecting the mathematical task to the work of teaching children is an element of the work of enacting the mathematical task.

Engaging teachers in mathematical talk. There are several ways to engage teachers in mathematics. Teachers may participate in whole group or small group discussions about the mathematics. This work might require asking teachers to review lectured material independently or with others. It may mean posing a new problem for teachers to work through. For this case, Suh posed a problem, $\frac{3}{4}$ of $\frac{16}{7}$. He engaged the teachers in two ways. First, he paused the lecture and asked the teachers to solve the problem individually. Second, he worked with individual teachers as they attempted to solve the problem. Upon talking with Lyn, Suh probed Lyn's work by asking him questions that guided Lyn through the steps Suh had produced on the board. Third, Suh asked the teachers to share their answers. The teachers agreed that the answer was $\frac{12}{7}$.

Then, Suh asked the teachers to share their solution processes. During this lesson segment, Suh had the teachers provide their solutions in a recitation manner. However, he did not let the direction of the recitation be subject to chance. He was direct about the use of the point on the number line interpretation for this work. He forced the connection of point on the number line interpretation to fraction multiplication. He prompted the teachers to "[b]egin with the number line. That is a good place to begin. You cannot go wrong" (Suh transcript, 070307, p. 2). Engaging teachers with the mathematics offers them opportunities to think through, talk about, reflect on the mathematics, as well as generate questions when they have difficulty. It requires listening to and responding to

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teachers mathematical ideas as well as formulating ways of talking about mathematics that makes the mathematics accessible to teachers.

Assessing. Monitoring teachers' engagement with the mathematics and their assimilation of the intended mathematics of the lesson entails assessing teachers' mathematical work. This may shape the course of the lesson. For this case, Suh maintained his practice of editing teachers' responses for precision. When Suh asked the teachers to describe the process of taking $\frac{3}{4}$ of $\frac{16}{7}$, Susan suggested that Suh "divide the segments." Suh interrupted and clarified, "You divide the segments between consecutive whole numbers into seven equal parts" (Suh transcript, 070307, p. 2). Suh continued this practice during whole group discussions and when working with individual teachers. The practice of assessing teachers' work provides feedback to teachers as they work to learn the mathematics and informs the mathematics teacher educator as the lesson develops.

Engaging in meta-talk. The mathematical content of mathematics teacher education includes knowledge of mathematical content, knowledge about mathematical practices, and knowledge about teaching mathematics. Stepping out of the work to address the immediate instructional goals and discussing topics that augment the lesson may be an element of the work to engage teachers with mathematics. I name this *meta-talk*. For this case, Suh used meta-talk for two purposes: discussing the structure of mathematical work and mathematical instruction with children. His work to address the disciplinary knowledge of mathematics is evidenced by his commentary on the use of theorems. When discussing the use of equivalent fractions (labeled the Fundamental Fact

of Fraction-Pairs[16]), he asked the teachers to put this aside as they worked to develop

some sense making about fraction multiplication. Suh presented this analogy.

> Do we have to use equivalent fractions? No, use common sense. Suppose for your birthday present I buy you a hammer. Would you go through the house and use the hammer? No, that is what little kids do. When you have a tool you use it when you need it. It is good because you have something to rely on. It is nonsense to say that you use the theorem all the time. It is your extra insurance. You have more ways to deal with a situation (Suh transcript, 070307, p. 2).

In this meta-talk segment, Suh asked the teachers to approach their work with fractions

by reasoning through the problem and applying what they know, to use definitions, and to

think carefully about when they need to apply theorems. Suh's goal to foster teachers'

disciplinary knowledge of mathematics extended to teachers' work with children. A

second meta-talk segment is found in Suh's attention to explanation. He asked the

teachers

> …what does it mean to explain something to somebody? When you say you want to explain something to somebody, you want to explain in terms of something that person already knows. Do you all agree? If I explain something to you using something you do not know about, do you know what I have explained? No, so now you are not explaining this to us, to yourself. You are explaining this to your young charges. Do you think they know how to multiply fractions? …you are only allowed to use what they already know (Suh transcript, 070307, p. 3).

Here Suh pressed on an important piece of mathematical work. He asked the teachers to

think carefully about what explanation means. He noted the importance of beginning with

what the learner knows. He used this to argue his work to examine three ways of

approaching *taking part of* as the beginning work with fraction multiplication. Meta-talk

facilitates work that supports the development of tasks of mathematical knowledge that

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[16] The Fundamental Fact of Fraction-Pairs (FFFP) states: Any two fractions can be denoted by fraction symbols which have the same denominator. Precisely, if the given fractions are $\dfrac{k}{l}$ and $\dfrac{m}{n}$, then they are respectively equal to $\dfrac{kn}{ln}$ and $\dfrac{lm}{ln}$.

supports the learning of mathematics (e.g., the knowledge about mathematics, how mathematicians work) and the teaching of mathematics (e.g., the knowledge of what it means to explain mathematics).

I summarize the elements of the work to selecting interpretations and representations, selecting examples, and managing the enactment of mathematical tasks in extended table is found in Appendix 4.2.

Conclusion

In this chapter, I began to construct the conceptual framework for the work of teaching mathematics to teachers. I proposed three tasks of this work, selecting interpretations and representations, selecting examples, and managing the enactment of the tasks, and justified my selection. This is the skeleton of my framework.

I presented two cases of teaching mathematics to teachers. I analyzed each for the elements of each task. For some elements, there appeared to be more fine-grained sub-elements of that work.

In Chapter 5, I continue my investigation of the work of teacher education by analyzing two additional cases. These cases are taught by a different teacher educator, one whose professional training and experiences vary from the first. The content of the cases are whole number division and decimal multiplication, respectively. The students are prospective teachers. Given these varied qualities, I examine the identified tasks of teacher education for elements entailed by this work and for some, sub-elements of those.

Chapter 5

The Work of Teaching Mathematics to Teachers:
Two Additional Cases

In Chapter 4, I presented two cases of the work of teaching practicing teachers the concept of fraction and fraction multiplication and analyzed each for the work to attend to three tasks of mathematics teacher education selecting of interpretations and representations, selecting examples, and managing the enactment of mathematical tasks for finely grained parts of the work. I wondered whether the teaching of teachers at different stages of their careers, prospective and practicing, by teacher educators from different educational and professional experiences might produce a different set of elements for this work.

In this chapter, I supplement the data and analysis presented in Chapter 4 with two cases of teaching mathematics to *prospective* teachers by a second mathematics teacher educator. These cases are distinctive from the first in several ways. First, the educational and professional backgrounds of the teacher educators are different. Second, the teachers are different, practicing and prospective. Third, the teaching environments differ, one being a summer professional institute and the other an undergraduate mathematics methods course. These varied cases offer insights into the mathematical work of mathematics teacher education and provide an opportunity to look for patterns or differences across the two settings. I focus on the teaching of interpretations of whole number division and decimal multiplication to prospective elementary teachers.

The Teaching of Two Interpretations of Whole Number Division

In the following vignette, I present Ball's teaching of the partitive and measurement interpretations of whole number division. I preface the vignette with an examination of the MMPG planning work.

Overview of the Goals: Interpretations and Applications

The MMPG worked to develop and refine lessons over several years. For the work on whole number division, MMPG partitioned the lesson into three segments. They designed segments on the two interpretations of division, story problems for $38 \div 4$, and meanings of quotients and remainders. Ball and her colleagues intended to examine the partitive and measurement interpretations of division and have the student teachers demonstrate these interpretations; they planned to attend to the mathematical language and notation commonly used for division; and, they sought to have the student teachers practice writing mathematically different story problems and analyzing the story problems for these differences. The class used their story problems to discuss interpretations of division, the contexts for story problems, the nature of quotients, and meanings of remainders.

In the following vignette, I present a brief portion of the work of teaching interpretations of whole number division. The narrative of the teaching of $38 \div 4$ is found in Appendix 5.1. I foreground the emphasis on interpretations and representations of whole number division and attempt to make visible the work to manage the task so that the mathematical details emerge from the student teachers' discussion.

Interpretations and Representations of Division

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We drop into Ball's mathematics methods class at the beginning of a lesson on interpretations of whole number division. Ball had asked the student teachers to work in pairs using colored tiles to represent the partitive and measurement interpretations of division using 12 ÷ 6 as the division fact.

Engaging with teachers and their mathematical work. Ball circulated around the room observing and posing questions about the representations and the role each number played in the representations. Ball asked Sharon to describe her representation. Sharon's response, "two groups of six", described a representation of the multiplication of 2 x 6. Ball adjusted her question focusing Sharon on the division of 12 ÷ 6. She probed to determine whether Sharon could identify the role of six in each representation. When a precise description was offered, Ball drew Sharon's attention to her use of correct language of partitive division and affirmed Sharon's work by stating, "Do you see that? You are using partitive language."

Ball shifted her attention to another student teacher, Brian, who was appeared to be struggling. Ball questioned Brian to learn about the nature of his difficulties.

Ball: What do you have?

Brian: 12 divided by 6.

Ball: So how does that relate to that? [Ball points to the two representations Brian had made.] How does that relate to 12 divided by 6?

Brian: That is what I am confused about. (Ball transcript, 111406, p. 1)

When Ball arrived at Brian's table, Brian had created several tile configurations. Ball focused on two. Brian conveyed his confusion about *how* the representations portrayed the two interpretations of division and the role of the divisor, six. To help Brian, Ball studied his work and identified two correct representations. She asked Brian

to think about the representations, sort out the role of six in each, and identify one as partitive and the other as measurement. Then, she left him to work independently.

Teachers' attention to interpretations and representations. After several minutes, Ball transitioned to a whole class discussion. Sharon and Brandy volunteered to present their work. They reproduced their representations with magnetic color tiles on the chalk board. Ball asked the teachers to explain how their arrays represented 12 divided by 6 and to focus on the role the number "6" was playing in the problem. See figures 5.1 and 5.2 for replications of the tile arrangements.

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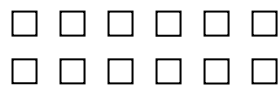
Figure 5.1. Sharon's representation of 12 ÷ 6.

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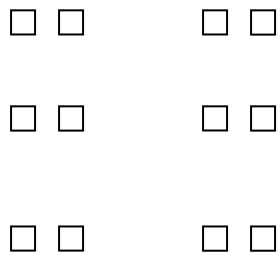
Figure 5.2. Brandy's representation of 12 ÷ 6.

Sharon traced each row of six tiles and noted that she had two groups of six tiles. When Ball asked Sharon to describe her representation, Sharon noted that 12 divided by 6 is 2. She confirmed this by stating, "We have 12 total and 2 groups" (Ball transcript, 111408, p. 1). Ball posed a question continuing the conversation about the representation.

She mediated the conversation to move between Brian and Sharon. After much

discussion, Sharon tried to clarify that she intended to create the *groups* as the *rows* in her

arrangement of the tiles. Ball noted that the conversation surfaced an important aspect of

teaching. That is, representations must depict clearly the mathematics *and* this must be

apparent to the learner. Ball continued the conversation to emphasize this point.

Ball: So explain how you see six in each group. That is what Brian is questioning.

Sharon: Oh, the row. Is that what you mean? I was thinking about the row.

Ball: Okay, so I think the important thing to see is that Brian might be making a suggestion about the row. It helps to push them together. But what you have is a group of six objects and you've made two groups of six objects. There is nothing wrong mathematically with what Sharon did the first time. Does that make sense? *It is getting clear about what kind of comment we make to each other is important. Do you see that?* (Ball transcript, 111408, p. 2)

In this interaction, it appears that Ball attempted to analyze the student teachers'

representations of 12 ÷ 6 and their conversations to explain the representations. Then,

Ball asked Sharon to explain her representation for a second time. Ball questioned Brian

to determine if he understood Sharon's work. The student teachers continued discussing

the representation and clarifying misunderstandings that Brian had raised. Ball managed

this conversation through a set of questions probing the student teachers' thinking about

the representations. She attempted to make visible the representations' details.

Ball refocused the discussion asking Brandy to explain her work. Brandy

presented 12 ÷ 6 as six groups of two. Then Ball discussed the student teachers'

explanation of division.

Ball: So there you have 6 groups. Can somebody explain what Brandy just said? Ellie?

Ellie: You have 12 and you want to make separate groups. So she made separate groups. And there are 2 in each group.

Ball: What kind of groups? *Separate* groups? Is that what you said? Did you say *separate* groups?

Ball repeated Ellie's description of separate groups and attempted to refine the student teachers' language around an important aspect of division. She arranged the square tiles in groups with different numbers of tiles in each group. She asked the student teachers if the groups she had arranged were acceptable. After the student teachers decided that the representations were inappropriate for the problem, a student teacher, Mary asked a question of one of her peers.

Mary: So how did you decide that was the right way? Obviously that is the right answer, but how did you know?

Ball: Good question.

Brandy: I guess that just making six groups. And then we want to see how many will go into each group if they are equal. [Brandy "deals out" the 12 tiles into 6 groups one at a time as she talks.]

Ball: Nice Brandy. So that makes clear that she makes 6 groups. But theoretically she might not have known how much she would end up with. Good question and a good answer.

Ball allowed the teachers to accept Brandy's representation. However, Ball pursued an important part of the work with division, that of equal groups. Although Brandy's representation included six groups with two tiles in each group, the fact that the groups contained equal numbers of tiles was not explicit. Using the word "separate" did not make that important detail clear. It appeared that Ball sought to raise the student teachers' awareness to this detail. She challenged the description of "separate groups." She rearranged the groups to include 6 groups of separate but unequal numbers of tiles that added to 12 tiles. Then she asked the teachers to revise their description of division.

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After Sharon's and Brandy's representations were reviewed, Ball asked the teachers to

analyze the representations to identify similarities and differences. Nikole began the

conversation by stating that each representation included 12 tiles.

Ball: They both have 12, okay. Anything else?

Charrese: The answer is 2 in each.

Ball: How can you see the result of 2 in each?

Charrese: Sharon has groups of 6 and there are 2 groups of 6. Brandy has 6 groups
             and there are 2 in each group.

Ball: Okay so that is the beginning of moving to what is different. Charrese is
             moving us toward that. Jenna?

Jenna: Sharon is looking for how many groups of 6. And Brandy is looking for 6
             groups, how many are in each group.

Ball: You caught yourself. What were you about to say? That is interesting. It
             is interesting to think about it because you want to use the 6 both times.
             So Sharon is making groups of 6 and Brandy is making 6 groups. What
             else is different? Do you two see anything else that is different besides
             what has been said? Anyone else? Okay, so do you want to talk a bit about
             what we call these and the notation we use? (Ball transcript, 111406, p. 4)

Ball acknowledged Jenna's correct response and restated the critical details of

Sharon's and Brandy's representations. "Sharon is making groups of 6 and Brandy is

making 6 groups" (Ball transcript, 111406, p. 4). She ended the conversation about the

representations and transitioned to a discussion about the vocabulary used specifically for

division and the two interpretations of division commonly found in school curricula.

In this vignette Ball and student teachers represented and discussed the

interpretations of whole number division. It appeared that this work allowed important

mathematical aspects of teaching whole number division to emerge such as the

importance of depicting and presenting representations that clearly portray the

mathematics and using precise language to describe the mathematics. I step away from the presentation of Ball's teaching of whole number division to analyze the work of selecting interpretations and representations, selecting examples, and managing the enactment of mathematical tasks.

Analysis of Tasks of Teaching Two Interpretations of Whole Number Division

Consistent with the cases discussed in Chapter 4, the work of teaching interpretations of whole number division appeared to entail careful consideration of interpretations and representations, selecting examples, and managing enactment of mathematical tasks.

Work of Selecting Interpretations and Representations Used for Teaching Mathematical Knowledge for Teaching

It appears that Ball's work to select interpretations and representations for whole number division entailed several elements: surveying, deliberating, and deciding.

Surveying. Selecting interpretations and representations begins with gathering those reasonable for teaching mathematical knowledge for teaching. Surveying interpretations and representations of whole number division entailed gathering interpretations and representations for both division and multiplication that supported teaching this content to teachers. Also, selecting representations required surveying those that supported interpretations.

Deliberating. Once interpretations and representations have been gathered, deliberating about which are most helpful for instruction involves several elements. Initially, reviewing instructional goals makes visible intended mathematical content of the lesson. Parsing this content reveals underlying mathematical ideas intended for exploration and study. For this particular case, an examination of partitive and

measurement division to raise teachers' awareness of the distinctions between the two and to develop teachers' abilities to create and discuss models were the instructional goals. Because of the specificity of the goal, these interpretations were necessary for instruction. Representations for these interpretations might have included a variety of materials or drawn pictures. Deliberating about representations includes considering those used for other operations, specifically multiplication, and how these representations foster connections between prior mathematical work and the mathematics of this lesson.

Deciding. The final selection of interpretations entails weighing the instructional goals of the lesson as well as the course, and choosing interpretations that foster those goals. In this particular case, Ball noted that

> the big emphasis in the course is that operations have different interpretations. … Teachers need to be able to make sure that they are being diverse enough in what they offer kids so that kids are getting a sense of meaning. They have to be clear about representations they are using (Ball interview, 112806, pp.5-6).

The selection of colored tiles to create representations of the partitive and measurement involved deciding on representations that fostered the instructional goal of work with the selected interpretations of whole number division and linking these interpretations to a fourth grade mathematics lesson from *Everyday Mathematics*. Ball and her colleagues suggested that student teachers use base ten blocks or color tiles (MMPG lesson plan, week 11, page 2). These data indicate that the choice of using color tiles may have been an in-the-moment decision.

The work of selecting interpretations and representations appears to involve gathering interpretations and representations appropriate for the instructional goals, deliberating about the use of these for the lesson and their connectedness to other lessons and the overarching course goals, and deciding on those that fostered the course and

lesson goals. Use of interpretations and representations for instruction is supported by

selected examples.

Work of Selecting Examples Used for Teaching Mathematical Knowledge for Teaching

Interpretations and representations are intended to foster instructional goals,

unpacking detailed mathematical ideas used for teaching mathematics children. However,

interpretations and representations do not work in isolation. Examples support the

teaching of mathematical knowledge for teaching and are an integral component of the

lesson. The work of selecting examples for teaching mathematical knowledge for

teaching appears to include several elements.

Reviewing. Selecting examples entails reviewing instructional goals. For this

particular case, unpacking the meaning of the partitive and measurement interpretations

of whole number division was paramount. During an interview, Ball noted that

> $35 \div 5$ can look two different ways. What is going on there is the divisor functions
> to do two different things. The dividend is the same every time. …It has to do
> with the divisor, the relation to what the divisor is doing to the dividend and what
> they contribute to the answer. …With division, we are trying to challenge and
> solidify [student teachers'] understanding of division (Ball interview, 112806, p.
> 6).

Reviewing instructional goals establishes the mathematics which examples must unpack

in concert with selected interpretations and representations.

*Checking. S*electing examples requires that these examples be usable for

instruction. For the case of whole number division, the dividend, divisor, and quotient

had to be varied enough so that the visual representation did not confound the divisor and

quotient. For the example of $12 \div 6$, partitive division resulted in six groups of two.

Measurement division resulted in six tiles in two groups. The variation between two and

six was visible. An example like $9 \div 3$ would not have made the distinctions between the

groups and the sizes of each group visible. For $9 \div 3$, the partitive division would have resulted in three groups each consisting of three tiles. Measurement division would have resulted in partitioning the nine tiles into three groups of three tiles. The equality of the divisor and quotient would not make the distinction between the divisor and quotient visible. Also, an example such as $10 \div 3$ would have generated a remainder which would have been premature at this stage of examining interpretations of whole number division and the roles of dividend, divisor, and quotient. A second consideration was the usability of the examples. For this particular lesson, numbers that were small, yet different enough suited the instructional purposes and required manageable numbers of tiles.

Sequencing. For some instructional goals like the work to develop the point on the number line interpretation of fraction, layered learning may be fostered by the sequencing of examples. However, for this particular lesson, the detailed understanding of the partitive and measurement interpretations of whole number was the focus. Consequently, reversing the divisor and quotient in the example $12 \div 2$, challenged the teachers to recreate representations with two as the divisor and discuss carefully their representations. Choosing any other example would not have presented the same challenge. That is, by interchanging the roles of six and two, the student teachers were challenged to focus on the role of the divisor. When six was the divisor, the partitive interpretation asks, "Given 12 tiles, how many tiles are divided equally into six groups?" The measurement interpretation asks, "Given 12 tiles, how many groups of six tiles are there?" Interchanging the roles of six and two, the student teachers are asked to think and speak carefully. When two was the divisor, the partitive interpretation asks, "Given 12 tiles, how many tiles are divided equally into two groups?" The measurement

interpretation asks, "Given 12 tiles, how many groups of two tiles are there?" Having student teachers create representations of a different problem, say 12 divided by three would have generated quotients of 4 and the comparison between the roles of divisor and quotient would not have produced the exchange of numerical value for the roles. Although the exchange of 2 for 6 was intended to challenge teachers to distinguish between divisors and quotients, this may not be a choice for children for exactly that reason. Such a choice might be confusing for new learners.

Ball concluded the lesson with the example $8 \div b$. This example moved the student teachers to think abstractly about the divisor. This appeared to foster precise language to describe the interpretations. The selection of three examples for this lesson, $12 \div 6$, $12 \div 2$, and $8 \div b$, appeared to foster to the instructional goal of unpacking the partitive and measurement interpretations, making visible the role of the divisor for each, and providing an exercise in which the student teachers thought and spoke carefully about the interpretations of division, and abstracted division using a variable as divisor.

Creating in-the-moment examples. Ball generated an example not in the lesson plan to respond to Ellie's definition of division as being of "separate groups." For this demonstration, Ball represented six groups with varying numbers of tiles in the groups, the sum being 12. This example made visible the inadequacy of the language of "separate groups" and motivated the more precise language of "equal groups."

For this case, the work of selecting examples appeared to include reviewing instructional goals, selecting examples that were usable for teaching to these goals, sequencing examples to support instructional goals which were the careful exposition of partitive and measurement interpretations and the roles of dividend, divisor, and quotient

for these interpretations. Although there is no evidence of a repository of several additional examples, Ball's interview provided evidence that she generated examples for this work in-the-moment. Her example of $35 \div 5$ during that interview indicated that she has specific criteria for these examples.

The Work of Managing the Enactment of Mathematical Tasks Used for Teaching Mathematical Knowledge for Teaching

Launching. The task is introduced during the launch. For this particular case, Ball posed the task, "remember that in the lesson you read there were two different interpretations of division. Use the color tiles in front of you to model this problem [$12 \div 6$] in these two different ways" (MMPG field notes, week 11, p. 1). During this launch the teacher educator asked the student teachers to use information from curriculum materials they reviewed as an assignment to model the division of $12 \div 6$. It linked the use of these materials used in some field placements to the mathematics lesson. This work became the focus of the later discussion.

