Arithmetic of the Yoshida Lift

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
2010

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To Mom and Dad

Acknowledgements

I am in debt to my thesis advisor, Chris Skinner, for his steadfast support and for suggesting this interesting problem. His keen insights and wealth of knowledge were pivotal for overcoming numerous obstacles; his encouragements kept the work going through pressing times.

I am equally in debt to Stephen DeBacker, whose generosity and kindness are unparalleled. His often humorous but always incisive career advice and mathematical suggestions have made me a better mathematician and a better person.

There are countless other individuals without whom this work would have been simply impossible. Among them I would like to mention in particular Brian Conrad, whose single-minded devotion to mathematics has profoundly shaped my mathematical temperaments.

Lastly I would like to thank the Mathematics Department at the University of Michigan, as well as numerous other mathematical organizations, for the generous support and the opportunities they provided. I like to especially thank the American Institute of Mathematics for hosting the Workshop on Generalizing the Theta Correspondence during the summer of 2008. It is while attending this workshop that I learned many of the recent results used in this work.

Contents

Abstract

This thesis concerns the arithmetic properties of the Yoshida lift, $\mathbf{Y}$, which is a scalar-valued holomorphic Siegel modular form of degree 2 obtained as the theta lift of a pair of automorphic forms $\mathbf{f}_1, \mathbf{f}_2$ on $D^\times$, where D is a definite quaternion algebra over $\mathbf{Q}$.

Specifically, we define a refined version of the Yoshida lift, $\mathbf{Y}$, which has the special property that it preserves $\mathfrak{p}$-integral structures and is not identically zero under mild conditions. For $\mathfrak{p}$-integrality, we compute a formula for the Fourier coefficients a^T of $\mathbf{Y}$ by exploiting an inherent freedom in the definition of $\mathbf{Y}$. The formula for a^T in turn allows us to compute the Bessel model of the Yoshida lift, and apply an argument of Cornut–Vatsal to conclude that $\mathbf{Y}$ is non-zero. Furthermore, if we assume Artin's conjecture on primitive roots, then we show that $\mathbf{Y}$ is in fact not zero modulo $\mathfrak{p}$.

Introduction

The present work is the first part of an ongoing project aimed at understanding and affirming some deep connections between special values of L-functions and certain Selmer groups. The main object we study here is a certain *theta lift*. Since the subject could get a bit technical, we begin with some informal excerpts from the elementary aspects of the subject to set the context for the non-experts.

0.0.1 Theta functions.

The simplest example of a theta function is the holomorphic function on the upper-half plane $\mathfrak{H} = \{z \in \mathbf{C} : \mathrm{Im}(z) > 0\}$ given by the series

$$\theta(z) = \sum_{n \in \mathbf{Z}} e^{2\pi \imath n^2 z}.$$

It is an automorphic form on for the congruence subgroup

$$\Gamma_0(4) = \left\{ \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbf{Z}) : c \in 4\mathbf{Z} \right\}$$

of weight $\frac{1}{2}$, in that it satisfies the transformation law [Iwa97, §2 (2.73)]

$$\theta(\gamma z) = (*) \cdot (cz + d)^{\frac{1}{2}} \cdot \theta(z)$$

for all $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(4)$ and $z \in \mathfrak{H}$.* Also note that the Fourier coefficients $a(m)$ of θ is 1 exactly when $m \in \mathbf{Z}^2$ and 0 otherwise, which so happens to be equal to the number of ways that we can represent m by a square.

The point to take home is that theta functions, such as $\theta(z)$, are in automorphic forms; moreover, their Fourier expansions are relatively tractable and carry interesting arithmetic information.

*We have suppressed a constant $*$ dependent on c and d to keep the presentation elementary.

0.0.2 Automorphic forms.

Roughly speaking, automorphic forms are functions that satisfy strong invariant properties under certain group actions. For us, the most exciting aspect of automorphic forms is that they are expected to give rise to L-functions, and can be used to study the analytic properties of these L-functions. Take $\theta(z)$ for example, one can show that [GM04, §2.1 (2.10)]

$$\pi^{-\frac{s}{2}} \cdot \Gamma\left(\frac{s}{2}\right) \cdot \zeta(s) = \int_1^\infty (t^{\frac{s}{2}-1} + t^{\frac{1-s}{2}-1})(\frac{1}{2} \cdot \theta(it) - \frac{1}{2})\, dt - \frac{1}{s} - \frac{1}{1-s}$$

where $\Gamma(s)$ is the usual Gamma function and $\zeta(s) = \sum_{n=1}^\infty n^{-s}$ is the Riemann zeta function. This integral representation of $\zeta(s)$ together with Jacobi's transformation law

$$\theta(\imath t) = \frac{1}{\sqrt{t}} \cdot \theta(\frac{\imath}{t})$$

allow us to establish a functional equation for $\zeta(s)$.

In fact, all L-functions that have Euler product factorizations and functional equations should be the L-functions associated with automorphic forms. Since Galois representations arising from geometry or otherwise give rise naturally to L-functions, one expects automorphic forms to be associated with Galois representations. This is one of the aims of the Langlands program.

0.0.3 Integrality.

In addition to their relations to zeta functions, the Fourier coefficients of automorphic forms (most notably those on GL_2) have shown to carry a lot of number-theoretic information. For example, let $Q(m_1, \ldots, m_r) = \sum_{i=1}^r \alpha_i z_i^2$ be a positive-definite quadratic form with $\alpha_i \in \mathbf{Z}_{>0}$. We define a theta function

$$\theta(z, Q) = \sum_{\underline{m} \in \mathbf{Z}^r} e^{2\pi\imath \cdot Q(\underline{m}) z}$$

for $z \in \mathfrak{H}$. It is an automorphic form for $\Gamma_0(2N)$ of weight $k = \frac{r}{2}$ [Iwa97, §11.3]. It has a Fourier expansion

$$\theta(z, Q) = \sum_{n \in \mathbf{Z}} r(n, Q) e^{2\pi\imath z}$$

where $r(n, Q)$ is the representation number of n by Q, that is, the number of ways we can right $n = Q(\underline{m})$ for some $\underline{m} \in \mathbf{Z}^r$. In particular, we see that these Fourier coefficients $r(n, Q)$ are all *integers*.

The integrality of these Fourier coefficients is not only interesting in its own right, but it is also important in constructing Galois representations. In fact, it is one of the (abide minor) ingredients used in Wiles's proof of Fermat's Last Theorem.

0.0.4 Theta lifts.

Let $\chi\colon (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^1$ be a Dirichlet character of conductor N. It turns out that χ is naturally an automorphic form for the group $\mathrm{SO}_2 \simeq S^1$. We define a theta lift of χ to the group GL_2 by

$$\theta_\chi(z) = \sum_{i=-n}^{n} \chi(n) \cdot n^\nu \cdot e^{2\pi\imath n^2 z}$$

where $\nu = 0$ if $\chi(-1) = 1$ and 1 otherwise. It is an automorphic form for the group $\Gamma_0(4)$ of weight $\nu + \frac{1}{2}$. So we have *lifted* the automorphic form χ on SO_2 to the group GL_2.

This simple example is actually quite representative of the general phenomenon. The theta lift which we study, $\mathbf{Y}$, called the *Yoshida lift*, can be expressed as a sum of a similar shape

$$\mathscr{Y}(Z) = \sum_{i=1}^{r} \frac{1}{e_i} \sum_{x \in h_i^{-1}\cdot \mathscr{X}_f \cap \mathbf{X}_{\mathbf{Q}}^T} \mathbf{f}_2^\bullet(\beta_i) \cdot \langle \tilde{\mathbf{P}}_k^\bullet(x), \mathbf{f}_1^\bullet(\alpha_i)\rangle_{2k,0} \cdot e^{2\pi\imath \operatorname{tr}(T_x Z)},^*$$

where $\mathscr{Y}$ is the holomorphic Siegel modular form of degree 2 associated to $\mathbf{Y}$. Note this expression also give a Fourier expansion

$$\mathscr{Y}(Z) = \sum_{T \in \mathscr{T}} a(T) \cdot e^{2\pi\imath \operatorname{tr}(TZ)}$$

of $\mathscr{Y}$ where the index T runs over the set of semi-definite symmetric 2×2 matrices with entries in $\mathbf{Q}$, and $Z \in \mathfrak{H}_2$.

The goal of this thesis is to study the arithmetic properties of these Fourier coefficients $a(T)$, and also to show that the Yoshida lift $\mathbf{Y}$ is not trivial, that is, not identically zero (or zero modulo a prime) as a function.

0.1 Motivations

As we mentioned at the beginning, the current work is not just an end in itself. In fact, we see it really as a stepping stone for achieving our goal of exhibiting evidence

* Here the h_i's are elements in an orthogonal group, e_i is a positive integer, and $h_i^{-1} \cdot \mathscr{X}_f \cap \mathbf{X}_{\mathbf{Q}}^T$ is a finite collection of integral maps for each i. Also $\tilde{\mathbf{P}}_k^\bullet$ is a vector-valued harmonic polynomial which we will define precisely.

for some deep conjectures in number theory. Since casting this work in this larger context helps one to better appreciate its relevance and utility, let us outline the vista.

0.1.1 The classical Yoshida lift.

To explain this, let D be a definite quaternion algebra over $\mathbf{Q}$. In [Yos80], Yoshida studied the theta lift taking a pair of automorphic forms, $\mathbf{f}_i$, $i = 1,2$ on $D^\times$ to a holomorphic automorphic form $\mathbf{Y}$ on GSp_4. To do this, he first showed that the product $\mathbf{f}_1 \otimes \mathbf{f}_2$ is an automorphic form on the orthogonal similitude group $\mathrm{GSO}(D)$ of D. This is an immediate consequence of the fact that $\mathrm{GSO}(D)$ is essentially $D^\times \times D^\times$.

Then the theory of Weil representations give rise to a collections of automorphic forms, $\Theta_{\boldsymbol{\varphi}}$, on a subgroup of $\mathrm{GSO}(D) \times \mathrm{GSp}_4$. They are naturally indexed by a choice of a Bruhat–Schwartz function $\boldsymbol{\varphi}$. The Yoshida lift

$$\mathbf{Y} = \langle \Theta_{\boldsymbol{\varphi}}, \mathbf{f}_1 \otimes \mathbf{f}_2 \rangle_{\mathrm{GSO}(D)}$$

is essentially the Petersson inner product of $\Theta_{\boldsymbol{\varphi}}$ with $\mathbf{f}_1 \otimes \mathbf{f}_2$ over the group $\mathrm{GSO}(D)$.

Moreover, Yoshida computed the Satake parameters of $\mathbf{Y}$ in terms of those of $\mathbf{f}_i$ at the unramified places. This allows us to study the L-functions associated with $\mathbf{Y}$ in terms of the L-functions associated with the $\mathbf{f}_i$'s.

0.1.2 Elements in a Selmer group.

By a standard construction, we can associate to $\mathbf{Y}$ a holomorphic Siegel modular form $\mathscr{Y}$ on the Siegel upper-half space $\mathfrak{H}_2$ of degree 2. Under favorable conditions, $\mathscr{Y}$ occurs in the cohomology of $\mathfrak{H}_2$, then we can associate to $\mathscr{Y}$ a p-adic Galois representation [Tay93], [Wei05]

$$\rho_{\mathscr{Y}} : \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(\mathbf{Q}_p).$$

What makes $\rho_{\mathscr{Y}}$ interesting is that it is expected to be semi-simple but *not* irreducible [Art04, pg. 78]. By constructing a congruence

$$\mathscr{Y} \equiv \mathscr{F} \pmod p$$

between $\mathscr{Y}$ and a *stable* Siegel modular form $\mathscr{F}$ whose Galois representation is irreducible, one can then (in theory) follow a well-known procedure dating back to the work of Ribet [Rib76], to construct elements in a Bloch–Kato Selmer group

$$H^1_f = H^1_f(\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}), (\mathscr{M}_1 \otimes \mathscr{M}_2)/p)$$

attached to $\mathbf{f}_1 \otimes \mathbf{f}_2$, where $\mathscr{M}_i$ denotes a $\mathbf{Z}_p$-lattice in the p-adic Galois representation associated with $\mathbf{f}_i$ [Tay89] for $i = 1, 2$.

0.1.3 Rankin–Selberg L-functions.

On the other hand, one can measure the congruence $\mathscr{Y} \equiv \mathscr{F} \pmod p$ between $\mathscr{Y}$ and stable Siegel modular forms in terms of the p-divisibility of a special value of the Rankin–Selberg L-function of $\mathbf{f}_1 \otimes \mathbf{f}_2$. Let us explain how to do this.

Since $\mathbf{Y}$ is a theta lift, we have at our disposal the *Rallis inner product formula*, which connects L-functions with ratios of the Petersson norms of automorphic forms. Without going into the details, the Rallis inner product formula applied to $\mathbf{Y}$ has the shape [GI, Lemma 7.11]

$$\frac{\langle \mathbf{Y}, \mathbf{Y}\rangle}{\|\mathbf{f}_1\|\|\mathbf{f}_2\|} = \frac{L^S(1, \boldsymbol{\pi}_1 \times \boldsymbol{\pi}_2) \cdot Z_S(\boldsymbol{\varphi}, \mathbf{f}_1, \mathbf{f}_2)}{\Omega},^{*} \tag{1.3.1}$$

which potentially relates the p-divisibility of $\frac{L^S(1,\boldsymbol{\pi}_1\times\boldsymbol{\pi}_2)}{\Omega}$ with that of $\frac{\langle \mathbf{Y},\mathbf{Y}\rangle}{\|\mathbf{f}_1\|\|\mathbf{f}_2\|}$.

By a principle stemming from [Hid81], the ratio $\frac{\langle \mathbf{Y},\mathbf{Y}\rangle}{\|\mathbf{f}_1\|\|\mathbf{f}_2\|}$ may be considered as a certain ratio of discriminants (a *congruence period*), and is a measurement of the amount of congruence $\mathscr{Y} \equiv \mathscr{F} \pmod p$ between $\mathscr{Y}$ and stable Siegel modular forms.

The upshot is that we can now phrase the existence of these congruences in terms of the p-divisibility of the algebraic L-value $\frac{L^S(1,\boldsymbol{\pi}_1\times\boldsymbol{\pi}_2)}{\Omega}$ on the right-hand side. This congruence, then, would give rise to elements in the Selmer group H^1_f.

Currently, however, the formula (1.3.1) is only available in the representation-theoretic setting, and does not yield much information in terms of arithmetic. Also, we have yet to make good sense of a congruence $\mathscr{Y} \equiv \mathscr{F} \pmod p$. This is where the current work comes in—to develop an arithmetic version of these results.

0.2 An overview of the results and methods

We now offer some broad sketches of the results obtained and the methods used.

*Here $\langle \mathbf{Y}, \mathbf{Y}\rangle$ and $\|\mathbf{f}_i\|$ are the Petersson norms of $\mathbf{Y}$ and $\mathbf{f}_i$ respectively. The $\boldsymbol{\pi}_i$'s denote cuspidal automorphic representations corresponding to $\mathbf{f}_i$, and $L^S(1, \boldsymbol{\pi}_1 \times \boldsymbol{\pi}_2)$ is the partial Rankin–Selberg L-function for the $\boldsymbol{\pi}_i$'s. Finally $Z_S(\boldsymbol{\varphi}, \mathbf{f}_1, \mathbf{f}_2)$ is a product of zeta integral at the "bad" primes in S defined with respect to the choice of some Bruhat–Schwartz function $\boldsymbol{\varphi}$, and Ω is a certain period.

0.2.1 A $\mathfrak{p}$-integral Yoshida lift.

The first step is to develop a version of the Yoshida lift which preserves integral structures. To explain this, let us fix once and for all a prime $p \geq 5$ in $\mathbf{Q}$ and let $\mathfrak{p}$ be a fixed prime ideal in $\bar{\mathbf{Q}}$ lying over p. By working with cuspidal automorphic representations $\boldsymbol{\pi}_i$ of $D^\times$ instead of automorphic forms, one gains the freedom to choose which $\mathbf{f}_i^\bullet$ in the representation space of $\boldsymbol{\pi}_i$ to lift, and we use $\bullet$ to index this choice. By choosing the Bruhat–Schwartz function $\boldsymbol{\varphi}^\bullet$ to align with $\mathbf{f}_i^\bullet$, one obtains the *same* Yoshida lift, $\mathbf{Y}$, independent of the choices made. This simple observation allows us to compute an expression (2.5.8)

$$a_{t_f}(T) = (-\det(T))^{\frac{k}{2}} \cdot \sum_{h_f \in \bar{E}_{x,f}} \frac{1}{e_h} \sum_{w \in \mathrm{Cl}(\mathscr{O}_h^x)} \langle \mathbf{t}_0^k, \mathbf{f}_1^x(\dot{t}\dot{w}\alpha_h) \rangle \cdot \mathbf{f}_2^x(x_1^{-1}\dot{w}x_1\beta_h)$$

for the Fourier coefficient $a_{t_f}(T)$ of the Siegel modular form $\mathscr{Y}$ associated with $\mathbf{Y}$ for all indices T.[†]

By finding possible representatives (α_f, β_f) for all elements in $\bar{E}_{x,f}$, we can verify that $a_{t_f}(T)$ is always $\mathfrak{p}$-integral assuming that the $\mathbf{f}_i$'s are. This gives our first main result:

Theorem 4.5.3. *If the compatible families $\{\mathbf{f}_i^\bullet\}$ attached to $\boldsymbol{\pi}$ are $\mathfrak{p}$-integral (Definition 2.3.4) for $i = 1, 2$, then the arithmetic Yoshida lift $\mathbf{Y}(\{\mathbf{f}_1^\bullet\}, \{\mathbf{f}_2^\bullet\})$ is $\mathfrak{p}$-integral in the sense that $\mathscr{Y} = \mathscr{Y}(\{\mathbf{f}_1^\bullet\}, \{\mathbf{f}_2^\bullet\})$ has all its Fourier coefficients $a(T)$ contained in*

$$\left(K \cap \frac{1}{k!}\mathscr{O}_{\mathbf{C}_p}\right) \subset \left(\bar{\mathbf{Q}} \cap \frac{1}{k!}\mathscr{O}_{\mathbf{C}_p}\right)$$

where $K = K(\{\mathbf{f}_1\}, \{\mathbf{f}_2\})$ is an algebraic extension of $\mathbf{Q}$ dependent on the $\{\mathbf{f}_i^\bullet\}$'s.

Moreover, if $p > k$, then the Tth Fourier coefficient $a(T)$ of $\mathscr{Y}$ lies in the local ring $\mathscr{O}_{F,(\mathfrak{p}_F)}$ over some number field $F = F^T$ dependent on T and $\{\mathbf{f}^\bullet\}$.

Here $\mathbf{f}_i^\bullet$ being $\mathfrak{p}$-integral means roughly that $\mathbf{f}_i^\bullet$ takes values in a $\bar{\mathbf{Z}}_{(\mathfrak{p})}$-lattice, where $\bar{\mathbf{Z}}_{(\mathfrak{p})} \subset \bar{\mathbf{Q}}$ is the localization of the ring of all algebraic integers at $\mathfrak{p}$.

[†]Here $\bar{E}_{x,f}$ is a finite set of *integrality-preserving isometries*, e_h is half the number of units in an imaginary quadratic field K_h, and $\mathrm{Cl}(\mathscr{O}_h^x)$ is a class group of K_h. Also we have $h_f = (\alpha_f, \beta_f)$ and $\langle \mathbf{t}_0^k, \mathbf{f}_1^x \rangle$ is a matrix coefficient of the *vector-valued* automorphic form $\mathbf{f}_1^x$.

0.2.2 Non-vanishing and non-vanishing modulo $\mathfrak{p}$.

For applications, one also needs to know that $\mathbf{Y}$ is non-zero (modulo $\mathfrak{p}$) in order to construct non-zero elements in H^1_f.[1] For this, we compute the Bessel models $B^{T,\chi}_{\mathbf{Y}}$ of $\mathbf{Y}$. For good choices of an index T and a character χ, $B^{T,\chi}_{\mathbf{Y}}(1)$ is on the one hand a linear combination of Fourier coefficients of $\mathbf{Y}$, and is on the other hand a product of two of character sums (1.6.2)

$$\sum_{i=1}^{h_n} \bar{\chi}_n(t_i) \cdot a_{t_i}(T_n) = (-l^{2n}\frac{\Delta}{4})^{\frac{k}{2}} \left(\sum_{w \in \mathrm{Cl}(\mathscr{O}_n)} \chi(w) \cdot \mathbf{f}_2^{x_n}(\dot{w}) \right) \cdot \left(\sum_{t \in \mathrm{Cl}(\mathscr{O}_n)} \bar{\chi}(t) \cdot \langle \mathbf{t}_0^k, \mathbf{f}_1^{x_n}(\dot{t}) \rangle \right).$$

We can then address the question of non-vanishing (modulo $\mathfrak{p}$) of a general character sum

$$\sum_{t \in \mathrm{Cl}(\mathscr{O}_n)} \bar{\chi}(t) \cdot \langle \mathbf{t}_0^k, \mathbf{f}^{x_n}(\dot{t}) \rangle \qquad (2.2.2)$$

using results of Cornut–Vatsal [CV07]. Let us briefly sketch the argument.

First we separate χ into a product of a "tame" character χ_t, and a "wild" character χ_w that is defined on a cyclic group of l-power order. For a fixed χ_t, we average the character sum 2.2.2 over all primitive χ_w's of conductor l^n. After some further massaging, we reduce sum (2.2.2) to (2.12.9)

$$\langle \mathbf{t}_0^k, \sum_{\tau \in R} \bar{\chi}_t(\tau) \cdot \mathbf{f}^*(\dot{\tau}\dot{s} \cdot \alpha_{-n}) \rangle.$$

Next we show this sum is not zero (modulo $\mathfrak{p}$).

By [CV07, Proposition 5.6], for $n \gg 0$, and any x, y in the domain of $\mathbf{f}^*$, we can choose t and u so that $\dot{\tau}_i \dot{t} \cdot \alpha_{-n} = \dot{\tau}_i \dot{u} \cdot \alpha_{-n}$ for all $i \neq 1$, and

$$x = \dot{\tau}_i \dot{t} \cdot \alpha_{-n} \neq \dot{\tau}_i \dot{u} \cdot \alpha_{-n} = y.$$

Now if we choose x and y so that $\mathbf{f}^*(x) \neq \mathbf{f}^*(y)$, then it follows that the character sum (2.12.9) is not zero for either $s = t$ or $s = u$.

To finish the argument, we observe that a certain Galois action permutes the characters χ_w while fixing other all terms. This allows us to prove that there exists a common character χ for which both characters sums in (1.6.2) are not zero.

Theorem 4.5.5. *Suppose that the central characters ε_i of the cuspidal automorphic*

[1] We can scale the Fourier coefficients of $\mathbf{Y}$ by some power of a uniformizer in the complete p-adic ring $\hat{\mathscr{O}}_{F,(\mathfrak{p}_F)}$ so it is non-zero modulo $\mathfrak{p}_F$; doing so, however, introduces additional unwanted p-divisibility in the Petersson norm of $\mathbf{Y}$ which we cannot control.

representations $\boldsymbol{\pi}_i$ are trivial. Then the arithmetic Yoshida lift $\mathbf{Y}(\{\mathbf{f}_1^\bullet\},\{\mathbf{f}_2^\bullet\})$ is not identically zero.

We should mention that the non-vanishing of the Yoshida lift has been studied by Yoshida [Yos80] and by Böcherer–Schulze-Pillot [BSP91], [BSP97]. Our result consider some cases not addressed in these works. Also our proof is fundamentally different from theirs.

For non-vanishing modulo $\mathfrak{p}$, however, we can only apply such an argument using Galois conjugation assuming Artin's conjecture on primitive roots. This gives the following conditional result:

Theorem 4.5.6. *Suppose that*

- *$p > k$;*
- *the cuspidal automorphic representations $\boldsymbol{\pi}_i$'s have level dN relatively prime to p (so $r = 0$);*
- *the central characters $\varepsilon_{\boldsymbol{\pi}_i}$ are trivial;*
- *and that Conjecture 6.2.6 holds.*

Under these assumptions, if the $\mathfrak{p}$-integral compatible families of automorphic forms $\{\mathbf{f}_i^\bullet\}$ attached to $\boldsymbol{\pi}_i$ are non-Eisenstein at $\mathfrak{p}$ in the sense of Definition 2.3.5 for $i = 1,2$, then the arithmetic Yoshida lift $\mathbf{Y}(\{\mathbf{f}_1^\bullet\},\{\mathbf{f}_2^\bullet\})$ is non-zero modulo $\mathfrak{p}$ in the sense that the image of $\mathscr{Y} = \mathscr{Y}(\{\mathbf{f}_1^\bullet\},\{\mathbf{f}_2^\bullet\})$ under the reduction map

$$H^0(\bar{\mathscr{M}}_{2,N},\bar{\omega}^{\otimes k}\otimes_{\mathbf{Z}[\frac{1}{N},\zeta_N]}\bar{\mathbf{Z}}_{(\mathfrak{p})}) \to H^0(\bar{\mathscr{M}}_{2,N},\bar{\omega}^{\otimes k}\otimes_{\mathbf{Z}[\frac{1}{N},\zeta_N]}(\bar{\mathbf{Z}}_{(\mathfrak{p})}/\mathfrak{p}))$$

is not zero.

Here the condition that $\mathbf{f}_i^\bullet$ is non-Eisenstein at $\mathfrak{p}$ essentially means it is not constant modulo $\mathfrak{p}$.

Skinner has suggested an idea for removing the condition on Artin's conjecture in some cases. We will consider this in an subsequent work.

0.2.3 Other highlights.

Besides the new results obtained above, we also recover the results (c.f. Theorem 4.5.1 and Remark 4.5.2) obtained by Yoshida [Yos80, Theorem 2.7] and Böcherer–Schulze-Pillot [BSP91, Corollary 6.1] characterizing $\mathbf{Y}$ and its L-functions in terms of the pair of automorphic forms $\mathbf{f}_i^\bullet$ on $D^\times$. Our proofs of these results employ some fairly recent results from the theory of Howe duality as well as the local Langlands correspondence for GSp_4. In this regards, our proofs are perhaps more illuminating as they manifest the results as special instances of general theories.

0.3 A synopsis of the contents

We provide a roadmap of the thesis to orient the readers and to single out some points of interest.

0.3.1 Chapter 1.

The first chapter concerns the structure theory of definite quaternion algebras and their multiplicative groups over $\mathbf{Q}$. We introduce here the notion of an *S-basis* for a definite quaternion algebra D (Definition §1.1.1). With respect to such an S-basis $(\delta, \jmath)$, we define a *weight embedding* (§1.1.5)

$$\epsilon_{\mathrm{wt}}^{\delta,\jmath}\colon D_F = D \otimes_{\mathbf{Q}} F_\jmath \hookrightarrow \mathrm{M}_2(K_\jmath^\delta)$$

for $F_\jmath$ a totally real field and $K_\jmath^\delta$ a CM-field over $F_\jmath$, as well as some *arithmetic embeddings* (§1.3.2 and §1.3.3)

$$\epsilon_{\mathrm{ar}}^{\delta,\jmath}\colon D_l = D \otimes_{\mathbf{Q}} \mathbf{Q}_l \hookrightarrow \mathrm{M}_2(\mathbf{Q}_l)$$

at each prime l in S. The weight embedding is used to define representations $\varrho^{\delta,\jmath}$ of $D_F^\times$ (§1.4); the arithmetic embeddings, on the other hand, are used to fix local *Eichler orders* $\mathscr{D}_l^{\delta,\jmath,S}$ at the primes $l \in S$ (§1.3). We also describe some specific models for $\varrho = \varrho_{2k}$, which germinate from representations of $\mathrm{PGL}_2(\mathbf{Z})$ (§1.2). Lastly, we fix some p-integral lattices inside $\varrho \otimes \mathbf{C}_p$, which will provide integral structures for the automorphic forms on $D^\times$. There are some more technical sections where we check that different weight embeddings are conjugate to each other (§1.4.2), and relate the weight embedding to the arithmetic embeddings at the special prime p (§1.4.4).

0.3.2 Chapter 2.

The second chapter provides the backgrounds on automorphic representations $\boldsymbol{\pi} \simeq \otimes' \pi_v$ on $D^\times$ (§2.2). The representations $\varrho^{\delta,\jmath}$ from Chapter Chapter 1. provide models for the archimedean component π_∞ of $\boldsymbol{\pi}$, and the specific *automorphic forms* $\mathbf{f}^{\delta,\jmath}$ on

$D^\times$ (2.3) that we compute with are vectors in $\boldsymbol{\pi} \otimes (\check{\varrho}_\infty^{\delta,\jmath})$ where $\check{\varrho}_\infty^{\delta,\jmath}$ is the contragredient of $\varrho_\infty^{\delta,\jmath} = \varrho^{\delta,\jmath} \otimes \mathbf{C}$. Moreover, we choose $\mathbf{f}^{\delta,\jmath}$ so it is invariant under right-translation by elements in $\mathscr{D}_l^{\delta,\jmath,S,\times}$ at all primes $l \in S$, and it turns out these conditions make the choice of $\mathbf{f}^{\delta,\jmath}$ unique up to scaling.

This does not yet account for the choice of the S-basis $(\delta, \jmath)$, and we check that the automorphic forms chosen above for different S-bases are indeed (up to scaling) conjugate to each other (§2.3.3). This motivates the notion of a *compatible set* of automorphic forms (Definition 2.3.2).

We describe the notions of a compatible set of algebraic (Definition 2.3.3) and $\mathfrak{p}$-integral (Definition 2.3.4) automorphic forms in the remaining sections. Some work is needed to ensure these notions are well-defined (§2.3.7). We finally conclude by introducing a mild condition on the $\mathfrak{p}$-integral automorphic forms needed for showing the non-vanishing modulo $\mathfrak{p}$ of the Yoshida lift (§2.3.8).

0.3.3 Chapter 3.

The third chapter has two parts. The first part focuses on the four different groups of orthogonal type associated with D (§3.1). We reduce the representation theory of the special orthogonal similitude group $\mathrm{GSO}(D)$ to that of $D^\times$ (§3.3), and fix a root systems together with a set of simple roots for the maximal compact subgroup $\mathrm{SO}(D)_\infty$ of $\mathrm{GSO}(D)$ (3.3.5). The key to take away is that (automorphic) representations on $\mathrm{GSO}(D)$ are simply a product of two (automorphic) representations on $D^\times$ whose central character are the reciprocal of each other (§3.3.1). This allows us to describe the Satake parameters of automorphic representations on $\mathrm{GSO}(D)$ in terms of those of automorphic representations on $D^\times$ (§3.3.7). For the sake of completeness, we describe how to go from representations on $\mathrm{GSO}(D)$ to those on $\mathrm{GO}(D)$ (§3.3.3); we also check that the Petersson pairing on $\mathrm{GO}(D)$ is compatible with that on $\mathrm{GSO}(D)$ under restriction (§3.3.6).

The second part of this chapter revolves around GSp_4, the sympletic group of rank 2 over $\mathbf{Q}$ (§3.4.1). We introduce the specific kinds of automorphic forms on GSp_4 (§3.4.6) that we consider, and discuss their representation theory (§3.4.7) with emphasis on the archimedean place (§3.4.8). This is used later on to verify that the Yoshida lift is holomorphic. We also describe the Satake parameters and the two types of partial L-functions (§3.4.9) associated with these automorphic representations.

Next we discuss the classical theory of Siegel modular forms of degree 2 (§3.4.11), as well as their Fourier coefficients (§3.5.1). The Bessel model (§3.5.5) is also introduced, and we relate its value at the identity with a linear combination of Fourier coefficients (§3.5.6).

We conclude this chapter with some tidbits in the arithmetic theory of Siegel modular forms, with emphasis on the q-expansion principle (§3.6).

0.3.4 Chapter 4.

The fourth chapter begins with a survey of the theory of Weil representations (§4.1). In view of future work, we discuss this in slightly greater generality than what is needed. The materials in this section are not necessary for understanding either the main results or the proofs.

What we do use extensively is the restriction $\boldsymbol{\omega} = \boldsymbol{\omega}_\psi$ of the Weil representation to the product $\mathrm{Sp}_4(\mathbf{A}) \times \mathrm{O}(D)_{\mathbf{A}}$ (§4.2). We write down the transformation laws for the Schrödinger model of $\boldsymbol{\omega}$ (§4.2.2), and extend these formulas to a larger group (§4.2.3). Such an extension is necessary for the Yoshida lift to live on $\mathrm{GSp}_4(\mathbf{A})$ and not just $\mathrm{Sp}_4(\mathbf{A})$, and only then can we discuss the Bessel model for the Yoshida lift.

The extended $\boldsymbol{\omega}$ defines a bijection between some admissible representations of $\mathrm{Sp}_4(\mathbf{Q}_v)$ and those of $\mathrm{GO}(D)_v$ (§4.2.4), and the theta lifts (§4.2.5) and its vector-valued version (§4.2.6) are in some sense a global realization of this bijection. There is a wealth of knowledge on the representation-theoretic aspects of the space generated by these lifts, and we assemble some of the results that are relevant to our case (§4.3). In particular, we describe the Satake parameters of the Yoshida lift (as a representation) in terms of the Satake parameters of the automorphic representation $\boldsymbol{\pi}_i$ on $D^\times$ being lifted.

Next we pick out a specific theta kernel $\Theta_{2k}^{\delta,\jmath,\mathscr{D}}$ (§4.4), which depends a number of choices, among which is the choice of an S-basis $(\delta, \jmath)$. This is expected since we want Θ to carry some $\varrho_\infty^{\delta,\jmath}$-action by $\mathrm{SO}(D)_\infty$ and this representation $\varrho_\infty^{\delta,\jmath}$ is cooked up from representations of $D_\infty^\times$.

Finally, we reap the fruits of our ground work—we define the Yoshida lift, verify that it is indeed independent of all the choices, and state the main theorems (§4.5). The proofs of these theorems occupy Chapter 5, Chapter 6, and also earlier sections of Chapter 4 (§4.3).

0.3.5 Chapter 5 and Chapter 6.

Since we have discussed the methods of proof earlier in this introduction (§0.2), we do not repeat it here. It is worth mentioning, in case the readers are concerned with the vast amount of integrations performed in Chapter 5, that these integrals are in a sense a red herring. In essence, they represent nothing more than a choice of exposition, since the groups we integrate over are either compact or compact modulo center, and the function we integrate over are often right-invariant by some open subgroup; consequently, we could potentially replace the integrals by finite sums, but this probably would do more harm than good.

0.4 Notations and conventions

0.4.1 Fields.

We fix an algebraic closure $\bar{\mathbf{Q}}$ of $\mathbf{Q}$. All algebraic field extensions of $\mathbf{Q}$ will be viewed as subfields of $\bar{\mathbf{Q}}$. We also fix an embedding $\bar{\mathbf{Q}} \hookrightarrow \mathbf{C}$. Note that the image of $\mathrm{Aut}(\mathbf{C}/\mathbf{R})$ gives an order 2 element in $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$. To avoid unnecessary confusion later, for any integer n we fix a choice of $\sqrt{n}$ in $\bar{\mathbf{Q}}$. Set $\imath = \sqrt{-1}$.

For a prime q in $\mathbf{Q}$, we fix an embedding of $\bar{\mathbf{Q}} \to \mathbf{C}_q$ and, unless otherwise specified, denote by the corresponding fraktur character $\mathfrak{q}$ the maximal ideal cut out by the the open unit disk in $\mathbf{C}_q$. Here $\mathbf{C}_q$ is the completion of $\bar{\mathbf{Q}}_q$ with respect to the absolute value normalized so that $|q| = \frac{1}{q}$. For a field extension $L/\mathbf{Q}$ in $\mathbf{C}_q$, we set $\mathfrak{q}_L = \mathfrak{m}_{\mathbf{C}_q} \cap \mathscr{O}_L$ and denote by $L_{\mathfrak{q}_L}$ or simply $L_{\mathfrak{q}}$ the completion of L in $\mathbf{C}_q$. This in particular applies to algebraic extensions of $\mathbf{Q}$.

At a prime q, denote by Frob_q the geometric Frobenius element in $\mathrm{Gal}(\bar{\mathbf{Q}}_q/\mathbf{Q}_q)^{\mathrm{ab}}$. The Weil group W_q of $\mathbf{Q}_q$ is the extension

$$1 \to I_q \to W_q \to \mathrm{Frob}_q^{\mathbf{Z}} \to 1$$

of the free abelian group $\mathrm{Frob}_q^{\mathbf{Z}}$ by the inertia subgroup $I_q \subset \mathrm{Gal}(\bar{\mathbf{Q}}_q/\mathbf{Q}_q)$. We equip W_q with the topology so that I_q is open.

By "primes", we will always mean non-archimedean (or finite) primes. The archimedean ones will be referred to as "archimedean places".

0.4.2 Groups and algebras.

Given G a linear algebraic group (scheme) defined over $\mathbf{Z}$, we use Z_G to denote the center of G. Given any $\mathbf{Z}$-algebra R, we denote the R-valued points of G by $G(R)$ or simply by G_R when there is no risk of confusing it with $G \times \mathrm{Spec}(R)$. The topology on $G(R)$ (and G_R) will be the one inherited from the topology on R. In particular, if $R \simeq \mathbf{Q}_v$, then we set $G_v = G(\mathbf{Q}_v)$.

For (ρ, V_ρ) a representation of $G(F)$, $(\check{\rho}, \check{V}_\rho)$ will be the corresponding contragredient representation. We denote by $V_\rho^{G(F)}$ the subspace of V_ρ fixed point-wise by $G(F)$ under ρ. When there is no risk of confusion, we suppress the actual representation and denote by $g \cdot v$ the action of $g \in G(F)$ on $v \in V_\rho$ through ρ.

We denote the $m \times n$ matrices with entries in a commutative ring R by $\mathrm{M}_{m\times n}(R)$, or simply $\mathrm{M}_n(R)$ if $m = n$. We use $\mathrm{SM}_n(R)$ to denote the subset of $n \times n$ symmetric matrices in $\mathrm{M}_n(R)$. Given any matrix $A \in \mathrm{M}_{m\times n}$, ${}^tA$ denotes the transpose of A. If R is equipped with an involution $x \mapsto \bar{x}$, then we use ${}^t\bar{A}$ to denote the conjugate transpose of A obtained by applying the involution to each of the entries of ${}^tA$.

We denote by $\mathrm{diag}[a_1, \dots, a_n]$ the diagonal matrix

$$\begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix}.$$

For convenience, we shall refer to an injective homomorphism of R-algebras as an *embedding*. We also frequently suppress subscripts and superscript on objects (e.g., a bilinear forms $(\bullet, \bullet)_V$ on a vector space V, the center Z_G of G, etc.) when there is no danger of confusion.

0.4.3 Adeles and the standard character

We denote by $\mathbf{A}$ the adeles of $\mathbf{Q}$, and $F_{\mathbf{A}}$ for the adeles of a number field $F/\mathbf{Q}$. We use v to denote a general place of F. For a $\mathbf{Q}$-vector space V, we put $V_{\mathbf{A}} = V \otimes_{\mathbf{Q}} \mathbf{A}$ and $V_f = V \otimes \mathbf{A}_f$ where $\mathbf{A}_f$ is the finite adeles of $\mathbf{Q}$. Similarly, for a $\mathbf{Z}$-module B, we put $B_f = B \otimes_{\mathbf{Z}} \hat{\mathbf{Z}}$.

We denote by $\psi\colon \mathbf{A}/\mathbf{Q} \to \mathbf{C}^1$ the *standard additive character* on $\mathbf{A}$ trivial on $\mathbf{Q}$. So at a finite place $v = p$, $\psi_p\colon \mathbf{Q}_p \to \mathbf{Q}_p/\mathbf{Z}_p \to \mathbf{C}^1$ is the character given by

$$\psi_p(a) = \exp(-2\pi\imath \cdot \mathrm{pr}(a))$$

where $\mathrm{pr}(a)$ is the image of a in $\mathbf{Q}_p/\mathbf{Z}_p$; and at the infinite place we have

$$\psi_\infty(a) = \exp(2\pi\imath a).$$

Given any $\mathbf{Q}$-vector space V together with a linear map $T\colon V \to \mathbf{Q}$, the composition $\psi \circ T$ defines a character on $V_{\mathbf{A}}/V$, which we frequently abbreviate as ψ_T. In particular, this applies to $\mathrm{M}_n(\mathbf{Q})$ as well as $\mathrm{SM}_n(\mathbf{Q})$ where we take T to be the standard trace map.

0.5 Measures

Although the normalization of Haar measures and invariant measures is not essential to the present work, we nevertheless specify them in view of subsequent works on computing the Rallis inner product. All the measures are chosen to be *right-invariant* under the corresponding group transformations (although this is immaterial as we will only be integrating over unimodular groups).

We begin with the various measures associated with a number field F.

0.5.1 Local additive measures.

Let v be a place of F lying above the place w of $\mathbf{Q}$. We fix an additive Haar measure on F_v in the standard way. Namely,

$$dx_v = \begin{cases} \text{the Haar measure normalized so } \operatorname{vol}(\mathscr{O}_{F,v}) = 1 & \text{if } v \text{ is non-archimedean;} \\ \text{the Lebesgue measure } dx & \text{if } v = \mathbf{R},\ x_v \mapsto x; \\ |dx \wedge d\bar{x}| = 2dx_1 dx_2 & \text{if } v = \mathbf{C} \text{ and } x_v \mapsto x_1 + \imath x_2. \end{cases}$$

Remark 0.5.1. The Haar measure chosen above is *self-dual* with respect to the standard character $\psi_{F,v} = \psi_w \circ \operatorname{tr}_{F/\mathbf{Q}}$ of F_v in the sense that the Fourier transform

$$\hat{\varphi}(x_v) = \int_{F_v} \varphi(y_v)\psi_{F,v}(x_v y_v)\, dy_v$$

for any Bruhat–Schwartz function φ satisfies the identity $\hat{\hat{\varphi}}(x_v) = \varphi(-x_v)$.

0.5.2 Local multiplicative measures.

We also fix the Haar measure $d^\times x_v$ on $F_v^\times$ by

$$d^\times x_v = \begin{cases} \zeta_v(1)\frac{dx_v}{|\mathrm{n}(x_v)|} & \text{if } v \text{ is non-archimedean;} \\ \frac{dx_v}{|x_v|} = \frac{dx}{|x|} & \text{if } v = \mathbf{R},\ x_v \mapsto x; \\ \frac{dx_v}{|x_v|^2} = \frac{2dx_1dx_2}{x_1^2+x_2^2} & \text{if } v = \mathbf{C},\ x_v \mapsto x_1 + \imath x_2. \end{cases}$$

Here in the non-archimdean case $\zeta_v(s) = \frac{1}{1-N_v^{-s}}$, where $N_v = [\mathscr{O}_v : \varpi_v \mathscr{O}_v]$ is the cardinality of the residue field of F at v. Also $\mathrm{n}\colon F_v^\times \to \mathbf{Q}_w^\times$ is the norm map. We note that $\operatorname{vol}(\mathscr{O}_{F,v}^\times) = 1$ with respect to this normalization.

0.5.3 Global measures.

The Tamagawa measure dx on $F_{\mathbf{A}}$ (resp. $d^\times x$ on $F_{\mathbf{A}}^\times$) is then the unique Haar measure that induces on each standard open subgroup $\prod_{v\in S} F_v \times \prod_{v\notin S} \mathscr{O}_v$ (resp. $\prod_{v\in S} F_v^\times \times \prod_{v\notin S} \mathscr{O}_v^\times$) the product measure $\sqrt{\mathrm{disc}(F)^{-1}}\prod_v dx_v$ (resp. $\sqrt{\mathrm{disc}(F)^{-1}}\prod_v d^\times x_v$) where $\mathrm{disc}(F)$ is the discriminant of F. Under these normalizations, $\mathrm{vol}(F_{\mathbf{A}}/F) = 1$ and

$$\mathrm{vol}(F_{\mathbf{A}}^1/F^\times) = \frac{2^{r_1}(2\pi)^{r_2}hR}{e\cdot \mathrm{disc}(F)^{\frac{1}{2}}} = \operatorname*{res}_{s=1}\zeta_F(s)$$

where r_1 (resp. r_2) is the number of real (resp. complex) embeddings, h is the class number of F, R is the regulator of F, and e is the number of roots of unity in $\mathscr{O}_F$ [Wei95, §VII-6, Proposition 12]. In particular, for $F = \mathbf{Q}$, we have $\mathrm{vol}(\mathbf{A}^1/\mathbf{Q}^\times) = 1$.

0.5.4 Compatible measures.

We state the following theorem for the record.

Theorem 0.5.1. *Let G be any locally compact topological group and H a closed subgroup of G. Let Δ_G and Δ_H be the corresponding modulus characters. Then a necessary and sufficient condition for G/H to admit a nonzero G-invariant Borel measure is that the restriction of Δ_G to H equals Δ_H. In this case such a measure $d\bar{g}$ is uniquely determined up to a scalar, and it can be normalized so that*

$$\int_G f(g)\,dg = \int_{H\backslash G}\left(\int_H f(hg)\,dh\right)d\bar{g}$$

or symbolically,

$$dg = dh d\bar{g},$$

for all $f \in L^1(G)$.

Proof. [Wei40, §9]. □

We shall refer to the right invariant measure $d\bar{g}$ thus obtained as the measure *compatible* with dg and dh. In particular, the theorem applies in the case that G and H are unimodular.

We will make frequent use of the above theorem in the following context. Suppose N is a closed subgroup of G and we have a short exact sequence of topological groups:

$$1 \to N \xrightarrow{i} G \xrightarrow{j} H \to 1,$$

where i is the natural inclusion. We can normalize the Haar measure dn on N so that it is compatible with the Haar measures dg and dh on G and H respectively, that is,

$$dg = dn\,dh.$$

Caution 0.5.2. We caution the reader that this normalization depends on the short exact sequence. If we post-compose j with an automorphism σ of H, the normalization will change by the modulus of the automorphism σ.

As such, we denote by dx (resp. $d^\times x$) the invariant measure on $F_{\mathbf{A}}/F$ (resp. $F_{\mathbf{A}}^\times/F^\times$) compatible with the Haar measure dx on $F_{\mathbf{A}}$ (resp. $d^\times x$ on $F_{\mathbf{A}}^\times$) and the discrete measure on F (resp. $F^\times$). We have that $\mathrm{vol}(F_{\mathbf{A}}/F) = 1$.

Chapter 1. Quaternion Algebras

1.1 Structures of quaternion algebras

We recall some facts concerning quaternion algebras and introduce some notations in passing.

1.1.1 Definite quaternion algebras.

Let d be a product of an odd number of distinct primes, and let D be the unique definite quaternion algebra defined over $\mathbf{Q}$ with discriminant d. We denote by $x \mapsto \bar{x}$ the main involution on D. Let $D^\times$ be the multiplicative group of D viewed as an algebraic group over $\mathbf{Q}$. The reduced norm $\mathrm{n}(x) = x\bar{x}$ defines a homomorphism from $D^\times$ to $\mathbf{G}_m$, and we denote its kernel by D^1. Correspondingly, the reduced trace $\mathrm{tr}(x) = x + \bar{x}$ is a linear map from $D = \mathrm{Lie}(D^\times)$ to $\mathbf{Q} = \mathrm{Lie}(\mathbf{G}_m)$ with kernel $D^{(0)} = \mathrm{Lie}(D^1)$.

