UNIVERSITY OF MICHIGAN
ENGINEERING RESEARCH INSTITUTE

ELECTRONIC DEFENSE GROUP TECHNICAL MEMORANDUM NO. 19

SUBJECT: Shunt Capacitance and Maximally-Flat Filter Design

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1. INTRODUCTION

We are here concerned with the capacitances C_0 and C_n of the filters of the type shown in Fig. 1 (as well as the realization of such filters), adjusted to provide a maximally-flat n-pole transfer function. Before treating the specific filters, the elements of the Darlington method will be set down.

The Darlington synthesis procedure gives

$$\rho(p)\rho(-p) = 1 - 4(R_1/R_2)F(p)F(-p) = \left(\frac{R_1 - Z(p)}{R_1 + Z(p)}\right)\left(\frac{R_1 - Z(-p)}{R_1 + Z(-p)}\right) \tag{1}$$

where $F(p) = F(j\omega) = e_2/e_1$ is the transfer function and where $Z(p)$ is the input impedance to a lossless network terminated in R_2 as shown in Fig. 2. The reflection coefficient $\rho(p)$ is given by

$$\left|\rho(j\omega)\right| = \left|\frac{R_1 - Z(j\omega)}{R_1 + Z(j\omega)}\right| \tag{2}$$

Given $F(p)$, R_1, and R_2, $\rho(p)\rho(-p)$ may be calculated as

$$\rho(p)\rho(-p) = H^2 \frac{(p - z_1)(p + z_1)(p - z_2)(p + z_2) \ldots}{(p - p_1)(p + p_1)(p - p_2)(p + p_2) \ldots} \tag{3}$$

where H^2 is a constant multiplier and where $p + z_1$ and $p - z_1$ refer to a zero in the left half-plane and its negative, respectively.

The poles and zeros of $\rho(p)$ are chosen from those of Eq. (3). The poles of $\rho(p)$ are those lying in the left half-plane. The zeros of $\rho(p)$ are selected in complex-conjugate pairs and either a zero or its negative, but not both, are taken. Thus,

$$\rho(p) = \pm H \frac{(p \pm z_1)(p \pm z_2) \ldots}{(p + p_1)(p + p_2) \ldots} = \frac{N}{D} \tag{4}$$

in which each $\pm$ sign indicates a choice. From Eq. (2) we find

$$Z(p) = R_1 \frac{D - N}{D + N} \tag{5}$$

after which the network is synthesized as an input impedance.

2. THE MAXIMALLY-FLAT LOW-PASS FILTER

For the low-pass filter of unity radian bandwidth,

$$F(p)F(-p) = \frac{\left[R_2/(R_1 + R_2)\right]^2}{\pm p^{2n} + 1} \tag{6}$$

where the positive sign is used for n even and the negative sign for n odd. Thus,

$$\rho(p)\rho(-p) = \frac{\pm p^{2n} + (1 - T)}{\pm p^{2n} + 1} \tag{7}$$

where

$$T = \frac{4R_1R_2}{(R_1 + R_2)^2} \tag{8}$$

is the transmission coefficient -- it is the ratio of actual to maximum possible power dissipation in the load R_2 at $\omega = 0$.

The poles of Eq. (7) lie with equal angular spacing on the unit circle. The zeros lie similarly on a circle of radius $(1 - T)^{1/2n}$. Using

$$F(p) = \frac{R_2/(R_1 + R_2)}{p^n + b_{n-1}\, p^{n-1} + \ldots + b_1 p + 1} \tag{9}$$

we have

$$\rho(p) = \pm \frac{p^n + a_{n-1}\, p^{n-1} + \ldots + a_1 p + \sqrt{1 - T}}{p^n + b_{n-1}\, p^{n-1} + \ldots + b_1 p + 1} \tag{10}$$

The $\pm$ multiplier leads to the dual networks. In order that we get a ladder having a shunt capacitance as a leading element, the positive sign must be chosen. Then,

$$Y(p) = G_1 \frac{2p^n + (b_{n-1} + a_{n-1})p^{n-1} + \ldots + (b_1 + a_1)p + (1 + \sqrt{1 - T})}{(b_{n-1} - a_{n-1})p^{n-1} + \ldots + (b_1 - a_1)p + (1 - \sqrt{1 - T})} \tag{11}$$