Making mathematical ideas visible. The instructional goals for this lesson segment include careful attention to the partitive and measurement interpretations of whole number division along with representations that could facilitate examining the different roles of the divisor in each interpretation. Making these mathematical ideas visible may be accomplished in several ways. For this lesson segment, mathematical ideas were introduced to the student teachers through their reading of a fourth grade lesson on whole number division. Based on this reading, student teachers produced their initial understandings by creating models of the division problem $12 \div 6$ in two different ways. Probing these initial understandings, Ball interacted with individuals and pairs of student teachers to discuss their models and to answer questions. Finally, samples of

student teachers' work were presented providing examples for class discussion. For this particular case, mathematical ideas were introduced; initial understandings were modeled and discussed; samples were displayed, presented, and probed. The work of making mathematical ideas visible included several phases of work and, at each phase, engaging student teachers with the mathematics appeared to provide student teachers opportunities to develop mathematical ideas from initial to more developed understandings.

Engaging student teachers in conversation about mathematics. A means of making mathematical ideas visible in this particular case appeared to go beyond "getting the mathematics out there" via lecture to engaging the student teachers in individual and paired work to read about, model, and discuss the whole number division interpretations. Engaging student teachers in conversation took on three forms: engaging individual student teachers in conversations, engaging individual students in presentations of mathematical work, and managing large group discussions.

First, the student teachers modeled their perceptions of the interpretations and discussed these with a partner. As Ball circulated around the room, she paused to question individuals. Her interaction with Sharon took on two forms: initial questioning and deeper probing. Pointing to Sharon's work, Ball asked, "What is this?" When Sharon responded, "two groups of six," Ball confirmed Sharon's response and asked her to think more analytically about her response. Ball asked Sharon to "relate [the model] back to the $12 \div 6$. In this representation, what does the six represent?" Ball continued to press Sharon to think more analytically about the representations and to connect the representations with the interpretations. She concluded by *explicitly* drawing Sharon's attention to her use of partitive language. Thus, in this interaction, the work of engaging

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the student teacher in conversation about her models involved initial questioning, prompting her to make connections between representation and interpretations, and raising her awareness to use of language.

Ball's interaction with a second student revealed additional elements of this type of engagement. Brian built several models and when asked to identify which modeled the partitive and measurement interpretations, he stated that he was confused and "not sure how to think about the numbers." Ball scaffolded the work. She identified correct representations. She identified a potential error when she asked, "you want to call that 12 ÷ 2, right?" Then, rather than providing an explanation, Ball asked Brian to use the information she had provided to think about the representations and attempt to explain why the models represented 12 ÷ 6 and the role the divisor played for each representation. Creating a task that Ball appeared to find manageable for Brian provided him with a starting point, identification of a potential error, and a prompt to work towards the connections between interpretations and representations.

During these exchanges, Ball engaged student teachers with mathematics to affirm or disaffirm student teachers' thinking, move their thinking into a more analytic space, make a mathematical point visible, and scaffold their work to help them analyze and connect the interpretations and representations.

Second, student teachers presented their work to their classmates to provide mathematical work for discussion. In this particular case, Sharon and Brandy created representations with magnetic colored tiles and described these to the student teachers. Brian continued to puzzle over the representations, noting that he did not *see* how they depicted the partitive and measurement interpretations. After much probing, Sharon

clarified that for the partitive interpretation, the rows of tiles represented the *groups*. This statement made clear how groups had been depicted. It allowed Brian to *see* how the representation modeled the partitive interpretation. It made visible a critical component of the work of teaching. That is, this exchange raised the student teachers' awareness to the importance of creating representations that were clear to students, not just to themselves. This work required analyzing student teachers' representations for the mathematical details needed to model the partitive and measurement interpretations. Further, it entailed providing support to student teachers as they thought through and communicated their understandings about the representations and interpretations.

Third, managing whole group conversation entailed selecting presenters who might offer mathematical work that would support constructive conversation, posing questions to make visible the mathematical work, and orchestrating conversation among student teachers in a way that fostered exchange of ideas among them. Prominent in this exchange was the repetition of descriptions and explanations. With each iteration, details were added to prior descriptions which eventually manifested a solution to Brian's dilemma. Also, student teachers' definitions were probed and refined as with Ellie's response that Brandy had created "separate groups." Ball applied Ellie's definition, demonstrating its lack of precision and motivating refinement from "separate groups" to "equal groups." Managing discussion, Ball orchestrated interaction among student teachers and affirmed both questions and answers that generated mathematical ideas. As with Mary's question about the creation of the example that revealed the weakness of Ellie's use of "separate groups," Ball affirmed the question and had Brandy respond, affirming her response as well. This orchestration of conversation among the student

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teachers fostered exchanges of ideas, provided opportunities for the teacher educator to

listen and respond to student teachers' mathematical talk, affirming or modifying when

appropriate. This is evident in the excerpt between Mary and Brandy.

Mary: So how did you decide that was the right way? Obviously that is the right
           answer, but how did you know?

Ball: Good question.

Brandy: I guess that just making six groups. And then we want to see how many
           will go into each group if they are equal. [Brandy "deals out" the 12 tiles
           into 6 groups one at a time as she talks.]

Ball: Nice Brandy. So that makes clear that she makes 6 groups. But
           theoretically she might not have known how much she would end up with.
           Good question and a good answer.

Fourth, Ball provided time for the student teachers to summarize their work in

their notebooks allowing them opportunities to sum up and reflect on their shared

mathematical work, reinforcing their understandings and raising questions about ideas

that were not yet formulated.

Assessing. Ongoing evaluation of student teachers' mathematical work provides

information about their developing mathematical knowledge for teaching. Evidence of

Ball's assessing student teachers' mathematical work is seen across this lesson segment.

Ball assessed and responded to Sharon's and Brian's representations of the partitive and

measurement interpretations; monitored the discussion of Sharon's and Brandy's public

work, assessing the representations and language used to describe the representations;

and, assessed Mary's questions and Brandy's response, affirming the important nature of

the questions and correctness of the response. Assessing student teachers' mathematical

work maintained the correctness of the work and it guided the instruction, informing

instructional decisions.

I step away from the analysis of the work to teach student teachers the interpretations of whole number division and examine one final case, the case of teaching decimal multiplication. This case provides evidence of the fine-grained work to select interpretations and representations. It offers an extensive set of examples used to unpack several layers of mathematical knowledge compressed within the algorithm for decimal multiplication. And, the case demonstrates a multifaceted approach to engage teachers with mathematics. I summarize elements of this work in appendix 5.3. I turn next to present the case of the teaching of decimal multiplication.

The Teaching of Interpretations of Decimal Multiplication

The following vignette describes a class session during which Ball modeled decimal multiplication. Just prior to this lesson, she worked with the teachers to make visible mathematical details of the 10 x 10 grid. For the lesson on decimal multiplication, Ball used the 10 x 10 grid to represent both the repeated addition and area interpretations of multiplication and to explain the mathematical characteristics of this operation. Ball introduced the lesson by modeling the decimal multiplication 0.7 x 0.1 using the area model of multiplication. For this model, the product of 0.7 x 0.1 is by definition the area of the rectangle having side measures of 0.7 and 0.1.

Introducing the Concept of Decimal Multiplication

Launching the task. Ball launched this lesson segment by asking the student teachers to multiply 0.7 x 0.1 in their notebooks. After they performed the calculation, Ball used the 10 x 10 grid representation to model the multiplication and to justify the product. First, she demonstrated how the factors were represented by the grid. She guided her finger along the vertical edge of the grid until she traced the edge of seven small

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squares in an attempt to represent 0.7. Then she traced the horizontal edge of the upper

left small square in an attempt to represent 0.1. She traced the perimeter of the resulting

rectangle and designated it as the rectangular region for 0.7 x 0.1. She counted the

squares that comprised the resulting area and stated that because the region was made up

of seven small squares, the product of 0.7 x 0.1 was seven hundredths. She reviewed the

work with the student teachers.

Ball: So I know from what we just did [in deploying the 10 x 10 grid] that each one of these segments is one tenth. I need seven tenths. One, two, three, four, five, six, seven. So the side length of this rectangle has a side length of seven tenths. Can you see that okay? Do you see where I get seven tenths? And I want the width of my rectangle to be one tenth. And I know that one of these has a side length of one tenth. So now I have made the rectangle with two sides. I can complete it by drawing the additional two sides of the rectangle. Now I have a rectangle that is seven tenths by one tenth. Okay? Do you see my rectangle? What is the area of my rectangle? What is the area of my rectangle? Sarah?

Sarah: Seven hundredths.

Ball: Seven hundredths?

Sarah: There are seven little boxes and each little box has area of one hundredth. So the area is seven hundredths.

Ball: So there are seven little squares. There are one hundred small squares in this drawing and we agreed earlier that each small square has an area of one hundredth. We have heard a couple of different ways to explain that. One way is that it is one hundredth of the whole. And Jenna said that it is one tenth times one tenth which is one hundredth. So here we have one, two, three, four, five, six, seven hundredths. So my problem seven tenths times one tenth equals seven hundredths. Why don't you do the problem seven tenths times one tenth as a written problem in your notebook right now? Not as a fraction, but as a decimal. So point seven times point one. What is the answer that you get?

In this segment, Ball attempted to make visible the rectangular region that

represents 0.7 x 0.1 and to engage the student teachers in drawing the representation and

using the representation to find the product of 0.7 x 0.1. She asked the student teachers to

find the product and to explain their answers. A student teacher, Sarah, offered the answer of seven-hundredths. She explained that the product is found by counting the number of "boxes" in the rectangle. She identified the value of each "box" as one one-hundredth and stated that the resulting area of seven hundredths. Ball restated Sarah's response, editing it to include agreed upon language. Further, she added to Sarah's response by including the justification for one small square representing one hundredth.

Ball asked the student teachers to reproduce their co-created work of the multiplication of 0.7 x 0.1 in their notebooks. She walked around the room and appeared to assess the student teachers' thinking by reading their written work. She spoke with individual student teachers, probing their understanding of decimal multiplication and the use of the 10 x 10 grid. When she asked the student teachers about the product, one responded that the answer was "point zero seven." Ball appeared dissatisfied with the response. She asked the student teachers for the answer using correct mathematical language. A student teacher responded, "seven hundredths." Ball checked that the teachers were in agreement.

Ball: Does every one have that? So computationally you know the way to get
                that answer. Now you have that if you have a rectangle with side length of
                seven tenths and a side length of one tenth, the area is seven hundredths
                which fits with what you came up with from your calculation. (Ball
                transcript, 110706, p. 3)

During this brief commentary, Ball attempted to connect the calculation of the product to the representation on the grid. She challenged the teachers to map the computation to the representation. She asked them to map seven tenths to a side length of seven small squares along the vertical edge and one tenth to a side length of one small square along the horizontal edge. Then, she asked them to map the product of seven

hundredths to the area of the rectangle made of seven small squares. This work required that student teachers map carefully between the two representations of this mathematics, the calculation and the grid.

Ball paused to check the student teachers' understanding. She asked if they understood why the product was seven hundredths. One teacher, Shannon, reflected back to how she multiplied decimals as a student and asked why her strategy was not correct.

Shannon: I haven't done this in so long. Like the point zero seven. As a student, how do you know? I was taught to move the decimal point the number of places to the, I don't know. Since it is point seven times point one, why not point seven? Point seven times point one, then it is point seven. (Ball transcript, 110706, p. 3)

Ball's work to map between the representations appeared to make visible a common error for decimal operations. That is, the addition of decimal algorithm, lining up the place values and adding place value by place value is applied to decimal multiplication. Shannon raised an important issue. That is, she and many student teachers had not multiplied decimals in a long time. And for many, they may not have done this operation without the use of a calculator. Also, Ball's attempt to deploy the 10 x 10 grid representation and use it to explain the product of 0.7 x 0.1 appeared to have generated the teachers' *thinking* about the standard algorithm for multiplication of decimals.

Ball launched the lesson with the problem: Multiply 0.7 x 0.1. With this task, a common error was made visible. Confusion about the placement of the decimal point in the product of two decimal numbers became the focus of the lesson. With this accomplished, Ball began a carefully orchestrated lesson on decimal multiplication. She began with whole number multiplication, an operation the class had studied previously. Then, she segued to an examination of decimal multiplication.

Revealing place value in decimal multiplication layer by layer. Ball noted that student teachers like Shannon have used algorithms to perform calculations without understanding why these algorithms work. Consequently, she thinks it important that student teachers come to understand the mathematical underpinnings of algorithms as well as other mathematics that are foundational to the mathematics that teachers teach children (Ball interview, 112808). Ball sought to provide a mathematical explanation for the placement of the decimal point in the product of two decimal numerals. She selected a series of multiplication problems beginning with the multiplication of two whole numbers, 7 x 1 and ending with the multiplication of 0.7 x 0.5 to accomplish this goal. She worked through this series of problems demonstrating the multiplication with the 10 x 10 grid. Further, she attempted to demonstrate the multiplication using both the repeated addition and area interpretations of the operation. Table 5.1 summarizes the multiplication problems Ball used to unpack mathematical knowledge for teaching decimal multiplication.

Multiplication Problem	Repeated Addition	Area
7 x 1	Modeled	Not modeled
7 x 0.1	Modeled	Modeled
0.7 x 0.1	Modeled	Modeled
0.7 x 0.5	Modeled	Modeled

Table 5.1. Summary of multiplication problems and interpretations used by Ball to scaffold the work for decimal multiplication.

Layered examples to teach decimal multiplication. Ball and the student teachers had used the area interpretation of multiplication and the 10 x 10 grid to discuss decimal

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multiplication. They sought to confirm the placement of the decimal point in products where one tenth is multiplied by one tenth using the 10 x 10 grid. In the following vignette, Ball proposed a second explanation for the decimal multiplication algorithm. She used the repeated addition interpretation of multiplication to find products and corroborate results of problems done with the area interpretation.

Beginning with what the teachers know: whole number multiplication. Ball presented a multiplication problem beginning with whole number multiplication. She asked the student teachers to represent 7 x 1 using the repeated addition interpretation in their notebooks. The student teachers drew seven units in their notebooks. They wrote 7 x 1 = 1 + 1 + 1 + 1 + 1 + 1 + 1. See figure 5.3 for the replication of Ball's work.

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Repeated Addition Interpretation

☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐

7 x 1 = 1 + 1 + 1 + 1 + 1 + 1 + 1

_____

Figure 5.3. The repeated addition interpretation of 7 x 1.

With this review of whole number multiplication complete, Ball adjusted the mathematical nature of the multiplication by making one factor a decimal. She asked the teachers to multiply 7 times 0.1. She began with the operand and identified the one tenth as one column in the 10 x 10 grid. She shaded that column and designated it as one tenth. Then she used the repeated addition interpretation of multiplication. She counted off seven of the columns in an effort to represent seven one-tenths of the grid. She shaded the region. See figure 5.4.

■ represents 0.1　　　 ▇ represents 0.6　　　 total shaded region represents 0.7

Figure 5.4. Ball's representation of the repeated addition interpretation of 7 x 0.1 = 0.7.

Ball paused to allow the student teachers ample time to complete their work. The student teachers recorded the multiplication of 7 x 0.1 in their notebooks. When the student teachers seemed to be in agreement with this representation and the answer to this calculation, Ball presented an example with decimals as both factors.

Ball demonstrated the multiplication of 0.7 x 0.1. She began by shading in one tenth of the 10 x 10 grid. She attempted to demonstrate that the multiplication of one tenth by seven tenths is different from multiplying by any whole number. To do this, Ball showed that one tenth (the column representing one tenth of the 10 x 10 grid) is partitioned into 10 equal small squares. Seven of these small squares represent seven tenths of the column that represented one tenth. She counted seven small squares and noted that the product was 0.07. See figure 5.5 for a replication of Ball's drawing.

| | 0.7 x 0.1 = 0.07 | | entire shaded region is 0.1 |

Figure 5.5. Ball's representation of the repeated addition interpretation of 0.7 x 0.1 = 0.07.

Ball confirmed the product of 0.07 by presenting the area interpretation of the multiplication of seven tenths times one tenth. She drew the rectangle for this problem and identified a vertical length of seven tenths and a horizontal length of one tenth. She identified the area of the resulting rectangle as the product of 0.7 x 0.1. She reasoned that since seven small squares were shaded and each small square represented one hundredth, the product was seven hundredths. To designate the region that represented the product, Ball made a bold outline of the rectangle and placed a small circle in each small square counting the number of small squares as she progressed. See figure 5.6 for Ball's representation.
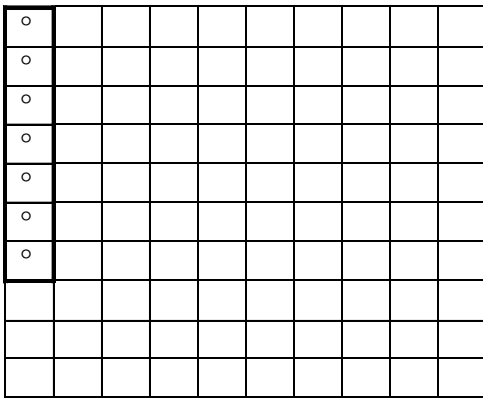
Figure 5.6. Ball's representation of the area interpretation of 0.7 x 0.1 = 0.07.

Ball examined the multiplication of 0.7 x 0.5 to exemplify a product with two non-zero digits to the right of the decimal point. Then, she discussed the roles of operator and operand for decimal multiplication. Because of the length of this vignette, I pause the presentation. The complete vignette is in appendix 5.3.

Analysis of Tasks of Teaching Interpretations of Decimal Multiplication

The decimal multiplication lesson included attention to two interpretations of multiplication and the use of the 10 x 10 grid to represent this operation. In the following section, I analyze the work of teaching this lesson for the elements of attending to the tasks of teacher education I focus on for this investigation.

Work of Selecting Interpretations and Representations Used for Teaching Mathematical Knowledge for Teaching

Surveying. Gathering all interpretations and representations that may be reasonable for the work of teaching mathematics to teachers involves a survey of interpretations and representations that apply to the concepts of the lesson, those that challenge teachers' prior knowledge and offer them a broader perspective about concepts used for teaching. For this particular case, collecting interpretations and representations

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for both multiplication and decimals was necessary. The work of selecting interpretations and representations begins with gathering those reasonable for teaching teachers providing a panoramic view of possibilities.

Deliberating. Weighing which interpretations and representations support the mathematics of the lesson entails examining several elements of work. First, examining the instructional goals exposes the mathematics to be addressed in the lesson. For this particular case, the goals were to model multiplication of decimals and build a correspondence between the arithmetic and geometric representations. These goals were supported by the objectives: (1) develop understanding of multiplication as producing an area where multipliers are the lengths of two sides of a rectangle; and, (2) work to make careful and explicit correspondences between representations (MMPG lesson plan, class 10, 2006). In addition, Ball confirmed products using the area and repeated addition interpretations for each example; she made explicit the roles of operator and operand.

Deliberating representations included considering the usability of materials to model the representations. Although base 10 blocks and 10 x 10 grids suit this purpose, 10 x 10 grids were more usable for instruction. That is, the 10 x 10 grid was easily reproduced for projection and use in student teachers' notebooks. These grids could be outlined or shaded to demonstrate factors and products without damaging costly materials. The 10 x 10 grids attended to the geometric aspect of instruction *and* were usable for instruction. Ball discussed a complication with base 10 blocks. She noted that student teachers often lay out base 10 blocks to form the factors. Then, they create a rectangle using the blocks with the dimensions of the factors. Ball noted

> you build an area model for three tenths times seven tenths. They run smack into the problem. They will know the answer is twenty-one hundredths. And they will

not be counting the side length. They will use three little blocks to build the side length for the three tenths. That's what they will be counting. … So you get twenty-one hundredths. First you call them tenths. And now you call them hundredths. You change them from one second to the next. What is going on? (Ball interview, 102406, p. 6).

It appeared that using the grid prevented the use of area representations for factors. However, student teachers had to be reminded that the factors were represented by the side lengths rather than the squares that formed the perimeter of the rectangle. Considering how interpretations and representations attended to instructional goals and how the interpretations and representations fostered conceptions and misconceptions involves attending to the details of the interpretations and representations.

Likewise, considering the repeated addition, area, Cartesian product, or tree diagram interpretations for multiplication included the study of which of these were suitable for work with decimals. This eliminated the Cartesian product or tree diagram interpretations. Of further importance was considering the connections among past and future mathematical work. The repeated addition and area interpretations had been emphasized for instruction on whole number multiplication. Ball noted that "we tend to fore ground the repeated addition and area models and background the Cartesian product. … Those two characterize what they really need to know" (Ball interview, 102406, p. 1). Maintaining both of these interpretations for decimal multiplication demonstrated a connectedness between whole number and decimal operations. Also, each interpretation fostered different mathematical concepts which MMPG embraced. The area interpretation facilitated the connections between arithmetic and geometric representations. The repeated addition interpretation supported the distinctive roles of operator and operand (MMPG debriefing field notes, 110706, p. 3). Ball and her

colleagues hypothesized that by using both interpretations of multiplication would provide the student teachers with opportunities to view factors and products differently. They saw these multiple experiences as opportunities for the student teachers to develop a deeper understanding of multiplication (MMPG debriefing field notes, 110706, p. 3). Considering interpretations and representations require careful deliberation about instructional goals, characteristics of interpretations and representations, how these support each other and contribute to developing teachers' mathematical knowledge for its use for teaching.

Deciding. Determining how interpretations and representations support instructional goals is an element in deciding which interpretations and representations are right for instruction. These are criteria for the selection or elimination. For this particular lesson, the goal to engage with both the arithmetic and geometric representations of decimal multiplication required the use of a geometric representation. Selection of the 10 x 10 grid was made because of its geometric properties and its usability for teaching. Mathematically, the representation makes use of the properties of a geometric representation, a square, to justify a numeric property (MMPG lesson plan, class 10, Fall 2006). Further, deciding to use *both* the area and repeated addition interpretations of multiplication offered two ways to think about multiplication, provided opportunities for products to be checked using an alternative interpretations, and afforded the opportunity to examine the roles of operator and operand. Deciding on interpretations and representations cemented the instructional terrain.

An alternative selection. Selecting interpretations and representations influences the trajectory of the lesson. To demonstrate this, suppose that Ball and her colleagues

chose to use decimal fractions as an alternate representation. If these were selected,

converting decimals to decimal fractions would be the first step in the work. For this

example, 0.7 equals $\dfrac{7}{10}$ and 0.1 equals $\dfrac{1}{10}$. Both denominators are 10^1. When the

fractions are multiplied, the product of the denominators is 10^{1+1} which is equal to 10^2.