1.1.2 Sub-algebras in D.

We can decompose D according to its sub-algebras as follows. Let

$$m(X) = X^2 + bX + c$$

be an irreducible monic polynomial with coefficients in $\mathbf{Q}$. Denote by $\Delta_m = b^2 - 4c$ the discriminant of $m(X)$, and let $\sqrt{\Delta_m} = \sqrt{b^2 - 4c}$ be the square root of Δ_m in $\bar{\mathbf{Q}}$ fixed in §0.4.1, then $K^m = \mathbf{Q}(\sqrt{\Delta_m})$ is the splitting field of $m(X)$ in $\bar{\mathbf{Q}}$. By the theory of local embeddings [Vig80, Ch. III, Théorème 3.8], K^m embeds into D as a sub-algebra if and only if K^m is imaginary quadratic and is not split at each of the primes dividing $d = \mathrm{disc}(D)$.

1.1.3 Basis for D.

Suppose K^m embeds into D, then such an embedding is equivalent to a choice of a root $\delta = \delta^m$ of $m(X)$ in D. Since $m(X)$ does not determine $\delta \in D$ uniquely, we set $\Delta_\delta = \Delta_m$, $\dot{\sqrt{\Delta_\delta}} = \delta - \bar{\delta}$, and $K^\delta = \mathbf{Q}(\dot{\sqrt{\Delta_\delta}}) = \mathbf{Q}(\delta)$ for bookkeeping. We have an orthogonal decomposition

$$D = K^\delta \perp K^{\delta,\perp}$$

with respect to the symmetric bilinear form $(x, y) = \mathrm{tr}(x\bar{y})$. Note that $K^{\delta,\perp} = \jmath K^\delta$ for any element $\jmath \in K^{\delta,\perp}$. Since δ and $\jmath$ determine the basis $\{1, \delta, \jmath, \jmath\delta\}$ of D as a vector space over $\mathbf{Q}$, we shall refer the pair $(\delta, \jmath)$ as a *basis* for D. We distinguish a special class of such bases for D:

Definition 1.1.1. Let S be a finite set of primes not dividing d. The pair $(\delta, \jmath)$ defined above is an S-*basis* for D if it satisfies the following condition at every prime $l \in S$: if l is not split in K^δ, then $-\mathrm{n}(\jmath)$ is square in $\mathbf{Q}_l$.*

Since D is split at the primes in S, such an S-basis $(\delta, \jmath)$ exists. Indeed, for any choice of $\jmath$, the Hilbert symbol $(-\mathrm{n}(\jmath), -\mathrm{n}(\jmath)\mathrm{n}(\delta))_l$ is equal to 1 for all $l \in S$ [O'M00, 57:9]. In other words,

$$-\mathrm{n}(\jmath) \cdot x_l^2 - \mathrm{n}(\jmath)\mathrm{n}(\delta) \cdot y_l^2 = z_l^2$$

for some $x_l, y_l, z_l \in \mathbf{Q}_l$ with $z_l \in \mathbf{Z}_l^\times$. It is a direct consequence of weak approximation that given any $\epsilon > 0$, there are rational numbers $x, y \in \mathbf{Q}$ such that $|x^2 - x_l^2| < \epsilon$ and $|y^2 - y_l^2| < \epsilon$ for all $l \in S$. When $\epsilon \le \frac{1}{l}$, we see that $-\mathrm{n}(\jmath) \cdot x^2 - \mathrm{n}(\jmath)\mathrm{n}(\delta) \cdot y^2 \equiv z_l \pmod{l}$, which implies that $X^2 = -\mathrm{n}(\jmath) \cdot x^2 - \mathrm{n}(\jmath)\mathrm{n}(\delta) \cdot y^2$ admits a solution† in $\mathbf{Z}_l$ by Hensel's lemma; thus replacing $\jmath$ by $\jmath(x + y \cdot \delta)$ gives an element in $K_l^{\delta,\perp}$ whose norm is the negative of a square in $\mathbf{Q}_l$.

We adopt the following convention to keep the notations under control.

Convention 1.1.2. When working with a *fixed* basis $(\delta, \jmath)$, we denote by $\dot{K}$ the subalgebra K^δ of D and by K the imaginary quadratic subfield K^m of $\bar{\mathbf{Q}}$. Similarly, given an element $t \in K = K^m$, we denote by $\dot{t}$ its image in $\dot{K} = K^\delta$ under the embedding $K \to D$ sending $\sqrt{\Delta_m}$ to $\dot{\sqrt{\Delta_\delta}}$. We also suppress the quantifier $(\delta, \jmath)$ on all notations in this situation as there is no risk for confusion. Similarly, we suppress specifying a choice of S when this choice has no bearing on the discussion.

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*See §1.3.3 for the motivation behind this condition.

†In the case $l = 2$, we need to choose $\epsilon \le \frac{1}{8}$, then $-\mathrm{n}(\jmath) \cdot x^2 - \mathrm{n}(\jmath)\mathrm{n}(\delta) \cdot y^2 \in 1 + 8\mathbf{Z}_2$ and is therefore a square [FT93, (3.8)].

1.1.4 $D^\times$ as a unitary similitude group.

The two-dimensional right K-vector space $D = \dot{K} \perp \jmath\dot{K}$ comes equipped with a hermitian form (with values in K) given by

$$(x,y)_{\delta,\jmath} = \mathrm{pr}(\bar{y}x)$$

for $x, y \in D$. Here $\mathrm{pr}\colon D = \dot{K} \perp \jmath\dot{K} \to \dot{K} \simeq K$ is the projection onto the first summand. Denote by $\bar{\mathrm{G}}\mathrm{U}(D, (\bullet,\bullet)_{\delta,\jmath})$ the algebraic group over $\mathbf{Q}$ defined by

$$\begin{aligned}&\bar{\mathrm{G}}\mathrm{U}(D,(\bullet,\bullet)_{\delta,\jmath})(A)\\ &\qquad = \{g \in \mathrm{GL}_{K\otimes A}(D\otimes A) : (g\cdot x, g\cdot y)_{\delta,\jmath} = \det(g)\cdot(x,y)_{\delta,\jmath} \text{ for all } x,y \in D\otimes A\}\end{aligned}$$

for any commutative $\mathbf{Q}$-algebra A. The left multiplication by $D^\times$ on D defines an injective homomorphism $D^\times \to \bar{\mathrm{G}}\mathrm{U}(D,(\bullet,\bullet)_{\delta,\jmath})$ which is in fact an isomorphism of algebraic groups over $\mathbf{Q}$ [Gro04, §5].

1.1.5 The weight embedding.

Set $F = \mathbf{Q}(\sqrt{\mathrm{n}(\jmath)})$. For a $\mathbf{Q}$-algebra A, denote by $A_\jmath = A \otimes F$ the F-algebra obtained by base change.* Set

$$\bar{\mathrm{G}}\mathrm{U}_2^{\delta,\jmath} = \bar{\mathrm{G}}\mathrm{U}(D,(\bullet,\bullet)_{\delta,\jmath}) \times_{\mathrm{Spec}(\mathbf{Q})} \mathrm{Spec}(F),$$

it is naturally the rational unitary similitude group (over F) attached to the two-dimensional right $K_\jmath$-hermitian vector space $D_\jmath$. As in the previous section, we have $D_\jmath^\times \simeq \bar{\mathrm{G}}\mathrm{U}_2^{\delta,\jmath}$ as algebraic groups over F.

Set $\tilde{\jmath} = \jmath/\sqrt{\mathrm{n}(\jmath)}$, it is an element in $D_\jmath$ of reduced norm 1. With respect to the ordered orthonormal basis $\{1,\tilde{\jmath}\}$ of $D_\jmath$ as a (right) $K_\jmath$-vector space, the hermitian form $(\bullet,\bullet)_{\delta,\jmath}$ corresponds to the identity matrix, and we have a matrix representation

$$\bar{\mathrm{G}}\mathrm{U}_2^{\delta,\jmath}(A) \simeq \{g \in \mathrm{GL}_{2,K_\jmath\otimes A}(K_\jmath \otimes A) : {}^t g\bar{g} = \det(g)\}$$

for any commutative F-algebra A. We shall refer to the induced injective homomorphism

$$\epsilon_{\mathrm{wt}}^{\delta,\jmath}\colon D^\times \to \mathrm{GL}_{2,K_\jmath}$$

*See §1.4.2 for motivation behind introducing this auxiliary real quadratic field.

as the *weight embedding*. It is explicitly given by

$$\dot{\alpha} + \jmath\dot{\beta} \mapsto \begin{bmatrix} \alpha & -\sqrt{\mathrm{n}(\jmath)} \cdot \bar{\beta} \\ \sqrt{\mathrm{n}(\jmath)} \cdot \beta & \bar{\alpha} \end{bmatrix}$$

for $\dot{\alpha} + \jmath\dot{\beta} \in D = \dot{K} \perp \tilde{\jmath}\dot{K}$.

1.1.6 $PD^\times$ as an orthogonal group.

The adjoint action by $\alpha \in D^\times$ on $x \in D^{(0)} = \mathrm{Lie}(D^1)$, $\mathrm{Ad}(\alpha) \cdot x = \alpha x \alpha^{-1}$, factors through the center $Z_{D^\times} \simeq \mathbf{G}_m$ and preserves the reduced norm. We thus obtain an injective homomorphism of algebraic groups over $\mathbf{Q}$,

$$PD^\times \xrightarrow{\mathrm{Ad}} \mathrm{SO}(D^{(0)}, \mathrm{n}),$$

where $PD^\times = Z_{D^\times} \backslash D^\times$ and $\mathrm{SO}(D^{(0)}, \mathrm{n})$ is the special orthogonal group for the quadratic space $(D^{(0)}, \mathrm{n})$. Since $\mathrm{SO}(D^{(0)}, n)$ is connected and has the same dimension as $PD^\times$, we see this is in fact an isomorphism.

Denote by $\boldsymbol{\iota}$ the subgroup of order 2 in $\mathrm{O}(D^{(0)}, \mathrm{n})$ generated by the main involution on D. If ι is the non-trivial element in $\boldsymbol{\iota}$, then we have

$$\iota \cdot \mathrm{Ad}(\bar{\alpha}^{-1}) \cdot (x) = \mathrm{Ad}(\alpha) \cdot (\iota \cdot x)$$

for all $\alpha \in D^\times$ and $x \in D^{(0)}$. It follows that $PD^\times \rtimes \boldsymbol{\iota} \simeq \mathrm{O}(D^{(0)}, \mathrm{n})$ with respect to the multiplication

$$(\alpha \rtimes \iota) \cdot (\beta \rtimes \kappa) = \alpha\beta^\iota \rtimes \iota\kappa,$$

with $\alpha^\iota = \bar{\alpha}^{-1}$ if ι is the non-trivial element.

1.2 Representations of PGL_2

We describe two families of finite-dimensional representations of $\mathrm{PGL}_2 = \mathrm{PGL}_2(\mathbf{Z})$ with coefficients in $\mathbf{Z}$. We chose to not use the language of group schemes since such a discussion would introduce confounding subtleties and unnecessary technicalities. It is helpful, however, to keep in mind that the representations we define are functorial in their coefficient rings. We fix an integer $k \in \mathbf{Z}_{\geq 0}$.

1.2.1 Induced representation of the Lie algebra.

The (algebraic) Lie algebra of PGL_2 is $\mathrm{M}_2^{(0)} = \mathrm{M}_2^{(0)}(\mathbf{Z})$ and has a basis over $\mathbf{Z}$ given by

$$H = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \qquad Y^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad Y^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Let (ϱ, V_ϱ) be a finite-dimensional representation of PGL_2 defined over a commutative ring R, and let $R[\epsilon]$ be the corresponding ring of dual numbers obtained by adjoining to R an element ϵ such that $\epsilon^2 = 0$. The $\mathrm{M}_2^{(0)}$-action on $V_\varrho \otimes R[\epsilon]$ formally defined by

$$\varrho(Y)v = \frac{\varrho(1 + \epsilon Y)v - v}{\epsilon}$$

preserves V_ϱ, thus induces an action by $\mathrm{M}_2^{(0)}$ on V_ϱ which we denote by $(d\varrho, V_\varrho)$.

1.2.2 Symmetric powers.

Let $\mathbf{Z}[X]_{\deg\leq 2k}$ be the $\mathbf{Z}$-module of polynomials of degree at most $2k$ with integral coefficients. We follow [Che05] and define two actions by $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2 = \mathrm{GL}_2(\mathbf{Z})$ on $\mathbf{Z}[X]_{\deg\leq 2k}$. First define two functions from GL_2 to $\mathbf{Z}[X]_{\deg\leq 2k}$,

$$j(g) = (bX + d)^2 \qquad \text{and} \qquad \check{j}(g) = (-cX + a)^2.$$

Then given $f(X) \in \mathbf{Z}[X]_{\deg\leq 2k}$, we define

$$\sigma_{2k}(g) \cdot f(X) = \det(g)^{-k} j(g)^k f\left(\frac{aX + c}{bX + d}\right),$$

and

$$\check{\sigma}_{2k}(g) \cdot f(X) = \det(g)^{-k} \check{j}(g)^k f\left(\frac{dX - b}{-cX + a}\right).$$

As the notation suggests, $\check{\sigma}_{2k}$ is dual to σ_{2k} (but only after base change to $\mathbf{Z}[\frac{1}{k!}]$). Furthermore, in each case the action factors through the center of GL_2 and therefore defines a representation of PGL_2. We denote these representations by $(\sigma_{2k}, \mathscr{V}_{2k})$ and $(\check{\sigma}_{2k}, \check{\mathscr{V}}_{2k})$ respectively.

The monomials,

$$\mathbf{t}_i^k = X^{k+i}$$

for $i = -k, \cdots, k$, define a basis of $\mathscr{V}_{2k}$ and $\check{\mathscr{V}}_{2k}$ over $\mathbf{Z}$. With respect to this basis, the unique PGL_2-equivariant pairing on $\mathscr{V}_{2k} \times \check{\mathscr{V}}_{2k}$, normalized so that $\langle \mathbf{t}_0^k, \mathbf{t}_0^k \rangle_{2k} = 1$, is given by

$$\langle \sum_{i=-k}^{k} a_i \mathbf{t}_i^k, \sum_{i=-k}^{k} b_i \mathbf{t}_i^k \rangle_{2k} = \sum_{i=-k}^{k} (-1)^i \frac{(k+i)!(k-i)!}{(k!)^2} a_i b_i. \tag{2.2.1}$$

Note this pairing takes values in $\frac{1}{k!}\mathbf{Z}$.

1.2.3 Harmonic polynomials.

Denote by $\mathbf{Poly}_k$ the space of homogeneous polynomials of degree k on $\mathrm{M}_2^{(0)}$ with coefficients in $\frac{1}{2^k k!}\mathbf{Z}$. It is again a $\mathbf{Z}$-module. The adjoint action by $g \in \mathrm{PGL}_2$ on $\mathrm{M}_2^{(0)}$ extends to an action on $P \in \mathbf{Poly}_k$ by

$$g \cdot P(Y) = P(\mathrm{Ad}(g)^{-1} \cdot Y) = P(g^{-1} Y g).$$

Let $Y = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ be an element in $\mathrm{M}_2^{(0)}$. The quadratic form on $\mathrm{M}_2^{(0)}$ given by $-\det(Y) = a^2 + bc$ is invariant under the adjoint action of PGL_2. A change of basis over $\mathbf{Z}[\frac{1}{2}]$ given by

$$x = a \qquad y = \frac{b+c}{2} \qquad z = \frac{b-c}{2}$$

identifies $-\det(\bullet)$ with the quadratic form $q(x,y,z) = x^2 + y^2 + z^2$. From this, we see that the differential

$$\Delta = q(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

is invariant under PGL_2 in the sense that $\Delta(g \cdot P) = \Delta(P)$ for all $g \in \mathrm{PGL}_2$.

Reverting back to the original coordinate system $\{a, b, c\}$, we find that

$$\Delta = \frac{\partial^2}{\partial a^2} + 4\frac{\partial^2}{\partial b \partial c}$$

and that

$$\mathbf{Harm}_k = \{P \in \mathbf{Poly}_k : \Delta P = 0\}$$

is stable under the adjoint action by PGL_2.

1.2.4 Highest weight theory for $\mathrm{PGL}_2(\mathbf{C})$.

Proposition 1.2.1. *Every complex irreducible representation of* $\mathrm{PGL}_2(\mathbf{C})$ *of dimension* $2k+1$ *is equivalent to* $(\sigma_{2k,\mathbf{C}}, \mathscr{V}_{2k} \otimes \mathbf{C})$.

Proof. Set $\mathbf{t}^k_{k+1} = 0 = \mathbf{t}^k_{-k-1}$, a direct computation shows that

- $d\sigma_{2k}(H) \cdot \mathbf{t}^k_i = 2i \cdot \mathbf{t}^k_i$;
- $d\sigma_{2k}(Y^+) \cdot \mathbf{t}^k_i = (k-i) \cdot \mathbf{t}^k_{i+1}$;
- $d\sigma_{2k}(Y^-) \cdot \mathbf{t}^k_i = (k+i) \cdot \mathbf{t}^k_{i-1}$.

With respect to the basis $\{\frac{(2k)!}{(k+i)!} \cdot \mathbf{t}^k_i\}$ of $\mathscr{V}_{2k} \otimes \mathbf{C}$, we see that $d\sigma_{2k,\mathbf{C}}$ is equivalent to the representation $(\rho_k, F^{(k)})$ of $\mathrm{M}_2^{(0)}(\mathbf{C}) \simeq \mathfrak{sl}_2(\mathbf{C})$ from [GW09, Proposition 2.3.3]. The proposition loc. cit. implies that all irreducible representations of $\mathfrak{sl}_2(\mathbf{C})$ of dimension $2k+1$ are equivalent to $(d\sigma_{2k,\mathbf{C}}, \mathscr{V}_{2k} \otimes \mathbf{C})$. By [GW09, Proposition 2.3.5], we conclude that all irreducible representations of $\mathrm{SL}_2(\mathbf{C})$ of dimension $2k+1$ are equivalent to $(\sigma_{2k,\mathbf{C}}, \mathscr{V}_{2k,\mathbf{C}})$ viewed as a representation of $\mathrm{SL}_2(\mathbf{C})$. The proposition then follows since any irreducible representation of $\mathrm{PGL}_2(\mathbf{C})$ of dimension $2k+1$ is also an irreducible representation of $\mathrm{SL}_2(\mathbf{C})$. □

In particular, we have $(\sigma_{2k,\mathbf{C}}, \mathscr{V}_{2k,\mathbf{C}}) \simeq (\check{\sigma}_{2k,\mathbf{C}}, \check{\mathscr{V}}_{2k,\mathbf{C}})$ as representations of $\mathrm{PGL}_2(\mathbf{C})$ by sending X^i to $(-X)^i$ for $i = 0, \cdots, 2k$.

Remark 1.2.1. In fact, the above proposition holds in greater generality with $\mathbf{C}$ replaced by any field [Jan03, Chapter II.2].

1.2.5 The irreducible submodule $\mathscr{H}_k$.

The preceding proposition shows that $(\mathrm{Ad}_{\mathbf{C}}, \mathbf{Harm}_{k,\mathbf{C}}) \simeq (\check{\sigma}_{2k,\mathbf{C}}, \check{\mathscr{V}}_{2k,\mathbf{C}})$. We like to construct such an isomorphism and distinguish the $\mathbf{Z}$-lattice in $\mathbf{Harm}_{k,\mathbf{C}}$ that corresponds to $\mathscr{V}_k$. To begin, define

$$M_{l,m,n}\left(\begin{bmatrix} a & b \\ c & -a \end{bmatrix}\right) = a^l \left(\frac{b}{2}\right)^m \left(\frac{c}{2}\right)^n.$$

The action by the Lie algebra $\mathrm{M}_2^{(0)}$ on these monomials are given by

- $H \cdot M_{l,m,n} = 2(n-m) \cdot M_{l,m,n}$;
- $Y^+ \cdot M_{l,m,n} = m \cdot M_{l+1,m-1,n} - 2l \cdot M_{l-1,m,n+1}$;

- $Y^- \cdot M_{l,m,n} = 2l \cdot M_{l-1,m+1,n} - n \cdot M_{l+1,m,n-1}$.

In view of action by the Lie algebra, set

$$\nu = \begin{cases} \frac{k}{2} & \text{if } k \text{ is even,} \\ \frac{k-1}{2} & \text{if } k \text{ is odd;} \end{cases}$$

and define the *weight 0* polynomial to be

$$P_0^k = \sum_{i=0}^{\nu} (-1)^i \binom{k}{2i} \binom{2i}{i} M_{k-2i,i,i}.$$

It is up to scaling the unique vector in $\mathbf{Harm}_k$ that is annihilated by the derivation defined by H. The other weight vectors are then given by

$$P_i^k = \begin{cases} \frac{(k+i)!}{k!} (-Y^+)^{-i} P_0^k & \text{if } i < 0, \\ \frac{(k-i)!}{k!} (-Y^-)^{i} P_0^k & \text{if } i > 0. \end{cases}$$

Since $M_{l,m,n}(\mathrm{diag}[a,-a]) = 0$ unless $m = n = 0$, we see that

$$P_i^k \left(\begin{bmatrix} a & \\ & -a \end{bmatrix} \right) = \begin{cases} a^k & \text{if } i = 0; \\ 0 & \text{otherwise.} \end{cases} \tag{2.5.2}$$

We denote by $\mathscr{H}_k$ the free $\mathbf{Z}$-module of rank $2k+1$ spanned by P_i^k. As we shall see below, it is stable under PGL_2. Furthermore, since $\mathscr{H}_{k,\mathbf{C}} = \mathbf{Harm}_{k,\mathbf{C}}$ is irreducible as a representation of $\mathrm{PGL}_2(\mathbf{C})$, $\mathscr{H}_k$ is also irreducible. We denote the corresponding representation of $\mathrm{PGL}_2(\mathbf{Z})$ by $(\rho_k, \mathscr{H}_k)$.

1.2.6 An explicit intertwining map.

Proposition 1.2.2. *The map*

$$\mathbf{t}_i^k \mapsto P_i^k$$

defines a $\mathbf{Z}$*-linear isomorphism* $\check{\mathscr{V}}_{2k} \simeq \mathscr{H}_k$ *that is equivariant for the* PGL_2*-actions* $\check{\sigma}_{2k}$ *and* ρ_k*.*

Proof. Since this map sends a basis of $\check{\mathscr{V}}_{2k}$ to that of $\mathscr{H}_k$, it defines an isomorphism of free $\mathbf{Z}$-modules. Furthermore, $\mathscr{H}_k$ is constructed so that this map intertwines the representations $(d\check{\sigma}_{2k}, \check{\mathscr{V}}_{2k})$ and $(d\rho_k, \mathscr{H}_k)$ of $\mathrm{M}_2^{(0)}(\mathbf{Z})$.† It follows that $(\check{\sigma}_{2k,\mathbf{C}}, \check{\mathscr{V}}_{2k,\mathbf{C}})$

†Indeed, we have

and $(\rho_{k,\mathbf{C}}, \mathscr{H}_{k,\mathbf{C}})$ are equivalent as representations of $\mathrm{PGL}_2(\mathbf{C})$. Consequently, the $\mathbf{Z}$-lattice $\mathscr{H}_k$ in $\mathscr{H}_{k,\mathbf{C}}$ corresponding to the image of $\check{\mathscr{V}}_{2k}$ is PGL_2 stable and is irreducible. □

The dual pairing between σ_{2k} and $\check{\sigma}_{2k}$ induces an isomorphism (over $\mathbf{Z}[\frac{1}{k!}]$)

$$\mathrm{Hom}_{\mathrm{PGL}_2}(\check{\mathscr{V}}_{2k}, \mathscr{H}_k) \simeq (\mathscr{V}_{2k} \otimes \mathscr{H}_k)^{\mathrm{PGL}_2},$$

under which the intertwining map defined above corresponds to the vector

$$\mathbf{P}_k = \sum_{i=-k}^{k} (-1)^i \frac{(k!)^2}{(k+i)!(k-i)!} P_i^k \otimes \mathbf{t}_i$$

in $\frac{1}{(2k)!}(\mathscr{H}_k \otimes \mathscr{V}_{2k})^{\mathrm{PGL}_2}$.

Remark 1.2.2. The representations $(\sigma_{2k}, \mathscr{V}_{2k})$ and $(\check{\sigma}_{2k}, \check{\mathscr{V}}_{2k})$ are used to define an integral structures for automorphic forms on $D^\times$ in §2.3.6; whereas we use the representation $(\rho_k, \mathscr{H}_k)$ to define a good archimedean Schwartz function for theta lifting in §4.4.1.

1.3 Eichler orders

Fix a basis $(\delta, \jmath)$ of D. We will choose a finite set of primes S later on so that $(\delta, \jmath)$ is an S-basis as in Definition 1.1.1. We distinguish a class of Eichler orders in D attached to $(\delta, \jmath)$. Let $p \geq 5$ be a fixed prime.

1.3.1 Review of Eichler orders.

Given a prime l and an integer $n \geq 0$, we define an order in $\mathrm{M}_2(\mathbf{Q}_l)$ by

$$I_l(n) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{M}_2(\mathbf{Z}_l) : \mathrm{ord}_l(c) \geq n \right\}.$$

- $d\check{\sigma}_{2k}(H) \cdot \mathbf{t}_i^k = -2i \cdot \mathbf{t}_i^k$;
- $d\check{\sigma}_{2k}(Y^+) \cdot \mathbf{t}_i^k = -(k+i) \cdot \mathbf{t}_{i-1}^k$;
- $d\check{\sigma}_{2k}(Y^-) \cdot \mathbf{t}_i^k = -(k-i) \cdot \mathbf{t}_{i+1}^k$.

From this we see that

$$\mathbf{t}_i^k = \begin{cases} \frac{(k+i)!}{k!}(-Y^+)^{-i}\mathbf{t}_0^k & \text{if } i < 0, \\ \frac{(k-i)!}{k!}(-Y^-)^{i}\mathbf{t}_0^k & \text{if } i > 0. \end{cases}$$

Let N be an integer relatively prime to dp. Given a non-negative integer r, we denote by $\mathscr{D} = \mathscr{D}(dNp^r)$ an *Eichler order* of level dNp^r in D. Recall this means that $\mathscr{D}$ is an order of D such that for a prime l, $\mathscr{D}_l$ is

- the unique maximal order $\mathscr{D}_l = \{x \in D_l : \mathrm{n}(x) \in \mathbf{Z}_l\}$ of D_l for l dividing d;
- conjugate to $I_l(\mathrm{ord}_l(Np^r))$ under some isomorphism $\epsilon_l \colon D_l \simeq \mathrm{M}_2(\mathbf{Q}_l)$ otherwise.

It follows from the definition that any two Eichler orders of level dNp^r in D are locally conjugate.

We now specify such an isomorphism ϵ_l at each prime $l \nmid d$. We consider the cases in which l splits in K and l does not split in K separately.

1.3.2 l splits in K.

We have $K \otimes \mathbf{Q}_l = K_{\mathfrak{l}} \times K_{\bar{\mathfrak{l}}}$ where $\mathfrak{l} = \mathfrak{l}_K$ is the prime of K above l fixed in §0.4.1 and $\bar{\mathfrak{l}}$ is its conjugate in K. We have that

$$\mathrm{GL}_{K\otimes\mathbf{Q}_l}(D \otimes \mathbf{Q}_l) = \mathrm{GL}_{K_{\mathfrak{l}}}(\dot{K}_{\mathfrak{l}} \perp \jmath \dot{K}_{\mathfrak{l}}) \times \mathrm{GL}_{K_{\bar{\mathfrak{l}}}}(\dot{K}_{\bar{\mathfrak{l}}} \perp \jmath \dot{K}_{\bar{\mathfrak{l}}}) \simeq \mathrm{GL}_2(\mathbf{Q}_l) \times \mathrm{GL}_2(\mathbf{Q}_l)$$

where the last map is defined with respect to the basis $\{1, \jmath\}$ of D_l as a right module over $K_{\mathfrak{l}} \times K_{\bar{\mathfrak{l}}} \simeq \mathbf{Q}_l \otimes \mathbf{Q}_l$. Let $\bar{\mathrm{G}}\mathrm{U} = \bar{\mathrm{G}}\mathrm{U}(D, (\bullet,\bullet)_{\delta,\jmath})$ be the unitary similitude group over $\mathbf{Q}$ defined in §1.1.4. A computation* shows that the image of $\bar{\mathrm{G}}\mathrm{U}(\mathbf{Q}_l)$ in $\mathrm{GL}_2(\mathbf{Q}_l) \times \mathrm{GL}_2(\mathbf{Q}_l)$ is exactly

$$\left\{(g, \det(g) \cdot M_\jmath^{-1t} g^{-1} M_\jmath) : g \in \mathrm{GL}_2(\mathbf{Q}_l)\right\}$$

where $M_\jmath$ is the diagonal matrix $\mathrm{diag}[1, \mathrm{n}(\jmath)]$. By projecting onto the first factor, we obtain an isomorphism

$$\epsilon_{\mathrm{ar},l}^{\delta,\jmath} \colon D_l^\times \simeq \bar{\mathrm{G}}\mathrm{U} \xrightarrow{\mathrm{pr}} \mathrm{GL}_2(\mathbf{Q}_l),$$

*A pair of matrices (g_1, g_2) lies in the image of $\bar{\mathrm{G}}\mathrm{U}(\mathbf{Q}_l)$ if and only if

$$^t(g_1,g_2) \cdot \left(\begin{bmatrix} 1 & \\ & \mathrm{n}(\jmath) \end{bmatrix}, \begin{bmatrix} 1 & \\ & \mathrm{n}(\jmath) \end{bmatrix}\right) \cdot \overline{(g_1,g_2)} = \det(g_1,g_2) \cdot \left(\begin{bmatrix} 1 & \\ & \mathrm{n}(\jmath) \end{bmatrix}, \begin{bmatrix} 1 & \\ & \mathrm{n}(\jmath) \end{bmatrix}\right).$$

The claim then follows from the equality

$$^t g_1 \cdot \begin{bmatrix} 1 & \\ & \mathrm{n}(\jmath) \end{bmatrix} \cdot g_2 = \det(g_1) \cdot \begin{bmatrix} 1 & \\ & \mathrm{n}(\jmath) \end{bmatrix}$$

by projecting onto the first factor.

which we shall refer to as the *arithmetic embedding* at l. It is explicitly determined by

$$\delta \mapsto \begin{bmatrix} \delta_l & \\ & \bar{\delta}_l \end{bmatrix} \qquad \jmath \mapsto \begin{bmatrix} & -\mathrm{n}(\jmath) \\ 1 & \end{bmatrix}$$

where δ_l (resp. $\bar{\delta}_l$) is the root of the monic polynomial $m(X)$ (c.f. §1.1.2) in $\mathbf{Q}_l$ corresponding to the embedding $K \to K_{\mathfrak{l}} \simeq \mathbf{Q}_l$ (resp. $K \to K_{\bar{\mathfrak{l}}} \simeq \mathbf{Q}_l$). This map extends to an isomorphism $D_l \simeq \mathrm{M}_2(\mathbf{Q}_l)$ of $\mathbf{Q}_l$-algebras, which we also denote by $\epsilon^{\delta,\jmath}_{\mathrm{ar},l}$.

1.3.3 l does not split in K.

In this case $\dot{K}_l = \mathbf{Q}_l + \mathbf{Q}_l\delta$ is a two-dimensional vector space over $\mathbf{Q}_l$ on which $\dot{K}_l$ acts linearly by left multiplication. To define an isomorphism $D_l \simeq \mathrm{End}_{\mathbf{Q}_l}(K_l)$, it suffices to specify how $\jmath$ acts on $\dot{K}_l$. Suppose for the moment that $\sqrt{-\mathrm{n}(\jmath)} \in \mathbf{Q}_l$, then $\jmath/\sqrt{-\mathrm{n}(\jmath)}$, as a linear transformation on $\dot{K}_l$, has order 2, trace 0, determinant -1 and satisfies the relation $\jmath\delta = \bar{\delta}\jmath$. Since the involution $t \mapsto \bar{t}$ on $\dot{K}_l$ satisfies all these conditions, we may let $\jmath/\sqrt{-\mathrm{n}(\jmath)}$ act on $\dot{K}_l$ by involution. This defines an isomorphism $D_l \simeq \mathrm{End}_{\mathbf{Q}_l}(\dot{K}_l)$ as $\mathbf{Q}_l$-algebras. We can explicitly describe this isomorphism in terms of matrices as follows. Denote by $m(X) = X^2 - aX + b$ the minimal polynomial for δ, and set $\delta^{(0)} = \frac{\delta-\bar{\delta}}{2} = \frac{\sqrt{\dot{\Delta}_\delta}}{2}$ to be the trace-zero part of δ; then with respect to the ordered basis $\{\delta^{(0)}, 1\}$ of $\dot{K}_l$, we have that

$$\delta \mapsto \begin{bmatrix} \frac{a}{2} & 1 \\ \frac{a^2-4b}{4} & \frac{a}{2} \end{bmatrix} \qquad \jmath \mapsto \sqrt{-\mathrm{n}(\jmath)} \cdot \begin{bmatrix} -1 & \\ & 1 \end{bmatrix}.$$

Analogous to the split case, we shall refer to the isomorphism

$$\epsilon^{\delta,\jmath}_{\mathrm{ar},l} \colon D_l \simeq \mathrm{M}_2(\mathbf{Q}_l)$$

and its induced isomorphism $D_l^\times \simeq \mathrm{GL}_2(\mathbf{Q}_l)$ as the *arithmetic embeddings* at l. We emphasize that $\epsilon^{\delta,\jmath}_{\mathrm{ar},l}$ is only defined when $-\mathrm{n}(\jmath)$ is a square in $\mathbf{Q}_l$. This explains the extra condition in the definition of an S-basis.

1.3.4 Local conditions on $\mathscr{D}^{\delta,\jmath,S}(dNp^r)$.

Let S be a finite set of primes l subjected to the conditions that

- D_l is split, and
- if l is inert in K, then $-\mathrm{n}(\jmath)$ is a square in $\mathbf{Q}_l$.

Then $(\delta, \jmath)$ is an S-basis for D and the arithmetic embedding is defined at all primes in S. For each $l \in S$, set $\mathscr{D}_l$ to be the inverse image of

$$I_l(\mathrm{ord}_l(Np^r)) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{M}_2(\mathbf{Z}_l) : \mathrm{ord}_l(c) \geq \mathrm{ord}_l(Np^r) \right\}$$

under $\epsilon^{\delta,\jmath}_{\mathrm{ar},l}$. In particular, if $l \in S$ is inert in K and $l \nmid Np^r$, then

$$\mathscr{D}_l \simeq \mathrm{End}_{\mathbf{Z}_l}(\mathbf{Z}_l \oplus \mathbf{Z}_l \delta^{(0)})$$

under the arithmetic embedding $\epsilon^{\delta,\jmath}_{\mathrm{ar},l}$.

It a consequence of weak approximation [Vig80, Ch. III, Proposition 5.1] that there is an Eichler order $\mathscr{D}^{\delta,\jmath,S}(dNp^r)$ such that $\mathscr{D}^{\delta,\jmath,S}(dNp^r)_l$ is not just isomorphic, but exactly *equal* to $\mathscr{D}_l$ for every prime $l \in S$.

Remark 1.3.1. We do not (in fact, cannot) specify precisely the local Eichler order at all places using the above procedure. In fact, we know that all lattices in D are locally identical at almost all primes [Vig80, Ch. III, Proposition 5.1].

1.3.5 Normalizer of $\mathscr{D}_l^\times$.

Denote by $N_l(n) = N(I_l(n)^\times)$ the normalizer of $I_l^\times$ in $\mathrm{GL}_2(\mathbf{Q}_l)$. For a description of $N_l(n)$, first observe that the elements of $N_l(n)$ stabilize the order $I_l(n)$, and that $I_l(n)$ is uniquely the intersection of the maximal orders $\mathrm{M}_2(\mathbf{Z}_l)$ and $w_l(n) \cdot \mathrm{M}_2(\mathbf{Z}_l) \cdot w_l(n)^{-1}$ where

$$w_l(n) = \begin{bmatrix} & -1 \\ l^n & \end{bmatrix}.$$

Since any element $x \in D_l^\times$ conjugates a maximal order to a maximal order, we see that $x \in N_l(n)$ must either stabilize both $\mathrm{M}_2(\mathbf{Z}_l)$ and $w_l(n) \cdot \mathrm{M}_2(\mathbf{Z}_l) \cdot w_l(n)^{-1}$ or permute them. It follows that $N_l(n)$ is generated by $I_l(n)^\times$, $w_l(n)$, and the center $\mathbf{Q}_l^\times$ as a group. We caution the reader that $N_l(n)$ is *not* compact as a subgroup of $\mathrm{GL}_2(\mathbf{Q}_l)$, even though its image in $\mathrm{GL}_{\mathbf{Z}_l}(I_l)$ is necessarily compact.

We now consider the normalizer $\mathscr{N}_l = \mathscr{N}^{\delta,\jmath,S}(dNp^r)_l$ of $\mathscr{D}^{\delta,\jmath,S}(dNp^r)_l^\times$ in $D_l^\times$. For $l \mid Np$, set $\varpi_l = \varpi_l(\mathrm{ord}_l(Np^r))$ to be the element in $D_l^\times$ corresponding to $w_l(\mathrm{ord}_l(Np^r))$ under $\epsilon^{\delta,\jmath}_{\mathrm{ar},l}$; for $l \mid d$, take ϖ_l to be a uniformizer of $D_l^\times$. By the preceding discussion, we have

$$\mathscr{N}_l = \begin{cases} \mathbf{Q}_l^\times \cdot \langle \mathscr{D}_l^\times, \varpi_l \rangle & \text{if } l \mid Np, \\ D_l^\times & \text{if } l \mid d, \text{ and} \\ \mathbf{Q}_l^\times \cdot \mathscr{D}_l^\times & \text{otherwise.} \end{cases}$$

1.3.6 Double-coset decompositions.

Denote by $K_l^\times$ the image of $\dot{K}_l^\times$ in $\mathrm{GL}_2(\mathbf{Q}_l)$ under the arithmetic embedding $\epsilon_{\mathrm{ar},l}$. For each $l \in S$, we describe the double-coset decompositions of $K_l^\times \backslash \mathrm{GL}_2(\mathbf{Q}_l)/\mathrm{GL}_2(\mathbf{Z}_l)$ depending on whether l is split in K or not.

l **splits in** K**:** In this case the Iwasawa decomposition gives

$$\mathrm{GL}_2(\mathbf{Q}_l) = \coprod_{n \leq 0} K_l^\times \cdot u_l(n) \cdot \mathrm{GL}_2(\mathbf{Z}_l) \tag{3.6.3}$$

where $u_l(n) = \begin{bmatrix} 1 & l^n \\ & 1 \end{bmatrix}$.

l **does not split in** K**:** In this case $\mathrm{GL}_2(\mathbf{Q}_l)$ is identified with $\mathrm{GL}_{\mathbf{Q}_l}(\dot{K}_l)$ via the basis $\{\delta^{(0)}, 1\}$ of K_l. Given a matrix $M \in \mathrm{GL}_2(\mathbf{Q}_l) \simeq \mathrm{GL}_{\mathbf{Q}_l}(\dot{K}_l)$, the lattice $\mathscr{I} = M \cdot \mathbf{Z}_l[\delta^{(0)}]$ in K_l is a principal ideal over $\mathscr{O}_l(\mathscr{I}) = \{t \in K_l : t \cdot \mathscr{I} \subseteq \mathscr{I}\}$ [Iha67, §Prop. 1]. Let ς be the conductor† of $\mathbf{Z}_l[\delta^{(0)}]$ and m the conductor of $\mathscr{O}_l(\mathscr{I})$, then we have

$$\mathscr{I} = t \cdot \mathscr{O}_l(\mathscr{I}) = t \cdot d_l(m - \varsigma) \cdot \mathbf{Z}_l[\delta^{(0)}]$$

where $d_l(n) = \mathrm{diag}[l^n, 1]$ and $t \in K_l^\times$. Consequently, $M \in K_l^\times \cdot d_l(n) \cdot \mathrm{GL}_{\mathbf{Z}_l}(\mathbf{Z}_l[\delta^{(0)}])$ and we obtained the following double-coset decomposition

$$\mathrm{GL}_2(\mathbf{Q}_l) = \coprod_{n \geq 0} K_l^\times \cdot d_l(n - \varsigma) \cdot \mathrm{GL}_2(\mathbf{Z}_l). \tag{3.6.4}$$

We also state for the record the following coset decomposition [Kna92, pg.259] of $\mathrm{GL}_2(\mathbf{Z}_l)$:

$$\mathrm{GL}_2(\mathbf{Z}_l) = \left(\coprod_{0 \leq s < l^{r-1}} \begin{bmatrix} 1 & \\ sl & 1 \end{bmatrix} I_l(r)^\times \right) \coprod \left(\coprod_{0 \leq t < l^r} \begin{bmatrix} t & 1 \\ -1 & \end{bmatrix} I_l(r)^\times \right). \tag{3.6.5}$$

†Recall the conductor of an order $\mathscr{O}_l$ of K_l is the integer n such that $\mathscr{O}_l = \mathbf{Z}_l + l^n \cdot \mathscr{O}_{K,l}$ where $\mathscr{O}_{K,l}$ is the maximal order of K_l.

1.4 Representations of $D^\times$ and $\mathscr{D}_p^\times$

We describe the representations of $D^\times$ and $\mathscr{D}_p^\times$ obtained from the representations of PGL_2 described in §1.2. The notations are from §1.1.

1.4.1 Families of representations of $D^\times$.

For a representation (ϱ, V_ϱ) of PGL_2 defined over $\mathbf{Z}$, denote by $(\varrho_A, V_{\varrho,A})$ the representation of $\mathrm{PGL}_2(A)$ obtained via base change to the $\mathbf{Z}$-algebra A. Now given a basis $(\delta, \jmath)$ for D, the representation $\varrho_{K_\jmath}$ pulls back to a representation of $D_{\mathbf{Q}}^\times$ through the isomorphism

$$\epsilon_{\mathrm{wt}}^{\delta,\jmath}\colon D_{\mathbf{Q}}^\times \simeq \bar{\mathrm{G}}\mathrm{U}_2^{\delta,\jmath}(\mathbf{Q}).$$

We denote the resulting representation of $D_{\mathbf{Q}}^\times$ by $(\varrho_{\mathrm{wt}}^{\delta,\jmath}, V_\varrho^{\delta,\jmath})$. It is defined over the field $K_\jmath = \mathbf{Q}(\sqrt{\Delta_\delta}, \sqrt{\mathrm{n}(\jmath)})$.

To keep matters precise, we now describe the theory of highest weight for $D_\infty^\times$. Fix an S-basis $(\delta, \jmath)$ for D and set $\tilde{\delta} = \frac{\delta}{\sqrt{\mathrm{n}(\delta)}}$ and $\tilde{\jmath} = \frac{\jmath}{\sqrt{\mathrm{n}(\jmath)}}$. The weight embedding $\epsilon_{\mathrm{wt},\infty}^{\delta,\jmath}\colon D_\infty^\times \simeq \bar{\mathrm{G}}\mathrm{U}_2^{\delta,\jmath}(\mathbf{R})$ induces an isomorphism $PD_\infty^\times \simeq \mathrm{P}\bar{\mathrm{G}}\mathrm{U}_2^{\delta,\jmath}(\mathbf{R})$, whose derivative defines an isomorphism $D_\infty^{(0)} \simeq \mathfrak{su}_2$ of Lie algebras explicitly given by*

$$\tilde{\delta} \mapsto \begin{bmatrix} \imath & \\ & -\imath \end{bmatrix} \qquad \tilde{\jmath} \mapsto \begin{bmatrix} & -1 \\ 1 & \end{bmatrix} \qquad \tilde{\jmath}\tilde{\delta} \mapsto \begin{bmatrix} & \imath \\ \imath & \end{bmatrix}.$$

Let $\mathfrak{t} = \mathbf{R}\tilde{\delta}$ be the Cartan sub-algebra of $D_\infty^{(0)}$ and let e_D be the character in $\mathfrak{t}_{\mathbf{C}}^\vee$ that send $\imath \cdot \tilde{\delta} = \mathrm{diag}[1, -1]$ to 1. We fix the unique set of simple roots for $\mathfrak{t}_{\mathbf{C}}$ in $D_{\mathbf{C}}^{(0)} \simeq \mathrm{M}_2^{(0)}(\mathbf{C})$ to be $\{e_D\}$.

Corollary 1.4.1. *Every irreducible complex representation of $PD_\infty^\times \simeq \mathrm{SO}(D^{(0)}, \mathrm{n})_{\mathbf{R}}$ of dimension $2k+1$ is isomorphic to $(\sigma_{2k,\mathrm{wt},\infty}^{\delta,\jmath}, \mathscr{V}_{2k,\infty}^{\delta,\jmath}) = (\sigma_{2k,\mathrm{wt},\mathbf{C}}^{\delta,\jmath}, \mathscr{V}_{2k,\mathbf{C}}^{\delta,\jmath})$.*

Proof. This a consequence of the highest weight theory for compact connected Lie groups [Kna02, Theorem 5.110]. We can also deduce this from Proposition 1.2.1 as follows. The weight embedding $\epsilon_{\mathrm{wt}}^{\delta,\jmath}$ induces an isomorphism between $D_{\mathbf{C}}^{(0)}$ and

*To see this, simply observe that filling this map in on the left makes the diagram

$$\begin{tikzcd} D_\infty^{(0)} \arrow[r, "\exp"] \arrow[d] & D_\infty^1/\{\pm 1\} \arrow[d, "\epsilon_{\mathrm{wt},\infty}^{\delta,\jmath}"] \\ \mathfrak{su}_2 \arrow[r, "\exp"] & \mathrm{P}\bar{\mathrm{G}}\mathrm{U}_2^{\delta,\jmath}(\mathbf{R}) \end{tikzcd}$$

commute. Here the exponential map $D_\infty^{(0)} \xrightarrow{\exp} D_\infty^1$ given by $\exp(x_1\tilde{\delta} + x_2\tilde{\jmath} + x_3\tilde{\jmath}\tilde{\delta}) = e^{x_1\tilde{\delta}} \cdot e^{x_2\tilde{\jmath}} \cdot e^{x_3\tilde{\jmath}\tilde{\delta}}$.

$\mathrm{M}_2^{(0)}(\mathbf{C})$; thus all irreducible representations of $D_{\mathbf{C}}^{(0)}$ of dimension $2k+1$ are equivalent to $(d\sigma_{2k,\mathrm{wt},\infty}^{\delta,\jmath}, \mathscr{V}_{2k,\infty}^{\delta,\jmath})$. Since the exponential map is surjective for any compact connected Lie group, the corollary follows. □

We shall refer to such an irreducible representation in the corollary as one of the highest weight $2k$.