The first term of the continued-fraction expansion gives C_0 as

$$C_0 = \frac{2G_1}{b_{n-1} - a_{n-1}} \tag{12}$$

The number b_{n-1} is the negative of the sum of the real parts of the pole positions of $\rho(p)$. Thus

$$b_{n-1} = \begin{cases} 1 + 2 \sum_{k=1}^{(n-1)/2} \cos \frac{k\pi}{n} & n \text{ odd} \\ 2 \sum_{k=1}^{n/2} \cos \frac{(2k-1)\pi}{2n} & n \text{ even} \end{cases} \tag{13}$$

The capacitance C_0 is a maximum when a_{n-1} is a maximum. This implies that we take the zeros of $\rho(p)$ all in the left half-plane to give

$$a_{n-1} = (1 - T)^{1/2n} b_{n-1} \tag{14}$$

By substituting for T in Eq. (11) at $p = 0$, it is found that Eq. (11) is self consistent for $R_1 \geq R_2$ but not for $R_1 \leq R_2$. Since in general the Darlington procedure gives a ratio R_1/R_2 equal to the ratio R_2/R_1 for opposite choices of the zeros of $\rho(p)$, we therefore conclude

$$C_0 = \begin{cases} \dfrac{2G_1}{b_{n-1}\left[1-(1-T)^{1/2n}\right]} & R_1 \geq R_2 \\ \dfrac{2G_1}{b_{n-1}\left[1+(1-T)^{1/2n}\right]} & R_1 \leq R_2 \end{cases} \tag{15}$$

C_n may be found by simply reversing the network such that R_2 becomes the source and R_1 the load.

$$C_n = \begin{cases} \dfrac{2G_2}{b_{n-1}\left[1 + (1 - T)^{1/2n}\right]} & R_1 \geq R_2 \\ \dfrac{2G_2}{b_{n-1}\left[1 - (1 - T)^{1/2n}\right]} & R_1 \leq R_2 \end{cases} \tag{16}$$

The ratio of C_n to C_0 for $R_1 \geq R_2$ can easily be found from preceding expressions. This ratio, after substituting for T (valid only for n odd) becomes

$$\frac{C_n}{C_0} = \left(\frac{R_1}{R_2}\right) \frac{1 - \left(\dfrac{1 - R_2/R_1}{1 + R_2/R_1}\right)^{\frac{1}{n}}}{1 + \left(\dfrac{1 - R_2/R_1}{1 + R_2/R_1}\right)^{\frac{1}{n}}} \tag{17}$$

For $R_1 \leq R_2$, reverse the subscripts in Eq. (17) and reverse the signs of the factors to the 1/n power. Eq. (17) is plotted in Fig. 3 for a few values of n.

Of particular interest are limiting expressions of the preceding relations. Then, resistance exists at only one end of the network as when the network is driven with an ideal current generator. Using power series expansions where necessary, we get

$$\begin{aligned} &\lim_{R_1 \to \infty} R_2 C_0 = \frac{n}{b_{n-1}} \\ &\lim_{R_2 \to \infty} R_1 C_n = \frac{n}{b_{n-1}} \\ &\lim_{R_1/R_2 \to \infty} \frac{C_n}{C_0} = \frac{1}{n} \qquad \text{n odd} \end{aligned} \tag{18}$$

The most important conclusion to be drawn from Fig. 3 is the rapidity with which the ratio C_n/C_0 reaches its limiting value as the ratio R_2/R_1 departs from unity.

The value of R_2C_0 is plotted as a function of n (unity bandwidth) in Fig. 4 which assumes $R_1 \rightarrow \infty$. The rapidity with which the value approaches the Bode limit of $\pi/2$ (for a rectangular band of unity bandwidth) should be noted. Of course, the maximally-flat function does not have such a very impressive behavior in this regard if the bandwidth is defined at the one or two db frequency rather than the three db frequency, although the Bode limit is approached with n regardless of the tolerance bandwidth used as a criterion. However, if n is at all large, the one and three db frequencies are almost negligibly different--for example, for n = 5, the one db frequency is 0.874 when the three db frequency is unity. If for this example the one db frequency had been used to define the bandwidth rather than the three db frequency, the capacitance C_0 would still only be about fourteen percent less than the Bode limit. A curve based on the one db bandwidth is shown dotted in Fig. 4. This second curve is also characteristic of maximally-flat filters with some types of optimum mismatch in that something other than the half-power bandwidth is used as a criterion.