To multiply the fractions, $\dfrac{7}{10^1} \times \dfrac{1}{10^1} = \dfrac{7 \times 1}{10^1 \times 10^1} = \dfrac{7}{10^2}$. This representation makes visible

the work with exponents in the denominators. It parallels the addition of the factors'

place values to the right of the decimal point in the standard algorithm. That is, suppose

Ball chose to have the student teachers change the decimals to fractions and perform the

computation. The product would be: $\dfrac{7}{10} \times \dfrac{1}{10} = \dfrac{7}{100}$. Choosing this instructional

trajectory makes two connections to other mathematical content. First, the connection

between decimals and fraction representations is made by expressing the decimals as

fractions. Second, the laws of exponents are applied and the connection between the

powers of 10 in the denominators and the algorithm for placement of the decimal point in

the product may be made. However, it does not connect the algorithm to a visual

representation and the geometric justification for the product of these decimals, a

productive way to help teachers unpack the reasoning behind the algorithm.

The two representations are different and have the potential of unpacking

mathematical knowledge for teaching. On the one hand, the use of the visual

representations may provide opportunities to connect decimal multiplication with

geometric representations, their attributes, and language associated with these. It provides

a mathematical justification for the placement of the decimal point in products for

decimal multiplication. The use of the 10 x 10 grid supports the work of multiplication and decimals.

Also, the use of fraction representations may provide opportunities to connect fractions and decimals and to apply laws of exponents to the algorithms for decimal multiplication. However, this strategy assumes that the learner understands fraction multiplication and laws of exponents. There are mathematical concepts that remain compressed in this alternative approach to teaching decimal multiplication. The decision to use either representation creates very different learning opportunities.

Work of Selecting Examples Used for Teaching Mathematical Knowledge for Teaching

Just as interpretations and representations reveal mathematical concepts or underpinnings of procedures, examples are intended to support the interpretations and representations and unpack mathematical ideas. The number of layers of mathematical detail might be determined by the examples that are selected and used. Examples working in concert with interpretations and representations may determine the mathematics that becomes available to the learner.

The work of selecting examples entails several stages. Identifying the mathematical ideas within the instructional goals, determining compressed mathematics to be unpacked, assessing the limitations of interpretations and representations, and organizing or sequencing these in a way that allows the mathematics to evolve throughout the lesson all guide the selection of examples. For this particular case, Ball had a pedagogical goal for examples. She designed an example to challenge the student teachers' algorithmic knowledge of decimal multiplication. Table 5.2 summarizes the

intended mathematics to be addressed by each example.

Multiplication Problem	Area	Repeated Addition
7 x 1	Review whole number multiplication	Not modeled
7 x 0.1	Introduce decimal as factor	Introduce decimal as operand. Confirm product using area interpretation.
0.7 x 0.1	Challenge student teachers' understanding of place value. Why not 0.7 x 0.1 = 0.7? Product has more decimal places than each factor. Define what it means to take 0.7 of something.	Introduce decimal as operator. Confirm product using area interpretation.
0.7 x 0.5	Product has two non-zero digits to the right of the decimal point.	Product has two non-zero digits to the right of the decimal point. Confirm product using area interpretation.

Table 5.2. Summary of examples and instructional goals used by Ball to scaffold the work for decimal multiplication.

Reviewing. The mathematical details of instructional goals are reviewed raising an awareness of mathematical details which examples need to make visible. For the case of decimal multiplication, examples were used to unpack multiplication as an area and repeated addition, to facilitate the careful demonstration of the correspondences between area representations and products, to develop explanations for the relationship between the decimal values for factors and products, and to make visible distinctions between operators and operands (MMPG lesson plan 10, 2006). Reviewing instructional goals to identify compressed mathematical ideas is an initial step in selecting examples.

Fitting. Assessing the affordances and limitations of interpretations and representations guides selection of examples. For this particular case, selection of the 10 x 10 grid as the geometric representation of decimals *and* the definition of the 10 x 10 grid as "one" determined potential values. The side length of the 10 x 10 grid square was taken to be the unit of length, and the total 10 x 10 grid square was taken to be the unit of area. Likewise, the side lengths and areas of the small squares were determined to be one tenth and one hundredth respectively. These dimensions limited examples to those having one decimal place of length, and two of area. It appeared that Ball and her colleagues also restricted selection to numbers for which the products were between zero and one (except the product of 7 x 1, where different units were used). This restriction avoided the need to use more than one grid for each computation. Assessing the affordances and limitations of representations and interpretations entails consideration of numbers appropriate for those and numbers that do not generate unnecessary complications.

Sequencing. Organizing the examples permits the unpacking of the mathematics and the evolution of the mathematics as the examples are revealed. For this case, examples were used to raise questions about student teachers' incoming misconceptions (0.7 x 0.1), generate an explanation for the addition of digits to the right of the decimal point in the factors to arrive at the number of digits to the right of the decimal point in the product using the geometry of the grid (7 x 0.1, 0.7 x 0.1, and 0.7 x 0.5), confirm the products and explore the roles of the operator and operand (repeat examples using repeated addition interpretation). Using the same digits (7 and 1) in the first three examples minimized variation and focused attention on place value for decimal multiplication. Organizing the examples allows for a step-by-step unpacking of the

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decimal place concepts for decimal multiplication and the roles of the operator and operand.

Selecting examples requires reviewing instructional goals, assessing selected interpretations and representations for their affordances and limitations, creating examples to unpack the mathematical details, and sequencing examples to permit evolution of mathematical ideas.

Work of Managing the Enactment of Mathematical Tasks Used for Teaching Mathematical Knowledge for Teaching

Launching. The mathematical work begins with the launch of the lesson. Launching may serve several purposes. In this particular case, Ball asked the student teachers to multiply 0.7 x 0.1. She appeared to work towards two goals: modeling a description of decimal multiplication and justification for the place value of the product *and* raising the student teachers' awareness of a common misconception. First, Ball proceeded with a detailed demonstration using the area interpretation of multiplication. In doing so, she reviewed the geometric details of the 10 x 10 grid. She modeled a precise demonstration of a description for this process. She justified the product. In the launch, it appeared that Ball established the level of detail, the process of demonstrating, and the language which she and the student teachers would use for this work. Second, she challenged the student teachers' understanding of decimal multiplication. Shannon asked, "Since it is point seven times point one, why not point seven?" (Ball transcript, 110706, p. 3). The careful mapping of the geometric representation to the arithmetic procedure generated a common misconception. Launching the mathematical task establishes the mathematical content and the expectations for the work. It may begin with student

teachers' incoming knowledge and initiate work to develop that knowledge to mathematical knowledge for teaching.

Developing mathematical ideas. Presenting the lesson entails decisions about modes of engaging student teachers with mathematics. Various modes of engagement include lectures that provide direct delivery of mathematics, dialogue between the teacher educator and student teachers, student work time that permits student teachers opportunities to engage with mathematics either individually or with other student teachers, or student presentation of content that provides student teachers opportunities to discuss their mathematical work. For this particular case, Ball presented the lesson using a variety of modes of engagement.

Lecturing is one form of engaging teachers with the mathematics. For this case, Ball lectured as a means of delivering content and modeling use of materials, language, and representations. Throughout the case of decimal multiplication, Ball used pre-drawn representations to present visual representations for her work. She simultaneously described where the factors were located on the grid, outlining the edges of the grid, emphasizing the linear nature while she drew the representations of the factors on the 10 x 10 grid. Then, Ball described how the product was represented by the squares comprising the rectangular region. She justified the product by counting the number of small squares in the region and designated a value, the product to that region. Lecturing allows for attention to mathematical details and modeling mathematical work.

Maintaining a dialogue about mathematics with student teachers is a second mode of engaging them with mathematics. After presenting an example, Ball asked the student teachers to either provide a product with an explanation or reproduce her work. For the

problem 0.7 x 0.1, Ball asked, "What is the area of my rectangle?" (Ball transcript, 110706, p. 3). Sarah's response of seven hundredths caused Ball to probe for an explanation for this product. Upon hearing Sarah's explanation that "there are seven little boxes and each little box has area of one hundredth" (Ball transcript, 110706, p. 3), Ball restated Sarah's explanation changing the use of "boxes" to the agreed upon term "squares." Ball reminded the teachers that the measure of one hundredth for each small square was the result of work they had co-produced. She continued to discuss the ways the group had produced that result. Maintaining a dialogue provides a vehicle by which student teachers become part of the mathematical conversation. It is an opportunity for student teachers to share their understandings and practice using mathematical language; it is an opportunity for the mathematics teacher educator to clarify language and understandings as well as review prior work.

Pausing the work of the entire class provided time for student teachers to work individually or in small groups. For this particular case, Ball provided opportunities to student teachers to record each example in their notebooks and to summarize key mathematical ideas. These opportunities allowed the student teachers to reproduce the example, write about the mathematical work, and question and formulate understandings. From the perspective of the mathematics teacher educator, it provided Ball time to walk around the classroom, checking teachers' work, answering questions, posing questions, and assessing teachers' learning. Pausing the class for individual and group work allows student teachers opportunities to think through and write about the mathematics of the lesson. Also, it provides the teacher educator with opportunities to interact with individual student teachers, to answer questions, and to assess their learning.

Parsing, organizing, layering, connecting. Making the mathematics of the lesson visible involves parsing, organizing, and layering the mathematical details using instructional decisions made intentionally for work with student teachers. Also, the work of making mathematics visible to student teachers includes explicitness (tracing, emphasizing, repetition), and moving among several instructional modes.

The work to enact the lesson on decimal multiplication involved parsing the multiplication in two ways, the number of decimal factors and the interpretation of multiplication. Introducing decimal factors one at a time allowed for examination of the effect of each factor on the geometric representation and numeric product. Using both area and repeated addition interpretations of multiplication allowed for attention to place value and the work of the operator and operand when working with decimal multiplication.

Organizing tasks to raise student teachers' awareness of fragile or faulty understandings, unpack mathematical ideas foundational to mathematical knowledge for teaching children, reinforce or confirm findings from a prior task shapes the evolution of the lesson. For this case, Ball raised student teachers' awareness to a common misconception involving the place value of the product of two decimals. With this misconception raised at the launch, examples were organized to elicit several products and their geometric representations to justify the addition of factors' place values to determine the place value of the product. Finally, the work to multiply using repeated addition confirmed the products using area and made visible the roles of the operator and operand. This organization was intentional. Ball shared that

> when we talk about the repeated addition model we are effectively letting areas be
> the things counted. …We really want to pay attention to [the areas] as discrete

items. So when we say three tenths of a group of five tenths. Five tenths is an area. We are cutting three tenths of the area of that. But we are actually trying to get them to think closer to the repeated addition model where there is addition of three fives. So that is kind of, not wrong, but it is like we are hiding something from the students on purpose to come up more saliently. And we chose to do the area model first. So we did the area model first because we have had trouble. The main problem is getting [the student teachers] to attend carefully to the notion that the factors are linear and that the product is an area" (Ball interview 112806, p. 3)

Organizing tasks may promote the generation of a need for greater understanding of mathematics and the evolution of mathematical ideas to foster this understanding.

Connecting mathematical ideas within and across domains fosters notions that mathematics is a science of interconnected and developing concepts, skills, and ideas among many domains. For this case, Ball attempted to make explicit the connections between the arithmetic and geometric meanings of 0.7 x 0.1. She demonstrated this in two ways. First, she identified one tenth as one column on the 10 x 10 grid. Then, she used the definition of seven tenths or seven parts out of 10 equally partitioned parts of a unit. Using this definition, Ball took seven tenths of one column of the grid or seven small squares. The result was seven hundredths. By using the representation of seven tenths, Ball performed this multiplication. She then used the area interpretation of multiplication to confirm the product. Ball demonstrated that the resulting area was equal to the region found by using the definition of seven tenths. Also, she connected the area of the resulting region with the product of 0.7 x 0.1. Ball applied the area and repeated addition interpretations of multiplication across both whole number and decimal fractions. When asked during an interview about connecting decimal multiplication to whole number multiplication, Ball responded, "To me it is all multiplication. …I want them to know it well enough so that they can do it with all place value representations of numbers" (Ball interview, 112806, p. 3). Demonstrating mathematical connections within

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and across mathematical domains fosters knowledge of the relationships within and across mathematical domains and promotes the vision that mathematics is an interconnected discipline as opposed to a set of isolated facts and skills.

Making the mathematics of a lesson visible entails parsing the mathematics for its underlying concepts, presenting these as a well-organized set of tasks, and connecting mathematical ideas within and across mathematical domains. Along with unpacking fundamental mathematical ideas, presenting mathematics to student teachers entails raising student teachers' awareness of this fundamental mathematics.

Raising awareness of compressed mathematical ideas. Raising student teachers' awareness of mathematical knowledge that is foundational to the mathematics they learned as children is found in the work of managing the enactment of tasks. In this case, two techniques were used to generate this awareness. First, as noted above, Ball's question, "What is 0.7 x 0.1?" challenged Shannon's knowledge of decimal multiplication and generated her question about the place value of the product of two decimals. This question presented an opportunity to pursue a multi-layered exposition of decimal multiplication using several examples and two interpretations of multiplication. Ball noted that her goal was to have the student teachers "know [multiplication] well enough so that they can do it with all place value representations" (Ball interview, 112806, p. 3). Second, explicit instruction or making mathematics visible to student teachers provided opportunities for their increased awareness of mathematical knowledge needed for teaching. It involved challenging their mathematical knowledge (0.7 x 0.1), modeling ways of describing, explaining, and justifying mathematics for instruction, and emphasizing the precision at which mathematics is spoken and demonstrated. Alerting

teachers to compressed mathematics within school mathematics is an element of the work of managing the enactment of mathematical tasks.

Assessing. Managing the enactment of the task requires monitoring student teachers' mathematical work. Assessing this work occurs during individual and group work as well as whole group discussion. Assessing entails several elements of work. First, assessing entails listening to, hearing, and modifying teachers' conversation about mathematics. For this case, a student teacher's statement that the product of seven tenths times one tenth was "point zero seven" (Ball transcript, 110706, p.3) prompted Ball's probe and elicitation of the response "seven hundredths" (Ball transcript, 110706, p. 3). Second, assessing entails probing teachers to make public their misunderstandings. When asking teachers if they understood why the product was seven hundredths, Shannon shared that she was confused.

Shannon: I haven't done this in so long. Like the point zero seven. As a student, how do you know? I was taught to move the decimal point the number of places to the, I don't know. Since it is point seven times point one, why not point seven? Point seven times point one, then it is point seven.
             (Ball transcript, 110706, p. 3)

Probing student teachers' understandings facilitates their verbalizing concerns and confusions. Assessing involves making teachers' understandings and misunderstandings visible by eliciting their ideas, and listening to, hearing, and modifying teachers' mathematical work.

Managing the enactment of mathematical tasks for teaching mathematical knowledge for teaching entails launching, developing mathematical ideas, engaging student teachers in conversation about mathematics, and assessing student teachers' evolving mathematical knowledge. In this section, I have analyzed the work of selecting

interpretations and representations, selecting examples, and enacting a lesson on decimal multiplication for the elements of this work. I summarize these findings in appendix 5.2.

Conclusion

In Chapter 5, I presented two cases of teaching mathematics for teaching to student teachers, two interpretations of whole number division and interpretations of decimal multiplication and analyzed each case for the elements of the work of selecting interpretations and representations, selecting examples, and managing the enactment of mathematical tasks entailed by the work of teaching mathematical knowledge for teaching.

In Chapter 6, I take my analysis further. I examine the four cases of teaching teachers mathematics used for teaching children to expose elements of this work and to present a framework from which further investigation can begin. I analyze the mathematical knowledge entailed by the work of teaching mathematical knowledge for teaching and make claims about a domain of mathematical knowledge entailed by this work.

Chapter 6

Mathematical Work and Knowledge Demands of Teacher Education

In Chapters 4 and 5, I presented a provisional framework for the work of teaching mathematics to teachers and I analyzed four cases of teaching mathematics to teachers, both prospective and practicing. From the analyses of these cases, it appears that the mathematical work of teaching teachers mathematics entails several specific tasks. I begin this chapter by situating my work within current research on teacher education and elaborate two important findings, the work and mathematical knowledge entailments of teacher education. I synthesize my findings and present a framework for the study of the work of teacher education. Subsequently, I look across my analysis and identify types of mathematical knowledge entailed by this work. I propose a domain of mathematical knowledge distinctive to the work of teacher education that appears to be entailed by the teaching in these data. I begin this chapter by situating my investigation.

Of the many tasks of teacher education, I focused on three: (1) selecting interpretations and representations, (2) selecting examples, and (3) managing the enactment of mathematical tasks all for the purpose of teaching mathematical knowledge for teaching. My analysis revealed elements that appeared to be central to the work of each task. Some elements were visible in each case; others appeared to be specific to one or two. This fine grained analysis is comparable to the study of functions when mathematics students zoom in on small intervals to analyze the behavior of local maxima

or minima and find interesting functional behavior. This fine grained analysis promoted a detailed vision of the three selected tasks and permitted an identification of the mathematical knowledge entailed by this work.

Another part of functional analysis is zooming out, or studying the behavior of a function across its domain. Similarly, a broader examination of the mathematical work of teacher education might be the examination of the *purposes* of teacher education as they relate to this dissertation. Teacher education is focused on teaching teachers to *think* and *know* like a teacher (Feiman-Nemser, 2008, p. 698).[17] The work of teacher education includes transitioning peoples' ways of thinking about and knowing mathematics from that of students to that of teachers. Teacher education is a career-long process with a plethora of opportunities for developing mathematical knowledge into mathematical knowledge for teaching as depicted in figure 6.1.

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┌─────────────────┐                              ┌─────────────────┐
│  Mathematical   │────────────────────────────▶ │  Mathematical   │
│   knowledge     │                              │  knowledge for  │
│                 │      Teacher Education       │    teaching     │
└─────────────────┘                              └─────────────────┘
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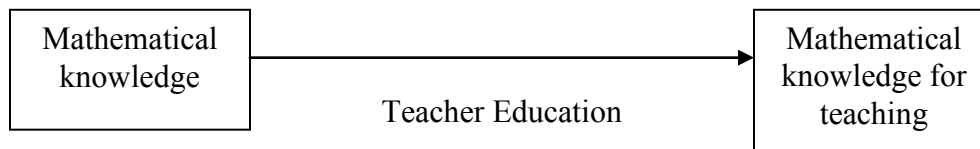
Figure 6.1. One purpose of teacher education is to foster the development of mathematical knowledge for one's own use to mathematical knowledge for teaching.

It is important to notice that the major purpose of the work of teacher education is beginning with people who already know some mathematics and developing that knowledge into mathematical knowledge for teaching. This development begins with challenging pre-service teachers' mathematical knowledge to identify misconceptions

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[17] A discussion of the purposes of teacher education is found in Chapter 2.

and fragile understandings (Ball, 1988) and shoring up this mathematical knowledge. Then, mathematical knowledge for teaching is developed to address four criteria for knowing mathematics for this work. First, mathematical knowledge for teaching fosters the decompression or unpacking of mathematical concepts, skills, and procedures. Second, it enables connections of mathematical ideas within and across mathematical domains. Third, mathematical knowledge for teaching supports the use of language that maintains mathematical integrity and is usable with children. Fourth, this mathematical knowledge fosters the use of practices germane to the discipline of mathematics (Ball & Bass, 2003b). The work of teacher education extends mathematical knowledge and transitions it into mathematical knowledge that is more unpacked, connected, articulate, and grounded in mathematical practices – *mathematical knowledge for teaching*.

Across the four cases, the teaching is focused on developing mathematical knowledge into mathematical knowledge for teaching. Some instructional goals addressed the actual mathematics that teachers teach; others focused on knowledge of mathematics that informs and shapes their teaching; and, still others made explicit knowledge about mathematical ways of work.

This work of developing mathematical knowledge for teaching is different from the work of teaching mathematics to children in at least three ways. First, unlike teachers who have spent many years learning mathematics and come to teacher education with some knowledge of mathematics, children have had fewer formal opportunities to learn mathematics and may come to school with at best tacit mathematical understandings. The work to teach children mathematics involves developing these understandings into mathematical proficiency (NRC, 2001). The work to teach teachers involves shoring up

mathematical knowledge and developing it into mathematical knowledge for teaching. Second, the content is different. Whereas, teachers teach mathematics, teacher educators teach mathematical knowledge for teaching. And third, the purposes are different. Whereas, children's mathematical proficiency is for their use as children continue through school or in their daily lives, teachers use mathematical knowledge for teaching to teach children mathematics. It is mathematical knowledge beyond that taught in school, unpacked for this special work. I claim that there appears to be a unique domain of mathematical knowledge entailed by this work. This being an initial empirical study of the mathematical knowledge entailed by the work of teacher education, my claims are intended to lay the groundwork for additional study into the mathematical work and knowledge demands of the work of teacher education.

In this chapter, I synthesize the case analyses and make claims about the three tasks of teacher education on which this investigation is focused. Then, I address the question: What is the mathematical knowledge entailed by these tasks? From my analyses, I have identified a distinctive domain of mathematical knowledge used for the teaching of the cases in my data – mathematical knowledge for teaching teachers (MKTT). This distinctive mathematical knowledge appears to be specialized knowledge of the subject matter sub-domains of mathematical knowledge for teaching. I provide evidence from the cases to support this claim and elaborate the distinctive qualities of MKTT.

Cross-Case Analysis of the Mathematical Work of Teaching
Mathematical Knowledge for Teaching

Next I turn to a discussion of the mathematical work of teaching teachers. Throughout this discussion I am trying to emphasize that the identified tasks may be

common with other mathematics teaching. However, the students, content, and purposes of teaching mathematical knowledge for teaching are distinctive in many ways. I begin with a cross case analysis of the elements that appear to comprise the work of selecting interpretations and representations for teaching mathematical knowledge for teaching.

The Work of Selecting Interpretations and Representations Used for Teaching Mathematical Knowledge for Teaching

My data provided evidence that teacher educators use interpretations and representations to unpack details compressed within mathematical concepts and procedures. Selecting interpretations and representations appeared to entail gathering those usable for teaching mathematical knowledge for teaching and reviewing mathematical ideas compressed within concepts; deliberating the mathematical details interpretations and representations fostered, limitations of their use, and usability for teaching; and deciding which interpretations and representations developed targeted mathematical knowledge for teaching These data indicate that the work of selecting interpretations and representations was focused specifically on the work of teaching mathematical knowledge for teaching. Some were chosen as they are typically known and used by teachers; others were chosen as transitional representations -- representations meant to help teachers modify their incoming mathematical ideas; and, others were selected to unpack mathematical ideas developing mathematical knowledge into mathematical knowledge for teaching.

