Computations in stable motivic homotopy theory

by

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To Mom & Dad,
who are stably nontrivial.

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CHAPTER I

Introduction

The cast of characters in motivic homotopy theory is well-known among specialists, but its unique mix of concepts from algebraic geometry, algebraic topology, and abstract homotopy theory makes it a formidable edifice when confronted by the casual observer. In this introduction, I aim to introduce the players and state foundational results that will be used in the rest of this thesis. This serves the dual purpose of establishing notation and giving the non-expert a toehold on the concepts and literature of motivic homotopy theory.

In §1.1, I introduce the basic objects and constructions of motivic homotopy theory. Over a base field k, motivic homotopy theory is built from the category of k-spaces, denoted $\mathrm{Spc}(k)$, which is the category of simplicial presheaves on Sm/k, the category of smooth k-schemes. I describe a model structure on $\mathrm{Spc}(k)$ that, after localization, produces the motivic homotopy category, $\mathcal{H}(k)$. The basic idea is to impose the properties of simplicial homotopy theory with a first model category that also captures the Nisnevich topology on smooth k-schemes, and then formally localize that category with respect to $* \to \mathbb{A}^1$.

I then go on to describe stable motivic homotopy theory, $\mathcal{SH}(k)$, in which smash product with the projective line is invertible. I arrive at this theory by produc-

ing a model structure on the category of k-spectra, Spt(k). (Here the topological nomenclature dominates that of algebraic geometry: a spectrum will be a sequence of k-spaces (X_n) and bonding maps $\mathbb{P}^1 \wedge X_n \to X_{n+1}$, not an affine scheme.) Stable motivic homotopy is a natural home for many cohomology theories of interest to algebraic geometers. These include motivic cohomology, algebraic K-theory, algebraic cobordism, and Hermitian K-theory, and I will sketch the construction of all of these.

In §1.2, I describe Voevodsky's fundamental calculations in motivic homotopy theory. First, I sketch how the resolution of the Milnor conjecture determines motivic cohomology in terms of Milnor K-theory. I then describe the motivic Steenrod algebra. Some care is taken here to connect results on reduced power operations from the '90s and motivic Eilenberg-MacLane spaces from the '00s that, combined, give a full description of the motivic Steenrod algebra (as a Hopf algebroid) and its dual.

In §1.3, I introduce the basic objects motivating this research, the stable motivic homotopy groups of spheres. I describe some of the topological background and history of these objects and Morel's contributions to their study.

In §1.4, I introduce my main computational tool, a class of motivic Adams-type spectral sequences. Based on the topological version in [Rav86, Chapter 2], I construct their motivic analogues and discuss their convergence properties. I discuss the tri-grading of the spectral sequence and the manner in which I will draw their diagrams.

In §1.5, I discuss some of the arithmetic structure of p-adic fields. The rest of this thesis concerns itself with computations in $\mathcal{SH}(F)$, where F is a p-adic field ($p > 2$, see Convention 1.36), and I determine explicit formulas for all of the input

data needed later in terms of the arithmetic of F. This includes computations of $F^\times$, the (Grothendieck-)Witt ring of F, the mod 2 Milnor K-theory of F, the motivic cohomology of F, and the Milnor-Witt K-theory of F.

In the final section of this introduction, §1.6, I outline the remaining chapters and highlight the main results contained therein.

1.1 Motivic homotopy theory

Motivic homotopy theory was introduced by Fabien Morel and Vladimir Voevodsky in [MV99]. Informally, the theory enriches the category of smooth schemes over a base field so it also admits simplicial constructions, and then imposes a homotopy-theoretic structure in which the affine line $\mathbb{A}^1$ plays the role of the unit interval. The formal constructions depend heavily on the theory of model categories, and I refer to [DS95, Hir03] for the concepts from model category theory.

Since its introduction, the basic constructions of motivic homotopy theory have been recast by other authors to create (model theoretically) equivalent categories suited to various purposes [Jar00, Dug01, Hu03]. In this thesis, I follow Jardine's definitions in [Jar00], which contains a very clean presentation of the basic objects and proves various equivalences between his model, the Morel-Voevodsky model, and a certain folk model of motivic homotopy.

The following technical jargon will be necessary to define the structures of motivic homotopy. Let Δ denote the simplicial category with objects $\mathbf{n} = \{0, 1, \ldots, n\}$, $n \in \mathbb{N}$, and morphisms order-preserving functions $\mathbf{n} \to \mathbf{m}$. For any category $\mathcal{D}$, let $\Delta^{\mathrm{op}}\mathcal{D}$ be the category of simplicial objects in $\mathcal{D}$. In other words, $\Delta^{\mathrm{op}}\mathcal{D}$ is the category of functors $\Delta^{\mathrm{op}} \to \mathcal{D}$ with arrows natural transformations. It is well known that, for instance, $\Delta^{\mathrm{op}}\mathrm{Set}$ is a combinatorial model for topological spaces via the

singularization and realization functors. For any locally small category $\mathcal{C}$, let $\mathrm{Pre}(\mathcal{C})$ denote presheaves on $\mathcal{C}$, i.e., functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ for Set the category of sets and functions. It is possible to combine these constructions by considering the category of *simplicial presheaves* on $\mathcal{C}$, denoted $\Delta^{\mathrm{op}} \mathrm{Pre}(\mathcal{C})$.

Note that objects in $\Delta^{\mathrm{op}} \mathrm{Pre}(\mathcal{C})$ may be considered either as simplicial objects in the category of presheaves on $\mathcal{C}$, or as presheaves of simplicial sets on $\mathcal{C}$. Moreover, the category of simplicial presheaves on $\mathcal{C}$ contains both $\mathcal{C}$ and $\Delta^{\mathrm{op}} \mathrm{Set}$ as full subcategories. For any $C \in \mathcal{C}$, we have the representable presheaf $\mathcal{C}(\cdot, C)$ with value on $D \in \mathcal{C}$ the hom-set of morphisms in $\mathcal{C}$ from D to C. By forming the constant simplicial presheaf which takes the value $\mathcal{C}(\cdot, C)$ for every $n \in \mathbb{N}$, we get a simplicial Yoneda embedding $\mathcal{C} \hookrightarrow \Delta^{\mathrm{op}} \mathrm{Pre}(\mathcal{C})$. Now consider a simplicial set S and form the constant simplicial presheaf which takes the value S on each object of $\mathcal{C}$. This produces an embedding $\Delta^{\mathrm{op}} \mathrm{Set} \hookrightarrow \Delta^{\mathrm{op}} \mathrm{Pre}(\mathcal{C})$.

Fix a base field k and let Sm/k denote the category of smooth schemes over $\mathrm{Spec}(\mathsf{k})$. We are now ready to describe the category of k-spaces.

Definition 1.1. The category of k-spaces is

$$\mathrm{Spc}(\mathsf{k}) := \Delta^{\mathrm{op}} \mathrm{Pre}(\mathrm{Sm}/\mathsf{k}).$$

Remark 1.2. As noted above, $\mathrm{Spc}(\mathsf{k})$ contains copies of both Sm/k and $\Delta^{\mathrm{op}} \mathrm{Set}$ via the simplicial Yoneda embedding and the constant simplicial presheaf functors, respectively. By abuse of notation, I will consider smooth k-schemes and simplicial sets as objects in $\mathrm{Spc}(\mathsf{k})$ without reference to these functors.

The model category structure on $\mathrm{Spc}(\mathsf{k})$ with homotopy category the motivic homotopy category over k is constructed in two stages. First, we impose the *local model structure* on $\mathrm{Spc}(\mathsf{k})$. Consider the Nisnevich topology on Sm/k in which coverings

are given by collections of étale maps admitting splittings over points [Nis89]. It may be helpful to think of the Nisnevich topology as a sort of halfway house between the étale and Zariski topologies. Henceforth, I always consider Sm/k to be a Grothendieck site with respect to the Nisnevich topology. This site has enough points, so we can form stalks in $\mathrm{Spc}(\mathsf{k})$; note that each stalk of a k-space is a simplicial set. The *local weak equivalences* in $\mathrm{Spc}(\mathsf{k})$ are the maps of simplicial presheaves which induce weak equivalences of simplicial sets in all stalks. The *cofibrations* are simply the monomorphisms, and the *global fibrations* are defined via the usual right lifting property. It is a theorem of Jardine's [Jar87] that this produces a proper closed simplicial model structure on $\mathrm{Spc}(\mathsf{k})$.

The local model structure on $\mathrm{Spc}(\mathsf{k})$ has an associated homotopy category, denoted $\mathcal{H}_s(\mathsf{k})$, but this is not the motivic homotopy category we are after. The next step is to Bousfield localize the local model structure with respect to a rational point $* \to \mathbb{A}^1$. This has the effect of making all projections $X \times \mathbb{A}^1 \to X$ equivalences, and it is the sense in which motivic homotopy theory replaces the unit interval of topology with the affine line. Call this new model structure on $\mathrm{Spc}(\mathsf{k})$ the *motivic model structure*. This forms a proper closed simplicial model category by [Jar00, Theorem 1.1]. The associated homotopy category is denoted $\mathcal{H}(\mathsf{k})$, and this is the motivic homotopy category.

It is worth noting that we can be more explicit about the motivic model structure, but a definition is in order first.

Definition 1.3. If $X, Y \in \mathrm{Spc}(\mathsf{k})$, the *function complex* $\mathbf{hom}(X, Y)$ is the simplicial set with n-simplices consisting of maps of simplicial presheaves $X \times \Delta^n \to Y$.

In the motivic model category on $\mathrm{Spc}(\mathsf{k})$, define *motivic fibrant* objects to be those

with the right lifting property with respect to all cofibrations

$$(\mathbb{A}^1 \times A) \cup_A B \to \mathbb{A}^1 \times B$$

induced by $* \to \mathbb{A}^1$ and all cofibrations $A \to B$. A *motivic weak equivalence* is then a map $X \to Y$ inducing weak equivalences of simplicial sets

$$\mathbf{hom}(Y, Z) \to \mathbf{hom}(X, Z)$$

for all motivic fibrant objects Z. A *cofibration* is still a monomorphism of simplicial presheaves, and a map is a *motivic fibration* if it satisfies the right lifting property with respect to all motivic weak equivalences which are also cofibrations.

Remark 1.4. One of the features of motivic homotopy theory is that it admits a bigraded family of spheres, $S^{m+n\alpha}$. Here S^1 is the simplicial set S^1 (thought of as a "topological" circle and pointed by its unique 0-simplex) and $S^\alpha := \mathbb{A}^1 \smallsetminus 0$ (thought of as a "geometric" or "twisted" circle and pointed by 1). The $m+n\alpha$-sphere is then the smash product of m copies of S^1 and n copies of S^α.

One can use the pushout square

$$\begin{array}{ccc} \mathbb{A}^1 \smallsetminus 0 & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \end{array}$$

and the motivic contractibility of $\mathbb{A}^1$ to prove that $\mathbb{P}^1$ is motivic weakly equivalent to $S^{1+\alpha}$ and the "Tate object" $\mathbb{A}^1/\mathbb{A}^1 \smallsetminus 0$. (The geometrically minded reader may find it helpful to think in terms of complex points.)

Remark 1.5. The above discussion has concerned itself with unpointed k-spaces, but there is an equally satisfactory theory for pointed k-spaces in which the theory goes through *mutatis mutandis*.

Most of the work in this thesis does not deal directly with motivic homotopy theory, but rather a stable variant. These ideas were first discussed in [Voe98] and later codified in [Jar00, Hu03, Mor04b]. Again, I will follow Jardine [Jar00].

Definition 1.6. A k-*spectrum* is a sequence of pointed k-spaces $X = (X_n)_{n \in \mathbb{N}}$ equipped with pointed bonding maps $\sigma : S^{1+\alpha} \wedge X_n \to X_{n+1}$. A map of k-spectra $f : X \to Y$ is a sequence of simplicial presheaf maps $X_n \to Y_n$ commuting with bonding maps in the obvious way:

$$
\begin{array}{ccc}
S^{1+\alpha} \wedge X_n & \xrightarrow{S^{1+\alpha} \wedge f} & S^{1+\alpha} \wedge Y_n \\
\sigma \downarrow & & \downarrow \sigma \\
X_{n+1} & \xrightarrow{\quad f \quad} & Y_{n+1}.
\end{array}
$$

The resulting category of k-spectra is denoted $\mathrm{Spt}(\mathsf{k})$.

The stable motivic model structure on $\mathrm{Spt}(\mathsf{k})$ depends on an auxiliary *level model structure*. Here the word "level" indicates a spacewise model structure defined in terms of the motivic model structure on $\mathrm{Spc}(\mathsf{k})$. Namely, *level fibrations* have component maps motivic fibrations, *level equivalences* have component maps motivic weak equivalences, and *cofibrations* have the left lifting property with respect to level acyclic fibrations.

At this point, we can define suspension, loop, and shift spectra as in topology, and I refer the interested reader to [Jar00, §2.3] for details. The gist of the construction of the stable motivic model structure is that a stabilization functor Q is applied to the level fibrant replacement JX of a k-spectrum X. A map g of k-spectra is a *stable equivalence* if it induces a level equivalence QJg. *Stable cofibrations* are just the cofibrations of the previous paragraph, and *stable fibrations* have the right lifting property with respect to stably acyclic cofibrations. According to [Jar00, Theorem 2.9], these definitions give $\mathrm{Spt}(\mathsf{k})$ a proper closed simplicial model category structure.

Stable equivalences are difficult to recognize via the above definition, and it will be useful to have an alternate description of stable equivalences in $\mathrm{Spt}(\mathsf{k})$. This can be formulated in terms of presheaves of (bigraded) homotopy groups.

Definition 1.7. For $m, n \in \mathbb{Z}$, $X \in \mathrm{Spt}(\mathsf{k})$, and $U \in \mathrm{Sm}/\mathsf{k}$, form the inductive system

$$[S^{m+n\alpha} \wedge U_+, X_0] \to [S^{m+n\alpha+(1+\alpha)} \wedge U_+, X_1] \to [S^{m+n\alpha+2(1+\alpha)} \wedge U_+, X_2] \to \cdots$$

where U_+ denotes U with a disjoint basepoint and $[\cdot, \cdot] = \mathcal{H}(\mathsf{k})(\cdot, \cdot)$ denotes motivic homotopy classes of maps. Define $\pi_{m+n\alpha} X(U)$ to be the colimit of the above sequence. This makes $\pi_{m+n\alpha} X$ into a presheaf on Sm/k called the *presheaf of stable motivic homotopy groups* of X.

Remark 1.8. By abuse of notation, I will often denote $\pi_{m+n\alpha} X(\mathrm{Spec}\,\mathsf{k})$ by just $\pi_{m+n\alpha} X$. In fact, unless explicitly identified as a presheaf of homotopy groups, I will always mean $\pi_{m+n\alpha} X(\mathrm{Spec}\,\mathsf{k})$ when I write $\pi_{m+n\alpha} X$.

The importance of this definition becomes apparent in light of the following lemma.

Lemma 1.9. ([Jar00, Lemma 3.7]) *A map $X \to Y$ in $\mathrm{Spt}(\mathsf{k})$ is a stable equivalence iff it induces isomorphisms of presheaves $\pi_{m+n\alpha} X \to \pi_{m+n\alpha} Y$ for all $m, n \in \mathbb{Z}$.* $\square$

Remark 1.10. This highlights the importance of considering the full class of bigraded spheres in Remark 1.4. The situation is distinguished from that of stable topology in the following two fashions: (1) homotopy groups are replaced by presheaves of homotopy groups, and (2) the groups are now bigraded. In all motivic contexts, the wildcard $*$ will refer to this bigrading instead of the usual $\mathbb{Z}$-grading.

One of the trademarks of spectra in topology is that they represent generalized

cohomology theories. Motivic spectra also produce cohomology theories on the category of smooth k-schemes via a recipe very familiar to topologists.

Definition 1.11. Given a k-spectrum E and smooth k-scheme U, define the $(m + n\alpha)$-*th E-cohomology* of U to be

$$E^{m+n\alpha}(U) = [U_+, \Sigma^{m+n\alpha}X]$$

where $\Sigma^{m+n\alpha}$ the $(m + n\alpha)$-th suspension functor on $\mathrm{Spt}(\mathsf{k})$.

The $(m + n\alpha)$-*th E-homology* of U is

$$E_{m+n\alpha}(U) = [S^{m+n\alpha}, E \wedge U_+].$$

Remark 1.12. The group $E^{m+n\alpha}(U)$ is commonly denoted $E^{m+n,n}(U)$ in other literature on motivic homotopy theory. The grading here is inspired by $\mathbb{Z}/2$-equivariant homotopy theory (see [HK01, HKOa]).

Remark 1.13. Both E-cohomology and E-homology can — and will — be extended to take values on k-spectra. For an E-spectrum X, define

$$E^*X = [X, \Sigma^{m+n\alpha}X]$$

$$E_*X = [S^{m+n\alpha}, E \wedge X] = \pi_{m+n\alpha}(E \wedge X).$$

The smash product of k-spectra $E \wedge X$ appearing in the second formula is by no means a small technical point. See [Voe98, Theorem 5.6] or [Jar00, §4] for details.

Remark 1.14. When evaluated on the sphere spectrum $\mathbb{1} = (S^0, S^{1+\alpha}, \ldots)$, E-cohomology and E-homology are dual in the sense that $E^{m+n\alpha}(\mathbb{1}) = E_{-m-n\alpha}(\mathbb{1})$. These groups are called the *coefficients* of E and are often denoted E^* and E_*.

I conclude this section by describing some cohomology theories represented by k-spectra important throughout the rest of this thesis.

(1) **Motivic cohomology** [Voe98, §6.1]. Friedlander-Suslin-Voevodsky motivic cohomology [FV00, SV96] with integer coefficients is represented in $\mathcal{SH}(\mathsf{k})$ by a k-spectrum $H\mathbb{Z}$. Its construction is motivated by the Dold-Thom theorem, its constituent k-spaces being motivic analogues of infinite symmetric products. I will only describe the construction in the case when the characteristic of k is zero, since this is all that is needed in this thesis. The n-th space of $H\mathbb{Z}$ is

$$(1.1) \qquad L((\mathbb{P}^1)^{\times n})/(\sum_{i=1}^{n} L(\mathbb{P}_i^1))$$

where $(\mathbb{P}^1)^{\times n}$ is the n-fold scheme-theoretic product of $\mathbb{P}^1$ and $\mathbb{P}_i^1$ is the subscheme of $(\mathbb{P}^1)^{\times n}$ with $\mathbb{P}^1$ in the i-th coordinate and $*$ elsewhere. For $X \in \mathrm{Sm}/\mathsf{k}$, $L(X)$ is the presheaf of Abelian groups whose value on $U \in \mathrm{Sm}/\mathsf{k}$ is the free Abelian group generated by closed irreducible subschemes of $U \times X$ that are finite and surjective over X. The quotient in (1.1) is taken in the category of Nisnevich sheaves of Abelian groups and then forgotten to k-spaces. External product of relative cycles induces maps

$$H\mathbb{Z}_n \wedge H\mathbb{Z}_m \to H\mathbb{Z}_{n+m},$$

and noting that $\mathbb{P}^1 \hookrightarrow L(\mathbb{P}^1) = H\mathbb{Z}_1$ produces the bonding maps for the k-spectrum $H\mathbb{Z}$.

The mod 2 motivic cohomology spectrum $H\mathbb{Z}/2$ is constructed as $H\mathbb{Z} \wedge \mathbb{1}/2$ where $\mathbb{1}/2$ is the mod 2 motivic Moore spectrum, i.e., the cofiber of $2 : \mathbb{1} \to \mathbb{1}$.

(2) **Algebraic K-theory** [Voe98, Hor05]. The stable representability of higher algebraic K-theory is covered in [Voe98, §6.2] and [Hor05, Theorem 3.1 and §5]. The basic idea is to let K denote the simplicial presheaf with value on U the 0-space of the topological spectrum representing the higher algebraic K-theory of U. (If

$U = \operatorname{Spec} A$ is affine, this is just $BGL(A)^+ \times \mathbb{Z}$.) Unstably, algebraic K-theory is represented by aK_f, the Nisnevich sheafification of a fibrant replacement of K in $\operatorname{Spc}(\mathsf{k})$. It is a consequence of Quillen's projective bundle formula that there is a motivic equivalence $\mathbb{P}^1 \wedge aK_f \to aK_f$, and this produces a $(1 + \alpha)$-periodic spectrum KGL representing algebraic K-theory in $\mathcal{SH}(\mathsf{k})$. See [Voe98, Theorem 6.9] or [Hor05, Remark 5.9].

(3) **Algebraic cobordism** [Voe98, §6.3]. Two types of algebraic cobordism are currently in vogue: the Thom spectrum construction of Voevodsky and the oriented cohomology theory of Levine-Morel [LM07]. Levine discusses the close relationship between these two theories in [Levb, Leva], but I will only concern ourselves with the Thom spectrum version in this thesis.