The purpose of the curves of Fig. 4 is not to demonstrate that the maximally-flat function is an exceptionally efficient function. In fact, in terms of the Bode limit, it is appreciably inferior to filters based upon Chebyshev functions. However, the mathematics of maximally-flat functions are considerably simpler than those associated with Chebyshev functions which makes the maximally-flat function excellent for practical purposes. (For example, even though relatively poor, constant-k filters are widely used because they are so easy to design. Unfortunately, constant-k filter theory assumes at least an approximate image impedance match). In addition, many

problems are very poorly solved using Chebyshev functions, for example, those requiring a fairly linear phase function of frequency or those involving transient behavior. Such problems are better solved (although still perhaps not best solved) using maximally-flat functions where the concept of the Bode limit is not especially pertinent.

The limiting values of C_0 and C_n for $T \to 0$ must be handled with caution, particularly if the limiting case is approached by allowing either the source or the load resistance to become zero. For example, if R_1 is fixed and T is caused to become very small by reducing R_2, Eqs. (15) and (16) show that both C_0 and C_n increase without bound while their ratio tends to a constant $1/n$. Although such increasingly large capacitance values are not surprising when associated with a vanishingly small source or load resistance, the phenomenon must be carefully handled when deriving optimum mismatch conditions.

3. FILTER DESIGN

The input admittance of the network is given by Eq. (11). We shall restrict our interest to cases when $R_1 \geq R_2$. If one need have $R_1 \leq R_2$, he can design the network for $R_1 \geq R_2$ and then turn the network end for end.

Since the zeros of the reflection coefficient lie on a circle of radius $(1 - T)^{1/2n}$, and since we have chosen only the left half-plane zeros of $\rho(p)$ as belonging to the input admittance (in order to maximize C_0), we have

$$\begin{aligned} a_{n-1} &= b_{n-1} (1 - T)^{1/2n} \\ a_{n-2} &= b_{n-2} (1 - T)^{2/2n} \\ &\vdots \\ a_{n-k} &= b_{n-k} (1 - T)^{k/2n} \end{aligned} \tag{19}$$

Then, Eq. (11) becomes

$$Y(p) = G_1 \frac{2p^n + b_{n-1}\left[1 + (1 - T)^{1/2n}\right]p^{n-1} + \ldots + \left[1+\sqrt{1 - T}\right]}{b_{n-1}\left[1 - (1 - T)^{1/2n}\right]p^{n-1} + \ldots + \left[1-\sqrt{1 - T}\right]} \qquad (20)$$

The coefficients b_{n-k} are found from the positions of the poles of $\rho(p)$ on the unit circle in the left half-plane. In order that Eq. (20) expand into partial fractions and give the desired network, it is necessary that

$$b_{n-2} = \frac{(b_{n-1})^2}{2} \qquad (21)$$

which somewhat simplifies obtaining the polynomial representing the poles of the transfer function. Another simplification is given by observing that these polynomials are symmetric. Some of the polynomials are

$$\begin{aligned}
&n = 1: \quad p + 1 \\
&n = 2: \quad p^2 + (2)^{1/2} p + 1 \\
&n = 3: \quad p^3 + 2p^2 + 2p + 1 \\
&n = 4: \quad p^4 + 2.6132p^3 + 3.41p^2 + 2.6132p + 1 \\
&n = 5: \quad p^5 + 3.236p^4 + 5.236p^3 + 5.236p^2 + 3.236p + 1
\end{aligned} \qquad (22)$$

The synthesis amounts to putting the input admittance of Eq. (20) into numerical terms and expanding it into a continued fraction. Then, the element values are easily identified. It is important to note that except for the lower values of n, a high order of numerical accuracy must be maintained in obtaining the continued-fraction expansion -- in other words, slide-rule accuracy is not sufficient. A desk calculator is required for the unsymmetric network for n equal to or larger than five, even five-place logarithms not being adequate.