Surveying. Selecting interpretations and representations entails surveying those appropriate for teaching concepts and operations in ways that unpack mathematical details. Across the cases, surveying appeared to be an initial gathering of those that might be usable for teaching mathematical knowledge for teaching. Evidence from Suh's

monographs indicate that his work to collect interpretations and representations involved a search of the literature on teaching mathematics to children and teachers, surveying mathematics for elementary teachers' texts, and examining school curricula. In Ball's work, too, it appears that interpretations and representations used for previous instruction or research were gathered. Once gathered, reviewing them for the mathematical ideas they support and targeted mathematical ideas within instructional goals provided information on which decisions could be made. Across all cases, surveying interpretations and representations – gathering and reviewing - appeared to be an initial element of the work of selecting interpretations and representations.

Deliberating. Candidates for interpretations and representations appeared to be analyzed on several criteria. First was the mathematical content itself. It entailed examining instructional goals and parsing these for mathematical details. Deliberating entailed studying interpretations and representations to see how they work together to reveal mathematical ideas; examining mathematical content for one particular lesson *and* looking across many lessons to determine the feasibility of threading interpretations and representations across lessons; and examining the mathematical trajectory which interpretations and representations supported and the intended mathematical terrain to determine a fit. Second, deliberating involved attention to the nature of the learners. For the work of teaching mathematical knowledge for teaching, deliberating required considering knowledge of the mathematics in school curricula, attending to the ways teachers learn, and the mathematical details to be unpacked. It required examination of the development of mathematical ideas to determine a pathway from mathematical knowledge to mathematical knowledge for teaching. In particular, evidence of this micro-

analysis is found in Suh's use of three interpretations of fraction, each interpretation chosen to develop mathematical knowledge for teaching. Third, deliberating required considering the usability of interpretations and representations for instruction. This includes examining how representations could be used for presentations and individual teacher use, the fit on projection devices, ease of use, and availability of materials to create representations (e.g., Ball's selection of the 10 x 10 grid, overhead transparencies for whole group presentation and drawings for student teachers' notebook). Fourth, deliberating included considering the amount of mathematical content (span of school years) and level of detail (fine-grained detail to unpack mathematical ideas) that are compressed in a brief instructional period. Deliberating appeared to involve considering mathematical details that foster mathematical knowledge for teaching, overarching trajectory of course goals, needs of teachers as learners, usability for instruction, and available instruction time.

The final selection of interpretations and representations appeared to require weighing several factors and making a final decision. Deciding on interpretations and representations appeared to involve attention to the mathematical entailments of immediate and long-term instructional goals and usability with teachers. Suh's selection of three interpretations and his choice *not* to use fractions as ordered pairs and Ball's selection of repeated addition and area interpretations *not* Cartesian product emphasized the crafting of selections for teaching mathematical knowledge for teaching. Deciding on interpretations and representations involved weighing how candidates address mathematical and pedagogical considerations of teaching the specialized content and selecting those that best suit this work.

The Work of Selecting Examples Used for Teaching Mathematical Knowledge for Teaching

Examples were selected to support and make visible the mathematics embedded within the chosen interpretations and representations. For one case, selected examples were used to scrutinize interpretations of division. For three cases, intricate sets of examples were used to unpack in a layered manner many mathematical details. Some examples drew attention to minute details such as the coordination between representing factors and the dimensions of the rectangle, a skill important to teaching; other sets of examples moved teachers through many years of school mathematics, zooming in to expose details of one example, fast forwarding to another school year, and refocusing on details of an example. This movement between examining fine-grained mathematical details in one example, moving quickly across many years in the same domain, refocusing and examining minute details in another example appeared across cases of multiplication and fractions. The selection of examples appeared to attend to the needs of special learners, teachers. They made visible teachers' fragile understandings and misconceptions; they unpacked mathematical details used for teaching but not taught to children; they fostered a vision of content taught across several grades. The work to select examples in these cases appeared to entail reviewing instructional goals, identifying categories of examples, fitting examples to interpretations and representations, sequencing examples, reviewing criteria for use, and developing repositories of examples. I continue with a synthesis of the analyses of the work to select examples beginning with reviewing instructional goals.

Reviewing. These cases indicate that instructional goals appeared to guide the mathematical work of the lesson. In each case, a piece of mathematical knowledge was

identified and developed by unpacking details about that mathematics. In each case, it appears that instructional goals targeted fostering mathematical knowledge into a more detailed, robust mathematical knowledge for teaching. The decimal multiplication lesson exemplifies a lesson laden with instructional goals (e.g., unpacking place value, using two interpretations of multiplication to justify an algorithm, and demonstrating the roles of operators and operands) that were addressed through a series of well-designed, layered examples. Instructional goals appeared to shape the selection of examples.

Identifying. Examples were used in each case and each example targeted a mathematical detail important to an operation or algorithm. Identifying each mathematical detail of the mathematical task allowed for creation of examples to unpack mathematical details. In addition, identifying mathematical details supported the categorization of examples (e.g., Suh's use of $\frac{1}{3} \cdot \frac{6}{5}$ to unpack *taking part of* or Ball's use of 0.7×0.1 to challenge place value of a product). Examples have identifiable mathematical characteristics that appeared to permit an unpacking of the underlying mathematics of the lesson. Identifying mathematical details permitted the development of examples and the identification of categories of examples for further development.

Fitting. With the mathematical details identified, attention to the affordances and limitations of interpretations and representations demand careful selection of examples. As an example, the 10x 10 grid selected by Ball as the geometric representation of decimal multiplication limited factors to having at most one decimal place. This required the *fitting* of examples to this representation. Assessing the affordances and limitations of representations and interpretations entails consideration of numbers that make visible the

intended mathematical ideas, are appropriate for the interpretations and representations that do not generate unnecessary complications.

Sequencing. When multiple layers of mathematical detail are compressed within a concept, operation, or algorithm, examples were selected and sequenced to unpack the mathematics layer by layer. Suh's sequencing of examples was intended to make visible the role of the operator in the fraction multiplication case. Careful sequencing three of examples (e.g., *taking part of,* concatenating parts, and using equivalent fractions) unpacked fraction multiplication intended for use in teaching children.

Further, when several instructional goals are the foci of the lesson, examples appeared to be sequenced to teach to these goals in a way that provided order and meaning to the lesson. As an example, Ball's sequenced examples to unpack place value, to use two interpretations of multiplication to corroborate findings, and to manifest the role of operator and operand. Sequencing examples appeared to be an important element of the work of selecting examples intended to foster mathematical knowledge into a more developed mathematical knowledge for teaching.

Checking. Instructional goals, appropriateness for teaching mathematical knowledge for teaching, and usability for instruction appeared to be criteria for selection of examples. Checking examples for each of these criteria ensures the development of mathematical ideas throughout the lesson. Instructional goals appeared to focus on unpacking mathematical details of operations and algorithms, mathematical ideas important for teaching. Attending to these goals, goals designed specifically for teaching special learners, teachers, examples appeared to be selected to foster the development of mathematical knowledge for teaching. Examples challenged mathematical knowledge,

fostered mathematical knowledge used for teaching, *and* made explicit mathematical details, practices, and structures. Also, examples had to be usable. That is, numbers had to be selected that made details visible (e.g., different enough to make the roles of divisor and dividend visible); they had to be selected to be usable for instruction (e.g., big enough to make the work visible, but small enough to use manageable numbers of materials); they had to be selected to be usable for teaching mathematical knowledge for teaching (e.g., carefully selected to make visible compressed understandings— "Are factors the measure of the edge of a rectangle or the border row of squares?") or provoke misconceptions (Why not 0.1 x 0.7 = 0.7?). Examples are checked to determine whether they meet the criteria of supporting instructional goals, meeting the needs of teaching mathematical knowledge for teaching, and being usable for instruction.

Developing repositories. The identification of mathematical details and categorizing examples based on these details appeared to allow for the creation of repositories of examples. These repositories might exist in written form as in Suh's monographs published for the professional development institute or in non-written form as the examples generated by Ball to modify a student teacher's definition of division. Repositories of examples appeared to be prepared and ready for teacher educator use for each case.

Selecting interpretations and representations and selecting examples appeared to be integral to the development of lessons for fostering mathematical knowledge into mathematical knowledge for teaching. In the next section, I provide a cross case analysis of managing the enactment of the mathematical tasks for the four cases. As with the prior

two tasks, I highlight aspects of the work that appear to be distinctive to the teaching of mathematical knowledge for teaching.

Managing the Enactment of Mathematical Tasks Used for Teaching Mathematical Knowledge for Teaching

Across the cases, managing the enactment of mathematical tasks aimed at developing mathematical knowledge into mathematical knowledge for teaching appeared to entail launching, developing mathematical ideas, engaging teachers in mathematical conversation, assessing and meta-talk.

Launching. Engaging teachers with mathematics often began with an introduction or launch. In the data, launching appeared to have several purposes: reviewing mathematics, challenging mathematical knowledge, modeling ways of mathematical work, or presenting new mathematical ideas. Launching sometimes re-introduced mathematics from previous lessons or established a base for extending that mathematical content. It might entail posing a question or problem that challenged the teachers' mathematical knowledge. Launching sometimes involved presenting unfamiliar content for discussion. Launching seemed to be used to establish the mathematical trajectory of the lesson.

Across each case, launching contextualized the mathematical knowledge *within the work of teaching.* For the whole number division case, the launch engaged student teachers in creating representations of two interpretations of division based on those used in school curriculum materials. In the remaining cases, the launch engaged teachers in a mathematical task that provoked an error, misconception, or weakness in mathematical knowledge. The data indicate that launching mathematical tasks in teacher education situates the task within the context of teaching children or begins with mathematical

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knowledge. In each case, the launch appeared to direct the lesson toward the development

of mathematical knowledge for teaching.

Developing mathematical ideas. Studying the ways mathematical ideas were

developed required close analysis of the interactions among the teacher educator and

teachers. Across the cases, mathematical ideas were developed beginning with

mathematical knowledge of the tasks. Table 6.1 summarizes incoming mathematical

knowledge and targeted mathematical knowledge for teaching in each case.

	Concept of fraction	Fraction multiplication	Decimal multiplication	Whole number division
Anticipated incoming knowledge	Fraction as part of the whole	Algorithmic knowledge of fraction multiplication	Algorithmic knowledge of decimal multiplication (sometimes erred)	Two interpretations of whole number division (sometimes fragile)
Targeted mathematical knowledge for teaching	Fraction as point on the number line	Taking part of	Justification for algorithm	Representations of two interpretations
		Repeated addition	Connections between geometric and algebraic representations	Roles of divisors for each interpretation
		Concatenation	Connections between repeated addition and area interpretations	
		Connections across three interpretations	Roles of operator and operand	

Table 6.1. Summary of anticipated incoming mathematical knowledge and targeted
mathematical knowledge for each case.

In these cases, teacher educators' work to develop mathematical ideas appeared to take on two pedagogical strategies: directed, lecture-style teaching and guided, interactive teaching. For the concept of fraction, fraction multiplication and decimal multiplication cases, the lessons were directive. Lectures appeared to support the mathematical work of unpacking mathematical details of the mathematical tasks, supporting correct understandings, dispelling misconceptions, and fostering mathematical knowledge for teaching. The instructors' use of tasks appeared to be parsed into mathematical details that were developed in a carefully sequenced set of examples revealing the mathematics layer by layer. The whole number division case appeared different. For this case, student teachers' representations of two interpretations of division were analyzed and discussed at the individual level and in whole group discussion. Student teachers' individual work, interaction with Ball, and whole group discussion appeared to be used to develop a fine-grained knowledge of whole number division. Directed, lecture style lessons and student teacher focused, interactive lessons appeared to begin with student teachers' mathematical knowledge and work with it to repair misunderstandings, strengthen fragile understandings, and unpack compressed mathematical knowledge.

Engaging teachers in conversation about mathematics. The teacher educators appeared to use four ways of engaging teachers in conversations about mathematics in these cases. First, paired or small group conversations about mathematics (e.g., student teachers' work on interpretations of division) demonstrated one type of mathematical conversation used in these cases. Second, while in small groups, student teachers appeared to engage in conversation about mathematics as teacher educators listened. At times, the teacher educator appeared to strengthen and reinforce correct mathematical

conversation and probe, scaffold, and redirect that which was weak or incorrect. In particular, Ball's appeared to press for detailed attention to interpretations of division. She examined Sharon's work probing the response and making visible the mathematics in the student teacher's response. She questioned Brian to learn the nature of his misunderstanding seemingly to move his thinking forward. Questioning, responding, and further probing of student teachers' productions were evidence of this type of conversation about mathematics. Third, it appeared that recitation style exchanges between teachers and teacher educators were used to develop and reinforce mathematical definitions and concepts as demonstrated by Suh's work to reinforce the point on the number line interpretation of fraction and Ball's work with interpretations of division. Fourth, it appeared that student teachers' presentations of their mathematical work (e.g., Sharon and Brandy's demonstration of whole number division representations) provided objects for class discussion. Throughout the cases, conversations engaged teachers with developing mathematical ideas.

Teacher educators' work to engage teachers in mathematical conversations appeared to involve special ways of management. Several elements of this work were visible. First, the work appeared to require the listening to, hearing, and responding to teachers' mathematical work. Second, managing conversations seemed to entail selecting ideas to pursue and focusing the conversation around those ideas. It appeared that Ball *chose* to pursue Brian's quandary about Sharon's and Brandy's representations which manifested a critical issue, making representations visible to the learner. Third, capitalizing on conversations appeared to involve summarizing teachers' contributions, highlighting correct statements, and managing conversation around incorrect statements.

In particular, Ball orchestrated conversation between Mary and Brandy allowing Brandy to address Mary's question. Fourth, it appeared that fostering correct use of mathematical language involved editing teachers' mathematical talk. Suh interrupted teachers, modeled correct language, and had teachers repeat the corrected statement. Ball produced in-the-moment tasks that manifested incorrect mathematical language (e.g., an example of *separate* groups was used to motivate *equal* groups). Fifth, it appeared that engaging in mathematical talk established and maintained a level of mathematical detail beyond that used for teaching children (e.g., locating fractions on the number line, multi-digit multiplication). Finally, it appeared that capitalizing on the conversations required making mathematical ideas explicit (e.g., Suh's definitive statement that the *exact* equivalent fraction needed for the multiplication of $\frac{3}{4} \cdot \frac{15}{7}$ is $\frac{4}{4}$). Engaging teachers in many types of conversation about mathematics appeared to be a prominent element in the task of managing engagement in mathematical tasks for teacher education.

Assessing. Developing teachers' mathematical language appeared to entail assessing their work. In-the-moment, ongoing assessment was evident as teacher educators engaged with teachers' responses. Teacher educators observed, listened to, questioned, and probed teachers' mathematical productions during paired and small group work. Suh assessed and edited teachers' language in-the-moment. Ball assessed, probed, and created tasks to assist teachers' work. Assessment of teachers' mathematical work appeared to inform the development of on-going mathematical work across the cases.

Meta-talk. Across the cases, teacher educators appeared to step out of instruction to address issues tangential to the mathematical focus. This technique, *meta-talk* appeared

to be used for two purposes: making a mathematical point explicit and connecting the mathematical work to the work of teaching. First, it appeared that meta-talk was used to make explicit the mathematical nature of the lesson. For the concept of fraction case, meta-talk appeared to be used to emphasize the critical nature of definition and attention to details for mathematical work. Second, meta-talk appeared to be used to connect mathematical work with the work of teaching as with Suh's justification of parsing and layering mathematical ideas compressed within the fraction multiplication algorithm and connecting this with the way children reason. It appeared that meta-talk provided opportunities for conversations about the nature of mathematical work and how this mathematical work supported teachers' mathematical knowledge for teaching children.

The work to teach mathematical knowledge for teaching appeared to involve elements that address establishing mathematical work or launching the lesson, making mathematics visible, engaging teachers in mathematical discussions, assessing teachers' mathematical work, and engaging in meta-talk. Close examination of the work indicated that the lessons appeared to be coherent. The work began with a mathematical idea and systematically unpacked mathematics within that idea in a set of well-organized, connected steps. Also, the work appeared to be done with intentionality. It seemed that each case was designed to make visible specific mathematical ideas (e.g., use of the 10 x 10 grid to justify place value for decimal multiplication rather than converting decimals to fractions). Across the cases, the work of teacher education appeared to be coherent and intentional.

The teacher educators' work to teach mathematics for the purpose of fostering mathematical knowledge for teaching appears to require attention to mathematical

knowledge beyond that taught in schools, consideration of the special kind of learners, teachers, and the purposes for which the mathematics taught would be used, teaching children. In the next section, I claim that the mathematical knowledge for teaching mathematical knowledge for teaching is distinctive to this work. I continue this chapter with an examination of the mathematical knowledge that appeared to be entailed by the work of teacher education.

<div align="center">Mathematical Knowledge Entailed by the Work of
Teaching Mathematical Knowledge for Teaching</div>

My argument has been that although the tasks under investigation might appear in other mathematics teaching, these tasks entailed consideration of the purposes for which the work is undertaken. These special considerations – special learners, teachers – special content, mathematical knowledge for teaching – and special purposes, teaching mathematical knowledge for teaching to be used to teach children - shape the mathematical demands of this kind of teaching.

Analytic Framework

My data indicate that the mathematical demands of the work to teach mathematics to teachers are extensive and substantial. They indicate robust claims about the mathematical skill involved in the work to teach mathematical knowledge for teaching.

We have defined mathematical knowledge for teaching (MKT) as "mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students" (Ball et al., 2008, p. 399). This knowledge includes "the perspective, habits of mind, and appreciation that matter for effective teaching of the discipline" (Ball et al., 2008, p. 399). We claim that MKT includes subject matter knowledge and pedagogical content knowledge. As seen in figure 6.2, we have pictured MKT as an "egg" that includes

several sub-domains. As our work evolves, we have hypothesized about and focused our study on several sub-domains of these.[18] I focus this investigation on the *mathematical* work of teacher education, precisely the subject matter knowledge sub-domains of the MKT construct.

To justify the focus on MKT as the content of instruction I analyzed the mathematics taught at both sites and interview data. Because Ball is the lead researcher for our team, it is a reasonable assumption that she was teaching mathematical knowledge that is part of the subject matter knowledge region of the egg. Although Suh did not claim that he was teaching MKT as our research team defines it, he consistently discussed the mathematical knowledge in his course as the mathematical knowledge teachers use to teach school mathematics. Further, his consistent attention to interpretations and representations of mathematical concepts, use of language, unpacking mathematical ideas to make visible underlying concepts, and modeling and making explicit ways of mathematical work are evidence of his work to develop MKT. There appears to be consistency between the subject matter knowledge to which Suh attended and mathematical knowledge we claim to be MKT. Therefore, the *subject matter knowledge sub-domains* of MKT are the focus of my study.

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[18] It is important to note that our research team is developing *horizon content knowledge* as I am writing this dissertation.
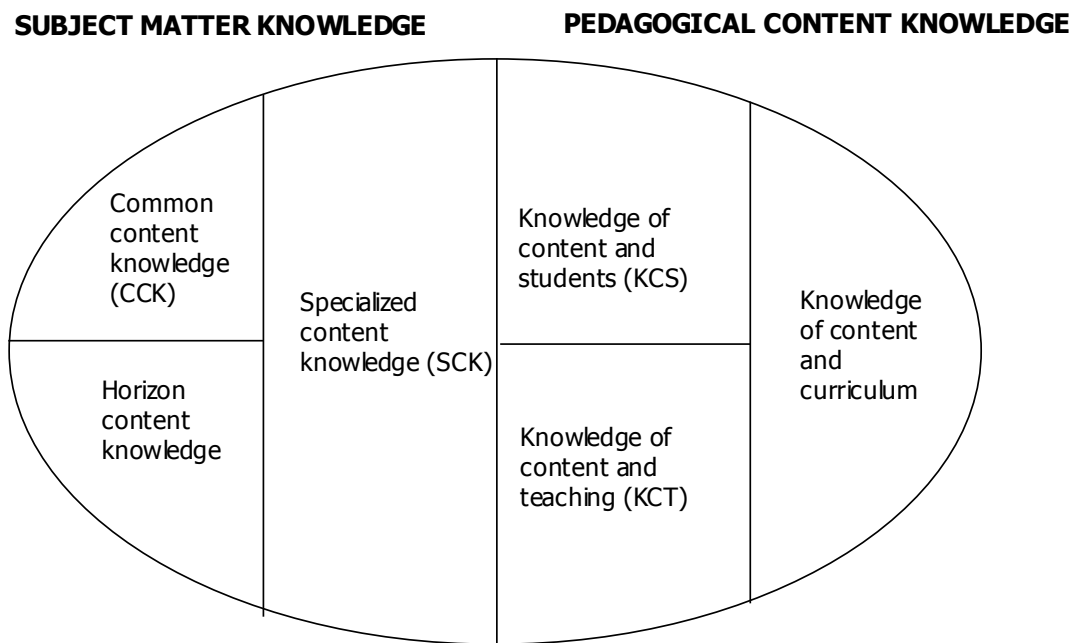
Mathematical Knowledge for Teaching

SUBJECT MATTER KNOWLEDGE   **PEDAGOGICAL CONTENT KNOWLEDGE**

Common content knowledge (CCK)

Specialized content knowledge (SCK)

Horizon content knowledge

Knowledge of content and students (KCS)

Knowledge of content and teaching (KCT)

Knowledge of content and curriculum

Figure 6.2. The MKT "egg" as depicted by Ball and Bass (2009).

Mathematical Knowledge Entailed by Teaching Mathematical Knowledge for Teaching

The mathematical knowledge entailed by the teaching in the four cases included several types: mathematical content knowledge (e.g., 0.7 x 0.1 = 0.07), disciplinary knowledge of mathematics (e.g., definitions are critical to mathematical work), mathematical knowledge for teaching (e.g., the area representing 0.7 x 0.1 is seven small squares in the 10 x 10 grid representing 0.07), mathematical knowledge beyond mathematical knowledge for teaching that supports its teaching (e.g., multiplication of two scalar factors produces a square factor), knowledge of teachers as mathematics learners (e.g., some teachers have procedural knowledge of algorithms without conceptual understanding), knowledge of pedagogical strategies for teachers (e.g., create an example to manifest fragile understandings).

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The mathematical knowledge demands appear to focus on two important aspects of mathematical work: development of mathematical knowledge for teaching specifically specialized content knowledge, and development of mathematical knowledge about ways of doing mathematical work. My work to analyze the mathematical knowledge demands of the teaching leads me to conjecture that there is a unique domain of mathematical knowledge entailed by the work of teacher education, mathematical knowledge for teaching teachers (MKTT). This mathematical knowledge for teaching teachers appears to be used prominently during the work to teach specialized content knowledge.