Recall that, for a vector bundle $E \xrightarrow{\xi} B$ in $\operatorname{Sm}/\mathsf{k}$ we can define the Thom space B^ξ (denoted $\operatorname{Th}(E)$ in [Voe98]) to be the simplicial presheaf quotient

$$B^\xi := E/(E \smallsetminus X)$$

where we view X as a subscheme of E via the 0-section of ξ. Let $\operatorname{Gr}(n, N)$ denote the Grassmannian of n-planes in $\mathbb{A}^N$, and let $E(n, N) \xrightarrow{\gamma(n,N)} \operatorname{Gr}(n, N)$ be the associated universal bundle. Taking colimits with respect to N, we get a universal bundle $E(n, \infty) \xrightarrow{\gamma_n} \operatorname{Gr}(n) = \operatorname{Gr}(n, \infty)$ over the infinite Grassmannian and its associated Thom space $\operatorname{Gr}(n)^{\gamma_n}$. The inclusions $i : \operatorname{Gr}(n) \to \operatorname{Gr}(n+1)$ induce maps $\operatorname{Gr}(n)^{i^*\gamma_{n+1}} \to \operatorname{Gr}(n+1)^{\gamma_{n+1}}$. Now $i^*\gamma_n = 1 \oplus \gamma_n$ where 1 is the trivial line bundle over $\operatorname{Gr}(n)$, and it is clear that, in general,

$$B^{1 \oplus \xi} = \mathbb{A}^1/(\mathbb{A}^1 \smallsetminus 0) \wedge B^\xi.$$

Hence we have maps

$$\mathbb{A}^1/(\mathbb{A}^1 \smallsetminus 0) \wedge \operatorname{Gr}(n)^{\gamma_n} \to \operatorname{Gr}(n+1)^{\gamma_{n+1}}.$$

By Remark 1.4, there is an equivalence $S^{1+\alpha} \to \mathbb{A}^1/(\mathbb{A}^1 \smallsetminus 0)$, and we have constructed a k-spectrum with n-th space $\mathrm{Gr}(n)^{\gamma_n}$. This is the algebraic cobordism spectrum, denoted MGL, and its associated cohomology theory (see Definition 1.11 and Remark 1.13) is algebraic cobordism.

Algebraic cobordism has a close relative which will play a central role in this thesis. For each rational prime p, there is an algebraic Brown-Peterson spectrum, denoted BPGL. The construction of this k-spectrum is detailed in [HK01, Vez01], where in each case it is constructed via an analogue of the Quillen idempotent. In this thesis, if no reference is made to the prime p, BPGL will automatically mean the algebraic Brown-Peterson spectrum at $p = 2$.

(4) **Hermitian K-theory** [Hor05]. Hermitian K-theory is the higher K-theory of quadratic forms over a ring. The topologist may think of Hermitian K-theory as an algebraic version of real K-theory, while algebraic K-theory is a version of complex K-theory.

The version of Hermitian K-theory of affine schemes defined in topology take the following form. For a ring A and $n \geq 0$, the n-th Hermitian K-group of A is

$$\pi_n(BO(A)^+ \times GW(A))$$

where $GW(A)$ is the Grothendieck-Witt ring of quadratic forms over A (see §1.3 for a definiton) and $BO(A)^+$ is the +-construction (or group completion) of the classifying space of the infinite orthogonal group of A. There is an obvious simplicial presheaf on affine k-schemes taking $\mathrm{Spec}\, A$ to $BO(A)^+ \times GW(A)$, and we can extend this to all of Sm/k by using the functorial version of Jouanalou's device in [Wei89]. Up to homotopy, this construction consists of choosing an affine vector bundle torsor $\mathrm{Spec}(A)$ over a smooth k-scheme U and defining the

value of the Hermitian K-theory simplicial presheaf on U to be

$$KO(U) := BO(A)^+ \times GW(A).$$

Unstable representability of Hermitian K-theory is then handled by considering the Nisnevich sheafification of a fibrant replacement of KO, aKO_f. See [Hor05, Corollary 3.4].

Stable representability of Hermitian K-theory is more nuanced that that of algebraic K-theory because the resulting spectrum is $4(1 + \alpha)$-periodic. There is an obvious hyperbolization map $H : K(A) \rightarrow KO(A)$ and its homotopy fiber is the U-theory of A; U may be extended to a k-space by the same device as above. There are also negative versions of KO and U in which Hermitian forms are replaced by skew Hermitian forms; these are denoted $_-KO$ and $_-U$. The Hermitian K-theory spectrum then has constituent k-spaces $\mathsf{KO}_{4k} := aKO_f$, $\mathsf{KO}_{4k+1} := a_-U_f$, $\mathsf{KO}_{4k+2} := a_-KO_f$, and $\mathsf{KO}_{4k+3} := aU_f$. The bonding maps are constructed in the proof of [Hor05, Theorem 5.5], where it is also shown that the adjoints of the bonding maps are equivalences. This theory of Hermitian K-theory will play an important role in Chapter VI.

1.2   Motivic cohomology and the motivic Steenrod algebra

One of the original applications of motivic homotopy was the resolution of the Milnor conjecture, identifying the mod 2 Galois cohomology of a field with its mod 2 Milnor K-theory. By results of Suslin-Voevodsky [SV96], this further specifies the mod 2 motivic cohomology of $\mathrm{Spec}(\mathsf{k})$.

Milnor defines and determines the basic properties of what is now called Milnor K-theory in his seminal paper [Mil70]. The Milnor K-theory of a field k is the graded

algebra

$$K_*^M(\mathsf{k}) = T(\mathsf{k}^\times)/(a \otimes (1 - a) \mid a \neq 0, 1)$$

where $T(\mathsf{k}^\times)$ is the tensor algebra on the Abelian group $\mathsf{k}^\times$. For obvious reasons, it is easy to confuse the group operation on $\mathsf{k}^\times$ and the tensor operation in $K_*^M(\mathsf{k})$. Hence it is convenient to consider the n-th Milnor K-group to be generated by "Milnor symbols" $\{a_1, \ldots, a_n\}$, $a_i \in \mathsf{k}^\times$. Here $\{\cdot\} : \mathsf{k}^\times \to K_1^M(\mathsf{k})$ is an isomorphism such that $\{ab\} = \{a\} + \{b\}$ and $\{a_1, \ldots, a_n\} = a_1 \otimes \cdots \otimes a_n$. Following common usage, let $k_*^M(\mathsf{k})$ denote mod 2 Milnor K-theory $K_*^M(\mathsf{k})/2$.

The main result of [Voe03a] is the following theorem.

Theorem 1.15 ([Voe03a])**.** *Mod 2 motivic cohomology of* $\mathrm{Spec}(\mathsf{k})$ *takes the form*

$$H^*(\mathrm{Spec}(\mathsf{k}); \mathbb{Z}/2) = k_*^M(\mathsf{k})[\tau]$$

where $|k_1^M(\mathsf{k})| = \alpha$ *and* $|\tau| = -1 + \alpha$.

In [Voe03b] and [Voe], Voevodsky determines the stable operations on mod 2 motivic cohomology. Let H denote the mod 2 motivic cohomology k-spectrum $H\mathbb{Z}/2$.

Definition 1.16. The *motivic Steenrod algebra* is the algebra of stable operations on H,

$$\mathcal{A}^* := H^*H.$$

Voevodsky shows in [Voe03b] that $\mathcal{A}^*$ contains the Bockstein β and power operations P^i, but he does not prove that they generate all of $\mathcal{A}^*$ in that paper. [Voe] fills the gap and proves the following structure theorem about $\mathcal{A}^*$.

Theorem 1.17 ([Voe03b, Voe])**.** *The motivic Steenrod algebra is generated by* β *and* P^i, $i \geq 0$. $\qquad\square$

$\mathcal{A}^*$ has the structure of a Hopf algebroid over H^*. (See [Rav86, Appendix A1] for the theory Hopf algebroids.) In this thesis, I will be more concerned with the dual to the motivic Steenrod algebra, $\mathcal{A}_* = H_*H$ which is a Hopf algebroid over $H_* = H^{-*}$.

Theorem 1.18 ([Voe03b, Voe]). *The dual motivic Steenrod algebra is a commutative free H_*-algebra isomorphic to*

$$H_*[\tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots]/(\tau_i^2 - \tau\xi_{i+1} - \rho(\tau_{i+1} + \rho\tau_0\xi_{i+1})).$$

Here, by a standard abuse of notation, $\tau \in H_{1-\alpha}$ is the dual of $\tau \in H^{-1+\alpha}$, ρ is the class of -1 in $H_{-\alpha} = k_1^M(\mathsf{k}) = \mathsf{k}^\times/(\mathsf{k}^\times)^2$, $|\tau_i| = (2^i - 1)(1 + \alpha) + 1$, and $|\xi_i| = (2^i - 1)(1 + \alpha)$. $\square$

We will also need to know the Hopf algebroid structure of $\mathcal{A}_*$ which is given in the following theorem.

Theorem 1.19 ([Voe03b, Voe]). *In the Hopf algebroid $\mathcal{A}_*$, the ξ_i and τ_i support the same structure maps as in topology, elements of $H_{0+*\alpha} = k_*^M(\mathsf{k})$ are primitive, and τ is not primitive in general. In particular, $\mathcal{A}_*$ has the following structure:*

(1.2)
$$\eta_L \tau = \tau$$
$$\eta_R \tau = \tau + \rho\tau_0$$
$$\Delta\xi_k = \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \xi_i$$
$$\Delta\tau_k = \tau_k \otimes 1 + \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \tau_i.$$

$\square$

Remark 1.20. It follows that $\mathcal{A}^*$ has a Milnor basis of elements of the form $Q_I(r_1, \ldots, r_n)$ as in topology with the degree shift $|Q_n| = 2^n(1 + \alpha) - \alpha$.

Certain quotient Hopf algebroids of $\mathcal{A}_*$ will be useful in my analysis of MGL (see Chapter II). The following definition is due to Mike Hill [Hill, Definition 1].

Definition 1.21. Let $\mathcal{E}(n)$, $0 \leq n < \infty$, denote the quotient Hopf algebroid

$$\mathcal{E}(n) := \mathcal{A}_*//(\xi_1, \xi_2, \ldots, \tau_{n+1}, \tau_{n+2}, \ldots)$$

$$= H_*[\tau_0, \ldots, \tau_n]/(\tau_i^2 - \rho\tau_{i+1} \mid 0 \leq i < n) + (\tau_n^2).$$

If $n = \infty$, let

$$\mathcal{E}(\infty) := \mathcal{A}_*//(\xi_1, \xi_2, \ldots)$$

$$= H_*[\tau_0, \tau_1, \ldots]/(\tau_i^2 - \rho\tau_{i+1} \mid 0 \leq i).$$

The Hopf algebroid $\mathcal{E}(n)$ is dual to the sub-Hopf algebroid of $\mathcal{A}^*$ generated by the Milnor primitives Q_i, $i \leq n$.

1.3 Stable homotopy groups of spheres

Recall that the motivic sphere spectrum $\mathbb{1}$ has constituent spaces $S^{n(1+\alpha)}$ and bonding maps the canonical isomorphisms $S^{1+\alpha} \wedge S^{n(1+\alpha)} \to S^{(n+1)(1+\alpha)}$. Its homotopy groups are, for $m, n \in \mathbb{Z}$,

$$\pi_{m+n\alpha} := \pi_{m+n\alpha}\mathbb{1} = \varinjlim_k [S^{m+n\alpha+k(1+\alpha)}, S^{k(1+\alpha)}].$$

The main goal of this thesis is to determine information about $\pi_*\mathbb{1}$; I will use this section to describe what is already known about it.

The two main known results on $\pi_*\mathbb{1}$ deal with (1) connectivity and (2) the structure of $\pi_{0+*\alpha}\mathbb{1}$. Both are due to Morel and their proofs can be found in [Mor05] and [Mor04b], respectively.

Theorem 1.22 (Connectivity Theorem [Mor05], see also [Mor04a, §6]). *For arbitrary $n \in \mathbb{Z}$ and any $m < 0$,*

$$\pi_{m+n\alpha}\mathbb{1} = 0.$$

□

This result says that $\pi_* \mathbb{1}$ vanishes in an entire half-plane and should be thought of as a motivic analogue of the (-1)-connectivity of the topological sphere spectrum.

The structure theorem for $\pi_{0+*\alpha} \mathbb{1}$ has a more involved statement, but is probably the easier result. I need to recall a few definitions from the algebraic theory of quadratic forms in order to proceed.

By a *form* I will always mean a nonsingular quadratic form over a given base field. The notation $\langle a_1, \ldots, a_n \rangle$ refers to the diagonal form $a_1 x_1^2 + \cdots + a_n x_n^2$. Form the commutative cancellation monoid (and semiring) of isomoetry classes of forms under the operations $\oplus$ (orthogonal sum) and $\otimes$ (tensor product). Note that all forms are diagonalizable and on diagonal forms

$$\langle a_1, \ldots, a_n \rangle \oplus \langle b_1, \ldots, b_m \rangle = \langle a_1, \ldots, a_n, b_1, \ldots, b_m \rangle,$$

$$\langle a_1, \ldots, a_n \rangle \otimes \langle b_1, \ldots, b_m \rangle = \langle \ldots, a_i b_j, \ldots \rangle.$$

The *Grothendieck-Witt ring* of the base field k, denoted $GW(\mathsf{k})$, is the Grothendieck construction applied to this monoid. Let $\mathbb{H}$ denote the isometry class of the hyperbolic form $x^2 - y^2 = \langle 1, -1 \rangle$. An easy induction shows that the ideal generated by $\mathbb{H}$ in $GW(\mathsf{k})$ is the same as $\mathbb{Z} \cdot \mathbb{H}$, i.e., $q \cdot \mathbb{H} = (\dim q)\mathbb{H}$. The *Witt ring* of k is

$$W(\mathsf{k}) := GW(\mathsf{k})/\mathbb{Z} \cdot \mathbb{H}.$$

The fundamental ideal $I(\mathsf{k})$ of $W(\mathsf{k})$ is the ideal of even-dimensional forms, i.e.,

$$0 \to I(\mathsf{k}) \to W(\mathsf{k}) \xrightarrow{\dim_0} \mathbb{Z}/2 \to 0$$

where $\dim_0$ is the mod 2 reduction of the dimension map $\dim : GW(\mathsf{k}) \to \mathbb{Z}$.

The Grothendieck-Witt ring can be specified by generators and relations as follows.

Theorem 1.23 (e.g., [Lam05, Theorem I.4.1]). *The Grothendieck-Witt ring is isomorphic to the free commutative ring on generators $\langle a \rangle$, $a \in k^\times$ subject to the relations*

$$\langle 1 \rangle = 1,$$

(1.3)
$$\langle a \rangle \langle b \rangle = \langle ab \rangle,$$

$$\langle a \rangle + \langle b \rangle = \langle a + b \rangle (1 + \langle ab \rangle)$$

whenever they make sense. □

Corollary 1.24. *Imposing the additional relation $1 + \langle -1 \rangle = 0$ recovers the Witt ring.* □

This presentation of $W(k)$ and the definition of Milnor K-theory in §1.2 may go some way towards motivating the following definition of Morel's Milnor-Witt K-theory.

Definition 1.25. Let $T_*^{MW}(k)$ denote the free $\mathbb{Z}$-graded associative algebra with generators $[u]$ in degree 1 for each $u \in k^\times$ and generator η of degree -1. The *Milnor-Witt K-theory* of k is the quotient of $T_*^{MW}(k)$ by the ideal generated by the relations

$$[ab] = [a] + [b] + \eta[a][b],$$

$$[a][1 - a] = 0,$$

(1.4)
$$[u]\eta = \eta[u],$$

$$\eta(2 + [-1]\eta) = 0$$

whenever they make sense.

Remark 1.26. The analogy with Milnor K-theory is clear if one thinks of the $[u]$ as Milnor symbols, in which case the second relation is the Steinberg relation defining

$K_*^M(\mathsf{k})$ and the first relation is a sort of "η-twisting" of the relation $\{ab\} = \{a\} + \{b\}$ in $K_*^M(\mathsf{k})$.

The analogy with the Witt ring becomes more transparent if we make the auxiliary definition

$$\langle u \rangle := 1 + \eta[u] \in K_0^{MW}(\mathsf{k}).$$

Then the final relation can be rewritten as

$$\eta(1 + \langle -1 \rangle) = 0,$$

which should remind us of quotienting out by the hyperbolic plane.

This analogy with Milnor K-theory and the Witt ring can be made precise. Recall that, by the norm residue isomorphism theorem, $k_n^M(\mathsf{k})$ is canonically isomorphic to $I(\mathsf{k})^n / I(\mathsf{k})^{n+1}$. Hence we can form a pullback square

$$\begin{array}{ccc} J(\mathsf{k})^n & \longrightarrow & K_n^M(\mathsf{k}) \\ \downarrow & & \downarrow \\ I(\mathsf{k})^n & \longrightarrow & I(\mathsf{k})^n / I(\mathsf{k})^{n+1} \end{array}$$

Theorem 1.27 ([Mor04b, Theorem 6.3.4]). *There is a canonical isomorphism*

$$K_n^{MW}(\mathsf{k}) \to J(\mathsf{k})^n = I(\mathsf{k})^n \times_{I(\mathsf{k})^n / I(\mathsf{k})^{n+1}} K_n^M(\mathsf{k}).$$

This holds for all $n \in \mathbb{Z}$ if we set $I(\mathsf{k})^n = W(\mathsf{k})$ for $n \leq 0$ and $K_n^M(\mathsf{k}) = 0$ for $n < 0$.

There is now some suggestive notation for a few elements of $\pi_* \mathbb{1}$: for $u \in \mathsf{k}^\times$, let $[u] \in \pi_{-\alpha} \mathbb{1}$ denote the element represented unstably by $S^0 \to \mathbb{A}^1 \smallsetminus 0$ sending the non-basepoint to u; let $\eta \in \pi_\alpha$ denote the element represented unstably by the Hopf map $\mathbb{A}^2 \smallsetminus 0 \to \mathbb{P}^1$ taking (x, y) to $[x : y]$. (There is an obvious pushout square which implies that $\mathbb{A}^2 \smallsetminus 0 \simeq S^{1+2\alpha}$.)

Theorem 1.28 (Structure Theorem, [Mor04b, Theorem 6.2.1]). *There is a ring isomorphism* $K_*^{MW}(\mathsf{k}) \to \pi_{0+*\alpha}\mathbb{1}$ *sending* $[u]$ *to* $[u]$ *and* η *to* η. $\qquad\square$

Remark 1.29. Note carefully that the map flips degrees so that $\pi_{n\alpha}\mathbb{1} \cong K_{-n}^{MW}(\mathsf{k})$.

1.4 Motivic Adams-type spectral sequences

I now introduce my main computational tools, motivic analogues of Adams-type spectral sequences from topology. For the background and applications of the Adams spectral sequence (ASS) in topology, see [Ada74, Rav86].

Fix a (not necessarily highly structured) ring k-spectrum E and a k-spectrum X. The *canonical E-Adams resolution* of X is

$$
\begin{array}{ccccccc}
X = X_0 & \longleftarrow & X_1 & \longleftarrow & X_2 & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
K_0 & & K_1 & & K_2 & &
\end{array}
$$

in which $K_s = E \wedge X_s$ and X_{s+1} is the homotopy fiber of the map $X_s \to K_s$. The associated fibration long exact sequences patch together to produce the exact couple inducing the spectral sequence [Rav86, HKOa, DI].

To be precise, the fiber sequences $X_{s+1} \to X_s \to K_s$ induce long exact sequences in stable motivic homotopy groups

$$
\cdots \to \pi_{m+1+n\alpha}K_s \xrightarrow{\partial} \pi_{m+n\alpha}X_{s+1} \to \pi_{m+n\alpha}X_s \to \pi_{m+n\alpha}K_s \to \cdots .
$$

Let $D_1^{s,m+n\alpha} = \pi_{m+n\alpha-s}X_s$ and let $E_1^{s,m+n\alpha} = \pi_{m+n\alpha-s}K_s$. An exact couple

$$
\begin{array}{ccc}
D_1 & \xrightarrow{\ i_1\ } & D_1 \\
& \nwarrow_{k_1} \quad \swarrow_{j_1} & \\
& E_1 &
\end{array}
$$

is formed by letting i_1 be induced by $X_{s+1} \to X_s$, j_1 be induced by $X_s \to K_s$, and k_1 be induced by ∂.

Definition 1.30. The spectral sequence induced by the above exact couple is the *motivic E-Adams spectral sequence*, or motivic E-ASS, for X.