Two very important special cases of Eq. (20) are worth individual study. For the matched case where $R_1 = R_2 = R$, T is unity and Eq. (20) becomes

$$Y(p) = R\,\frac{2p^n + b_{n-1}\,p^{n-1} + \ldots + b_1 p + 1}{b_{n-1}p^{n-1} + b_{n-2}p^{n-2} + \ldots + b_1 p + 1} \tag{23}$$

When $R_1 \to \infty$, the form of Eq. (20) is indeterminant. Therefore, limiting values must be obtained as

$$\lim_{R_1 \to \infty} R_1\left[1 - (1 - T)^{k/2n}\right] = \frac{2k}{n}\,R_2 \tag{24}$$

in which case Eq. (20) becomes

$$Y(p) = G_2\,\frac{p^n + b_{n-1}\,p^{n-1} + b_{n-2}\,p^{n-2} + \ldots + b_1 p_1 + 1}{\frac{b_{n-1}}{n}\,p^{n-1} + \frac{2b_{n-2}}{n}\,p^{n-2} + \ldots + \frac{(n-1)b_1}{n}\,p + 1} \tag{25}$$

4. OPTIMIZATION PROCEDURES

There exist almost as many optimum filters as there exist students of the subject for the simple reason that there are so many criteria that can be used in specifying some "best" filter. It is the purpose here to point out a few of the various criteria that admit of optimization and work out some of the more promising ones--of course, all as associated with the maximally-flat function.

The half-power bandwidth of the maximally-flat filter is given through a modification of Eq. (15) as

$$B = \frac{2G_1}{b_{n-1}\,C_0\left[1 \pm (1 - T)^{1/2n}\right]} \tag{26}$$

The product BT is a power gain times bandwidth product. Its maximization under certain restraints is of obvious importance to students of filter theory. However, to the designer of vacuum-tube amplifiers, the product $B\sqrt{T}$ may be of more interest--it is the product of bandwidth and voltage gain rather than bandwidth and power gain. There is no reason to suspect that the maximization

of BT and $B\sqrt{T}$ give the same networks. More generally, one might maximize some product of the form BT^k where k is a weighting factor describing the importance of having a large output at $\omega = 0$.

The product BT^k is obtained by multiplying Eq. (26) by T^k. However, unless the source or load impedance is specified, there can exist no optimum situation--it is the product of resistance and capacitance rather than resistance alone or capacitance alone that is important. Therefore, if R_1C_0 is also used as a parameter in Eq. (26), a more meaningful relation to study is

$$R_1C_0BT^k = \frac{2T^k}{b_{n-1}\left[1 - (1 - T)^{1/2n}\right]} \quad , \qquad R_1 \geq R_2 \tag{27}$$

where R_1 has been assumed larger than R_2--this obviously gives a larger value to Eq. (27) than when R_2 is larger than R_1.

Eq. (27) has a maximum of infinity at $T = 0$ unless $k > 1$. Thus, relatively heavy weighting must be given to T in order to find a unique optimum. The maximum value of Eq. (27) can be found in the usual manner by setting its derivative with respect to T equal to zero. The result is the relation

$$(1 - T_0)^{1/2n} = \frac{1}{1 + \dfrac{T_0}{2kn(1 - T_0)}} \quad , \qquad \begin{cases} R_1 \geq R_2 \\ k > 1 \end{cases} \tag{28}$$

where T_0 is the optimum value of T.

If one were interested in having a large value of R_2 as well as a large value of C_0, one should maximize the relation

$$R_2C_0BT^k = \frac{2T^{k-1}\left[2 - T - 2(1 - T)^{1/2}\right]}{b_{n-1}\left[1 - (1 - T)^{1/2n}\right]} \tag{29}$$

The optima for the examples above (as well as similar optima in general) are rather broad. Therefore, determining the optimum mismatch to a high accuracy with the subsequent requirement for precision network elements is not necessary.

So far, we have specified optimum criteria only for the half-power bandwidth; that is, a bandwidth specified by the frequency where the output from the filter is three decibels below that at $\omega = 0$. It is possible to obtain an additional set of criteria using a prescribed tolerance bandwidth. In this, the insertion power ratio is used. This ratio, for a filter with unity half-power bandwidth is

$$\frac{P_{20}}{P_2} = \frac{1 + \omega^{2n}}{T} \tag{32}$$

which is a minimum of unity at $\omega = 0$ only if $R_1 = R_2$. A sketch of this ratio is given in Fig. 5. Let the bandwidth B' be defined at the frequency where $P_{20}/P_2 = 1/\beta$ where it is necessary that $\beta < T$. Then,

$$B' = (T/\beta - 1)^{1/2n} \tag{33}$$

One can use the bandwidth B' in the preceding relations rather than the bandwidth B. With this modification, Eqs. (27) and (30) become