Our research group characterizes SCK as unique mathematical knowledge and skill demanded by the work of teaching children. It includes "mathematical knowledge beyond that being taught to children" (Ball et al., p. 400); it involves "the use of decompressed mathematical knowledge that might be taught directly to students as they develop understanding … and fluency with compressed mathematical knowledge" (Ball et al., p. 400). Of equal importance to the amount and the characterization of mathematical knowledge needed for this work is the manner in which the knowledge is held and deployed. For teachers' work, they must be able to "hold unpacked mathematical knowledge" for use in making mathematical ideas "visible to and learnable by students," for talking explicitly about and using mathematical language, for selecting and using mathematical representations effectively, and for describing, explaining, and justifying mathematical ideas (Ball et al., p. 400).

The work of mathematics teacher educators appears to differ from that of teachers because teacher educators begin with teachers' compressed mathematical knowledge and attempt to decompress it for the work of teaching children. Because of the nature of this

work, a backing into the mathematics, the teacher educator begins with compressed mathematical knowledge which may be fragile or error ridden. This appears to entail challenging teachers' existing knowledge, addressing misconceptions and errors, and unpacking the mathematics within teachers' compressed knowledge of school mathematics.

Just as the mathematical knowledge for teaching children is a more detailed and unpacked mathematical knowledge that is beyond the mathematics taught to children, these dissertation data indicate that the mathematical knowledge entailed by the work to teach subject matter sub-domains of mathematical knowledge for teaching[19] is a more detailed and unpacked mathematical knowledge used to make visible the mathematical knowledge teachers use for teaching – unpacked, connected, language focused, and discipline oriented. I address the mathematical knowledge entailed by the work to teach to each of these characteristics of mathematical knowledge for teaching in turn.

First, for the work of *unpacking* MKT, it appeared that fine-grained knowledge of interpretations and representations was used. For the concept of fraction case, knowledge of many interpretations of fraction and ways these supported each other informed the use of three to guide teachers' part of the whole interpretation to the point on the number line interpretation. For the decimal multiplication case, detailed knowledge of the 10 x 10 grid, knowledge of the units represented by the side lengths and area of the whole grid and the small squares, the affordances and limitations of the grid, how it can be used to model both the repeated addition and area interpretations of multiplication and the connections between these allowed for the unpacking of mathematical knowledge which supports decimal multiplication. Moreover, knowledge of conventions (e.g., the operator

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[19] Henceforth, for brevity I use MKT to reference the subject matter sub-domains of MKT.

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is represented by the vertical dimension and the operand is represented by the horizontal) and mathematics beyond that of the lesson (e.g., a scalar measure times a scalar measure equals a square measure which can be extended to dimensional analysis) appeared to inform the mathematics of the lesson.

Also, the unpacking of mathematical knowledge for teaching entailed fine-grained knowledge of mathematical details to select examples that manifest these details. Both Suh and Ball used sequences of examples to unpack and develop fraction and decimal multiplication, respectively. The mathematical knowledge of the mathematical ideas compressed within mathematical knowledge for teaching and the knowledge of an order in which to reveal and layer these ideas were entailed by the work of decompressing and unpacking mathematical knowledge for teaching.

Second, developing mathematical knowledge for teaching, a *connected* mathematical knowledge entailed knowledge within and across domains, a fine-grained, yet broad knowledge of mathematical knowledge for teaching that supported selection of details and threading these across mathematical ideas. Suh's use of fraction as point on the number line throughout the fraction unit and previously in the whole number unit (i.e., whole number as point on the number line) threaded this interpretation within and across domains. Also, for fraction multiplication, Suh used three ways of thinking about multiplication, (1) the definition of fraction explaining the work of the operator, (2) parsing the operand into six *copies of* one fifth, and (3) parsing the operand into three copies of two fifths, connecting each to the other. Ball's use of repeated addition and area interpretations of multiplication across each multiplication lesson maintained a connected vision of this operation. Further, Ball's use of rectangular regions connected algebraic

and geometric representations, connections across domains. Mathematical knowledge within and across domains supported the teaching of connected mathematical knowledge.

Third, developing precise mathematical language entailed mathematical knowledge of the concepts under study and of the broader mathematical terrain and language that maintained the integrity of the mathematics and was usable with teachers. This development entailed the mathematical knowledge to: listen to, hear, and respond to teachers' mathematical talk; think through teachers' mathematical talk, discern what was correct and develop that language as well as discern what was incorrect and modify that into mathematically correct language; create an in-the-moment example to make visible an error (e. g., Ellen's *separate* groups for division) and make explicit correct language (e.g., do you hear that – you are using partitive language); foster mathematical language usable within and across mathematical domains (e.g., divide the unit into 3 equal segments which was later used for fraction multiplication). The mathematical knowledge entailed by developing mathematical language for teaching appears to be fine-grained and broad knowledge of mathematical ideas that unpack mathematical knowledge, connect it within and across domains and knowledge of mathematical language that maintains mathematical integrity and is usable with teachers.

The work to develop and manage the mathematical work of these cases entailed the mathematical knowledge of interpretations and representations, mathematical knowledge of examples that fostered the development of the mathematics, knowledge of multiplication, including algorithmic knowledge, common misconceptions, and errors. Further, it entailed knowledge of school mathematics and the mathematical knowledge used to teach the mathematics in school curricula. This mathematical knowledge includes

common content knowledge (e.g., how to perform fraction multiplication), and specialized content knowledge (e.g., knowledge of interpretations, use of examples, and ways of managing the enactment of the mathematical tasks), and a domain of mathematical knowledge to begin with mathematical knowledge and foster it into mathematical knowledge for teaching. This domain of mathematical knowledge appears to be a robust form of subject matter sub-domains of MKT to unpack these, most importantly SCK for the work of teacher education.

Fourth, developing mathematical knowledge for teaching that includes and uses mathematical structures and ways of work entailed a robust knowledge of these. Across the cases, a mathematical knowledge to unpack the specialized knowledge for teaching using interpretations and representations and emphasizing precise mathematical language, to attend to mathematical details beyond those taught to children (the product of two scalars is a square measure), and to develop a connected set of mathematical knowledge for teaching appears to be required by this work. As cited earlier in this document, our research teams' conceptualization of the MKT sub-domains includes the definition of *specialized content knowledge* (SCK) as "the mathematical knowledge and skill unique to teaching." (Ball et al., 2008, p. 400). These tasks include "evaluating the plausibility of students' claims, giving or evaluating mathematical explanations, [and] choosing and developing useable definitions" (Ball et al, 2008, p. 400). Looking further in this document, an example of use of definition or use of mathematical language was the work to distinguish between the meaning of *edge* for everyday purposes and for mathematical purposes. Also, an example of explaining and justifying mathematical ideas is the explanation of "why you invert and multiply to divide fractions" ((Ball et al, 2008,

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p. 400). In these dissertation data, I found mathematical work that appears to be beyond use of definition and explanation. This work emphasized a deeper level of the knowledge of the epistemology of mathematics -- robust knowledge of ways of producing and justifying mathematical knowledge -- that appears to be used in these cases. This work entailed knowledge of mathematical structures, ways of reasoning, and ways of mathematical work. Across these cases there appeared to be evidence of the use of mathematical structures and mathematical reasoning to produce mathematical knowledge for teaching.

First, definitions of concepts were established and threaded throughout the work with that concept. In the concept of fraction case, fractions were defined as *points on the number line.* A precise definition was established and used to define fractions from constant values such as $\frac{1}{3}$ and $\frac{1}{4}$ to rational expressions such as $\frac{4n}{4}$. As the teaching of fraction operations progressed, this agreed upon definition was threaded throughout the development of operations. As seen in the case of fraction multiplication, development of this operation was rooted in the definition of fraction and *taking part of.* For the decimal multiplication case, fundamental roles of operator and operand were used to make visible the repeated addition interpretation of multiplication of decimals.

Further development of mathematical ideas appeared to rely on the development and application of properties and theorems. The concept of fraction and fraction multiplication cases provided evidence of this. Beginning with the definition of fraction, properties of fractions were discussed. Theorems were stated and proven. As work evolved, properties and theorems were used to justify additional mathematical ideas. For the concept of fraction case, the development of the "Fundamental Fact of Fraction Pairs"

provided evidence of this. Patterns were used to establish that $\frac{mk}{nk} = \frac{m}{n}$ for any whole

numbers k, m, where n and k were not equal to 0. This "Fundamental Fact of Fraction

Pairs" was used for subsequent mathematical work such as multiplying $\frac{3}{4} \cdot \frac{15}{7}$ when

taking part of was insufficient for fraction addition (Suh, June 21, 2007, p. 29).

Developing an agreed-upon definition and building mathematics on that definition was

evident in these cases. The development of mathematical statements, proof of these

statements, and continued use of these statements modeled ways of mathematical work.

The mathematical demands of this work entailed knowledge of mathematics, knowledge

about mathematical structures, and about ways of mathematical work.

Second, ways of reasoning mathematically appeared to be prominent in the work

of these cases. First, representational reasoning seemed to rely on representations to

reveal mathematical ideas, properties, or relationships. The data provided evidence that

multiple interpretations, physical and symbolic representations were used to develop

mathematical ideas (e.g., area and repeated addition interpretations to justify and confirm

place value for decimal multiplication, application of area and repeated addition

interpretations for multiplication throughout instruction on this operation, multiple

interpretations of fraction). Second, inductive reasoning appeared to be used to examine

examples and draw generalizations. For the concept of fraction, several series of fractions

were examined. In particular, a sequence of fourths was identified and the pattern

$\frac{4}{4}, \frac{8}{4}, \frac{12}{4}$ was generalized to $\frac{4n}{4} = 4$. For work with whole number division, analysis of

word problems appeared to lead to the creation of categories of word problems that

produced particular types of quotients.[20] Third, deductive reasoning appeared to be used to "reason out" the meaning and algorithm for fraction multiplication. Examples involving *taking part of* and concatenation unpacked the meaning of fraction multiplication. Examples that required equivalent fractions to use those concepts generated the need for the fraction multiplication algorithm. Likewise, application of interpretations of multiplication and the geometric characteristics of the 10 x 10 grid supported the deductive process justifying products of decimal factors. Fourth, algebraic reasoning appeared to be used when patterns of numeric values were explored to create abstractions. For work with fractions, the sequence $\frac{4}{4}, \frac{8}{4}, \frac{12}{4}$, and $\frac{16}{4}$ was used to generalize that $\frac{4n}{4} = n$. Knowledge of mathematical ways of reasoning appeared to be used for the work of teaching mathematical knowledge to teachers (Long, DeTemple, & Millman, 2009).

Third, connections among mathematics from different domains appeared to be used to unpack mathematical ideas. Specifically, properties of geometric figures were used to justify algebraic properties. Line segments were used to model fraction multiplication. Rectangular regions were used to justify place value for decimal multiplication. Mathematics from multiple domains was used to unpack fundamental mathematics of whole number and rational number arithmetic.

Use of knowledge of the epistemology of mathematics, ways of producing mathematical ideas and justifying mathematical truths appear to be threaded throughout the mathematical work in the selected cases. Mathematical structures, ways of reasoning,

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[20] See the case of 38 ÷ 4 in Appendix 5.1.

connections of mathematical ideas across domains, all ways of working mathematically appear to be used across the work to teach mathematical knowledge for teaching. At the level of teaching mathematical knowledge for teaching, MKTT might be considered applied mathematical knowledge honed to address the special needs of teaching teachers.

Distinctive nature of MKTT. This investigation indicates that there is a domain of mathematical knowledge entailed by the work of teacher education beyond the mathematical knowledge used for teaching children. My claim is not that the knowledge used by teacher educators may not be used by other professionals. In fact, use of the epistemology of mathematics is the backbone of research mathematics and mathematical knowledge for teaching is the mathematical knowledge used by teachers to teach children. The distinctive nature, the uniqueness of MKTT exists in the level of the mathematics, the purposes and ways it is used.

Mathematical knowledge (knowledge of and about mathematics) and mathematical knowledge for teaching are nested within MKTT. In addition, MKTT includes mathematical knowledge of these that is more developed, more fundamental, and focused on the teaching of mathematical knowledge for teaching. It addresses three distinctive foci of teacher education: special teaching task (to develop mathematical knowledge into mathematical knowledge for teaching), special learners (teachers), and special focus (application of mathematical knowledge for teaching to the work of teaching). These foci appear to entail mathematical knowledge demands that enlist fundamental and specialized mathematical knowledge for teaching and shape it to satisfy the distinctive needs of this special work.

Mathematical knowledge for teaching teachers includes a robust knowledge of the epistemology of mathematics, mathematical structures and ways of work. In addition, it includes the knowledge to apply this knowledge to the teaching of mathematical knowledge for teaching. In particular, Suh attends to epistemology in two ways: modeling and meta-talk. He worked deliberately to foreground definition. Once the definition of fraction was established, he threaded it throughout the development of equivalent fractions and fraction operations.[21] Along with modeling mathematical ways of work, Suh was explicit about his attention to mathematical ways of work. He stepped back from the mathematical content of the lesson and discussed how he was modeling mathematical ways of work. In particular, he prefaced classes stressing the importance of details in mathematical work; he emphasized that "definition is the bedrock of mathematics;" and, he encouraged teachers to use the fundamental theorem of fractions once the class had developed it. Thus, disciplinary knowledge, mathematical structures and ways of work were used to do mathematical work and were made explicit through meta-talk. The mathematical knowledge entailed by using mathematical structures and ways of work and making these usable for teaching are evident in these cases.

A distinctive aspect of the mathematical work in these cases is the attention to mathematical content across many years of school mathematics. In particular, Ball teaches several lessons on multiplication beginning with interpretations prevalent in school mathematics, repeated addition and area, moving through multi-digit multiplication, and ending with decimal multiplication. She moves from focusing on minute detail, conventions and mathematical language to fast forwarding across years to

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[21] Suh had used the point on the number line interpretation of whole number, extending his use of this interpretation of number across numeration systems.

settle on a move advanced topic. This moving across years of school mathematics and attending to mathematical details that teachers may not teach but will use to teach entails broad, yet comprehensive mathematical knowledge for teaching. Mathematical knowledge that fosters the identification and threading of key concepts within and across mathematical domains; mathematical knowledge of conventions that might not be taught to children but are used to make instruction more precise; and, mathematical knowledge to unpack mathematical knowledge to make it more visible to teachers for their own mathematical knowledge and for their use of mathematical knowledge for teaching.

Teacher educators' attention to teachers' mathematical talk appeared to involve mathematical knowledge to listen to, hear, and respond to teachers. Mathematical knowledge of specialized content knowledge and acute level of attention to mathematical detail appeared to permit teacher educators' work to develop mathematical knowledge for teaching. In particular, Ball made Sharon listen to her talk and attempted to raise Sharon's awareness to the mathematical nuances of her description of a representation (Do you hear that? You are using partitive language.). This very precise attention to mathematical detail appeared to entail a highly developed mathematical knowledge for teaching.

The distinctive nature of MKTT seems to include disciplinary knowledge of mathematics honed to the work of teacher education, to teaching mathematical knowledge to teachers for their knowledge and for their use of this knowledge for teaching. Also, MKTT appears to include mathematical knowledge of mathematical knowledge for teaching that fosters teachers' decompression and unpacking of

mathematical knowledge to develop it into the mathematical knowledge used for teaching.

Characteristics of Mathematical Knowledge for Teaching Teachers

Leinhardt and Steele (2005) note that how teachers hold and use mathematical knowledge for the work of teaching is important. Throughout the analysis of these cases the ways mathematical knowledge was used seemed to demonstrate several characteristics.

Panoramic. Across the cases, it appears that the work to teach mathematical knowledge for teaching entailed a panoramic knowledge, a knowledge that captured the broad landscape of mathematical knowledge for teaching and mathematical knowledge foundational to mathematical knowledge for teaching. This included knowledge of a mathematical topic from the fundamental concept through school mathematics and beyond. Suh's teaching of fractions began with development of the concept of fraction and proceeded through fraction operations. Suh threaded the concept of fraction throughout the work. Additionally, he applied concepts used to teach whole number operations for this work. Ball's work with multiplication provides additional evidence of panoramic knowledge. It began with basic interpretations of multiplication, worked through multi-digit multiplication and decimal multiplication. This work passed through many years of school mathematics, threading interpretations throughout instruction. During an interview, Ball shared that teachers

> teach at various levels when they introduce multiplication and my sense is that they think about multiplication uni-dimensionally and about repeated addition. … It would be important at high school too because multiplication is a significant operation in algebra and calculus. There too being able to interpret statements or being able to work from situations to model or interpret them. It is still important. It is still a foundation to be able to think later. So I guess I think that definitions

and interpretations about the meaning of operations are fundamental to the teaching of operations (Ball interview, 102506, p. 3).

It appears that a panoramic knowledge of mathematics was entailed by this work.

Connected. Mathematical knowledge used for the teaching in these cases appeared to make connections within and across mathematical domains. For the concept of fraction, geometric representations beginning with teachers' two-dimensional interpretation (fraction rectangles), transitioning to a one-dimensional interpretation (segment on the number line), and selecting the zero-dimensional interpretation (point-on-the-number line interpretation) entailed the knowledge of these interpretations, the affordances and limitations of each. For the teaching of multiplication, one representation, the rectangular grid, was modified to support the teaching of the concept of multiplication, multi-digit multiplication, and decimal multiplication. Further, connections were made across mathematical domains. The point on the number line interpretation was threaded throughout instruction on whole numbers and rational numbers. For decimal multiplication, the geometry of the 10 x 10 grid was used to justify place value. Robust mathematical knowledge, knowledge connected within and across several domains appeared to be used for the work of teacher education.

Fluent. The work to teach multiplication to prospective teachers appears to entail both panoramic and detailed mathematical knowledge in a fluid, seemingly effortless manner. I characterize this as *fluent.* It appears that a fluent mathematical knowledge was required for two purposes: moving back and forth between panoramic and detailed mathematical work and attending to in-the-moment responses and decision making. First, examining a series of lessons on multiplication, Ball began by teaching two interpretations of multiplication selected specifically because "these are the

interpretations [teachers] use for teaching" (Ball interview, 102506, p.1). This work appeared to be focused and detailed. Then, the work fast forwarded to multi-digit multiplication. Again, the work to unpack the mathematics within the standard algorithm and partial product algorithm appeared to require detailed work to represent these algorithms using the area interpretation and connect the representation to the algorithms and their written representations. After this detailed work, there was a stepping back, moving forward, and re-focusing on decimal multiplication with the same intensity as the prior sets of detailed attention to multiplication. Panoramic knowledge of multiplication appeared to allow for work across grade levels. Detailed knowledge of multiplication appeared to permit focused, discriminating study. Connected knowledge appeared to foster the threading of interpretations and representations across domains. Using these in concert and with agility appeared to support the teacher educators' work.

Second, responding to teachers' work and making in-the-moment instructional decisions appeared to entail panoramic, yet comprehensive mathematical knowledge that was robust and available instantaneously. In the case of the concept of fraction, responding to a teacher's query about the use of definitions, properties, and theorems entailed an immediate discussion about the structure of mathematical work. In the case of multi-digit multiplication in Chapter 1, listing to, hearing, and responding to student teachers' mathematical productions appeared to entail the ability of examine mathematical work and make in-the-moment decisions about directions in which the lesson could move. Further, synthesizing three student teachers' work, identifying positive aspects of each, and building a spontaneous production appeared to entail robust mathematical knowledge.

Deliberate, intentional. For each case, mathematical work had been planned from the writing of instructional goals, to the selecting of interpretations and representations, the selecting of examples, and the enacting of the lesson. These cases were taught from teacher educator developed materials. Suh wrote, reviewed, and revised monographs for the teaching of the professional development summer institute over several years. Ball and the MMPG research team developed material over several iterations of the mathematics methods course. At each site, lessons appeared to be developed and delivered with deliberateness and intentionality. This seemed to entail mathematical knowledge of and about mathematics (Ball 1989), mathematical knowledge used for teaching, and mathematical knowledge to begin with the domain of school mathematics, identify the compressed mathematical knowledge within that domain, and unpack the mathematical knowledge to develop it into mathematical knowledge for teaching.

Knowledge of Mathematical Knowledge for Teaching and Knowledge of Teachers as Learners

Although this investigation focused on the subject matter knowledge demands entailed by the work of teacher education, I find it impossible to ignore the way the work of teacher education appeared to be shaped by addressing the needs of the special learners, teachers. As noted several times, the work appears to be distinctive because the content was mathematical knowledge for teaching, the learners were *teachers,* and the purpose was teaching mathematics to children. It appears that a sub-domain of MKTT that melds mathematical knowledge of mathematical knowledge for teaching *and* knowledge of teachers as learners and teaching teachers was used in the cases. This sub-domain appears to include knowledge of: mathematical knowledge teachers typically possess – what they find challenging, typical misconceptions and errors; school

mathematics – the mathematics teachers teach; mathematical ideas that help teachers

transition from what they know to mathematical knowledge for teaching. In addition to

this mathematical knowledge, there appeared to be knowledge about ways that teachers

learn mathematics. Because pedagogical content knowledge was not the focus of this

study, I discuss these findings in more detail in Chapter 7 and propose additional research

to confirm and develop these findings.

Conclusion

In this chapter, I summarized the findings of this dissertation of which there are two.

First, I claim that the work of teaching mathematical knowledge for teaching is special. It

entails many tasks, three of which are selecting interpretations and representations,

selecting examples, and managing the enactment of mathematical tasks for the work of

teaching mathematical knowledge for teaching. Analysis of the data revealed many

elements of each task, elements that appeared focused on the work of teaching special

learners, teachers, for their unique work, teaching. The work of selecting interpretations

and representations included surveying those available, deliberating which are

appropriate for use for teaching mathematical knowledge for teaching, and deciding

which would optimally develop mathematical knowledge to mathematical knowledge for

teaching. The work of selecting examples included reviewing instructional goals for

mathematical details, checking examples to ensure that they criteria for use, generating

categories of examples, sequencing examples for layered unpacking of mathematical

details, developing repositories of examples for use. Managing the enactment of

mathematical tasks entailed launching the lesson by contextualizing it within the work of

teaching, developing mathematical ideas specific to unpack mathematical knowledge for

teaching, engaging teachers in mathematical talk to develop mathematical ideas and teachers' abilities to communicate these, assessing teachers' mathematical work to inform instruction, and engaging in meta-talk to make explicit mathematical ideas and to connect mathematical work to teaching. Elements of each task were adapted to attend to the work of developing mathematical knowledge to mathematical knowledge for teaching.

Second, I claim that there is a domain of mathematical knowledge used for the work of teaching mathematical knowledge for teaching. I label this MKTT.[22] I discuss distinctive characteristics of the mathematical knowledge entailed by the work of teacher education in these dissertation data.