1.4.2 Conjugation between bases.

Let $(\delta, \jmath)$ and (γ, κ) be two bases for D. Define an auxiliary totally real number field $F = \mathbf{Q}(\sqrt{-\Delta_\delta}, \sqrt{-\Delta_\gamma}, \sqrt{\mathrm{n}(\jmath)}, \sqrt{\mathrm{n}(\kappa)})$ and set $D_F = D \otimes F$. Let $K = K_{\jmath,\kappa} = F(\imath)$ be the CM-field obtained by adjoining $\imath = \sqrt{-1}$ to F. The $K_{\jmath,\kappa}$-vector space

$$V_{\varrho,K} = V_\varrho^{\delta,\jmath} \otimes K_{\jmath,\kappa} = V_\varrho^{\gamma,\kappa} \otimes K_{\jmath,\kappa}$$

affords two representations of $D_F^\times$, namely $\varrho_{\mathrm{wt},K}^{\delta,\jmath}$ and $\varrho_{\mathrm{wt},K}^{\gamma,\kappa}$; we proceed to study how they are related. We have an F-linear automorphism ϕ of D_F uniquely determined by

$$\dot{\sqrt{\Delta_\delta}} \xrightarrow{\phi} \sqrt{\Delta_\delta/\Delta_\gamma} \cdot \dot{\sqrt{\Delta_\gamma}} \qquad \text{and} \qquad \jmath \xrightarrow{\phi} \sqrt{\mathrm{n}(\jmath)/\mathrm{n}(\kappa)} \cdot \kappa.$$

By Skolem-Noether [Vig80, Ch. I, Théorème 2.1], there exists an element $\alpha \in D_F^\times$ such that

$$\phi(x) = \alpha \cdot x \cdot \alpha^{-1}$$

for all $x \in D_F$. This leads to the following proposition.

Proposition 1.4.2. *The pair of bases $(\delta, \jmath)$ and (γ, κ) of D are conjugate over the totally real number field $F = \mathbf{Q}(\sqrt{-\Delta_\delta}, \sqrt{-\Delta_\gamma}, \sqrt{\mathrm{n}(\jmath)}, \sqrt{\mathrm{n}(\kappa)})$ in the sense there exists an $\alpha \in D_F^\times$ such that*

$$\epsilon_{\mathrm{wt},F}^{\gamma,\kappa} \circ \mathrm{Ad}(\alpha)(x) = \epsilon_{\mathrm{wt},F}^{\delta,\jmath}(x)$$

for all $x \in D_F^\times$.

Proof. The preceding discussion shows that there exists $\alpha \in D_F^\times$ such that

$$\epsilon_{\mathrm{wt},F}^{\gamma,\kappa}(\alpha \cdot \dot{\sqrt{\Delta_\delta}} \cdot \alpha^{-1}) = \sqrt{\Delta_\delta/\Delta_\gamma} \cdot \begin{bmatrix} \sqrt{\Delta_\gamma} & \\ & -\sqrt{\Delta_\gamma} \end{bmatrix} = \epsilon_{\mathrm{wt},F}^{\delta,\jmath}(\sqrt{\Delta_\delta})$$

and

$$\epsilon_{\mathrm{wt},F}^{\gamma,\kappa}(\alpha \cdot \jmath \cdot \alpha^{-1}) = \sqrt{\mathrm{n}(\jmath)/\mathrm{n}(\kappa)} \cdot \begin{bmatrix} & -\sqrt{\mathrm{n}(\kappa)} \\ \sqrt{\mathrm{n}(\kappa)} & \end{bmatrix} = \epsilon_{\mathrm{wt},F}^{\delta,\jmath}(\jmath)$$

as matrices in $\mathrm{M}_2(\bar{\mathbf{Q}})$. Since the weight embeddings $\epsilon_{\mathrm{wt}}^{\bullet}$ are maps of algebras, the claim follow. $\square$

Corollary 1.4.3. *The pair of bases $(\delta,\jmath)$ and (γ,κ) of D are conjugate over the CM-field $K_{\jmath,\kappa} = F(\imath)$, with $F = \mathbf{Q}(\sqrt{-\Delta_\delta},\sqrt{-\Delta_\gamma},\sqrt{\mathrm{n}(\jmath)},\sqrt{\mathrm{n}(\kappa)})$, in the sense that there exists an $\alpha \in D_F^\times$ such that*

$$\varrho^{\gamma,\kappa}_{\mathrm{wt},K_{\jmath,\kappa}} \circ \mathrm{Ad}(\alpha)(x)\cdot v = \varrho^{\delta,\jmath}_{\mathrm{wt},K_{\jmath,\kappa}}(x)\cdot v$$

for all $x \in D_F^\times$ and $v \in V_{\varrho,K_{\jmath,\kappa}} = V^{\delta,\jmath}_{\varrho,K_\jmath}\otimes K_{\jmath,\kappa} = V^{\gamma,\kappa}_{\varrho,K_\kappa}\otimes K_{\gamma,\kappa}$.

Proof. This follows from the definition of $\varrho^{\gamma,\kappa}_{\mathrm{wt},K_{\jmath,\kappa}}$ and $\varrho^{\delta,\jmath}_{\mathrm{wt},K_{\jmath,\kappa}}$. $\square$

A consequence of the corollary is that the representations $(\varrho^{\delta,\jmath}_{\mathrm{wt},F}, V^{\delta,\jmath}_{\varrho,K_\jmath})$, for different $S^{\delta,\jmath}$-basis $(\delta,\jmath)$ for D, are all mutually conjugate over the compositum $K^{\mathrm{quad}} = \mathbf{Q}(\sqrt{q} : q \in \mathbf{Z})$ of all quadratic extensions of $\mathbf{Q}$ in $\bar{\mathbf{Q}}$. In view of this, we set

$$V_{\varrho,\mathrm{quad}} = V^{\delta,\jmath}_{\varrho,K_\jmath}\otimes K^{\mathrm{quad}} \qquad V_{\varrho,\bar{\mathbf{Q}}} = V^{\delta,\jmath}_{\varrho,K_\jmath}\otimes \bar{\mathbf{Q}} \qquad V_{\varrho,\infty} = V^{\delta,\jmath}_{\varrho,K_\jmath}\otimes \mathbf{C}$$

and ignore the quantifier $(\delta,\jmath)$ on the representation space from now on.

1.4.3 A distinguished p-integral lattice.

Fix a S-basis $(\delta,\jmath)$ for D and let $(\varrho_l, V_{\varrho,l})$ be a representation of $\mathrm{PGL}_2(\mathbf{Q}_l)$ with coefficients in $\mathbf{C}_l$. Analogous to the archimedean case, the representation $(\varrho_l, V_{\varrho,l})$ pulls back to a representation $(\varrho_{\mathrm{ar},l}, V_{\varrho,\mathrm{ar},l})$ of $D_l^\times$ using the arithmetic embedding $\epsilon^{\delta,\jmath}_{\mathrm{ar},l}\colon D_l^\times \simeq \mathrm{GL}_2(\mathbf{Q}_l)$.

We now consider the case that $l = p$ and $\varrho_p = \check{\sigma}_{2k,p}$ acting on $V_{\varrho,p} = \check{\mathscr{V}}_{2k,p} = \mathbf{C}_p[T]_{\le 2k}$. For each integer $n \ge 0$, denote by $\mathscr{M}_{2k,p}(n)$ the $\mathscr{O}_{\mathbf{C}_p}$-lattice in $\check{\mathscr{V}}_{2k,p}$ generated by

$$\check{\sigma}_{2k,p}(I_p(n)^\times)\cdot \mathbf{t}_0^k$$

over $\mathscr{O}_{\mathbf{C}_p}$. We emphasize that $\mathscr{M}_{2k,p}(n)$ is independent of $(\delta,\jmath)$ and $D^\times$. Also, note that $\mathscr{M}_{2k,p}(0) = \mathscr{O}_{\mathbf{C}_p}[X]_{\deg\le 2k}$.

Let $\mathscr{D}_p = \mathscr{D}^{\delta,\jmath,\{p\}}(dNp^r)_p$ be a local Eichler order at p fixed in §1.3, its action on $\check{\mathscr{V}}_{2k,\mathrm{ar},p} = \check{\mathscr{V}}_{2k,p}$ preserves the lattice $\mathscr{M}_{2k,p} = \mathscr{M}_{2k,p}(r)$ by construction. Furthermore, we have that

$$\langle \mathscr{O}_{\mathbf{C}_p}[X]_{\deg\le 2k}, \mathscr{O}_{\mathbf{C}_p}[X]_{\deg\le 2k}\rangle_{2k} \subseteq \frac{1}{k!}\mathscr{O}_{\mathbf{C}_p}$$

and therefore $\langle \mathscr{M}_{2k,p}, \mathscr{M}_{2k,p}\rangle_{2k} \subseteq \frac{1}{k!}\mathscr{O}_{\mathbf{C}_p}$.

Let $\mathscr{N}_p = \mathscr{N}^{\delta,\jmath,\{p\}}(dNp^r)_p$ be the normalizer of $\mathscr{D}_p^\times$ in $D_p^\times$ as in §1.3.5. It is generated by $\mathscr{D}_p^\times$, $\varpi_p = (\epsilon_{\mathrm{ar},p}^{\delta,\jmath})^{-1}(w_p(r))$, and the center $\mathbf{Q}_p^\times$. Since $\mathbf{t}_0^k$ is fixed† by $\check{\sigma}_{2k,p}(w_p(r))$, we see that $\mathscr{M}_{2k,p}(r)$ is stable under the $\check{\sigma}_{2k,\mathrm{ar},p}^{\delta,\jmath}$-action by $\mathscr{N}_p$. It is also for this reason that we did not choose a larger lattice, e.g., $\mathscr{O}_{\mathbf{C}_p}[X]_{\deg\le 2k}$, when $r>0$.

1.4.4 Transition matrices.

Again fix $(\delta,\jmath)$ an S-basis for D. To a representation (ϱ, V_ϱ) of PGL_2, and every prime $l\in S$, we have attached two representations

$$\varrho_{\mathrm{wt}}\colon D^\times \xrightarrow{\epsilon_{\mathrm{wt}}} \mathrm{GL}_2(K_\jmath) \curvearrowright V_\varrho\otimes K_\jmath \qquad \text{and} \qquad \varrho_{\mathrm{ar}}\colon D_l^\times \xrightarrow{\epsilon_{\mathrm{ar}}} \mathrm{GL}_2(\mathbf{Q}_l) \curvearrowright V_\varrho\otimes \mathbf{C}_l$$

defined over $K_\jmath$ and $\mathbf{C}_l$ respectively. By base change, ϱ_{wt} also gives rise to a representation of $D_l^\times$ with coefficients in $\mathbf{C}_l$. We like to describe how to conjugate one representation to the other, which amounts to finding a matrix $C = C_l^{\delta,\jmath}$ in $\mathrm{GL}_2(\mathbf{C}_l)$ that makes the following diagram

$$\begin{tikzcd}
D_l^\times \arrow[r, "\epsilon_{\mathrm{ar}}"] \arrow[d, equal] & \mathrm{GL}_2(\mathbf{Q}_l) \arrow[r] & \mathrm{GL}_2(\mathbf{C}_l) \arrow[d, "\mathrm{Ad}(C)"] \\
D_l^\times \arrow[r, "\epsilon_{\mathrm{wt}}"] & \mathrm{GL}_2(K_\jmath) \arrow[r] & \mathrm{GL}_2(\mathbf{C}_l)
\end{tikzcd}$$

commute. (Here $\mathrm{Ad}(C)\cdot g = C\cdot g\cdot C^{-1}$ for $g\in \mathrm{GL}_2(K_{\jmath,l})$.) A direct computation shows that

$$C_l^{\delta,\jmath} = \begin{cases} \begin{bmatrix} \frac{1}{\sqrt{\mathrm{n}(\jmath)}} & \\ & 1 \end{bmatrix} & \text{if } l \text{ splits in } K; \\ \begin{bmatrix} \frac{\sqrt{\Delta_\delta}}{2} & 1 \\ \frac{\sqrt{-\Delta_\delta}}{2} & -\sqrt{-1} \end{bmatrix} & \text{if } l \text{ does not split in } K \end{cases}$$

has this property. We note that $C_l^{\delta,\jmath}$ lies in $\mathrm{GL}_2(K_\jmath(\sqrt{-1}))$. Also note that

$$\varrho_{\mathrm{wt}} = \varrho(C_l^{\delta,\jmath})\cdot\varrho_{\mathrm{ar}}\cdot\varrho(C_l^{\delta,\jmath})^{-1}$$

†Indeed, we have that

$$\begin{aligned} \check{\sigma}_{2k,p}(w_p(r))\cdot\mathbf{t}_0^k &= \check{\sigma}_{2k,p}\left(\begin{bmatrix} & -1 \\ p^r & \end{bmatrix}\right)\cdot\mathbf{t}_0^k \\ &= \left(\frac{1}{p^r}\right)^k\cdot(-p^rT)^{2k}\cdot\left(\frac{1}{p^rT}\right)^k \\ &= \mathbf{t}_0^k. \end{aligned}$$

as representations of $D_l^\times$ on $V_\varrho \otimes \mathbf{C}_l$.

1.4.5 Comparing transition matrices.

Given $(\delta, \jmath)$ an S-basis and (γ, κ) an S'-basis for D, we want to compare the associated transition matrices $C_l^{\delta,\jmath}$ and $C_l^{\gamma,\kappa}$ for $l \in S \cap S'$.* To begin, observe that by the definition of an Eichler order, there exists an element $\alpha_l \in D_l^\times$ such that

$$\alpha_l \cdot \mathscr{D}^{\delta,\jmath,S}(dNp^r)_l \cdot \alpha_l^{-1} = \mathscr{D}^{\gamma,\kappa,S'}(dNp^r)_l.$$

It follows that

$$\mathrm{M}_2(\mathbf{Q}_l) \xrightarrow{(\epsilon_{\mathrm{ar},l}^{\delta,\jmath})^{-1}} D_l \xrightarrow{\mathrm{Ad}(\alpha_l)} D_l \xrightarrow{\epsilon_{\mathrm{ar},l}^{\gamma,\kappa}} \mathrm{M}_2(\mathbf{Q}_l)$$

is an automorphism of $\mathrm{M}_2(\mathbf{Q}_l)$ that preserves the Eichler order $I_l(n)$ where $n = \mathrm{ord}_l(Np^r)$. By Skolem–Noether [Vig80, Ch. I, Théorème 2.1], this automorphism is given by conjugation by some invertible matrix $M = M^{\delta,\jmath,\gamma,\kappa}$ in $\mathrm{M}_2(\mathbf{Q}_l)$. From this we see that

$$\epsilon_{\mathrm{ar},l}^{\gamma,\kappa}(\alpha_l x \alpha_l^{-1}) = M \cdot \epsilon_{\mathrm{ar},l}^{\delta,\jmath}(x) \cdot M^{-1}$$

for all $x \in D_l$. Moreover, we must have $M \in N_l(n)$ in order to stabilize $I_l(n)$.

In view of this relation between the arithmetic embeddings, let us consider the equation

$$\begin{aligned} \epsilon_{\mathrm{wt},l}^{\gamma,\kappa}(\alpha_l x \alpha_l^{-1}) &= C_l^{\gamma,\kappa} \cdot \epsilon_{\mathrm{ar},l}^{\gamma,\kappa}(\alpha_l x \alpha_l^{-1}) \cdot (C_l^{\gamma,\kappa})^{-1} \\ &= C_l^{\gamma,\kappa} M \cdot \epsilon_{\mathrm{ar},l}^{\delta,\jmath}(x) \cdot M^{-1}(C_l^{\gamma,\kappa})^{-1}. \end{aligned}$$

By Proposition 1.4.2, there exists a totally real number field F and an element $\alpha_\infty \in D_F^\times$ such that $\epsilon_{\mathrm{wt},F}^{\gamma,\kappa} \circ \mathrm{Ad}(\alpha_\infty) = \epsilon_{\mathrm{wt},F}^{\delta,\jmath}$. Plug this into the equation above, we get:

$$\epsilon_{\mathrm{wt},l}^{\delta,\jmath}(x) = (M_\alpha^{-1} C_l^{\gamma,\kappa} M) \cdot \epsilon_{\mathrm{ar},l}^{\delta,\jmath}(x) \cdot (M^{-1}(C_l^{\gamma,\kappa})^{-1} M_\alpha)$$

where $M_\alpha = \epsilon_{\mathrm{wt},l}^{\delta,\jmath}(\alpha_\infty^{-1}\alpha_l)$. This equation says that $M_\alpha^{-1} C_l^{\gamma,\kappa} M$ has the same properties as $C_l^{\delta,\jmath}$, which again by Skolem–Noether implies the two must differ by some scalar matrix. In summary, we have that

$$\begin{aligned} C_l^{\delta,\jmath} &= A \cdot M_\alpha^{-1} C_l^{\gamma,\kappa} M \\ &= A \cdot \epsilon_{\mathrm{wt},l}^{\delta,\jmath}(\alpha_\infty^{-1}\alpha_l) \cdot C_l^{\gamma,\kappa} M \end{aligned} \tag{4.5.6}$$

*This is used in §2.3.7.

for some scalar matrix $A \in \mathbf{C}_l^\times \subset \mathrm{GL}_2(\mathbf{C}_l)$.

Chapter 2. Automorphic Forms on $D^\times$

2.1 Measures

We follow [Vig80] in normalizing the various Haar measures.

2.1.1 Local additive measures.

For the additive group D_v, we set the Haar measure dx_v by

$$dx_v = \begin{cases} \text{the Haar measure giving } \mathscr{D}_l \text{ a volume of } \frac{1}{l} & \text{if } v = l \text{ and } l \mid d, \\ \text{the Haar measure giving } \mathscr{D}_l \text{ a volume of } 1 & \text{if } v = l \text{ and } l \nmid d, \\ 4 \cdot dx_1\, dx_2\, dx_3\, dx_4 & \text{if } v = \infty. \end{cases}$$

Here $\mathscr{D}_l$ is a maximal order at l, and at the archimedean place, we have identified x_v with its image $x_1 + x_2\imath + x_3\jmath + x_4\imath\jmath$ under any isomorphism $D_\infty \simeq \mathbf{H}$.*

Remark 2.1.1. Again, the Haar measure chosen above is self-dual for the the Fourier transform

$$\hat{\varphi}(x_v) = \int_{D_v} \varphi(y_v)\psi_v(\mathrm{tr}(x_v\bar{y}_v))\, dy_v.$$

This also explains the dependency on the discriminant of D_v [Vig80, Ch. II, §4].

2.1.2 Local multiplicative measures.

For the multiplicative group $D_v^\times$, set $d^\times x_v$ by

$$d^\times x_v = \begin{cases} \zeta_v(1) \cdot \frac{dx_v}{|n(x_v)|^2} & \text{if } v = l, \\ \frac{dx_v}{|n(x_v)|^2} & \text{if } v = \infty. \end{cases}$$

Here $\zeta_l(s) = (1 - l^{-s})^{-1}$ is the usual Euler factor at l.

*The Haar measure dx_∞ is invariant under conjugation by $D_\infty^\times$, hence it does not depend on the choice of the isomorphism. Similarly all maximal orders $\mathscr{D}_l$ are conjugate, so the normalization at l is independent of this choice.

With respect to this normalization, we have that for $\mathscr{D}_l$ a maximal order in D_l [Vig80, Ch. II, §4]:

$$\mathrm{vol}(\mathscr{D}_l^\times) = \begin{cases} \frac{\zeta_l(1)}{l\zeta_l(2)} & \text{if } l \mid d, \text{ and} \\ \frac{1}{\zeta_l(2)} & \text{otherwise.} \end{cases}$$

We also know that the maximal compact at the archimedean place, $D_\infty^1 \simeq \mathbf{H}^1$, has volume $2\pi^2$.

2.1.3 Global measures.

Since the product $\prod_l \mathrm{vol}(\mathscr{D}_l^\times)$ is finite, we can take the Tamagawa measure dx on $D_{\mathbf{A}}$ (resp. $d^\times x$ on $D_{\mathbf{A}}^\times$) to be the unique Haar measure that induces on each $\prod_{v\in S} D_v \times \prod_{v\notin S} \mathscr{D}_v$ (resp. $\prod_{v\in S} D_v^\times \times \prod_{v\notin S} \mathscr{D}_v^\times$) the product measure $\prod_v dx_v$ (resp. $\prod_v d^\times x_v$).

We also denote by dx (resp. $d^\times x$) the invariant measure on $D\backslash D_{\mathbf{A}}$ (resp. $D^\times\backslash D_{\mathbf{A}}^\times$) compatible with dx (resp. $d^\times x$) and the discrete measure on D (resp. $D^\times$). We have that $\mathrm{vol}(D\backslash D_{\mathbf{A}}) = 1$ [Vig80, Ch. III, Théorème 2.3].

2.1.4 Induced measures.

We give D_v^1 the Haar measure $d^1 x_v$ which is compatible for the short exact sequence

$$1 \to D_v^1 \to D_v^\times \xrightarrow{\mathrm{n}} \left(\begin{cases} \mathbf{R}_{>0} & \text{if } v = \infty, \\ \mathbf{Q}_v^\times & \text{otherwise} \end{cases} \right) \to 1$$

and the Haar measures on $D_v^\times$ and $\mathbf{Q}_v^\times$ fixed earlier. As before, we obtain a Tamagawa measure $d^1 x$ on $D_{\mathbf{A}}^1$ such that $\mathrm{vol}(D^1\backslash D_{\mathbf{A}}^1) = 1$ [loc. cit.].

Finally we turn our attention to the projective group $Z_{D^\times}\backslash D^\times$ where $Z = Z_{D^\times} \simeq \mathbf{G}_m$ is the center of $D^\times$. For a place v, we normalize the Haar measure $\bar{d}^\times x_v$ on $Z_v\backslash D_v^\times$ so it is compatible for the Haar measures on $Z_v \simeq \mathbf{Q}_v^\times$ and that on $D_v^\times$. The Tamagawa measure $\bar{d}^\times x_v$ in this case gives $D^\times \mathbf{A}^\times\backslash D_{\mathbf{A}}^\times$ a volume of 2 [loc. cit.].

2.2 Automorphic representations on $D^\times$

We describe the automorphic forms that we will be working with.

2.2.1 Automorphic representations.

Denote by $\mathscr{A}(D^\times)$ the $\mathbf{C}$-vector space of all smooth function[†]

$$f\colon D_{\mathbf{A}}^\times \to \mathbf{C}$$

satisfying the following conditions for all $x \in D_{\mathbf{A}}^\times$:

- $f(\gamma x) = f(x)$ for all $\gamma \in D_{\mathbf{Q}}^\times$;
- $f(z_\infty x) = f(x)$ for all $z_\infty \in Z_\infty$;
- $f(xu_f) = f(x)$ for all $u_f \in U_f$ for some compact open subgroup U_f of $D_f^\times$; and
- $x \cdot f$ is $D_\infty^\times$-finite, i.e., the function $u_\infty \mapsto f(xu_\infty)$ generates a finite-dimensional $\mathbf{C}$-vector space under right translation by $D_\infty^\times$.

Remark 2.2.1. The requirement that f is invariant under $Z_\infty \simeq \mathbf{R}^\times$ together with the fact that $Z_\infty \backslash D_\infty^\times$ is compact shows that f is of moderate growth.

$\mathscr{A}(D^\times)$ is a representation space of $D_{\mathbf{A}}^\times$ under the right regular action. An *automorphic representation* $(\boldsymbol{\pi}, V_{\boldsymbol{\pi}})$ of $D^\times$ is then an irreducible subquotient of $\mathscr{A}(D^\times)$.

2.2.2 The Petersson inner product and cuspidal representations.

Since $D_{\mathbf{Q}}^\times Z_\infty \backslash D_{\mathbf{A}}^\times$ is compact, we have the $D_{\mathbf{A}}^\times$-invariant Petersson inner product on $\mathscr{A}(D^\times)$ defined by

$$\langle f, g\rangle = \int_{Z_\infty D_{\mathbf{Q}}^\times \backslash D_{\mathbf{A}}^\times} f(x)\bar{g}(x)\, d^\times x.$$

Under this inner product, $\mathscr{A}(D^\times)$ decomposes into a discrete direct sum of irreducible unitary representations of $D_{\mathbf{A}}^\times$ each occurring with multiplicity one [JL70, Lemma 14.1].

The one-dimensional automorphic representations of $D_{\mathbf{A}}^\times$ are exactly the (unitary) characters

$$\chi_D\colon D_{\mathbf{Q}}^\times Z_\infty \backslash D_{\mathbf{A}}^\times \xrightarrow{\mathrm{n}} \mathbf{Q}^\times \mathbf{R}_{>0} \backslash \mathbf{A}^\times \xrightarrow{\chi} \mathbf{C}^1$$

where χ is a finite-order Hecke character of $\mathbf{A}^\times$. The orthogonal complement of the sum of these one-dimensional representations under the Petersson inner product is stable under $D_{\mathbf{A}}^\times$, and we denote it by $\mathscr{A}^0(D^\times)$. An automorphic representation $\boldsymbol{\pi}$ is *cuspidal* if it lies in $\mathscr{A}^0(D^\times)$.

[†]That is, $\mathbf{f}(x_f x_\infty)$ is a smooth function on $D_\infty^\times$ for any fixed $x_f \in D_f^\times$ and is a locally constant function on $D_f^\times$ for any fixed $x_\infty \in D_\infty^\times$.

2.2.3 Factorization.

Let $\boldsymbol{\pi}$ be an automorphic representation of $D^\times$, there is an abstract isomorphism $\boldsymbol{\pi} \simeq \otimes_v' \pi_v$ that factors $\boldsymbol{\pi}$ into irreducible admissible representations of $D_v^\times$ (with each component π_v uniquely determined up to isomorphism) [Fla79, Theorem 4]. We recall some useful invariants for characterizing $\boldsymbol{\pi}$.

2.2.4 The central character.

The center $Z_f \simeq \mathbf{A}_f^\times$ acts on $V_{\boldsymbol{\pi}}$ through a finite order Hecke character $\varepsilon_{\boldsymbol{\pi}}: \mathbf{Q}^\times \backslash \mathbf{A}_f^\times \to \mathbf{C}^1$. We call $\varepsilon = \varepsilon_{\boldsymbol{\pi}}$ the *central character* of $\boldsymbol{\pi}$. It factors as $\varepsilon = \otimes_l \varepsilon_l$. The conductor of ε_l, $c(\varepsilon_l)$, is 0 if $\mathbf{Z}_l^\times \subset \ker(\varepsilon_l)$ and is equal to the smallest integer $m > 0$ such that $1 + l^m \mathbf{Z}_l \subset \ker(\varepsilon_l)$ otherwise.

Since $\mathbf{Q}^\times \backslash \mathbf{A} \simeq \mathbf{Z}_f^\times$, and $\mathbf{C}^1$ has no small subgroups, ε is in fact a finite order character, and therefore takes values in the number field $\mathbf{Q}(\varepsilon) \subset \bar{\mathbf{Q}}$ generated by the values of ε. It follows that $c(\varepsilon_l) = 1$ for almost all l. We refer to the product $c(\varepsilon) = \prod_l c(\varepsilon_l)$ as the conductor of ε.

2.2.5 The level.

The *level* (or *conductor*) of $\boldsymbol{\pi}$ is the product of the level of the local components π_l,

$$c(\boldsymbol{\pi}) = \prod_{l<\infty} l^{c(\pi_l)},$$

where $c(\pi_l) \geq c(\varepsilon_l)$ is defined using the following recipe.

D **splits at** l**:** Identify $D_l^\times$ with $\mathrm{GL}_2(\mathbf{Q}_l)$ and view π_l as a smooth representation of $\mathrm{GL}_2(\mathbf{Q}_l)$. For $n \geq c(\varepsilon_l)$, we can extend ε_l to a character of the compact open subgroup $I_l(n)^\times \subset \mathrm{GL}_2(\mathbf{Z}_l)$ from §1.3.1 by

$$\varepsilon_l\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{cases} 1 & \text{if } n = c(\varepsilon_l) = 0, \text{ and} \\ \varepsilon_l(d) & \text{if } n \geq 1. \end{cases}$$

We then define the level (or conductor) of π_l to be the smallest integer n such that the space of $(\varepsilon_l^{-1} \otimes \pi_l)(I_l(n)^\times)$-invariant vectors

$$\{f_l \in V_{\pi_l} : u \cdot f_l = \varepsilon_l(u) \cdot f_l \text{ for all } u \in I_l(n)^\times\},$$

is non-zero.

D **ramifies at l:** In this case we define a compact open subgroup

$$U_l^D(n) = \begin{cases} \mathscr{D}_l^\times & \text{if } n = 1, \text{ and} \\ 1 + \varpi_l^{n-1}\mathscr{D}_l & \text{if } n > 1, \end{cases}$$

where $\mathscr{D}_l$ is the maximal order of D_l and ϖ_l is a uniformizer. Then $c(\pi_l) = n$ where n is the least integer such that the space of $U_l^D(n)$-invariant vectors in V_{π_l} is non-zero.

2.2.6 The weight.

By Corollary 1.4.1, the irreducible finite-dimensional representation π_∞ of $PD_\infty^\times$ is equivalent to $(\sigma_{2k,\mathrm{wt},\infty}^{\delta,\jmath}, \mathscr{V}_{2k,\infty})$ for some integer $2k \geq 0$. (Indeed, all irreducible representations of $PD_\infty^\times \simeq \mathrm{SO}(D^{(0)}, \mathrm{n})_{\mathbf{R}}$ are of odd dimension [GW09, Lemma 3.2.15].) Furthermore, this equivalence is unique up to scaling by Schur's lemma. We refer to the integer $2k$ attached to π_∞ as the *weight* of $\boldsymbol{\pi}$. It is independent of the S-basis $(\delta, \jmath)$.

2.2.7 The Jacquet–Langlands correspondence.

The (global) Jacquet–Langlands correspondence refers to the unique bijection between the automorphic representations on $D^\times$ and the automorphic representations on GL_2 that are discrete series* at all places v where D is ramified [Gel75, Theorem 10.2]. We denote by $\boldsymbol{\pi}^{\mathrm{JL}}$ the representation corresponding to $\boldsymbol{\pi}$ on $D^\times$ under this bijection.

Fix factorizations $\boldsymbol{\pi} \simeq \otimes' \pi_v$ and $\boldsymbol{\pi}^{\mathrm{JL}} \simeq \hat{\otimes}' \pi_v^{\mathrm{JL}}$. We have that $\pi_v \simeq \pi_v^{\mathrm{JL}}$ as admissible smooth representations of $D_v^\times \simeq \mathrm{GL}_2(\mathbf{Q}_v)$ at all place v where $D_v \simeq \mathrm{M}_2(\mathbf{Q}_v)$ is split [Lub10, Appendix, Theorem 3.3].

2.2.8 L-parameters and local constants.

Given a prime l, the local Langlands correspondence [BH06, §33.1] attaches to π_l^{JL} a $\mathrm{GL}_2(\mathbf{C})$-conjugacy class of admissible representations

$$\phi = \phi_{\pi_l^{\mathrm{JL}}} : W_l \times \mathrm{SL}_2(\mathbf{C}) \to \mathrm{GL}_2(\mathbf{C})$$

*That is, their matrix coefficients are *essential square-integrable* in the sense that a twist of the matrix coefficient by a character of $\mathbf{Q}_v^\times$ is in the $L^2(Z_{\mathrm{GL}_2}(\mathbf{Q}_v)\backslash \mathrm{GL}_2(\mathbf{Q}_v))$.

called an *L-parameter* (or a *Langlands parameter*) with $\det \circ \phi$ identified with ε_l through local class field theory. Here W_l is the Weil group at l defined in §0.4.1. In addition, ϕ satisfies the properties

$$L(s, \pi_l^{\mathrm{JL}} \otimes \chi) = L(s, \phi \otimes \chi) \qquad \text{and} \qquad \epsilon(s, \pi_l^{\mathrm{JL}} \otimes \chi, \psi') = \epsilon(s, \phi \otimes \chi, \psi')$$

for all characters χ of $\mathbf{Q}_l^\times$ and characters $\psi' \neq 1$ of $\mathbf{Q}_l$, and these properties determine the correspondence uniquely. Here

$$L(s, \phi \otimes \chi) = \det(1 - (\phi \otimes \chi)^{I_l}(\mathrm{Frob}_l) \cdot l^{-s})^{-1}$$

is the standard L-factor attached to the representation $(\phi \otimes \chi)|_{W_l}$.[†] The epsilon factors are essentially generalized Gauss sums used for establishing functional equations [BH06, §24.2, §29.4] for the respective L-factors, we will not mention them in the remaining sections. For convenience, we shall refer to ϕ as the L-parameter for π_l.

In the case that l does not divide the level $c(\boldsymbol{\pi})$ of $\boldsymbol{\pi}$, the representation $\pi_l = \pi_l^{\mathrm{JL}} \simeq \pi(\mu, \mu')$ is a spherical principal series for some characters μ and μ' of $\mathbf{Q}_l^\times$ that are trivial on $\mathbf{Z}_l^\times$. In this case, $\phi \simeq \mu \oplus \mu'$ where we view μ and μ' as unramified characters of W_l through local class field theory. In particular, we see that ϕ uniquely determined by its image on Frob_l,

$$\phi(\mathrm{Frob}_l) = \mathrm{diag}[\beta_l, \beta_l'],$$

where $\beta_l = \mu(\mathrm{Frob}_l)$ and $\beta_l' = \mu'(\mathrm{Frob}_l)$. We shall refer to $\{\beta_l, \beta_l'\}$ as the *Satake parameters* for π_l. Note that $\mu \cdot \mu' = \varepsilon_l$ and

$$L(s, \phi \otimes \chi) = \frac{1}{(1 - \chi(l)\beta_l \cdot l^{-s})(1 - \chi(l)\beta_l' \cdot l^{-s})}.$$

Finally, we set

$$L^{(c(\boldsymbol{\pi}))}(s, \boldsymbol{\pi}) = \prod_{l \nmid c(\boldsymbol{\pi})} L(s, \phi_l)$$

(so taking $\chi = 1$) and refer to it as the *partial L-function* for the automorphic representation $\boldsymbol{\pi}$.

[†] Recall that $I_l \subset W_l$ is the inertia subgroup and $\mathrm{Frob}_l \in W_l / I_l$ is the geometric Frobenius element.

2.3 Automorphic forms on $D^\times$

Let $\boldsymbol{\pi}$ be an automorphic representation of level dNp^r, weight $2k$, and character ε. Note the restriction on the level implies that ε is unramified at all primes not dividing Np^r. We attach some (vector-valued) automorphic forms to a given automorphic representation $\boldsymbol{\pi}$ of $D^\times$.

2.3.1 The recipe.

Each of the automorphic forms that we attach to $\boldsymbol{\pi}$ depends on the choice of

- a finite set of primes S containing those dividing Np but not those dividing d,
- an S-basis $(\delta, \jmath)$ for D as in §1.1.3, and
- an Eichler order $\mathscr{D} = \mathscr{D}^{\delta,\jmath,S}(dNp^r)$ of level dNp^r as in §1.3.4.

Definition 2.3.1. An *automorphic form on* $D^\times$, $\mathbf{f} = \mathbf{f}_{\boldsymbol{\pi}}^{\delta,\jmath}$, attached to $\boldsymbol{\pi}$ with respect to $(\delta, \jmath)$ and $\mathscr{D}$ is a vector in the one-dimensional vector space [Gel75, Theorem 4.24]

$$\mathscr{A}^{\delta,\jmath}(\boldsymbol{\pi}) = (V_{\boldsymbol{\pi}} \otimes \check{\mathscr{V}}_{2k,\infty})^{\mathscr{D}_f^\times \times D_\infty^\times},$$

where $\mathscr{D}_f^\times$ acts* by $\varepsilon^{-1}\boldsymbol{\pi}_f \otimes 1$ and $D_\infty^\times$ acts by $\boldsymbol{\pi}_\infty \otimes \check{\sigma}_{2k,\mathrm{wt},\infty}^{\delta,\jmath}$.

As such, it satisfies the transformation law

$$\mathbf{f}(\gamma x u_f u_\infty) = \varepsilon(u_f) \cdot \check{\sigma}_{2k,\mathrm{wt},\infty}^{\delta,\jmath}(u_\infty)^{-1} \cdot \mathbf{f}(x) \tag{3.1.1}$$

for all $\gamma \in D_{\mathbf{Q}}^\times$, $u_f \in \mathscr{D}_f^\times$, and $u_\infty \in D_\infty^\times$. Moreover, we can express $\mathbf{f}$ as a function

$$\mathbf{f} = \sum_{i=-k}^{k} f_i \otimes \mathbf{t}_i^k \colon D_{\mathbf{A}}^\times \to \check{\mathscr{V}}_{2k,\infty},$$

for some $f_i \in V_{\boldsymbol{\pi}}$.

2.3.2 Hecke operators.

Let $\mathscr{D}_l = \mathscr{D}^{\delta,\jmath,S}(dNp^r) \otimes \mathbf{Z}_l$ be the local Eichler order as in the previous section. For any prime l, define the Hecke algebra $\mathscr{H}(\mathscr{D}_l^\times)$ to be the $\mathbf{C}$-algebra of locally-constant, compactly supported functions on $D_l^\times$ that are bi-invariant under the compact open

*Here we extend ε to a character on $\mathscr{D}_f^\times$ as in §2.2.5 after identifying $\mathscr{D}_l^\times$ with $I_l(\mathrm{ord}_l(n))^\times$ through $\epsilon_{\mathrm{ar},l}^{\delta,\jmath}$.

subgroup $\mathscr{D}_l^\times$. It acts on π_l by convolution. We distinguish some special Hecke operators arithmetic significance.

$l \nmid dNp^r$**:** Let $\varpi_l \in D_l^\times$ be an element of norm l, and denote by $T_l \in \mathscr{H}(\mathscr{D}_l^\times)$ the normalized characteristic function $\mathbf{1}_{l\mathbf{Z}_l^\times} = \frac{1}{\mathrm{vol}(\mathscr{D}_l^\times)} \cdot \mathbf{1}_{\mathscr{D}_l^\times \varpi_l \mathscr{D}_l^\times}$ on the set of elements in $D_l^\times$ with norms in $l\mathbf{Z}_l^\times$. The convolution action by T_l on V_{π_l} preserves the one-dimensional subspace $V_{\pi_l}^{\mathscr{D}_l^\times}$, and hence

$$T_l \cdot f_l = a_l(\boldsymbol{\pi}) \cdot f_l$$

for some $a_l(\boldsymbol{\pi}) \in \mathbf{C}$ and all $f_l \in V_{\pi_l}^{\mathscr{D}_l^\times}$. We can describe the induced action on $\mathscr{A}^{\delta,\jmath}(\boldsymbol{\pi})$ explicitly by

$$\begin{aligned}(T_l \cdot \mathbf{f})(x) &= \frac{1}{\mathrm{vol}(\mathscr{D}_l^\times)} \cdot \int_{D_l^\times} \mathbf{1}_{l\mathbf{Z}_l^\times}(y^{-1}x) \cdot \mathbf{f}(y)\, d^\times y \\ &= \sum_i \mathbf{f}(l^{-1} \cdot x \cdot \varpi_{i,l}) = \varepsilon_l(l)^{-1} \sum_{i=0}^{l} \mathbf{f}(x \cdot \varpi_{i,l})\end{aligned}$$

with respect to any decomposition $\mathscr{D}_l^\times \varpi_l \mathscr{D}_l^\times = \bigcup_{i=0}^l \varpi_{i,l} \cdot \mathscr{D}_l^\times$.† Setting $a_l(\mathbf{f}) = a_l(\boldsymbol{\pi})$, we have $T_l \cdot \mathbf{f} = a_l(\mathbf{f}) \cdot \mathbf{f}$.

$l \mid dNp^r$**:** Denote by $w_l \in \mathscr{H}(\mathscr{D}_l^\times)$ the normalized characteristic function $\frac{1}{\mathrm{vol}(\mathscr{D}_l^\times)} \mathbf{1}_{\mathscr{D}_l^\times \varpi_l \mathscr{D}_l^\times}$, where ϖ_l is the element of norm l^{r_l} in $\mathscr{N}^{\delta,\jmath,S}(dNp^r)_l$ (here $r_l = \mathrm{ord}_l(dNp^r)$) as defined in §1.3.5. Since $\varpi_l^2 = -l^{r_l} \in \mathbf{Q}_l^\times$, we have that

$$w_l^2 \cdot f_l = \varepsilon_l(-l^{r_l}) \cdot f_l$$

for all $f_l \in V_{\pi_l}$. We shall refer to these operators w_l as the *Atkin–Lehner operators*. Note that they are involutions when $\varepsilon_l = 1$. As before, denote by $a_l(\boldsymbol{\pi})$ (resp. $a_l(\mathbf{f})$) the eigenvalue of w_l on π_l (resp. $\mathbf{f} \in \mathscr{A}^{\delta,\jmath}(\boldsymbol{\pi})$). We see it is either $\sqrt{\varepsilon_l(-l^{r_l})}$ or $-\sqrt{\varepsilon_l(-l^{r_l})}$.

We note that for $l \nmid dNp^r$, the Satake parameters $\{\beta_l, \beta_l'\}$ attached to π_l are roots of the Hecke polynomial

$$\det(X - T_p) = X^2 - a_l(\mathbf{f})X + l^{2k-2}\varepsilon(l).$$

†The last equality follows since $\mathbf{1}_{l\mathbf{Z}_l^\times}(y^{-1}x)$ is supported on $\{y \in x\mathscr{D}_l^\times \varpi_l^{-1} \mathscr{D}_l^\times = l^{-1} \cdot x \mathscr{D}_l^\times \varpi_l \mathscr{D}_l^\times\}$.

2.3.3 Compatibility with conjugation.

Let $(\delta, \jmath)$ be an S-basis and (γ, κ) an S'-basis for D for two possibly different finite sets of primes S and S'. We maintain the assumption that S and S' contain primes dividing Np and do not contain those dividing d. We now compare the two automorphic forms $\mathbf{f}^{\delta,\jmath} = \mathbf{f}_{\boldsymbol{\pi}}^{\delta,\jmath}$ and $\mathbf{f}^{\gamma,\kappa} = \mathbf{f}_{\boldsymbol{\pi}}^{\gamma,\kappa}$ attached to the same $\boldsymbol{\pi}$.

Let $\alpha = (\alpha_v) \in D_{\mathbf{A}}^\times$ be an element such that

- $\check{\sigma}_{2k,\mathrm{wt},\infty}^{\gamma,\kappa} \circ \mathrm{Ad}(\alpha_\infty) = \check{\sigma}_{2k,\mathrm{wt},\infty}^{\delta,\jmath}$ as representations of $D_\infty^\times$, and
- $\alpha_l \cdot \mathscr{D}^{\delta,\jmath,S}(dNp^r)_l \cdot \alpha_l^{-1} = \mathscr{D}^{\gamma,\kappa,S'}(dNp^r)_l$ for each l.

By Corollary 1.4.3, α_∞ not only exists but can be chosen to lie in $D_F^\times$ where $F = \mathbf{Q}(\sqrt{-\Delta_\delta}, \sqrt{-\Delta_\gamma}, \sqrt{\mathrm{n}(\jmath)}, \sqrt{\mathrm{n}(\kappa)})$. The existence of α_l follows from the fact that all local Eichler orders of a given level n are conjugate.

Consider the automorphic form $\boldsymbol{\pi}(\alpha^{-1}) \cdot \mathbf{f}^{\gamma,\kappa}$. It transforms under $u_\infty \in D_\infty^\times$ by

$$\begin{aligned}(\boldsymbol{\pi}(\alpha^{-1}) \cdot \mathbf{f}^{\gamma,\kappa})(xu_\infty) &= \mathbf{f}^{\gamma,\kappa}(x\alpha_f^{-1}\alpha_\infty^{-1} \cdot \alpha_\infty u_\infty \alpha_\infty^{-1}) \\ &= \check{\sigma}_{2k,\mathrm{wt},\infty}^{\delta,\jmath}(u_\infty)^{-1} \cdot \mathbf{f}^{\gamma,\kappa}(x\alpha^{-1}) \\ &= \check{\sigma}_{2k,\mathrm{wt},\infty}^{\delta,\jmath}(u_\infty)^{-1} \cdot (\boldsymbol{\pi}(\alpha^{-1}) \cdot \mathbf{f}^{\gamma,\kappa})(x);\end{aligned}$$

and transforms under $u_f \in \mathscr{D}^{\delta,\jmath,S}(dNp^r)_f^\times$ by

$$\begin{aligned}(\boldsymbol{\pi}(\alpha^{-1}) \cdot \mathbf{f}^{\gamma,\kappa})(xu_f) &= \mathbf{f}^{\gamma,\kappa}(x\alpha_\infty^{-1}\alpha_f^{-1} \cdot \alpha_f u_f \alpha_f^{-1}) \\ &= \varepsilon(u_f) \cdot \mathbf{f}^{\gamma,\kappa}(x\alpha^{-1}) \\ &= \varepsilon(u_f) \cdot (\boldsymbol{\pi}(\alpha^{-1}) \cdot \mathbf{f}^{\gamma,\kappa})(x).\end{aligned}$$

We see that

$$\boldsymbol{\pi}(\alpha^{-1}) \cdot \mathbf{f}^{\gamma,\kappa} \in (V_{\boldsymbol{\pi}} \otimes \check{\mathscr{V}}_{2k,\infty})^{\mathscr{D}_f^\times \times D_\infty^\times}$$

where $\mathscr{D}_f = \mathscr{D}^{\delta,\jmath,S}(dNp^r)_f$. Since $(V_{\boldsymbol{\pi}} \otimes \check{\mathscr{V}}_{2k,\infty})^{\mathscr{D}_f^\times \times D_\infty^\times}$ is one-dimensional, we conclude that

$$\mathbf{f}^{\delta,\jmath} = \mu \cdot \boldsymbol{\pi}(\alpha^{-1}) \cdot \mathbf{f}^{\gamma,\kappa}$$

for some $\mu \in \mathbf{C}^\times$. This guarantees that the following definition is not vacuous.

Definition 2.3.2. Given an S-basis $(\delta, \jmath)$ and an S'-basis (γ, κ) for D, we say the automorphic forms $\mathbf{f}_{\boldsymbol{\pi}}^{\delta,\jmath}$ and $\mathbf{f}_{\boldsymbol{\pi}}^{\gamma,\kappa}$ are *compatible* if there exists some $\alpha = (\alpha_v) \in D_{\mathbf{A}}^\times$, with $\alpha_\infty \in D_F^\times \subset D_\infty^\times$ for the totally real number field $F = \mathbf{Q}(\sqrt{-\Delta_\delta}, \sqrt{-\Delta_\gamma}, \sqrt{\mathrm{n}(\jmath)}, \sqrt{\mathrm{n}(\kappa)})$, such that

$$\mathbf{f}_{\boldsymbol{\pi}}^{\delta,\jmath} = \boldsymbol{\pi}(\alpha^{-1}) \cdot \mathbf{f}_{\boldsymbol{\pi}}^{\gamma,\kappa}$$

as vectors in $(V_{\boldsymbol{\pi}} \otimes \check{\mathscr{V}}_{2k,\infty})^{\mathscr{D}^{\delta,\jmath,S}(dNp^r)_f^\times \times D_\infty^\times}$. More generally, if $\{(\delta,\jmath)\}$ is a set of $S^{\delta,\jmath}$-bases for D, then a corresponding collection of automorphic forms $\{\mathbf{f}_{\boldsymbol{\pi}}^{\delta,\jmath}\}$, indexed by these $S^{\delta,\jmath}$-basis $(\delta,\jmath)$ of D, is a *compatible set of automorphic forms* attached to $\boldsymbol{\pi}$ if any two members are compatible.