$$R_1 C_0 B' T^k = \frac{2T^k (T/\beta - 1)^{1/2n}}{b_{n-1}\left[1 - (1 - T)^{1/2n}\right]} \quad , \qquad \begin{matrix} R_1 \gtrless R_2 \\ T > \beta \end{matrix} \tag{34}$$

$$(R_1 C_0 + R_2 C_n) B' T^k = \frac{4T^k (T/\beta - 1)^{1/2n}}{b_{n-1}\left[1 - (1 - T)^{1/n}\right]} \quad , \qquad \begin{matrix} R_1 \gtrless R_2 \\ T > \beta \end{matrix} \tag{35}$$

which, because T cannot be smaller than β, has a maximum for a finite T for all positive values of the weighting k including zero. It would appear

reasonable to use a weighting of zero in the above relations because β is a parameter that can be specified.

The general optimization relation corresponding to Eq. (34) is

$$(1 - T_0)^{1/2n} = \frac{1}{1 + \dfrac{(T_0 - \beta)/(1 - T_0)}{2nk(1 - \beta/T_0) + 1}} \tag{36}$$

An example will be drawn from this relation for $n = 3$, $\beta = 0.5$, and $k = 0$. One gets $T_0 = 0.565$ giving the surprisingly large resistance ratio $R_1/R_2 = 4.87$. The tolerance bandwidth is $B' = 0.712$ and $R_1C_0 = 7.71$. Also, $R_1C_0B' = 5.49$. The matched filter having the same tolerance bandwidth has $R_1C_0 = 1.406$ and $R_1C_0B = 1.0$. Evidently, the advantages of an appreciable mismatch are rather impressive.

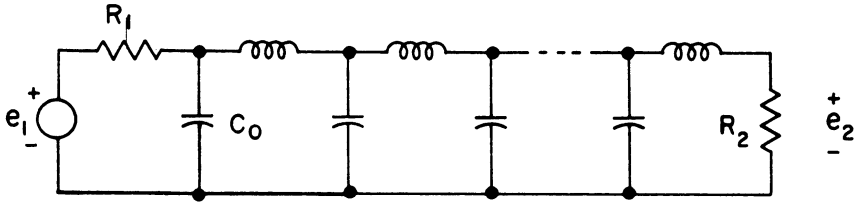
5. EXAMPLES

The five-pole matched filter with $R_1 = R_2$ and with unity half-power bandwidth is shown in Fig. 6a. This can be compared to the similar constant-k filter shown in Fig. 6b. Values are in ohms, henrys, and farads.

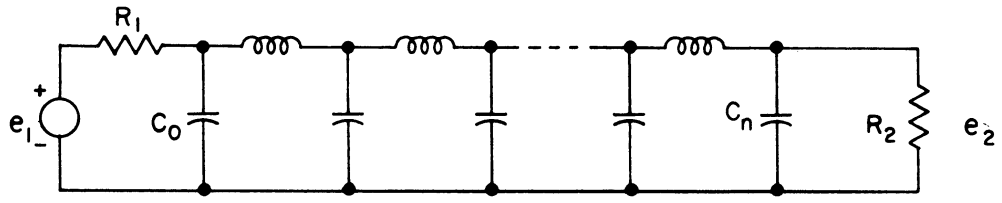
As a second example, let us obtain the circuit for the optimum three-pole mismatched filter having an optimum T_0 as given by Eq. (36) and the example following that equation. If R_2 is normalized to one ohm, the input admittance is

$$Y(p) = \frac{0.41028p^3 + 0.76743p^2 + 0.72117p + 0.34045}{0.25908p^2 + 0.48460p + 0.34045} \tag{37}$$

The corresponding network is shown in Fig. 7a which can be compared to the matched maximally-flat filter shown in Fig. 7b (which is also a constant-k filter section).