Mathematical knowledge for teaching teachers appeared to include a specialized knowledge of mathematical knowledge for teaching, knowledge of mathematical knowledge for teaching that appeared panoramic, connected, and fluent. This robust knowledge of mathematical knowledge for teaching seemed to contribute to the mathematical environment created by the mathematics teacher educator and the teachers. It appeared to be a resource drawn upon to respond to teachers' questions and mathematical talk.

Mathematical knowledge for teaching teachers appeared to include a robust knowledge of the discipline of mathematics. It appeared to involve: knowledge about mathematical structures such as definitions, properties, theorems, and lemmas and how these are used to *do* mathematics; knowledge about descriptions, explanations, justifications, and proof and how these are used for mathematical work. Disciplinary

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[22] Professor Hyman Bass is studying a domain of mathematical knowledge used by teacher educators he has labeled as Mathematical Knowledge for Mathematics Education (MKME). MKTT is a subset of this domain focused specifically on the work of teaching teachers.

knowledge of mathematics seemed to support work to develop or use planned definitions

of mathematical concepts to develop properties and theorems and then to use the

properties and theorems for subsequent work.

Having concluded this chapter, I turn next to Chapter 7 in which I pose several

challenges to this work and suggest directions for research that will build on this study.

Chapter 7

Posing Questions and Extending This Study

My investigation into the mathematical work of and knowledge entailed by teacher education produced a microanalysis of three tasks of teacher education and the mathematical knowledge demands entailed by these tasks. The work of teaching mathematical knowledge for teaching appears to include distinctive qualities such as using multiple interpretations for a mathematical idea, challenging mathematical knowledge to identify fragile understandings or misconceptions, movement across several years of school mathematics within a compressed amount of class time, and explicit attention to fine-grained mathematical details. Likewise, the mathematical knowledge entailed by the mathematical work of teacher education appears to include a level of mathematical knowledge beyond mathematical knowledge used for teaching children, a mathematical knowledge that I claim includes specialized knowledge of mathematical knowledge for teaching. My investigation is the first empirical study into the mathematical work and knowledge entailed by teacher education. And so, my claims are intended to begin a conversation.

Potential Questions to Extend This Study

This conversation may begin with objections to this study. I propose several questions that might be raised and initiate responses to these.

As a Member of a Research Team Studying MKT, is the Author's Work too Situated within the Work of MKT?

For initial analytical phases, my analytical framework was MKT theory. These analyses focused on examining data on Ball's teaching for evidence of MKT and mathematical knowledge that appeared to be different from MKT. Because Ball professes to be teaching MKT, it was reasonable to use this framework. However, the extent to which the argument was developed on Ball's work, the limited use of data from the second observation site appeared to skew the findings. Even with this emphasis on data from Ball's work, findings that appeared to be reasonable and worth pursuing were two: three tasks of teacher education appeared to be important to the work of developing mathematical knowledge into mathematical knowledge for teaching and the work of teaching mathematical knowledge for teaching appeared to entail mathematical knowledge beyond that being taught. To resolve the concern that analyses were riveted to the MKT, I shifted my focus from a direct analysis of data for use of MKT to a finer-grained analysis of the work of teacher education for elements of that work. I then analyzed the elements of the tasks of teacher education for the mathematical knowledge entailed by that work.

In subsequent analyses, I examined Suh's work for the mathematical knowledge being taught. Suh stated explicitly that the purpose of the summer institute was teachers' development of mathematical knowledge for teaching. Moreover, examination of the mathematical content of these lessons reveals that there was a careful unpacking of mathematical concepts and operations using interpretations and representations paying close attention to mathematical language and mathematical ways of work. The mathematical knowledge of these lessons appears to be like specialized content knowledge. Consequently, it seems reasonable to discuss the subject matter knowledge

in each case as at least similar to subject matter knowledge sub-domains (CCK and SCK) of our research team's concept of MKT.

Had Different Cases Been Selected, Would Findings Have Been Different?

I selected the four cases quite deliberately. Across the two sites, I selected cases that were parallel in topic: an introductory concept (fraction and whole number division) and an operation on rational numbers (fraction multiplication and decimal multiplication). This selection was intended to test whether the elements of the tasks of teacher education were similar or different across comparable mathematics content. My selection of these cases was made to study in a fine-grained, purposeful manner the identified tasks of teacher education.

The data include cases about: fraction addition and subtraction, decimal addition and subtraction, applications of whole number multiplication and division. In particular, the multi-digit multiplication and the 38 divided by 4 cases found in Chapter 1 and Appendix 5.1, respectively provide much important data on interactions between teacher educator, student teachers, and mathematical knowledge. Additional cases from these data provide evidence of teacher educator work to provide opportunities for student teachers to do mathematical work found in teaching, present that work, and interact with peers to critique this work. These cases provide further evidence of mathematical knowledge demands entailed by the work of teacher education, support findings of this study, and provide additional insights into the work of teacher education.

Had the Subjects Been Different, Would the Findings Have Been Different?

These subjects were selected because of their very different educational and professional backgrounds anticipating that the work would look very different. Two

factors that were similar were: course materials were developed by the teacher educator (Suh) or a team of researchers that included the teacher educator (Ball) and a research mathematician was involved in the materials development. These factors might prompt two questions. First, would the work and mathematical knowledge demands appear different if the courses were taught from commercially developed materials? Second, would the work and mathematical demands of teacher education appear different if research mathematicians were not contributing to the development of the course? Specifically, would the attention to disciplinary nature of mathematical work be visible? These questions are of particular interest to me and I hope to pursue them.

Are There Sub-Domains of Mathematical Knowledge for Teaching Teacher?

I claim that there is a distinctive domain of mathematical knowledge use for teaching mathematical knowledge for teaching to teachers. For this investigation, the data provide robust evidence of the use of mathematical knowledge beyond mathematical knowledge for teaching. However, just as our research team has identified sub-domains of MKT, I wonder if there are sub-domains of MKTT. Further research is needed to identify sub-domains.

<center>*Future Research*</center>

Additional Studies

This dissertation study was conducted at two university sites at which teacher educators developed and taught the teacher education courses. Extending this study to investigate teacher education at different types of teacher education institutions – liberal arts colleges, teacher education institutions not focused on research and community colleges – is important to developing this study. Investigations at a variety of sites might

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support the tasks of teacher education identified for this study and provide other tasks deemed important for this work. Thus, variation of teacher education institution might offer additional insights into the work of teacher education and the mathematical demands of that work.

At both sites, research mathematicians were engaged in the course development. Additional studies of teacher education courses in which research mathematicians contribute to curriculum materials or teaching might offer further insights into the work of teacher education and the mathematical knowledge entailed by this work. In addition, studies of sites where research mathematicians are not involved in course development and teaching of teacher education courses would offer important insights into the role of mathematicians in this work. These studies might support and add to the findings of this dissertation study.

Although the teacher educators in this study come from different research paradigms, evidence confirmed their work to develop mathematical knowledge into mathematical knowledge for teaching. Investigations at several more sites where teacher educators propose different theories of mathematical knowledge for teaching or whose research interests focus on different aspects of teacher education might provide additional insights into the mathematical work of teacher education and the knowledge demands.

Additional studies that vary the teacher education sites, research interests of teacher educators, and presence of research mathematicians might offer additional insights into the mathematical work and knowledge demands entailed by that work and strengthen the findings of this dissertation.

Studies of Pedagogical Content Knowledge Entailed by the Work of Teacher Education

The focus of my dissertation is the *mathematical work and knowledge* entailed by teacher education. Another critical focus is the *pedagogical content knowledge* entailed by the work of teacher education. There has been some investigation of pedagogical strategies for teacher education (Franke, et al., 2001; Goldsmith & Schifter, 1995; Putnam & Borko, 2000; Simon, 1994; Sowder, 1998; Stein & Brown, 1995).

Although my investigation focused on the subject matter knowledge demands entailed by the work of teacher education, it appears that this study provides evidence that the work of teacher education is shaped by addressing the needs of the special learners, teachers. It appears that a sub-domain of MKTT that melds mathematical knowledge of mathematical knowledge for teaching *and* knowledge of teachers as learners and teaching teachers was used in the cases. This sub-domain appears to include knowledge of: mathematical knowledge teachers typically possess – what they find challenging, typical misconceptions and errors; school mathematics – the mathematics teachers teach; mathematical ideas that help teachers transition from what they know to mathematical knowledge for teaching. In addition to this mathematical knowledge, there appeared to be knowledge about ways that teachers learn mathematics.

There is evidence of teacher educators' use of knowledge of teachers' incoming knowledge to launch the lesson across the cases. At times, the work appeared to challenge to teachers' knowledge to identify a gap or misconception (Goldsmith & Schifter, 1995) as see in both the fraction and decimal multiplication cases. For fraction multiplication, Suh posed a problem: "Take two thirds of a bag of rice means that you divide the bag of rice into three equal parts and you take two" (Suh transcript, 070307, p. 1). He seemed to anticipate that teachers would suggest fraction multiplication to solve

the problem. He suggested that they delay that and first examine the meaning of *taking part of*. For decimal multiplication, Ball posed the problem "multiply 0.7 x 0.1" (Ball transcript, 110706, p. 3). There appeared to be two potential errors: placement of the decimal point in the product and use of area rather than linear representations to depict the factors. Knowledge of the school mathematics, typical errors and misconceptions, the mathematical knowledge used to teach children, and the mathematical knowledge to foster the former into the later appeared to be entailed by the teaching throughout the cases.

The data suggests that mathematical ideas were made *explicit*. This appeared to entail mathematical knowledge of mathematical details important for mathematical knowledge for teaching and knowledge of the importance of being explicit. Suh noted for fraction multiplication that multiplying $\frac{15}{7}$ by $\frac{4}{4}$ was *the* equivalent fraction important for the work. Ball emphasized the linear nature of factors by tracing the side lengths when demonstrating factors. Teacher educators appeared to make visible mathematical ideas critical to teachers' mathematical work.

The data indicates two strategies to *contextualize* mathematics for instruction: use of meta-talk and situating mathematical tasks within the work of teaching children (Putnam & Borko, 2000). First, the use of meta-talk seemed to allow a stepping back from the mathematics of a lesson to connect the mathematical work with teaching children (e.g., Suh's discussion of threading mathematical ideas to create a connected vision of mathematics for children). Second, mathematical work was situated within teaching (e.g., interpretations of division are taken from fourth grade curricula). Contextualizing the work of teacher education appeared to entail knowledge of

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mathematical knowledge for teaching and knowledge of this strategy for teaching teachers.

The data appears to provide evidence that *discourse communities* were used for the work of teacher education. Suh had teachers work in pairs to practice their talk about fractions. Ball had teachers work in pairs to explore interpretations of whole number division. Knowledge of this strategy for teacher education appeared to have informed the work in these cases.

An amalgam of mathematical knowledge to unpack mathematical knowledge for teaching and knowledge about teachers as learners of mathematics appeared to be used by the teacher educators in these cases. Evidence of the use of such knowledge suggests that a study of the pedagogical content knowledge of teacher educators is a viable research topic (Doerr, 2002; Geddis & Wood, 1997; John, 2002; Zavlavsky 2005) and analysis of these data might contribute to this body of literature.

Continued investigation into both the subject matter and pedagogical knowledge entailed by teacher education would provide insights into the nature of this work and offer ways to develop and implement teacher education programs focused on the teaching of mathematical knowledge for teaching.

Studies of Novice Mathematics Teacher Educators

Like novice teachers, novice mathematics teacher educators begin their work with a set of knowledge and skills. Studies of novice mathematics teacher educators could provide interesting data about these knowledge and skills. Two types of investigations might provide foundational research for an agenda on MKTT. First, a survey of novice mathematics teacher educators would include professional and educational backgrounds,

a survey of mathematical knowledge for teaching and mathematical knowledge for

teaching teachers. In addition to these data, learning about the path by which the

mathematics teacher educator came to this work would be instructive. Second, case study

research of novice mathematics teacher educators to identify their entry level knowledge

of MKT and their mathematical knowledge for teaching teachers, to document their use

of mathematical knowledge in their work to teach teachers, to trace an evolution of the

level of mathematical knowledge used over a period of several years, to examine the

evolution of mathematical knowledge to mathematical knowledge for teaching teachers.

Findings from such studies might inform the development of career pathways for future

teacher educators.

Mixed Methods Research

In this study I used qualitative methods to explore the existence of a domain of

mathematical knowledge for teaching teachers. Qualitative studies using a variety of

subjects would generate theories that would either confirm or disaffirm the results of my

study. In order to develop the MKTT theory further, a research agenda that includes both

qualitative and quantitative methods is in order. As stated by the National Research

Council (2002) "knowledge is generated through a sequence of interrelated descriptive

and causal studies, through a constant process of refining theory and knowledge" (p.

123). I suggest a research agenda using mixed methods that includes measures

development, extensive qualitative studies, and validation work. The research design of

the Learning Mathematics for Teaching Project (Ball, et al, 2005; Hill, Rowan, & Ball,

2005; Hill, Schilling, & Ball, 2004; Hill & Ball, 2004) is one that could be adapted for

this research. Mixed methods research would develop the findings of this study and offer insights for continued research.

Implications for Professional Development of Mathematics Teacher Educators

My initial investigation into the mathematical work of teacher education and its mathematical knowledge demands is but a glimpse into this topic. More research is required before we have a well-formed image of the work and knowledge demands of this field. As this research progresses, one might ask how an evolving image of teacher education might influence the practice of teacher education. Will knowledge about this work influence the structure of teacher education? Will it cause the field to look carefully at the preparation of teacher educators? More specifically, will knowledge about the mathematical work and the knowledge entailed by this work encourage leaders in teacher education to examine the mathematical preparation of teacher educators? And beyond the scope of this dissertation, will knowledge about the pedagogical work of teacher educators shape this preparation of teacher educators? The study of the many learning opportunities offered teachers throughout teacher education – the mathematical and pedagogical work and the knowledge demands of this work – might shape teacher education programs in the future. This dissertation provides an initial image of the mathematical work of teacher education and the knowledge demands entailed by this work. Additional investigations may provide findings that will add detail to the findings of this study providing a sharper image. Further, an examination of coursework in programs designed to prepare mathematics teacher educators is necessary. Finally, a thoughtful conversation among mathematics teacher educators who teach prospective

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mathematics teacher educators about appropriate mathematics coursework might shape the programs from which mathematics teacher educators come.

Conclusion

My initial investigation into the work of teacher education and its mathematical knowledge demands offers preliminary findings. Continued research into my dissertation questions, refocus and refinement of these questions to guide additional study may inform the field about this very important work.

Appendices

Appendix 3.1

Interview Protocols

Ball Interview
Mathematics Methods
October 24, 2006 Multiplication

1. I want to start with your thoughts on how the lesson was designed. So can you tell me a little bit about that?
2. So what are the major mathematical ideas that you want them to come away with?
3. You began by asking the pre-service teachers to show 2 x 4 in as many ways as possible. Were there any particular reasons for the 2 x 4? Why did you choose those numbers?
4. Would you mind drawing the representations?
5. What are the three meanings of multiplication that you address in the course? What is the mathematical significance of the meanings?
6. How important are the meanings to the study of multiplication? And maybe more carefully, the multiplication the teachers are going to teach?
7. So along with division and that very nice relationship between multiplication and division, you've talked about combinatorics. Are there other mathematical topics this connects to and do you intend to talk about them in the course?
8. So you moved from 2 x 4 to 25 times 35. Why did you choose that? Are there other mathematical ideas that you see?
9. Could you to build the area representation for 35 x 25 and go through the work of that?
10. What mathematical challenges does the problem present?
11. How are you going to extend this to show decimal multiplication?
12. Extending this. Having teachers think ahead to algebra and multiplication of binomials, do you see how this fits with that?
13. Are there things that you don't address in this course and why? You have to make decisions. There is a lot more to do with multiplication. There are properties, there are applications, and so on. You have to make a conscious decision about what you attend to or not.
14. What are your sources? What knowledge of mathematics are you drawing on?
15. How do you see the knowledge you use to teach math methods to be different from the knowledge you use to teach children?

Ball Interview
Mathematics Methods
November 7, 2006

Multiplication
1. The work on multiplication began with the problem 15 x 32. What did you hope to accomplish through this problem?
   a. The work on multiplication began with Lauren's work. What were your thoughts about her work to multiply 15 x 32? If you had been working with her individually, how might you have redirected her work?

   b. Amy presented a second presentation of this problem. How did her work differ from Lauren's? How did her work add to the work of the class? …. Or take away from the work of the class? Amy began to break up the 300 as 100 + 100 + 100. Where do you think she was going with this – or- why do you think she pursued this line of reasoning?

   c. Both students' work challenges the teacher educator to respond. As the teacher educator, what are different ways you might have responded and what would you hope to "get" from those responses? What are the mathematical challenges of this work?

   d. Then, you presented the work of multiplying 15 x 32. What did you hope to convey to the pre-service teachers through your demonstration?

   e. Did you accomplish your goals for this segment of the work on multiplication? If the work fell short, how? Did anything surprise you about this segment of the evening's work?

2. You prefaced the multiplication of decimals by studying the 10 x 10 grid. What important mathematical ideas did you intend to surface?

   a. You used both repeated addition and area to explicate the multiplication for the 0.7 x 0.1 example and 0.7 x 0.5. Why did you use both models?

   b. What do you expect the pre-service teachers will take from this lesson to their teaching?

   c. What is hard for the pre-service teachers?

3. Thinking through this lesson, how is this different from what you would do with children?

4. Thinking through this lesson, what will be the goals of the next lesson? What will be important to communicate to the students?

Ball Interview
Math Methods Class
November 14, 2006

1. The November 14th class began with work on division. You asked the prospective teachers to show 12 ÷ 6 in two ways, to identify how the ways were the same and how they were different. What mathematics were you probing? Why is this important for elementary teachers? What are you trying to make explicit that will be important for the prospective teachers' teaching? What mathematics are you drawing upon?

2. You discussed partitive and measurement models at length. You noted that the "basic structure is to look at different reasons for the divisor and quotient." Of what importance is this to teaching prospective teachers about division? … for teaching children about division? Can you distinguish between your knowledge of this domain as a teacher educator as compared to your knowledge of this domain as a teacher?

3. The prospective teachers wrote story problems for the problem 38 ÷ 4. Is this an activity you would do with children? If so, would it differ from what you did with the prospective teachers? How?

4. The prospective teachers provided the following story problems.
   a. Mandy has 38 inches of ribbon. She cuts 4 inch pieces. How many pieces will she have?
   b. There are 38 cookies. If every child receives 4 cookies, how many children are there? How many cookies are left over?
   c. Cookie Monster has 38 cookies. He will eat 4 each day. How many days until they are gone?
   d. Bob ran a total of 38 miles. If he ran only 4 days of the week and an equal number of miles each day, how many miles did he run each day?

What do you need to know to enact this task? What did you hope to elicit from it? If all possible results (interpretations) of the 38 ÷ 4 problem do not emerge, how do you catalyze those? What misconceptions or confusions did you anticipate? Did you see anything that you have not seen with past prospective teachers? Have you done this type of activity with children? If so, describe how you would enact it with children. How is that different from the task with prospective teachers?

5. HB identified this as an important mathematical space. He noted the importance of maintaining the integrity of the mathematics, that there be no distortions of the mathematics. What types of distortions have you seen – or are aware of? How are you able to correct these?

6. When I think about the strands of MKT identified by LMT, I am not certain that I am probing for CCK, SCK, KCS, and KCT. Is there a way (or a generic set of probes) that I could use to get at this for any given lesson?

Suh Interview
Professional Development Institute
July 16, 2007

1. Can you tell me a bit about the mathematics you attend to during this professional development institute and your reasons for attending to this?
2. The purpose of my work is to study the work of mathematics teacher educators. Can you tell me a bit about that?
       a. How do you plan for the institute?
       b. How do you decide what you are going to teach?
       c. How do you choose examples?
3. You were tending to mathematics very carefully. You were defining fraction on a number line and my guess is that is typically not the first way it is addressed in the.
4. When working with fractions, you got to $4 = 4n / n$. Then, you looked at examples and you talked about $10/3 => 10 = (3 \times 3) + 1$. And then you moved to $k/5$.
5. You used a variable in the numerator and then defined k to be between 11 and 14. Why did you move in that direction?
6. You must have thought carefully about these examples. What did you want to elicit from these? Do you think that is a big transition for them? Do you think it is hard for them? You used 5 over here?
7. You seemed to stun the teachers with the use of variables. Did you intend on doing that? Why did you introduce the inequality?
8. I noticed that a lot of your work with fractions rested on equivalent fractions. What is your purpose?
9. The first morning, you were discussing the number line interpretation of fraction. You seem to want the teachers to know the definition perfectly. What is your purpose?
10. Did you do all the work using the number line representation? Did you demonstrate each operation on the number line?
11. So if a teacher goes back to definitions and properties and refers back to then this is a greater enabler of children's learning?
12. I am thinking in another way. One of my hypotheses is that when people teach teachers, they come to that work differently than if they teach other people. When you teach other math courses, do you approach them differently?