Note that the E-ASS is tri-graded. I will denote the r-th page of the E-ASS by $E_r^{*,*}$ where the first $*$ is an integer called the *homological degree*, and the second $*$ is a bigrading of the form $m + n\alpha$ called the *motivic degree*. For a tri-grading $(s, m + n\alpha)$, I call the bigrading $m + n\alpha - s = (m - s) + n\alpha$ the *total motivic degree* or *Adams grading*; sometimes Adams grading will also refer to the tri-degree $(s, m + n\alpha - s)$. The differentials in the motivic E-ASS take the form

$$d_r : E_r^{s,m+n\alpha} \rightarrow E_r^{s+r,m+n\alpha+r-1}.$$

In other words, the r-th differential increases homological degree by r and decreases Adams grading by 1.

In order to describe the convergence properties of the motivic E-ASS, I first must digress and introduce the concept of completion in $\mathrm{Spt}(\mathsf{k})$ and $\mathcal{SH}(\mathsf{k})$. Experts will recognize the completion I am intersted in as a type of Bousfield localization. The machinery of Bousfield localization for topological spectra was introduced in [Bou79]. Hirschhorn expands on the theory and generalizes it to cellular model categories in [Hir03]. The application of this theory in the motivic context has been developed by Hornbostel in [Hor06].

Definition 1.31. The *2-completion* of a k-spectrum X, $X\widehat{}_2$, is the Bousfield localization $L_{\mathbb{1}/2}X$ of X with respect to the class of stable cofibrations of k-spectra $E \rightarrow F$ which become stable motivic equivalences after smashing with $\mathbb{1}/2$.

The homotopy groups of a 2-completion are related to the homotopy groups of the original k-spectrum by the following short exact sequence.

Theorem 1.32 ([HKOa]). *If X is a k-spectrum and $X\widehat{\,}_2$ its 2-completion, then the motivic stable homotopy groups of X fit into split short exact sequences*

$$0 \to \mathrm{Ext}(\mathbb{Z}/2^\infty, \pi_{m+n\alpha}X) \to \pi_{m+n\alpha}X\widehat{\,}_2 \to \mathrm{Hom}(\mathbb{Z}/2^\infty, \pi_{m-1+n\alpha}) \to 0$$

for all $m, n \in \mathbb{Z}$. □

Using the language of completion, I can state the following analogue of [Rav86, Theorem 2.2.3].

Theorem 1.33. *Fix a p-adic field F (see §1.5), let $E = H$ or BPGL, and let X be a cell spectrum of finite type. Then the E_2-term of the motivic E-ASS is*

$$\mathrm{Ext}_{E_*E}(E_*, E_*X)$$

and the motivic E-ASS converges to $\pi_*X\widehat{\,}_2$ where permanent cycles in tri-degree $(s, m + n\alpha)$ represent elements of $\pi_{m+n\alpha-s}X\widehat{\,}_2$.

Proof. This is an immediate corollary of the more general result in [HKOb]. □

Remark 1.34. If $E = H$, the mod 2 motivic cohomology spectrum, then the motivic H-Adams spectral sequence for X will simply be called the the *motivic Adams spectral sequence*, or motivic ASS, for X. For X cell of finite type, the motivic ASS for X takes the form

$$\mathrm{Ext}_{\mathcal{A}_*}(H_*, H_*X) \implies \pi_*X\widehat{\,}_2.$$

If $E = \mathsf{BPGL}$, the motivic Brown-Peterson spectrum at the prime 2, then the motivic BPGL-Adams spectral sequence for the sphere spectrum $\mathbb{1}$ will be called the *motivic Adams-Novikov spectral sequence*, or motivic ANSS. The motivic ANSS takes the form

$$\mathrm{Ext}_{\mathsf{BPGL}_*\mathsf{BPGL}}(\mathsf{BPGL}_*, \mathsf{BPGL}_*) \implies \pi_*\mathbb{1}\widehat{\,}_2.$$

1.5 Structure of p-adic fields

The computations in this thesis all occur over p-adic fields, and I will use this section to survey arithmetic invariants of p-adic fields important to motivic homotopy theory and specify the structure of some of the objects introduced in §1.2 and §1.3 over p-adic fields.

Definition 1.35. A *p-adic field* is a complete discrete valuation field of characteristic 0 with finite residue field.

It is well-known that every p-adic field is a finite extension of the ℓ-adic rationals $\mathbb{Q}_\ell$ for some rational prime ℓ. If p is a specified rational prime, then the term "p-adic field" will refer to a finite extension of $\mathbb{Q}_p$.

Convention 1.36. The structure of p-adic fields differs in the cases $p = 2$ and $p > 2$: for instance, $\mathbb{Q}_2$ has 8 square classes (i.e. $|\mathbb{Q}_2^\times/(\mathbb{Q}_2^\times)^2| = 8$) while p-adic fields have 4 square classes for every $p > 2$. In order to avoid a great many minor modifications, I will only deal with p-adic fields for which $p > 2$ in this thesis. Henceforth, the term p-adic field will only refer to *nondyadic* p-adic fields, i.e., p-adic fields for which $p > 2$; moreover, the letter F will always refer to a p-adic field unless stated otherwise.

Let $v : F \to \mathbb{Z} \cup \infty$ denote the valuation on F. F has a *ring of integers* $\mathcal{O} :=$ $\{x \in F \mid v(x) \geq 0\}$. It is trivial to check that $\mathcal{O}$ is a domain, and $F = \operatorname{Frac}\mathcal{O}$, the field of fractions of $\mathcal{O}$. Moreover, $\mathcal{O}$ is a local ring with maximal ideal $\mathfrak{m} :=$ $\{x \in F \mid v(x) \geq 1\}$. A *uniformizer* of F is an element $\pi \in F$ such that $v(\pi) = 1$; note that for any choice of uniformizer π, $(\pi) = \mathfrak{m}$.

The *residue field* of F is

$$\mathbb{F} := \mathcal{O}/\mathfrak{m}.$$

Note that $\mathbb{F}$ is necessarily a finite field; let $q := |\mathbb{F}|$ and call q the *residue order* of F.

Of course, q is a prime power p^m where F is a p-adic field.

One of the fundamental properties of p-adic fields is that they satisfy Hensel's

lemma, an approximation lemma which I now state.

Lemma 1.37 (Hensel's Lemma, e.g., [Cas86, Lemma 3.1]). *For a polynomial $f \in$*

$\mathcal{O}[x]$, *let $a_0 \in \mathcal{O}$ satisfy $v(f(a_0)) > 2v(f'(a_0))$. Then there exists $a \in \mathcal{O}$ such that*

$f(a) = 0$.

At variance with the nomenclature of number theorists, I will call $F^\times = F \smallsetminus 0$

the *units* of F; they are the multiplicative group of nonzero elements of F, and not

the number theorists' $\mathcal{O}^\times$, which I will call the units in the valuation ring of F.

As a consequence of Hensel's lemma, the units of a p-adic field F are equipped

with a *Teichmüller lift* $\mathbb{F}^\times \hookrightarrow F^\times$.

Proposition 1.38. *Let F be a p-adic field with chosen uniformizer π. Identify $\mathbb{F}^\times$*

with its image under the Teichmüller lift. Then

$$F^\times = \pi^{\mathbb{Z}} \times \mathbb{F}^\times \times (1 + \mathfrak{m}).$$

Proof. Every element $x \in F^\times$ can be written in the form $\pi^{v(x)}u$ for $u \in \mathcal{O}^\times$ of

valuation 0. Observe that $\mathcal{O}^\times/(1 + \mathfrak{m}) = \mathbb{F}^\times$, and the Teichmüller lift provides a

splitting. $\square$

Definition 1.39. For an arbitrary field k, the group of *square classes* of k is $\mathsf{k}^\times/(\mathsf{k}^\times)^2$

where $(\mathsf{k}^\times)^2 = \{x^2 \mid x \in \mathsf{k}\}$.

Corollary 1.40. *Let F be a p-adic field with chosen uniformizer π and choose u to*

be a nonsquare in the Teichmüller lift $\mathbb{F}^\times$. The the square classes of F are

$$F^\times/(F^\times)^2 = \pi^{\mathbb{Z}/2} \times u^{\mathbb{Z}/2}.$$

When $q = |\mathbb{F}^{\times}| \equiv 3$ (4), we may choose u to be -1; when $q \equiv 1$ (4), the image of -1 in the square classes of F is zero. $\qquad\square$

Every discretely valued field (E, v) comes equipped with a *tame symbol*

$$\left(\frac{\cdot,\cdot}{E}\right) : E^{\times} \times E^{\times} \to \mathbb{E}^{\times}$$

defined by the formula

$$\left(\frac{x,y}{E}\right) = (-1)^{v(x)v(y)} x^{v(y)} y^{-v(x)} \quad \mod \mathfrak{m}.$$

Lemma 1.41 ([Mil71, Lemma 11.5]). *The tame symbol is a Steinberg symbol and hence induces a homomorphism $K_2^M(E) \to \mathbb{E}^{\times} = K_1^M(\mathbb{E})$.*

As a consequence of Lemma 1.41 and [Mil70, Example 1.7], I can determine the mod 2 Milnor K-theory of a p-adic field.

Proposition 1.42. *Fix a p-adic field F, a uniformizer π, and a nonsquare $u \in \mathbb{F}^{\times}$. As a $\mathbb{Z}$-graded $\mathbb{Z}/2$-algebra,*

$$(1.5) \qquad k_*^M(F) = \begin{cases} \mathbb{Z}/2[\{u\}, \{\pi\}]/(\{u\}^2, \{\pi\}^2) & \text{if } q \equiv 1 \quad (4), \\[2mm] \mathbb{Z}/2[\{u\}, \{\pi\}]/(\{u\}^2, \{\pi\}(\{u\} - \{\pi\})) & \text{if } q \equiv 3 \quad (4) \end{cases}$$

where $|\{\pi\}| = |\{u\}| = 1$.

Proof. Abusing notation, I will write x for $\{x\} \in K_1^M(F)$ or $k_1^M(F)$ whenever the context does not admit confusion.

Since $k_1^M(F)$ is the group of square classes of F, Corollary 1.40 implies that

$$k_1^M(F) = \pi^{\mathbb{Z}/2} \times u^{\mathbb{Z}/2}.$$

Moreover, by [Mil70, Example 1.7(2)], $k_2^M(F)$ has dimension 1 as a $\mathbb{Z}/2$-vector space. By the same reference, we also know that $K_n^M(F)$ is divisible for every $n \geq 3$, and it follows that $k_n^M(F) = 0$ for all $n \geq 3$.

We still must determine the multiplicative structure of $k_*^M(F)$, which amounts to determining the products $u^2, u\pi, \pi^2 \in k_2^M(F)$. First note that

$$\left(\frac{u, \pi}{F}\right) = (-1)^0 u^1 \pi^0 = u \in K_1^M(\mathbb{F}),$$

which reduces to the nontrivial generator u of $k_1^M(\mathbb{F})$. By Lemma 1.41, it follows that $u\pi \neq 0 \in k_2^M(F)$.

The argument above also proves that, after reduction mod 2, the tame symbol is an isomorphism $k_2^M(F) \to k_1^M(\mathbb{F})$. Hence to compute the products u^2 and π^2, it suffices to compute

$$\left(\frac{u, u}{F}\right) \quad \text{and} \quad \left(\frac{\pi, \pi}{F}\right).$$

These symbols are 1 and -1, respectively, so $u^2 = 0 \in k_1^M(F)$ while π^2 is nontrivial iff $q \equiv 3 \ (4)$. This determines the multiplicative structure given in (1.5).

$\square$

As a corollary to this computation, Theorems 1.15 and 1.18 allow a complete description of the coefficients of mod 2 motivic cohomology and the dual Steenrod algebra over a p-adic field.

Theorem 1.43. *Over a p-adic field F, the coefficients of mod 2 motivic homology are*

$$H_* = k_*^M(F)[\tau]$$

where $|\tau| = 1 - \alpha$, $|k_n^M(F)| = -n\alpha$, and $k_*^M(F)$ has the form given in Proposition 1.42.

The dual motivic Steenrod algebra has the form

$$\mathcal{A}_* = H_*[\tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots]/(\tau_i^2 - \tau\xi_{i+1} - \rho(\tau_{i+1} + \rho\tau_0\xi_{i+1})).$$

The class ρ is trivial iff $q \equiv 1$ (4). In this case,

$$\mathcal{A}_* = H_*[\tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots]/(\tau_i^2 - \tau\xi_{i+1}) \cong \mathcal{A}_*^{\mathbb{C}} \otimes_{H_*^{\mathbb{C}}} k_*^M(F)$$

where $(H_*^{\mathbb{C}}, \mathcal{A}_*^{\mathbb{C}})$ is the dual motivic Steenrod algebra over $\mathbb{C}$, which has the structure

$$H_*^{\mathbb{C}} = \mathbb{Z}/2[\tau],$$

$$\mathcal{A}_*^{\mathbb{C}} = \mathbb{Z}/2[\tau, \tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots]/(\tau_i^2 - \tau\xi_{i+1}).$$

Proof. Most of the theorem is a concatenation of results in Theorems 1.15 and 1.18 and Proposition 1.42. The form of $(H_*^{\mathbb{C}}, \mathcal{A}_*^{\mathbb{C}})$ is obvious after noting that $k_*^M(\mathbb{C})$ is trivial outside of degree 0. The class ρ is trivial iff -1 is a square in $\mathbb{F}^\times = \mathbb{F}_q^\times$; it is standard that this is the case iff $q \equiv 1$ (4). $\qquad\square$

The structure of H_* over F is depicted in Figure 1.1. Here the horizontal axis measures the $\mathbb{Z}$-component of the motivic bigrading, while the vertical axis measures the $\mathbb{Z}\alpha$-component. Each "diamond" shape is a copy of $k_*^M(F)$, and the diagonal arrows of slope -1 represent τ-multiplication.

I now turn to the theory of quadratic forms over p-adic fields. This material will be important to the study of Hermitian K-theory over a p-adic field (see Chapter VI) and is related to motivic cohomology via the Milnor conjecture.

In the case of a p-adic field F, the Grothendieck-Witt and Witt rings are well-known.

Theorem 1.44. (e.g., [Lam05, Theorem VI.2.2]) *Assume $q \equiv 1$ (4). Then*

$$GW(F) \cong \mathbb{Z} \oplus (\mathbb{Z}/2)^3 \text{ as Abelian groups, and}$$

$$W(F) \cong \mathbb{Z}/2[\mathbb{Z}/2 \times \mathbb{Z}/2] \text{ as rings.}$$

Moreover, the direct summands of $GW(F)$ are generated by

$$\langle 1 \rangle, \ \langle u \rangle - \langle 1 \rangle, \ \langle \pi \rangle - \langle 1 \rangle, \text{ and } \langle u\pi \rangle - \langle \pi \rangle,$$
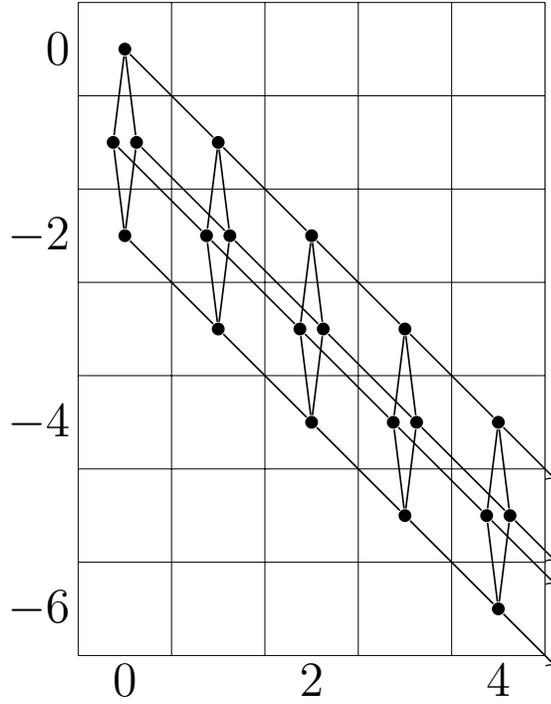
Figure 1.1: Mod 2 motivic homology over F

respectively.

Assume $q \equiv 3$ (4). Then

$$GW(F) \cong \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2 \text{ as Abelian groups, and}$$

$$W(F) \cong \mathbb{Z}/4[\mathbb{Z}/2] \text{ as rings.}$$

Moreover, the direct summands of $GW(F)$ are generated by

$$\langle 1 \rangle, \ \langle \pi \rangle - \langle 1 \rangle, \ and \ \langle -1 \rangle - \langle 1 \rangle.$$

$\square$

Recall that Milnor-Witt K-theory decomposes in terms of the fundamental ideal of the Witt ring and Milnor K-theory. Since we know both of these objects over a p-adic field, Theorems 1.28 and 1.27 put us in a position to determine a relatively explicit form for the stable homotopy groups $\pi_{0+n\alpha}\mathbb{1}$.

Theorem 1.45. *The motivic stable homotopy groups over F in dimensions $0 + n\alpha$ have the form*

$$\pi_{0+n\alpha}\mathbb{1} \cong K_{-n}^{MW}(F) \cong \begin{cases} W(F) & \text{if } n > 0, \\ GW(F) & \text{if } n = 0, \\ K_1^M(F) \oplus I(F)^2 & \text{if } n = -1, \\ K_n^M(F) & \text{if } n < -1. \end{cases}$$

Proof. The groups $K_n^{MW}(F)$ for $n \leq 0$ are known by [Mor04b]. By Theorem 1.27 there are split short exact sequences

$$0 \to I(F)^{n+1} \to K_n^{MW}(F) \to K_n^M(F) \to 0.$$

By theorem 1.44, we can observe that $I(F)^n = 0$ for $n \geq 3$. This proves the result. $\square$

1.6 Outline of the thesis

The main goal of this thesis is to describe the behavior of an analogue of the alpha family in the motivic ANSS over a p-adic field ($p > 2$, see Convention 1.36). In order to access this family, I need a description of the E_2-term of the motivic ANSS, and hence an understanding of the Hopf algebroid $(\mathsf{BPGL}_*, \mathsf{BPGL}_*\mathsf{BPGL})$. This is accomplished via the motivic ASS for BPGL. The computations for BPGL inspired similar ones for kgl, a connective version of KGL, which are presented in parallel since they may be of independent interest to algebraic K-theorists. The thesis also contains digressions on Massey products and the motivic J-homomoprhism since these are needed to understand the alpha family in the motivic ANSS.

Chapter II discusses preliminaries to the motivic ASSs for BPGL and kgl. It begins by constructing kgl as a BPGL-algebra. Once basic properties of kgl are established,

Theorems 2.8 and 2.12 determine the $\mathcal{A}_*$-comodule structure of the mod 2 motivic homology of BPGL and kgl, respectively.

This information is necessary to determine the E_2-term of the motivic ASSs for BPGL and kgl, which is done in Chapter III. The pertinent Ext algebras are recognized as the cohomology of the Hopf algebroids $\mathcal{E}(\infty)$ and $\mathcal{E}(1)$, respectively. These algebras are computed via a filtration spectral sequence in Theorems 3.4 and 3.6.

Chapter IV determines the coefficients of the 2-complete motivic Brown-Peterson spectrum over a p-adic field via the motivic ASS. Using a result of Morel on the coefficients of the algebraic cobordism spectrum and May's "higher Leibniz rule", the computation reduces to understanding the behavior of the Tate twist τ in the spectral sequence. This computation is presented as Theorem 4.9. As a corollary to these computations, I determine the E_2-term of the motivic ANSS over a p-adic field in terms of basic arithmetic data and the topological ANSS E_2-term; this result is Theorem 4.11. I also show how the same techniques reproduce known computations of 2-complete algebraic K-theory in Theorem 4.17.

Chapter V is a digression on Massey products necessary to control some of the terms appearing in Theorem 4.11. I recast some basic homological algebra in order to identify certain derived terms in E_2 of the motivic ANSS as Massey products. I also import the construction of Toda brackets to the motivic context and review results of Moss that exhibit how Massey products in the ANSS represent Toda brackets.

Chapter VI is prepatory to my analysis of the motivic ANSS over a p-adic field. It recalls some important facts about Hermitian K-theory, determines a half-plane worth of coefficients of KO, and constructs the motivic J-homomorphism. I also discuss the motivic e-invariant and the image of J.

In Chapter VII, I conclude by exploring the behavior of a motivic analogue of the

alpha family in the motivic ANSS. As in the topological ANSS, the J-homomorphism determines differentials on the α-family, but novel behavior appears over p-adic fields, such as nontrivial differentials at the E_2-level that kill Massey product "shadows" of τ-multiplication. My main results on the motivic alpha family over a p-adic field are Theorems 7.4 and 7.11.

CHAPTER II

Comodules over the dual motivic Steenrod algebra

In this chapter only, I work over a general characteristic 0 field k. Let H denote the mod 2 motivic cohomology k-spectrum. I determine the $\mathcal{A}_*$-comodule structure on $H_*\mathsf{BPGL}$ and $H_*\mathsf{kgl}$. For the definition of BPGL, see §1.1(3) and [HK01, Vez01]. The spectrum kgl is a connective variant of the algebraic K-theory k-spectrum KGL (see §1.1(2) and [Voe98, §6.2]) which has not been discussed in the literature previously. I begin by defining kgl as a BPGL-algebra and determining a few of its basic properties. I then use a result of Borghesi [Bor03] (a natural algebraicization of the Steenrod algebra-module structure on the mod 2 singular cohomology of the topological Brown-Peterson spectrum) to determine the $\mathcal{A}_*$-comodule structure on $H_*\mathsf{BPGL}$. The same theorem of Borghesi and techniques of Wilson [Wil75] produce the $\mathcal{A}_*$-comodule structure on $H_*\mathsf{BPGL}$. I also track similar results for a version of motivic Johnson-Wilson cohomology in a series of remarks. Througout, I work at the prime 2.