We emphasize that the choice $\alpha = (\alpha_v)$ is independent of the representation $\boldsymbol{\pi}$. Indeed, we see from the definition that α_l depends only on the choice of the local Eichler orders for a finite prime l; and $\alpha_\infty \in D_F^\times$ is used to conjugate the weight embeddings as in Proposition 1.4.2. Also note that since

$$\begin{aligned}\mathrm{Ad}(\alpha_l^{-1}) \cdot \mathscr{H}_l(\mathscr{D}^{\gamma,\kappa,S'}(dNp^r)_l^\times) &= \{x \mapsto \phi(\alpha_l x \alpha_l^{-1}) : \phi \in \mathscr{H}_l(\mathscr{D}^{\gamma,\kappa,S'}(dNp^r)_l^\times)\} \\ &= \mathscr{H}_l(\mathscr{D}^{\delta,\jmath,S}(dNp^r)_l^\times),\end{aligned}$$

compatible automorphic forms for $\boldsymbol{\pi}$ have the same Hecke eigenvalues $a_l(\boldsymbol{\pi})$ at all primes l.

2.3.4 Algebraic automorphic forms.

Define $\mathbf{f}^{\mathrm{alg}}: D_f^\times \to \check{\mathscr{V}}_{2k,\infty}$ to be the restriction of $\mathbf{f}$ to $D_f^\times$. It satisfies the transformation law

$$\mathbf{f}^{\mathrm{alg}}(\gamma^\infty x u_f) = \mathbf{f}(\gamma x u_f \gamma_\infty^{-1}) = \varepsilon(u_f) \cdot \check{\sigma}_{2k,\mathrm{wt},\infty}^{\delta,\jmath}(\gamma) \cdot \mathbf{f}^{\mathrm{alg}}(x)$$

for all $\gamma = \gamma^\infty \gamma_\infty \in D_{\mathbf{Q}}^\times \subset D_{\mathbf{A}}^\times = D_f^\times \times D_\infty^\times$, $x \in D_f^\times$, and $u_f \in \mathscr{D}_f^\times$. Since the action of $D_{\mathbf{Q}}^\times$ under $\check{\sigma}_{2k,\mathrm{wt},\infty}^{\delta,\jmath}$ is in fact defined over the number field $K_\jmath$, the following definition makes sense.

Definition 2.3.3. An automorphic form $\mathbf{f}_{\boldsymbol{\pi}}^{\delta,\jmath}$ is *algebraic* if $\mathbf{f}_{\boldsymbol{\pi}}^{\delta,\jmath}(D_f^\times) \subset \check{\mathscr{V}}_{2k,\bar{\mathbf{Q}}}$. A compatible set of automorphic forms $\{\mathbf{f}_{\boldsymbol{\pi}}^{\delta,\jmath}\}$ attached to $\boldsymbol{\pi}$ is *algebraic* if one (and hence all) of its members are algebraic.

Let us verify the assertion made in the parenthesis, that is, every member of an algebraic compatible set $\{\mathbf{f}_{\boldsymbol{\pi}}^{\delta,\jmath}\}$ satisfies the condition that $\mathbf{f}_{\boldsymbol{\pi}}^{\delta,\jmath}(D_f^\times) \subset \check{\mathscr{V}}_{2k,\bar{\mathbf{Q}}}$. Indeed, let $\mathbf{f}_{\boldsymbol{\pi}}^{\gamma,\kappa}$ be an automorphic form satisfying this condition as in the definition; then given another automorphic form $\mathbf{f}_{\boldsymbol{\pi}}^{\delta,\jmath}$ in the compatible family, we have, for some $\alpha \in D_{\mathbf{A}}^\times$ with $\alpha_\infty \in D_F^\times \subset D_\infty^\times$, that

$$\begin{aligned}\mathbf{f}_{\boldsymbol{\pi}}^{\delta,\jmath}(D_f^\times) = (\boldsymbol{\pi}(\alpha^{-1}) \cdot \mathbf{f}_{\boldsymbol{\pi}}^{\gamma,\kappa})(D_f^\times) &= \check{\sigma}_{2k,\mathrm{wt},\infty}^{\gamma,\kappa}(\alpha_\infty) \cdot \mathbf{f}_{\boldsymbol{\pi}}^{\gamma,\kappa}(D_f^\times) \\ &\subset \check{\sigma}_{2k,\mathrm{wt},\infty}^{\gamma,\kappa}(\alpha_\infty) \cdot \check{\mathscr{V}}_{2k,\bar{\mathbf{Q}}} = \check{\mathscr{V}}_{2k,\bar{\mathbf{Q}}}\end{aligned} \tag{3.4.2}$$

as claimed (noting that $\alpha_\infty \in D_F^\times$ implies $\epsilon_{\mathrm{wt},\infty}^{\gamma,\kappa}(\alpha_\infty) \in \mathrm{GL}_2(\bar{\mathbf{Q}})$).

Now since $D_{\mathbf{Q}}^{\times}\backslash D_f^{\times}/\mathscr{D}_f^{\times}$ is finite [Vig80, Ch. III, Théorème 5.4], we see that an algebraic automorphic form $\mathbf{f}_{\boldsymbol{\pi}}^{\delta,\jmath,\mathrm{alg}} = \mathbf{f}^{\delta,\jmath}|_{D_f^{\times}}$ in fact takes values in $\check{\mathscr{V}}_{2k,K_\jmath}^{\delta,\jmath} \otimes K$ for some number field $K \supseteq K_\jmath$. We refer to the intersection of all such K's as the *field of definition* of $\mathbf{f}_{\boldsymbol{\pi}}^{\delta,\jmath}$ and denote it by $K(\mathbf{f}_{\boldsymbol{\pi}}^{\delta,\jmath})$.

Finally, by (3.4.2), we see that two compatible algebraic automorphic forms $\mathbf{f}_{\boldsymbol{\pi}}^{\delta,\jmath}$ and $\mathbf{f}_{\boldsymbol{\pi}}^{\gamma,\kappa}$ are both defined over the compositum $K(\mathbf{f}_{\boldsymbol{\pi}}^{\delta,\jmath}) \cdot F = K(\mathbf{f}_{\boldsymbol{\pi}}^{\gamma,\kappa}) \cdot F$ where $F = \mathbf{Q}(\sqrt{-\Delta_\delta}, \sqrt{-\Delta_\gamma}, \sqrt{\mathrm{n}(\jmath)}, \sqrt{\mathrm{n}(\kappa)})$ is the totally real field over which the element α_∞ is defined. As a result, we see that all members of an algebraic compatible set of automorphic forms $\{\mathbf{f}_{\boldsymbol{\pi}}^{\delta,\jmath}\}$ are defined over $K(\{\mathbf{f}_{\boldsymbol{\pi}}^{\bullet}\}) = K^{\mathrm{quad}} \cdot F$, where K^{quad} is the compositum $\mathbf{Q}(\sqrt{q} : q \in \mathbf{Z})$ of all quadratic extension of $\mathbf{Q}$ as in §1.4.2.

2.3.5 Existence of algebraic automorphic forms.

It remains to address the existence of such algebraic (compatible sets of) automorphic forms. The following proposition shows they exist abundantly.

Proposition 2.3.1. *For every automorphic representation $\boldsymbol{\pi}$, there exists an algebraic compatible set of automorphic forms $\{\mathbf{f}_{\boldsymbol{\pi}}^{\delta,\jmath}\}$ attached to $\boldsymbol{\pi}$.*

Proof. This is essentially [Gro99, Prop. 8.3]. We supply a proof in this special case for convenience. To ease the notations, we suppress the superscript $(\delta,\jmath)$. Let $\varepsilon = \varepsilon_{\boldsymbol{\pi}}$ be the central character of $\boldsymbol{\pi}$, recall from §2.2.4 that it is a finite character with values in the number field $\mathbf{Q}(\varepsilon)$ generated by its values. Denote by $M_{2k}(\mathscr{D}_f;\varepsilon)$ the space of functions* $\phi\colon D_{\mathbf{Q}}^{\times}\backslash D_f^{\times} \to \check{\mathscr{V}}_{2k,\mathbf{Q}(\varepsilon)}$ such that $\phi(xu_f) = \varepsilon(u_f)\cdot\phi(x)$ for all $x \in D_{\mathbf{Q}}^{\times}\backslash D_f^{\times}$ and $u_f \in \mathscr{D}_f^{\times}$. Since $D_{\mathbf{Q}}^{\times}\backslash D_f^{\times}/\mathscr{D}_f^{\times}$ is finite, we see that $M_{2k}(\mathscr{D}_f;\varepsilon)$ is finite-dimensional over $\mathbf{Q}(\varepsilon)$.

Let $\mathbf{T} = \mathbf{Z}[T_l : l \nmid dNp]$ be the sub-algebra of $\prod_{l\nmid dNp}\mathscr{H}(\mathscr{D}_l^{\times})$ generated by the Hecke operators T_l. Note that $\mathbf{T}$ is commutative, and that it acts on $M_{2k}(\mathscr{D}_f;\varepsilon)$ by convolution just as in §2.3.2. Consequently $M_{2k}(\mathscr{D}_f;\varepsilon)$ decomposes as a direct sum of finitely many simultaneous $\mathbf{T}$-eigenspaces $V_{\underline{a}}$ indexed by a collection of Hecke eigenvalues $\underline{a} = \{a(T_l) \in \bar{\mathbf{Q}} : l \nmid dNP\}$ of T_l:

$$M_{2k}(\mathscr{D}_f;\varepsilon)\otimes\bar{\mathbf{Q}} \simeq \bigoplus_{\underline{a}} V_{\underline{a}}.$$

Moreover, each $V_{\underline{a}} \simeq \oplus\bar{\mathbf{Q}}\cdot\phi$ is an orthogonal direct sum of one-dimensional eigenspaces $\bar{\mathbf{Q}}\cdot\phi$.

*The space $M_{2k}(\mathscr{D}_f;\varepsilon)$ is sometimes referred to as the space of automorphic forms of weight $2k$, level dNp^r, and character ε on $D^{\times}$ [Buz04, §4].

Now $\mathbf{f} = \mathbf{f}_{\boldsymbol{\pi}}^{\delta,\jmath}$ defines a vector in $V_{\underline{a}} \otimes_{\bar{\mathbf{Q}}} \mathbf{C}$ where $\underline{a} = \{a_l(\boldsymbol{\pi})\}$; conversely, every element ϕ in $M_{2k}(\mathscr{D}_f;\varepsilon)$ is a linear combination of such automorphic forms.† By multiplicity one [DI95, Corollary 6.3.1] for elliptic modular forms together with the Jacquet–Langlands correspondence, we see that the eigenspace $V_{\underline{a}}$ corresponding to $\mathbf{f}$ is one-dimensional, thus $\mathbf{f}$ is a multiple of $\phi \in M_{2k}(\mathscr{D}_f;\varepsilon) \otimes \bar{\mathbf{Q}}$. Consequently, we can scale $\mathbf{f}$ so that $\mathbf{f}(D_f^\times) \subset \check{\mathscr{V}}_{2k,\bar{\mathbf{Q}}}$, and then conjugate it to obtain an algebraic compatible set $\{\mathbf{f}^{\delta,\jmath}\}$. □

2.3.6 $\mathfrak{p}$-integral automorphic forms.

Let $D_p^\times$ act on the $\mathbf{C}_p$-vector space $\check{\mathscr{V}}_{2k,p} = \check{\mathscr{V}}_{2k,\bar{\mathbf{Q}}} \otimes \mathbf{C}_p$ by $\check{\sigma}_{2k,\mathrm{wt},p} = \check{\sigma}_{2k,\mathrm{wt}}^{\delta,\jmath} \otimes_{K_\jmath} \mathbf{C}_p$. For an algebraic automorphic representation $\boldsymbol{\pi}$, consider the function $\mathbf{f}_{\boldsymbol{\pi}}^p\colon D_f^\times \to \check{\mathscr{V}}_{2k,p}$ given by

$$\mathbf{f}_{\boldsymbol{\pi}}^p(x) = \mathbf{f}^p(x) = \check{\sigma}_{2k,\mathrm{wt},p}(x_p)^{-1} \cdot \mathbf{f}_{\boldsymbol{\pi}}^{\mathrm{alg}}(x).$$

It satisfies the transformation law

$$\begin{aligned}\mathbf{f}^p(\gamma x u_f^p u_p) &= \varepsilon(u_f) \cdot \check{\sigma}_{\mathrm{wt}}(\gamma x_p u_p)^{-1} \cdot \mathbf{f}^{\mathrm{alg}}(\gamma x)\\ &= \varepsilon(u_f) \cdot \check{\sigma}_{\mathrm{wt}}(u_p)^{-1} \cdot \check{\sigma}_{\mathrm{wt}}(x_p)^{-1} \cdot \check{\sigma}_{\mathrm{wt}}(\gamma)^{-1}\check{\sigma}_{\mathrm{wt}}(\gamma) \cdot \mathbf{f}^{\mathrm{alg}}(x)\\ &= \varepsilon(u_f) \cdot \check{\sigma}_{\mathrm{wt}}(u_p)^{-1} \cdot \mathbf{f}^p(x)\end{aligned}$$

for all $\gamma \in D_{\mathbf{Q}}^\times \subseteq D_f^\times$, $x \in D_f^\times$, $u_f^p \in \prod_{l\neq p} \mathscr{D}_l^\times$, $u_p \in \mathscr{D}_p^\times$, and $u_f = u_f^p u_p$. Let $\mathscr{M} = \mathscr{M}_{2k,p}(r)$ be the $\mathscr{O}_{\mathbf{C}_p}$-integral lattice in $\check{\mathscr{V}}_{2k,p}$ defined in §1.4.3. It is stable under the $\check{\sigma}_{2k,\mathrm{ar},p} = \check{\sigma}_{2k,\mathrm{ar},p}^{\delta,\jmath}$-action by $\mathscr{D}_p^\times$. As $\check{\sigma}_{2k,\mathrm{ar},p}$ and $\check{\sigma}_{2k,\mathrm{wt},p}$ are conjugated under the matrix $C_p^{\delta,\jmath}$ from §1.4.4, we see that the lattice

$$\mathscr{M}^{\delta,\jmath} = \check{\sigma}_p(C_p^{\delta,\jmath}) \cdot \mathscr{M}$$

is stable under the $\check{\sigma}_{2k,\mathrm{wt},p}$-action by $\mathscr{D}_p^\times$. This motivates the following definition.

Definition 2.3.4. An algebraic automorphic form $\mathbf{f}_{\boldsymbol{\pi}} = \mathbf{f}_{\boldsymbol{\pi}}^{\delta,\jmath}$ attached to $\boldsymbol{\pi}$ is $\mathfrak{p}$-*integral* if $\mathbf{f}_{\boldsymbol{\pi}}^p$ takes values in $\mathscr{M}^{\delta,\jmath}$, or, equivalently, if

$$\mathbf{f}_{\boldsymbol{\pi}}(x) \in \check{\sigma}_{2k,\mathrm{wt},p}^{\delta,\jmath}(x_p) \cdot \check{\sigma}_p(C_p^{\delta,\jmath}) \cdot \mathscr{M}$$

for all $x \in D_f^\times$. A compatible set of automorphic forms $\{\mathbf{f}^{\delta,\jmath}\}$ is $\mathfrak{p}$-*integral* if one (hence all, as we will see shortly) of its members is $\mathfrak{p}$-integral.

†Indeed, $(\mathscr{A}(D^\times) \otimes \check{\mathscr{V}}_{2k,\infty})^{\mathscr{D}_f^\times \times D_\infty^\times} \simeq M_{2k}(\mathscr{D}_f;\varepsilon) \otimes \mathbf{C}$.

Since an algebraic automorphic form $\mathbf{f}_{\boldsymbol{\pi}}^{\delta,\jmath,\mathrm{alg}} = \mathbf{f}_{\boldsymbol{\pi}}^{\delta,\jmath}|_{D_f^\times}$ takes values in $\check{\mathscr{V}}_{2k}^{\delta,\jmath} \otimes F(\mathbf{f}_{\boldsymbol{\pi}}^{\delta,\jmath})$, we see that it is $\mathfrak{p}$-integral for almost all prime ideals $\mathfrak{p}$ in $\bar{\mathbf{Q}}$.

2.3.7 $\mathfrak{p}$-integrality and S-basis.

We would like to check that every member of a $\mathfrak{p}$-integral compatible set of automorphic forms $\{\mathbf{f}_{\boldsymbol{\pi}}^{\delta,\jmath}\}$ is $\mathfrak{p}$-integral. Let $(\delta,\jmath)$ be a S-basis and (γ,κ) a S'-bases for D, and let $\mathbf{f}^{\delta,\jmath} = \mathbf{f}_{\boldsymbol{\pi}}^{\delta,\jmath}$ and $\mathbf{f}^{\gamma,\kappa} = \mathbf{f}_{\boldsymbol{\pi}}^{\gamma,\kappa}$ be a corresponding pair of compatible automorphic forms attached to $\boldsymbol{\pi}$. Assuming $\mathbf{f}^{\gamma,\kappa}$ is $\mathfrak{p}$-integral, we check that $\mathbf{f}^{\delta,\jmath}$ is also $\mathfrak{p}$-integral. Let $\alpha \in D_{\mathbf{A}}^\times$ be an element that conjugates $\mathbf{f}^{\gamma,\kappa}$ to $\mathbf{f}^{\delta,\jmath}$, i.e., $\boldsymbol{\pi}(\alpha^{-1})\cdot\mathbf{f}^{\gamma,\kappa} = \mathbf{f}^{\delta,\jmath}$. We need to verify that $(\boldsymbol{\pi}(\alpha^{-1})\cdot\mathbf{f}^{\gamma,\kappa})(x) = \mathbf{f}^{\gamma,\kappa}(x\cdot\alpha^{-1})$ lies in

$$\check{\sigma}_{2k,\mathrm{wt},p}^{\delta,\jmath}(x_p)\cdot\check{\sigma}_{2k,p}(C_p^{\delta,\jmath})\cdot\mathscr{M} = \check{\sigma}_{2k,p}(\epsilon_{\mathrm{wt},p}^{\delta,\jmath}(x_p)\cdot C_p^{\delta,\jmath})\cdot\mathscr{M} \tag{3.7.3}$$

for all $x \in D_f^\times$. Since $\mathbf{f}^{\gamma,\kappa}$ is $\mathfrak{p}$-integral, and $\alpha_\infty \in D_F$ for some number field F, we have that

$$\begin{aligned}\mathbf{f}^{\gamma,\kappa}(x\cdot\alpha^{-1}) &= \check{\sigma}_{2k,\mathrm{wt},F}^{\gamma,\kappa}(\alpha_\infty)\cdot\mathbf{f}^{\gamma,\kappa}(x\cdot\alpha_f^{-1})\\ &\in \check{\sigma}_{2k,p}(\epsilon_{\mathrm{wt},F}^{\gamma,\kappa}(\alpha_\infty)\cdot\epsilon_{\mathrm{wt},p}^{\gamma,\kappa}(x_p\cdot\alpha_p^{-1})\cdot C_p^{\gamma,\kappa})\cdot\mathscr{M}\end{aligned}$$

where $\epsilon_{\mathrm{wt},F}^{\gamma,\kappa}(\alpha_\infty)\cdot\epsilon_{\mathrm{wt},p}^{\gamma,\kappa}(x_p\cdot\alpha_p^{-1})\cdot C_p^{\gamma,\kappa}$ is a matrix in $\mathrm{GL}_2(F_p)$ for the finite extension $F_p/\mathbf{Q}_p$ in $\mathbf{C}_p$. It suffices to show that this image lattice

$$\check{\sigma}_{2k,p}(\epsilon_{\mathrm{wt},F}^{\gamma,\kappa}(\alpha_\infty)\cdot\epsilon_{\mathrm{wt},p}^{\gamma,\kappa}(x_p\cdot\alpha_p^{-1})\cdot C_p^{\gamma,\kappa})\cdot\mathscr{M}$$

is equal to the lattice on the right of (3.7.3) above; or equivalently, that the matrix

$$\begin{aligned}&(\epsilon_{\mathrm{wt},p}^{\delta,\jmath}(x_p)\cdot C_p^{\delta,\jmath})^{-1}\cdot\epsilon_{\mathrm{wt},F}^{\gamma,\kappa}(\alpha_\infty)\cdot\epsilon_{\mathrm{wt},p}^{\gamma,\kappa}(x_p\cdot\alpha_p^{-1})\cdot C_p^{\gamma,\kappa}\\ &= (C_p^{\delta,\jmath})^{-1}\cdot(\epsilon_{\mathrm{wt},p}^{\delta,\jmath}(x_p))^{-1}\cdot\epsilon_{\mathrm{wt},p}^{\delta,\jmath}(x_p)\cdot\epsilon_{\mathrm{wt},F}^{\gamma,\kappa}(\alpha_\infty)\cdot\epsilon_{\mathrm{wt},p}^{\gamma,\kappa}(\alpha_p^{-1})\cdot C_p^{\gamma,\kappa}\\ &= (C_p^{\delta,\jmath})^{-1}\cdot\epsilon_{\mathrm{wt},F}^{\gamma,\kappa}(\alpha_\infty)\cdot\epsilon_{\mathrm{wt},p}^{\gamma,\kappa}(\alpha_p^{-1})\cdot C_p^{\gamma,\kappa}\end{aligned}$$

stabilizes $\mathscr{M}$. By (4.5.6) from § 1.4.5, we have

$$(C_p^{\delta,\jmath})^{-1}\cdot\epsilon_{\mathrm{wt},\bar{\mathbf{Q}}}^{\gamma,\kappa}(\alpha_\infty)\cdot\epsilon_{\mathrm{wt},p}^{\gamma,\kappa}(\alpha_p^{-1})\cdot C_p^{\gamma,\kappa} = AM$$

for some scalar matrix $A \in \mathrm{GL}_2(\mathbf{C}_p)$ and $M \in N_p(r)$, which indeed stabilizes $\mathscr{M}$ under the $\check{\sigma}_{2k,p}$-action.

2.3.8 Non-Eisenstein at $\mathfrak{p}$.

We distinguish a class of $\mathfrak{p}$-integral automorphic forms that are important for arithmetic applications.

Definition 2.3.5. A $\mathfrak{p}$-integral automorphic form $\mathbf{f} = \mathbf{f}_{\boldsymbol{\pi}}^{\delta,\jmath}$ attached to a cuspidal automorphic representation $\boldsymbol{\pi}$ is *non-Eisenstein at* $\mathfrak{p}$ if $\langle \mathbf{t}_0^k, \mathbf{f}(x)\rangle_{2k}$ is not constant modulo $\mathfrak{p}$ as a function of $x \in D_f^\times$.

Let us check this definition is not vacuous:

Proposition 2.3.2. *There exist* $\mathfrak{p}$*-integral automorphic forms* $\mathbf{f}_{\boldsymbol{\pi}}^{\delta,\jmath}$ *that are non-Eisenstein at* $\mathfrak{p}$*.*

Proof. We have that $\mathbf{f} = \mathbf{f}_{\boldsymbol{\pi}}^{\delta,\jmath} = \sum_{i=-k}^{k} f_i \otimes \mathbf{t}_i^k$ where each $f_i \in V_{\boldsymbol{\pi}}$ is an eigenvector for all the Hecke operators. Now $\langle \mathbf{t}_0^k, \mathbf{f}(x)\rangle_{2k} = f_0(x)$; so if this matrix coefficient is constant modulo $\mathfrak{p}$ as a function in $x \in D_f^\times$, then by §2.3.2, we have

$$(T_l \cdot f_0)(x) = \varepsilon_l(l)^{-1} \sum_{j=0}^{l} f_0(x \cdot \varpi_{j,l}) \equiv \varepsilon_l(l)^{-1} \cdot (l+1) \cdot \alpha \pmod{\mathfrak{p}}$$

for some constant $\alpha \in \bar{\mathbf{Q}} \cap \mathscr{O}_{\mathbf{C}_p}$. This forces the Hecke eigenvalue $a_l(\boldsymbol{\pi})$ to be equivalent to $\varepsilon(l)^{-1} \cdot (l+1)$ modulo $\mathfrak{p}$ for all $l \nmid dNp$.

This, however, implies that the Galois representation associated with $\mathbf{f}$, or equivalently, associated with a newform in $\boldsymbol{\pi}^{\mathrm{JL}}$, is of the form $\varepsilon \cdot (1 \oplus \chi_{\mathrm{cyc}})$ modulo $\mathfrak{p}$, and is in particular not irreducible. Consequently, we see that any automorphic form $\mathbf{f}$ whose associated Galois representation is irreducible modulo $\mathfrak{p}$ is non-Eisenstein at $\mathfrak{p}$ in the above sense, and there are such residually irreducible Galois representations by [Lan95, Part XI, Theorem 3.4]. □

Chapter 3. $\mathrm{GSO}(D)$ and GSp_4

3.1 From $D^\times$ to $\mathrm{GSO}(D)$

We now consider D as a four-dimensional vector space over $\mathbf{Q}$ equipped with the symmetric bilinear form $(x, y)_D = \mathrm{tr}(x\bar{y})$. The associated quadratic form is twice the reduced norm $(x, x)_D = 2\mathrm{n}(x)$.

3.1.1 Orthogonal similitude groups.

We have the corresponding group of orthogonal similitudes,

$$\mathrm{GO}(D) = \{h \in \mathrm{GL}(D) : \mathrm{n}(h \cdot x) = \lambda(h) \cdot \mathrm{n}(x) \text{ for all } x \in D\},$$

where λ is the multiplier character.

Let $D^\times \times D^\times$ act on D by

$$(\alpha, \beta) \cdot x \mapsto \alpha x \beta^{-1}.$$

This gives a homomorphism $D^\times \times D^\times \to \mathrm{GO}(D)$ with kernel the subgroup $Z^\triangle$ given by $Z = Z_{D^\times}$ diagonally embedded inside $D^\times \times D^\times$.* The image of the homomorphism is the connected component $\mathrm{GSO}(D)$ of $\mathrm{GO}(D)$ defined by the condition $\det(h) = \lambda(h)^2$. In other words, we have an isomorphism

$$\tilde{H} = Z^\triangle \backslash D^\times \times D^\times \simeq \mathrm{GSO}(D).$$

The main involution is an isometry on D and has determinant -1. It follows that

$$\tilde{H}^+ = \tilde{H} \rtimes \boldsymbol{\iota} \simeq \mathrm{GO}(D)$$

*Alternatively, one can let $D^\times \times D^\times$ act on D by $(\alpha, \beta) \cdot x \mapsto \alpha x \bar{\beta}$ as in [Rob01] and [GT]. The difference between the two isomorphisms is a matter of language. We decided to use the isomorphism above because it appears better suited for arithmetic.

under the multiplication given by

$$(\alpha,\beta)\rtimes\iota\cdot(\alpha',\beta')\rtimes\iota' = (\alpha,\beta)\cdot(\alpha',\beta')^{\iota'}\rtimes\iota\iota'.$$

We note that the non-trivial element $\iota \in \boldsymbol{\iota}$ acts on $\tilde{H}$ by $(\alpha,\beta)^{\iota} = (\bar{\beta}^{-1},\bar{\alpha}^{-1})$.

The center $Z^{\tilde{H}^+} = Z^{\tilde{H}}$ of $\tilde{H}^+$ and $\tilde{H}$ is the image of $Z_{D^\times}\times Z_{D^\times}$. We identify $Z^{\tilde{H}} \simeq \mathbf{G}_m$ through its projection onto the first factor.

3.1.2 Orthogonal groups.

The kernel of λ in $\mathrm{GO}(D)$ (resp. $\mathrm{GSO}(D)$) is the subgroup $\mathrm{O}(D)$ (resp. $\mathrm{SO}(D)$). The kernel of λ in $\tilde{H}$, on the other hand, is the subgroup

$$H = \{(\alpha,\beta)\in\tilde{H}^+ : \mathrm{n}(\alpha) = \mathrm{n}(\beta)\}.$$

We then have $H \simeq \mathrm{SO}(D)$ and $H^+ = H\rtimes\boldsymbol{\iota} \simeq \mathrm{O}(D)$.

The multiplier character pulls back to the character $\lambda(\alpha,\beta) = \mathrm{n}(\alpha)\mathrm{n}(\beta)^{-1}$ on $\tilde{H}$. Since $\lambda(\iota) = 1$, we see that

$$\lambda(\tilde{H}^+_{\mathbf{A}}) = \mathrm{n}(D^\times_{\mathbf{A}}) = \mathbf{A}_f^\times\cdot\mathbf{R}_{>0}$$

and $\lambda(\tilde{H}^+_{\mathbf{Q}}) = \mathbf{Q}_{>0}$ [Vig80, Ch. III, Théorème 4.1]. Now for $z\in Z_\infty\simeq\mathbf{R}^\times$, we have $\lambda(z,1) = \mathrm{n}(z) = z^2$. It follows that

$$\mathrm{GSO}(D)_\infty = Z^{\tilde{H}}_\infty\cdot\mathrm{SO}(D)_\infty \quad\text{and}\quad \mathrm{GO}(D)_\infty = Z^{\mathrm{GO}}_\infty\cdot\mathrm{O}(D)_\infty$$

since we can factor an element $h\in\tilde{H}_\infty$ as $h = z_h\cdot\frac{h}{z_h}$ with $z_h = (\sqrt{\lambda(h)},1)\in Z^{\tilde{H}}_\infty$. Form this we conclude that the maximal compact subgroup of $\mathrm{GSO}(D)_\infty \simeq \tilde{H}_\infty$ (resp. $\mathrm{GO}(D)_\infty \simeq \tilde{H}^+_\infty$) is $\mathrm{SO}(D)_\infty\simeq H_\infty$ (resp. $\mathrm{O}(D)_\infty \simeq H^+_\infty$).

3.1.3 Rank-one tori in $\tilde{H}$.

Fix $(\delta,\jmath)$ an S-basis for D, and consider the two-dimensional subspace $\dot{K} = K^\delta\subset D$. Denote by T^δ the subgroup of $\tilde{H}\simeq\mathrm{GSO}(D)$ that fixes $\dot{K}$ point-wise. Using the decomposition $D = \dot{K}\perp\jmath\dot{K}$, we see that

$$T^\delta = \triangle(\mathbf{Q}^\times\backslash\dot{K}^\times)$$

is the image of $\dot{K}^\times$ in $\tilde{H}$ under the diagonal embedding. Note that T^δ in fact lies in H.

3.1.4 Compact open subgroups.

For a prime l, the maximal compact open subgroups of $\mathrm{GL}(D)_l$ are given by

$$\mathrm{GL}(\mathscr{D}_l) = \{g \in \mathrm{GL}(D)_l : g \cdot \mathscr{D} \subseteq \mathscr{D}\}$$

for lattice $\mathscr{D} \subseteq D$.

The intersection $\tilde{U}_l^{\mathscr{D}} = \tilde{H}_l \cap \mathrm{GL}(\mathscr{D}_l)$ (resp. $U_l^{\mathscr{D}} = H_l \cap \mathrm{GL}(\mathscr{D}_l)$) is then a compact subgroup of $\tilde{H}_l = \tilde{H}(\mathbf{Q}_l)$ (resp. $H_l = H(\mathbf{Q}_l)$) which is maximal for almost all primes l. In particular, if $\mathscr{D}_l = \mathscr{D}^{\delta,\jmath,S}(dNp^r)_l$ is an Eichler order attached to the S-basis $(\delta, \jmath)$ of D in §1.3.4, then by §1.3.5, $\tilde{U}_l^{\mathscr{D}}$ is the subgroup of $\tilde{H}_l$ generated by the image of $\mathscr{D}_l^\times \times \mathscr{D}_l^\times$ and the element $(w_l(n), w_l(n))$ since we factor through $Z_l^\triangle$ in defining $\tilde{H}$. Similarly, we see that $U_l^{\mathscr{D}}$ is generated by the image of $\{(\alpha, \beta) \in \mathscr{D}_l^\times \times \mathscr{D}_l^\times : \mathrm{n}(\alpha) = \mathrm{n}(\beta)\}$ and $(w_l(n), w_l(n))$.

Set $\tilde{U}_f^{\mathscr{D}} = \prod_{l<\infty} \tilde{U}_l^{\mathscr{D}}$ and $U_f^{\mathscr{D}} = \prod_{l<\infty} U_l^{\mathscr{D}}$, then $\tilde{U}^{\mathscr{D}} = \tilde{U}_f^{\mathscr{D}} \times H_\infty$ (resp. $U^{\mathscr{D}} = U_f^{\mathscr{D}} \times H_\infty$) is a compact subgroup of $\tilde{H}_{\mathbf{A}}$ (resp. $H_{\mathbf{A}}$).

Since the main involution on D leaves $\mathscr{D}$ invariant, $\tilde{U}_l^{\mathscr{D},+} = \tilde{U}_l^{\mathscr{D}} \rtimes \boldsymbol{\iota}_l$ and $U_l^{\mathscr{D},+} = U_l^{\mathscr{D}} \rtimes \boldsymbol{\iota}_l$ define compact open subgroups of $\tilde{H}_l^+$ and H_l^+ which are maximal for all but finitely many primes l. We define $\tilde{U}_f^{\mathscr{D},+}$ and $U_f^{\mathscr{D},+}$ (resp. $\tilde{U}^{\mathscr{D},+}$ and $U^{\mathscr{D},+}$) of $\tilde{H}_f^+$ and H_f^+ (resp. of $\tilde{H}_{\mathbf{A}}^+$ and $H_{\mathbf{A}}^+$) as before.

3.2 Measures

We make precise our normalizations of Haar measures on the various groups along the same lines as in [Wei82].

3.2.1 Decompositions of $\tilde{H}$ and H.

We have a commutative diagram of algebraic groups

$$\begin{tikzcd}
1 \arrow[r] & D^1 \arrow[r, "\mathrm{in}"] \arrow[d] & H \arrow[r, "\mathrm{pr}"] \arrow[d] & Z\backslash D^\times \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & D^\times \arrow[r, "\mathrm{in}"] & \tilde{H} \arrow[r, "\mathrm{pr}"] & Z\backslash D^\times \arrow[r] & 1
\end{tikzcd}$$

where

$$\mathrm{in}\colon \alpha \mapsto (1,\alpha) \quad \text{and} \quad \mathrm{pr}\colon (\alpha,\beta) \mapsto \alpha.$$

The diagram admits a section $Z\backslash D^\times \to H$ given by $\alpha \mapsto (\alpha,\alpha)$, and we obtain a realization of H and $\tilde{H}$ as semi-direct products:

$$\tilde{H} \simeq (Z\backslash D^\times) \ltimes D^\times \quad \text{and} \quad H \simeq (Z\backslash D^\times) \ltimes D^1$$

by identifying $\alpha \ltimes \beta$ with $(\alpha, \alpha\beta)$ in $\tilde{H}$. The multiplication in the semi-product is given by

$$\alpha \ltimes \beta \cdot \alpha' \ltimes \beta' = (\alpha\alpha') \ltimes (\alpha'^{-1}\beta\alpha'\beta).$$

3.2.2 Local measures.

For each place v, we normalize the Haar measure $d\tilde{h}_v$ (resp. dh_v) on $\tilde{H}_v$ (resp. H_v) so it is compatible for the Haar measures on $D_v^\times$ (resp. D_v^1) and $Z_v\backslash D_v^\times$ that we fixed in §2.1.

Remark 3.2.1. Alternately, one can appeal to the definition of $\tilde{H}_v$ and normalize the Haar measure so it respects the Haar measures on $Z_v^\triangle \simeq \mathbf{Q}_v^\times$ and $D_v^\times \times D_v^\times$. That this procedure gives the same normalization follows from the commutative diagram

$$\begin{tikzcd} Z^\triangle \arrow[r, "z\mapsto(z,1)"] \arrow[d, equal] & D^\times \times D^\times \arrow[d, "(\alpha,\beta)\mapsto(\alpha,\alpha\beta)"] \\ Z^\triangle \arrow[r, "z\mapsto(z,z)"'] & D^\times \times D^\times \end{tikzcd}$$

and the fact that $D^\times$ is unimodular. The same goes for H_v.

We mention in passing that $\mathrm{vol}(\tilde{H}_\infty) = \mathrm{vol}(D_\infty^1)\cdot\mathrm{vol}(\mathbf{R}^\times\backslash D_\infty^\times) = 2\pi^3$.

3.2.3 Global measures.

Denote by $d\tilde{h}$ and dh the resulting Tamagawa measures on $\tilde{H}_{\mathbf{A}}$ and $H_{\mathbf{A}}$. We have that

$$\mathrm{vol}(H_{\mathbf{Q}}\backslash H_{\mathbf{A}}) = \mathrm{vol}(D^\times\mathbf{A}^\times\backslash D_{\mathbf{A}}^\times)\cdot\mathrm{vol}(D^1\backslash D_{\mathbf{A}}^1) = 2$$

as expected [Wei82, §3.7(c)].

3.2.4 Modulus of some outer automorphisms.

Consider the automorphism on $H_{\mathbf{A}} = \{(\alpha, \beta) : \mathrm{n}(\alpha) = \mathrm{n}(\beta)\}$ induced by conjugation by $(\gamma, 1)$ for some $\gamma \in D_{\mathbf{A}}^{\times}$. Using the semi-direct product representation $H_{\mathbf{A}} \simeq Z_{\mathbf{A}} \backslash D_{\mathbf{A}}^{\times} \ltimes D_{\mathbf{A}}^{1}$, we see that this automorphism corresponds to conjugation by $\gamma \ltimes \gamma^{-1}$. Now

$$(\gamma \ltimes \gamma^{-1}) \cdot (\alpha \ltimes \beta) \cdot (\gamma^{-1} \ltimes \gamma) = \alpha^{\gamma} \ltimes ((\alpha^{\gamma})^{-1} \alpha \beta)$$

where we set $\alpha^{\gamma} = \gamma \alpha \gamma^{-1}$. Since $(\alpha^{\gamma})^{-1} \alpha$ is in $D_{\mathbf{A}}^{1}$, and $Z_{\mathbf{A}} \backslash D_{\mathbf{A}}^{\times}$, $D_{\mathbf{A}}^{1}$ are unimodular, we conclude that the modulus of the conjugation automorphism $(\gamma, 1)$ is 1.

3.2.5 Measures on $\tilde{H}^{+}$ and H^{+}.

To define measures on the groups $\tilde{H}_v^{+}$ and H_v^{+}, it suffices to specify a measure on the the two-torsion subgroup $\boldsymbol{\iota}$ generated by the main involution $x \mapsto \bar{x}$. Let $d\iota_v$ be the Haar measure on $\boldsymbol{\iota}_v$ such that $\mathrm{vol}(\boldsymbol{\iota}_v) = 1$. The product measure $d\iota = \prod_v d\iota_v$ is a Tamagawa measure on $\boldsymbol{\iota}_{\mathbf{A}}$ giving $\boldsymbol{\iota}_{\mathbf{Q}} \backslash \boldsymbol{\iota}_{\mathbf{A}}$ a volume of $\frac{1}{2}$ (since $\mathrm{vol}(\boldsymbol{\iota}_{\mathbf{A}}) = 1$ and $\# \boldsymbol{\iota}_{\mathbf{Q}} = 2$). We denote by $d^{+}\tilde{h}_v$ and $d^{+}h_v$ the normalized Haar measures thus obtained on $\tilde{H}_{\mathbf{A}}^{+}$ and $H_{\mathbf{A}}^{+}$.

3.3 Some representation theory of $\mathrm{GSO}(D)$

We discuss some representation theory of $\mathrm{GSO}(D)$ and its siblings using representation theory of $D^{\times}$ and the isomorphism $\mathrm{GSO}(D) \simeq \tilde{H}$.

3.3.1 Decomposing representations.

The isomorphism $\mathrm{GSO}(D) \simeq \tilde{H}$ shows that every complex representation (ϱ, V_{ϱ}) of $\tilde{H}$ is isomorphic to a product

$$(\varrho_1 \boxtimes \varrho_2, V_{\varrho,1} \otimes V_{\varrho,2})$$

of representations $(\varrho_i, V_{\varrho,i})$, $i = 1, 2$, of $D^{\times}$. Furthermore, starting with a pair of irreducible complex representation ϱ_i of $D^{\times}$ with the central characters ω and ω^{-1} respectively, the product representation

$$\varrho_1 \boxtimes \varrho_2$$

of $\mathrm{GSO}(D)$ is also irreducible with central character ω [GW09, Proposition 4.2.5]. If $\langle \bullet, \bullet \rangle_i$ is a $D^{\times}$-invariant $\mathbf{C}$-bilinear pairing on $V_{\varrho_i} \times \check{V}_{\varrho_i}$ for $i = 1, 2$, then $\langle \bullet, \bullet \rangle_1 \otimes \langle \bullet, \bullet \rangle_2$

is a $\tilde{H}$-invariant bilinear pairing on $V_\varrho \times \check{V}_\varrho$.

3.3.2 Automorphic representations on $\mathrm{GSO}(D)$.

Given two cuspidal automorphic representations $\boldsymbol{\pi}_i$ of $D^\times$, $i = 1, 2$, with the central characters $\varepsilon_{\boldsymbol{\pi}_1} = \varepsilon$ and $\varepsilon_{\boldsymbol{\pi}_2} = \varepsilon^{-1}$, we obtain a representation

$$\boldsymbol{\pi}_{1,2} = \boldsymbol{\pi}_1 \boxtimes \boldsymbol{\pi}_2$$

of $\tilde{H} \simeq \mathrm{GSO}(D)$ with central character ε. It is cuspidal automorphic in the sense that it is infinite-dimensional and occurs in $L^2(Z_\infty^H \tilde{H}_{\mathbf{Q}} \backslash \tilde{H}_{\mathbf{A}})$.

Let $\{\mathbf{f}_{\boldsymbol{\pi}_i}^{\delta,\jmath}\}$ be a compatible set of automorphic forms attached to $\boldsymbol{\pi}_i$ in the sense of Definition 2.3.2, then the collection $\{\mathbf{f}_{1,2}^{\delta,\jmath}\}$ where

$$\mathbf{f}_{1,2}^{\delta,\jmath}(\alpha,\beta) = \mathbf{f}_{\boldsymbol{\pi}_1}^{\delta,\jmath}(\alpha) \otimes \mathbf{f}_{\boldsymbol{\pi}_2}^{\delta,\jmath}(\beta)$$

is a *compatible set* of automorphic forms on $\tilde{H}$ attached to $\boldsymbol{\pi}_{1,2}$ in that

$$\mathbf{f}_{1,2}^{\delta,\jmath} = \boldsymbol{\pi}_{1,2}(h_\alpha^{-1}) \cdot \mathbf{f}_{1,2}^{\gamma,\kappa}$$

where $h_\alpha = (\alpha, \alpha) \in H_{\mathbf{A}}$ and α is the element chosen in §2.3.3.

3.3.3 Extension to $\tilde{H}^+$.

The factorization $\boldsymbol{\pi}_i \simeq \otimes_v' \pi_{i,v}$ from §2.2.3 for $i = 1, 2$ give a factorization $\boldsymbol{\pi}_{1,2} \simeq \otimes_v' \pi_{(1,2),v}$ where $\pi_{(1,2),v} = \pi_{1,v} \boxtimes \pi_{2,v}$. We can extend $\pi_{(1,2),v}$ to a representation of $\tilde{H}_v^+$ using the following recipe [Rob01, §3]:

$\pi_{1,v} \not\simeq \pi_{2,v}$: In this case the (compact) induction $\mathrm{Ind}_{\tilde{H}_v}^{\tilde{H}_v^+} \pi_{1,2,v}$ is irreducible. Set $\pi_{(1,2),v}^+ = \mathrm{Ind}_{\tilde{H}_v}^{\tilde{H}_v^+} \pi_{(1,2),v}$; we can realize it on the space $V_{\pi_{1,2}}^+ = V_{\pi_{1,2}} \oplus V_{\pi_{1,2}}$ explicitly by

$$\pi_{(1,2),v}^+(h \rtimes 1) \cdot (w, w') = (\pi_{(1,2),v}(h) \cdot w, \pi_{(1,2),v}(h^\iota) \cdot w')$$

and $\pi_{(1,2),v}^+(1 \rtimes \iota) \cdot (w, w') = (w', w)$ for the non-trivial element $\iota \in \boldsymbol{\iota}$.

$\pi_{1,v} \simeq \pi_{2,v}$: In this case

$$\mathrm{Ind}_{\tilde{H}_v}^{\tilde{H}_v^+} \pi_{(1,2),v} \simeq \pi_{(1,2),v}^+ \oplus \pi_{(1,2),v}^-$$

is a direct sum of two inequivalent representations of $\tilde{H}_v^+$ where $\pi_{(1,2),v}^\pm$ acts on

$$V_{\pi_{1,2}}^\pm = \{(w \otimes w', \pm w' \otimes w) : w, w' \in V_{\pi_1} \simeq V_{\pi_2}\}$$

via the formula defined in the first case.

We note that when $\pi_{i,l}$ is a spherical representation, then so is[†] $\pi^{+}_{(1,2),l}$; hence $\boldsymbol{\pi}^{+}_{1,2} = \otimes'_{v}\pi^{+}_{(1,2),v}$ is well-defined and is in fact a cuspidal automorphic representation of $\tilde{H}^{+}$ occurring in $L^2(Z_{\infty}^{\tilde{H}^{+}}\tilde{H}^{+}_{\mathbf{Q}}\backslash\tilde{H}^{+}_{\mathbf{A}})$.

More concretely, we can extend the compatible set of automorphic forms $\{\mathbf{f}^{\delta,\jmath}_{1,2}\}$ on $\tilde{H}$ to a compatible set $\{\mathbf{f}^{\delta,\jmath,+}_{1,2}\}$ by

$$\mathbf{f}^{\delta,\jmath,+}_{1,2}(h \rtimes 1) = \mathbf{f}^{\delta,\jmath}_{1,2}(h) \oplus \mathbf{f}^{\delta,\jmath}_{1,2}(h^{\iota}) \in \check{\mathscr{V}}^{+}_{2k_1,2k_2,\infty}$$

where $\iota \in \iota_{\mathbf{Q}} \subset \iota_{\mathbf{A}}$ and

$$\mathbf{f}^{\delta,\jmath,+}_{1,2}(1 \rtimes \iota_v) = \begin{cases} \mathbf{f}^{\delta,\jmath,+}_{1,2}(1) & \text{if } v \leq \infty, \text{ and} \\ \check{\sigma}^{\delta,\jmath,+}_{2k_1,2k_2}(1 \rtimes \iota_{\infty}) \cdot \mathbf{f}^{\delta,\jmath,+}_{1,2}(1) & \text{if } v = \infty. \end{cases}$$

Here $(\check{\sigma}^{\delta,\jmath,+}_{2k_1,2k_2}, \check{\mathscr{V}}^{+}_{2k_1,2k_2,\infty})$ is the representation of H^{+}_{∞} associated with $(\check{\sigma}^{\delta,\jmath}_{2k_i}, \check{\mathscr{V}}_{2k_i,\infty})$ for $i = 1, 2$ using the procedure from the previous section.