Appendix 4.1
Elements of Tasks of Teacher Education (Suh)
A Working Document

Concept of Fraction		
Selecting Interpretations / Representations	**Selecting Examples**	**Managing Enactment**

1. Reviewing all possible interpretations / representations
   a. Surveying or gathering all feasible interpretations / representations
   b. Examining other mathematics educators' use of interpretations / representations
	1. Reviewing instructional goals
	   a. make visible shortcomings of teachers' incoming knowledge
	   b. make connections between whole number and fraction operations
	   c. develop teacher' abilities to work with abstraction
		1. Launching
		   a. goal: challenge, review, present

2. Deliberating about which interpretations / representations to use
   a. considering the mathematical focus of the lesson
   b. parsing the task for each mathematical element
   c. considering how available interpretations / representations attend to the mathematics or what mathematical details they reveal, or not
   d. considering how the interpretations / representations build on prior mathematical work
   e. considering how teachers often teach the mathematical concept – what they know from teaching school mathematics, curricula
	2. Test examples – do they satisfy criteria for use (are they achieve instructional goals, are they usable for teaching	2. Generating mathematical ideas
		   a. presenting in lecture
		   b. engaging in mathematical conversation
		    --deciding the format of conversation
		      --- paired or small group work
		      ---large group work
		   -- deciding the nature of conversation
		      ---recitation of precisely worded definitions, procedures
		      ---exploration of teachers' thinking
		      ---practice of modeling, algorithms
		   -- orchestration of conversation
		      ---questioning
		      ---responding
		      ---what to pursue, what to let go (this determines the mathematics that is

		taken up)

3. Deciding on which interpretations / representations to use
  a. choosing the interpretations / representations that support the instructional goals.
  b. choosing the interpretations/ representations that allow the work to begin with teachers' incoming knowledge and foster it to the identified mathematical knowledge for teaching.
	3. Sequencing examples
	  a. considering the layers of mathematical knowledge within the lesson
	  b. which to be used for introducing, unpacking and layering, reinforcing, extending
		3. Assessing
		  a. listening and responding to teachers' work in whole group
		  b. listening and responding to teachers' work in small group or individual work

Fraction Multiplication		
Selecting Interpretations / Representations	**Selecting Examples**	**Managing Enactment**

1. Reviewing all possible interpretations / representations
  a. . Parsing the task for each mathematical component
  b. Surveying or gathering all feasible interpretations / representations *for this case, the interpretations / representations of both multiplication and fractions*
  c. Examining other mathematics educators' use of interpretations / representations[23]
	1. Reviewing instructional goals
	  a. make visible shortcomings of teachers' incoming knowledge
	  b. make connections between whole number and fraction operations
	  c. develop teacher' abilities to work with abstraction
		1. Launching
		  a. generating work on or conversation about mathematics
		  b. challenging teachers to think differently about mathematics

2. Deliberating about which interpretations / representations to use
  a. parsing the task for each mathematical component
	2. Testing examples – do they satisfy criteria for use
	  --- achieve instructional goals
	  ---connect with prior and future mathematics
		2. Developing mathematical ideas / producing the mathematics
		  a. presenting in lecture format
		  b. engaging in

---

[23] This is evident in this case.

b. considering the mathematical focus of the lesson c. considering how available interpretations / representations attend to the mathematics or what mathematical details they reveal d. considering how the interpretations / representations build on prior mathematical work e. considering how teachers often teach the mathematical concept – what they know from teaching school mathematics, curricula f. considering how the interpretations and representations unpack the mathematics g. considering how these interpretations and representations foster the development of later mathematical ideas	(threading concepts within and across mathematical domains) --- usable for teaching	mathematical conversation --deciding the format of conversation --- paired or small group work ---large group work -- deciding the nature of conversation ---recitation of precisely worded definitions, procedures ---exploration of teachers' thinking ---practice of modeling, algorithms -- orchestration of conversation ---questioning ---responding ---what to pursue, what to let go (this determines the mathematics that is taken up) Parsing Organizing Layering Threading all done with intentionality Raising teachers' awareness to SCK Explicitness Making mathematical sense Raising teachers' awareness to this sensemaking Connecting mathematics to the work of teaching children - Engaging teachers with mathematics

		-individual solution to computation
		 - work with individual teachers
		  -sharing of work in whole group
		  - beginning the conversation by providing the "first step" – begin with the number line…
		small group or pairs
		  - listening & responding to teachers
		  - guiding teachers work

3. Deciding on which interpretations / representations to use
  a. choosing interpretations / representations that support the instructional goals. (integrity of math, unpack mathematics, develop knowledge about mathematics – work from definitions, properties, etc)
  b. choosing interpretations/ representations that allow the work to begin with teachers' incoming knowledge and foster it to the identified mathematical knowledge for teaching.
  c. choosing interpretations / representations that build on prior lessons
  d. choosing interpretations / representations that support
	3. Sequencing examples
	  a. considering the layers of mk packed within the lesson
	  c. sequencing to foster layered learning
	  b. which to be used for introducing, unpacking and layering, reinforcing, extending
		3. Assessing
		  a. listening and responding to teachers' work in whole group
		  b. listening and responding to teachers' work in small group or individual work
		  c. editing teachers' work

Vignette of Thirty-Eight Divided By Four

The Case of 38 ÷ 4

After developing the interpretations of whole number division, Ball transitions to

a new task. She asks the teachers to find the quotient for 38 ÷ 4. Once they have done

that she asks the teachers to write two different story problems that provide appropriate

contexts for the computation. In this instructional segment, Ball asks the student teachers

to apply their knowledge of interpretations of whole number division by having them

write story problems for the purpose of applying their knowledge of partitive and

measurement interpretations. Also, she asks the student teachers to consider the nature of

the quotients and remainders of the story problems. I begin the discussion of the 38 ÷ 4

task by presenting Ball's instructional moves and continue with a few of the teachers'

story problems.

Story Problems for 38 ÷ 4

The launch. Ball instructs the student teachers to "first, do the problem so that

when we write our story problems our stories actually go with the answer to the problem.

The problem is 38 divided by 4" (Ball transcript, 111406, p. 7). She polls the class for

their answers. The student teachers state that their quotients are: 9 remainder 2, 9 ½, and

9.5. Ball seeks to elaborate the instructions for writing story problems. She asks the

teachers to

> write as many different stories as you can for which 38 divided by 4 is the
> arithmetic problem that you would write. But we want you to write stories that are
> in some way *different* from one another. Work with a partner or alone to write
> story problems that you could make a claim are different from one another. And
> the difference can't be that one is about soda and one is about popcorn, one is
> about your friend, but different in some mathematical way. And when we read

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them off and talk about them we will give you practice to listen to these and hear what is different about them. (Ball transcript, 111406, p. 7).

Checking in. Ball's task asked the student teachers to write story problems that were different in mathematical ways. She did not prescribe *what* mathematical differences to include. Ball circulated around the room as the teachers wrote story problems. After one minute, she asked the teachers to share their work.

Ball: So just for an example, can someone share one and see what it sounds like. Madison?

Madison: Thomas has 38 suckers and 4 friends. How many suckers can each friend get?

Ball: So the first question is: does that go with 38 divided by 4? Where is the 38?

Taylor: Number of suckers.

Ball: Where is the four?

Taylor: Number of friends.

Ball: And what is the question that Taylor is asking? What is the question you have to solve for this problem?

Taylor: The number of suckers each person gets if Thomas gives them out equally to the four friends.

Ball: So what interpretation of division is that? (Ball transcript, 111406, p. 7).

Several student teachers responded. Some respond that the problem was a partitive division; others responded that it was measurement interpretation. Ball pressed to have the teachers distinguish between the interpretations. One teacher justified the problem as partitive division because the suckers were being divided into four equal groups. Ball moved to clarify the next part of the task. She confirmed that Madison's story problem modeled partitive division and that Madison's next task was to write a

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problem that was different in a way other than context. Ball asked the student teachers to continue their work.

Ball's work with small groups of student teachers. Ball circulated around the room assessing the student teachers' story problems. She restated the task and attempted to emphasize the writing of *mathematically* different problems. She worked with a group having difficulty writing problems for both interpretations. She proposed that the student teachers write problems that used the same context but had different roles for the divisor. She suggested that this process to help the student teachers write problems with different interpretations.

Ball moved on to observe a second group. After several seconds of observation, she read their story problem and probed the nature of the quotient and remainder. Ball and the students talked through the remainders for the problems and appropriate ways to communicate the answers.

Ball: You have a 38 inch shelf. You want to store books that are 4 inches wide. How many books can you store? What answer makes sense? In the problem we talked about as a class, the answer is 9 remainder 2. What is interesting about your problem is that that remainder is not 2 books, right? So maybe you should try writing a problem that is about what ever you were dividing up so you can see the difference in the remainders.

Beth: I did that with cookies. There are 38 cookies to be divided among 4 students. How many cookies will each student get and how many will be left over?

Ball: In that case you chose not to cut up the left over cookies. But since they were cookies you could cut them up, right? So you could write – without cutting any cookies up. You would get 9 with 2 left over. That is different from your bookshelf problem. That is good. That is exactly what we want you to do. We want you to notice that your answers turn out a little different depending on the problem. So see if you can write a problem where the answer is 10. Just like the answer can be 9 it can also be 10 in the real world. You are on the way with the problem you have. Now think about a problem where the answer is going to be 10. (Ball transcript,

111408, p. 8)

During her interactions with these student teachers, Ball acknowledged the student teachers' progress. She asked if they have written a problem that had a different *type* of remainder. Ball suggested a second treatment of the remainder and proposed that the student teachers continue working.

Ball walked over to another group. In this group, a student teacher read her story problem. "I walk 38 blocks. Every 9 ½ blocks I stop for a break" (Ball transcript, 111406, p. 8). Ball interrupted the student teacher to point out that 9 ½ is not a number in the problem 38 ÷ 4 and that the teacher needed to use 4 as the divisor. In this instance, the student teachers lost sight of the original problem on which they were to work. Ball guided them back to the original task.

Ball asked the student teachers to read through the problems they had written and determine the interpretation of division for each problem. After a few minutes the class shared their work. Ball itemized the many details to which the student teachers had addressed. She provided the student teachers with a checklist. First, they were to check that each problem could be solved by finding the answer 38 divided by 4. Second, they needed to find the answer to the problem. Third, Ball asked the student teachers to notice which ones appeared similar and which seemed different (Zaslavsky, 2008). Finally, she asked the student teachers to be attentive to the remainder, its form, and what it represented.

Whole group discussion of 38 ÷4. The group began the discussion with a measurement problem. Jenna read her story problem, "Mandy has 38 inches of ribbon. She cut them into 4 inch long strips. How many ribbons does she end up with?" Ball

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attempted to clarify that Jenna was asking about the number of pieces of ribbon. Then she

asked Jenna two questions. First, is the answer to the problem found by computing 38

divided by 4? Second, what interpretation of division does the problem model? Jenna

stated that the problem is a measurement interpretation problem, Ball revoiced this fact

and asked the student teachers about the quotient and the remainder.

Ball: Yes, this fits nicely with measurement because you have 38 inches of
             something and you are measuring off 4 inch pieces. So this might be one
             of the easier ones because you have measurement right in the problem. So
             what is the answer to this problem?

Jenna: 9 remainder 2

Ball: What does that mean? 9 remainder 2 pieces. What does that mean? How
             many pieces would she end up with?

Jenna: 9 and one half.

Ball: Okay, think about it really. You ask how many pieces does she have? She
             Is cutting them up into 4 inch pieces.

Jenna: 10 pieces

Ball: How is she going to get 10 pieces?

Jenna: She would have 9 pieces of 4 inches.

Ball: 9 pieces of?

Jenna: 9 pieces of equal length.

Ball: What length are they?

Jenna: 4 inches.

Ball: Okay, 9 pieces of 4 inch ribbon. How about we say 9 pieces of 4 inch
             ribbon and one piece of 2 inch ribbon? Can you hear that? (Ball
             transcript, 111406, p. 8)

        Ball attempted to get the student teachers to think carefully about the quotient and

remainder. When Jenna offered that the answer was nine and one half, Ball probed

Jenna's understanding of her problem. She asked her to clarify the meaning of the answer nine and one half. The student teachers clarified that the problem asked for the number of pieces of ribbon. Jenna changed her answer to 10 pieces. Ball sought an explanation for the answer of 10. Jenna attempted to edit her answer and stated that there were nine pieces of equal length. Ball attempted to revise the answer. She noted that there were "9 pieces of 4 inch ribbon and one piece of 2 inch ribbon" (Ball transcript, 111406, p. 9). She asked, "Can you hear that?" (p. 9). It appeared that Ball tried to use this initial problem to model the analytical process for this work of examining answers to problems, specifically analyzing quotients and remainders. She attempted to model the precise language that she expected the student teachers to use. With the analysis and language for the answer 9 pieces of 4 inch ribbon and 1 piece of 2 inch ribbon established, Ball shifted the conversation to alternative numerical answers.

After working on ways of expressing the answers to problems, Ball asked the student teachers for alternative ways to state the answer. Brian asked, "Wouldn't it be nine and a half pieces? They have four inch pieces so there would be a half of a four" (Ball transcript, 111406, p. 9). Ball probed Brian's thinking about this problem.

Ball: So once she's cut the 9 pieces how many inches of ribbon has she used up? How many?

Brian: 36 inches.

Ball: How do you know that?

Brian: 9 times 4 is 36.

Ball: So what is left? Can you picture that?

Brian: 2 inches.

Ball: 2 inches. So you have a choice. What are the choices for the answer to

this problem? You could say that the answer is nine because the question asks how many of these 4 inch pieces does she end up with. What would you say about the other ribbon? A half of a four inch piece, or 2 inches of ribbon left over. Okay, so this question of what to do with a remainder is an important question. It says that you have 2 inches left over. Why don't we say it that way since you were asking about 4 inch pieces? Why don't we say with 2 inches left over? So the answer that Jessica wrote on the board, does this problem go with that? [The answer written on the board is 9 ½.]

Ball: There is a way of thinking about it as 9 and a half. What is a way of thinking about it as 9 and a half?

Sandy: When there are 9 – 4 inch pieces and a half of a 4 inch piece.

Ball: And half of a 4 inch piece. Any comments about this one? (Ball transcript, 111406, p. 9).

Ball attempted to address Brian's solution in a way that confirmed its plausibility and analyzed the solution to make visible why it was an alternate way to represent the solution. At this point in the conversation, Ball and the student teachers had attempted to examine one story problem. They determined that it demonstrated the measurement interpretation of division. Further, they worked to discuss two ways to represent the solution and tried to perfect the language to communicate the solution. Ball paused to ask the student teachers if they "see how we analyze this?" (p. 9). Along with the solution to this specific problem, Ball sought to model the analysis of the solution so that the student teachers could apply this procedure to other problems. She moved the student teachers along to consider a new story problem for $38 \div 4$.

Ball and the student teachers continued to discuss story problems for $38 \div 4$. They attempted to review the cookie problem that asked how many children would receive had four cookies distributed evenly among children. Jenna began the conversation by trying to compare the ribbon and cookie problems.

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Jenna:	For the ribbon, the answer is 9 pieces of ribbon and 2 inches of ribbon left over. For the cookies, 9 children get cookies and there are 2 cookies left over.
Ball:	Does everyone see what Jenna is noticing? The main answer is in terms of beyond their numerical component. The answers have a meaning that relates to the story problem and it is important for this meaning be communicated. (Ball transcript, 111408, p. 10)

Ball attempted to elicit story problem for which the answer is 9.5 or 9 or a problem and that had a different interpretation? The student teachers worked on problems to produce an answer of nine. Using cookies as a context, a student posed the problem, "Cookie Monster has 38 cookies. He will eat 4 each day. How many days until they are gone?" (Ball transcript, 111408, p. 10). Ball confirmed that this is an example of measurement division. After an extended discussion about the solution to the problem, a student teacher, Amanda, stated that it would take 9.5 days for the cookies to be eaten. The student teachers debated the solution.

Amanda:	Well, we could think about the problem as Cookie Monster eating one cookie every six hours so it would take 9.5 days.
Brian:	That might be 10 days.
Jenna:	I don't understand why that wouldn't be 9 remainder 2.
Amanda:	Wouldn't you use remainder 2 in the problem?
Jenna:	I don't understand why it would be 9.5
Ball:	What about 10?
Jenna:	Ya, if you have enough for 9 days and you would have 2 cookies left over.
Lindsey:	If you think about measurement, you are like if you are eating them every 6 hours, you would be eating them for 12 hours which is one half of a day. So that is .5. So do you have to put equally in there?
Ball:	Where do you want it?

Diana: In the question. So I was thinking it could be 9 remainder 2.

Sandy: I know how you can get it. If you say everyday he has 4 cookies, how many days does he have 4 cookies?

Brian: That would be 9.

Lindsey: True.

Ball: So you could edit this to read: on how many days could she eat 4 cookies and how many would be left? But that is not how it was originally written. So the way it was originally written. How many days – it would be 10 days. What interpretation did we say this was? Measurement? (Ball transcript, 111406, p. 10)

The student teachers attempted to think through several ways to express the solution to this problem. Ball worked with them to find an appropriate wording of the problem and to analyze how the wording affects the solution.

Ball: So the 4 tells you how many groups you have. So if the 4 tells you how many are in each group. It is partitive. Why?

Sarah: You have a total of 38 miles.

Ball: Why were you so careful with your wording? Why did you say equal number of miles each day? Why does that matter?

Janice: It has to be 9.5.

Ball: If Janice had written this – he ran 4 days, how many miles did he run each day? Well, one day he could have run 1 mile, another day he could have run 9 miles, whatever. (Ball transcript, 111406, p. 11)

Ball seeks to make visible the distinction between partitive and measurement division. She appears to emphasize that partitive division is identifiable by the problem creating a number of groups designated by the divisor that are of equal size. Likewise, measurement division is identifiable by the problem providing the number of elements in each group and asking for the number of groups. When a student teacher identifies the ribbon problem as an example of measurement division, Ball agrees and seeks to make

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the connection between the fact that the ribbon problem is about measuring ribbon so that it can get cut into small pieces of a particular size and the interpretation is about measuring out equal size portions.

Analysis of the work to analyze story problems: quotients and remainders. Along with the distinctions between partitive and measurement division, Ball attempted to raise the student teachers' awareness to another aspect of whole number division, the nature of the quotient and remainder. I revisit the story problem work to extract and analyze Ball's work to attend to the quotients of the story problems, particularly those that raised differences in the nature of quotients. I identify two distinctions. The first distinction addresses the nature of the quotients and remainders. Some problems have quotients and remainders that are of the same object; other problems have quotients and remainders that are of different objects. The second distinction is of problems whose answers may take on a variety of forms such as a whole number quotient with a remainder, a fraction or decimal, or a whole number that requires the interpretation of the problem to determine which integer value is most appropriate for the answer.

My analysis of the first two problems in the teaching segment indicates that they may demonstrate the first distinction. The first problem is: Thomas has 38 pieces of candy and 4 friends. If Thomas gives the pieces of candy out equally, how many pieces of candy can each friend get? The answer to this problem is: each friend receives 4 pieces of candy and there are two pieces of candy left. The second problem is: You have a shelf that is 38 inches long. You want to store books that are 4 inches wide. How many books can you store? The answer to this problem is: You can store 9 books and there are 2 inches of shelf length left over. In the first problem, the quotient and remainder are both

numbers of pieces of candy; in the second problem, the quotient is the number of books and the remainder is the number of inches of shelf length left over. Ball seeks to make this distinction explicit. During the teaching episode, Ball and the student teachers work through the first four problems summarized in Table 5.3[24]. She paused and asked the student teachers to think about the ways the problems are similar and different. Sharon offered a response.

Sharon: For the ribbon, the answer is 9 pieces of ribbon and 2 inches of ribbon left over. For the cookies, 9 children get cookies and there are 2 cookies left over.

Ball: Does everyone see what Sharon is noticing? The main answer is in terms of one thing and the left over is in terms of something else. Two inches of ribbon left over not pieces of ribbon. Two cookies left over not children left over. Does everyone see that? That is very nice. That is something you have to be really careful with and thinking about what the numbers mean. These two problems turn out to be largely the same. One is about cookies and one about ribbon. (Ball transcript, 111406, p. 10).

In this work, Ball attempted to make an important characteristic of quotients visible to the teachers. She used two of the student generated examples as problems where the quotients and remainders are either like quantities, that is, they quantified an amount of the same thing (9 pieces of candy with 2 pieces of candy left over)or of unlike quantities (9 pieces with 2 inches left over). After this characteristic was discussed, Ball and the student teachers examined each student generated example to determine whether the quotients and remainders represented like or unlike quantities. With this characteristic of quotients and remainders examined, Ball appeared to use the ribbon problem to demonstrate that there may be several ways to answer some division problems. I turn now to examine Ball's use of the ribbon problem as an example and the variety of answers she elicited.

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[24] These are the pieces of candy, book shelf, cookie, and ribbon problems.

Ball attempted to demonstrate through the student teachers' examples the variety of ways division problems may be answered. When analyzing the ribbon problem, the student teachers arrived at several possible ways to think about and express the answer to this problem. The first answer, 9 remainder 2, was appropriate for the answer to the calculation 38 ÷ 4. Ball sought to elicit answers that communicate some connection to the story. She asked the teachers to read the problem and attempted to provide an answer that more closely communicated an answer for the number of pieces of ribbon. Student teachers provided two responses: 9 ½ pieces and 10 pieces of ribbon. When Ball asked the student teachers to state how many *four inch pieces of ribbon* Mandy could cut, a student teacher responded that there were 9 - 4 inch pieces of ribbon that could be cut from the 38 inch strip. Work with this story problem generated several possible answers. Therefore, Ball was able to examine it and help the teachers consider a variety of answers and the ways the answers fit the problem.

This brief analysis of the ribbon problem examined the variety of answers Ball elicited from the student teachers. It might be informative to review the goals Ball and her colleagues had in designing this problem. During an interview, Ball noted several purposes for this task, the writing of story problems for the division problem 38 ÷ 4. First, she noted that this is an activity that teachers do for the work of teaching. That is, teachers "create or interpret story problems" as part of their work to teach children. Second, she stated that the story problems may have contextualized meanings that differ from the numerical meaning of the calculation 38 ÷ 4. Third, the story problems may provide contexts for discussion of continuous or discrete numbers and the implications for the problem and its answer. Fourth, the activity may provide opportunities to examine

similarities and differences across the story problems and their solutions. Fifth, Ball saw this activity as one which fosters skills student teachers may use for teaching. This activity provides teachers with opportunities to listen "to other people's work and evaluate it, learn to be more careful about distinctions, practice making and practice listening" to others' work (Ball interview, 112806, p. 7). Ball perceived this activity as one that fostered the teachers' MKT for their own knowledge and for their use of MKT as teachers.

I present the list of student generated examples written and discussed by the student teachers, the interpretations they represent, and their quotients and remainders in Table 5.3.

Story Problems and Their Characteristics for 38 ÷ 4		
Story problem	Interpretation Dividend/divisor	Quotient & remainder
Thomas has 38 pieces of candy and 4 friends. If Thomas gives the pieces of candy out equally, how many pieces of candy can each friend get? You have a 38 inch shelf. You want to store books that are 4 inches wide. How many books can you store?	Partitive Discrete / discrete Measurement Continuous/discrete	9 pieces of candy and 2 pieces of candy left 9 books with 2 inches of shelf length left over
There are 38 cookies to be divided among 4 students. How many cookies will each student get and how many will be left over?	Partitive Discrete / discrete	9 cookies with 2 cookies left over
I walk 38 blocks. Every 9 ½ blocks I stop for a break.	Measurement Discrete /discrete	Expected answer: 4 breaks and 2 blocks left over
Mandy has 38 inches of ribbon. She will cut them into 4 inch long strips. How many ribbons does she end up with?	Measurement Continuous /continuous	9 – 4 inch strips of ribbon and 1 – 2 inch strip of ribbon 10 pieces of ribbon 9 pieces of equal length 9 ½ pieces
Cookie Monster has 38 cookies. He will eat 4 each day. How many days until the cookies are gone?	Measurement Discrete / discrete Discrete / continuous	9.5 days 10 days 9 days with 2 cookies left
Bob runs 38 miles each week. If he runs 4 days a week and an equal number of miles each day, how many miles does he run each day?	Partitive Continuous /discrete	9.5 miles each day

Table 1. A summary of the examples to discuss partitive and measurement division through the writing of story problems for the calculation 38 ÷ 4.