Recall that there are canonical elements $v_1, v_2, \ldots \in \mathsf{BPGL}_*$ that appear in dimensions $|v_i| = (2^i - 1)(1 + \alpha)$ [HK01]. Also let $v_0 = 2 \in \mathsf{BPGL}_0$. These elements are the images of $v_i \in \mathsf{BP}_{2(2^i-1)}$ induced by the Lazard ring isomorphism $\mathsf{MU}_* \to \mathsf{MGL}_{*(1+\alpha)}$. As a model for "connective" algebraic K-theory, I make the following definition

suggested by Mike Hopkins and communicated to me by Mike Hill.

Definition 2.1. Define a motivic spectrum kgl as the quotient of BPGL by $(v_2, v_3, \ldots)$.

Note that kgl is well-defined by motivic Landweber exactness [NØS].

Remark 2.2. More generally, one could define the *motivic Johnson-Wilson spectra* $\mathsf{BPGL}\langle n \rangle$ to be the quotient of BPGL by the ideal $(v_{n+1}, v_{n+2}, \ldots)$, in which case $\mathsf{kgl} = \mathsf{BPGL}\langle 1 \rangle$. Note that these spectra fit into cofiber sequences

$$\Sigma^{|v_n|}\mathsf{BPGL}\langle n \rangle \xrightarrow{v_n} \mathsf{BPGL}\langle n \rangle \to \mathsf{BPGL}\langle n-1 \rangle.$$

The following theorem relates the coefficients of kgl with established objects of interest, the algebraic K-groups of the ground field.

Convention 2.3. Throughout the rest of this chapter, let KGL denote the 2-localization of the full k-spectrum KGL.

Theorem 2.4. *Let $_{v_1}\mathsf{kgl}_*$ denote the v_1-power torsion in the coefficients of kgl, i.e., the elements $x \in \mathsf{kgl}_*$ such that there exists $n \in \mathbb{N}$ such that $v_1^n x = 0 \in \mathsf{kgl}_*$. (I will sometimes refer to these elements simply as v_1-torsion.) Then there is an exact sequence*

$$0 \to {}_{v_1}\mathsf{kgl}_* \to \mathsf{kgl}_* \to \mathsf{KGL}_*.$$

Moreover, if $\overline{\mathsf{KGL}_*}$ denotes the subalgebra of KGL_* consisting of elements in degree $m + n\alpha$, $m \geq 0$, then there is a short exact sequence

$$0 \to {}_{v_1}\mathsf{kgl}_* \to \mathsf{kgl}_* \to \overline{\mathsf{KGL}_*} \to 0.$$

Proof. By the motivic Conner-Floyd theorem [NØS], $\mathsf{KGL}_* = v_1^{-1}\mathsf{kgl}_*$. The first exact sequence is then a basic fact of localization.

Clearly, though, the map $\mathsf{kgl}_* \to \mathsf{KGL}_*$ is not surjective since KGL_* is Bott $= v_1$-periodic. Note, though, that KGL_* is generated by v_1 of dimension $1 + \alpha$ and elements

of degree $m+n\alpha$, $m \geq 0$. (In fact, we could restrict the second collection of generators to degrees $0 + n\alpha$, $n \leq 0$.) Again by the motivic Conner-Floyd theorem, it is now a straightforward combinatorial check that $\mathsf{kgl}_* \to \mathsf{KGL}_*$ is surjective in dimensions $m + n\alpha$, $m \geq 0$. $\qquad\square$

Remark 2.5. The k-spectrum kgl is connective in the sense that $\mathsf{kgl}_{m+n\alpha} = 0$ for all $m < 0$. We will see that there is a rich class of v_1-torsion in KGL_*, so it is the case that kgl_* is bigger than KGL_* in its nonvanishing dimensional range. Still, producing computations of kgl_* explicit enough to capture its v_1-torsion will determine KGL_* in a meaningful dimensional range by the second exact sequence. In particular,

$$(\mathsf{kgl}_*/_{v_1}\mathsf{kgl}_*)_{m+0\alpha} = \mathsf{KGL}_{m+0\alpha}$$

for $m \geq 0$.

I now turn to determining the $\mathcal{A}_*$-comodule structure of $H_*\mathsf{BPGL}$ and $H_*\mathsf{kgl}$. To access these, I will actually determine the $\mathcal{A}^*$-module structure of $H^*\mathsf{BPGL}$ and $H^*\mathsf{kgl}$ first.

Recall the Milnor primitives $Q_i \in \mathcal{A}^*$, $|Q_i| = 2^i(1+\alpha) - \alpha$ from §1.2. The following theorem of Borghesi should appear quite familiar to topologists.

Theorem 2.6 ([Bor03, Proposition 6]). *The mod 2 motivic cohomology of* MGL *takes the form*

$$H^*\mathsf{MGL} = (\mathcal{A}^*/(Q_0, Q_1, \ldots))[m_i \mid i \neq 2^n - 1]$$

as an $\mathcal{A}^*$-*module where* $|m_i| = i(1 + \alpha)$.

Corollary 2.7. *The mod 2 motivic cohomology of* BPGL *takes the form*

$$H^*\mathsf{BPGL} = \mathcal{A}^*/(Q_0, Q_1, \ldots)$$

as an $\mathcal{A}^*$-*module.* $\qquad\square$

Recall Definition 1.21 which defines the $\mathcal{A}_*$-algebras $\mathcal{E}(n)$, $0 \leq n \leq \infty$. In particular, we have

$$\mathcal{E}(\infty) = \mathcal{A}_* // (\xi_1, \xi_2, \ldots)$$

$$= H_*[\tau_0, \tau_1, \ldots] / (\tau_i^2 - \rho \tau_{i+1} \mid i \geq 0),$$

(2.1)

$$\mathcal{E}(1) = \mathcal{A}_* // (\xi_1, \xi_2, \ldots, \tau_2, \tau_3, \ldots)$$

$$= H_*[\tau_0, \tau_1] / (\tau_0^2 - \rho \tau_1, \tau_1^2).$$

These algebras are dual to $\mathcal{A}^*/(Q_0, Q_1, \ldots)$ and $\mathcal{A}^*/(Q_0, Q_1)$, respectively.

There is a general yoga of passing from $\mathcal{A}^*$-module structure on cohomologies to $\mathcal{A}_*$-comodule structure on homologies. Applied to the above situation, I get the following theorem describing the $\mathcal{A}_*$-comodule structure on $H_*\mathsf{BPGL}$.

Theorem 2.8. *As an $\mathcal{A}_*$-comodule algebra,*

$$H_*\mathsf{BPGL} = \mathcal{A}_* \,\square_{\mathcal{E}(\infty)}\, H_*.$$

$\square$

To determine the $\mathcal{A}_*$-comodule structure of $H_*\mathsf{kgl}$ I will first determine $H^*\mathsf{kgl}$ as an $\mathcal{A}^*$-module and then apply the same yoga. My determination of $H^*\mathsf{kgl} = H^*\mathsf{BPGL}\langle 1 \rangle$ is modeled on the topological calculation [Wil75]. The calculation depends on Corollary 2.7 and the following unpublished result of Mike Hopkins and Fabien Morel, communicated to me by Mike Hill.

Theorem 2.9 ([HM]). *Recall that* $\mathsf{BPGL}\langle 0 \rangle = \mathsf{BPGL}/(v_1, v_2, \ldots)$. *After 2-completion, this* k-*spectrum is the 2-complete integral motivic cohomology spectrum,*

$$\mathsf{BPGL}\langle 0 \rangle\widehat{\,}_2 = H\mathbb{Z}_2.$$

This enables me to make the following computation of $H^*\mathsf{kgl}$.

Theorem 2.10. *As an $\mathcal{A}^*$-module algebra,*

$$H^*\mathsf{kgl} = \mathcal{A}^*/(Q_0, Q_1).$$

Proof. As in Remark 2.2, there is a cofiber sequence

$$\Sigma^{1+\alpha}\mathsf{kgl} \xrightarrow{v_1} \mathsf{kgl} \to \mathsf{BPGL}\langle 0 \rangle.$$

This induces a long exact sequence in cohomology

$$H^0\mathsf{kgl} \xrightarrow{\partial^*} H^{2+\alpha}\mathsf{BPGL}\langle 0 \rangle \to H^{2+\alpha}\mathsf{BPGL}\langle 1 \rangle \to H^1\mathsf{BPGL}\langle n \rangle \to \cdots$$

Note that the canonical map $\mathsf{BPGL} \to \mathsf{kgl}$ induces, in cohomology, the map

$$H^*\mathsf{kgl} \to \mathcal{A}^*/(Q_0, Q_1, \ldots)$$

by Corollary 2.7. By a simple dimension count, the map on coefficients $\mathsf{BPGL}_* \to \mathsf{kgl}_*$ is an isomorphism in dimensions $m + n\alpha$ with $n \leq 6 - m$. It follows that the map in cohomology is an isomorphism in the same dimensional range. Now $|Q_1| = 2 + \alpha$, which falls in the dimensional range, so we know that $Q_1(1) = 0 \in H^*\mathsf{kgl}$.

By the above long exact sequence, it follows that $1 \in H^0\mathsf{kgl}$ maps to $\lambda Q_1(1)$ in $H^{2+\alpha}\mathsf{BPGL}\langle 0 \rangle$. By Theorem 2.9, $H^*\mathsf{BPGL}\langle 0 \rangle = \mathcal{A}^*/(Q_0)$, so $\lambda Q_1(1) \neq 0$. As in [Wil75, Proposition 1.7], it follows that $H^*\mathsf{kgl} = \mathcal{A}^*/(Q_0, Q_1)$, as desired. $\qquad\square$

Remark 2.11. In general, $|Q_n| = |v_n| + 1$, so the cofiber sequence of Remark 2.2 allows us to run the cohomology long exact sequence argument of [Wil75, Proposition 1.7] for general n. A dimensional analysis similar to the one above still works, and we can conclude that

$$H^*\mathsf{BPGL}\langle n \rangle = \mathcal{A}^*/(Q_0, \ldots, Q_n)$$

as $\mathcal{A}^*$-modules for all $0 \leq n \leq \infty$.

Since $\mathcal{E}(1)$ is dual to $\mathcal{A}^*/(Q_0, Q_1)$, the following theorem follows immediately.

Theorem 2.12. *As an $\mathcal{A}_*$-comodule algebra,*

$$H_*\mathsf{kgl} = \mathcal{A}_* \,\square_{\mathcal{E}(1)}\, H_*.$$

$\square$

Remark 2.13. By Remark 2.11, it also follows that

$$H_*\mathsf{BPGL}\langle n \rangle = \mathcal{A}_* \,\square_{\mathcal{E}(n)}\, H_*$$

for all $0 \le n \le \infty$.

CHAPTER III

Computing motivic Ext-algebras

Theorems 2.8 and 2.12 identify the homology of BPGL and kgl in the category of $\mathcal{A}_*$-comodules. By Theorem 1.33, this data forms part of the input to the E_2-term of the motivic ASS for BPGL and kgl. In fact, both E_2-terms take the form

$$\mathrm{Ext}_{\mathcal{A}_*}(H_*, \mathcal{A}_* \,\square_{\mathcal{E}(n)}\, H_*).$$

The following change of rings theorem will identify this algebra with the cohomology of $\mathrm{Ext}_{\mathcal{E}(n)}(H_*, H_*)$.

Theorem 3.1 ([Rav86, Theorem A1.3.12]). *For $0 \leq n \leq \infty$, the map of Hopf algebroids $(H_*, \mathcal{A}_*) \to (H_*, \mathcal{E}(n))$ induces an isomorphism*

$$\mathrm{Ext}_{\mathcal{A}_*}(H_*, \mathcal{A}_* \,\square_{\mathcal{E}(n)}\, H_*) \cong \mathrm{Ext}_{\mathcal{E}(n)}(H_*, H_*).$$

$\square$

Remark 3.2. While Theorem 3.1 identifies the E_2-term of the motivic ASS for $\mathsf{BPGL}\langle n \rangle$ as the cohomology of $\mathcal{E}(n)$, I will only compute this E_2-term in the cases $n = 1, \infty$ over a p-adic field F, i.e., the cases of $\mathsf{BPGL}\langle 1 \rangle = \mathsf{kgl}$ and $\mathsf{BPGL}\langle \infty \rangle = \mathsf{BPGL}$ over F.

Fix a p-adic field F (see §1.5) with residue order q. In this section, I begin by computing $\mathrm{Ext}_{\mathcal{E}(\infty)}(H_*, H_*)$ over F; this is the E_2-term for the motivic ASS com-

puting $(\mathsf{BPGL}\widehat{2})_*$. This work was antecedent to Mike Hill's paper [Hill] in which he performs similar computations over the field of real numbers $\mathbb{R}$.

Recall that when $q \equiv 1$ (4), Remark 1.43 implies that $Ext_{\mathcal{E}(\infty)}(H_*, H_*)$ is easily computable in terms of its complex counterpart $\mathrm{Ext}_{\mathcal{E}(\infty)^\mathbb{C}}(H_*^\mathbb{C}, H_*^\mathbb{C})$. In fact, $(H_*^F, \mathcal{E}(\infty)^F) = (H_*^\mathbb{C}, \mathcal{E}(\infty)^\mathbb{C}) \otimes_{H_*^\mathbb{C}} H_*^F$ as Hopf algebroids, so, by change of base,

$$\mathrm{Ext}_{\mathcal{E}(\infty)^F}(H_*^F, H_*^F) = \mathrm{Ext}_{\mathcal{E}(\infty)^\mathbb{C}}(H_*^\mathbb{C}, H_*^\mathbb{C}) \otimes_{H_*^\mathbb{C}} H_*^F$$

when $q \equiv 1$ (4). Moreover, since $\rho = 0$, $\mathcal{E}(\infty)^\mathbb{C} = \mathcal{E}(\infty)^{\mathrm{top}} \otimes_{\mathbb{Z}/2} H_*^\mathbb{C}$. Here $\mathcal{E}(\infty)^{\mathrm{top}}$ is the analogous quotient of the topological dual Steenrod algebra, but degree-shifted so that elements usually in degree $2m$ appear in dimension $m(1 + \alpha)$. Hence, again by change of base, I can compute the E_2-term of the motivic ASS for $\mathsf{BPGL}^\mathbb{C}$. To be precise,

$$\mathrm{Ext}_{\mathcal{E}(\infty)^\mathbb{C}}(H_*^\mathbb{C}, H_*^\mathbb{C}) = \mathrm{Ext}_{\mathcal{E}(\infty)^{\mathrm{top}}}(H_*^{\mathrm{top}}, H_*^{\mathrm{top}}) \otimes_{\mathbb{Z}/2} H_*^\mathbb{C}$$

$$= \mathbb{Z}/2[v_0, v_1, \ldots] \otimes_{\mathbb{Z}/2} H_*^\mathbb{C}.$$

(See [Rav86, Corollary 3.1.10] for the computation in topology.)

This yields, for $q \equiv 1$ (4), the computation

$$\mathrm{Ext}_{\mathcal{E}(\infty)^F}(H_*^F, H_*^F) = H_*^F[v_0, v_1, \ldots]$$

where $|v_i| = (1, (2^i - 1)(1 + \alpha) + 1)$.

When $q \equiv 3$ (4), $\mathcal{E}(\infty)$ does not split over $\mathcal{E}(\infty)^\mathbb{C}$. In order to deal with the extra complexity introduced by the relation $\tau_i^2 = \rho\tau_{i+1}$, I filter by powers of ρ and consider the associated filtration spectral sequence [Rav86, Theorem A1.3.9]. (In [Hill], Hill refers to this spectral sequence (over $\mathbb{R}$) as the "ρ-Bockstein spectral sequence.")

Since $\mathcal{E}(\infty)_*^F//(\rho) = \mathcal{E}(\infty)_*^{\mathbb{C}} \oplus \pi\mathcal{E}(\infty)_*^{\mathbb{C}}$, this spectral sequence takes the form

$$E_1 = \begin{pmatrix} \mathrm{Ext}_{\mathcal{E}(\infty)^{\mathbb{C}}}(H_*^{\mathbb{C}}, H_*^{\mathbb{C}}) \\ \oplus \\ \pi\,\mathrm{Ext}_{\mathcal{E}(\infty)^{\mathbb{C}}}(H_*^{\mathbb{C}}, H_*^{\mathbb{C}}) \end{pmatrix} [\rho]/(\rho^2) \implies \mathrm{Ext}_{\mathcal{E}(\infty)^F}(H_*^F, H_*^F).$$

Since $\eta_R(\tau) - \eta_L(\tau) = \rho\tau_0$ in $\mathcal{E}(\infty)^F$, τ supports the d_1-differential

$$d_1\tau = \rho\tau_0.$$

Since they are on the boundary, the generators $\pi, \rho, v_0, v_1, \ldots$ do not support differentials, and we have determined the E_2 page of the filtration spectral sequence:

$$E_2 = \begin{matrix} k_*^M(F)[\tau^2, v_0, \ldots]/(\rho v_0 = 0) \\ \oplus \\ \rho\tau k_*^M(F)[\tau^2, v_0, \ldots] \end{matrix}.$$

A dimensional analysis shows that there are no nontrivial d_2 differentials. Since $\rho^2 = 0$, the spectral sequence collapses here.

In order to fully determine $\mathrm{Ext}_{\mathcal{E}(\infty)}(H_*, H_*)$ when $q \equiv 3$ (4), I must address hidden extensions in $E_2 = E_\infty$.

Proposition 3.3. *For $q \equiv 3$ (4), any hidden extensions in the associated graded E_∞ of $\mathrm{Ext}_{\mathcal{E}(\infty)}(H_*, H_*)$ are contained in the ideal $(\pi\rho\tau v_1 - \tau^2 v_0)$.*

Proof. A quick glance at tri-degrees shows that we only need concern ourselves with candidate relations of the form

$$\pi\rho\tau^{2r+1}\prod_{a \in A} v_{i_a} = \tau^{2r+2}\prod_{b \in B} v_{j_b}.$$

Note that π, ρ, τ all have Ext-degree 0 while all of the v_i have Ext-degree 1. It follows that $|A| = |B|$. Computing the total dimension of both sides of the equation, we

derive the equality

$$\sum_{a \in A} 2^{i_a} = 1 + \sum_{b \in B} 2^{j_b}.$$

It follows that the formal quotient

$$\frac{\prod_{a \in A} v_{i_a}}{\prod_{b \in B} v_{j_b}} = \frac{v_1}{v_0},$$

so any extant relations are of the form

$$\rho \pi \tau^{2r} v_1 = \tau^{2r+1} v_0.$$

□

This, combined with the filtration spectral sequence computation, proves the following theorem.

Theorem 3.4. *Over a p-adic field F,*

$$\text{Ext}_{\mathcal{A}_*}(H_*, H_* \text{BPGL}) = \begin{cases} \bigwedge(\pi, u)[\tau, v_0, \ldots] & \text{if } q \equiv 1 \quad (4), \\ \\ k_*^M(F)[\tau^2, v_0, \ldots]/(\rho v_0) & \\ \\ \qquad \qquad \oplus & \text{if } q \equiv 3 \quad (4) \\ \\ \rho \tau k_*^M(F)[\tau^2, v_0, \ldots] & \end{cases}$$

up to possible relations in the ideal $(\pi \rho \tau v_1 - \tau^2 v_0)$ in the $q \equiv 3$ (4) case. □

Remark 3.5. Note that by Proposition 1.42, the exterior algebra $\bigwedge(\pi, u)$ is shorthand for $k_*^M(F)$ when $q \equiv 1$ (4). Moreover, the same proposition specifies the mod 2 Milnor K-theory of F as

$$\mathbb{Z}/2[\pi, \rho]/(\rho^2, \pi(\rho - \pi)).$$

In the next chapter, we will see that any candidate relations are killed by the Adams spectral sequences for BPGL, so any extra precision in our understanding of $\text{Ext}_{\mathcal{E}(\infty)}(H_*, H_*)$ is not completely necessary for applications to BPGL.

The same methodology employed to determine the cohomology of $\mathcal{E}(\infty)$ works for $\mathcal{E}(1)$. We record the result in the following theorem.