3.3.4 The level.

Let $\boldsymbol{\pi}_i$ be as before for $i = 1, 2$. Suppose furthermore that $\boldsymbol{\pi}_i$'s share the same level dNp^r and that their eigenvalues under the Atkin–Lehner operators are reciprocals, i.e., $a_l(\boldsymbol{\pi}_1) \cdot a_l(\boldsymbol{\pi}_2) = 1$ for all $l \mid dNp$. Under these hypotheses, we see that, for any choice of an Eichler order $\mathscr{D}$ of level dNp^r, the subspace fixed by the compact open subgroup $\tilde{U}^{\mathscr{D}}$, $V^{\tilde{U}^{\mathscr{D}}}_{\boldsymbol{\pi}_{(1,2),f}}$, is one-dimensional over $\mathbf{C}$. In fact, we have that

$$V^{\tilde{U}^{\mathscr{D}}}_{\boldsymbol{\pi}_{(1,2),f}} = \{f_1 \otimes f_2 : f_i \in V^{\mathscr{D}^{\times}_f}_{\boldsymbol{\pi}_{i,f}}\},$$

and $V^{\mathscr{D}^{\times}_f}_{\boldsymbol{\pi}_{i,f}}$ is by definition one-dimensional. We shall refer to the compact open subgroup $\tilde{U}^{\mathscr{D}}$ as the *level* of $\boldsymbol{\pi}_{1,2}$. We note that the automorphic form $\mathbf{f}^{\delta,\jmath}_{1,2}$ defined

[†]Indeed, let $f_i \in V_{\pi_{i,l}}$ that is fixed by some maximal compact open subgroup $U_{i,l} \subset D^{\times}_{l}$. Then the vector

$$f^{+}_{1,2} = \begin{cases} (f_1 \otimes f_2, f_1 \otimes f_2) & \text{if } \pi_{1,l} \not\simeq \pi_{2,l}, \text{ and} \\ (f_1 \otimes f_1, f_1 \otimes f_1) & \text{if } \pi_{1,l} \not\simeq \pi_{2,l} \end{cases}$$

is invariant under the subgroup in $\tilde{H}^{+}_{l}$ generated by $U_{1,l} \times U_{2,l}$ and the main involution ι_l. This is a maximal compact subgroup of $\tilde{H}^{+}_{l}$ since we have $U_{1,l} \simeq \mathrm{GL}_2(\mathbf{Z}_l) \simeq U_{2,l}$ and the maximal compact open subgroups of $\tilde{H}^{+}_{l} \simeq \mathrm{GO}_{2,2}(\mathbf{Q}_l)$ are exactly the ones generated by $\mathrm{GL}_2(\mathbf{Z}_l) \times \mathrm{GL}_2(\mathbf{Z}_l)$ together with the involution ι_l.

in the previous section spans the one-dimensional $\tilde{U}^{\mathscr{D}^{\delta,\jmath}} \times \tilde{H}_\infty$-invariant subspace of $V_{\boldsymbol{\pi}_{1,2}} \otimes (\check{\mathscr{V}}_{2k_1,\infty} \otimes \check{\mathscr{V}}_{2k_2,\infty})$ (where $2k_i$ denotes the weight of $\boldsymbol{\pi}_i$).

The same observation applies to the automorphic form $\mathbf{f}_{1,2}^{\delta,\jmath,+}$ is right-invariant under $\tilde{U}_f^{\mathscr{D},+}$. In fact, $V^+_{\boldsymbol{\pi}_{(1,2)},f}$ has a unique one-dimensional subspace fixed by $\tilde{U}_f^{\mathscr{D}^{\delta,\jmath},+}$ generated by any matrix coefficient of the form $\langle \mathbf{f}_{1,2}^{\delta,\jmath,+}, v\rangle^+_{2k_1,2k_2}$ for $v \in \check{\mathscr{V}}^+_{2k_1,2k_2,\infty}$.

3.3.5 Weights for H_∞ and H_∞^+.

As we are mostly interested in representations of $\mathrm{GSO}(D)_\infty \simeq \tilde{H}_\infty$ which are trivial on the center, it suffices to consider the irreducible unitary representations of $\mathrm{SO}(D)_\infty \simeq H_\infty$ trivial on $Z_\infty^H \simeq \{\pm 1\}$ by §3.1.2. Now

$$Z_\infty^H \backslash H_\infty = (PD_\infty^\times) \times (PD_\infty^\times),$$

hence we are reduced to considering pairs of irreducible representations of $PD_\infty^\times$. The notations are from §1.4.1.

To begin, fix an S-basis $(\delta,\jmath)$ for D and let $\mathfrak{t} = \mathbf{R}\tilde{\delta}$ be the Cartan sub-algebra of $D_\infty^{(0)} = \mathrm{Lie}(PD_\infty^\times)$ fixed in §1.4.1. It is the Lie algebra of the torus T in $PD_\infty^\times$ given by the image of $\exp(\mathbf{R}\tilde{\delta}) = \{t_1 + t_2\tilde{\delta} : t_1^2 + t_2^2 = 1\}$. Then $\mathfrak{t} \times \mathfrak{t}$ is a Cartan sub-algebra of $\mathrm{Lie}(H_\infty)$. Denote by $\{e_{D,1}, e_{D,2}\}$ the union of the simple roots associated with the $(D_{\mathbf{C}}^{(0)}, \mathfrak{t}_{\mathbf{C}})$'s.

Given two complex representations $(\varrho_{2k_i}, V_{2k_i})$ of $PD_\infty^\times$ for $i = 1,2$ of highest weight $2k_i$ (with respect to the sets of simple root $\{e_{D,i}\}$), we denote by

$$(\varrho_{2k_1,2k_2}, V_{2k_1,2k_2}) = (\varrho_{2k_1} \boxtimes \varrho_{2k_2}, V_{2k_1} \otimes V_{2k_2})$$

the corresponding representation on H_∞. We like to determine its weight with respect to a set of simple roots for $\mathrm{SO}(D)_\infty$ which we now describe.

Fix an isomorphism $\mathrm{SO}(D)_\infty \simeq \mathrm{SO}_4(\mathbf{R})$ with respect to the ordered orthonormal basis $\{1, \tilde{\delta}, \tilde{\jmath}, \tilde{\jmath}\tilde{\delta}\}$ of D. Note the norm form corresponds to the identity matrix $\mathrm{diag}[1,1,1,1]$. The torus in $(PD_\infty^\times) \times (PD_\infty^\times)$ given by the image of

$$\exp(\mathbf{R}\tilde{\delta}) \times \exp(\mathbf{R}\tilde{\delta}) = \{(s_1 + s_2\tilde{\delta}, t_1 + t_2\tilde{\delta}) : s_i, t_i \in \mathbf{R}^1, s_1^2 + s_2^2 = 1 = t_1^2 + t_2^2\}$$

is mapped to

$$\left\{\begin{bmatrix} a & b & & \\ -b & a & & \\ & & c & d \\ & & -d & c \end{bmatrix} : \begin{matrix} a = s_1 t_1 + s_2 t_2 \\ b = s_1 t_2 - s_2 t_1 \\ c = s_1 t_1 - s_2 t_2 \\ d = s_1 t_2 + s_2 t_1 \end{matrix}\right\} \simeq \mathrm{SO}_2(\mathbf{R}) \times \mathrm{SO}_2(\mathbf{R})$$

in $\mathrm{SO}_4(\mathbf{R})/\{\pm 1\}$. Now the Lie algebra

$$\mathfrak{so}_2 = \left\{ W_u = \begin{bmatrix} & u \\ -u & \end{bmatrix} : u \in \mathbf{R} \right\}$$

is abelian and $\exp(\mathfrak{so}_2) = \mathrm{SO}_2(\mathbf{R})$, hence $\mathfrak{t} \times \mathfrak{t}$ is mapped to the Cartan sub-algebra $\mathfrak{h} = \mathbf{R}X_1 \oplus \mathbf{R}X_2$ in $\mathfrak{so}_4$ where

$$X_1 = \begin{bmatrix} W_1 & \\ & 0 \end{bmatrix} \qquad \text{and} \qquad X_2 = \begin{bmatrix} 0 & \\ & W_1 \end{bmatrix}.$$

Let e_i be the linear form sending X_i to 1 and X_j to 0 for $j \neq i$, then $\mathfrak{h}^\vee = \mathbf{R}e_1 \oplus \mathbf{R}e_2$. The corresponding roots of $\mathfrak{h}_{\mathbf{C}}$ in $\mathfrak{so}_{4,\mathbf{C}}$ are then given by $\{\pm e_1 \pm e_2\}$, and we fix a set of simple roots to be $\{e_1 \pm e_2\}$. With respect the partial ordering on $\mathfrak{h}_{\mathbf{C}}^\vee$ induced by this choice of simple roots, the irreducible representations of $\mathrm{SO}_4(\mathbf{R})$ are indexed by the dominant integral weights $a_1 e_1 + a_2 e_2$ in $\mathfrak{h}_{\mathbf{C}}^\vee$ with $a_i \in \mathbf{Z}$ which we refer to as representation of H_∞ of the highest weight (a_1, a_2).

Now under the isomorphism $\mathfrak{t} \times \mathfrak{t} \simeq \mathfrak{so}_2 \times \mathfrak{so}_2$, the element $(\tilde{\delta}, 0)$ is mapped to (W_1, W_1) and $(0, \tilde{\delta})$ is mapped to $(W_1, -W_1)$. It follows that

$$e_1 = \frac{e_{D,1} + e_{D,2}}{2} \qquad \text{and} \qquad e_2 = \frac{e_{D,1} - e_{D_2}}{2}.$$

Consequently, we see that if ϱ_{2k_i} is of the highest weight $2k_i$, then $\varrho_{2k_1,2k_2}$ is an irreducible representation of H_∞ of the highest weight $\underline{k} = (k_1 + k_2, k_1 - k_2)$. Also, we refer to the irreducible representation $V^+_{2k_1,2k_2}$ of H^+_∞ obtained from $V_{2k_1,2k_2}$ in §3.3.3 as a representation of H^+_∞ of the highest weight $(k_1 + k_2, k_1 - k_2; +)$.

Finally we denote by $\langle \bullet, \bullet \rangle_{2k_1,2k_2} = \langle \bullet, \bullet \rangle_{2k_1} \otimes \langle \bullet, \bullet \rangle_{2k_2}$ the invariant bilinear pairing on $V_{2k_1,2k_2} \times \check{V}_{2k_1,2k_2}$ and $\langle \bullet, \bullet \rangle^+_{2k_1,2k_2}$ its natural extension to $V^+_{2k_1,2k_2} \times \check{V}^+_{2k_1,2k_2}$.

3.3.6 The Petersson pairing.

Let (ϱ^+, V_ϱ^+) be an irreducible complex representation of H_∞^+ on which the center $Z_\infty^{H^+}$ acts trivially. Fix a H_∞^+-invariant bilinear pairing $\langle \bullet, \bullet \rangle^+$ between ϱ^+ and its contragredient $\check{\varrho}^+$. In preparation for theta lifts, we define the Petersson pairing to be the induced bilinear pairing $\langle \bullet, \bullet \rangle_{H^+}$ between $\mathbf{f}^+ \in L^2(Z_\infty^{H^+} H_{\mathbf{Q}}^+ \backslash H_{\mathbf{A}}^+) \otimes V_\varrho^+$ and $\mathbf{g}^+ \in L^2(Z_\infty^{H^+} H_{\mathbf{Q}}^+ \backslash H_{\mathbf{A}}^+) \otimes \check{V}_\varrho^+$ given by

$$\langle \mathbf{f}^+, \mathbf{g}^+ \rangle_{H^+} = \int_{H_{\mathbf{Q}}^+ \backslash H_{\mathbf{A}}^+} \langle \mathbf{f}^+(h), \mathbf{g}^+(h) \rangle^+ \, d^+h.$$

If $\mathbf{f}^+$ (resp. $\mathbf{g}^+$) is obtained from $\mathbf{f} \in L^2(Z_\infty^H H_{\mathbf{Q}} \backslash H_{\mathbf{A}}) \otimes V_\varrho$ (resp. $\mathbf{g} \in L^2(Z_\infty^H H_{\mathbf{Q}} \backslash H_{\mathbf{A}}) \otimes \check{V}_\varrho$), then

$$\begin{aligned}
\langle \mathbf{f}^+, \mathbf{g}^+ \rangle_{H^+} &= \int_{\boldsymbol{\iota}_{\mathbf{Q}} \backslash \boldsymbol{\iota}_{\mathbf{A}}} \int_{H_{\mathbf{Q}} \backslash H_{\mathbf{A}}} \langle \varrho_\infty^+(\iota_\infty^{-1}) \cdot \mathbf{f}^+(h \rtimes \iota_f), \check{\varrho}_\infty^+(\iota_\infty^{-1}) \cdot \mathbf{g}^+(h \rtimes \iota_f) \rangle^+ \, dh \, d\iota \\
&= \int_{\boldsymbol{\iota}_{\mathbf{Q}} \backslash \boldsymbol{\iota}_{\mathbf{A}}} \int_{H_{\mathbf{Q}} \backslash H_{\mathbf{A}}} \langle \mathbf{f}(h), \mathbf{g}(h) \rangle + \langle \mathbf{f}(h^\iota) + \mathbf{g}(h^\iota) \rangle \, dh \, d\iota \\
&= \int_{H_{\mathbf{Q}} \backslash H_{\mathbf{A}}} \langle \mathbf{f}(h), \mathbf{g}(h) \rangle \, dh \\
&= \langle \mathbf{f}, \mathbf{g} \rangle_H
\end{aligned}$$

since $\mathrm{vol}(\boldsymbol{\iota}_{\mathbf{Q}} \backslash \boldsymbol{\iota}_{\mathbf{A}}) = \frac{1}{2}$ by §3.2.5.

3.3.7 L-parameters and local constants.

Recall that the dual group of $\mathrm{GSO}(D)$ is

$$\mathrm{GSpin}_4(\mathbf{C}) = \{(g, g') \in \mathrm{GL}_2(\mathbf{C}) \times \mathrm{GL}_2(\mathbf{C}) : \det(g) = \det(g')\}.^*$$

By Langlands functoriality, the local L-parameter attached to $\pi_{(1,2),l}$ is then

$$\phi_{\pi_{(1,2),l}} = \phi_{\pi_{1,l}} \oplus \varepsilon_l \cdot \phi_{\pi_{2,l}} : W_l \times \mathrm{SL}_2(\mathbf{C}) \to \mathrm{GSpin}_4(\mathbf{C}) \rtimes W_l.$$

*Indeed, the maps $\mathrm{SO}(D) \to \mathrm{GSO}(D)$ and $D^\times \times D^\times \to \mathrm{GSO}(D)$ induce homomorphisms between the respective dual groups

$$\widehat{\mathrm{GSO}(D)} \to \mathrm{SO}_4(\mathbf{C}) \qquad \text{and} \qquad \widehat{\mathrm{GSO}(D)} \to \mathrm{GL}_2(\mathbf{C}) \times \mathrm{GL}_2(\mathbf{C}).$$

By [Bor79, §2.2], $\widehat{\mathrm{GSO}(D)}$ injects into $\mathrm{GL}_2(\mathbf{C}) \times \mathrm{GL}_2(\mathbf{C})$ and is therefore a subgroup of rank 3 with a central isogeny (i.e. a surjective homomorphism with kernel contained in the center) to $\mathrm{SO}_4(\mathbf{C})$, hence $\widehat{\mathrm{GSO}(D)} = \mathrm{GSpin}_4(\mathbf{C})$.

Note that we have to twist by ε_l (viewed as a character on W_l through local class field theory) on the second factor in order for the image to land in $\mathrm{GSpin}_4(\mathbf{C})$.

Let $c_i = c(\boldsymbol{\pi})$ be the level of $\boldsymbol{\pi}_i$. For l not dividing c_1c_2, $\pi_{(1,2),l} = \pi(\mu_1, \mu_1') \boxtimes \pi(\mu_2, \mu_2')$ is spherical; therefore it is uniquely determined by the Satake parameters $\beta_{i,l} = \mu_i(\mathrm{Frob}_l)$ and $\beta_{i,l}' = \mu_i'(\mathrm{Frob}_l)$ as in §2.2.8. Note that $\mu_1\mu_1' = \varepsilon_l = \mu_2^{-1}\mu_2'^{-1}$. In this case, the corresponding L-parameter is unramified and is uniquely determined by its image on Frob_l,

$$\phi_{\pi_{(1,2),l}}(\mathrm{Frob}_l) = \begin{bmatrix} \beta_{1,l}\cdot\beta_{2,l}' & & & \\ & \beta_{1,l}\cdot\beta_{2,l} & & \\ & & \beta_{1,l}'\cdot\beta_{2,l}' & \\ & & & \beta_{1,l}'\cdot\beta_{2,l} \end{bmatrix}.^{\dagger}$$

Set

$$\begin{aligned} & L^{(c_1c_2)}(s, \boldsymbol{\pi}_1 \boxtimes \boldsymbol{\pi}_2, \mathrm{std}) \\ &= \prod_{l \nmid c_1c_2} \det(1 - \phi_{\pi_{(1,2),l}}(\mathrm{Frob}_l)\cdot l^{-s})^{-1} \\ &= \prod_{l \nmid c_1c_2} \frac{1}{(1-\beta_{1,l}\beta_{2,l}'\cdot l^{-s})(1-\beta_{1,l}\beta_{2,l}\cdot l^{-s})(1-\beta_{1,l}'\beta_{2,l}'\cdot l^{-s})(1-\beta_{1,l}'\beta_{2,l}\cdot l^{-s})}. \end{aligned}$$

We refer to it as the *partial Rankin–Selberg L-function* for the automorphic representation $\boldsymbol{\pi}_1 \boxtimes \boldsymbol{\pi}_2$.

3.4 Automorphic forms on GSp_4

We review some basics of the theory of Siegel modular forms, focusing on the case of degree 2.

3.4.1 Symplectic groups.

Let $L = \mathbf{Q}\mathbf{w}_1 \oplus \mathbf{Q}\mathbf{w}_2$ be a two-dimensional $\mathbf{Q}$-vector space and denote by $L^\vee = \mathbf{Q}\check{\mathbf{w}}_1 \oplus \mathbf{Q}\check{\mathbf{w}}_2$ its linear dual. We emphasize that L (resp. $L^\vee$) comes equipped with a preferred

† Indeed, the standard representation of

$$\widehat{\mathrm{GSO}(D)} = \{(g, g') \in \mathrm{GL}_2(\mathbf{C}) \times \mathrm{GL}_2(\mathbf{C}) : \det(g) = \det(g')\}$$

into $\mathrm{GL}_4(\mathbf{C}) \simeq \mathrm{GL}(\mathrm{M}_2(\mathbf{C}))$ given by the action $(g, g')\cdot A = gAg'^{-1}$ for a matrix $A \in \mathrm{M}_2(\mathbf{C})$ takes an element $(\mathrm{diag}[s,t], \mathrm{diag}[s',t'])$ in the split torus of $\widehat{\mathrm{GSO}(D)}$ to $\mathrm{diag}[\frac{s}{s'}, \frac{s}{t'}, \frac{t}{s'}, \frac{t}{t'}]$ in $\mathrm{GL}_4(\mathbf{C})$ with respect to the basis $\{e_{11}, e_{12}, e_{21}, e_{22}\}$ of $\mathrm{M}_2(\mathbf{C})$. (Here e_{ij} is the 2×2 matrix with 1 in the ijth position and 0 elsewhere.)

choice of an ordered-basis $\{\mathbf{w}_i\}$ (resp. $\{\check{\mathbf{w}}_i\}$).

The direct sum of these two spaces,

$$W = L \oplus L^\vee = \mathbf{Q}\mathbf{w}_1 \oplus \mathbf{Q}\mathbf{w}_2 \oplus \mathbf{Q}\check{\mathbf{w}}_1 \oplus \mathbf{Q}\check{\mathbf{w}}_2$$

has a natural choice of an alternating form $(\bullet, \bullet)_W$ given by

$$(x_1 \oplus \check{y}_1, x_2 \oplus \check{y}_2)_W = \check{y}_2(x_1) - \check{y}_1(x_2)$$

for $x_i \in L$ and $\check{y}_i \in L^\vee$. We have the group

$$\mathrm{GSp}_4 = \{g \in \mathrm{GL}(W) : (g \cdot v, g \cdot w)_W = \lambda'(g)(v,w)_W, \forall v, w \in W\},$$

where $\lambda' \colon \mathrm{GSp}_4 \to \mathbf{G}_m$ is the multiplier. We note that λ' is surjective*, and its kernel is the symplectic group Sp_4.

As we will be working with matrices, we fix an ordered $\mathbf{Q}$-basis $\{\mathbf{w}_1, \mathbf{w}_2, \check{\mathbf{w}}_1, \check{\mathbf{w}}_2\}$ for W, which amounts to an embedding $\mathrm{GSp}_4 \to \mathrm{GL}_4$. Note that the alternating form has the matrix representation $\begin{bmatrix} & 1_2 \\ -1_2 & \end{bmatrix}$ with respect to this choice of basis.

3.4.2 The Siegel parabolic.

Let P be the Siegel parabolic subgroup of GSp_4 stabilizing the flag $0 \subset L \subset W$. Let M be the Levi subgroup of P stabilizing $L^\vee$, and N the unipotent radical of P, so $P \simeq M \ltimes N$. We have that $M \simeq \mathrm{GL}(L) \times \mathbf{G}_m$ and $N \simeq \mathrm{Hom}^+_{\mathbf{Q}}(L^\vee, L) \simeq \mathrm{SM}_2$, where $\mathrm{Hom}^+_{\mathbf{Q}}(L^\vee, L)$ denotes the self-dual maps from $L^\vee$ to L (with respective to *any* linear isomorphism $L \simeq L^\vee$). We set $P_1 = P \cap \mathrm{Sp}_4$ and $M_1 = M \cap \mathrm{Sp}_4$.

With respect to the basis $\mathbf{w}_1, \mathbf{w}_2$ of L, we have the following matrix representations:

$$M = \left\{ \begin{bmatrix} a & \\ & \alpha \cdot {}^t a^{-1} \end{bmatrix} : a \in \mathrm{GL}(L) \simeq \mathrm{GL}_2, \alpha \in \mathbf{G}_m \right\} \quad \text{and} \quad N = \left\{ \begin{bmatrix} 1_2 & S \\ & 1_2 \end{bmatrix} : S \in \mathrm{SM}_2 \right\}.$$

3.4.3 Compact subgroups.

We define some compact subgroups of $\mathrm{GSp}_4(\mathbf{Q}_v)$ and $\mathrm{Sp}_4(\mathbf{Q}_v)$ for each place v.

*The linear automorphism $x + y \mapsto \alpha \cdot x + y$ for $x \in L$ and $y \in L^\vee$ defines an element in GSp_4 that has similitude $\alpha \in \mathbf{G}_m$.

For $v = \infty$**:** We define the maximal compact subgroup U_∞^{Sp} of $\mathrm{Sp}_4(\mathbf{R})$ to be

$$U_\infty^{\mathrm{Sp}} = \left\{ u = \begin{bmatrix} A & B \\ -B & A \end{bmatrix} : u \in \mathrm{Sp}_4(\mathbf{R}) \right\}.$$

Note that U_∞^{Sp} is also a maximal compact subgroup of

$$\mathrm{GSp}_4(\mathbf{R})^+ = \{ g \in \mathrm{GSp}_4(\mathbf{R}) : \lambda'(g) > 0 \};$$

whereas the corresponding maximal compact subgroup of $\mathrm{GSp}_4(\mathbf{R})$, denoted by U_∞^{GSp}, is the extension of U_∞^{Sp} by the order two element $\eta = \mathrm{diag}[1, 1, -1, -1]$.

For $v = l$**:** For an integer $n \geq 0$, we define a compact open subgroup of $\mathrm{GSp}_4(\mathbf{Q}_l)$ by

$$U_l^{\mathrm{GSp}}(n) = \mathrm{GSp}_4(\mathbf{Z}_l) \cap \begin{bmatrix} \mathrm{M}_2(\mathbf{Z}_l) & \mathrm{M}_2(\mathbf{Z}_l) \\ l^n \mathrm{M}_2(\mathbf{Z}_l) & \mathrm{M}_2(\mathbf{Z}_l) \end{bmatrix}.$$

Given a positive integer $N = \prod_l l^{n_l}$, we set $U^{\mathrm{GSp}}(N)_f = \prod_{l<\infty} U_l^{\mathrm{GSp}}(n_l)$. It is a compact open subgroup of $\mathrm{GSp}_4(\mathbf{A}_f)$. We denote by $U^{\mathrm{GSp}}(N)$ the product $U^{\mathrm{GSp}}(N)_f \times U_\infty^{\mathrm{GSp}}$.

3.4.4 Representations of U_∞^{Sp} and U_∞^{GSp}.

Let $\mathfrak{k}$ be the Lie algebra of U_∞^{Sp}, we have that

$$\mathfrak{k} = \left\{ \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \in \mathrm{M}_2(\mathbf{R}) : A = -{}^t A, B = {}^t B \right\}.$$

The elements

$$T_1 = -\imath \begin{bmatrix} 0 & \mathrm{diag}[1,0] \\ -\mathrm{diag}[1,0] & 0 \end{bmatrix} \quad \text{and} \quad T_2 = -\imath \begin{bmatrix} 0 & \mathrm{diag}[0,1] \\ -\mathrm{diag}[0,1] & 0 \end{bmatrix}$$

span the Cartan sub-algebra $\mathfrak{t} = \mathbf{R}T_1 \oplus \mathbf{R}T_2$ of $\mathfrak{k}$. Let e_i be the linear form on $\mathfrak{t}_{\mathbf{C}} = \mathfrak{t} \otimes \mathbf{C}$ which sends T_i to 1 and T_j to 0 for $j \neq i$. We have that $\mathfrak{t}_{\mathbf{C}}^\vee = \mathbf{C}e_1 \oplus \mathbf{C}e_2$, and the roots of $\mathfrak{t}_{\mathbf{C}}$ in $\mathfrak{k}_{\mathbf{C}} = \mathfrak{k} \otimes \mathbf{C}$ are $\Delta_c = \{e_1 - e_2, e_2 - e_1\}$. We fix a set of simple roots to be $\{e_1 - e_2\}$, this in turn defines a partial ordering on $\mathfrak{t}_{\mathbf{C}}^\vee$ and the dominant integral weights are element $k_1 e_1 + k_2 e_2$ with $k_i \in \mathbf{Z}$ such that $k_1 \geq k_2$. By the theory of highest weight [Kna02, Theorem 5.110], all irreducible finite-dimensional representations of

U_∞^{Sp} are up to equivalence in bijection with these tuples of integers $\underline{k} = (k_1, k_2)$ with $k_1 \geq k_2$. We shall refer to any representation corresponding to $\underline{k}$ as an irreducible representation of U_∞^{Sp} of the highest weight $\underline{k}$.

Let $(\tau_{\underline{k}}, \mathscr{W}_{\underline{k}})$ be such a representation of the highest weight $\underline{k}$. The induction $\mathrm{Ind}_{U_\infty^{\mathrm{Sp}}}^{U_\infty^{\mathrm{GSp}}} \mathscr{W}_{\underline{k}}$ is isomorphic to the direct sum

$$\mathscr{W}_{\underline{k}} \oplus \mathscr{W}_{\underline{k}}$$

where $u \in U_\infty^{\mathrm{Sp}}$ acts by $u \cdot (w \oplus w') = \tau_{\underline{k}}(u) \cdot w \oplus \tau_{\underline{k}}(\bar{u}) \cdot w'$ and the element $\mathrm{diag}[1, 1, -1, -1]$ in U_∞^{GSp} acts by swapping the two vectors. From this we see that this induced representation is the direct sum $\mathscr{W}_{\underline{k}}^+ \oplus \mathscr{W}_{\underline{k}}^-$ of two irreducible isomorphic submodules defined by

$$\mathscr{W}_{\underline{k}}^+ = \{(w, \bar{w}) : w \in \mathscr{W}_{\underline{k}}\} \qquad \mathscr{W}_{\underline{k}}^- = \{(w, -\bar{w}) : w \in \mathscr{W}_{\underline{k}}\}.$$

We focus on the irreducible representation $(\tau_{\underline{k}}^+, \mathscr{W}_{\underline{k}}^+)$ of U_∞^{GSp} and refer to it as an irreducible representation of U_∞^{GSp} of the highest weight $\underline{k}$.

3.4.5 Cartan decomposition.

Let $\mathfrak{g} = \mathfrak{sp}_4$ be the Lie algebra of $\mathrm{Sp}_4(\mathbf{R})$. Under the Cartan involution $X \mapsto -{}^tX$, $\mathfrak{g}$ decomposes as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k} = \mathrm{Lie}(U_\infty^{\mathrm{Sp}})$ is the $(+1)$-eigenspace and $\mathfrak{p}$ is the (-1)-eigenspace. It follows at once from the definition that $\mathfrak{p}_{\mathbf{C}} = \mathfrak{p} \otimes \mathbf{C}$ is stable under the adjoint action by $\mathrm{Sp}_4(\mathbf{R})$; moreover, it decomposes into a direct sum $\mathfrak{p}_{\mathbf{C}} \simeq \mathfrak{p}_{\mathbf{C}}^+ \oplus \mathfrak{p}_{\mathbf{C}}^-$ of stable subspaces under $\mathrm{Ad}(U_\infty^{\mathrm{Sp}})$ where

$$\mathfrak{p}_{\mathbf{C}}^{\pm} = \left\{ \begin{bmatrix} A & \pm\imath A \\ \pm\imath A & -A \end{bmatrix} \in \mathrm{M}_2(\mathbf{C}) : A = {}^tA \right\}.$$

3.4.6 Automorphic forms on GSp_4.

Let $(\tau_{\underline{k}}^+, \mathscr{W}_{\underline{k}}^+)$ be an irreducible representation of U_∞^{GSp} of the highest weight $\underline{k} = (k_1, k_2)$ with $k_1 \geq k_2 \geq 2$ and $k_1 \equiv k_2 \pmod 2$ so it factors through $\{\pm 1\} \subset U_\infty^{\mathrm{GSp}} \cap Z_{\mathrm{GSp}_4}(\mathbf{R})$. Also let ε a character of $\mathbf{Q}^\times \backslash \mathbf{A}_f^\times$, and $U^{\mathrm{GSp}} = U^{\mathrm{GSp}}(N)$ for some integer $N > 0$ as above.

Definition 3.4.1. A holomorphic *automorphic form on* GSp_4 of level N, weight $\underline{k}$, and central character ε is a smooth function $\mathbf{F} \colon \mathrm{GSp}_4(\mathbf{A}) \to \mathscr{W}_{\underline{k}}^+$ such that† $\mathfrak{p}_{\mathbf{C}}^- \cdot \mathbf{F} = 0$

†Here $\mathfrak{p}_{\mathbf{C}}^- \cdot \mathbf{F} = \{X \cdot \mathbf{F} : X \in \mathfrak{p}_{\mathbf{C}}^-\}$ where $(X \cdot \mathbf{F})(g) = \frac{d}{dt}\big|_{t=0} \mathbf{F}(g \cdot \exp(tX))$ is the usual Lie action.

and

$$\mathbf{F}(\gamma z g u_\infty u_f) = \varepsilon(z)\cdot\tau_{\underline{k}}^+(u_\infty)^{-1}\cdot\mathbf{F}(g),$$

for all $\gamma \in \mathrm{GSp}_4(\mathbf{Q}), z \in Z_{\mathrm{GSp}_4}(\mathbf{A}), g \in \mathrm{GSp}_4(\mathbf{A})$, and $u_\infty u_f \in U^{\mathrm{GSp}}$.

Remark 3.4.1. The holomorphy condition $\mathfrak{p}_{\mathbf{C}}^- \cdot \mathbf{F} = 0$ together with the Koecher principle [AZ95, II, §3.2] imply that $\mathbf{F}$ is of moderate growth.

We denote the space of automorphic forms on GSp_4 of level N, weight $\underline{k}$, and central character ε by $\mathscr{A}_\varepsilon(\mathrm{GSp}_4; N, \underline{k})$. We say an automorphic form $\mathbf{F} \in \mathscr{A}_\varepsilon(\mathrm{GSp}_4; N, \underline{k})$ is *cuspidal* if

$$\int_{N'(\mathbf{Q})\backslash N'(\mathbf{A})} \mathbf{F}(ng)\, dn = 0, \text{ for all } g \in \mathrm{GSp}_4(\mathbf{A}), \tag{4.6.1}$$

where we integrate over all possible unipotent radicals N' of each proper parabolic subgroup of GSp_4. We denote by $\mathscr{A}_\varepsilon^0(\mathrm{GSp}_4; N, \underline{k})$ the subspace of cuspidal automorphic forms (or just cusp forms) in $\mathscr{A}_\varepsilon(\mathrm{GSp}_4; N, \underline{k})$.

When $k_1 = k = k_2$, $(\tau_{\underline{k}}^+, \mathscr{W}_{k,k}) \simeq \det^k$ is one-dimensional, and consequently the automorphic forms in $\mathscr{A}_\varepsilon(\mathrm{GSp}_4; N, (k,k))$ are scalar-valued.

3.4.7 Automorphic representations on GSp_4.

Denote by $\mathscr{A}(\mathrm{GSp}_4)$ the vector space of functions on $\mathrm{GSp}_4(\mathbf{A})$ generated by the matrix coefficients

$$\{\mathbf{F}_v = \langle v, (g\cdot\mathbf{F})\rangle_{\underline{k}}\}$$

running over all $v \in \mathscr{W}_{\underline{k}}$, all $\mathbf{F} \in \mathscr{A}_\varepsilon(\mathrm{GSp}_4; N, \underline{k})$ for all possible central characters, levels, and weights, and all $g \in \mathrm{GSp}_4(\mathbf{A})$ with $(g\cdot\mathbf{F})(g') = \mathbf{F}(g'g)$. We also define $\mathscr{A}^0(\mathrm{GSp}_4)$ in the same way but only taking $\mathbf{F} \in \mathscr{A}_\varepsilon^0(\mathrm{GSp}_4; N, \underline{k})$. It is immediate from the definition that $\mathscr{A}(\mathrm{GSp}_4)$ and $\mathscr{A}^0(\mathrm{GSp}_4)$ are stable under the right translation by $\mathrm{GSp}_4(\mathbf{A})$. For us, an *automorphic representation* (resp. *cuspidal automorphic representation*) on GSp_4 is then an irreducible sub-quotient $\mathbf{\Pi}$ of $\mathscr{A}(\mathrm{GSp}_4)$ (resp. $\mathscr{A}^0(\mathrm{GSp}_4)$) viewed as a $\mathrm{GSp}_4(\mathbf{A})$-module. Although this rather naive approach excludes many representations of $\mathrm{GSp}_4(\mathbf{A})$ that are considered automorphic by [BJ79], it is sufficient for our purposes.

We have a factorization $\mathbf{\Pi} \simeq \hat{\otimes}_v' \Pi_v$ where Π_l is an irreducible admissible smooth representation of $\mathrm{GSp}_4(\mathbf{Q}_l)$ for $l < \infty$ and Π_∞ is an admissible representation of $\mathrm{PGSp}_4(\mathbf{R})$. Similarly, the central character factors as $\varepsilon = \otimes_v \varepsilon_v$. The automorphic representations defined above are exactly the ones whose archimedean component Π_∞ is either a holomorphic discrete series or to a limit of discrete series of $\mathrm{PGSp}_4(\mathbf{R})$

[AS01, §4.5]. We now review how to parametrize these (limits of) holomorphic discrete series in terms of Harish-Chandra parameters.

3.4.8 Holomorphic discrete series of $\mathrm{PGSp}_4(\mathbf{R})$.

To begin, observe that the Cartan sub-algebra $\mathfrak{t}_{\mathbf{C}}$ of $\mathfrak{k}$ is also a Cartan sub-algebra of $\mathfrak{g}_{\mathbf{C}}$, and the set of roots of $\mathfrak{t}_{\mathbf{C}}$ in $\mathfrak{g}_{\mathbf{C}}$ is $\Delta = \Delta_{\mathrm{c}} \cup \Delta_{\mathrm{nc}}$ where $\Delta_{\mathrm{c}} = \{e_1 - e_2, e_2 - e_1\}$ as before is the subset of compact roots with roots space in $\mathfrak{k}_{\mathbf{C}}$, and $\Delta_{\mathrm{nc}} = \{e_1 + e_2, -e_1 - e_2, \pm 2e_1, \pm 2e_2\}$ is the subset of non-compact roots with roots spaces in $\mathfrak{p}_{\mathbf{C}}$.

By the works of Harish-Chandra as described in [Pau05, §3.1], the (limits of) discrete series of $\Pi \in \mathrm{Sp}_4(\mathbf{R})$ are parametrized pairs (λ_d, Ψ) where $\lambda_d = a_1 e_1 + a_2 e_2 \in \mathfrak{t}^\vee, a_i \in \mathbf{Z}$, and Ψ is the corresponding set of positive roots subjected to a number of conditions.* Since we have fixed a choice of positive compact roots, we may refer to λ_d as the *Harish-Chandra parameter* of Π.

In particular, if we take $\Psi = \{e_1 - e_2, e_1 + e_2, 2e_1, 2e_2\}$, then the holomorphic discrete series correspond to Harish-Chandra parameters $\lambda_d = a_1 e_1 + a_2 e_2$ with $a_1 > a_2 > 0$ and we denote it by Π_{a_1,a_2}. The irreducible representation $\tau_{\underline{k}}$ of the highest weight $\underline{k} = (a_1 + 1, a_2 + 2)$ occurs in Π_{a_1,a_2} with multiplicity one, and is the *lowest K-type* of Π_{a_1,a_2}. Since Π_{a_1,a_2} is irreducible, it factors through the center $Z_{\mathrm{Sp}_4}(\mathbf{R}) \simeq \{\pm 1\}$ exactly when $\tau_{\underline{k}}$ does, and this happens exactly when $a_1 + 1 \equiv a_2 + 2 \pmod 2$. In this case, $\mathrm{Ind}_{\mathrm{PSp}_4(\mathbf{R})}^{\mathrm{PGSp}_4(\mathbf{R})} \Pi_{a_1,a_2}$ is a discrete series of $\mathrm{PGSp}_4(\mathbf{R})$ which we denote by $\Pi^+_{a_1,a_2}$. These discrete series (and limits of such) exhaust all the infinity types of the automorphic representations from §3.4.7 as we can check using [HK92, Table 2.2.1].

3.4.9 L-parameters and L-factors.

Let $\mathbf{\Pi} \simeq \hat{\otimes}' \Pi_v$ be an automorphic representation of GSp_4 as in §3.4.7. Given a prime l, the local Langlands correspondence [GT] attaches to Π_l a L-parameter, i.e., a $\mathrm{GSp}_4(\mathbf{C})$-conjugacy class of admissible representations

$$\phi_{\Pi_l} : W_l \times \mathrm{SL}_2(\mathbf{C}) \to \mathrm{GSp}_4(\mathbf{C})$$

*Namely that

- $\Delta_{\mathrm{c}}^+ = \{e_1 - e_2\} \subset \Psi$;
- λ_d is dominant with respect to Ψ; and
- for all simple roots $\alpha \in \Psi$ we have that if $\langle \lambda_d, \alpha \rangle = 0$ then $\alpha \in \Delta_{\mathrm{nc}}$ is non-compact.

such that $\lambda' \circ \phi_{\Pi_l}$ is identified with ε_l through local class field theory.[†]

Suppose now that $\mathbf{\Pi}$ contains a vector $\langle v, \mathbf{F}\rangle_{\underline{k}}$ for some $\mathbf{F} \in \mathscr{A}_\varepsilon(\mathrm{GSp}_4; N, \underline{k})$ and $v \in \mathscr{W}_{\underline{k}}$; then for all primes $l \nmid N$, $\Pi_l \simeq \Pi(\chi_0, \chi_1, \chi_2)$ is the spherical representation attached to an unramified character

$$\mathrm{diag}[t_1, t_2, t_1^{-1}t_0, t_2^{-1}t_0] \mapsto \chi_0(t_0) \cdot \chi_1(t_1) \cdot \chi_2(t_2)$$

of the standard maximal torus T [AS01, §2.2]. As such, Π_l is uniquely characterized by its Satake parameters $a_{i,l} = \chi_i(\mathrm{Frob}_l)$, where we view χ_i as a character on W_l/I_l via local class field theory. Given such a $\mathbf{\Pi}$, we define its *level* to be the smallest integer N for which it satisfies the property just mentioned.

We can define two different L-factors in this case,

$$\begin{aligned} L(s, \phi_{\Pi_l}, \mathrm{std}) &= \det(1 - (\mathrm{std} \circ \phi_{\Pi_l})(\mathrm{Frob}_l) \cdot l^{-s})^{-1} \\ &= \frac{1}{(1 - l^{-s})(1 - a_{1,l}l^{-s})(1 - a_{2,l}l^{-s})(1 - a_{1,l}^{-1}l^{-s})(1 - a_{2,l}^{-1}l^{-s})} \end{aligned} \tag{4.9.2}$$

corresponding to the standard representation $\mathrm{GSp}_4(\mathbf{C}) \xrightarrow{\mathrm{std}} \mathrm{SO}_5(\mathbf{C})$ and

$$\begin{aligned} L(s, \phi_{\Pi_l}, \mathrm{spin}) &= \det(1 - (\mathrm{spin} \circ \phi_{\Pi_l})(\mathrm{Frob}_l) \cdot l^{-s})^{-1} \\ &= \frac{1}{(1 - a_{0,l}l^{-s})(1 - a_{0,l}a_{1,l}l^{-s})(1 - a_{0,l}a_{2,l}l^{-s})(1 - a_{0,l}a_{1,l}a_{2,l}l^{-s})} \end{aligned} \tag{4.9.3}$$

corresponding to the spin representation $\mathrm{GSp}_4(\mathbf{C}) \xrightarrow{\mathrm{spin}} \mathrm{GSpin}_5(\mathbf{C})$ [AS01, 2.4].

Finally, with N equal to the level of $\mathbf{\Pi}$ as above, we define its *partial standard L-function* (resp. *partial spinor L-function*) to be

$$L^{(N)}(s, \mathbf{\Pi}, \bullet) = \prod_{l \nmid N} L(s, \phi_{\Pi_l}, \bullet)$$

where $\bullet = \mathrm{std}$ (resp. $\bullet = \mathrm{spin}$).

3.4.10 The Siegel upper-half space of degree 2.

Let $\mathrm{SM}_2(\mathbf{C})$ be the set of symmetric matrices with complex entries and let

$$\mathfrak{H}_2 = \{Z \in \mathrm{SM}_2(\mathbf{C}) : \mathrm{Im}(Z) > 0\}$$

[†]Since GSp_4 is split over $\mathbf{Q}$, we may safely ignore the factor W_l in ${}^L\mathrm{GSp}_4$.

be the Siegel upper-half space of degree 2 and $\bar{\mathfrak{H}}_2 = \{\bar{Z} : Z \in \mathfrak{H}_2\}$ the Siegel lower-half space. Then $\mathrm{GSp}_4(\mathbf{R})$ (resp. $\mathrm{GSp}_4(\mathbf{R})^+$ and $\mathrm{Sp}_4(\mathbf{R})$) acts transitively on $\mathfrak{H}_2 \cup \bar{\mathfrak{H}}_2$ (resp. $\mathfrak{H}_2$) as a group of holomorphic automorphisms by

$$Z \xrightarrow{g} g\langle Z\rangle = (aZ+b)(cZ+d)^{-1}$$

for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GSp}_4(\mathbf{R})$ (resp. $\mathrm{GSp}_4(\mathbf{R})^+$ and $\mathrm{Sp}_4(\mathbf{R})$). The stabilizer of the point $\mathbf{i} = \mathrm{diag}[\iota,\iota,\iota,\iota] \in \mathfrak{H}_2$ under the action by $\mathrm{Sp}_4(\mathbf{R})$ is U_∞^{Sp}, and $\mathfrak{H}_2$ is diffeomorphic to $\mathrm{Sp}_4(\mathbf{R})/U_\infty^{\mathrm{Sp}}$. For $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GSp}_4(\mathbf{R})^+$ and $Z \in \mathfrak{H}_2$, we define the $\mathrm{GL}_2(\mathbf{C})$-valued automorphy factor $J(g,Z)$ on $\mathrm{GSp}_4(\mathbf{R})^+ \times \mathfrak{H}_2$ by

$$J(g,Z) = cZ + d.$$

Note that J induces the isomorphism $U_\infty^{\mathrm{Sp}} \simeq \mathrm{U}_2(\mathbf{R})$ by $u \mapsto J(u,\mathbf{i})$.

Remark 3.4.2. Denote by $d\rho$ the differential of the diffeomorphism $\mathrm{Sp}_4(\mathbf{R})/U_\infty^{\mathrm{Sp}} \to \mathfrak{H}_2$ sending g to $d\langle \mathbf{i}\rangle$. It defines an isomorphism

$$\mathfrak{p} \xrightarrow{d\rho} T_{\mathbf{i}}(\mathfrak{H}_2)$$

where $\mathfrak{p}$ is the Lie algebra of $P_1(\mathbf{R}) = P(\mathbf{R}) \cap \mathrm{Sp}_4(\mathbf{R})$ from §3.4.5, and $T_{\mathbf{i}}(\mathfrak{H}_2)$ is the tangent space of $\mathfrak{H}_2$ at the point $\mathbf{i}$. Moreover, the induced isomorphism $d\rho_{\mathbf{C}}\colon \mathfrak{p}_{\mathbf{C}} \to T_{\mathbf{i}}(\mathfrak{H}_2) \otimes_{\mathbf{R}} \mathbf{C}$ identifies the element of $\mathfrak{p}_{\mathbf{C}}^+$ with the holomorphic differential operators in $T_{\mathbf{i}}(\mathfrak{H}_2) \otimes_{\mathbf{R}} \mathbf{C}$ [AS01, §4.2].