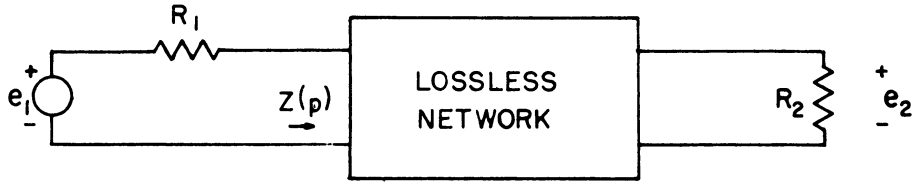
(a) n EVEN



(b) n ODD

FIG. 1
LOW-PASS LADDER FILTER

FIG. 2
THE GENERAL LOSSLESS NETWORK WITH TERMINATIONS



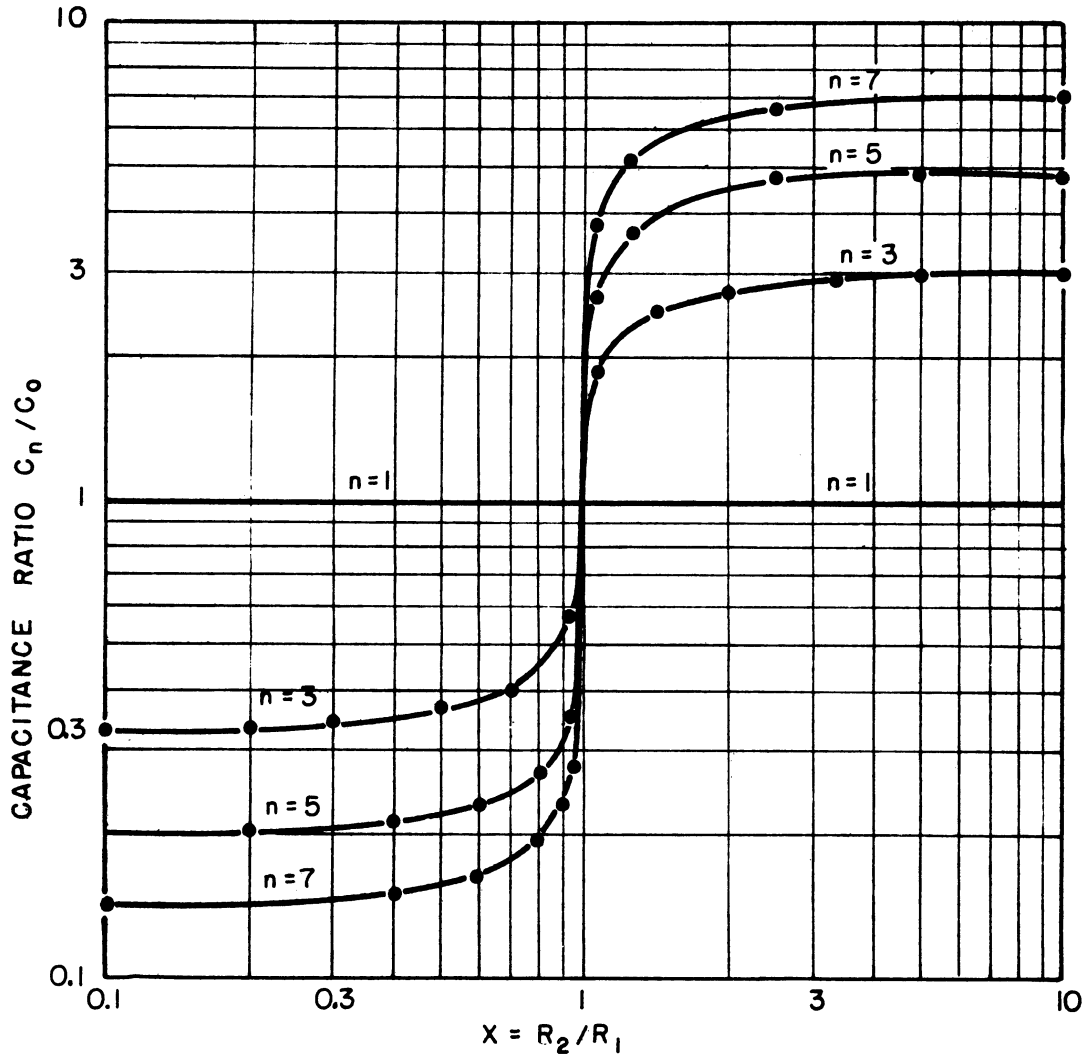


FIG. 3. CAPACITANCE RATIO FOR n ODD.