Theorem 3.6. *Over a p-adic field F,*

$$
\text{Ext}_{\mathcal{A}_*}(H_*, H_* \text{kgl}) = \begin{cases} \bigwedge(\pi, u)[\tau, v_0, v_1] & \text{if } q \equiv 1 \quad (4), \\ \\ k_*^M(F)[\tau^2, v_0, v_1]/(\rho v_0) \\ \\ \qquad\qquad \oplus & \text{if } q \equiv 3 \quad (4) \\ \\ \rho\tau k_*^M(F)[\tau^2, v_0, v_1] \end{cases}
$$

up to possible relations in the ideal $(\pi\rho\tau v_1 - \tau^2 v_0)$ in the $q \equiv 3$ (4) case. $\qquad\square$

Proof. This follows the exact same structure as the $\mathcal{E}(\infty)$ case. Start by noting [Rav86, Theorem 3.1.16] that the topological $\mathcal{E}(1)$ has cohomology

$$
\text{Ext}_{\mathcal{E}(1)^{\text{top}}}(H_*^{\text{top}}, H_*^{\text{top}}) = \mathbb{Z}/2[v_0, v_1].
$$

When $q \equiv 1$ (4), $\mathcal{E}(1)^F$ splits as

$$
\mathcal{E}(1)^F = \mathcal{E}(1)^{\mathbb{C}} \otimes_{H_*^{\mathbb{C}}} H_*^F
$$
$$
= \mathcal{E}(1)^{\text{top}} \otimes_{\mathbb{Z}/2} H_*^F.
$$

Hence, when $q \equiv 1$ (4),

$$
\text{Ext}_{\mathcal{E}(1)}(H_*, H_*) = \bigwedge(\pi, u)[\tau, v_0, v_1]
$$

by Proposition 1.42 and Theorem 1.43.

The case $q \equiv 3$ (4) remains and I again filter by powers of ρ and run a filtration spectral sequence. The E_1-term is

$$E_1 = \begin{pmatrix} \operatorname{Ext}_{\mathcal{E}(1)^{\mathbb{C}}}(H_*^{\mathbb{C}}, H_*^{\mathbb{C}}) \\ \oplus \\ \pi \operatorname{Ext}_{\mathcal{E}(1)^{\mathbb{C}}}(H_*^{\mathbb{C}}, H_*^{\mathbb{C}}) \end{pmatrix} [\rho]/(\rho^2) \implies \operatorname{Ext}_{\mathcal{E}(1)^F}(H_*^F, H_*^F).$$

Since $\eta_R(\tau) - \eta_L(\tau) = \rho\tau_0$ in $\mathcal{E}(1)^F$, τ supports

$$d_1\tau = \rho\tau_0$$

exactly as in the $\mathcal{E}(\infty)$ case. This determines the E_2-term of the filtration spectral sequence to be

$$E_2 = \begin{array}{c} k_*^M(F)[\tau^2, v_0, v_1]/(\rho v_0) \\ \oplus \\ \rho\tau k_*^M(F)[\tau^2, v_0, v_1] \end{array}.$$

Again, $\rho^2 = 0$ so the spectral sequence collapses here. Candidate relations are handled exactly as they are in Proposition 3.3, just with fewer v_is, completing the proof of the theorem. $\qquad\square$

In the next chapter, we will see that the candidate relations in $\operatorname{Ext}_{\mathcal{E}(1)}(H_*, H_*)$ are of no consequence after running the motivic ASS for kgl.

CHAPTER IV

Computations with the motivic Adams spectral sequence

Theorem 3.4 determines the E_2-term of the motivic ASS for BPGL (see also Theorem 1.33). This spectral sequence takes the form

$$\mathrm{Ext}_{\mathcal{A}_*}(H_*, H_*\mathsf{BPGL}) \implies \mathsf{BPGL}\widehat{_{2_*}}$$

where $\mathsf{BPGL}\widehat{_2}$ is the 2-completion (i.e., Bousfield localization at $\mathbb{1}/2$) of BPGL. My main tool in analyzing this spectral sequence is the following higher Leibniz rule due to May.

Proposition 4.1 ([May70, Proposition 6.8]). *If x supports a d_r-differential, then x^2 survives to E_{r+1} and*

$$d_{r+1}x^2 = xd_r(x)v_0.$$

$\square$

A small fact about the motivic ASS for kgl is needed in needed in order to run the motivic ASS for BPGL when $q \equiv 3$ (4). I will state this as a lemma here and prove it at the end of the section.

Lemma 4.2. *In the motivic ASS for kgl over a p-adic field F with residue order $q \equiv 3$ (4), τ^2 supports the differential*

$$d_2\tau^2 = \rho\tau v_0^2.$$

Via a map of spectral sequences induced by the canonical map $\mathsf{BPGL} \to \mathsf{kgl}$, Lemma 4.2 implies the following result.

Lemma 4.3. *In the motivic ASS for* BPGL *over a p-adic field* F *with residue order* $q \equiv 3$ (4), τ^2 *supports the differential*

$$d_2\tau^2 = \rho\tau v_0^2.$$

Proof. The canonical map of F-spectra $\mathsf{BPGL} \to \mathsf{kgl}$ induces a map of Adams resolutions and hence a map of motivic ASSs,

$$f : \{\text{motivic ASS for } \mathsf{BPGL}\} \to \{\text{motivic ASS for } \mathsf{kgl}\}\,.$$

On the E_2-term for BPGL given in Theorem 3.4, the map is simply reduction by $(v_2, v_3, \ldots)$. Since f is injective in dimension $(2, 3 - 2\alpha)$, Lemma 4.2 implies that $d_2\tau^2 = \rho\tau v_0^2$ in the motivic ASS for BPGL as well (when $q \equiv 3$ (4)). $\qquad\square$

The following proposition is an immediate corollary of Proposition 4.1 and Lemma 4.3.

Proposition 4.4. *If* $q \equiv 3$ (4)*, then there are differentials*

$$d_{1+v}\tau^{2^v} = \rho\tau^{2^v-1}v_0^{1+v}$$

for all $v \geq 1$ *in the motivic ASS for* BPGL. *By the usual Leibnitz rule, this implies that the* τ-*powers support differentials*

$$d_{1+\nu(s)}\tau^s = \rho\tau^{s-1}v_0^{1+\nu(s)}$$

where $\nu = \nu_2$ *is the 2-adic valuation on rational integers.* $\qquad\square$

In the $q \equiv 1$ (4) case the τ-poewrs support a similar family of differentials. To access these via the higher Leibniz rule, I must first determine what (if any) differential τ supports. To this end, I employ the following theorem of Morel.

Theorem 4.5 ([Mor04b]). *Over an arbitrary perfect field* k, *the coefficients of algebraic cobordism* MGL *in dimension* $0 - n\alpha$, $n \geq 0$, *are given by*

$$\mathsf{MGL}_{-n\alpha} = K_n^M(\mathsf{k}).$$

In particular, $\mathsf{MGL}_{-\alpha} = \mathsf{k}^\times$. Over our p-adic field F, the connectivity of MGL [Mor04a] and Theorem 1.32 implies that $(\mathsf{MGL}\widehat{_2})_{-\alpha} = \mathbb{Z}_2\{\pi\} \oplus \mathbb{Z}/2^k\{u\}$ where $k = \nu(q-1)$. Since v_0, which represents $2 + \rho\eta$ in $\pi_0 \mathbb{1}\widehat{_2}$, has image 2 in $(\mathsf{MGL}\widehat{_2})_0$, τ must support the differential

$$d_k\tau = uv_0^k;$$

otherwise, $(\mathsf{MGL}\widehat{_2})_{-\alpha}$ will not have the appropriate torsion. Again by the higher Leibniz rule, I have proved the following proposition.

Proposition 4.6. *If* $q \equiv 1$ (4) *and* $k = \nu(q-1)$, *then*

$$d_{k+v}\tau^{2^v} = u\tau^{2^v - 1}v_0^{k+v}$$

for all $v \geq 0$ *in the motivic ASS for* BPGL. *This implies that the* τ-*powers support differentials*

$$d_{k+\nu(s)}\tau^s = u\tau^{s-1}v_0^{k+\nu(s)}.$$

$\square$

Via Propositions 4.4 and 4.6 I derive the following unified description of the abutment of the motivic ASS for BPGL.

Theorem 4.7. *The* E_∞-*term of the motivic ASS for* BPGL *over a local field* F *is*

$$\Gamma'[v_1, v_2, \ldots]$$

where Γ' has additive structure

$$\Gamma' = \begin{cases} \mathbb{Z}/2[v_0] & in\ dimension\ 0, \\[1.5ex] \mathbb{Z}/2[v_0] \oplus \mathbb{Z}/2[v_0]/v_0^k & in\ dimension\ -\alpha, \\[1.5ex] \mathbb{Z}/2[v_0]/v_0^k & in\ dimension\ -2\alpha, \\[1.5ex] \mathbb{Z}/2[v_0]/v_0^{k+\nu(i)} & in\ dimension\ (i-1)(1-\alpha) - \epsilon\alpha\ for\ i \geq 1\ and\ \epsilon = 1\ or\ 2, \\[1.5ex] 0 & otherwise. \end{cases}$$

The multiplicative structure of Γ' is specified by the following comments: v_0-multiplication is already captured by the above description. The summands of $\Gamma'_{-\alpha}$ are generated by π and u, satisfying the relations in Proposition 1.42; in particular, πu generates $\Gamma'_{-2\alpha}$. Let γ_i denote an additive generator of $\Gamma'_{i(1-\alpha)-\alpha}$. Then $\pi\gamma_i$ generates $\Gamma'_{i(1-\alpha)-2\alpha)}$ while $u\gamma_i = 0$. All other products in Γ' are trivial by dimension reasons.

Proof. The generators $\pi, u, \pi u, v_0, v_1, \ldots$ do not support differentials since they are on the boundary of the E_2-term. Propositions 4.4 and 4.6 specify the differentials on the τ-powers, and this determines the entire spectral sequence. The structure of Γ' is a consequence of the same two propositions. $\qquad\square$

Remark 4.8. The behavior of this spectral sequence is depicted in Figure 4.1. Here elements in degree $(s, m + n\alpha)$ are depicted in total motivic degree $m + n\alpha - s)$ with the homological degree suppressed. (The horizontal axis measures $\mathbb{Z}$ while the vertical axis measures $\mathbb{Z}\alpha$.) Future diagrams of motivic spectral sequences will be drawn in the same fashion.

The diagonal arrows in the Figure 4.1 represent v_i-multiplication, $i \geq 1$. In the dimensional range shown, $E_{\nu(q-1)+3} = E_\infty$. Note that the v_0-towers in all pictures actually come out of the page, as do the v_i-multiplication arrows.
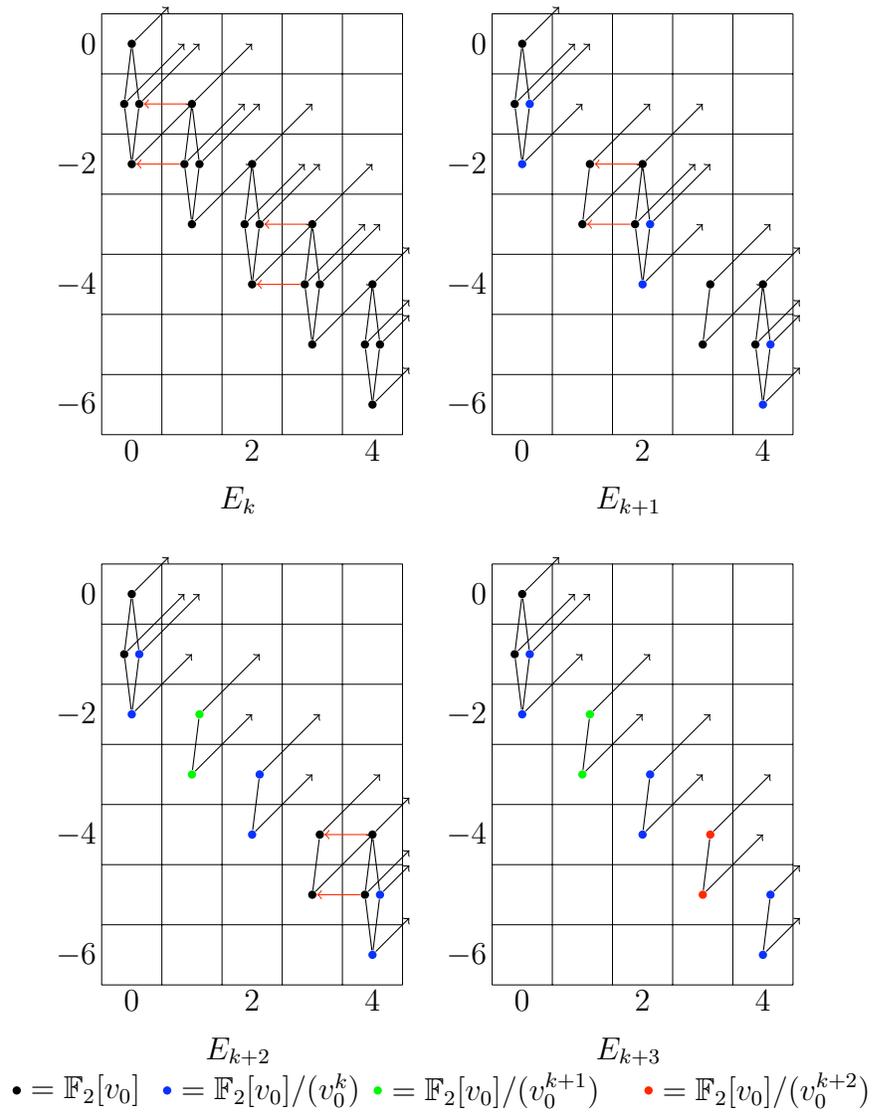
Figure 4.1: The motivic ASS for BPGL over F

A quick inspection of tri-degrees reveals that there are no hidden extensions except those created by v_0-multiplication. Indeed, the lines of slope 1 originating in the nonzero dimensions of Γ' do not overlap, so we only need to worry about v_0-towers. Since v_0-represents 2 in $(BPGL\widehat{2})_0$, any copies of $\mathbb{Z}/2[v_0]$ produce copies of the 2-adic integers $\mathbb{Z}_2$, and any copies of $\mathbb{Z}/2[v_0]/v_0^{k+\nu(s)}$ produce copies of $\mathbb{Z}/2^{k+\nu(s)}$. This proves the following theorem.

Theorem 4.9. *Let $w_i = 2^{k+\nu(i)}$. The coefficients of the 2-complete Brown-Peterson spectrum* $\mathsf{BPGL}\widehat{2}$ *over a p-adic field F are*

$$(BPGL\widehat{2})_* = \Gamma[v_1, v_2, \ldots]$$

where $|v_i| = (2^i - 1)(1 + \alpha)$ and, additively,

$$\Gamma = \begin{cases} \mathbb{Z}_2 & \text{in dimension } 0, \\\\ \mathbb{Z}_2 \oplus \mathbb{Z}/w_1 & \text{in dimension } -\alpha, \\\\ \mathbb{Z}/w_1 & \text{in dimension } -2\alpha, \\\\ \mathbb{Z}/w_i & \text{in dimension } (i-1)(1-\alpha) - \epsilon\alpha \text{ for } i \geq 1 \text{ and } \epsilon = 1 \text{ or } 2, \\\\ 0 & \text{otherwise.} \end{cases}$$

Multiplicative relations are the same as those in Theorem 4.7. □

Corollary 4.10. *The coefficients of the 2-complete algebraic cobordism spectrum over a p-adic field F are*

$$(\mathsf{MGL}\widehat{2})_* = \Gamma[v_1, v_2, \ldots, u_j | j \neq 2^n - 1]$$

where $|v_i| = (2^i - 1)(1 + \alpha)$ and $|u_j| = j(1 + \alpha)$. □

Before moving on to the motivic ASS for kgl, I record here a consequence of the identification of $(\mathsf{BPGL}\widehat{2})_*$ that will be vital to understanding the motivic ANSS over a p-adic field.

Theorem 4.11. *The Hopf algebroid for* $\mathsf{BPGL}\widehat{_2}$ *over a p-adic field* F *splits as*

$$(\mathsf{BPGL}\widehat{_{2*}}, \mathsf{BPGL}\widehat{_{2*}}\mathsf{BPGL}\widehat{_2}) = (\mathsf{BP}\widehat{_{2*}}, \mathsf{BP}\widehat{_{2*}}\mathsf{BP}\widehat{_2}) \otimes_{\mathbb{Z}_2} \Gamma.$$

Moreover, the E_2*-term of the motivic ANSS in homological degree* s *is*

$$\mathrm{Ext}^s_{\mathsf{BPGL}\widehat{_{2*}}\mathsf{BPGL}\widehat{_2}}(\mathsf{BPGL}\widehat{_{2*}}, \mathsf{BPGL}\widehat{_{2*}}) = \begin{array}{c} {}^{\mathrm{top}}E_2^s \otimes_{\mathbb{Z}_2} \Gamma \\[4pt] \oplus \\[4pt] \mathrm{Tor}_1^{\mathbb{Z}_2}({}^{\mathrm{top}}E_2^{s+1}, \Gamma) \end{array}.$$

Here ${}^{\mathrm{top}}E_2 = \mathrm{Ext}_{\mathsf{BP}\widehat{_{2*}}\mathsf{BP}\widehat{_2}}(\mathsf{BP}\widehat{_{2*}}, \mathsf{BP}\widehat{_{2*}})$ *with degrees shifted so that elements appearing in degree* $(s, 2m)$ *in topology appear in degree* $(s, m(1 + \alpha))$ *motivically.*

Proof. For typographical simplicity, I drop the 2-completion $(\ )\widehat{_2}$ from my notation in this proof.

The second statement is an easy consequence of the first via the cobar resolution computing $\mathrm{Ext}_{\mathsf{BPGL}_*\mathsf{BPGL}}(\mathsf{BPGL}_*, \mathsf{BPGL}_*)$ and the universal coefficient theorem.

As a consequence of motivic Landweber exactness, Naumann-Østvær-Spitzweck [NØS] deduce a splitting of the MGL Hopf algebroid as

$$(\mathsf{MGL}_*, \mathsf{MGL}_*\mathsf{MGL}) = (\mathsf{MU}_*, \mathsf{MU}_*\mathsf{MU}) \otimes_{\mathsf{MU}_*} \mathsf{MGL}_*$$

where MU_* is the coefficients of (topological) complex cobordism, the Lazard ring. This splitting passes to BPGL, so

$$(\mathsf{BPGL}_*, \mathsf{BPGL}_*\mathsf{BPGL}) = (\mathsf{BP}_*, \mathsf{BP}_*\mathsf{BP}) \otimes_{\mathsf{BP}_*} \mathsf{BPGL}_*$$

$$= (\mathsf{BP}_*, \mathsf{BP}_*\mathsf{BP}) \otimes_{\mathsf{BP}_*} (\mathsf{BP}_* \otimes_{\mathbb{Z}_2} \Gamma)$$

$$= (\mathsf{BP}_*, \mathsf{BP}_*\mathsf{BP}) \otimes_{\mathbb{Z}_2} \Gamma.$$

$\square$

This description of the E_2-term of the motivic ANSS over F already pays dividends in the form of a graded algebra with infinitely many nonzero components previously undiscovered in the stable stems $\pi_*\mathbb{1}\widehat{_2}$.

Theorem 4.12. *The algebra* Γ *is permanent and represents a copy of* Γ *in* $\pi_* \mathbb{1}\widehat{\,}_2$.

Proof. The elements of $\Gamma = E_2^{0,0} \otimes \Gamma$ are on the boundary of the spectral sequence and hence permanent. A simple dimensional analysis shows that they are not the targets of any differentials. □

I now turn to the analogous computations for kgl.

Proposition 4.13. *If $q \equiv 3$ (4), then there are differentials*

$$d_{1+v}\tau^{2^v} = \rho\tau^{2^v-1}v_0^{1+v}$$

for all $v \geq 1$ in the motivic ASS for kgl. *By the usual Leibniz rule, this implies that the τ-powers support differentials*

$$d_{1+\nu(s)}\tau^s = \rho\tau^{s-1}v_0^{1+\nu(s)}.$$

Proof. This is a consequence of Lemma 4.2 and the higher Leibniz rule, Proposition 4.1. □

Lemma 4.14. *If $q \equiv 1$ (4) and $k = \nu(q-1)$, then τ supports the differential*

$$d_k\tau = \rho v_0^k$$

in the motivic ASS for kgl *over F.*

Proof. This differential is the image of the analogous one in the motivic ASS for BPGL under the spectral sequence map induced by BPGL $\to$ kgl. □

Proposition 4.15. *If $q \equiv 1$ (4), then there are differentials*

$$d_{k+v}\tau^{2^v} = \rho\tau^{2^v-1}v_0^{k+v}$$

for all $v \geq 0$ in the motivic ASS for kgl. *By the usual Leibniz rule, this implies that the τ-powers support differentials*

$$d_{k+\nu(s)}\tau^s = \rho\tau^{s-1}v_0^{k+\nu(s)}.$$

Proof. This is an immediate consequence of Lemma 4.14 and Proposition 4.1. $\qquad\square$

Propositions 4.13 and 4.15 produce the following unified description of the abutment of the motivic ASS for kgl over F.