3.4.11 Classical Siegel modular forms of degree 2.

Let $k_1 = k = k_2$ and fix an integer $N \geq 0$ as before. Given any $g_f \in \mathrm{GSp}_4(\mathbf{A}_f)$, set

$$\Gamma_N^0(g_f) = \mathrm{GSp}_4(\mathbf{Q}) \cap (\mathrm{GSp}_4(\mathbf{R})^+ \times g_f U_f^{\mathrm{GSp}}(N) g_f^{-1}).$$

We shall consider $\Gamma_N^0(g_f)$ as a subgroup of $\mathrm{GSp}_4(\mathbf{Q})^+ \subset \mathrm{GSp}_4(\mathbf{R})^+$ and set $\Gamma_N^0 = \Gamma_N^0(1)$ for convenience. For an scalar-valued automorphic form $\mathbf{F} \in \mathscr{A}_\varepsilon(\mathrm{GSp}_4; N, (k,k))$, we define a function $\mathscr{F}_{g_f}(Z)$ on $\mathfrak{H}_2^+$ by

$$\mathscr{F}_{g_f}(Z) = \lambda'(g_\infty)^{-k} \cdot \det(J(g_\infty,\mathbf{i}))^k \cdot \mathbf{F}(g_\infty g_f) \tag{4.11.4}$$

where g_∞ is any element in $\mathrm{GSp}_4(\mathbf{R})^+$ such that $g_\infty\langle \mathbf{i}\rangle = Z$.* Under this identification, the left regular action by $g'_\infty \in \mathrm{GSp}_4(\mathbf{R})^+$ on $\mathscr{A}_\varepsilon^0(\mathrm{GSp}_4; N, (k,k))$,

$$(g'_\infty \cdot \mathbf{F})(g) = \mathbf{F}((g'_\infty)^{-1}g),$$

goes over to the action classical denoted by:

$$(\mathscr{F}_{g_f}\big|_k g'_\infty)(Z) = \lambda'(g'_\infty)^{-k}\cdot \det(J(g'_\infty, Z))^k \cdot \mathscr{F}_{g_f}({g'_\infty}^{-1}\langle Z\rangle). \tag{4.11.5}$$

Consequently, $\mathscr{F}_{g_f}(Z)$ satisfies the usual invariance condition:

$$(\mathscr{F}_{g_f}|_k\gamma)(Z) = \mathscr{F}_{g_f}(Z),^\dagger \tag{4.11.6}$$

for all $\gamma \in \Gamma_N^0(g_f)$.

Remark 3.4.3. Suppose that g_f is contained in the Levi subgroup $M(\mathbf{A}_f)$ of the Siegel parabolic. Since $M \simeq \mathrm{GL}_2\times \mathbf{G}_m$, we see that $g_f = \gamma\cdot u_f$ with $\gamma \in \mathrm{GL}_2(\mathbf{Q})\times\mathbf{Q}^\times$ and $u_f \in \mathrm{GL}_2(\mathbf{Z}_f)\times \mathbf{Z}_f^\times$ by strong approximation and class number one for $\mathbf{Q}$. Furthermore, u_f normalizes $U^{\mathrm{GSp}}(N)_f$. From this we see that $\Gamma_N^0(g_f) = \gamma^{-1}\Gamma_N^0\gamma$ is just a conjugate of Γ_N^0 in $\mathrm{GSp}_4(\mathbf{Q})^+$.

If $\mathscr{F}_{g_f}$ is holomorphic as a function on $\mathfrak{H}_2$, then we say it is a *Siegel modular form* of degree 2, level $\Gamma_N^0(g_f)$, weight k, and character ε. We denote the space of such Siegel modular forms by $M_k(\Gamma_N^0(g_f);\varepsilon)$.

---

*Let us check this is well-defined. Suppose $g'_\infty\langle\mathbf{i}\rangle = Z$, then $g'_\infty = g_\infty\cdot u_\infty z_\infty$ for some $u_\infty \in U_\infty^{\mathrm{Sp}}$ and $z_\infty \in Z_{\mathrm{GSp}_4}(\mathbf{R})$. We have

$$\begin{aligned}
&\lambda'(g'_\infty)^{-k}\cdot\det(J(g'_\infty,\mathbf{i}))^k\cdot\mathbf{F}(g'_\infty g_f)\\
&= \lambda'(g_\infty)^{-k}\cdot\lambda'(u_\infty)^{-k}\cdot z_\infty^{-2k}\cdot\det(J(g_\infty,\mathbf{i}))^k\cdot\det(J(u_\infty,\mathbf{i}))^k\cdot z_\infty^{2k}\cdot\mathbf{F}(g_\infty u_\infty z_\infty g_f)\\
&= \lambda'(g_\infty)^{-k}\cdot\det(J(g_\infty,\mathbf{i}))^k\cdot\det(J(u_\infty,\mathbf{i}))^k\cdot\det(J(u_\infty,\mathbf{i}))^{-k}\cdot\mathbf{F}(g_\infty)\\
&= \mathscr{F}_{g_f}(Z).
\end{aligned}$$

†Indeed, we have

$$\begin{aligned}
&(\mathscr{F}_{g_f}|_k\gamma)(Z)\\
&= \lambda'(\gamma)^{-k}\cdot\det(J(\gamma,Z))^k\cdot\mathscr{F}_{g_f}(\gamma_\infty^{-1}\langle Z\rangle)\\
&= \lambda'(\gamma)^{-k}\cdot\det(J(\gamma,Z))^k\cdot\lambda'(\gamma^{-1}g_\infty)^{-k}\cdot\det(J(\gamma^{-1}g_\infty,\mathbf{i}))^k\cdot\mathbf{F}(\gamma_\infty^{-1}g_\infty g_f)\\
&= \lambda'(g_\infty)^{-k}\cdot\det(J(g_\infty,\mathbf{i}))^k\cdot\mathbf{F}(g_\infty g_f) = \mathscr{F}_{g_f}(Z)
\end{aligned}$$

since $\mathbf{F}(\gamma_\infty^{-1}g_\infty g_f) = \mathbf{F}(\gamma^{-1}g_\infty\gamma_f g_f) = \mathbf{F}(g_\infty g_f)$.

3.5 Fourier coefficients and Bessel models

We recall some facts concerning the Fourier coefficients and Bessel models of Siegel modular forms.

3.5.1 Classical Fourier coefficients.

Let $\boldsymbol{\psi}$ be the standard character on $\mathrm{SM}_2(\mathbf{A})$ defined in §0.4.3. Let $\mathscr{T}$ be the set of all rational symmetric positive semi-definite 2×2 matrices. That is, the set of matrices

$$T = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

with $a, b, c \in \mathbf{Q}$ and $a, c > 0$ and $\det(T) \geq 0$. We shall refer to elements of $\mathscr{T}$ as *indices*, as they index the Fourier coefficients of Siegel modular forms of degree 2. Indeed, we have a Fourier expansion

$$\mathscr{F}_{g_f}(Z) = \sum_{T\in\mathscr{T}} a_{g_f}(T)\boldsymbol{\psi}_\infty(TZ) = \sum_{T\in\mathscr{T}} a_{g_f}(T)\cdot e^{2\pi\imath\cdot\mathrm{tr}(TZ)}.$$

Here the Tth Fourier coefficient $a_{g_f}(T)$ at the cusp $\mathbf{i}\infty$ is given by [Sug85, (1-17)]

$$a_{g_f}(T) = \frac{1}{\mathrm{vol}(N_{g_f})}\cdot\int_{L(g_f)\backslash\mathrm{SM}_2(\mathbf{R})} \mathscr{F}_{g_f}(X+\mathbf{i}Y)\boldsymbol{\psi}_\infty(-T(X+\mathbf{i}Y))\,dX, \qquad (5.1.7)$$

where $L(g_f)$ is the $\mathbf{Z}$-lattice in $\mathrm{SM}_2(\mathbf{Q})$ defined by

$$\left\{S\in\mathrm{SM}_2(\mathbf{Q}) : \begin{bmatrix} 1 & S \\ & 1 \end{bmatrix} \in N(\mathbf{Q})\cap g_f U^{\mathrm{GSp}}(N) g_f^{-1}\right\}$$

and $N_{g_f} = N(\mathbf{A}_f)\cap g_f U_f^{\mathrm{GSp}} g_f$. Here dX denotes the usual Euclidean measure on $\mathrm{SM}_2(\mathbf{R})\simeq\mathbf{R}^3$ fixed in §0.5.1; also note that $\mathrm{vol}(N_1) = \mathrm{vol}(\mathrm{SM}_2(\mathbf{Z}_f)) = 1$ when $g_f = 1$.

Remark 3.5.1. The isomorphism $d\rho\colon\mathfrak{p}\simeq T_{\mathfrak{i}}(\mathfrak{H}_2)$ identifies $L(g_f)\backslash\mathrm{SM}_2(\mathbf{R})$ with a neighborhood U of infinity in $\Gamma(g_f)\backslash\mathfrak{H}_2$ just as in the case of elliptic modular forms. Moreover, since the action by elements in $\mathfrak{p}$ on functions on $\mathfrak{H}_2$ go through the exponential map $\exp\colon\mathfrak{p}\to P_1(\mathbf{R})\subset\mathrm{Sp}_4(\mathbf{R})$, the standard Euclidean measure dX on $\mathrm{SM}_2(\mathbf{R})$ is identified with the invariant volume form $(2\pi\imath dx)\wedge(2\pi\imath dy)\wedge(2\pi\imath dz)$ on the image U, again similar to the case of elliptic modular forms.

Let $L(g_f)^\vee = \{S'\in\mathrm{SM}_2(\mathbf{Q}) : \mathrm{tr}(SS')\in\mathbf{Z}$ for all $S\in L(g_f)\}$ be the dual lattice to $L(g_f)$ under the trace form; we have that $a_{g_f}(T) = 0$ if $T\notin L(g_f)^\vee$. We note that for $g_f = 1$, the Fourier coefficients are indexed by the subset $\mathscr{T}_{\text{int}} \subset \mathscr{T}$ of symmetric matrices that are *semi-integral*, namely, the ones with the entries a and c in $\mathbf{Z}$ and b in $\frac{1}{2}\mathbf{Z}$.

We say a (classical) Siegel modular form $\mathscr{F}_{g_f}$ is *cuspidal* if $a_{g_f}(T) = 0$ for all indices $T \in \mathscr{T}$ that are semi-definite but not definite. The relation (5.2.9) below combined with the proof of [AS01, Lemma 5] imply that these are exactly the ones coming from the cuspidal automorphic forms.

Remark 3.5.2. We saw in Remark 3.4.3 that for $g_f = \gamma \cdot u_f \in M(\mathbf{A}_f) \simeq M(\mathbf{Q}) \times M(\mathbf{Z}_f)$, $\Gamma^0_N(g_f) = \gamma^{-1}\Gamma^0(N)\gamma$. Consequently, the Tth Fourier coefficient of $\mathscr{F}_{g_f} = \mathscr{F}\big|_k \gamma$ at the cusp $\mathbf{i}\infty$ is the same as the Tth Fourier coefficient of $\mathscr{F}$ at the cusp $\gamma \cdot \mathbf{i}\infty$.

3.5.2 Fourier functionals.

Similarly, given $\mathbf{F} \in \mathscr{A}_\varepsilon(\mathrm{GSp}_4; N, (k,k))$ and an index $T \in \mathscr{T}$, we define a function $\mathbf{a}^T = \mathbf{a}^T_{\mathbf{F}}$ on $\mathrm{GSp}_4(\mathbf{A})$ by

$$\mathbf{a}^T(g) = \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \mathbf{F}\left(\begin{bmatrix} 1 & S \\ & 1 \end{bmatrix} g\right) \boldsymbol{\psi}(-TS)\, dS \tag{5.2.8}$$

where we are integrating over the orbit space associated with the unipotent radical of the Siegel parabolic. To relate this back to (5.1.7), note that $\mathbf{a}^T(g)$ is right-invariant under N_{g_f}, and we have an isomorphism

$$N(\mathbf{Q})\backslash N(\mathbf{A})/N_{g_f} \simeq L(g_f)\backslash \mathrm{SM}_2(\mathbf{R})$$

for the $\mathbf{Z}$-lattice $L(g_f) \subset \mathrm{SM}_2(\mathbf{Q})$ defined in the preceding section. A comparison* with (5.1.7) yields the identity [Sug85, (1-19)]

$$\mathbf{a}^T(g_f g_\infty) = \lambda'(g_\infty)^k \cdot \det(J(g_\infty, \mathbf{i}))^{-k} \cdot a_{g_f}(T) \cdot \boldsymbol{\psi}_\infty(T \cdot g_\infty\langle \mathbf{i}\rangle), \tag{5.2.9}$$

*Indeed, we have

$$\begin{aligned}
\mathbf{a}^T(g) &= \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \mathbf{F}\left(\begin{bmatrix} 1 & S \\ & 1 \end{bmatrix} g\right) \boldsymbol{\psi}(-TS)\, dS \\
&= \lambda'(g_\infty)^k \cdot \det(J(g_\infty, \mathbf{i}))^{-k} \\
&\quad \cdot \int_{L(g_f)\backslash \mathrm{SM}_2(\mathbf{R})} \mathscr{F}_{g_f}(S + g_\infty\langle \mathbf{i}\rangle) \boldsymbol{\psi}_\infty(-T(S + g_\infty\langle \mathbf{i}\rangle)) \cdot \boldsymbol{\psi}(T \cdot g_\infty\langle \mathbf{i}\rangle)\, dS \\
&= \lambda'(g_\infty)^k \cdot \det(J(g_\infty, \mathbf{i}))^{-k} \cdot \boldsymbol{\psi}(T \cdot g_\infty\langle \mathbf{i}\rangle) \cdot a_{g_f}(T).
\end{aligned}$$

where $a_{g_f}(T)$ is the Tth Fourier coefficient of $\mathscr{F}_{g_f}$ corresponding to $\mathbf{F}$. In particular, we have

$$\mathbf{a}^T(g_f) = a_{g_f}(T) \cdot \boldsymbol{\psi}_\infty(T \cdot \mathbf{i}) = a_{g_f}(T) \cdot e^{-2\pi \cdot \mathrm{tr}(T)}. \tag{5.2.10}$$

3.5.3 Indices and quadratic forms.

An index $T = \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix} \in \mathscr{T}$ naturally defines a quadratic form on L by

$$Q^T(x\mathbf{w}_1 + y\mathbf{w}_2) = \begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + bxy + cy^2.$$

Set $m^T(X) = aX^2 + bX + c$; since

$$y^2 \cdot m^T(\frac{x}{y}) = Q^T(x\mathbf{w}_1 + y\mathbf{w}_2),$$

we see that Q^T (and hence T) is positive-definite on L if and only if m^T is irreducible, $4ac - b^2 > 0$ and $a, c \geq 0$.

3.5.4 Rank-one tori in GSp_4.

Suppose now that $T \in \mathscr{T}$ is positive-definite. We have the orthogonal similitude group

$$\mathrm{GSO}(T) = \{w \in \mathrm{GL}(L) : Q^T(w \cdot x) = \det(w) \cdot Q^T(x) \text{ for all } x \in L\}$$

which is naturally a subgroup of $M \subset \mathrm{GSp}_4$ via the embedding

$$t \mapsto \begin{bmatrix} t & \\ & \det(t) \cdot {}^t t^{-1} \end{bmatrix}.$$

On the other hand, denote by K^T the splitting field of $m^T(X) = aX^2 + bX + c$ in $\bar{\mathbf{Q}}$. It is imaginary quadratic. Denote by $\mathrm{n}(\bullet)$ the norm on $K^{T,\times}$, and let $\delta^T = \frac{-b+\sqrt{b^2-4ac}}{2a}$ be a fixed root of $m^T(X)$ in K^T. We have a linear isomorphism of $\mathbf{Q}$-vector spaces

$$L \xrightarrow{\phi^T} K^T \quad \text{given by} \quad \mathbf{w}_1 \mapsto 1 \text{ and } \mathbf{w}_2 \mapsto \delta^T,$$

which sends $Q^T(\bullet)$ to $a \cdot \mathrm{n}(\bullet)$, i.e., $Q^T(x\mathbf{w}_1 + y\mathbf{w}_2) = ax^2 + bxy + cy^2 = a \cdot \mathrm{n}(x + y \cdot \delta^T)$. Since the left multiplication by $t \in K^{T,\times}$ on K^T preserves $\mathrm{n}(\bullet)$ up to scaling by $\mathrm{n}(t)$, we see that ϕ^T induces an injective homomorphism

$$K^{T,\times} = \mathrm{Res}_{K/\mathbf{Q}}\mathbf{G}_m \to \mathrm{GSO}(T)$$

which is in fact an isomorphism of algebraic groups over $\mathbf{Q}$ by connectivity and dimension considerations. We shall use $K^{T,\times}$ and $\mathrm{GSO}(T)$ synonymously if there is no risk of confusion.

3.5.5 Definition of the Bessel model.

Recall that given a cusp form $\mathbf{F}$ in $\mathscr{A}^0_\varepsilon(\mathrm{GSp}_4; N, (k,k))$, we have attached to $\mathbf{F}$ (and T) a function $\mathbf{a}^T$ on $\mathrm{GSp}_4(\mathbf{A})$ in §3.5.2, which computes the T-th Fourier coefficient of the classical Siegel modular forms $\mathscr{F}_{g_f}$ associated to $\mathbf{F}$ up to a period.

Let $\chi: K^{T,\times}_{\mathbf{Q}}\backslash K^{T,\times}_{\mathbf{A}} \to \mathbf{C}^1$ be a Hecke character of $K^{T,\times}$ such that $\chi|_{\mathbf{A}^\times} = \varepsilon$, then the T-th *Bessel model* of $\mathbf{F}$ with respect to χ is the function

$$B^{T,\chi}_{\mathbf{F}}(g) = \int_{\mathbf{A}^\times K^{T,\times}_{\mathbf{Q}}\backslash K^{T,\times}_{\mathbf{A}}} \bar{\chi}(t)\cdot \mathbf{a}^T(tg)\, d^\times t. \tag{5.5.11}$$

Here the invariant measure $d^\times t$ is normalized so that it is compatible for the Tamagawa measures on $K^{T,\times}_{\mathbf{Q}}\backslash K^{T,\times}_{\mathbf{A}}$ and $\mathbf{Q}^\times\backslash\mathbf{A}^\times$. In particular, we have that $\mathrm{vol}(\mathbf{R}_{>0}\backslash K^\times_\infty) = 2\pi$.

3.5.6 Relation to Fourier coefficients.

For convenience, set $K = K^T$. Let us compute the value of the Bessel model at the identity element:

$$B^{T,\chi}_{\mathbf{F}}(1) = \int_{\mathbf{A}^\times K^\times_{\mathbf{Q}}\backslash K^\times_{\mathbf{A}}} \bar{\chi}(t)\cdot \mathbf{a}^T(t)\, d^\times t$$

From the definition of $\mathbf{a}^T(t)$, we see that the integrand is right-invariant under the compact open subgroup $U^\chi_f \subset U^{\mathrm{GSp}}_f(N)$ of $K^\times_f \subset \mathrm{GSp}_4(\mathbf{A}_f)$ given by the kernel of $\chi|_{K^\times_f}$. Let $h = h_\chi$ be the cardinality of the idele class group

$$\mathrm{Cl}(\chi) = K^\times_{\mathbf{Q}} K^\times_\infty\backslash K^\times_{\mathbf{A}}/U^\chi_f$$

determined by χ, and fix a set of representatives $t_1 = 1, \cdots, t_h$ with $t_{i,\infty} = 1$ for all i. We have a decomposition

$$\mathbf{A}^\times K^\times_{\mathbf{Q}}\backslash K^\times_{\mathbf{A}} = (\mathscr{O}^\times_\chi\cdot\mathbf{R}_{>0}\cdot\mathbf{Z}^\times_f)\backslash(\coprod_{i=1}^{r} K^\times_\infty\cdot t_i\cdot U^\chi_f)$$

where $\mathscr{O}_\chi^\times = K_\mathbf{Q}^\times \cap (K_\infty \cdot U_f^\chi)$ is a subgroup of the finite group $\mathscr{O}_K^\times$. Set $e_\chi = \#\mathscr{O}_\chi^\times$. Combining the relation (5.2.9) with this decomposition, we get

$$B_\mathbf{F}^T(1) = \frac{\mathrm{vol}(U_f^\chi)}{e_\chi} \cdot \left(\int_{\mathbf{R}_{>0}\backslash K_\infty^\times} \bar{\chi}_\infty(t_\infty)\, d^\times t_\infty \right) \cdot e^{-2\pi \mathrm{tr}(T)} \cdot \sum_{i=1}^{r} \bar{\chi}(t_i) \cdot a_{t_i}(T)$$

since $\lambda'(t_\infty)^k \cdot \det(J(t_\infty, \mathbf{i}))^{-k} = 1$ and $\mathrm{vol}(\mathbf{Z}_f^\times) = 1$. Now the archimedean integral

$$\int_{\mathbf{R}_{>0}\backslash K_\infty^\times} \bar{\chi}_\infty(t_\infty)\, d^\times t_\infty = \begin{cases} 0 & \text{if } \chi_\infty \neq 1, \\ \mathrm{vol}(\mathbf{R}_{>0}\backslash K_\infty^\times) = \mathrm{vol}(S^1) = 2\pi & \text{if } \chi_\infty = 1. \end{cases}$$

We arrive at the following expression of $B_\mathbf{F}^T(1)$ as a linear combination of translated Fourier coefficients [Sug85, (1-26)]:

$$B_\mathbf{F}^T(1) = \frac{2\pi \cdot \mathrm{vol}(U_f^\chi)}{e_\chi} \cdot e^{-2\pi \mathrm{tr}(T)} \cdot \sum_{i=1}^{h} \bar{\chi}(t_i) \cdot a_{t_i}(T). \tag{5.6.12}$$

3.6 Arithmetic theory of Siegel modular forms

It is impossible to do justice to the arithmetic theory of Siegel modular forms in just a few pages, even in the special case that we are considering. We repeat (almost word-for-word) the summary of this theory given in [BR89, §3] with additional input from [Ich09, §2]. All schemes considered here are locally noetherian.

3.6.1 Some terminology concerning abelian schemes.

We begin by fixing some notations.

- An *abelian scheme* $\mathscr{A}$ over the base S is a group scheme $\pi\colon \mathscr{A} \to S$ which is smooth and proper with connected (geometric) fibers [Mil86, §20]. Let $e\colon S \to \mathscr{A}$ be the identity section.

- $\omega_{\mathscr{A}/S}$ is the invertible sheaf $\bigwedge^g \pi_*(\Omega^1_{\mathscr{A}/S}) \simeq \bigwedge^g e^*(\Omega^1_{\mathscr{A}/S})$ where g is the relative dimension of $\mathscr{A} \to S$ and $\Omega^1_{\mathscr{A}/S}$ is the sheaf of relative differentials. We will take $g = 2$.

- A *principle polarization* of $\pi\colon \mathscr{A} \to S$ is an isomorphism $\lambda\colon \mathscr{A} \to {}^t\mathscr{A}$, where ${}^t\mathscr{A}$ is the dual abelian scheme and which arises by the procedure of [Mil86, §10].

3.6.2 Siegel moduli space of degree 2.

We repeat the discussion in [Ich09, §2] which summarizes some basic notions in the algebraic theory of Siegel moduli spaces for the reader's convenience. We have added some additional references but all credits should go to Ichikawa. We focus on the case $g = 2$.

Let n be a positive integer, and let ζ_N be a primitive Nth root of unity. Let $\mathscr{M}_{2,N}$ be the moduli stack (which becomes the fine moduli space scheme when $N \geq 3$) over $\mathbf{Z}[1/N, \zeta_N]$ principally polarized abelian schemes of relative dimension 2 with sympletic level N structure [Cha86, §1]. Then $\mathscr{M}_{2,N}(\mathbf{C})$ is a complex orbifold of dimension 3, and it is represented as the quotient space $\mathfrak{H}_2/\Gamma_N$ of the Siegel upper-half space $\mathfrak{H}_2$ from §3.4.10 by the integral symplectic group

$$\Gamma_N = \ker(\mathrm{Sp}_4(\mathbf{Z}) \to \mathrm{Sp}_4(\mathbf{Z}/N\mathbf{Z}))$$

which is sometimes referred to as the *principal sympletic* group of level N. Note that the group $\Gamma_N^0 \subset \mathrm{GSp}_4(\mathbf{Q})^+$ from §3.4.11 contains Γ_N. We set $\mathscr{M}_2 = \mathscr{M}_{2,1}$.

Let $\mathscr{A}$ be the universal abelian scheme with identity section e over $\mathscr{M}_{2,N}$ [Cha86, Theorem 1.4]. The invertible sheaf $\omega_{\mathscr{A}/\mathscr{M}_{2,N}}$ from §3.6.1 is sometimes referred to as the *Hodge line bundle*; it corresponds to the automorphy factor $J(\bullet, \bullet)$ over $\mathscr{M}_{2,N}(\mathbf{C})$.

3.6.3 A smooth compatification of $\mathscr{M}_2$.

We extract the discussion from [BR89, §3.3], which highlights some aspects of the procedure involved in constructing (a single point on the) boundaries used for compatifying $\mathscr{M}_2 = \mathscr{M}_{2,1}$ when $N = 1$.

Let $\mathrm{SM}_{\mathbf{Q}} = \mathrm{SM}_2(\mathbf{Q})$ be the space of symmetric 2×2 matrices with entries in $\mathbf{Q}$. The trace map $\mathrm{tr}\colon X \times Y \mapsto \mathrm{tr}(XY)$ identifies $\mathrm{SM}_{\mathbf{Q}}$ with its dual. Under this identification, the dual of $\mathrm{SM}_{\mathbf{Z}}$, the lattice of matrices in $\mathrm{SM}_{\mathbf{Q}}$ with integral entries, is the set of semi-integral symmetric matrices with off-diagonal entries in $\frac{1}{2}\mathbf{Z}$ and diagonal entries integral, which we denote by $\mathrm{SM}_{\mathbf{Z}}^{\vee}$. It contains the cone $\mathscr{T}_{\mathrm{int}}$ of semi-integral semi-definite symmetric matrices.

Let $s_1, \ldots, s_3$ be a basis of $\mathrm{SM}_{\mathbf{Q}}$ with each s_i positive definite. Let σ be the cone of all $\mathbf{Q}$-linear combinations of the s_i's with each coefficient strictly positive.

Define

$$\Sigma = \{X \in \mathrm{SM}_{\mathbf{Z}}^{\vee} : \mathrm{tr}(XY) \geq 0 \text{ for all } Y \in \sigma\}$$

and

$$\Sigma^+ = \{X \in \mathrm{SM}_{\mathbf{Z}}^\vee : \mathrm{tr}(XY) > 0 \text{ for all } Y \in \sigma\}.$$

Let R_σ be the ring generated by the symbols q^X, $X \in \Sigma$, with the relations $q^X q^Y = q^{X+Y}$ for $X, Y \in \Sigma$. Let $I_\sigma \subseteq R_\sigma$ be the ideal generated by the q^X's for $X \in \Sigma^+$. Let S_σ be the I_σ-adic completion of R_σ. Let K_σ be the quotient field of S_σ, then K_σ contains the quotient field of R_σ, and in particular, $q^X \in K_\sigma$ for all $X \in \mathrm{SM}_{\mathbf{Z}}^\vee$. Let $q_{ij} = q^{X_{ij}}$ where $X_{11} = \mathrm{diag}[1,0]$, $X_{22} = \mathrm{diag}[0,1]$, and $X_{12} = \begin{bmatrix} & \frac{1}{2} \\ \frac{1}{2} & \end{bmatrix} = X_{21}$.

Let $\mathbf{G}_m^2 = \mathbf{G}_m \times \mathbf{G}_m$ be the two-dimensional split torus over R_σ. For $i, j = 1, 2$, let $p_i \in \mathbf{G}_m^2(K_\sigma)$ be the point whose j-coordinate is q_{ij}. Let $P_\sigma \subseteq \mathbf{G}_m^2(K_\sigma)$ be the free abelian group generated by the p_i's. Then the rather intricate construction of Mumford [FC90, Chapter III, §4] provides a quotient of the formal completion $\hat{\mathbf{G}}_m^2$ by P_σ, which is a principally polarized semi-abelian scheme $(\mathscr{X}_\sigma \to S_\sigma, \lambda)$ such that $\mathscr{X}_\sigma \times K_\sigma$ is abelian [FC90, Chapter III, §5].

As describe in [FC90, Chapter IV, Theorem 5.7], we can glue the point† $(\mathscr{X}_\sigma \to S_\sigma, \lambda)$ to $\mathscr{M}_2$, and repeating roughly the same procedure for a collection of good cones σ' produces the boundary strata for a (smooth) toroidal integral compactification $\bar{\mathscr{M}}_2$ of $\mathscr{M}_2$. We then define $\bar{\mathscr{M}}_{2,N}$ to be the normalization of $\bar{\mathscr{M}}_2$ in $\mathscr{M}_{2,N}$ as on [FC90, pg. 128]. As such, the universal abelian scheme $\mathscr{A}$ extends to a universal semi-abelian scheme $\mathscr{X}$ over $\bar{\mathscr{M}}_{2,N}$ with the identity section again denoted by e. Moreover, the invertible sheaf $\bar{\omega} = \bigwedge^2 e^*(\Omega_{\mathscr{X}/\bar{\mathscr{M}}_{2,N}})$ gives an extension of the Hodge line bundle $\omega = \omega_{\mathscr{A}/\mathscr{M}_{2,N}}$ to $\bar{\mathscr{M}}_{2,N}$.

3.6.4 Geometric Siegel modular forms of degree 2.

Let R be a $\mathbf{Z}[\frac{1}{N}, \zeta_N]$-algebra, we define the R-module $M_{k,N}^{\mathrm{geom}}(R)$ of *geometric Siegel modular forms* over R of degree 2, weight k, and (principal symplectic) level N to be

$$M_{k,N}^{\mathrm{geom}}(R) = H^0(\bar{\mathscr{M}}_{2,N}, \bar{\omega}^{\otimes k} \otimes_{\mathbf{Z}[\frac{1}{N},\zeta_N]} R).$$

By Koecher's principle [FC90, Chapter V, Remark 1.2 (c)], we have that

$$H^0(\bar{\mathscr{M}}_{2,N}, \bar{\omega}^{\otimes k} \otimes_{\mathbf{Z}[\frac{1}{N},\zeta_N]} R) = H^0(\mathscr{M}_{2,N}, \omega^{\otimes k} \otimes_{\mathbf{Z}[\frac{1}{N},\zeta_N]} R).$$

By Serre's GAGA and Hartog's theorem [FC90, Chapter V, §1], $M_{k,N}^{\mathrm{geom}}(\mathbf{C})$ is identical to the space of holomorphic Siegel modular forms of level Γ_N and weight k

†More precisely, the point $(\mathscr{X}_\sigma \times U_\sigma, \lambda, \omega_{\mathrm{can}})$ from §3.6.5.

of degree 2. Since $\Gamma_N \subset \Gamma_N^0$, we see that this space contains the space $M_k(\Gamma_N^0, \varepsilon)$ of classical Siegel modular forms of level Γ_N^0, weight k, and character ε. In other words, we have

$$M_k(\Gamma_N^0, \varepsilon) \subset M_{k,N}^{\mathrm{geom}}(\mathbf{C}).$$

3.6.5 Geometric Fourier expansion.

We begin by describing the case $N = 1$, following again [BR89, §3.4]. Recall in §3.6.3 we have described a principal polarized semi-abelian scheme $(\mathscr{X}_\sigma, \lambda)$ with respect to the fixed cone $\sigma \subset \mathrm{SM}_{\mathbf{Q}}$. Let x_i be the character of $\mathbf{G}_m^2$ obtained by projecting onto the i-the factor. Then $\frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} = \omega_{\mathrm{can}}$ generates $\omega_{\mathscr{X}_\sigma \times \mathscr{U}_\sigma / \mathscr{U}_\sigma}$ where $\mathscr{U}_\sigma = \mathrm{Spec}(I_\sigma^{-1} S_\sigma)$ is obtained by localizing S_σ away from I_σ. The significance of $\mathscr{U}_\sigma$ is that it is the largest open subscheme on which $\mathscr{X}_\sigma$ is abelian. Using the moduli space interpretation of $\bar{\mathscr{M}}_2$, the tuple $(\mathscr{X}_\sigma \times U_\sigma, \lambda, \omega_{\mathrm{can}})$ defines a point on $\bar{\mathscr{M}}_2$, which we refer to as the *cusp at infinity* on $\bar{\mathscr{M}}_2$.

Given a geometric modular form $\mathscr{F}^{\mathrm{geom}} \in M_k^{\mathrm{geom}}(R)$, we can evaluate it at the cusp at infinity on $\bar{\mathscr{M}}_{2,N}$; the value $\mathscr{F}^{\mathrm{geom}}(\mathscr{X}_\sigma \times U_\sigma, \lambda, \omega_{\mathrm{can}})$ lies in $I_\sigma^{-1} S_\sigma \otimes_{\mathbf{Z}} R$ and is called the *q-expansion of $\mathscr{F}^{\mathrm{geom}}$* along the cusp at infinity. This expansion in fact can be computed using any *top-dimensional* rational cone in $\mathrm{SM}_2(\mathbf{R})$ and supported at the elements q^T with $T \in \mathscr{T}_{\mathrm{int}}$ positive semi-definite [FC90, Chapter V, Proposition 1.5]. In particular, if we choose σ to be the cone defined with respect to the basis X_{11}, $X_{12} + X_{21}$ and X_{22}, then $\Sigma^+ = \mathscr{T}_{\mathrm{int}}^+$ is the lattice of positive-definition semi-integral symmetric 2×2 matrices, and the q-expansion of $\mathscr{F}^{\mathrm{geom}}$ lies in

$$I_\sigma^{-1} S_\sigma \otimes_{\mathbf{Z}} R = \tilde{R}[\![q^T : T \in \mathscr{T}_{\mathrm{int}}^+]\!]$$

where $\tilde{R} = R[q^T]$ with T running over the semi-integral symmetric indefinite matrices in $\mathscr{T}_{\mathrm{int}} - \mathscr{T}_{\mathrm{int}}^+$.

For general N, we have the following q-expansion for $M_{k,N}^{\mathrm{geom}}(R)$:

Proposition 3.6.1. *For any $\mathbf{Z}[\frac{1}{N}, \zeta_N]$-algebra R and any 0-dimensional cusp c of $\bar{\mathscr{M}}_{2,N}$, there is a q-expansion homomorphism*

$$q\text{-}\exp(c; R) \colon M_{k,N}^{\mathrm{geom}}(R) \to \mathbf{Z}[\frac{1}{N}, \zeta_N][\![q^{\frac{T}{N}} : T \in \mathscr{T}_{\mathrm{int}}]\!] \otimes_{\mathbf{Z}[\frac{1}{N}, \zeta_N]} R \tag{6.5.13}$$

obtained from totally degenerate Mumford families with level-n structures over the cusp c.

Proof. [FC90, Chapter V, Proposition 1.8 (i)]. □

We see that the proposition implies that the Fourier expansion of a geometric Siegel modular form $\mathscr{F} \in M_{k,N}^{\mathrm{geom}}(R)$ at *any cusp* c are all defined over R. Next we have the q-expansion principle.

Theorem 3.6.2 (q-expansion principle)**.** *If $M_1 \subset M_2$ are two $\mathbf{Z}[\frac{1}{N}, \zeta_N]$-modules, then a geometric Siegel modular form of degree 2, level N, and weight k over M_2 is in fact in $H^0(\bar{\mathscr{M}}_{2,N}, \bar{\omega}^{\otimes k} \otimes_{\mathbf{Z}[\frac{1}{N},\zeta_N]} \underline{M_1})$ if (and only if) its q-expansion (at one cusp c) lies $\mathbf{Z}[\frac{1}{N}, \zeta_N][\![q^{\frac{T}{N}} : T \in \mathscr{T}_{\mathrm{int}}]\!] \otimes_{\mathbf{Z}[\frac{1}{N},\zeta_N]} M_1$.*[*]

Proof. [FC90, Chapter V, Proposition 1.8 (iii)]. Also see [Ich09, §2]. The "only if" part is immediate from the definition of the q-expansion q−exp above. □

Apply this theorem to the case that $M_1 = R$ is some $\mathbf{Z}[\frac{1}{N}, \zeta_N]$-algebra in $\bar{\mathbf{Q}} \subset \mathbf{C}$ and $R_2 = \mathbf{C}$, we obtain the following result.

Corollary 3.6.3. *If the classical Siegel modular form $\mathscr{F}$ of degree 2, level Γ_N^0, weight k, and character ε has all its Fourier coefficients $a(T)$ contained in R, then $\mathscr{F} \in H^0(\bar{\mathscr{M}}_{2,N}, \bar{\omega}^{\otimes k} \otimes_{\mathbf{Z}[\frac{1}{N},\zeta_N]} R)$.*

Proof. By §3.6.4, we have $M_k(\Gamma_N^0, \varepsilon) \subset M_{k,N}^{\mathrm{geom}}(\mathbf{C})$. The corollary follows by the q-expansion principle to $\mathscr{F}$ since we have normalized our volume form in Remark 3.5.1 on $\mathfrak{H}_2$ so the classical Fourier expansion and the geometric q-expansion coincide [FC90, pg. 141]. □

[*]Here $\underline{M_1}$ is the constant sheaf on $\mathrm{Spec}(\mathbf{Z}[\frac{1}{N}, \zeta_N])$ associated with M_1.

Chapter 4. The Yoshida Lift

4.1 Review of the Weil representation

We review some basic facts concerning the Weil representation.

4.1.1 The Metaplectic group and the Weil representation.

Let $\mathbf{W}$ be a vector space of dimension $2n$ over $\mathbf{Q}$, and let $(\bullet,\bullet)_{\mathbf{W}}$ be a non-degenerate alternating form on $\mathbf{W}$. Let ψ be the standard additive character on $\mathbf{Q}\backslash\mathbf{A}$ defined in §0.4.3. For every place v of $\mathbf{Q}$, the theory of Weil (a.k.a. oscillator) representations [How79] provides us with a unitary representation $\omega_{\psi,v}$ of the local metaplectic group* $\mathrm{Mp}(\mathbf{W}_v)$ of $\mathbf{W}$.

With respect to a complete polarization† $\mathbf{W} = \mathbf{Y} + \mathbf{X}$ of $\mathbf{W}$, we have a Schödinger model which realizes $\omega_{\psi,v}$ in the space $\mathscr{S}(\mathbf{X}_v)$ of Bruhat–Schwartz functions (i.e. the locally-constant compactly-supported functions) on $\mathbf{X}_v$ for $v = p$ finite and in the space of square-integrable functions $L^2(\mathbf{X}_\infty)$ of $\mathbf{X}_\infty$ for $v = \infty$.

Splicing these representations together in the usual fashion, one obtains a representation $\boldsymbol{\omega}_\psi$ of the global metaplectic group $\mathrm{Mp}(\mathbf{W}_\mathbf{A})$ on the restricted algebraic direct product space $\mathscr{S}(\mathbf{X}_\mathbf{A}) = \prod'_{l<\infty}\mathscr{S}(\mathbf{X}_l) \times L^2(\mathbf{X}_\infty)$. The universal property of this product allows us to extend elements in $\mathscr{S}(\mathbf{X}_\mathbf{A})$, which are initially defined on open subsets of $\mathbf{X}_\mathbf{A}$, to functions on all of $\mathbf{X}_\mathbf{A}$ [Rob01, §5].

4.1.2 Explicit formulas for the Schrödinger model.

For computational purposes, let us describe the Schrödinger model of the local Weil representation explicitly. Fix a complete polarization of $\mathbf{W} = \mathbf{Y} + \mathbf{X}$ as above. Let P be the parabolic subgroup of $\mathrm{Sp}(\mathbf{W})$ that stabilizes the flag $0 \subset \mathbf{Y} \subset \mathbf{W}$. Let

*For a prime l, we take $\mathrm{Mp}(\mathbf{W}_l)$ to be an extension of $\mathrm{Sp}(\mathbf{W}_l)$ by $\mathbf{C}^1$; for $v = \infty$, we take $\mathrm{Mp}(\mathbf{W}_\infty)$ to be a two-sheeted cover of $\mathrm{Sp}(\mathbf{W}_\infty)$. This convention is needed in order for $\mathrm{Mp}(\mathbf{W}_\mathbf{A})$ to split over a symplectic-orthogonal dual pair such as $\mathrm{Sp}_4(\mathbf{A}) \times \mathrm{O}(D)_\mathbf{A}$.

†That is, $\mathbf{X}$ and $\mathbf{Y}$ are maximal isotropic subspaces of $\mathbf{W}$ such that $\mathbf{Y} \simeq \mathbf{X}^\vee$ under the alternating form on $\mathbf{W}$.

$M \simeq \mathrm{GL}(\mathbf{Y})$ be the Levi subgroup of P that stabilizes the dual subspace $\mathbf{X}$. We have the Levi decomposition $P = MN$ where the unipotent radical N of P can be identified with the additive group of self-dual linear maps from $\mathbf{X}$ to $\mathbf{Y}$, $\mathrm{Hom}(\mathbf{X},\mathbf{Y})^{+}$. In terms of matrices, we can represent $p = a \cdot b$ for $a \in \mathrm{GL}(\mathbf{Y})$ and $b \in \mathrm{Hom}(\mathbf{X},\mathbf{Y})^{+}$ by

$$p = \begin{bmatrix} a & \\ & {}^{t}a^{-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & b \\ & 1 \end{bmatrix}.$$

For each place v, we have a unique splitting of $N(\mathbf{Q}_v)$ to $\mathrm{Mp}(\mathbf{W}_v)$. For $m \in M(\mathbf{Q}_v)$, denote by $\tilde{m}$ a lift (i.e. an element in the pre-image) of m to $\mathrm{Mp}(\mathbf{W}_v)$; then given $\varphi_v \in \mathscr{S}(\mathbf{X}_v)$, we have [Li00, §3.1]

$$\begin{aligned} (\omega_{\psi,v}\left(\widetilde{\begin{bmatrix} a & \\ & {}^{t}a^{-1} \end{bmatrix}}\right)\varphi_v)(x) &= \gamma(\tilde{a}) \cdot |\det(a)|_v^{\frac{1}{2}} \cdot \varphi_v({}^{t}ax), \text{ and} \\ (\omega_{\psi,v}\left(\begin{bmatrix} 1 & b \\ & 1 \end{bmatrix}\right)\varphi_v)(x) &= \psi_v(\frac{(bx,x)}{2}) \cdot \varphi_v(x), \end{aligned} \tag{1.2.1}$$

where $\gamma(\tilde{a})$ is a fourth root of unity depending on the particular polarization.

4.1.3 Action of a Weyl element.

It remains to specify the action of a Weyl element. For this, we fix a dualizing basis $\mathbf{x}_i, \mathbf{y}_i$ of $\mathbf{W}$ for $i = 1, \ldots, n$ so that $(\mathbf{y}_i, \mathbf{x}_j)_{\mathbf{W}} = \delta_{ij}$. Define a symmetric bilinear form on $\mathbf{X}$ by

$$(\sum_i a_i \mathbf{x}_i, \sum_j b_j \mathbf{x}_j)_{\mathbf{X}} = \sum_i a_i b_i.$$

We have the Fourier transform taking $\varphi_v \in \mathscr{S}(\mathbf{X}_v)$ to the function

$$\hat{\varphi}_v(x) = \int_{\mathbf{X}_v} \varphi_v(y)\psi_v((x,y))\,dy$$

in $\mathscr{S}(\mathbf{X}_v)$ where dy is the self-dual measure on $\mathbf{X}_v \simeq \mathbf{Q}_v^n$.

Define an element τ in $\mathrm{Sp}(\mathbf{W})$ by $\tau(\mathbf{x}_i) = \mathbf{y}_i$ and $\tau(\mathbf{y}_i) = -\mathbf{x}_i$. Then for $\varphi_v \in \mathscr{S}(\mathbf{X}_v)$, we have

$$\omega_{\psi,v}(\tau)\varphi_v(x) = \zeta\hat{\varphi}_v(x) \tag{1.3.2}$$

where ζ is a certain fourth root of unity, again depending on the polarization. Since τ and P generate $\mathrm{Sp}(\mathbf{W})$ by the Bruhat decomposition, this completes the description of the Schrödinger model.

Remark 4.1.1. The Weil representation enjoys the special property of being *minimal* [Li00], and there are only finitely many such minimal representations for the metaplectic group.

4.2 The reductive dual pair $(\mathrm{Sp}_4, \mathrm{O}(D))$

Let us now consider the case of interest where $\mathbf{W} = W \otimes D$ is the $\mathbf{Q}$-vector space equipped with the alternating form $(\bullet,\bullet)_{\mathbf{W}} = (\bullet,\bullet)_W \otimes (\bullet,\bullet)_D$. It has $\mathbf{X} = L^\vee \otimes D \simeq \mathrm{Hom}_{\mathbf{Q}}(L, D)$ as a maximal totally isotropic subspace and a complete polarization given by $\mathbf{W} \simeq \mathbf{X} \oplus \mathbf{X}^\vee$. The subgroups Sp_4 and $\mathrm{O}(D)$ of $\mathrm{Sp}(\mathbf{W})$ form a reductive dual pair in the sense that one is the centralizer of the other.

4.2.1 Linear maps attached to indices.

The norm form on D induces a bilinear map

$$\mathbf{X} \times \mathbf{X} = L^\vee \otimes D \times D \otimes L^\vee \xrightarrow{\mathrm{n}} L^\vee \otimes L^\vee \simeq \mathrm{Hom}_{\mathbf{Q}}(L, L^\vee)$$

which can be explicitly described as follows. Recall that $L = \mathbf{Q}\mathbf{w}_1 + \mathbf{Q}\mathbf{w}_2$ (resp. $L^\vee = \mathbf{Q}\check{\mathbf{w}}_1 + \mathbf{Q}\check{\mathbf{w}}_2$) comes equipped with the preferred choice of an ordered basis, $\{\mathbf{w}_i\}$ (resp. $\{\check{\mathbf{w}}_i\}$). Given $x \in \mathbf{X}$, set $x_i = x(\mathbf{w}_i)$, then $\mathrm{n}(x)$ is equal to the matrix

$$2 \cdot T_x = \begin{bmatrix} 2\mathrm{n}(x_1) & \mathrm{tr}(x_1\bar{x}_2) \\ \mathrm{tr}(x_1\bar{x}_2) & 2\mathrm{n}(x_2) \end{bmatrix}$$

with respect to the ordered basis of L and $L^\vee$. This also gives a concrete description of the restriction of $(\bullet,\bullet)_{\mathbf{W}}$ to $\mathbf{X}$, namely, $(x,y)_{\mathbf{W}} = \mathrm{tr}(2T_{x\bar{y}})$ for $x, y \in \mathbf{X}$.

Note that if x_1 and x_2 are integral in the sense that their minimal polynomials are integral monic polynomials, then T_x is a positive semi-definite semi-integral symmetric matrix, i.e., an index in $\mathscr{T}_{\mathrm{int}}$. Converse, given an index $T \in \mathscr{T}_{\mathrm{int}}$, we define $\mathbf{X}^T$ to be the set of elements $x \in \mathbf{X}$ such that $T_x = T$. Note that $\mathbf{X}^T$ may be empty.

Remark 4.2.1. The scaling by 2 on the entries in $2 \cdot T_x$ is due to the fact that $(x,x)_D = \mathrm{tr}(x\bar{x}) = 2\mathrm{n}(x)$.