Appendix 5.2

Analysis of Elements of the Work to Teach Mathematics to Teachers (Ball)
A Working Document

Whole Number Division		
Selecting Interpretations / Representations	**Selecting Examples**	**Managing Enactment**

1. Surveying possible interpretations / representations
  a. Surveying or gathering possible interpretations / representations for whole number division
	1. Reviewing instructional goals
	  -- identify the meaning of partitive and measurement division
		1. Reviewing or rehearsing (MMPG)
		  a. talking through the lesson
		    --how does it unfold
		    -- how does the math evolve, become unpacked, is layered
		    -- does the lesson maintain the integrity of the math *and* make it understandable for teachers
		  b. modifying
		  c. fostering ways to help teachers think abstractly

2. Deliberating about which interpretations / representations to use
  a. examining instructional goals of the course, the lesson, and MMPG
  b. considering which interpretations and representations support instructional goals
  c. considering materials to model representations
    --- availability
    --- connected to materials used to model other operations
  d. considering how the interpretations / representations build on prior and future mathematical work
	2. Ensuring usability for instruction
	  a. unpack mathematics intended by instructional goals
	  b. usable for teaching
		2. Launching
		  a. reviewing content foundational to this lesson
		  b. presenting new mathematical ideas
		  c. probing or challenging teachers' incoming knowledge
		
		  a. generating work on or conversation about mathematics
		    ---building on reading from homework
		    ---student teachers' work to create representations of partitive and measurement division

3. Deciding on which interpretations / representations to use
  a. weighing instructional goals of the lesson and the course
  b. choosing interpretations / representations that support the instructional goals
    --- integrity of math
    --- unpack mathematics
    --- link to school mathematics
	3. Sequencing examples
	  a. focusing attention on partitive and measurement interpretations
	  b. sequencing to challenge student teachers' abilities to distinguish between the role of the divisor for each interpretation
	  c. sequencing to unpack (12 ÷ 6), reinforce (12 ÷ 2), and extend (8 ÷ b)
		3. Making mathematical ideas visible
		  a. student teacher work to create representations of 12 ÷ 6 using partitive and measurement interpretations building on readings from homework
		  b. teacher educator interacting with individual student teachers to have them explain representations / engage with mathematics

		---questioning and probing --- student teachers explaining representations ---scaffolding teachers' difficulty / misunderstanding c. engage in talk about mathematics --teacher to teacher talk --teacher educator interaction ---large group work --teacher educator / teacher -- teacher / teacher d. interaction ---initial questioning ---deeper probing ---confirming response ---redirecting ---pressing to make connections ---what to pursue, what to let go (this determines the mathematics that is taken up) e. raising teachers' awareness to SCK ---explicitness ---detail of mathematical work Parsing Organizing Layering Threading all done with intentionality
	4. Using examples from a repository a. creation of "in the moment" example to challenge misstatement of definition (Ellie's "separate" groups refined to "equal" groups)—this may be from a repository also her use of $35 \div 5$ during interview may come from a repository	4. Assessing a. listening and responding to teachers' work in whole group probing b. listening and responding to teachers' work in small group or individual work c. editing teachers' work

Analysis of Elements of the Work to Teach Mathematics to Teachers Matrix

Decimal Multiplication		
Selecting Interpretations / Representations	**Selecting Examples**	**Managing Enactment**

1. Surveying possible interpretations / representations
  a. surveying or gathering possible interpretations / representations for this case, the interpretations / representations of <u>both</u> multiplication and decimals
	1. Reviewing instructional goals
	  a. identify mathematical ideas within instructional goals
		1. Launching
		  a. generating work on or conversation about mathematics
		  b. challenging student teachers' incoming knowledge
		  c. presenting an example
		  d. modeling
		  e. presenting mathematical ideas

2. Deliberating about which interpretations / representations to use
  a. examine instructional goals
  b. parsing the task for each mathematical component
  c. considering which interpretations and representations support instructional goals
  d. considering which are usable for teaching
   ---considering which are usable for the work of teaching – grids could be easily reproduced, drawn in notebooks
  ---considering which did or did not produce student teacher errors – base 10 blocks used as border a common error with student teachers
  e. considering how available interpretations / representations attend to the mathematics or what mathematical details they reveal
  f. considering how the interpretations / representations build on prior and future mathematical work
  g. considering how the interpretations and representations might foster misconceptions,
	2. Identifying compressed mathematics	2. Developing mathematical ideas
		  a. presenting in lecture format
		  b. engaging in mathematical conversation
		    --- paired or small group work
		    ---large group work
		    ---recitation of precisely worded definitions, procedures
		    ---exploration of teachers' thinking
		    ---practice of modeling, algorithms
		    --- orchestration of conversation
		      ---questioning
		      ---responding
		      ---what to pursue, what to let go (this determines the mathematics that is taken up)
		Parsing
		Organizing
		Layering
		Threading
		  all done with intentionality
		
		Raising teachers' awareness to SCK
		  Explicitness
		  Making mathematical sense
		  Raising teachers' awareness to this sensemaking
		
		Connecting mathematics to the work of teaching children

3. Deciding on which interpretations / representations to use
  a. choosing interpretations /
	3. Sequencing examples
	  a. considering the layers of mk packed within the lesson
	  c. sequencing to foster layered
		3. Engaging teachers with mathematics
		  -individual solution to computation

representations that support the instructional goals. (integrity of math, unpack mathematics, two interpretations to examine multiplication from different perspectives
 	
  b. choosing interpretations/ representations that allow the work to begin with teachers' incoming knowledge and foster it to the identified mathematical knowledge for teaching
 	
  c. choosing interpretations / representations that build on prior lessons and make connections across domains
 	
  d. working from hypothesis that using both interpretations of multiplication would provide opportunities to view factors and products differently and develop a deeper understanding of multiplication
	learning
	  b. which to be used for introducing, unpacking and layering, reinforcing, extending
		- work with individual teachers
		  -sharing of work in whole group
		  - beginning the conversation by providing the "first step" – begin with the number line…
		small group or pairs
		  - listening & responding to teachers
		  - guiding teachers work

	4. Assessing affordances and limitations of interpretations and representation	5. Assessing
		  a. listening and responding to teachers' work in whole group
		  b. listening and responding to teachers' work in small group or individual work
		   c. editing teachers' work

Appendix 5.3
The Case of Two Interpretations of Decimal Multiplication

The Teaching of Interpretations of Decimal Multiplication

The following vignette describes a class session during which Ball modeled decimal multiplication. Just prior to this lesson, she worked with the teachers to explicate the mathematical details of the 10 x 10 grid. For the lesson on decimal multiplication, Ball used the 10 x 10 grid to represent both the repeated addition and area interpretations of multiplication and to explain the mathematical characteristics of this operation. Ball introduced the lesson by modeling the decimal multiplication 0.7 x 0.1 using the area model of multiplication. For this model, the product of 0.7 x 0.1 is by definition the area of the rectangle having side measures of 0.7 and 0.1.

Introducing the Concept of Decimal Multiplication

Launching the task. Ball launched this lesson segment by asking the student teachers to multiply 0.7 x 0.1 in their notebooks. After they performed the calculation, Ball used the 10 x 10 grid representation to model the multiplication and to justify the product. First, she demonstrated how the factors were represented by the grid. She guided her finger along the vertical edge of the grid until she traced the edge of seven small squares in an attempt to represent 0.7. Then she traced the horizontal edge of the upper left small square in an attempt to represent 0.1. She traced the perimeter of the resulting rectangle and designated it as the rectangular region for 0.7 x 0.1. She counted the squares that comprised the resulting area and stated that because the region was made up of seven small squares, the product of 0.7 x 0.1 was seven hundredths. She reviewed the work with the student teachers.

Ball: So I know from what we just did [in deploying the 10 x 10 grid] that each

one of these segments is one tenth. I need seven tenths. One, two, three, four, five, six, seven. So the side length of this rectangle has a side length of seven tenths. Can you see that okay? Do you see where I get seven tenths? And I want the width of my rectangle to be one tenth. And I know that one of these has a side length of one tenth. So now I have made the rectangle with two sides. I can complete it by drawing the additional two sides of the rectangle. Now I have a rectangle that is seven tenths by one tenth. Okay? Do you see my rectangle? What is the area of my rectangle? What is the area of my rectangle? Sarah?

Sarah: Seven hundredths.

Ball: Seven hundredths?

Sarah: There are seven little boxes and each little box has area of one hundredth. So the area is seven hundredths.

Ball: So there are seven little squares. There are one hundred small squares in this drawing and we agreed earlier that each small square has an area of one hundredth. We have heard a couple of different ways to explain that. One way is that it is one hundredth of the whole. And Jenna said that it is one tenth times one tenth which is one hundredth. So here we have one, two, three, four, five, six, seven hundredths. So my problem seven tenths times one tenth equals seven hundredths. Why don't you do the problem seven tenths times one tenth as a written problem in your notebook right now? Not as a fraction, but as a decimal. So point seven times point one. What is the answer that you get?

In this segment, Ball attempted to make visible the rectangular region that represents 0.7 x 0.1 and to engage the student teachers in drawing the representation and using the representation to find the product of 0.7 x 0.1. She asked the student teachers to find the product and to explain their answer. A student teacher, Sarah, offered the answer of seven-hundredths. She explained that the product is found by counting the number of "boxes" in the rectangle. She identified the value of each "box" as one one-hundredth and stated that the resulting area of seven hundredths.

Ball restated Sarah's response, editing it to include agreed upon language. Further, she added to Sarah's response by including the justification for one small square representing one hundredth.

Ball asked the student teachers to reproduce their co-created work of the multiplication of 0.7 x 0.1 in their notebooks. She walked around the room and attempted to assess the student teachers' thinking by reading their written work. She spoke with individual student teachers, probing their understanding of decimal multiplication and the use of the 10 x 10 grid. When she asked the student teachers about the product, one responded that the answer was "point zero seven." Ball appeared to be dissatisfied with the response. She asked the student teachers for the answer using correct mathematical language. A student teacher responded, "seven hundredths." Ball checked that the teachers were in agreement.

Ball: Does every one have that? So computationally you know the way to get that answer. Now you have that if you have a rectangle with side length of seven tenths and a side length of one tenth, the area is seven hundredths which fits with what you came up with from your calculation. (Ball transcript, 110706, p. 3)

During this brief commentary, Ball attempted to connect the calculation of the product to the representation on the grid. She challenged the teachers to map the computation to the representation. She asked them to map seven tenths to a side length of seven small squares along the vertical edge and one tenth to a side length of one small square along the horizontal edge. Then, she asked them to map the product of seven hundredths to the area of the rectangle made of seven small squares. This work challenged the student teachers to map carefully between the two representations of this mathematics, the calculation and the grid.

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Ball paused to check the student teachers' understanding. She asked if they

understood why the product was seven hundredths. One teacher, Shannon, reflected back

to how she multiplied decimals as a student and asked why her strategy was not correct.

Shannon: I haven't done this in so long. Like the point zero seven. As a student, how
            do you know? I was taught to move the decimal point the number of
            places to the, I don't know. Since it is point seven times point one, why
            not point seven? Point seven times point one, then it is point seven.
            (Ball transcript, 110706, p. 3)

Ball's work to map between the representations appeared to make visible a

common error for decimal operations. That is, the addition of decimal algorithm, lining

up the place values and adding place value by place value is applied to decimal

multiplication. Sarah raised an important issue. That is, she and many student teachers

had not multiplied decimals in a long time. And for many, they may not have done this

operation without the use of a calculator. Also, Ball's attempt to deploy the 10 x 10 grid

representation and use it to explain the product of 0.7 x 0.1 appeared to have generated

the teachers' *thinking* about the standard algorithm for multiplication of decimals.

Ball launched the lesson with the problem: multiply 0.7 x 0.1. With this task, a

common error was made visible. Confusion about the placement of the decimal point in

the product of two decimal numbers became the focus of the lesson. With this

accomplished, Ball began a carefully orchestrated lesson on decimal multiplication. She

began with whole number multiplication, an operation the class had studied previously.

Then, she segued to an examination of decimal multiplication. I continue the presentation

of Ball's work to teach decimal multiplication. For this lesson, she parsed the

mathematics into three layers of difficulty: multiplication of two whole numbers,

multiplication of one whole number and one decimal number, and multiplication of two

decimal numbers. Further, she discussed multiplication using both the repeated addition and area interpretations of multiplication.

Revealing place value in decimal multiplication layer by layer. Ball noted that student teachers like Shannon have used algorithms to perform calculations without understanding why these algorithms work. Consequently, she thinks it important that student teachers come to understand the mathematical underpinnings of algorithms as well as other mathematics that are foundational to the mathematics that teachers teach children (Ball interview, 112808). Ball sought to provide a mathematical explanation for the placement of the decimal point in the product of two decimal numerals. It appears that she selected a series of multiplication problems beginning with the multiplication of two whole numbers, 7 x 1 and ending with the multiplication of 0.7 x 0.5 to accomplish this goal. She worked through this series of problems demonstrating the multiplication with the 10 x 10 grid. Further, she attempted to demonstrate the multiplication using both the repeated addition and area interpretations of the operation. Table 5.1 summarizes the multiplication problems Ball used to unpack mathematical knowledge for teaching decimal multiplication.

Multiplication Problem	Repeated Addition	Area
7 x 1	Modeled	Not modeled
7 x 0.1	Modeled	Modeled
0.7 x 0.1	Modeled	Modeled
0.7 x 0.5	Modeled	Modeled

Table 5.1. Summary of multiplication problems and interpretations used by Ball to scaffold the work for decimal multiplication.

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Ball attempted to use the grid as a geometric model to justify placement of the decimal point in the product of two numbers with digits in the tenths place. Specifically, she sought to develop the understanding that the product of seven tenths times one tenth equals seven hundredths. For this work, Ball began with whole number multiplication and worked through additional examples, adjusting one factor at a time. She multiplied 7 x 1, 7 x 0.1, 0.7 x 0.1, and 0.7 x 0.5. To demonstrate the scaffolded nature of this work, I present a summary of Ball's instruction and include figures that replicate the representations of multiplication produced by Ball using the 10 x 10 grid.

Layered examples to teach decimal multiplication. Ball and the student teachers had used the area interpretation of multiplication and the 10 x 10 grid to discuss decimal multiplication. They sought to confirm the placement of the decimal point in products where one tenth is multiplied by one tenth using the 10 x 10 grid. In the following vignette, Ball proposed a second explanation for the decimal multiplication algorithm. She reminded the teachers about the repeated addition interpretation of multiplication and she used this interpretation of multiplication to find products and corroborate results of problems done with the area interpretation.

Beginning with what the teachers know: Whole number multiplication. Ball presented a multiplication problem beginning with whole number multiplication. She asked the student teachers to represent 7 x 1 using the repeated addition interpretation in their notebooks. The student teachers drew seven units in their notebooks. They wrote 7 x 1 = 1 + 1 + 1 + 1 + 1 + 1 + 1. See figure 5.3 for the replication of Ball's work.
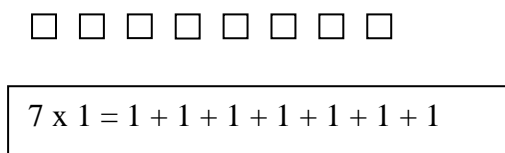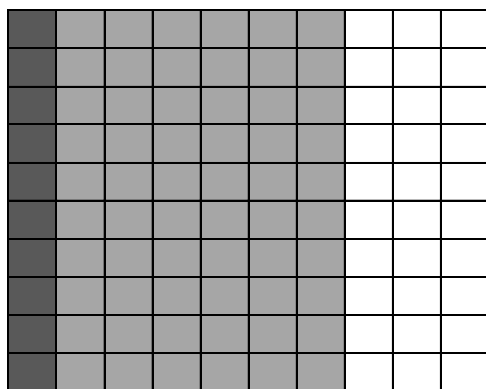

Repeated Addition Interpretation

☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐

| 7 x 1 = 1 + 1 + 1 + 1 + 1 + 1 + 1 |

Figure 5.3. The repeated addition interpretation of 7 x 1.

With this review of whole number multiplication complete, Ball adjusted the mathematical nature of the multiplication by making one factor a decimal. She asked the teachers to multiply 7 times 0.1. She began with the operand and identified the one tenth as one column in the 10 x 10 grid. She shaded that column and designated it as one tenth. Then she used the repeated addition interpretation of multiplication. She counted off seven of the columns in an effort to represent seven one-tenths of the grid. She shaded the region. See figure 5.4.
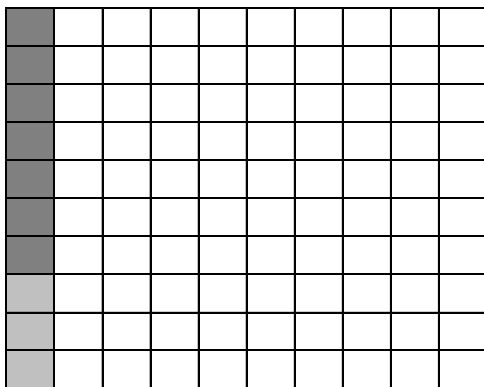
■ represents 0.1   ▦ represents 0.6   total shaded region represents 0.7

Figure 5.4. Ball's representation of the repeated addition interpretation of 7 x 0.1 = 0.7.

Ball paused to allow the student teachers ample time to complete their work. The student teachers attempted to replicate the multiplication of 7 x 0.1 in their notebooks. When the student teachers seemed to be in agreement with this representation and the answer to this calculation, Ball presented another example, one where two factors were decimals.

Ball demonstrated the multiplication of 0.7 x 0.1. She began by shading in one tenth of the 10 x 10 grid. She attempted to demonstrate that the multiplication of one tenth by seven tenths is different from multiplying by any whole number. To do this, Ball showed that one tenth (the column representing one tenth of the 10 x 10 grid) is partitioned into 10 equal small squares. Seven of these small squares represent seven tenths of the column that represented one tenth. She counted seven small squares and noted that the product was 0.07. See figure 5.5 for a replication of Ball's drawing.

█ 0.7 x 0.1 = 0.07   ▫ entire shaded region is 0.1

Figure 5.5. Ball's representation of the repeated addition interpretation of
0.7 x 0.1 = 0.07.

Ball confirmed the product of 0.07 by presenting the area interpretation of the multiplication of seven tenths times one tenth. She drew the rectangle for this problem

and identified a vertical length of seven tenths and a horizontal length of one tenth. She

identified the area of the resulting rectangle as the product of 0.7 x 0.1. She reasoned that

since seven small squares are shaded and each small square represents one hundredth, the

product is seven hundredths. To designate the region that represented the product, Ball

made a bold outline of the rectangle and placed a small circle in each small square

counting the number of small squares as she progressed. See figure 5.6 for Ball's
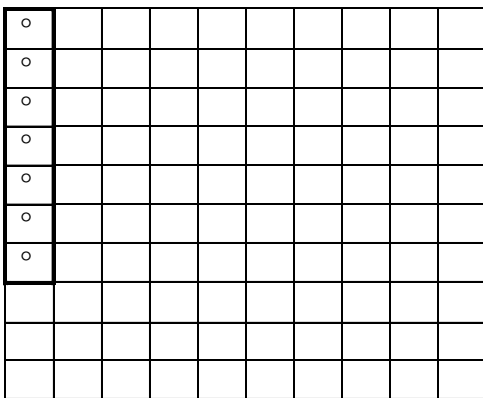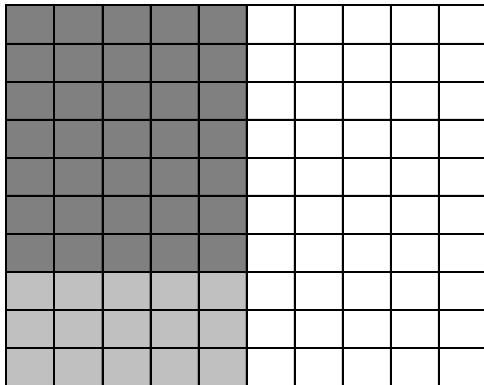
representation.

Figure 5.6. Ball's representation of the area interpretation of 0.7 x 0.1 = 0.07.

Ball adjusted this calculation by one more characteristic. She suggested that the

student teachers multiply 0.7 x 0.5. She continued by representing 0.7 x 0.5 using both

the repeated addition and area interpretations of multiplication. She began by shading the

operand, five tenths of the 10 x 10 grid or 5 columns. She demonstrated the process of

multiplying one of the one tenths by seven tenths. She shaded seven small squares of the

column or seven tenths of that left most column or one tenth. This replicated her work to

multiply 0.7 x 0.1. Because the problem asked for seven tenths of *five* tenths, she shaded

seven of the ten small squares in each of the columns that make up the five tenths. She

calculated the number of small squares; 7 x 5 = 35. She reasoned that because each small

square represented one hundredth, the product was 35 hundredths. See figure 5.7 for a

replication of the Ball's representation of 0.7 x 0.5 = 0.35.

0.7 x 0.5 = 0.35

Figure 5.7. Ball's representation of the repeated addition interpretation of 0.7 x 0.5.

       Ball followed the pattern of the previous examples. She attempted to confirm the

product of 0.35 through the use of the area interpretation. Consistent with the other

calculations, she traced the length of seven tenths along the vertical edge of the 10 x 10

grid. Then, she traced the length of five tenths along the horizontal edge. She drew the

rectangle defined by these segments and the segments parallel to them. She counted out

the small squares in the resulting rectangle to arrive at the area of 35 small squares or

0.35. See figure 5.8 for a replication of Ball's representation.
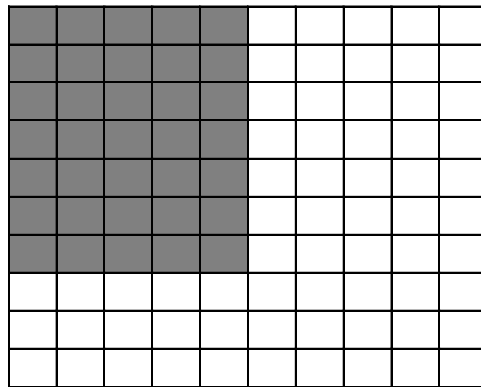
☐ 0.7 x 0.5 = 0.35

Figure 5.8. Ball's representation of the area interpretation of 0.7 x 0.5 = 0.35.

The lesson continued with the examination of the roles of operator and operand. Because of the length of this vignette, I pause the presentation and transition to analyze the work of teaching decimal multiplication to student teachers. In this analysis, I examine three tasks of teacher education for the finer details, the elements that make up this work. I ask, "What pieces of work must be done to select interpretations and representations, select examples, and manage the enactment of the task? What work is involved in beginning with the mathematical knowledge student teachers possess and fostering this into mathematical knowledge for teaching?" I begin by analyzing the work to select interpretations and representations. For decimal multiplication, this involved selection of interpretations and representations for *decimals* and *multiplication.*

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