Theorem 4.16. *The E_∞-term of the motivic ASS for* kgl *over a p-adic field F is*

$$\Gamma'[v_1]$$

where Γ' is as in Theorem 4.7.

Proof. This is a consequence of Propositions 4.13 and 4.15 along with simple dimensional accounting. $\qquad\square$

Theorem 4.17. *The coefficients of the 2-complete connective algebraic K-theory F-spectrum over a p-adic field F are*

$$(\mathsf{kgl}\widehat{_2})_* = \Gamma[v_1]$$

where Γ is as in Theorem 4.9.

Remark 4.18. I will comment without proof (the details would be redundant) that the algebra Γ is the coefficients of the 2-complete integral cohomology F-spectrum $H\mathbb{Z}_2$. This is a consequence of Theorem 2.9 and the motivic ASS for $\mathsf{BPGL}\langle 0 \rangle$.

It remains to prove Lemma 4.2. This is the only part of this thesis using outside calculations in higher algebraic K-theory (excluding, say, the construction of KGL via Quillen's projective bundle formula), and it would be satisfying to be able to remove this dependancy. I have not yet discovered a method for doing so, and it is still quite interesting that these methods only require input from KGL_3 in the case $q \equiv 3$ (4).

Proof of Lemma 4.2. Recall from, e.g., [RW00] that $(\mathsf{KGL}\widehat{_2})_3 = \mathbb{Z}/4$ when $q \equiv 3\ (4)$.

By Theorem 3.6 and dimensional accounting, $(\mathsf{kgl}\widehat{_2})_3$ is generated by $\rho\tau v_1^2$. Since v_0 represents 2, Theorem 2.4 implies that $\rho\tau v_1^2 v_0^2$ is v_1-torsion or 0, i.e., $\rho\tau v_0^2$ is v_1-torsion or 0. Suppose for contradiction that $\rho\tau v_0^2 \neq 0 \in (\mathsf{kgl}\widehat{_2})_{1-2\alpha}$ and $\rho\tau v_0^2 v_1 = 0$. Then a differential in the motivic ASS for kgl must hit $\rho\tau v_0^2 v_1^t$, but by Theorem 3.6 and dimensional accounting, there are no elements of low enough homological degree in Adams grading one more than the Adams grading of $\rho\tau v_0^2 v_1^t$. I conclude that $\rho\tau v_0^2 = 0 \in (\mathsf{kgl}\widehat{_2})_{1-2\alpha}$, and the only differential capable killing $\rho\tau v_0^2$ is

$$d_2\tau^2 = \rho\tau v_0^2.$$

$\square$

CHAPTER V

Massey products

Theorem 4.11 determines the E_2-term of the motivic ANSS over a p-adic field F in terms of Γ and the E_2-term of the topological ANSS. The precise statement is that

$$
{}^F E_2^s =
\begin{array}{c}
\Gamma \otimes_{\mathbb{Z}_2} {}^{\mathrm{top}} E_2^s \\
\oplus \\
Tor_1^{\mathbb{Z}_2}(\Gamma, {}^{\mathrm{top}} E_2^{s+1})
\end{array}
$$

where ${}^{\mathrm{top}} E_2$ is the E_2-term of the topological ANSS under an appropriate degree shift. Determining the entirety of ${}^{\mathrm{top}} E_2$ is an extremely difficult problem and no full, explicit computation is known. What is available are theorems about infinite families of elements in ${}^{\mathrm{top}} E_2$, which I will return to in Chapter VII. In this chapter, my primary goal is to introduce some structures which provide control over the second summand in my decomposition of ${}^F E_2$, $Tor_1^{\mathbb{Z}_2}(\Gamma, {}^{\mathrm{top}} E_2)$. I will refer to this summand as the *derived terms* of ${}^F E_2$. I will show that these terms are represented by Massey products. I will also discuss results of Moss [Mos70] identifying terms in $\pi_* \mathbb{1}\widehat{{}_2}$ represented by Massey products as Toda brackets.

I limit my discussion of Massey products to triple products, and will only mention the higher and matric Massey products of [May69] in passing. Recall that in a DGA C with differentials $d : C^s \to C^{s+1}$, the *Massey product* $\langle x, y, z \rangle$ for $x, y, z \in H^* C$

is defined whenever $xy = yz = 0$ by the following procedure: let $\tilde{x}, \tilde{y}, \tilde{z}$ denotes lifts of x, y, z to C. Choose $u, v \in C$ such that $du = (-1)^{1+|x|}\tilde{x}\tilde{y}$ and $dv = (-1)^{1+|y|}\tilde{y}\tilde{z}$. Then $(-1)^{1+|u|}u\tilde{z} + (-1)^{1+|x|}\tilde{x}v = (-1)^{|x|+|y|}u\tilde{z} + (-1)^{|x|}\tilde{x}v$ is a cocycle in C and represents an element of the Massey product $\langle x, y, z \rangle \subset H^{|x|+|y|+|z|-1}C$. Elements of $\langle x, y, z \rangle$ can differ by members of $xH^{|y|+|z|-1}C \oplus zH^{|x|+|y|-1}C$, and this is called the *indeterminacy* of $\langle x, y, z \rangle$.

The next proposition describes the "derived terms" appearing in the universal coefficient theorem in terms of Massey products. Let $C = C_0 \otimes_R M$ denote the tensor product of a flat R-DGA C_0 with an R-module M where R is a PID or, more generally, any ring for which projective modules are free. (By "flat R-DGA" I mean a DGA C of flat R-modules in which each submodule $B^{s+1} = d(C^s)$ is also flat.) Note that the cobar resolution computing the E_2-term of the motivic ANSS is an example of such a tensor product by Theorem 4.11.

The universal coefficient theorem applies in this scenario and implies that

$$H^s C = H^s C_0 \otimes_R M \oplus \operatorname{Tor}_1^R(H^{s+1}C_0, M).$$

If $x \in H^*C_0$ and $m \in M$ are torsion, let $\sigma_m x$ denote the Tor_1 term in H^*C corresponding to these elements.

Proposition 5.1. *Let R be a ring for which projective modules are free. Suppose $C = C_0 \otimes_R M$ where C_0^s is a free R-module for all s. Take torsion elements $x \in H^{s+1}C_0, m \in M$ and let $n = \max\{\operatorname{ord} x, \operatorname{ord} y\}$. Then*

$$\sigma_m x \in \langle m, n, x \rangle \subset H^s C.$$

Proof. The proof follows from inspection of your favorite proof of the universal coefficient theorem and an appropriate choice of splitting in the resulting short exact

sequence. I base my proof on the exposition in [Wei94, §3.6]. Let Z_0^s denote the cocycles in C_0^s and B_0^{s+1} denote the coboundaries in C_0^{s+1} so there is a short exact sequence $0 \to Z_0^s \to C_0^s \to B_0^{s+1} \to 0$. Since all the terms in the short exact sequence are free, $0 \to Z_0^s \otimes M \to C_0^s \otimes M \to B_0^{s+1} \otimes M \to 0$ is exact for every s. These result in an exact sequence of cochain complexes with associated cohomology long exact sequence

$$\cdots \to B_0^s \otimes M \to Z_0^s \otimes M \to H^s(C_0 \otimes M) \to B_0^{s+1} \otimes M \xrightarrow{\partial} Z_0^{s+1} \otimes M \to \cdots .$$

It follows that $\mathrm{Tor}_*(H^s(C_0), M)$ is the homology of

$$0 \to B_0^s \otimes M \xrightarrow{\partial} Z_0^s \otimes M \to 0,$$

providing the Künneth short exact sequence

$$0 \to H^s(C_0) \otimes M \to H^s(C_0 \otimes M) \to \mathrm{Tor}_1^R(H^{s+1}C, M) \to 0.$$

By assumption, every submodule of a free R-module is free. Since B_0^{s+1} is a submodule of C_0^{s+1}, it is also a free R-module. Hence $C_0^s \to B_0^{s+1}$ splits (noncanonically); fix a splitting $s : B_0^{s+1} \to C_0^s$. We have a decomposition $C_0^s \cong Z_0^s \oplus B_0^{s+1}$. Moreover, $\mathrm{Tor}_1^r(H^{s+1}C_0, M) \cong \ker \partial \subset B_0^{s+1} \otimes M$ can be considered as a submodule of $C_0^s \otimes M$ via $s \otimes M$. $\ker \partial$ is generated additively by elements of the form $dc \otimes m$ where $c \in C_0^s$, $dc = rc'$, and $rm = 0$. Since $nm = nx = 0$, $\sigma_m(x) \in \mathrm{Tor}_1^R(H^{s+1}C_0, M) \subset H^sC$ is represented on the chain level by some $s(dc) \otimes m \in C_0^s \otimes M$ with $dc = r\tilde{x}$, for $\tilde{x}$ a lift of x to C_0^s. This proves that $\sigma_m(x)$ is represented by $s(dc) \otimes m$, which visibly represents an element of in $\langle m, n, x \rangle$, as desired. $\qquad \square$

I will now record a juggling theorem for Massey products necessary to effectively manipulate them in calculations.

Proposition 5.2 ([May69, Corollary 3.2]). *If $\langle x, y, z \rangle$ is defined, then so is $\langle x, y, zw \rangle$ and*

$$\langle x, y, z \rangle w \subset \langle x, y, zw \rangle.$$

$\square$

The Massey products of algebra represent Toda brackets in homotopy. Toda brackets are standard constructions in topology, and their formation (see [MT68, Chapter 17] or [Koc96]) carries over to $\mathcal{SH}(\mathsf{k})$, k an arbitrary field.

For my purposes, it suffices to note that whenever

$$W \xrightarrow{\gamma} X \xrightarrow{\beta} Y \xrightarrow{\alpha} Z$$

are maps of k-spectra such that $\beta\gamma = \alpha\beta = 0$ in $\mathcal{SH}(\mathsf{k})$, there is an associated *Toda bracket*

$$\langle \alpha, \beta, \gamma \rangle \in [\Sigma W, Z]$$

where $[\cdot, \cdot] = \mathcal{SH}(\mathsf{k})(\cdot, \cdot)$ defined up to indeterminacy $\alpha_*[\Sigma W, Y] + (\Sigma\gamma)^*[\Sigma X, Z]$.

In particular, if $\alpha \in \pi_\lambda \mathbb{1}\,\widehat{_2}$, $\beta \in \pi_\mu \mathbb{1}\,\widehat{_2}$, and $\gamma \in \pi_\nu \mathbb{1}\,\widehat{_2}$ such that $\gamma\beta = \beta\alpha = 0$, then there is an associated Toda bracket

$$\langle \alpha, \beta, \gamma \rangle \in \pi_{\lambda+\mu+\nu+1} \mathbb{1}\,\widehat{_2}$$

with indeterminacy $\alpha\pi_{\mu+\nu+1}\mathbb{1}\,\widehat{_2} + \beta\pi_{\lambda+\nu+1}\,\widehat{_2}$.

Theorem 5.3. *Suppose x, y, z are permanent cycles in the r-th page of the motivic ANSS such that $xy = yz = 0$. Furthermore, suppose x, y, z represent $X, Y, Z \in \pi_* \mathbb{1}\,\widehat{_2}$ such that $XY = YZ = 0$. Suppose, moreover, that all elements of $E_{s+s'-n+1}^{n,i+i'+n+1}$ and $E_{s'+s''-m+1}^{m,i'+i''+m+1}$ are permanent, where X, Y, Z have dimensions i, i', i'', respectively, in $\pi_* \mathbb{1}\,\widehat{_2}$; x, y, z have filtration degrees s, s', s'', respectively, in the motivic ANSS; and $0 \le n \le s + s' - r$, $0 \le m \le s' + s'' - r$.*

Then the Massey product $\langle x, y, z \rangle$ *contains a permanent cycle representing the Toda bracket* $\langle X, Y, Z \rangle$.

Proof. This follows from [Mos70, Theorem 1.2]. □

Let α_1 denote the image of $\alpha_1 \in {}^{\text{top}} E_2^{1,1+\alpha}$ in E_2 of the motivic ANSS over F, and let $k = \nu(q-1)$, q the residue order of F. Theorem 5.3 produces the following computation.

Proposition 5.4. *The Massey product* $\langle u, 2^k, \alpha_1 \rangle$ *is a nonzero strictly defined permanent cycle in* E_2 *of the motivic ANSS representing the Toda bracket* $\langle u, 2^{k-1}(2 + \rho\eta), \eta \rangle$.

Proof. By Proposition 5.1 we can identify $\langle u, 2^k, \alpha_1 \rangle$ with $\sigma_u \alpha_1 \neq 0$, a derived term of ${}^F E_2^{0,1+0\alpha}$. The permanence of everything of degree $(0,1)$ is clear by a dimension count, and it is obvious that u and α_1 represent u and η, respectively. Moreover, 2 represents $2 + \rho\eta$ since $\rho\alpha_1$, which represents $\rho\eta$, is in a higher homological degree than 2. The fact that $(2 + \rho\eta)\eta = 0$ is one of the defining relations of Milnor-Witt K-theory (see Definition 1.25) and the other vanishing results are obvious, so the result follows from Theorem 5.3. □

Definition 5.5. Let

$$\vartheta_\eta := \langle u, 2^{k-1}(2 + \rho\eta), \eta \rangle,$$

the Toda bracket represented by $\sigma_u \alpha_1$.

Remark 5.6. The notation for ϑ_η should be reminiscent of $\theta\eta$ (the notation in [HKOa] for $\tau\eta$), a product that does not exist in $\pi_* \mathbb{1}_{\widehat{2}}$ over a p-adic field, but is very important in the computational study of motivic homotopy over an algebraically closed field (see [HKOa]). Note that both elements are in $\pi_1 \mathbb{1}_{\widehat{2}}$ over their respective fields. In

Chapter VII, I will show that ϑ_η often serves as a replacement for $\tau\eta$ when the latter is not available.

CHAPTER VI

The motivic J-homomorphism

Before determining some differentials in the motivic Adams-Novikov spectral sequence over the p-adic field F, I will compile some straightforward but important results on Hermitian K-theory and a motivic analogue of the J-homomorphism. These results will form the geometric input necessary to determine Adams-Novikov differentials on a motivic variant of the alpha family.

For the moment, let k be an arbitrary field. Let KO denote Hornbostel's Hermitian K-theory k-spectrum (see §1.2(4) and [Hor05]) and, as in §1.3, let $GW(\mathsf{k})$ and $W(\mathsf{k})$ denote the Grothendieck-Witt and Witt groups of quadratic forms over k, respectively. In [Hor05], Hornbostel computes the following coefficient groups of KO:

$$\mathsf{KO}_0 = GW(\mathsf{k}),$$

$$\mathsf{KO}_{-n} = 0 \text{ if } n \equiv 1, 2, 3 \quad (4),$$

$$\mathsf{KO}_{-n} = W(\mathsf{k}) \text{ if } n \equiv 0 \quad (4)$$

for $n > 0$. (In fact, Hornbostel more generally identifies the negative KO-groups of a ring with Balmer-Witt groups, but these have the above behavior over fields.)

Recall from [Hor05] that U-theory arises as the fiber of the hyperbolization map $\mathsf{KGL} \to \mathsf{KO}$,

$$U \to \mathsf{KGL} \to \mathsf{KO}.$$

Recall also that there are "negative" variants of KO and U in which Hermitian forms are replaced by skew Hermitian forms.

In the following theorem, I identify an entire half-plane of coefficient groups of KO.

Theorem 6.1. *Over an arbitrary field* k*, the coefficient group* $\mathsf{KO}_{m+n\alpha}$ *of Hermitian* K*-theory for* $n \geq m$ *has the following form:*

$$\mathsf{KO}_{4n(1+\alpha)} = GW(\mathsf{k}),$$

$$\mathsf{KO}_{4n(1+\alpha)+r\alpha} = W(\mathsf{k}) \text{ for } r > 0,$$

$$\mathsf{KO}_{(4n+1)(1+\alpha)} = \mathrm{coker}(U_{1+\alpha} \to \mathsf{KGL}_{1+\alpha}),$$

$$\mathsf{KO}_{(4n+2)(1+\alpha)} = \mathbb{Z},$$

$$\mathsf{KO}_{(4n+3)(1+\alpha)} = \mathrm{coker}(U_{3+3\alpha} \to \mathsf{KGL}_{3+3\alpha}),$$

and all other KO*-coefficient groups in this dimensional range vanish.*

Proof. By the $4(1 + \alpha)$-periodicity of KO (see §1.2(4)), it suffices to determine the coefficients $\mathsf{KO}_{m+n\alpha}$ when $0 \leq n \leq 3$ and $m \geq n$. Consider the fibration sequence

$$U \to \mathsf{KGL} \xrightarrow{H} \mathsf{KO}$$

where H is the hyperbolization map and KGL is the motivic algebraic K-theory spectrum (see §1.2(2)). This fibration produces long exact sequences

$$\cdots \to \mathsf{KO}_{m+(n-1)\alpha} \to U_{m-1+(n-1)\alpha} \to \mathsf{KGL}_{m-1+(n-1)\alpha} \to \mathsf{KO}_{m-1+(n-1)\alpha} \to \cdots$$

for each $n \in \mathbb{Z}$. According to [Hor05], $U \simeq \Omega^{1+\alpha}\mathsf{KO}$, so the long exact sequence can be rewritten as

$$(6.1) \qquad \cdots \to \mathsf{KO}_{m+(n-1)\alpha} \to \mathsf{KO}_{m+n\alpha} \to \mathsf{KGL}_{m-1+(n-1)\alpha} \to \mathsf{KO}_{m-1+(n-1)\alpha} \to \cdots .$$

The connectivity of algebraic K-theory of fields along with $(1 + \alpha)$ Bott-periodicity implies that $\mathsf{KGL}_{m+n\alpha} = 0$ for $n > m$, so when $n > m$ there are isomorphisms $\mathsf{KO}_{m+1+n\alpha} \cong \mathsf{KO}_{m+1+(n+1)\alpha}$. This, Horbostel's vanishing results on KO_{-n}, and $4(1 + \alpha)$-periodicity of KO imply that $\mathsf{KO}_{m+n\alpha} = 0$ for $n > m$ unless $m \equiv 0$ (4). It now suffices to determine KO_* on the $(1 + \alpha)$-diagonal and in dimensions $n + (n + 1)\alpha$, $n \equiv 0$ (4). All such computations can be done quite easily with (6.1), and I only illustrate the computation of KO_α and $\mathsf{KO}_{1+\alpha}$ here.

To compute KO_α, let $m = 0$, $n = 1$ in (6.1). The result is the exact sequence

$$\mathsf{KGL}_0 = \mathbb{Z} \xrightarrow{H} GW(\mathsf{k}) = \mathsf{KO}_0 \to \mathsf{KO}_\alpha \to 0 = \mathsf{KGL}_{-1}.$$

By definition, $W(\mathsf{k}) = \operatorname{coker}(\mathbb{H} : \mathbb{Z} \to GW(\mathsf{k}))$ which matches the hyperbolization map in these dimensions, so $\mathsf{KO}_\alpha = W(\mathsf{k})$.

To compute $\mathsf{KO}_{1+\alpha}$, let $m = 1$, $n = 2$ in (6.1). This produces the exact sequence

$$U_{1+\alpha} \to \mathsf{KGL}_{1+\alpha} \to \mathsf{KO}_{1+\alpha} \to \mathsf{KO}_{1+2\alpha},$$

and $\mathsf{KO}_{1+2\alpha} = 0$ by the above comments. Now $\mathsf{KO}_{1+2\alpha} = \mathsf{KO}_{-3-2\alpha}$ by $4(1 + \alpha)$-periodicity. This implies that $\mathsf{KO}_{1+\alpha} = \operatorname{coker}(U_{1+\alpha} \to \mathsf{KGL}_{1+\alpha})$. $\qquad \square$

Remark 6.2. Over the p-adic field F, $GW = GW(F)$ and $W = W(F)$ are known by Theorem 1.44. Recall that the abstract isomorphism type differs depending on whether -1 is a square in F, i.e., whether the residue order q is congruent to 1 or 3 modulo 4. To be precise,

$$GW = \begin{cases} \mathbb{Z} \oplus (\mathbb{Z}/2)^3 & \text{if } q \equiv 1 \quad (4), \\ \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2 & \text{if } q \equiv 3 \quad (4) \end{cases}$$

and

$$W = \begin{cases} (\mathbb{Z}/2)^4 & \text{if } q \equiv 1 \quad (4), \\ \\ (\mathbb{Z}/4)^2 & \text{if } q \equiv 3 \quad (4). \end{cases}$$

Hermitian K-theory is the domain of a motivic analogue of the real J-homomorphism. Similarly, a motivic analogue of the complex J-homomorphism has algebraic K-theory as its domain. To construct these homomorphisms, I produce a commutative diagram

(6.2)
$$\begin{array}{ccc} GL & \xrightarrow{\mathcal{J}_{\mathsf{KGL}}} & F \\ {\scriptstyle \mathcal{H}} \downarrow & \nearrow {\scriptstyle \mathcal{J}_{\mathsf{KO}}} & \\ O & & \end{array}$$

I now describe the terms in this diagram: GL is the colimit of the group schemes GL_r under the maps $GL_r \to GL_{r+1}$ sending $A \mapsto A \oplus 1$. Let O_{2r} denote the group scheme of $2r \times 2r$ matrices preserving the form $\sum_{i=1}^r x_i y_i$. Then O is the colimit of the O_{2r} under the maps $O_{2r} \to O_{2r+2}$ sending $A \mapsto A \oplus 1 \oplus 1$. If F_r is the space of self-equivalences on a fibrant replacement of $\mathbb{A}^r \smallsetminus 0$, then $F = \operatorname{colim} F_r$ under the obvious maps induced by the natural inclusions $\mathbb{A}^r \smallsetminus 0 \to \mathbb{A}^{r+1} \smallsetminus 0$.