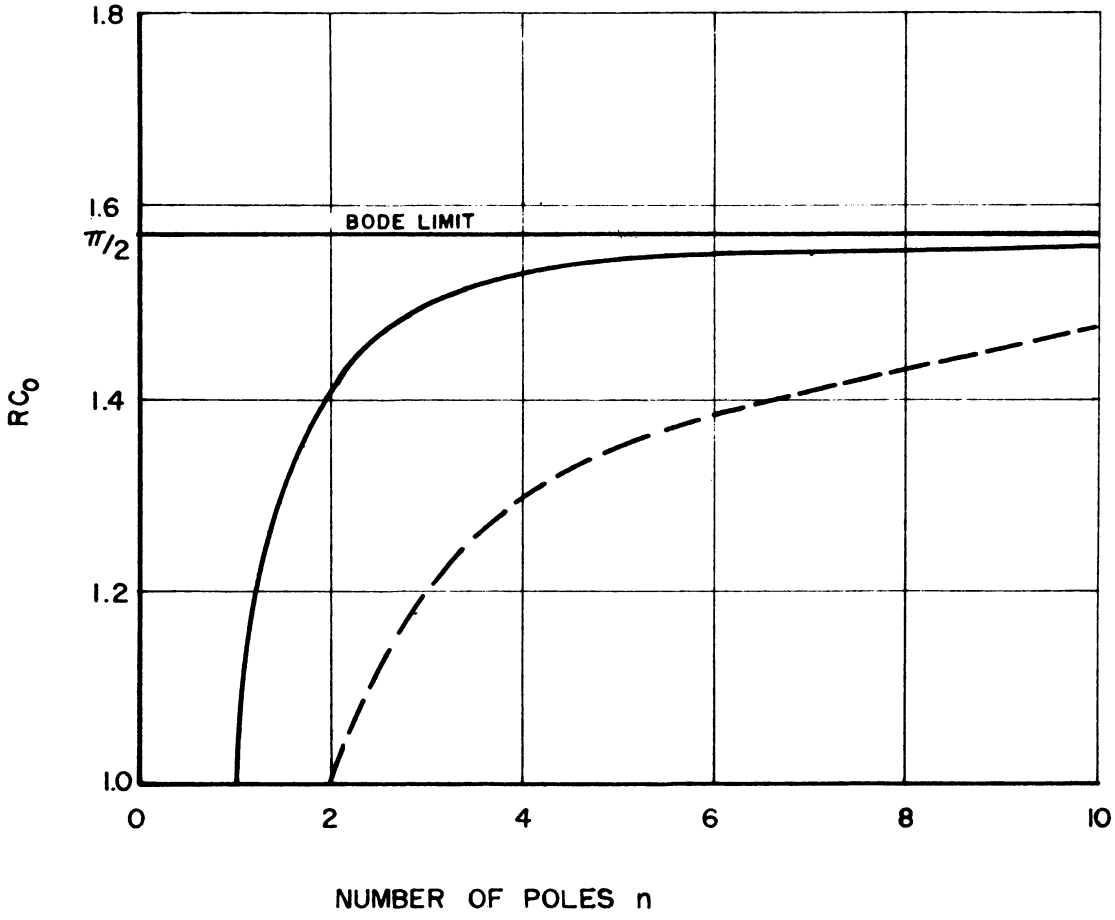


FIGURE 4

OUTPUT CAPACITANCE FOR $R_1 = \infty$.