4.2.2 The Schrödinger model.

Fix once and for all a splitting [Kud94, Theorem 3.1 (case 1_+)]

$$\mathrm{Sp}_4(\mathbf{A}) \times \mathrm{O}(D)_{\mathbf{A}} \hookrightarrow \mathrm{Mp}(\mathbf{W}_{\mathbf{A}}).$$

The Weil representation $\boldsymbol{\omega}_\psi$ restricts to a representation on $\mathrm{Sp}_4(\mathbf{A}) \times \mathrm{O}(D)_{\mathbf{A}}$ which we shall simply denote by $\boldsymbol{\omega}$. Consider the parabolic subgroup $\mathbf{P} \subseteq \mathrm{Sp}(\mathbf{W})$ stabilizing $L \otimes D$, then the intersection $\mathbf{P} \cap \mathrm{Sp}_4 \times \mathrm{O}(D)$ is the product $P \times \mathrm{O}(D)$, where P is the Siegel parabolic stabilizing L.

We now describe the Schrödinger model in this situation. Fix a place v of $\mathbf{Q}$, given $(g,h) \in \mathrm{Sp}_4(\mathbf{Q}_v) \times \mathrm{O}(D)_v$, and a Bruhat–Schwartz function $\varphi_v \in \mathscr{S}(\mathbf{X}_v)$ (or a smooth function $\varphi_\infty \in L^2(\mathbf{X}_\infty)$), we have

$$\omega_v(1,h)\varphi_v(x) = \varphi_v(h^{-1} \cdot x), \qquad h \in \mathrm{O}(D)_v; \tag{2.2.3a}$$

$$\left(\omega_v\left(\begin{bmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{bmatrix}, 1\right)\varphi_v\right)(x) = |\det(a)|_v^2 \cdot \varphi_v(x \cdot a), \qquad a \in \mathrm{GL}(L_v); \tag{2.2.3b}$$

$$\left(\omega_v\left(\begin{bmatrix} 1_2 & S \\ 0 & 1_2 \end{bmatrix}, 1\right)\varphi_v\right)(x) = \boldsymbol{\psi}_v(ST_x) \cdot \varphi_v(x), \qquad S \in \mathrm{SM}_2(\mathbf{Q}_v); \tag{2.2.3c}$$

$$\left(\omega_v\left(\begin{bmatrix} 0 & 1_2 \\ -1_2 & 0 \end{bmatrix}, 1\right)\varphi_v\right)(x) = \zeta\hat{\varphi}_v(x). \tag{2.2.3d}$$

Recall that $\boldsymbol{\psi} = \psi \circ \mathrm{tr}$. We also note that $h \in \mathrm{O}(D)$ acts on $\mathbf{X}$ by post-composition sending x to

$$L \xrightarrow{x} D \xrightarrow{h} D;$$

and $a \in \mathrm{GL}(L_v)$ acts by pre-composition sending x to

$$L \xrightarrow{a} L \xrightarrow{x} D.$$

To simplify notation, we write $\boldsymbol{\omega}(g)$ for $\boldsymbol{\omega}(g,1)$ for $g \in \mathrm{Sp}_4(\mathbf{Q}_v)$ from now on.

Remark 4.2.2. We tacitly made several simplifications in the above formulas. For instance, since the discriminant of the quadratic form $2\mathrm{n}(\bullet)$ is a square in $\mathbf{Q}$, a character that usually appears in (2.2.3b) always evaluates to 1, hence is left out. Also, by §4.2.1, we have $(S \cdot x, x)_{\mathbf{W}} = \mathrm{tr}(S(2 \cdot T_x))$ and so $\boldsymbol{\psi}_v(\frac{(S\cdot x,x)_{\mathbf{W}}}{2}) = \boldsymbol{\psi}_v(ST_x)$.

4.2.3 The theta correspondence for $(\mathrm{GSp}_4, \mathrm{GO}(D))$.

As we plan to compute the Bessel model of the Yoshida lift as an automorphic form on GSp_4, we now discuss the theta correspondence for the pair $(\mathrm{GSp}_4, \mathrm{GO}(D))$. This involves considering the subgroup of $\mathrm{GSp}_4 \times \mathrm{GO}(D)$ defined by

$$\mathrm{G}(\mathrm{Sp}_4 \times \mathrm{O}(D)) = \{(g,h) \in \mathrm{GSp}_4 \times \mathrm{GO}(D) : \lambda'(g) = \lambda(h)\}.$$

For every place v, we can extend the local Schrödinger model of ω_v to $\mathrm{G}(\mathrm{Sp}_4 \times \mathrm{O}(D))_v$ as follows. Given $(g,h) \in \mathrm{G}(\mathrm{Sp}_4 \times \mathrm{O}(D))_v$ and φ_v a Bruhat–Schwartz function on $\mathbf{X}_v$ (or a smooth function in $L^2(\mathbf{X}_\infty)$), we set

$$(\omega_v(g,h)\varphi_v)(x) = |\lambda(h)|^{-2}(\omega_v(g_1)\varphi_v)(h^{-1} \cdot x), \tag{2.3.4}$$

where

$$g_1 = g\begin{bmatrix} 1 & 0 \\ 0 & \lambda'(g) \end{bmatrix}^{-1} \in \mathrm{Sp}_4(\mathbf{Q}_v).$$

4.2.4 Howe duality.

For l a prime, and for $G_l = \mathrm{GO}(D)_l$ or $\mathrm{Sp}_4(\mathbf{Q}_l)$, denote by $\mathrm{Irr}(G_l)$ the set of equivalence classes of irreducible admissible smooth representations of G_l. We then define $\mathcal{R}(G_l)$ to be the set of $\pi_l \in \mathrm{Irr}(G_l)$ which occur in ω_l in the sense that $\mathrm{Hom}_{G_l}(\omega_l, \pi_l) \neq 0$. Following [Rob01, pg. 263], we define $\mathcal{R}(\mathrm{GSp}_4(\mathbf{Q}_l))$ to be the set of $\pi_l \in \mathrm{Irr}(\mathrm{GSp}_4(\mathbf{Q}_l))$ such that $\pi_l|_{\mathrm{Sp}_4(\mathbf{Q}_l)}$ is multiplicity-free and has an irreducible constituent in $\mathcal{R}(\mathrm{Sp}_4(\mathbf{Q}_l))$.*

For $v = \infty$, and for $G_\infty = \mathrm{GO}(D)_\infty$ or $\mathrm{Sp}_4(\mathbf{R})$, denote by $\mathrm{Irr}(G_\infty)$ the set of equivalence classes of irreducible $(\mathfrak{g}, K)$-modules† underlying the admissible representations of G_∞. We then define $\mathcal{R}(G_\infty)$ to be the set of equivalence classes of $(\mathfrak{g}, K)$-modules π_∞ of G_∞ which occur in ω_∞ in the sense that $\mathrm{Hom}_{(\mathfrak{g},K)}(\omega_\infty, \pi_\infty) \neq 0$ where ω_∞ now denotes the underlying $(\mathfrak{g}, K)$-module. As before, we define $\mathcal{R}(\mathrm{GSp}_4(\mathbf{R})^+)$ to be the set of $\pi_\infty \in \mathrm{Irr}(\mathrm{GSp}_4(\mathbf{R})^+)$ such that $\pi_\infty|_{\mathrm{Sp}_4(\mathbf{R})}$ is multiplicity-free and has an irreducible constituent in $\mathcal{R}(\mathrm{Sp}_4(\mathbf{R}))$. Finally, we set $\mathcal{R}(\mathrm{GSp}_4(\mathbf{R}))$ to be the set of $\pi_\infty \in \mathrm{Irr}(\mathrm{GSp}_4(\mathbf{R}))$ such that some irreducible constituent of $\pi_\infty|_{\mathrm{GSp}_4(\mathbf{R})^+}$ is contained in $\mathcal{R}(\mathrm{GSp}_4(\mathbf{R})^+)$.

The (partial) Howe duality in this special case refers to the following theorem:

Theorem 4.2.1 (Howe, Moeglin–Vignéras–Waldspurger, Waldspurger, B. Roberts)**.** *If $v \neq 2$, then*

$$\{(\pi_v^{\mathrm{GSp}}, \pi_v^{\mathrm{GO}}) \in \mathcal{R}(\mathrm{GSp}_4(\mathbf{Q}_v)) \times \mathcal{R}(\mathrm{GO}(D)_v) : \mathrm{Hom}_{\mathrm{G}(\mathrm{Sp}_4 \times \mathrm{O}(D))_v}(\omega_v, \pi_v^{\mathrm{GSp}} \otimes \pi_v^{\mathrm{GO}}) \neq 0\}$$

---

*We cannot directly define $\mathcal{R}(\mathrm{GSp}_4(\mathbf{Q}_l))$ because $\mathrm{GSp}_4(\mathbf{Q}_l) \times \mathrm{GO}(D)_l$ does not lie in $\mathrm{G}(\mathrm{Sp}_4 \times \mathrm{O}(D))_l$.

†Here $\mathfrak{g} = \mathrm{Lie}(G_\infty)$ and K is the maximal compact subgroup of G_∞ defined in §3.1.2 and §3.4.3 respectively.

is the graph of a bijection between $\mathcal{R}(\mathrm{GSp}_4(\mathbf{Q}_v))$ *and* $\mathcal{R}(\mathrm{GO}(D)_v)$*, and*

$$\dim_{\mathbf{C}} \mathrm{Hom}_{\mathrm{G}(\mathrm{Sp}_4\times\mathrm{O}(D))_v}(\omega_v, \pi_v^{\mathrm{GSp}} \otimes \pi_v^{\mathrm{GO}}) \le 1.$$

If $v = 2$*, then the same statements hold with* $\mathcal{R}(\mathrm{GSp}_4(\mathbf{Q}_2))$ *replaced by the subset of tempered representations* $\mathcal{R}(\mathrm{GSp}_4(\mathbf{Q}_2))_{\mathrm{temp}}$.

Proof. This is exactly [Rob01, Theorem 1.8] in the case that the discriminant of the quadratic space is a square (so $d = 1$ in the notations there), and $m = 4 = 2n$. □

Given $\pi_v \in \mathcal{R}(\mathrm{GO}(D)_v)$, we denote by $\theta(\pi_v)$ the unique element in $\mathcal{R}(\mathrm{GSp}_4(\mathbf{Q}_v))$ (or $\mathcal{R}(\mathrm{GSp}_4(\mathbf{Q}_2))_{\mathrm{temp}}$ when $v = 2$) such that $\dim_{\mathbf{C}} \mathrm{Hom}(\omega_v, \theta(\pi_v) \otimes \pi_v) = 1$.

Remark 4.2.3. We have a complete description of the local theta lifts $\theta(\pi^+_{(1,2),v})$ in terms of the representations $\pi_{1,v}$ and $\pi_{2,v}$ of $D_v^\times$ in [GT, Tables 2 and 3].

4.2.5 Theta kernels and theta lifts.

Identify $\mathrm{GO}(D)$ with $\tilde{H}^+$ and view $\boldsymbol{\omega}$ as a representation of $\mathrm{G}(\mathrm{Sp}_4 \times H^+)_{\mathbf{A}}$. Given a smooth vector $\varphi \in \mathscr{S}(\mathbf{X}_{\mathbf{A}})$, we define the *theta kernel*, Θ_φ, to be the function on $(g,h) \in \mathrm{G}(\mathrm{Sp}_4 \times H^+)_{\mathbf{A}}$ given by

$$\Theta_\varphi(g,h) = \sum_{x\in\mathbf{X}_{\mathbf{Q}}} (\boldsymbol{\omega}(g,h)\varphi)(x).$$

By [How79, Theorem 4.1], we have that Θ_φ is left-invariant under $\mathrm{G}(\mathrm{Sp}_4 \times H^+)_{\mathbf{Q}}$ and thus belongs to the space $L^2(\mathrm{G}(\mathrm{Sp}_4 \times H^+)_{\mathbf{Q}} \backslash \mathrm{G}(\mathrm{Sp}_4 \times H^+)_{\mathbf{A}})$. The theta lift of $f \in L^2(\tilde{H}^+_{\mathbf{Q}} \backslash \tilde{H}^+_{\mathbf{A}})$ with respect to φ is then the function in $L^2(\mathrm{GSp}_4(\mathbf{Q})\backslash \mathrm{GSp}_4(\mathbf{A}))$ given by

$$\theta_\varphi(f)(g) = \int_{H^+_{\mathbf{Q}}\backslash H^+_{\mathbf{A}}} \Theta_\varphi(g, hh_g)\cdot f(hh_g)\, d^+h,$$

for any $h_g \in \tilde{H}^+_{\mathbf{A}}$ with $\lambda(h_g) = \lambda'(g)$. A priori, such an h_g only exists for elements g in the subgroup

$$\begin{aligned} \mathrm{GSp}_4(\mathbf{A})^+ &= \{g \in \mathrm{GSp}_4(\mathbf{A}) : \lambda'(g) \in \lambda(\mathrm{GO}(D)_{\mathbf{A}}) = \mathbf{A}_f^\times \cdot \mathbf{R}_{>0}\} \\ &= \mathrm{GSp}_4(\mathbf{A}_f) \times \mathrm{GSp}_4(\mathbf{R})^+; \end{aligned}$$

however, since the element $\begin{bmatrix} & 1_2 \\ -1_2 & \end{bmatrix} \in \mathrm{GSp}_4(\mathbf{Q})$ has similitude -1, we see that $\theta_\varphi(f)$ extends uniquely [Rob01, pg. 289] to include $g_\infty \in \mathrm{GSp}_4(\mathbf{R})$ with $\lambda'(g_\infty) < 0$ so that

$$\theta_\varphi(f)(g_\infty) = \theta_\varphi(f)\left(\begin{bmatrix} & 1_2 \\ -1_2 & \end{bmatrix} g_\infty\right).$$

Given an automorphic representation $(\boldsymbol{\pi}^+, V_{\boldsymbol{\pi}}^+)$ on $\tilde{H}^+$, we define its theta lift to be the representation $\Theta(\boldsymbol{\pi}^+)$ of $\mathrm{GSp}_4(\mathbf{A})$ generated by $\theta_\varphi(f)$ for all $f \in V_{\boldsymbol{\pi}}^+$ and $\varphi \in \mathscr{S}(\mathbf{X}_{\mathbf{A}})$. It follows from the definition that $\Theta(\boldsymbol{\pi}^+)$ occurs in the space of automorphic forms on GSp_4 and hence is an automorphic representation*.

4.2.6 Vector-valued theta lifts.

For computational purposes, it is more convenient to fix models for the weights (or K-types) of $\boldsymbol{\pi}^+$ and $\boldsymbol{\omega}$, and integrate vector-valued automorphic forms. More precisely, let (ϱ^+, V_ϱ^+) be an irreducible representation of $\tilde{H}_\infty^+$ on which the center acts trivially. Suppose that $\boldsymbol{\pi}_\infty^+ \simeq \varrho^+$, then given a vector-valued automorphic form $\mathbf{f}^+ \in (V_{\boldsymbol{\pi}}^+ \otimes \check{V}_\varrho^+)^{\tilde{H}_\infty^+}$ and a vector-valued Bruhat–Schwartz function $\boldsymbol{\varphi}^+ = \sum_i \varphi_i \otimes v_i^+ \in (\mathscr{S}(\mathbf{X}_{\mathbf{A}}) \otimes V_\varrho^+)^{\tilde{H}_\infty^+}$, we define

$$\begin{aligned}\Theta_{\boldsymbol{\varphi}^+}(\mathbf{f}^+)(g) &= \langle \boldsymbol{\omega}(g, h_g) \cdot \Theta_{\boldsymbol{\varphi}^+}, \boldsymbol{\pi}^+(h_g) \cdot \mathbf{f}^+ \rangle_{H^+} \\ &= \int_{H_{\mathbf{Q}}^+ \backslash H_{\mathbf{A}}^+} \langle \Theta_{\boldsymbol{\varphi}^+}(g, hh_g) \cdot \mathbf{f}^+(hh_g) \rangle^+ \, d^+h\end{aligned} \tag{2.6.5}$$

where $\Theta_{\boldsymbol{\varphi}^+}(g,h) = \sum_{x \in \mathbf{X}_{\mathbf{Q}}} \sum_i (\boldsymbol{\omega}(g,h) \cdot \varphi_i)(x) \otimes v_i^+$ similar to before, and $\langle \bullet, \bullet \rangle^+$ is a $\tilde{H}_\infty^+$-invariant pairing on $V_\varrho \times \check{V}_\varrho$.

By §3.3.6, we see that $\Theta_{\boldsymbol{\varphi}^+}(\mathbf{f}^+)$ is equal to

$$\begin{aligned}\Theta_{\boldsymbol{\varphi}}(\mathbf{f})(g) &= \langle \boldsymbol{\omega}(g, h_g) \cdot \Theta_{\boldsymbol{\varphi}}, \boldsymbol{\pi}(h_g) \cdot \mathbf{f} \rangle_H \\ &= \int_{H_{\mathbf{Q}} \backslash H_{\mathbf{A}}} \langle \Theta_{\boldsymbol{\varphi}}(g, hh_g), \mathbf{f}(hh_g) \rangle \, dh\end{aligned} \tag{2.6.6}$$

where now $\boldsymbol{\varphi} = \sum_i \varphi_i \otimes v_i$ and v_i is the projection of v_i into an irreducible subrepresentation of H_∞ in V_ϱ^+ and $\mathbf{f}$ is similarly defined. From this we see that it suffices to work with integrals over $H_{\mathbf{Q}} \backslash H_{\mathbf{A}}$, and we shall do so from now on.

*Here we use the notions of an automorphic form and an automorphic representation are those from [BJ79].

4.3 Representation-theoretic aspects of the Yoshida lift

We summarize some results concerning the automorphic representation $\Theta(\boldsymbol{\pi}_{1,2}^{+})$ of GSp_4 generated by the Yoshida lift of the automorphic representation $\boldsymbol{\pi}_{1,2}^{+}$ on $\mathrm{GO}(D)$ from §3.3.3.

4.3.1 A characterization of $\Theta(\boldsymbol{\pi}_{1,2}^{+})$.

Now consider the automorphic representation $\boldsymbol{\pi}_{1,2}^{+}$ on $\tilde{H}^{+}$ obtained from the pair of automorphic forms $\boldsymbol{\pi}_i$ on $D^{\times}$, $i = 1,2$ as in §3.3.2 and §3.3.3. Recall that $\boldsymbol{\pi}_1$ has central character ε and $\boldsymbol{\pi}_2$ has central character ε^{-1} (and $\varepsilon_\infty = 1$), and that ε is also the central character of $\boldsymbol{\pi}_{1,2}^{+}$. The following theorem then characterizes $\Theta(\boldsymbol{\pi}_{1,2}^{+})$:

Theorem 4.3.1 ([Rob01, Theorem 8.3]). $\Theta(\boldsymbol{\pi}_{1,2}^{+})$ *is non-zero and is an irreducible unitary cuspidal automorphic representation*† *of* $\mathrm{GSp}_4(\mathbf{A})$ *with central character* ε*, and*

$$\Theta(\boldsymbol{\pi}_{1,2}^{+}) \simeq \otimes_v' \theta(\check{\pi}_{(1,2),v}^{+}) \simeq \otimes_v' \theta(\pi_{(1,2),v}^{\vee,+})$$

where $\theta(\pi_{(1,2),v}^{+})$ *is the representation of* $\mathrm{GSp}_4(\mathbf{Q}_v)$ *corresponding to* $\pi_{(1,2),v}^{+}$ *under the Howe duality. For all* v, $\theta(\pi_{(1,2),v}^{+})$ *is tempered.*

Proof. Since $\boldsymbol{\pi}_{1,2}^{+}$ is cooked up so (3) holds in [Rob01, Theorem 8.3], the theorem follows. □

4.3.2 Matching levels.

Let $\mathscr{D}$ be an Eichler order of level dNp^r in D. At each prime l, consider the Bruhat–Schwartz function given by

$$\varphi_l = \mathbf{1}_{\mathscr{X}_l}$$

where $\mathscr{X}_l = \mathscr{L}_l^{\vee} \otimes \mathscr{D}_l$ and $\mathscr{L}^{\vee} = \mathbf{Z}\check{\mathbf{w}}_1 \oplus \mathbf{Z}\check{\mathbf{w}}_2$ as before. Let

$$U_l^{\mathrm{GSp}}(n_l) = \mathrm{GSp}_4(\mathbf{Z}_l) \cap \begin{bmatrix} \mathrm{M}_2(\mathbf{Z}_l) & \mathrm{M}_2(\mathbf{Z}_l) \\ l^{n_l}\mathrm{M}_2(\mathbf{Z}_l) & \mathrm{M}_2(\mathbf{Z}_l) \end{bmatrix}$$

be a compact open subgroup of $\mathrm{GSp}_4(\mathbf{Q}_l)$ that we considered in §3.4.3. Here we take $n_l = \mathrm{ord}_l(dNp^r)$. We would like to check that φ_l is fixed by $U_l^{\mathrm{GSp}}(n_l)$ under the Weil representation ω_l. First observe that given an element $g \in U_l^{\mathrm{GSp}}(n_l)$ with similitude

†When the Yoshida lift from §4.5.1 is non-zero, the notion of a cuspidal automorphic representation is that from §3.4.7; in the exceptional case that the Yoshida lift is identically zero, the notion of a cuspidal automorphic representation is that from [BJ79].

$\alpha \in \mathbf{Z}_l^\times$, the element $h_g = (\alpha, 1) \in U_l^{\mathscr{D}}$ has the same similitude and fixes φ_l under ω_l. Consequently, we have $\omega(g, h_g) \cdot \varphi_l = \omega(g \cdot \mathrm{diag}[1, \alpha]^{-1}, 1) \cdot \varphi_l$. Since $g \cdot \mathrm{diag}[1, \alpha]^{-1}$ has similitude 1, it suffices to show that φ_l is fixed by $U^{\mathrm{Sp}}(n_l) = \mathrm{GSp}_4(\mathbf{Q}_l) \cap U^{\mathrm{GSp}}(n_l)$. This is [Yos84, Lemma 2.1], whose proof we reproduce here for convenience.

First note that $U^{\mathrm{Sp}}(n_l)$ is generated by its elements of the form

$$u^+(S) = \begin{bmatrix} 1 & S \\ & 1 \end{bmatrix} \qquad d(a) = \begin{bmatrix} a & \\ & {}^t a^{-1} \end{bmatrix} \qquad u^-(S) = \begin{bmatrix} 1 & \\ S & 1 \end{bmatrix}.$$

We have

$$(\omega_l(u^+(S)) \cdot \varphi_l)(x) = \boldsymbol{\psi}_l(ST_x) \cdot \varphi_l(x) = \varphi_l(x)$$

since $\boldsymbol{\psi}_l(ST_x) = 1$ for $x \in \mathscr{X}_l$. In the same vein, we have

$$(\omega_l(d(a)) \cdot \varphi_l)(x) = \varphi_l(x \cdot a) = \varphi_l(x)$$

since $x \cdot a$ is in $\mathscr{X}_l$ iff x is. Finally, we have

$$(\omega_l(u^-(S)) \cdot \varphi_l)(x) = (\omega_l \left(\begin{bmatrix} & -1 \\ 1 & \end{bmatrix} \cdot \begin{bmatrix} 1 & -S \\ & 1 \end{bmatrix} \cdot \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \right) \cdot \varphi_l)(x).$$

The Fourier transform

$$\hat{\varphi}_v(x) = \int_{\mathbf{X}_l} \varphi_l(y) \cdot \boldsymbol{\psi}_l(T_{x\bar{y}})\, dy$$

is the characteristic function $\mathrm{vol}(\mathscr{X}_l) \cdot \mathbf{1}_{\mathscr{X}_l^\vee}$ for the dual lattice

$$\mathscr{X}_l^\vee = \{x \in \mathbf{X} = L^\vee \otimes D : (x, y)_{\mathbf{W}} = \mathrm{tr}(T_{x\bar{y}}) \in \mathbf{Z}_l \text{ for all } y \in \mathbf{X}\}.$$

From this we see that $\hat{\varphi}_l$ is fixed under ω_l by the symmetric matrices $S \in l^{l_n}\mathrm{SM}_2(\mathbf{Z}_l)$. Consequently, for such matrices $S \in l^{l_n}\mathrm{SM}_2(\mathbf{Z}_l)$, we have

$$(\omega_l \left(\begin{bmatrix} 1 & -S \\ & 1 \end{bmatrix} \cdot \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \right) \cdot \varphi_l)(x) = (\omega_l \left(\begin{bmatrix} & -1 \\ 1 & \end{bmatrix} \right) \cdot \varphi_l)(x)$$

and therefore φ_l is fixed by $u^-(S)$.

4.3.3 Matching weights.

Let $\mathfrak{g}$ be the Lie algebra of $\mathrm{Mp}(\mathbf{W})_\infty$ and let K be a maximal compact subgroup of $\mathrm{Mp}(\mathbf{W})_\infty$ containing $H_\infty^+ \times U_\infty^{\mathrm{Sp}}$. The $(\mathfrak{g}, K)$-module underlying the archimedean

Weil representation $\omega_{\psi,\infty}$ has a realization on $\mathbf{Poly(X)} = \mathbf{Poly(X)_C}$, the space of polynomials on $\mathbf{X}$ with complex coefficients, known as the Fock model [How89, §2]. By our choice of K, we may restrict $\mathbf{Poly(X)}$ as a representation of $H_\infty^+ \times U_\infty^{\mathrm{Sp}}$. Following [HK92, §4], for $U = H_\infty^+$ or U_∞^{Sp}, and an irreducible representation ϱ of U, we let $\deg(\varrho)$ denote the the smallest integer d such that ϱ occurs in $\mathbf{Poly(X)}^d$, the space of polynomials of degree d. Also denote by $\mathbf{Poly(X)}_\varrho$ the ϱ-isotypic subspace of $\mathbf{Poly(X)}$, then the space of U*-harmonics* is

$$\mathbf{Harm}^U(\mathbf{X}) = \bigoplus_\varrho \mathbf{Poly(X)}_\varrho \cap \mathbf{Poly(X)}^{\deg(\varrho)}.$$

Since H_∞^+ and U_∞^{Sp} commute with each other in K, the space of simultaneous harmonics

$$\mathbf{Harm}^{H_\infty^+ \times U_\infty^{\mathrm{Sp}}}(\mathbf{X}) = \mathbf{Harm}^{H_\infty^+}(\mathbf{X}) \cap \mathbf{Harm}^{U_\infty^{\mathrm{Sp}}}(\mathbf{X})$$

is again a representation of $H_\infty^+ \times U_\infty^{\mathrm{Sp}}$. The archimedean Howe duality [How89, Theorem 2.1] in this context implies that given ϱ an irreducible representation of H_∞^+, the ϱ-isotypic component $\mathbf{Harm}^{H_\infty^+ \times U_\infty^{\mathrm{Sp}}}(\mathbf{X})_\varrho$ is either zero or is isomorphic to $\varrho \otimes \varsigma$ for some irreducible representation ς of U_∞^{Sp}. Futhermore, this correspondence $\varrho \mapsto \varsigma$ is a bijection between the representations of H_∞^+ and U_∞^{Sp} that occur in $\mathbf{Poly(X)}$.

We now describe this bijection explicitly in terms of the highest weights of the representations under consideration.

Proposition 4.3.2. *The irreducible representation ϱ of H_∞^+ occurs in $\mathbf{Poly(X)_C}$ if and only if it has the highest weight $(k_1 + k_2, k_1 - k_2; +)$ with $k_1 \geq k_2 \geq 0$ as defined in §3.3.5. In this case, ϱ corresponds to the irreducible representation ς of U_∞^{Sp} of the highest weight $(k_1 + k_2 + 2, k_1 - k_2 + 2)$. Furthermore, we have $\deg(\varrho) = 2k_1$ and $\deg(\varsigma) = 2k_1$.*

Proof. This is [Pau05, Proposition 4] in the case $p = 4$ and $q = 0$. □

4.3.4 Matching Harish-Chandra parameters.

We follow [Pau05] to describe now the Harish-Chandra parameters correspond for the dual pair $(H_\infty^+, \mathrm{Sp}_4(\mathbf{R}))$ focusing on the case of (limits of) holomorphic discrete series.

Proposition 4.3.3. *Let $\lambda_d = a_1 e_1 + a_2 e_2$ be the Harish-Chandra parameter associated to a representation Π of $\mathrm{Sp}_4(\mathbf{R})$ as described in §3.4.8. Suppose $a_1 \geq a_2 \geq 0$. Let $\theta(\Pi)$*

be the representation of $H_\infty^+ \simeq \mathrm{O}_4(\mathbf{R})$ *corresponding to* Π *under Howe duality. The possibilities for* $\theta(\Pi)$ *are as follows:*

$a_1 \geq a_2 > 0$**:** *In this case* Π *is a holomorphic discrete series and* $\theta(\Pi)$ *is of the highest weight* $(a_1 - 1, a_2; +)$.

$a_1 > a_2 = 0$**:** *In this case* Π *is a limit of holomorphic discrete series and* $\theta(\Pi)$ *is of the highest weight* $(a_1 - 1, 0; +)$.

Proof. We have a complete description of how the Harish-Chandra parameters for the limits of discrete series for $\mathrm{Sp}_4(\mathbf{R})$ and $\mathrm{O}_4(\mathbf{R})$ correspond under the archimdean Howe duality in [Pau05, Theorem 15]. Note that we have fixed our root systems in §3.3.5 for $\mathfrak{so}_4$ and in §3.4.8 for $\mathfrak{sp}_4$ to align with that in [Pau05, §2.1].

Since $\mathrm{O}_4(\mathbf{R})$ is compact, its irreducible representations are uniquely determined by their highest weights with respect to the simple roots fixed in §3.3.5. Suppose $\lambda_d = a_1 e_1 + a_2 e_2 = (a_1, a_2)$ is a Harish-Chandra parameter for $\mathrm{O}_4(\mathbf{R})$, then it corresponds to the irreducible representation of $\mathrm{O}_4(\mathbf{R})$ of the highest weight

$$\lambda_d - \frac{1}{2} \cdot ((e_1 + e_2) + (e_1 - e_2)) = \lambda_d - e_1$$

by [Pau05, §3.2].

Using the notations from [Pau05, Theorem 15], in the case $a_1 \geq a_2 > 0$ is a holomorphic discrete series we have $w = 0 = z$, $k = 2$, and $p_1 = 2$. This lands us in (1), and the Harish-Chandra parameter for $\theta(\Pi)$ is (a_1, a_2). On the other hand, if $a_1 > a_2 = 0$ is a limit of discrete series, then $w = 0$, $z = 1$, $k = 1$, and $p_1 = 1$. This lands us in (4), and the Harish-Chandra parameter for $\theta(\Pi)$ is $(a_1, 0)$.

Also there is a matter of sign since we are considering representations of $\mathrm{O}_4(\mathbf{R})$ and not just $\mathrm{SO}_4(\mathbf{R})$; in the cases that we are considering, this sign is always $+$. □

Since the archimedean Howe duality [How89, Theorem 1] defines a bijection between the admissible representations of $\mathrm{Sp}_4(\mathbf{R})$ and $\mathrm{O}(D)_\infty$ that occur in ω_∞, we have the following corollary:

Corollary 4.3.4. *We have that* $\theta_\infty(\mathscr{V}^+_{2k_1, 2k_2, \infty}) \simeq \Pi_{k_1+k_2+1, k_1-k_2}$.

Proof. Indeed, since $\mathscr{V}^+_{2k_1, 2k_2, \infty}$ is a representation of $\mathrm{O}(D)_\infty \simeq \mathrm{O}_4(\mathbf{R})$ of the highest weight $(k_1 + k_2, k_1 - k_2; +)$, we have $a_1 = k_1 + k_2 + 1$ and $a_2 = k_1 - k_2$ in the statement of the proposition. □

4.3.5 Matching L-parameters and L-functions.

We can embed GSpin_4 into GSp_4 as the subgroup

$$\{(g,g') \in \mathrm{GL}(\mathbf{Q}\mathbf{w}_1 \oplus \mathbf{Q}\check{\mathbf{w}}_1) \times \mathrm{GL}(\mathbf{Q}\mathbf{w}_2 \oplus \mathbf{Q}\check{\mathbf{w}}_2) : \det(g) = \det(g')\}.^*$$

Consequently, the L-parameter $\phi_{\pi_{(1,2),l}}$ associated with $\pi_{(1,2),l}$ induces an L-parameter

$$\tilde{\phi}: W_l \times \mathrm{SL}_2(\mathbf{C}) \to \mathrm{GSp}_4(\mathbf{C}),$$

which is associated with the representation $\theta(\pi^+_{(1,2),l})$ of $\mathrm{GSp}_4(\mathbf{Q}_l)$ [GT, §6]. (In other words, the local theta correspondence realizes the local Langlands functoriality in this case.)

Let $c_i = c(\boldsymbol{\pi}_i)$ be the level of the automorphic representation $\boldsymbol{\pi}_i$ on $D^\times$. For $l \nmid c_1c_2$, the spherical representations $\pi_{1,l} = \pi^{\mathrm{JL}}_{1,l} \simeq \pi(\mu_1,\mu_1')$ and $\pi_{2,l} = \pi^{\mathrm{JL}}_{2,l} \simeq \pi(\mu_2,\mu_2')$ are taken to the spherical representation $\theta(\pi^+_{(1,2),l}) \simeq \Pi(\mu_1, \varepsilon_l\mu_2'\mu_1^{-1}, \varepsilon_l\mu_2\mu_1^{-1}) = \Pi(\mu_1, \mu_2'\mu_1', \mu_2\mu_1')$ of $\mathrm{GSp}_4(\mathbf{Q}_l)$ defined in §3.4.9 by the local theta correspondence [GT, Theorem 9.5 (vi)].† It follows that the Satake parameters of $\theta(\pi^+_{(1,2),l})$ are

$$a_{0,l} = \beta_{1,l}, a_{1,l} = \beta'_{2,l}\beta'_{1,l}, a_{2,l} = \beta_{2,l}\beta'_{1,l}.$$

Since $\beta_{1,l}\beta'_{1,l} = \varepsilon_l(\mathrm{Frob}_l) = \beta_{2,l}^{-1}\beta'^{-1}_{2,l}$, we see that the partial standard L-function is

$$\begin{aligned}
&L^{(c_1c_2)}(s, \Theta(\boldsymbol{\pi}^+_{1,2}), \mathrm{std}) \\
&= \prod_{l\nmid c_1c_2} \frac{1}{(1-l^{-s})(1-\beta'_{2,l}\beta'_{1,l}\cdot l^{-s})(1-\beta_{2,l}\beta'_{1,l}\cdot l^{-s})(1-\beta'^{-1}_{2,l}\beta'^{-1}_{1,l}\cdot l^{-s})(1-\beta^{-1}_{2,l}\beta'^{-1}_{1,l}\cdot l^{-s})} \\
&= \zeta^{(c_1c_2)}(s)\cdot \prod_{l\nmid c_1c_2} \frac{1}{(1-\beta'_{2,l}\beta'_{1,l}\cdot l^{-s})(1-\beta_{2,l}\beta'_{1,l}\cdot l^{-s})(1-\beta_{2,l}\beta'_{1,l}\cdot l^{-s})(1-\beta'_{2,l}\beta_{1,l}\cdot l^{-s})} \\
&= \zeta^{(c_1c_2)}(s)\cdot L^{(c_1c_2)}(s, \boldsymbol{\pi}_1 \boxtimes \boldsymbol{\pi}_2, \mathrm{std}),
\end{aligned} \tag{3.5.7}$$

---

*In terms of matrices, the embedding of $\mathrm{GSpin}_4 \simeq \{(g,g') \in \mathrm{GL}_2 \times \mathrm{GL}_2 : \det(g) = \det(g')\}$ into GSp_4 is given by

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right) \mapsto \begin{bmatrix} a & & b & \\ & a' & & b' \\ c & & d & \\ & c' & & d' \end{bmatrix}.$$

†We have to twist the second factor by ε because of our choice of the isomorphism $\tilde{H} \simeq \mathrm{GSO}(D)$.

and the partial spinor L-function is

$$
\begin{aligned}
&L^{(c_1c_2)}(s,\Theta(\boldsymbol{\pi}_{1,2}^{+}),\mathrm{spin})\\
&=\prod_{l\nmid c_1c_2}\frac{1}{(1-\beta_{1,l}l^{-s})(1-\beta_{1,l}\beta'_{2,l}\beta'_{1,l}\cdot l^{-s})(1-\beta_{1,l}\beta_{2,l}\beta'_{1,l}\cdot l^{-s})(1-\beta_{1,l}\beta'_{2,l}\beta'_{1,l}\beta_{2,l}\beta'_{1,l}\cdot l^{-s})}\\
&=\prod_{l\nmid c_1c_2}\frac{1}{(1-\beta_{1,l}l^{-s})(1-\beta'_{1,l}l^{-s})}\cdot\frac{1}{(1-\beta_{2,l}^{-1}l^{-s})(1-\beta'^{-1}_{2,l}l^{-s})}\\
&=L^{(c_1c_2)}(s,\boldsymbol{\pi}_1)\cdot L^{(c_1c_2)}(s,\check{\boldsymbol{\pi}}_2)
\end{aligned}
\tag{3.5.8}
$$

which agree with [BSP91, Cor 6.1] when $\varepsilon=1$.

4.4 Good theta kernels

We describe the Bruhat–Schwartz functions and the resulting theta kernels used to define the Yoshida lift.

4.4.1 Recipe for a Bruhat–Schwartz function.

The Bruhat–Schwartz function depends on

- a finite set of primes S containing those dividing Np, but not those dividing d;
- an S-basis $(\delta,\jmath)$ for D as in §1.1.3;
- an Eichler order $\mathscr{D}=\mathscr{D}^{\delta,\jmath,S}(dNp^r)$ of level dNp^r as in §1.3.4; and
- an integer $k\geq 0$.

Given these data, the Bruhat–Schwartz function $\boldsymbol{\varphi}=\boldsymbol{\varphi}_{2k}^{\delta,\jmath,\mathscr{D}}=\otimes_v\varphi_{2k,v}^{\delta,\jmath,\mathscr{D}}$ is a factorizable vector in

$$(\mathscr{S}(\mathbf{X}_{\mathbf{A}})\otimes\mathscr{V}_{2k,0,K_\jmath}^{\delta,\jmath})^{U_f\times H_\infty}$$

where $U_f=U_f^{\mathscr{D}}$ acts by $\boldsymbol{\omega}_f\otimes 1$, and H_∞ acts by the representation $\boldsymbol{\omega}_\infty\otimes\sigma_{2k,0,\mathrm{wt},\infty}^{\delta,\jmath}$. The precise recipe is as follows:

For $v=l$ a prime: We take

$$\varphi_l(x)=\varphi_l^{\delta,\jmath,\mathscr{D}}(x)=\frac{1}{\mathrm{vol}(U_l^{\mathscr{D}})}\cdot\mathbf{1}_{\mathscr{X}_l}(x)$$

where $\mathscr{X}_l=\mathscr{L}_l^\vee\otimes\mathscr{D}_l$ and $\mathbf{1}_{\mathscr{X}_l}$ is the characteristic function on $\mathscr{X}_l$. Note that that $\varphi_f=\otimes_l\varphi_l$ is invariant under $U_f^{\mathscr{D}}$.

For $v = \infty$: Given $x \in \mathbf{X}_\infty = L_\infty^\vee \otimes D_\infty$, we set $x_i = x(\mathbf{w}_i)$ and denote by

$$x^{(0)} = (x_1\bar{x}_2)^{(0)} = \frac{x_1\bar{x}_2 - x_2\bar{x}_1}{2}$$

the trace-zero part of $x_1\bar{x}_2$. With the notation fixed, let us define a vector-valued polynomial $\tilde{\mathbf{P}}_{2k} = \tilde{\mathbf{P}}_{2k}^{\delta,\jmath}$ on $\mathbf{X}_\infty$ with values in $\mathscr{V}_{2k,0,K_\jmath}^{\delta,\jmath}$ by

$$\tilde{\mathbf{P}}_{2k}^{\delta,\jmath}(x) = \mathbf{P}_{2k}(\epsilon_{\mathrm{wt},\infty}^{\delta,\jmath}(x^{(0)})) = \sum_{i=-k}^{k}(-1)^i\frac{(k!)^2}{(k+i)!(k-i)!}\tilde{P}_i^k((x_1\bar{x}_2)^{(0)}) \otimes \mathbf{t}_i$$

where $\tilde{P}_i^k(x) = P_i^k(\epsilon_{\mathrm{wt},\infty}^{\delta,\jmath}(x))$ for $x \in D_\infty$ and $P_i^k \in \mathscr{H}_k$ is the harmonic polynomial defined in §1.2.5. We then set

$$\varphi_\infty(x) = \varphi_{2k,\infty}^{\delta,\jmath,\mathscr{D}}(x) = E(x)\cdot\tilde{\mathbf{P}}_{2k}^{\delta,\jmath}(x)$$

where $E(x) = e^{-2\pi\cdot\mathrm{tr}(T_x)} = e^{-2\pi(\mathrm{n}(x_1)+\mathrm{n}(x_2))}$ is the standard Gaussian.

The corresponding theta kernel is then

$$\Theta_{2k}^{\delta,\jmath,\mathscr{D}}(g,h) = \Theta_{\boldsymbol{\varphi}_{2k}^{\delta,\jmath,\mathscr{D}}}(g,h) = \sum_{x\in\mathbf{X}_\mathbf{Q}}(\boldsymbol{\omega}(g,h)\cdot\boldsymbol{\varphi}_{2k}^{\delta,\jmath,\mathscr{D}})(x).$$

Proposition 4.4.1. *$\tilde{\mathbf{P}}_{2k}^{\delta,\jmath}$ lies in the subspace of $\mathbf{Harm}^{H_\infty\times U_\infty^{\mathrm{Sp}}}(\mathbf{X})_{\sigma_{2k,0,\mathrm{wt},\infty}^{\delta,\jmath}} \otimes \mathscr{V}_{2k,0,\infty}$ fixed by H_∞ under $\omega_\infty \times \sigma_{2k,0,\mathrm{wt},\infty}^{\delta,\jmath}$.*

Proof. We suppress the superscript $(\delta,\jmath)$ and the subscript $(2k,0)$ in order to keep the discussion transparent. First note that the degree of $\tilde{\mathbf{P}}_{2k}$ is $2k = \deg(\sigma_{2k})$ since $x_1\bar{x}_2$ has coordinates of degree 2 (with respect to any choice of $\mathbf{X}_\infty \simeq \mathbf{R}^8$); hence $\tilde{\mathbf{P}}_{2k} \in \mathbf{Harm}^{H_\infty\times U_\infty^{\mathrm{Sp}}}(\mathbf{X})_{\sigma_{\mathrm{wt},\infty}} \otimes \mathscr{V}_\infty$ by Proposition 4.3.2. Now given $h = (\alpha,\beta) \in H_\infty$, we have that

$$\begin{aligned}
&\sigma_{\mathrm{wt},\infty}(h)\cdot(\omega_\infty(h)\cdot\tilde{\mathbf{P}}_{2k})(x)\\
&= \sum_{i=-k}^{k}(-1)^i\frac{(k!)^2}{(k+i)!(k-i)!}\tilde{P}_i^k((\alpha^{-1}x_1\bar{x}_2\alpha)^{(0)}) \otimes \sigma_{\mathrm{wt},\infty}(\alpha)\cdot\mathbf{t}_i\\
&= \sum_{i=-k}^{k}(-1)^i\frac{(k!)^2}{(k+i)!(k-i)!}\cdot(\rho_k(\alpha)\cdot\tilde{P}_i^k)((x_1\bar{x}_2)^{(0)}) \otimes \sigma_{\mathrm{wt},\infty}(\alpha)\cdot\mathbf{t}_i\\
&= \tilde{\mathbf{P}}_{2k}(x)
\end{aligned}$$

by §1.2.6. ◻

Again Proposition 4.3.2, we see that $\omega_\infty(U_\infty^{\mathrm{Sp}}) \cdot \tilde{\mathbf{P}}_{2k}^{\delta,\jmath}$ spans over $\mathbf{C}$ an irreducible representation of U_∞^{Sp} of the highest weight $(k+2, k+2)$.

Remark 4.4.1. The Bruhat–Schwartz functions we considered above belong to the space of Bruhat–Schwartz functions from [Yos80, pg. 202] and [Yos84, (2.11)-(2.14)].

4.4.2 S-basis attached to linear maps.

Given $x \in \mathbf{X} = \mathbf{X}_{\mathbf{Q}} = \mathrm{Hom}_{\mathbf{Q}}(L, D)$, set

$$\delta_x = x_2 \bar{x}_1$$

with $x_i = x(\mathbf{w}_i)$ as before. Note that δ_x is not in $\mathbf{Q} \subseteq D$ if and only if x is non-degenerate, i.e., $x(L) = \mathbf{Q}x_1 + \mathbf{Q}x_2$ is two-dimensional in D. Suppose that x is non-degenerate and lies in $\mathscr{X} = \mathscr{L}^\vee \otimes \mathscr{D}$, then δ_x lies in $\mathscr{D}$, and we can complete it to an S-basis $(\delta_x, \jmath)$ for D for any suitable choice of finite set of primes S as in §1.1.3. In this case, we shall refer to $(\delta_x, \jmath)$ as an *S-basis attached to x*.

4.4.3 A key simplification.

Let $x \in \mathscr{X}$ be non-degenerate, and let $(\delta_x, \jmath)$ be an S-basis attached to x. The following proposition is central to our calculation of Fourier coefficients.

Proposition 4.4.2.

$$\boldsymbol{\varphi}_{2k,\infty}^{\delta_x,\jmath}(x) = e^{-2\pi\cdot(\mathrm{n}(x_1)+\mathrm{n}(x_2))} \cdot (-\mathrm{n}(\delta_x^{(0)}))^{\frac{k}{2}} \cdot \mathbf{t}_0^k$$

Proof. First we note that

$$\epsilon_{\mathrm{wt},\infty}^{\delta_x,\jmath}(x^{(0)}) = \epsilon_{\mathrm{wt},\infty}^{\delta_x,\jmath}(\delta_x^{(0)}) = \begin{bmatrix} \delta_x^{(0)} & \\ & \bar{\delta}_x^{(0)} \end{bmatrix}$$

in $\mathrm{M}_2(\mathbf{C})$. By (2.5.2) from §1.2.5, we see that

$$\begin{aligned}
\tilde{\mathbf{P}}_{2k}^{\delta_x,\jmath}(x) &= \sum_{i=-k}^{k} (-1)^i \frac{(k!)^2}{(k+i)!(k-i)!} \tilde{P}_i^k((x_1\bar{x}_2)^{(0)}) \otimes \mathbf{t}_i \\
&= (\delta_x^{(0)})^k \cdot \mathbf{t}_0^k \\
&= (-\mathrm{n}(\delta_x^{(0)}))^k \cdot \mathbf{t}_0^k
\end{aligned}$$

as claimed. □

We emphasize that this key simplification occurs only when we align x and $(\delta_x, \jmath)$ in this manner, and this is one of the main reason for introducing these S-bases.

4.4.4 Compatibility with conjugation.

Let $(\delta, \jmath)$ be an S-basis and (γ, κ) be an S'-basis for D and let $\mathscr{D}$ and $\mathscr{D}'$ be the corresponding Eichler orders. Let us compare the Bruhat–Schwartz functions $\boldsymbol{\varphi}$ and $\boldsymbol{\varphi}_{2k}^{\delta,\jmath,\mathscr{D}}$ and $\boldsymbol{\varphi}_{2k}^{\gamma,\kappa,\mathscr{D}'}$.