The map $\mathcal{J}_{\mathsf{KGL}} : GL \to F$ is induced by maps $GL_r \to F_r$ induced by the natural action of GL_r on $\mathbb{A}^r \smallsetminus 0$, and $\mathcal{H} : GL \to O$ is induced by the hyperbolization maps $GL_r \to O_{2r}$ sending $A \mapsto A \oplus (A^t)^{-1}$. The map $\mathcal{J}_{\mathsf{KO}}$ is slightly more delicate. First note that O_{2r} acts naturally on V_r, the "unit sphere" variety in $\mathbb{A}^{2r}$ defined by the equation $\sum_{i=1}^r x_i y_i = 1$. The projection $V_r \to \mathbb{A}^r \smallsetminus 0$ sending $(x_1, \ldots, y_1, \ldots) \mapsto (x_1, \ldots)$ is an affine bundle and hence an equivalence. Moreover, these maps are compatible with the maps $F_r \to F_{r+1}$, so we get a compatible collection of map $O_{2r} \to F_r$, and these define $\mathcal{J}_{\mathsf{KO}}$.

Definition 6.3. Let a denote the Nisnevich sheafification functor and let $(-)_f$ denote

fibrant replacement. Applying the functor $\pi_* a(-)_f$ to (6.2) produces a commutative diagram

(6.3)
$$\begin{array}{ccc} \mathsf{KGL}_* & \xrightarrow{J_{\mathsf{KGL}}} & \pi_{*-1}\mathbb{1} \\ {\scriptstyle H}\downarrow & \nearrow{\scriptstyle J_{\mathsf{KO}}} & \\ \mathsf{KO}_* & & \end{array}$$

where $*$ is an index of the form $m+n\alpha$ with $m > 0$ and H is the usual hyperbolization map from algebraic K-theory to Hermitian K-theory.

Remark 6.4. The effect of $\pi_* a(-)_f$ on GL and O is described in [Hor05, Theorem 3.1].

Remark 6.5. The above construction is compatible with 2-completion and hence also produces 2-complete versions of the algebraic and Hermitian J-homomorphisms.

I now describe a small piece of the image of J_{KO} in the 2-complete motivic stable stems. My methods are dependent on computations over $\mathbb{C}$ joint with Po Hu and Igor Kriz; see [HKOa, Theorem 3].

I begin by defining the algebraic e-invariant, which is a homomoprhism

$$e : \pi_{n-1+n\alpha}\mathbb{1}\widehat{_2} \to E_2^{1,n(1+\alpha)}$$

where the right-hand side is the degree $(1, n(1+\alpha))$ component of the E_2-term of the motivic ANSS. (The construction will work over an arbitrary field k.) The following lemma makes the construction of e possible.

Convention 6.6. Throughout the rest of this thesis, let π_* denote $\pi_*\mathbb{1}\widehat{_2}$ and, abusing notation, let BPGL denote $\mathsf{BPGL}\widehat{_2}$. All sphere spectra are also implicitly 2-completed.

Any $f \in \pi_{n-1+n\alpha}$ can be represented by a k-spectrum map $S^{-1} \xrightarrow{f} S^{-n(1+\alpha)}$. Consider the cofiber sequence

$$S^{-1} \xrightarrow{f} S^{-n(1+\alpha)} \to Cf \to S^0 \to S^{-n+1-n\alpha} \to \cdots .$$

Applying the functor BPGL_0 produces a $\mathsf{BPGL}_*\mathsf{BPGL}$-comodule exact sequence

$$\mathsf{BPGL}_1 \xrightarrow{\mathsf{BPGL}_* f} \mathsf{BPGL}_{n(1+\alpha)} \to \mathsf{BPGL}_0 Cf \to \mathsf{BPGL}_0 \to \mathsf{BPGL}_{n-1+n\alpha}$$

(see Definition 1.11). Note that $\mathsf{BPGL}_{n-1+n\alpha} = 0$ because $\mathsf{BPGL}_{m+\ell\alpha} = 0$ whenever $m < \ell$, and, as in topology, $\mathsf{BPGL}_* f = 0$. Hence the above exact sequence is actually short exact,

$$(6.4) \qquad\qquad 0 \to \mathsf{BPGL}_{n(1+\alpha)} \to \mathsf{BPGL}_0 Cf \to \mathsf{BPGL}_0 \to 0.$$

Definition 6.7. The *algebraic e-invariant* of $f \in \pi_{n-1+n\alpha}$ (or just *e-invariant* of f if context does not allow confusion with the e-invariant from topology) is the extension $e(f) \in \mathrm{Ext}^{1,n(1+\alpha)}_{\mathsf{BPGL}_*\mathsf{BPGL}}(\mathsf{BPGL}_*, \mathsf{BPGL}_*)$ represented by (6.4). Of course, we may also consider $e(f) \in E_2^{1,n(1+\alpha)}$, the $(1, n(1+\alpha))$-degree component of the motivic ANSS.

Proposition 6.8. *The algebraic e-invariant is a homomoprhism*

$$\pi_{n-1+n\alpha} \to \mathrm{Ext}^{1,n(1+\alpha)}_{\mathsf{BPGL}_*\mathsf{BPGL}}(\mathsf{BPGL}_*, \mathsf{BPGL}_*).$$

Proof. This is purely a matter of translation from topology to the motivic setting; see [Ada66, Proposition 3.3]. The key idea is to show that for $f, g \in \pi_{n-1+n\alpha}$, $e(f \vee g)$ is the external sum $e(f) \oplus e(g)$ and then use basic properties of Ext-algebras and generalized cohomology theories. $\qquad\square$

The e-invariant, in its topological form, was Adams's main tool in proving his theorems on the image of the topological J-homomorphism (see [Ada66]), although his e-invariant was measured in (topological complex or real) K-theory instead of cobordism. The idea is to detect the image of the J-homomorphism by computing eJ, thus producing "lower bounds" on the image of J. A topological e-invariant based on BP is discussed in [Rav86, §5.3].

The following fact is known about the image of J_{KO} over $\mathbb{C}$. (I will continue my habit of dropping notation for 2-completion.)

Theorem 6.9 ([HKOa, Theorem 3]). *Over an algebraically closed field of characteristic 0, the image of the 2-complete J-homomorphism $J_{\mathsf{KO}} : \mathsf{KO}_{n(1+\alpha)} \to \pi_{n-1+n\alpha}$ is isomorphic to the 2-component of the topological real J-homomorphism $J : \mathsf{KO}_{2n}^{\mathrm{top}} \to \pi_{2n-1}$.* $\square$

Corollary 6.10. *Over an algebraically closed field of characteristic 0, the image of $eJ_{\mathsf{KGL}} : \mathsf{KGL}_{n(1+\alpha)} \to E_2^{1,n(1+\alpha)}$ matches the image of Ravenel's complex e-invariant [Rav86, §5.3] in $^{\mathrm{top}}E_2^{1,2n}$.* $\square$

Fix an embedding $F \hookrightarrow \mathbb{C}$ for $\mathbb{C}$ an algebraically closed field of characteristic 0. To analyze the the e-invariant and J-homomorphism over the p-adic field F, I consider how eJ_{KGL} transforms under base change with respect to $F \hookrightarrow \mathbb{C}$.

Lemma 6.11. *In degree $(1, n(1 + \alpha))$, the base change $F \hookrightarrow \mathbb{C}$ induces an isomoprhism of motivic ANSS E_2-terms $^{F}E_2^{1,n(1+\alpha)} = {}^{\mathbb{C}}E_2^{1,n(1+\alpha)}$. Moreover, both terms are isomorphic to $^{\mathrm{top}}E_2^{1,2n}$ (in topological grading).*

Proof. Recall that, in motivic grading, all elements of $^{\mathrm{top}}E_2$ are in grading $(s, n(1+\alpha))$ for nonnegative integers s, n. It is clear that none of the $\otimes\Gamma$ terms in Theorem 4.11 contribute to $^{F}E_2^{1,n(1+\alpha)}$ except for tensoring with elements of Γ_0. If there is a derived term $\sigma_\gamma x$ of degree $(1, n(1+\alpha))$, then γ would have to have motivic degree 0, in which case $\sigma_\gamma x = 0$. This proves that $^{F}E_2^{1,n(1+\alpha)} \cong {}^{\mathrm{top}}E_2^{1,2n}$ (in topological grading), and the base change isomorphism is now obvious in light of [HKOa, (36)]. $\square$

Theorem 6.12. *The image of eJ_{KGL} over F in $^{F}E_2^{1,n(1+\alpha)}$ matches the image of eJ_{KGL} over $\mathbb{C}$ in $^{\mathbb{C}}E_2^{1,n(1+\alpha)}$.*

Proof. Consider the diagram

$$(6.5)$$

$$
\begin{array}{ccc}
\mathsf{KGL}^F_{n(1+\alpha)} & \longrightarrow & \mathsf{KGL}^{\mathbb{C}}_{n(1+\alpha)} \\
{\scriptstyle J_{\mathsf{KGL}}}\big\downarrow & & \big\downarrow{\scriptstyle J_{\mathsf{KGL}}} \\
\pi^F_{n-1+n\alpha} & \longrightarrow & \pi^{\mathbb{C}}_{n-1+n\alpha} \\
{\scriptstyle e}\big\downarrow & & \big\downarrow{\scriptstyle e} \\
{}^F E_2^{1,n(1+\alpha)} & \longrightarrow & {}^{\mathbb{C}} E_2^{1,n(1+\alpha)}
\end{array}
$$

induced by base change with respect to $F \hookrightarrow \mathbb{C}$. The top map is an isomorphism since we are working over fields, and the bottom map is an isomorphism by Lemma 6.11. $\qquad\square$

Corollary 6.13. *The image of $e J_{\mathsf{KGL}} : \mathsf{KGL}_{n(1+\alpha)} \to {}^F E_2^{1,n(1+\alpha)}$ over F is isomorphic to the image of Ravenel's complex e-invariant [Rav86, §5.3] in ${}^{\mathrm{top}} E_2^{1,2n}$. In particular, $\mathrm{im}\, e J_{\mathsf{KGL}}$ has index 1 in ${}^F E_2^{1,n(1+\alpha)}$ for $n = 1, 2$ and index 2 for $n \geq 3$.*

Proof. An immediate consequence of Theorem 6.12, Corollary 6.10, and [Rav86, §5.3]. $\qquad\square$

The Hermitian J-homomorphism will be important to my analysis of the motivic ANSS over F when $n \equiv 0$ or 2 (4). For these cases, I must understand the precise nature of the 2-complete hyperbolization map $\mathsf{KGL}_{n(1+\alpha)} \to \mathsf{KO}_{n(1+\alpha)}$. First consider the case $n \equiv 0$ (4) in which case we are considering the hyperbolization map $\mathbb{H} : \mathbb{Z} \to GW$ (see Theorem 6.1). The completion short exact sequence Theorem 1.32 and Theorem 1.44 reveal that H is of the form

$$\mathbb{Z} \longrightarrow \mathbb{Z}_2 \oplus (\mathbb{Z}/2)^3$$

$$1 \longmapsto (2, 0, 0, 0)$$

when $q \equiv 1$ (4) and of the form

$$\mathbb{Z} \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2$$

$$1 \longmapsto (2, 0, 1)$$

when $q \equiv 3$ (4). If $q \equiv 1$ (4), it immediately follows that for $n \equiv 0$ (4),

$$(6.6) \qquad \operatorname{im} eJ_{\mathsf{KO}} = {}^F E_2^{1,n(1+\alpha)} = {}^{\mathrm{top}} E_2^{1,2n},$$

the same as in topology for the real J-homomorphism. If $q \equiv 3$ (4), then the image of the generator of $\mathsf{KGL}_{n(1+\alpha)}$ under H is not divisible by 2, so we must use a bit of care. For $q \equiv 3$ (4), it is the case that H sends twice the generator to $(4, 0, 0) \in GW(F)$. It is now important that ${}^F E_2^{1,n(1+\alpha)} = {}^{\mathrm{top}} E_2^{1,2n} = \mathbb{Z}/2^{\nu(4n)}$ by [Rav86, Theorems 5.2.6 and 5.3.6]. Since $n \equiv 0$ (4), the order of the generator of ${}^F E_2^{1,n(1+\alpha)}$ has order at least 16. It follows that the image of $(1, 0, 0) \in \mathsf{KO}_{n(1+\alpha)} = GW$ under eJ_{KO} is the generator of ${}^F E_2^{1,n(1+\alpha)}$, so (6.6) holds for $n \equiv 0$ (4) and $q \equiv 3$ (4) as well.

Now consider the case $n \equiv 2$ (4). In this case, since $\_W = 0$ for fields, the hyperbolization map $\mathsf{KGL}_{n(1+\alpha)} \to \mathsf{KO}_{n(1+\alpha)}$ is an isomorphism and $\operatorname{im} eJ_{\mathsf{KO}}$ has index 2 in ${}^F E_2^{1,n(1+\alpha)}$ whenever $n > 2$. (In the case $n = 2$, eJ_{KGL} is surjective, so eJ_{KO} is as well.) This proves the following theorem.

Theorem 6.14. *If $n \equiv 0$ (4), then $\operatorname{im} eJ_{\mathsf{KO}} = {}^F E_2^{1,n(1+\alpha)}$. If $n \equiv 2$ (4), then $\operatorname{im} eJ_{\mathsf{KO}}$ has index 2 in ${}^F E_2^{1,n(1+\alpha)}$ unless $n = 2$, in which case eJ_{KO} is surjective.* $\square$

Remark 6.15. The above theorem produces lower bounds on the image of $J_{\mathsf{KO}} : \mathsf{KO}_{n(1+\alpha)} \to \pi_{n-1+n\alpha}$ which are consistent with the 2-component of the image of the real topological J-homomorphism $J : \mathsf{KO}_{2n}^{\mathrm{top}} \to \pi_{2n-1}$. There is some chance that the image of J_{KO} over a p-adic field is larger than its topological counterpart, but it would be impossible to capture such data in what I will call the "alpha family" in the motivic ANSS; see the next chapter for a definition and analysis of this family.

CHAPTER VII

Computations in the motivic Adams-Novikov spectral sequence

In this chapter, I exploit my computations of eJ_{KO} (Theorem 6.14) to determine some differentials in the motivic ANSS over a p-adic field F. I isolate a collection of elements in the E_2-term of this spectral sequence which is closely related to the image of J_{KO} called the alpha family; it is a straightforward generalization of the topological alpha family dependent on Theorem 4.11. In addition to Theorem 6.14, I will use the motivic ANSS for the mod 2 Moore spectrum, $\mathbb{1}/2$, to determine differentials. Pictures are indispensable in understanding these differentials, and I have taken some care in preparing them.

Recall the alpha family inside the E_2-term of the topological ANSS. It consists of elements of the form $\alpha_k\alpha_1^a$, $k \geq 1$ and $a \geq 0$, where $|\alpha_k| = (1, 2k)$. My notation, which matches that of [HKOa], differs from Ravenel's [Rav86]. By α_k I mean the generator of $E_2^{1,2k}$; for $k \equiv 0$ (4) this is Ravenel's $\bar{\alpha}_k$, and for $k \equiv 2$ (4), this is his $\alpha_{k/3}$. Ravnel's notation is intended to remind us that, for $k \equiv 0$ (4), $E_2^{1,2k}$ has large order depending on the denominators of certain Bernoulli numbers, while for $k \equiv 0$ (4), $E_2^{1,2k}$ has order $8 = 2^3$; see [Rav86, Definition 5.1.4].

Under my notation, α_k generates $\mathrm{Ext}^{1,2k}_{\mathsf{BP}_*\mathsf{BP}}(\mathsf{BP}_*, \mathsf{BP}_*)$ and has order

$$
\mathrm{ord}\,\alpha_k = \begin{cases} 2^{\nu(4k)} & \text{if } k \equiv 0 \quad (4), \\[2ex] 2 & \text{if } k \equiv 1 \quad (4), \\[2ex] 8 & \text{if } k \equiv 2 \quad (4), \\[2ex] 2 & \text{if } k \equiv 3 \quad (4) \end{cases}
$$

where ν is the 2-adic valuation on rational integers.

By Theorem 4.11, the topological alpha family appears in the $^{\mathrm{top}}E_2 \otimes \Gamma_{0+0\alpha}$ factor of E_2, where $\alpha_k \otimes 1$ has degree $(1, k(1+\alpha))$.

Definition 7.1. In the motivic setting, over a p-adic field F, the *alpha family* is the collection of linear combinations of elements in $^F E_2$ of the form

$$
\alpha_k \otimes \gamma \qquad \text{or} \qquad \sigma_\gamma \alpha_k
$$

where $k \geq 1$ and $\gamma \in \Gamma$. As shorthand, I will denote $\alpha_k \otimes \gamma$ as $\gamma\alpha_k$.

Remark 7.2. Note that by standard Massey product juggling relations (see Proposition 5.2), the motivic alpha family includes products of elements of the above two forms.

Figure 7.1 depicts all elements in the alpha family generated by a particular α_k. I will call such a collection of elements the "α_k family". The top left entry in the picture is α_k. Note that the picture only represents the total motivic grading of the elements and suppresses homological grading, as in Figure 4.1. Here I have shaded the derived terms in gray to indicate that they have homological degree one lower than their non-derived counterparts. The vertical arrows indicate α_1-multiplication, and it is important to keep in mind that these towers extend out of the page, increasing homological degree, in addition to increasing total motivic degree by α.
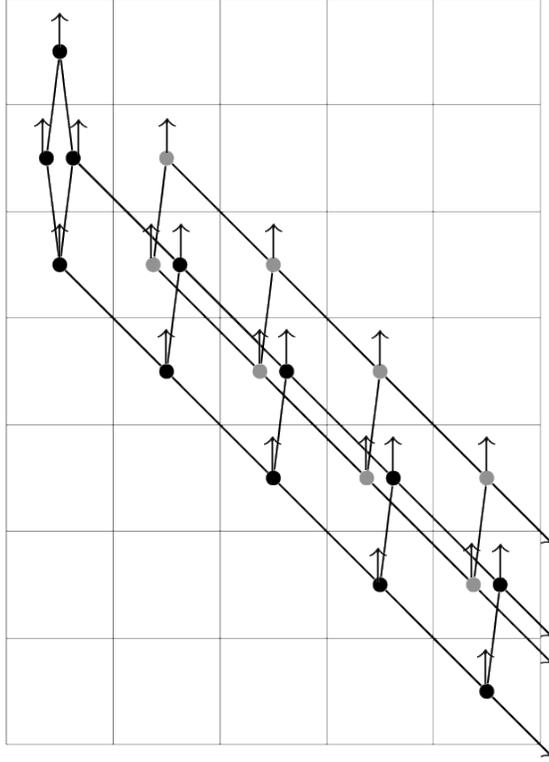
Figure 7.1: The α_k family over F

It is very enjoyable to compare Figure 7.1 to Figure 1.1. While τ-multiplication does not exist in the motivic ANSS, the comparison makes the nature in which derived terms substitute for τ-multiplication very clear.

Remark 7.3. The alpha family member α_1 is represented by Morel's $\eta \in \pi_\alpha$ (see Theorem 1.28); this class is represented by the Hopf map $S^{1+2\alpha} \simeq \mathbb{A}^2 \smallsetminus 0 \to \mathbb{P}^1$ unstably.

Recall that in topology [Rav86, §5.3] there are ANSS differentials

(7.1)
$$d_3\alpha_{4k+2} = \alpha_1^3\alpha_{4k},$$
$$d_3\alpha_{4k+3} = \alpha_1^3\alpha_{4k+1}.$$

Over $\mathbb{C}$ [HKOa, Theorem 1], a difference in twist and the topological realization

functor forces motivic ANSS differentials

(7.2)
$$d_3\alpha_{4k+2} = \tau\alpha_1^3\alpha_{4k},$$

$$d_3\alpha_{4k+3} = \tau\alpha_1^3\alpha_{4k+1}.$$

(Using this thesis's notation, $^{\mathbb{C}}E_2 = {}^{\text{top}}E_2[\tau]$ where $|\tau| = (0, 1 - \alpha)$.