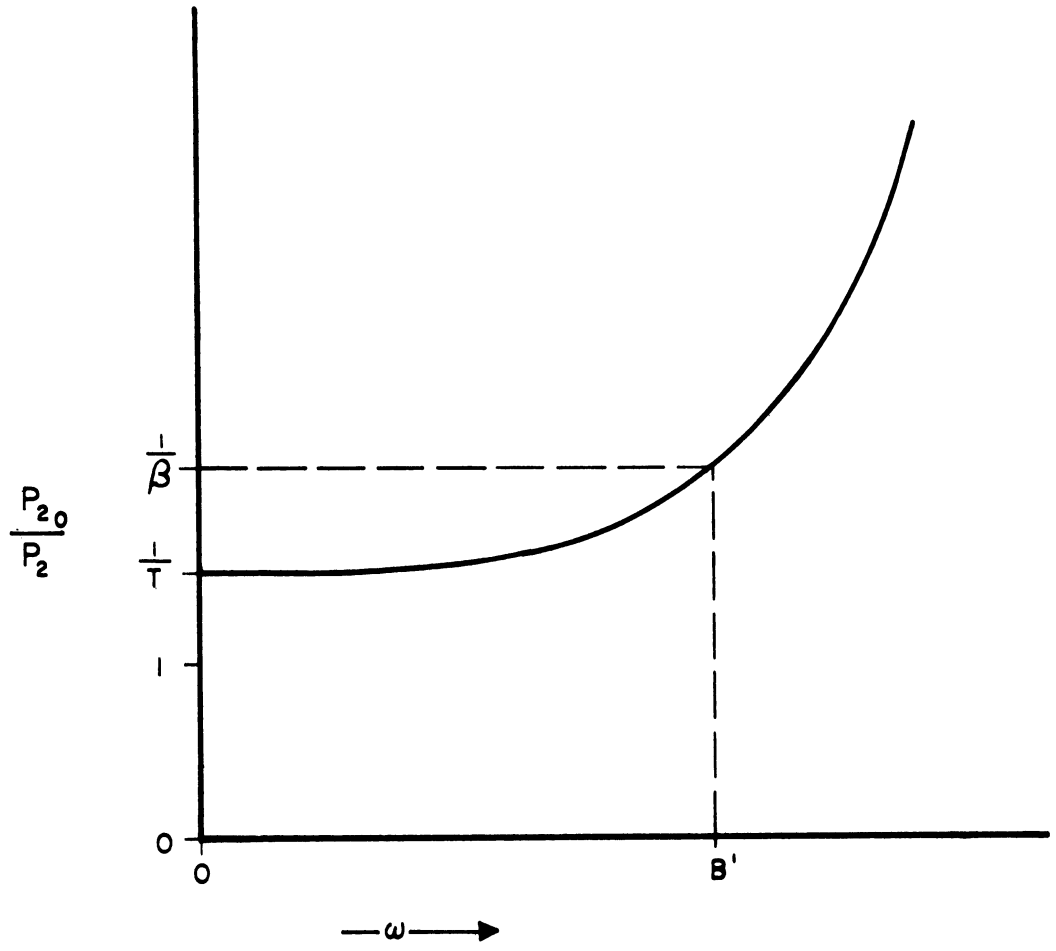
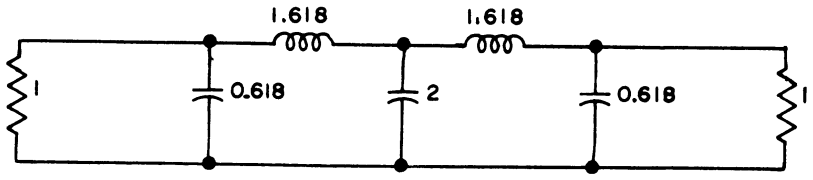
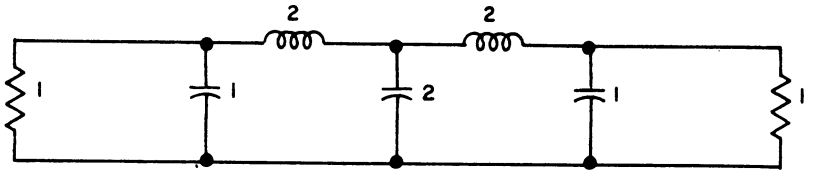


FIG. 5. INSERTION POWER RATIO AND DEFINITION OF BANDWIDTH.

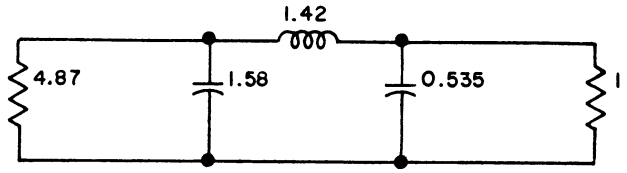


(a)

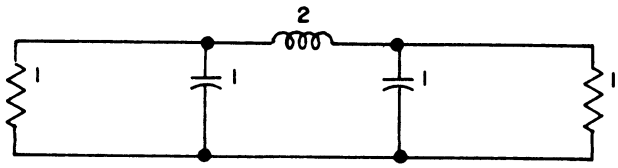


(b)

FIG. 6. COMPARISON OF MAXIMALLY-FLAT AND CONSTANT-k FIVE-POLE FILTERS.



(a)



(b)

FIG. 7. COMPARISON OF OPTIMUM AND MATCHED THREE-POLE FILTERS.

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