The following theorem specifies the motivic ANSS differentials over a p-adic field on the alpha family. This spectral sequence exhibits completely novel behavior in that d_2-differentials are necessary to capture the analogue of the τ-torsion created over $\mathbb{C}$ in (7.2). (Recall that in topology the E_2-term of the ANSS exhibits a "checkerboard sparsity" by which all even differentials are trivial. The same sparsity exists over $\mathbb{C}$, so this is genuinely new behavior observed over p-adic fields.)

Theorem 7.4. *The motivic ANSS over a p-adic field F exhibits the following behavior:*

(i) If $k \equiv 0$ or 1 (4), then $\alpha_k\alpha_1^a$ is a nonzero permanent cycle for all $a \geq 0$. If $k \equiv 0$ (4), then these elements represent elements of $\operatorname{im} J_{\mathsf{KO}}$, while for $k \equiv 1$ (4), they are not in $\operatorname{im} J_{\mathsf{KO}}$.

(ii) If $k \equiv 2$ or 3 (4), then

(7.3)
$$d_2\alpha_k = \sigma_u\alpha_{k-2}\alpha_1^3.$$

(iii) If $k \equiv 2$ (4), then $2\alpha_k$ is a permanent cycle represented by the image of J_{KO}.

The differntials of (ii) are depicted in Figure VII. In that picture, k is taken to be a postive integer congruent to 2 or 3 mod 4, and I only depict the α_k and α_{k-2} families. If $k \equiv 2$ (4), then, in the full spectral sequence diagram, the same picture is repeated with a shift of $1 + \alpha$. Including all this information makes the picture quite unreadable, so Figure VII only includes this limited amount of information.
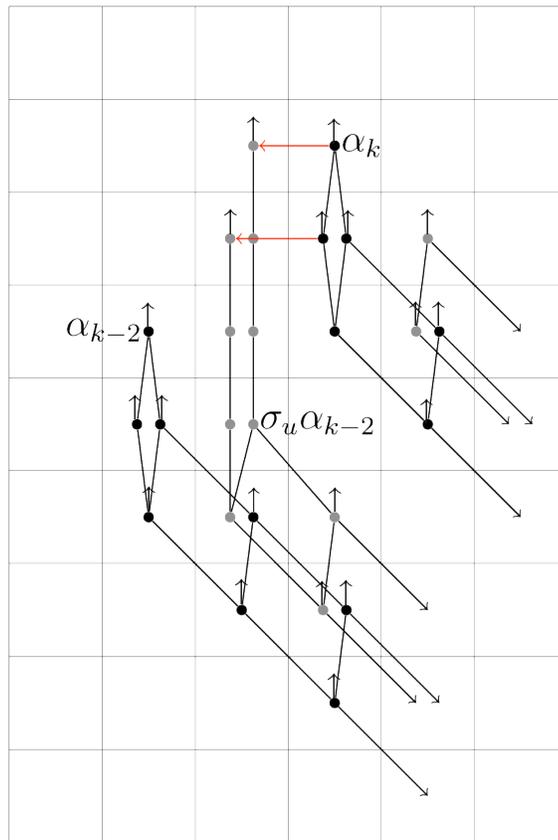
Figure 7.2: The alpha family in the E_2-term of the motivic ANSS over F

I prove Theorem 7.4 in a series of propositions and lemmas modelled on the techniques in [Rav86, Section 5.3]. I begin by proving that the d_2-differentials in (ii) are correct. While these differentials admit a unified description, they require separate arguments in the cases $k \equiv 2$ and $k \equiv 3$ (4).

Proposition 7.5. *If $k \equiv 2$ (4), then (7.3) holds.*

Proof. Note that by Theorem 6.1, $(\widehat{\mathsf{KO}_2})_{1+2\alpha} = 0$. Recall (Definition 5.5) that ϑ_η is the Toda bracket $\langle u, 2^{k-1}(2 + \rho\eta), \eta \rangle$ represented by the Massey product $\sigma_u \alpha_1 = \langle u, 2^k, \alpha_1 \rangle$. It follows that the image of $\vartheta_\eta \eta^2$ in $(\widehat{\mathsf{KO}_2})_{1+2\alpha}$ is 0. By Theorem 6.14, α_{k-2} is in the image of J_{KO} for $k \equiv 2$ (4). Moreover, $\sigma_u \alpha_{k-2} \alpha_1^3 = \sigma_u \alpha_1 \alpha_1^2 \alpha_{k-2}$ is represented by $\vartheta_\eta \eta^2 \alpha_{k-2}$, and $\vartheta_\eta \eta^2$ commutes with J_{KO}, so we can conclude that $\sigma_u \alpha_{k-2} \alpha_1^3$ dies in the ANSS. By a dimension count, it is clear that (7.3) is the only differential capable of killing $\sigma_u \alpha_{k-2} \alpha_1^3$. $\square$

Remark 7.6. The same proof does not apply to the $k \equiv 3$ (4) case because then α_{k-2} is not in the image of J_{KO}: if it were, then base change to $\mathbb{C}$ would imply that α_{k-2} was in the image of J_{KO} over $\mathbb{C}$, contradicting [HKOa, Theorem 3].

Proposition 7.7. *If $k \equiv 3$ (4), then (7.3) holds.*

The following lemma about the E_2-term of the motivic ANSS for $\mathbb{1}/2$ is necessary for the proof of Proposition 7.7. It will also be important to my analysis of the E_3-page of the motivic ANSS over F.

Lemma 7.8. *The E_2-term of the motivic ANSS for $\mathbb{1}/2$ takes the form*

$$^F \mathrm{Ext}_{\mathsf{BPGL}_* \mathsf{BPGL}}(\mathsf{BPGL}_*, \mathsf{BPGL}_*(\mathbb{1}/2)) =$$

$$= {}^{\mathbb{C}} \mathrm{Ext}_{\mathsf{BPGL}_* \mathsf{BPGL}}(\mathsf{BPGL}_*, \mathsf{BPGL}_*(\mathbb{1}/2)) \otimes_{\mathbb{Z}/2[\tau]} H_*^F$$

$$= {}^{\mathrm{top}} \mathrm{Ext}_{\mathsf{BP}_* \mathsf{BP}}(\mathsf{BP}_*, \mathsf{BP}_*(\mathbb{1}/2)) \otimes_{\mathbb{Z}/2} H_*^F.$$

Moreover, $^{F}\mathrm{Ext}_{\mathsf{BPGL}_*\mathsf{BPGL}}(\mathsf{BPGL}_*, \mathsf{BPGL}_*(\mathbb{1}/2))$ *contains a summand of the form*

$$(\mathbb{Z}/2[v_1, h_0] \otimes \{1, y\}) \otimes_{\mathbb{Z}/2} H_*^F$$

where $|v_1| = (0, 1 + \alpha)$, $|h_0| = (1, 1 + \alpha)$, *and* $|y| = (1, 4(1 + \alpha))$.

Proof. The final part of the lemma follows from [Rav86, Theorem 5.3.13(a)] once the first part is established. To this end, note that $v_0 = 0$ in the computations of Theorem 4.9 after passing to $\mathsf{BPGL}_*(\mathbb{1}/2)$. It follows that the ρ-power filtration spectral sequence collapses, and we get the computation

$$\mathsf{BPGL}_*(\mathbb{1}/2) = {}^{\mathbb{C}}\mathsf{BPGL}_*(\mathbb{1}/2) \otimes_{\mathbb{Z}/2[\tau]} H_*^F$$

over F. The identification of $^{\mathbb{C}}\mathsf{BPGL}_*(\mathbb{1}/2)$ with $\mathsf{BP}_*(\mathbb{1}/2)[\tau]$ follows from the analysis over algebraically closed fields in [HKOa]. Change of base completes the computation of Ext algebras. $\square$

I now identify a small class of differentials appearing in the motivic ANSS for $\mathbb{1}/2$ over F essential to computations in the motivic ANSS for $\mathbb{1}$.

Lemma 7.9. *In the motivic ANSS for* $\mathbb{1}/2$ *over* F *for* $k \equiv 2, 3$ (4) *there are differentials*

(7.4) $$d_3 v_1^k = \tau v_1^{k-2} h_0^3.$$

Proof. Over $\mathbb{C}$, topological realization and dimensional analysis implies that (7.4) in the motivic ANSS for $\mathbb{1}/2$. By the change of base field map of spectral sequences, (7.4) also holds over F. $\square$

Proof of Proposition 7.7. By Lemma 7.9, $\tau v_1^{k-2} h_0^3$ dies in the motivic ANSS for $\mathbb{1}/2$. It follows that $\tau \alpha_{k-2} \eta^3 = 0$ in $\pi_* \mathbb{1}/2$. Now $\tau \eta$ is the reduction of ϑ_η under the canonical map $\mathbb{1} \to \mathbb{1}/2$, so we also know that $\alpha_{k-2} \vartheta_\eta \eta^2$ is divisible by 2 in π_*. This

element is represented by $\sigma_u \alpha_{k-2} \alpha_1^3$, which has order 2 in E_2. If it is 0 in π_*, then (7.3) is all that can kill it in the motivic ANSS. We need to concern ourselves with elements of lower homological degree and the same total motivic degree and show that all such elements x do not satisfy $2x = \alpha_{k-2} \vartheta_\eta \eta^2$ in π_*, i.e. that they do not support extensions. Once this is proven, the proposition follows.

The elements we must concern ourselves with are in $E_2^{2,k(1+\alpha)}$, $E_2^{1,k(1+\alpha)-1}$, and $E_2^{0,k(1+\alpha)-2}$. The latter two groups are easily seen to be trivial by Theorem 4.11 and dimensional accounting. The first group, $E_2^{2,k(1+\alpha)}$ can be nontrivial, but (again by dimensional accounting) all contributions come from $^{\text{top}}E_2^{2,k(1+\alpha)}$. This group is described by Ravenel in [Rav86, Corollary 5.4.5]. It consists of a summand generated by $\alpha_{k-1}\alpha_1$ and, possibly, a summand generated by a member of the "beta family". Since $k \equiv 3 \ (4)$, Proposition 7.5 implies that $\alpha_{k-1}\alpha_1$ supports a differential, and hence does not survive to π_*. On the other hand, since k is odd, [Rav86, Corollary 5.4.5] implies that any beta element in $^{\text{top}}E_2^{2,k(1+\alpha)}$ is in the image of δ, the connecting map for the reduction $\mathbb{1} \to \mathbb{1}/2$. (Note that this is equivalent to $\phi(i,j) = 0$ in Ravenel's notation.) It follows that these particular βs, if permanent, must have order 2 in π_*, and hence we cannot have $2\beta = \sigma_u \alpha_{k-2} \alpha_1^3$ in E_∞. As explained above, this implies that $\sigma_u \alpha_{k-2} \alpha_1^3 = 0$ is the target of α_k under d_2 in the motivic ANSS.

$\square$

Propositions 7.5 and 7.7 establish (ii) in Theorem 7.4.

Proposition 7.10. *Part (iii) of Theorem 7.4 holds.*

Proof. Fix $k \equiv 2 \ (4)$. By Theorem 6.14, $2\alpha_k$ is in the image of $eJ_{\mathsf{KO}} = eJ_{\mathsf{KGL}}$, so it must be permanent. $\square$

Proof of Theorem 7.4. By Propositions 7.5, 7.7, and 7.10, it now suffices to prove

part (i) of Theorem 7.4. If $k \equiv 0$ (4), then α_k is permanent since it is the image of a generator of $\mathsf{KO}_{k(1+\alpha)}$ under eJ_{KO}. Since α_1 represents η it is a permanent cycle. It follows that v_1 is permanent in the motivic ANSS for $\mathbb{1}/2$ since $\delta v_1 = \alpha_1$. By v_1^4-periodicity, the element v_1^k in the motivic ANSS for $\mathbb{1}/2$ is also permanent, and it follows that $\delta v_1^k = \alpha_k$ is permanent.

I can now use comparison with the motivic ANSS over $\mathbb{C}$ to prove the permanence of $\alpha_k \alpha_1^a$, $k \equiv 0$ or 1 (4). By the Leibniz rule and the above paragraph, $d_r \alpha_k \alpha_1^a = 0$ for all $a \geq 0$. Base change to $\mathbb{C}$ produces a map of stable motivic homotopy groups $\pi_*^F \mathbb{1} \widehat{2} \to \pi_*^{\mathbb{C}} \mathbb{1} \widehat{2}$ sending α_k to α_k and α_1 to α_1. By [HKOa, Theorem 1], $\alpha_k \alpha_1^a$ is nonzero over $\mathbb{C}$, so the same must hold over F. $\qquad\square$

To complete my description of the alpha family in the motivic ANSS over F I must analyze the alpha family in the E_3-page.

Recall that γ_i generates $\Gamma_{i(1-\alpha)-\alpha}$ for $i \geq 1$, while, by convention, $\gamma_0 = u \in \Gamma_{-\alpha}$, where π and u generate the respective summands of $\Gamma_{-\alpha} = \mathbb{Z}_2 \oplus \mathbb{Z}/2^k$, $k = \nu(q-1)$. $\Gamma_{i(1-\alpha)-2\alpha}$ is generated by $\pi\gamma_i$.

Theorem 7.11. *In the motivic ANSS over F for $k \equiv 2$ or 3 (4),*

(7.5)
$$d_3 \gamma_i \alpha_k = \gamma_{i+1} \alpha_{k-2} \alpha_1^3,$$
$$d_3 \sigma_{\gamma_i} \alpha_k = \sigma_{\gamma_{i+1}} \alpha_{k-2} \alpha_1^3.$$

I will make some comments on this theorem before proceeding to give its proof.

Remark 7.12. By the structure of Γ, there are also differentials

$$d_3 \pi \gamma_i \alpha_k = \pi \gamma_{i+1} \alpha_{k-2} \alpha_1^3,$$

$$d_3 \sigma_{\pi\gamma_i} \alpha_k = \sigma_{\pi\gamma_{i+1}} \alpha_{k-2} \alpha_1^3$$

in the motivic ANSS over F, $k \equiv 2, 3$ (4).

Remark 7.13. The motivation for this theorem is straightforward: the differentials $d_2\alpha_k = \sigma_u\alpha_{k-2}\alpha_1^3$ for $k \equiv 2, 3$ (4) of Theorem 7.4(ii) are the p-adic "shadows" of the differentials $d_3\alpha_k = \tau\alpha_{k-2}\alpha_1^3$ in the motivic ANSS over $\mathbb{C}$ (see Remark 5.6). If τ lifted to $\pi_{1-\alpha}^F$ — and it does not: τ does not even lift to $^FE_2$ — then we would expect differentials $d_3\alpha_k = \tau\alpha_{k-2}\alpha_1^3$ as over $\mathbb{C}$, and hence differentials $d_3 u\tau^i\alpha_k = u\tau^{i+1}\alpha_{k-2}\alpha_1^3$. In Γ, γ_i is the shadow of $u\tau^i$ (see the proof of Theorem 4.7), so we should expect $d_3\gamma_i\alpha_k = \gamma_{i+1}\alpha_{k-2}$. (Note carefully that the shadow of $\tau\alpha_{k-2}\alpha_1^3$ lowers homological degree by 1, and this is why we see a class of d_2-differentials in Theorem 7.4(ii) rather than the d_3s that appear here.) The class of d_3s on derived terms follows the same pattern.

Figure 7.3 depicts the behavior of the E_3-page of the motivic ANSS over F in light of Theorem 7.11. Legibility becomes a major issue when trying to depict all the differentials at once. I have again only represented classes in the α_k and α_{k-2} family, $k \equiv 2$ or 3 (4), although I have also found it necessary to separate the non-derived and derived terms in my diagram of the E_3-page of the spectral sequence. The fearless reader may place these pictures one on top of the other in order to see the full behavior in the α_k and α_{k-2} families on E_3; the utterly brazen reader may then duplicate that picture and shift it by $1 + \alpha$ (altering labels as necessary) in order to see the entire alpha family in the motivic ANSS, or at least be impressed by a great number of intersecting and overlapping lines.

Proof of Theorem 7.11. By Lemma 7.9 says that there are differentials

$$d_3\tau^i uv_1^k = \tau^{i+1}uv_1^{k-2}h_0^3$$

in the motivic ANSS for $\mathbb{1}/2$ for $k \equiv 2, 3$ (4). By construction, the connecting map $\delta : E_2^s(\mathbb{1}/2) \to E_2^{s+1}(\mathbb{1})$ takes $\tau^i uv_1^{k-2}h_0^3$ to $\gamma_i\alpha_{k-2}\alpha_1^3$. Since δ commutes with
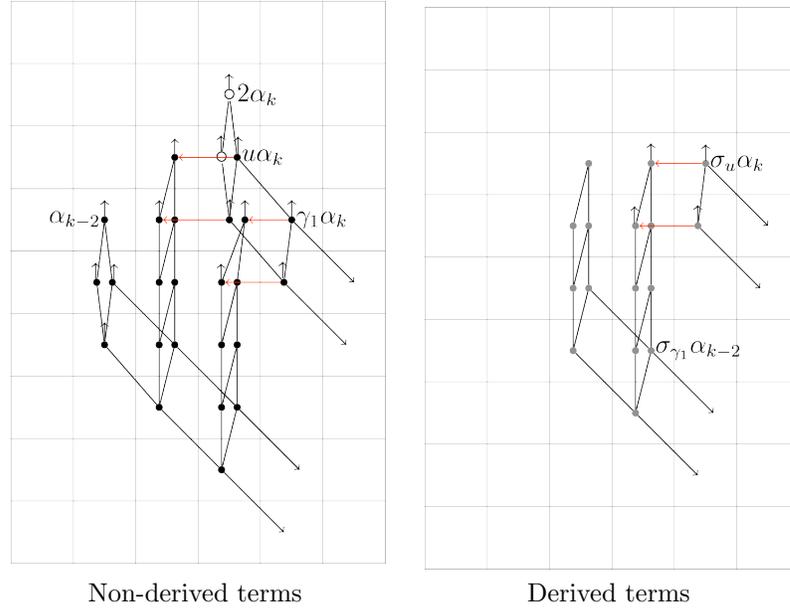
Figure 7.3: The alpha family in the E_3-term of the motivic ANSS over F

differentials, this implies that

$$d_3\gamma_i\alpha_k = \gamma_{i+1}\alpha_{k-2}\alpha_1^3.$$

Since the cobar representatives of the $\sigma_{\gamma_i}\alpha_k$ take the form $2^r\alpha_k \otimes \gamma$ (see Proposition 5.1), it it clear that we also have

$$d_3\sigma_{\gamma_i}\alpha_k = \sigma_{\gamma_{i+1}}\alpha_{k-2}\alpha_1^3.$$

$\square$

As Figure 7.4 depicts, Theorems 7.4 and 7.11 determine the alpha family's behavior in the motivic ANSS over F. In particular, when $k \equiv 3 \ (4)$ (in which case α_k has order 2), no remnant of the α_k family remains. When $k \equiv 2 \ (4)$, α_k has order 8, and $2\alpha_k \otimes \Gamma$ is a large, nontrivial algebra surviving to π_*; still, all α_1 multiples and derived terms support differentials and do not make it to E_∞. For $k \equiv 0, 1 \ (4)$, α_k still supports a nontrivial infinite α_1-tower in π_*, but all of the $\Gamma_{>0+n\alpha}$-multiples and derived terms of α_k only have nontrivial α_1-towers out to α_1^2.
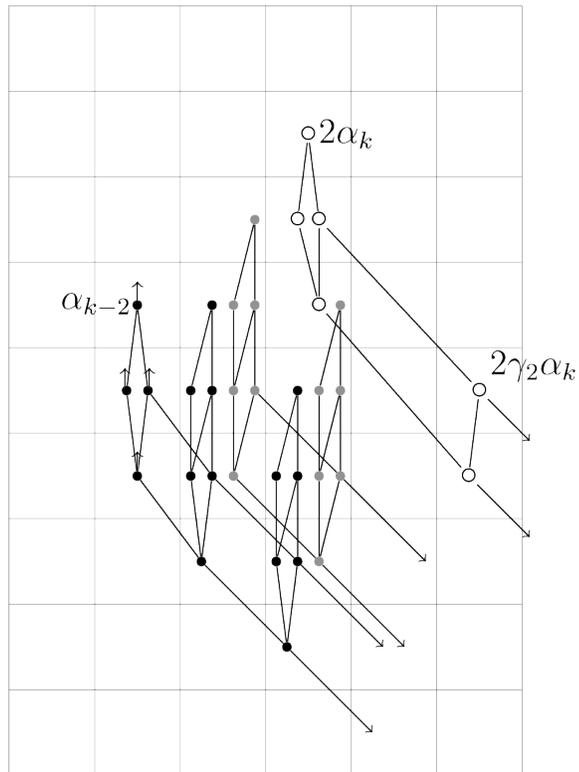
Figure 7.4: The alpha family in the E_∞-term of the motivic ANSS over F

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