Let $\alpha = (\alpha_v) \in D_{\mathbf{A}}^{\times}$ be an element defined as in §2.3.3. It enjoys the properties

$$\epsilon_{\mathrm{wt},\infty}^{\gamma,\kappa} \circ \mathrm{Ad}(\alpha_\infty) = \epsilon_{\mathrm{wt},\infty}^{\delta,\jmath} \qquad \text{and} \qquad \alpha_l \cdot \mathscr{D}_l \cdot \alpha_l^{-1} = \mathscr{D}_l'$$

for all l. It follows that

$$\tilde{\mathbf{P}}_{2k}^{\gamma,\kappa}(\alpha_\infty \cdot x \cdot \alpha_\infty^{-1}) = \tilde{\mathbf{P}}_{2k}^{\delta,\jmath}(x) \qquad \text{and} \qquad \varphi_l^{\gamma,\kappa,\mathscr{D}'}(\alpha_l \cdot x \cdot \alpha_l^{-1}) = \varphi_l^{\delta,\jmath,\mathscr{D}}(x)$$

where $\alpha_v \cdot x \cdot \alpha_v^{-1} = h_{\alpha_v} \cdot x$ is the post-composition action by $h_{\alpha,v} = (\alpha_v, \alpha_v) \in H_v$. As a result, we see $\boldsymbol{\omega}(h_\alpha^{-1}) \cdot \boldsymbol{\varphi}_{2k}^{\gamma,\kappa,\mathscr{D}'} = \boldsymbol{\varphi}_{2k}^{\delta,\jmath,\mathscr{D}}$ and therefore

$$\boldsymbol{\omega}(h_\alpha^{-1}) \cdot \Theta_{2k}^{\gamma,\kappa,\mathscr{D}'} = \Theta_{2k}^{\delta,\jmath,\mathscr{D}}. \tag{4.4.9}$$

4.5 An arithmetic Yoshida lift

We are ready to define an arithmetic Yoshida lift.

4.5.1 Definition of the Yoshida lift.

Let $\boldsymbol{\pi}_1$ (resp. $\boldsymbol{\pi}_2$) be a cuspidal automorphic form of

- weight $2k$ (resp. weight 0),
- level dNp^r (resp. level dNp^r), and
- central character ε (resp. central character ε^{-1}) such that $\varepsilon_\infty = 1$.

We further assume that $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ have reciprocal eigenvalue under all Atkin–Lehner operators w_l from §2.3.2, i.e., $a_l(\boldsymbol{\pi}_1)a_l(\boldsymbol{\pi}_2) = 1$ for all $l \mid dNp$. Let $\{\mathbf{f}_i^\bullet = \mathbf{f}_i^{\delta,\jmath}\}$ be a compatible set of automorphic forms attached to $\boldsymbol{\pi}_i$ for $i = 1, 2$. They give rise to the compatible set of automorphic forms $\{\mathbf{f}_{1,2}^\bullet = \mathbf{f}_1^\bullet \otimes \mathbf{f}_2^\bullet\}$ on $\tilde{H}$ attached to $\boldsymbol{\pi}_{1,2}$ as in §3.3.2.

Definition 4.5.1. The Yoshida lift of $\{\mathbf{f}_1^\bullet\}$ and $\{\mathbf{f}_2^\bullet\}$, denoted by $\mathbf{Y}(\{\mathbf{f}_1^\bullet\},\{\mathbf{f}_2^\bullet\})$, is a scalar-valued function on $\mathrm{GSp}_4(\mathbf{A})$ given by the theta lift

$$\begin{aligned}\mathbf{Y}(\{\mathbf{f}_1^\bullet\},\{\mathbf{f}_2^\bullet\})(g) &= \frac{1}{\mathrm{vol}(H_\infty)}\cdot\langle\boldsymbol{\omega}(g,h_g)\cdot\Theta_{2k}^\bullet,\boldsymbol{\pi}_{1,2}(h_g)\cdot\mathbf{f}_{1,2}^\bullet\rangle_H \\ &= \frac{1}{\mathrm{vol}(H_\infty)}\int_{H_{\mathbf{Q}}\backslash H_{\mathbf{A}}}\langle\Theta_{2k}^\bullet(g,hh_g),\mathbf{f}_{1,2}^\bullet(hh_g)\rangle_{2k,0}\,dh\end{aligned}\tag{5.1.10}$$

where h_g is any element in $\tilde{H}_{\mathbf{A}}$ with $\lambda(h_g)=\lambda'(g)$.

For simplicity, we sometimes denote the Yoshida lift by $\mathbf{Y}(\boldsymbol{\pi}_1,\boldsymbol{\pi}_2)$, keeping in mind that this actually depends on a compatible set of automorphic forms $\{\mathbf{f}_i^\bullet\}$ for each i.

4.5.2 Independence of basis.

Let us check $\mathbf{Y}=\mathbf{Y}(\{\mathbf{f}_1^\bullet\},\{\mathbf{f}_2^\bullet\})$ is well-defined, that is, it is independent of the actual automorphic form $\mathbf{f}_{1,2}^\bullet$ which we are lifting. To begin, let $(\delta,\jmath)$ be an S-basis and (γ,κ) be an S'-basis for two possibly different set of finite primes S and S'. Fix Eichler orders $\mathscr{D}=\mathscr{D}^{\delta,\jmath,S}(dNp^r)$ and $\mathscr{D}'=\mathscr{D}^{\gamma,\kappa,S'}(dNp^r)$ and denote by $\mathbf{f}_i^{\delta,\jmath}$ and $\mathbf{f}_i^{\gamma,\kappa}$ the compatible automorphic forms attached to $\boldsymbol{\pi}$ with respect to these choices following the recipe from §2.3.1 for $i=1,2$. Let $\alpha=(\alpha_v)\in D_{\mathbf{A}}^\times$ be an element defined in §2.3.3, and set $h_\alpha=(\alpha,\alpha)\in H_{\mathbf{A}}$, we have:

$$\begin{aligned}&\langle\boldsymbol{\omega}(g,h_g)\cdot\Theta_{2k}^{\gamma,\kappa,\mathscr{D}'},\boldsymbol{\pi}_{1,2}(h_g)\cdot\mathbf{f}_{1,2}^{\gamma,\kappa}\rangle_H\\ &=\int_{H_{\mathbf{Q}}\backslash H_{\mathbf{A}}}\langle\Theta_{2k}^{\gamma,\kappa,\mathscr{D}'}(g,h(h_gh_\alpha^{-1}h_g^{-1})h_g),\mathbf{f}_{1,2}^{\gamma,\kappa}(h(h_gh_\alpha^{-1}h_g^{-1})h_g)\rangle_{2k,0}\,dh\\ &=\int_{H_{\mathbf{Q}}\backslash H_{\mathbf{A}}}\langle(\boldsymbol{\omega}(h_\alpha^{-1})\cdot\Theta_{2k}^{\gamma,\kappa,\mathscr{D}'})(g,hh_g),(\boldsymbol{\pi}_{1,2}(h_\alpha^{-1})\cdot\mathbf{f}_{1,2}^{\gamma,\kappa})(hh_g)\rangle_{2k,0}\,dh\\ &=\int_{H_{\mathbf{Q}}\backslash H_{\mathbf{A}}}\langle\Theta_{2k}^{\delta,\jmath,\mathscr{D}}(g,hh_g),\mathbf{f}_{1,2}^{\delta,\jmath}(hh_g)\rangle_{2k,0}\,dh\\ &=\langle\boldsymbol{\omega}(g,h_g)\cdot\Theta_{2k}^{\delta,\jmath,\mathscr{D}},\boldsymbol{\pi}_{1,2}(h_g)\cdot\mathbf{f}_{1,2}^{\delta,\jmath}\rangle_H\end{aligned}$$

since $\boldsymbol{\pi}_{1,2}(h_\alpha^{-1})\cdot\mathbf{f}_{1,2}^{\gamma,\kappa}=\mathbf{f}_{1,2}^{\delta,\jmath}$ by §3.3.2 and $\boldsymbol{\omega}(h_\alpha^{-1})\cdot\Theta_{2k}^{\gamma,\kappa,\mathscr{D}'}=\Theta_{2k}^{\delta,\jmath,\mathscr{D}}$ by §4.4.4.

4.5.3 The arithmetic Yoshida lift as a theta function.

Let $\mathscr{Y}$ be the classical Siegel modular form to $\mathbf{Y}$ using the procedure from §3.4.11, namely,

$$\mathscr{Y}(Z)=\det(J(g_\infty,\mathbf{i}))^{k+2}\cdot\mathbf{Y}(g_\infty)$$

where $g_\infty \in \mathrm{GSp}_4(\mathbf{R})^+$ is any element such that $g_\infty\langle \mathbf{i}\rangle = Z$. Note that $\mathbf{Y}$ has weight $(k+2, k+2)$ as we saw in §4.4.1. Let us check that $\mathscr{Y}$ is in fact just a linear combination of theta functions along the lines indicated in [Yos80, (2.20)].

First note that it suffices to take $g_\infty \in \mathrm{Sp}_4(\mathbf{R})$ since $\mathrm{Sp}_4(\mathbf{R})$ acts transitively on $\mathfrak{H}_2$. By definition, we have

$$\begin{aligned}
\mathscr{Y}(Z) &= \frac{1}{\mathrm{vol}(H_\infty)} \cdot \det(J(g_\infty, \mathbf{i}))^{k+2} \cdot \int_{H_\mathbf{Q}\backslash H_\mathbf{A}} \langle \Theta^\bullet_{2k}(g_\infty, h), \mathbf{f}^\bullet_{1,2}(h)\rangle_{2k,0}\, dh \\
&= \frac{1}{\mathrm{vol}(H_\infty)} \cdot \det(J(g_\infty, \mathbf{i}))^{k+2} \cdot \int_{H_\mathbf{Q}\backslash H_\mathbf{A}} \langle \sum_{x\in \mathbf{X}_\mathbf{Q}} (\boldsymbol{\omega}(g_\infty, h) \cdot \boldsymbol{\varphi}^\bullet_{2k})(x), \mathbf{f}^\bullet_{1,2}(h)\rangle_{2k,0}\, dh \\
&= \frac{1}{\mathrm{vol}(H_\infty)} \cdot \det(J(g_\infty, \mathbf{i}))^{k+2} \cdot \int_{H_\mathbf{Q}\backslash H_\mathbf{A}} \langle \sum_{x\in \mathbf{X}_\mathbf{Q}} (\boldsymbol{\omega}(g_\infty) \cdot \boldsymbol{\varphi}^\bullet_{2k})(h^{-1}x), \mathbf{f}^\bullet_{1,2}(h)\rangle_{2k,0}\, dh.
\end{aligned}$$

We claim the integrand is invariant under the action by $U^\mathscr{D} = U^\mathscr{D}_f \times H_\infty$. Indeed, since $\boldsymbol{\pi}_i$'s have reciprocal eigenvalues under the Atkin-Lehner operators, we see that $\mathbf{f}^\bullet_{1,2}$ is right-invariant under $U^\mathscr{D}_f$. On the other hand, for $h_f \in U^\mathscr{D}_f$, we have that $h^{-1}\cdot x$ lies in the support $\mathscr{X}_f$ of $\boldsymbol{\varphi}^\bullet_{2k,f} = \otimes_{v<\infty}\varphi^\bullet_{2k,l}$ if and only if $x \in \mathscr{X}_f$, from which we see that $\boldsymbol{\varphi}^\bullet_{2k,f}$ is also right-invariant under $U^\mathscr{D}_f$. Furthermore, we have that

$$(\omega_\infty(h_\infty)\cdot \varphi^\bullet_{2k,\infty})(x) = \sigma^\bullet_{2k,0,\mathrm{wt}}(h_\infty^{-1}) \cdot \varphi^\bullet_{2k,\infty}(x)$$

by construction.* Likewise, $\mathbf{f}^\bullet_{1,2}(h_f h_\infty) = \check{\sigma}^\bullet_{2k,0,\mathrm{wt}}(h_\infty^{-1})\cdot \mathbf{f}^\bullet_{1,2}(h_f)$. Since the pairing $\langle \bullet, \bullet\rangle_{2k,0}$ is H_∞-invariant, the claim follows.

Now $H_\mathbf{Q} H_\infty \backslash H_{\mathbf{A}_f}$ is compact since D is definite, and so we have a double-coset decomposition

$$H_\mathbf{A} = \bigcup_{i=1}^r H_\mathbf{Q}\cdot h_i \cdot U^\mathscr{D}_f H_\infty$$

for a fixed set of representatives $h_1, \dots, h_r$ in H_f.

To continue, we note that by the Iwasawa decomposition of $\mathrm{Sp}_4(\mathbf{R}) = P_1(\mathbf{R})\cdot U^{\mathrm{Sp}}_\infty$, it suffices to consider an element g_∞ in the Siegel parabolic $P_1(\mathbf{R})$, which has the form

$$g_\infty = \begin{bmatrix} a & b \\ & {}^t a^{-1} \end{bmatrix} = \begin{bmatrix} a & \\ & {}^t a^{-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & a^{-1}b \\ & 1 \end{bmatrix}$$

*Indeed, we saw in the proof of Proposition 4.4.1 that

$$(\omega_\infty(h_\infty)\cdot \tilde{\mathbf{P}}^\bullet_{2k})(x) = \sigma^\bullet_{2k,0,\mathrm{wt}}(h_\infty^{-1}) \cdot \tilde{\mathbf{P}}^\bullet_{2k}(x).$$

The Gaussian $e^{-2\pi\cdot(\mathrm{n}(x_1)+\mathrm{n}(x_2))} = e^{-2\pi\cdot \mathrm{tr}(T_x)}$, on the other hand, is fixed by $\omega_\infty(h_\infty)$ since $\mathrm{tr}(T_{h_\infty^{-1}\cdot x}) = \mathrm{tr}(T_x)$.

with $a^{-1}b \in \mathrm{SM}_2(\mathbf{R})$ symmetric. We have $J(g_\infty, \mathbf{i}) = \det({}^t a^{-1}) = \det(a)^{-1}$ and that

$$\begin{aligned}\omega_\infty(g_\infty)\cdot\varphi^\bullet_{2k,\infty}(x) &= \det(a)^2\cdot\psi_\infty(\mathrm{tr}(a^{-1}b\cdot T_x))\cdot\varphi^\bullet_{2k,\infty}(x\cdot a)\\ &= \det(a)^2\cdot e^{2\pi\imath\cdot\mathrm{tr}(a^{-1}b\cdot T_x)}\cdot e^{-2\pi\cdot\mathrm{tr}(a\cdot T_x\cdot{}^t a)}\cdot\det(a)^k\cdot\tilde{\mathbf{P}}^\bullet_k(x)\\ &= \det(a)^{k+2}\cdot e^{2\pi\imath\cdot\mathrm{tr}(T_x\cdot({}^t aa\cdot\imath+a^{-1}b))}\cdot\tilde{\mathbf{P}}^\bullet_k(x)\end{aligned}$$

since $\psi_\infty(a) = \exp(2\pi\imath a)$ by §0.4.3, and a direct computation shows that $T_{x\cdot a} = a\cdot T_x\cdot{}^t a$ and $(x\cdot a)^{(0)} = \det(a)\cdot x^{(0)}$. Moreover, we see that ${}^t aa\cdot\imath + a^{-1}b = a^t a\cdot\imath + b^t a = g_\infty\langle\mathbf{i}\rangle$ as in [Yos80, (2.19)].

Substitute this in to the integral, we see that $\mathscr{Y}$ in fact a linear combination of theta functions:

$$\begin{aligned}\mathscr{Y}(Z) &= \sum_{i=1}^r \frac{1}{e_i}\sum_{x\in\mathbf{X}^T_{\mathbf{Q}}}\langle\boldsymbol{\varphi}^\bullet_{2k,f}(h_i^{-1}x)\cdot e^{2\pi\imath\cdot\mathrm{tr}(T_x\cdot Z)}\cdot\tilde{\mathbf{P}}^\bullet_k(x), \mathbf{f}^\bullet_{1,2}(h_i)\rangle_{2k,0}\\ &= \sum_{i=1}^r \frac{1}{e_i}\sum_{x\in h_i^{-1}\cdot\mathscr{X}_f\cap\mathbf{X}^T_{\mathbf{Q}}}\mathbf{f}^\bullet_2(\beta_i)\cdot\langle\tilde{\mathbf{P}}^\bullet_k(x), \mathbf{f}^\bullet_1(\alpha_i)\rangle_{2k,0}\cdot e^{2\pi\imath\mathrm{tr}(T_x Z)}\end{aligned}\tag{5.3.11}$$

where we set $h_i = (\alpha_i, \beta_i)$ and $e_i = \#(H_{\mathbf{Q}}\cap h_i U^{\mathscr{D}}_f h_i^{-1})$.

Remark 4.5.1. Because of the twist by h_i however and the presence of the (vector-valued) harmonic polynomial $\tilde{\mathbf{P}}^\bullet_k$, we cannot say much concerning this expression in terms of integrality. On the other hand, we can deduce the rationality of the Yoshida lift quite easily from this expression.

4.5.4 Properties of the arithmetic Yoshida lift.

We summarize the main properties of the Yoshida lift in the following generalization of [Yos80, Theorem 2.7]:

Theorem 4.5.1 (Yoshida)**.** *The arithmetic Yoshida lift* $\mathbf{Y} = \mathbf{Y}(\{\mathbf{f}^\bullet_1\}, \{\mathbf{f}^\bullet_2\})$ *is a scalar-valued holomorphic automorphic form on* $\mathrm{GSp}_4(\mathbf{A})$ *of level* dNp^r*, weight* $k+2$*, and central character* ε*. In other words, we have*

$$\mathbf{Y}(\gamma z g u_f u_\infty) = \varepsilon(z)\cdot\det(J(u_\infty, \mathbf{i}))^{-k-2}\cdot\mathbf{Y}(g),$$

for all $\gamma\in\mathrm{GSp}_4(\mathbf{Q}), z\in Z_{\mathrm{GSp}_4}(\mathbf{A}), g\in\mathrm{GSp}_4(\mathbf{A})$*, and* $u_f u_\infty\in U^{\mathrm{GSp}}(dNp^r)$*. The associated function on* $\mathfrak{H}_2$*,*

$$\mathscr{Y}(Z) = \det(J(g_\infty, \mathbf{i}))^{k+2}\cdot\mathbf{Y}(g_\infty)$$

for $g_\infty \in \mathrm{Sp}_4(\mathbf{R})$ such that $g_\infty\langle \mathbf{i}\rangle = Z$ is a holomorphic Siegel modular form of degree 2, character ε, level dNp^r, and weight $k+2$.

Proof. By §4.3.2 and our choice of the Bruhat–Schwartz function at the finite places, we see that $\mathbf{Y}$ has level dNp^r. The weight of $\mathbf{Y}$ is $k+2$ follows from Proposition 4.3.2 and Proposition 4.4.1. We see that the central character of $\mathbf{Y}$ is ϵ by apply $(z,1) \in Z_{\tilde{H}}(\mathbf{A}_f)$ to the Yoshida lift and noting that $\Theta^\bullet_{2k}$ is fixed by $(z,1)$.

The holomorphy of $\mathbf{Y}$ follows from the fact the irreducible automorphic representation $\Theta(\boldsymbol{\pi}^+_{1,2})$ generated by $\mathbf{Y}$ has infinity type $\boldsymbol{\Pi}_{k+1,k}$, which is a holomorphic discrete series when $k \geq 1$ (and a limit of such when $k = 0$), together with the fact that $\mathbf{Y}$ is a vector of the lowest K-type $\tau_{(k+2,k+2)}$. This is Corollary 4.3.4 together with §3.4.8 and Theorem 4.3.1, noting that $\breve{\pi}^+_{(1,2),\infty} \simeq \pi^+_{(1,2),\infty}$ as representations of $\tilde{H}^+_\infty$. Alternatively, we can expand $\mathscr{Y}$ into a linear combination of theta functions (5.3.11), which is holomorphic by inspection.

Finally, the cuspidality of $\mathbf{Y}$ follows from Theorem 4.3.1. □

Remark 4.5.2. We also described the partial standard and partial spinor L-functions associated with $\mathbf{Y}$ in §4.3.5. Note that since $c(\boldsymbol{\pi}_i) = dNp^r$ for $i = 1,2$, the Euler factors for these L-functions are defined at all primes not dividing dNp^r.

4.5.5 Rationality and integrality.

As the name implies, the arithmetic Yoshida lift $\mathbf{Y} = \mathbf{Y}(\{\mathbf{f}^\bullet_1\},\{\mathbf{f}^\bullet_2\})$ should carry some arithmetic content. To justify this, let $\mathscr{Y} = \mathscr{Y}(\{\mathbf{f}^\bullet_1\},\{\mathbf{f}^\bullet_2\})$ be the classical Siegel modular form associated with $\mathbf{Y}$ as in §3.4.11. It has a Fourier expansion (along the cusp at infinity) given by

$$\mathscr{Y} = \sum_{T\in\mathscr{T}} a(T)\cdot e^{2\pi\iota\cdot\mathrm{tr}(TZ)},$$

and by §3.6.5, the arithmetic properties of $\mathscr{Y}$ are reflected through the Fourier coefficients $a(T)$.

First we have the following proposition concerning the rationality of $\mathscr{Y}$.

Proposition 4.5.2. *If the compatible families $\{\mathbf{f}^\bullet_i\}$ attached to $\boldsymbol{\pi}$ are algebraic (Definition 2.3.3) for $i = 1,2$, then the Tth Fourier coefficient $a(T)$ of $\mathscr{Y} = \mathscr{Y}(\{\mathbf{f}^\bullet_1\},\{\mathbf{f}^\bullet_2\})$ lies in some number field $F = F^T$ which depends on T and $\{\mathbf{f}^\bullet_i\}$.*

Moreover, all the Fourier coefficients $a(T)$ of $\mathscr{Y}$ are contained in the compositum $K(\{\mathbf{f}_1\},\{\mathbf{f}_2\}) = K(\{\mathbf{f}_1\})\cdot K(\{\mathbf{f}_2\})$ where $K(\{\mathbf{f}_i\})$ is the field of definition of $\{\mathbf{f}_i\}$ as in §2.3.4.

Proof. This is Proposition 5.2.1 and Corollary 5.2.2 in the case $t_f = 1$. It follows from the formula (2.5.8) for $a(T)$. $\square$

In the same vein, we have the following result concerning the $\mathfrak{p}$-integrality of $\mathscr{Y}$.

Theorem 4.5.3. *If the compatible families $\{\mathbf{f}_i^\bullet\}$ attached to $\boldsymbol{\pi}$ are $\mathfrak{p}$-integral (Definition 2.3.4) for $i = 1, 2$, then the arithmetic Yoshida lift $\mathbf{Y}(\{\mathbf{f}_1^\bullet\}, \{\mathbf{f}_2^\bullet\})$ is $\mathfrak{p}$-integral in the sense that $\mathscr{Y} = \mathscr{Y}(\{\mathbf{f}_1^\bullet\}, \{\mathbf{f}_2^\bullet\})$ has all its Fourier coefficients $a(T)$ contained in*

$$\left(K \cap \frac{1}{k!}\mathscr{O}_{\mathbf{C}_p}\right) \subset \left(\bar{\mathbf{Q}} \cap \frac{1}{k!}\mathscr{O}_{\mathbf{C}_p}\right)$$

with $K = K(\{\mathbf{f}_1\}, \{\mathbf{f}_2\})$ as in the previous theorem.

Moreover, if $p > k$, then the Tth Fourier coefficient $a(T)$ of $\mathscr{Y}$ lies in the local ring $\mathscr{O}_{F,(\mathfrak{p}_F)}$ for the same number field $F = F^T$ from the previous theorem.

Proof. This is Theorem 5.3.3 in the case $t_f = 1$ together with the preceding proposition since the $\mathfrak{p}$-integral compatible sets $\{\mathbf{f}_i^\bullet\}$ are by definition algebraic. $\square$

The following corollary is crucial for attaching Galois representations to $\mathbf{Y}$, which we plan to take on in a subsequent work. Set $\bar{\mathbf{Z}}_{(\mathfrak{p})} = \bar{\mathbf{Q}} \cap \mathscr{O}_{\mathbf{C}_p}$.

Corollary 4.5.4. *Suppose $p > k$ and that the cuspidal automorphic representations $\boldsymbol{\pi}_i$ have level dN relatively prime to p (so $r = 0$), then $\mathscr{Y} = \mathscr{Y}(\{\mathbf{f}_1^\bullet\}, \{\mathbf{f}_2^\bullet\})$ defines a global section in $H^0(\bar{\mathscr{M}}_{2,N}, \bar{\omega}^{\otimes k} \otimes_{\mathbf{Z}[\frac{1}{N},\zeta_N]} \bar{\mathbf{Z}}_{(\mathfrak{p})})$.*

Proof. Since $p > k$, all the Fourier coefficients $a(T)$ of $\mathscr{Y}$ belong to $\bar{\mathbf{Z}}_{(\mathfrak{p})}$, the corollary then follows from Corollary 3.6.3. $\square$

4.5.6 Non-vanishing.

The question of the non-vanishing of the Yoshida lift has been studied in [Yos80], [BSP91], and [BSP97]. We strengthen some of these results by considering the case of Yoshida lifts of arbitrary level. This is our second main theorem:

Theorem 4.5.5. *Suppose that the central characters ε_i of the cuspidal automorphic representations $\boldsymbol{\pi}_i$ are trivial. Then the arithmetic Yoshida lift $\mathbf{Y}(\{\mathbf{f}_1^\bullet\}, \{\mathbf{f}_2^\bullet\})$ is not identically zero.*

Proof. We have shown in Theorem 6.2.5 that an infinite number of Fourier coefficients $a_{t_i}(\mathscr{Y})$ are not zero, consequently $\mathscr{Y}$ is identically not zero; therefore, $\mathbf{Y}(\boldsymbol{\pi}_1, \boldsymbol{\pi}_2)$ is also not identically zero. $\square$

Remark 4.5.3. We should mention that [BSP97] has obtained non-vanishing results in the case of lifts to higher-rank symplectic groups and the case of general weights. It is also worth pointing out that our methods of proof are fundamentally different.

4.5.7 Non-vanishing modulo $\mathfrak{p}$.

Assuming Artin's conjecture on primitive roots, then we have the following theorem concerning the non-triviality of $\mathscr{Y}$ modulo $\mathfrak{p}$:

Theorem 4.5.6. *Suppose that*

- $p > k$*;*
- *the cuspidal automorphic representations* $\boldsymbol{\pi}_i$*'s have level* dN *relatively prime to* p *(so* $r = 0$*);*
- *the central characters* $\varepsilon_{\boldsymbol{\pi}_i}$ *are trivial;*
- *and that Conjecture 6.2.6 holds.*

Under these assumptions, if the $\mathfrak{p}$*-integral compatible families of automorphic forms* $\{\mathbf{f}_i^\bullet\}$ *attached to* $\boldsymbol{\pi}_i$ *are non-Eisenstein at* $\mathfrak{p}$ *in the sense of Definition 2.3.5 for* $i = 1, 2$*, then the arithmetic Yoshida lift* $\mathbf{Y}(\{\mathbf{f}_1^\bullet\}, \{\mathbf{f}_2^\bullet\})$ *is non-zero modulo* $\mathfrak{p}$ *in the sense that the image of* $\mathscr{Y} = \mathscr{Y}(\{\mathbf{f}_1^\bullet\}, \{\mathbf{f}_2^\bullet\})$ *under the reduction map*

$$H^0(\bar{\mathscr{M}}_{2,N}, \bar{\omega}^{\otimes k} \otimes_{\mathbf{Z}[\frac{1}{N},\zeta_N]} \bar{\mathbf{Z}}_{(\mathfrak{p})}) \to H^0(\bar{\mathscr{M}}_{2,N}, \bar{\omega}^{\otimes k} \otimes_{\mathbf{Z}[\frac{1}{N},\zeta_N]} (\bar{\mathbf{Z}}_{(\mathfrak{p})}/\mathfrak{p}))$$

is not zero.

Proof. By Corollary 4.5.4, $\mathscr{Y}$ defines a global section in $H^0(\bar{\mathscr{M}}_{2,N}, \bar{\omega}^{\otimes k} \otimes_{\mathbf{Z}[\frac{1}{N},\zeta_N]} \bar{\mathbf{Z}}_{(\mathfrak{p})})$. By Proposition 3.6.1, the Fourier expansions of $\mathscr{Y}$ along other cusps $\gamma \cdot \mathbf{i}$ for $\gamma \in \mathrm{GSp}_4(\mathbf{Q})^+$ also have their coefficients belonging to $\bar{\mathbf{Z}}_{(\mathfrak{p})}$. The Remarks 3.4.3 and 3.5.2 explained that the Fourier coefficient $a_{t_f}(T)$ for any $t_f \in M(\mathbf{A}_f)$ with $M \subset \mathrm{GSp}_4$ the Levi subgroup of the Siegel parabolic, are in fact Fourier coefficients of $\mathscr{Y}$ along some cusp $\gamma\mathbf{i}$. Consequently, we see that $a_{t_f}(T)$, for all possible indices T, lie in $\bar{\mathbf{Z}}_{(\mathfrak{p})}$ as well.

Now we have shown in Theorem 6.2.7 that one of the Fourier coefficients $a_{t_i}(T_n)$ is not contained in $\mathfrak{p} \cdot \bar{\mathbf{Z}}_{(\mathfrak{p})}$. Apply Theorem 3.6.2 (or just Proposition 3.6.2) with $M_1 = \mathfrak{p} \cdot \bar{\mathbf{Z}}_{(\mathfrak{p})}$ and $M_2 = \bar{\mathbf{Z}}_{(\mathfrak{p})}$, we see that the section $\mathscr{Y}$ is *not* contained in

$$H^0(\bar{\mathscr{M}}_{2,N}, \bar{\omega}^{\otimes k} \otimes_{\mathbf{Z}[\frac{1}{N},\zeta_N]} \underline{\mathfrak{p} \cdot \bar{\mathbf{Z}}_{(\mathfrak{p})}})$$

(here $\underline{\mathfrak{p} \cdot \bar{\mathbf{Z}}_{(\mathfrak{p})}}$ denotes the constant sheaf on $\mathrm{Spec}(\mathbf{Z}[\frac{1}{N}, \zeta_N])$ associated with $\mathfrak{p}\bar{\mathbf{Z}}_{(\mathfrak{p})}$). Consequently its image in $H^0(\bar{\mathscr{M}}_{2,N}, \bar{\omega}^{\otimes k} \otimes_{\mathbf{Z}[\frac{1}{N}, \zeta_N]} (\bar{\mathbf{Z}}_{(\mathfrak{p})}/\mathfrak{p}))$ is not zero. $\square$

Chapter 5. Fourier Coefficients

5.1 An integral representation

Let $\mathbf{Y} = \mathbf{Y}(\{\mathbf{f}_1^\bullet\}, \{\mathbf{f}_2^\bullet\})$ be the arithmetic Yoshida lift of the compatible sets of automorphic forms $\{\mathbf{f}_i^\bullet\}$ as in Definition 4.5.1, and denote by $\mathscr{Y}$ the associated classical Siegel modular form. We rework the integral representation for the Fourier functional of $\mathscr{Y}$ in [Yos84] to fit into our framework. Since $\mathscr{Y}$ is a cusp form, we only need to consider the indices $T \in \mathscr{T}_{\mathrm{int}}$ which are semi-integral and positive definite.

5.1.1 Unfolding the theta kernel.

Fix an index $T \in \mathscr{T}_{\mathrm{int}}$ and assume that T is positive-definite. Let K^T be the imaginary quadratic field attached to T, and identify $K^{T,\times} \simeq \mathrm{GSO}(T)$ as a subgroup of $M \subset \mathrm{GSp}_4$ via the embedding describe in §3.5.4. Let us compute that value of $\mathbf{a}^T$ at

$$t = \begin{bmatrix} t & \\ & \mathrm{n}(t) \cdot {}^t t^{-1} \end{bmatrix} \in K_{\mathbf{A}}^{T,\times}.$$

We have:

$$\begin{aligned} \mathbf{a}^T(t) &= \int_{N_{\mathbf{Q}} \backslash N_{\mathbf{A}}} \mathbf{Y}\left(\begin{bmatrix} 1 & S \\ & 1 \end{bmatrix} t \right) \boldsymbol{\psi}(-TS)\, dS \\ &= \int_{N_{\mathbf{Q}} \backslash N_{\mathbf{A}}} \boldsymbol{\psi}(-TS) \cdot \frac{1}{\mathrm{vol}(H_\infty)} \int_{H_{\mathbf{Q}} \backslash H_{\mathbf{A}}} \langle \Theta_{\boldsymbol{\varphi}}^\bullet(St, h), \mathbf{f}_{1,2}^\bullet(hh_t) \rangle_{2k,0}\, d^1 h\, dS \end{aligned}$$

where h_t is an element in $H_{\mathbf{A}}$ of similitude $\lambda'(t) = \mathrm{n}(t)$ and $\bullet$ is an arbitrary choice of S-basis for D. For simplicity, we set $\langle \bullet, \bullet \rangle = \langle \bullet, \bullet \rangle_{2k,0}$.

By expanding $\Theta^\bullet_{\boldsymbol{\varphi}}$ and applying the Weil action for similitude groups, we find that

$$\Theta^\bullet_{\boldsymbol{\varphi}}(St, hh_t) = \sum_{x \in \mathbf{X_Q}} \boldsymbol{\psi}(T_x S) \cdot \boldsymbol{\varphi}^\bullet(h_t^{-1} h^{-1} x t), ^*$$

where $h \in H_{\mathbf{A}}$ and $t \in \mathrm{GL}(L)_{\mathbf{A}}$ act on $x \in \mathbf{X}$ by post and pre-composition, and T_x is the symmetric matrix associated to x in §4.2.1. Substituting this into the integral, we get

$$\begin{aligned}
\mathbf{a}^T(t) &= \frac{1}{\mathrm{vol}(H_\infty)} \int_{H_{\mathbf{Q}} \backslash H_{\mathbf{A}}} \sum_{x \in \mathbf{X_Q}} \langle \boldsymbol{\varphi}^\bullet(h_t^{-1} h^{-1} x t), \mathbf{f}^\bullet_{1,2}(hh_t) \rangle \int_{N_{\mathbf{Q}} \backslash N_{\mathbf{A}}} \boldsymbol{\psi}(T_x S - TS)\, dS\, d^1 h \\
&= \frac{1}{\mathrm{vol}(H_\infty)} \int_{H_{\mathbf{Q}} \backslash H_{\mathbf{A}}} \sum_{x \in \mathbf{X}^T_{\mathbf{Q}}} \langle \boldsymbol{\varphi}^\bullet(h_t^{-1} h^{-1} x t), \mathbf{f}^\bullet_{1,2}(hh_t) \rangle\, d^1 h,
\end{aligned}$$

since

$$\int_{N_{\mathbf{Q}} \backslash N_{\mathbf{A}}} \boldsymbol{\psi}(T_x S - TS)\, dS$$

is nonzero if and only if $\boldsymbol{\psi}((T_x - T)S) \equiv 1$, which only happens when $T_x = T$. This gives the following result.

Proposition 5.1.1. *$a_{t_f}(T) = 0$ if $\mathbf{X}^T_{\mathbf{Q}}$ is empty.*

Proof. If $\mathbf{X}^T_{\mathbf{Q}}$ is empty, then $\mathbf{a}^T(t) = 0$. The proposition follows by §3.5.2 (5.2.10). □

5.1.2 $\mathbf{X}^T$ as a homogeneous space.

To continue, observe that $H \simeq \mathrm{SO}(D)$ acts on $\mathbf{X}^T_{\mathbf{Q}} \subset \mathrm{Hom}_{\mathbf{Q}}(L, D)$ by post-composition. We have the following proposition.

Proposition 5.1.2. *If $\mathbf{X}^T_{\mathbf{Q}}$ is not empty, then it is a homogeneous space under the* SO(D).

---

*Indeed, set $\Theta_{\boldsymbol{\varphi}} = \Theta^\bullet_{\boldsymbol{\varphi}}$, we have

$$\begin{aligned}
\Theta_{\boldsymbol{\varphi}}(St, hh_t) &= \sum_{x \in \mathbf{X_Q}} (\omega(St, hh_t)\boldsymbol{\varphi})(x) \\
&= |\mathrm{n}(t)|^{-2} \sum_{x \in \mathbf{X_Q}} (\omega\left(S \cdot \begin{bmatrix} t & \\ & \mathrm{n}(t)^t t^{-1} \end{bmatrix} \begin{bmatrix} 1 & \\ & \mathrm{n}(t)^{-1} \end{bmatrix}, hh_t\right) \boldsymbol{\varphi})(x) \\
&= |\mathrm{n}(t)|^{-2} \sum_{x \in \mathbf{X_Q}} \boldsymbol{\psi}(T_x S) \cdot (\omega\left(\begin{bmatrix} t & \\ & \mathrm{n}(t)^t t^{-1} \end{bmatrix}, hh_t\right) \boldsymbol{\varphi})(x) \\
&= \sum_{x \in \mathbf{X_Q}} \boldsymbol{\psi}(T_x S) \cdot \boldsymbol{\varphi}(h_t^{-1} h^{-1} x t).
\end{aligned}$$

Proof. This is a consequence of Witt's decomposition theorem [O'M00, Theorem (42:17)]. Indeed, fix a representative $x \in \mathbf{X}_{\mathbf{Q}}^{T}$, and let y be an elements in $\mathbf{X}_{\mathbf{Q}}^{T}$ so that $T_y = T_x$. The linear isomorphism defined by sending $x_i \to y_i$ takes $x(L)$ to $y(L)$ while preserving the norm form. Witt's decomposition theorem then provides us with an isometry from $x(L)^{\perp}$ to $y(L)^{\perp}$, which we can choose to have the correct determinant so together with the isometry from $x(L) \to y(L)$, it defines an element in $\mathrm{SO}(D)$. Note this does not require x to be non-degenerate. □

We assume without loss of generality that $\mathbf{X}_{\mathbf{Q}}^{T}$ is non-empty in the following computations.

5.1.3 Orbit integral.

Fix a representative $x \in \mathscr{X} \cap \mathbf{X}_{\mathbf{Q}}^{T}$, and denote by St_x the stabilizer of x as a closed algebraic subgroup in H. We have that

$$\mathbf{X}_{\mathbf{Q}}^{T} \simeq \mathrm{St}_{x,\mathbf{Q}} \backslash H_{\mathbf{Q}}.$$

By substitution, we get

$$\begin{aligned}\mathbf{a}^{T}(t) &= \frac{1}{\mathrm{vol}(H_\infty)} \cdot \int_{H_{\mathbf{Q}} \backslash H_{\mathbf{A}}} \sum_{\gamma \in \mathrm{St}_{x,\mathbf{Q}} \backslash H_{\mathbf{Q}}} \langle \boldsymbol{\varphi}^{\bullet}(h_t^{-1} h^{-1} \gamma^{-1} x t), \mathbf{f}_{1,2}^{\bullet}(\gamma h h_t) \rangle \, d^1 h \\ &= \frac{1}{\mathrm{vol}(H_\infty)} \cdot \int_{\mathrm{St}_{x,\mathbf{Q}} \backslash H_{\mathbf{A}}} \langle \boldsymbol{\varphi}^{\bullet}(h_t^{-1} h^{-1} x t), \mathbf{f}_{1,2}^{\bullet}(h h_t) \rangle \, d^1 h.\end{aligned}$$

5.1.4 S-bases attached to indices.

We have associated a polynomial $m^T(X) = aX^2 + bX + c$ with $T \in \mathscr{T}_{\mathrm{int}}$ in §3.5.3. Let T be positive-definite as above. We define a related polynomial $\tilde{m}^T$ by

$$\tilde{m}^T(X) = X^2 + bX + ac.$$

Since $m^T(X) = \frac{1}{a} \cdot \tilde{m}^T(aX)$, m^T and $\tilde{m}^T$ define the same quadratic form and share the same splitting field K^T.

Let $x \in \mathbf{X}_{\mathbf{Q}}^{T}$ be the fixed representative above. As in §4.4.2, we can complete $\delta_x = x_2 \bar{x}_1 \in D$ to an S-basis $(\delta_x, \jmath)$ for D. Note that δ_x satisfies the polynomial $\tilde{m}^T$, and so we see that $K^{\delta_x} = \mathbf{Q}(\delta_x) \simeq K^T$ as imaginary quadratic fields. More importantly, we now have the following description of St_x:

$$\mathrm{St}_x = \{(\dot{t}, x_1^{-1}\dot{t}x_1) \in H : \dot{t} \in K^{\delta_x}\} \simeq T^{\delta_x},^{*} \tag{1.4.1}$$

where $T^{\delta_x} = \triangle(\mathbf{Q}^\times \backslash K^{\delta_x,\times}) \simeq \mathbf{Q}^\times \backslash \dot{K}^{T,\times}$ is the torus in H defined in §3.1.3.

5.1.5 Equivariant linear maps.

We have now two (right) actions by $t \in K^{T,\times}$ on $y \in \mathbf{X_Q} = \mathrm{Hom}_{\mathbf{Q}}(L, D)$. One is through the identification $L \xrightarrow{\phi^T} K^T$ from §3.5.4 together with the pre-composition $y \cdot t$:

$$L \xrightarrow{t} L \xrightarrow{y} D;$$

the other is through the map $t \mapsto (\dot{t}, 1) \in \tilde{H}$ together with the post-composition $(\dot{t}, 1) \cdot y$:

$$L \xrightarrow{y} D \xrightarrow{(\dot{t},1)} D.$$

Let us compare these two actions in the special case that y is the fixed representative $x \in \mathbf{X}^T_{\mathbf{Q}}$. Set $t = t_1 + t_2 \cdot \delta^T$, with $\delta^T \in K^T$ a fixed root of m^T as in §3.5.4, then $\dot{t} = t_1 + t_2 \cdot \frac{\delta_x}{a} = t_1 + t_2 \cdot \frac{\delta_x}{\mathrm{n}(x_1)}$ as an element in $K^{\delta_x} \subset D$. we have:

$$\begin{aligned} (x \cdot t)(\mathbf{w}_1) &= x((\phi^T)^{-1}(t \cdot 1)) \\ &= x(t_1 \mathbf{w}_1 + t_2 \mathbf{w}_2) \\ &= t_1 x_1 + t_2 \frac{\delta_x}{\mathrm{n}(x_1)} x_1 \\ &= \dot{t} \cdot x_1 \end{aligned} \qquad \begin{aligned} (x \cdot t)(\mathbf{w}_2) &= x((\phi^T)^{-1}(t \cdot \delta^T)) \\ &= x(-\frac{c}{a} t_2 \mathbf{w}_1 + (t_1 - \frac{b}{a} t_2)\mathbf{w}_2) \\ &= -\frac{c}{a} t_2 \cdot \frac{\bar{\delta}_x}{\mathrm{n}(x_2)} x_2 + (t_1 - \frac{b}{a} t_2) x_2 \\ &= \dot{t} \cdot x_2 \end{aligned}$$

where the final equality follows from the fact $\frac{\delta}{a} = -\frac{b}{a} - \frac{c}{a}\frac{\bar{\delta}_x}{c}$. Consequently, we find that

$$(\dot{t}, 1) \cdot x = x \cdot t. \tag{1.5.2}$$

---

*Indeed, given $(\dot{t}, x_1^{-1}\dot{t}x_1) \in \mathrm{St}_x$, we have $\dot{t} \cdot x_1 \cdot x_1^{-1}\dot{t}^{-1}x_1 = x_1$ and

$$\begin{aligned} \dot{t} \cdot x_2 \cdot x_1^{-1}\dot{t}^{-1}x_1 &= \dot{t} \cdot \delta_x \cdot \dot{t}^{-1} x_1 \cdot \frac{1}{\mathrm{n}(x_1)} \\ &= \delta_x \cdot x_1 \cdot \frac{1}{\mathrm{n}(x_1)} \\ &= x_2. \end{aligned}$$

5.1.6 Factoring out toral action.

We return to our computation of $\mathbf{a}^T(t)$ for $t \in K_{\mathbf{A}}^{T,\times}$. The element $(\dot{t},1) \in \tilde{H}_{\mathbf{A}}$ has similitude $\mathrm{n}(t)$, so we can set $h_t = (\dot{t},1)$. By (1.5.2), the integral expression for $\mathbf{a}^T(t)$ becomes:

$$\mathbf{a}^T(t) = \frac{1}{\mathrm{vol}(H_\infty)} \cdot \int_{\mathrm{St}_{x,\mathbf{Q}}\backslash H_{\mathbf{A}}} \langle \boldsymbol{\varphi}^\bullet(h_t^{-1}h^{-1}h_t \cdot x), \mathbf{f}_{1,2}^\bullet(hh_t)\rangle \, d^1h.$$

Now conjugation by h_t defines an automorphism on $H_{\mathbf{A}}$ that preserves the subgroup $\mathrm{St}_{x,\mathbf{A}}$. Furthermore, the modulus of the conjugation automorphism by $h_t = (\dot{t},1)$ is 1 by §3.2.4. It follows that

$$\begin{aligned}
\mathbf{a}^T(t) &= \frac{1}{\mathrm{vol}(H_\infty)} \cdot \int_{\mathrm{St}_{x,\mathbf{Q}}\backslash H_{\mathbf{A}}} \langle \boldsymbol{\varphi}^\bullet((h_t^{-1}hh_t)^{-1} \cdot x), \mathbf{f}_{1,2}^\bullet(h_t(h_t^{-1}hh_t))\rangle \, d^1h \\
&= \frac{1}{\mathrm{vol}(H_\infty)} \cdot \int_{\mathrm{St}_{x,\mathbf{Q}}\backslash H_{\mathbf{A}}} \langle \boldsymbol{\varphi}^\bullet(h^{-1} \cdot x), \mathbf{f}_{1,2}^\bullet(h_th)\rangle \, d^1h.
\end{aligned}$$

We are now in a position to break up the integral by the action of $\hat{w} = (\dot{w}, x_1^{-1}\dot{w}x_1) \in \mathrm{St}_{x,\mathbf{A}}$. More precisely, we have

$$\begin{aligned}
\mathbf{a}^T(t) &= \frac{1}{\mathrm{vol}(H_\infty)} \cdot \int_{\mathrm{St}_{x,\mathbf{Q}}\backslash \mathrm{St}_{x,\mathbf{A}}} \int_{\mathrm{St}_{x,\mathbf{A}}\backslash H_{\mathbf{A}}} \langle \boldsymbol{\varphi}^\bullet((\hat{w}h)^{-1} \cdot x), \mathbf{f}_{1,2}^\bullet(h_t\hat{w}h)\rangle \, d^1h \, d\hat{w} \\
&= \frac{1}{\mathrm{vol}(H_\infty)} \cdot \int_{\mathrm{St}_{x,\mathbf{A}}\backslash H_{\mathbf{A}}} \int_{\mathrm{St}_{x,\mathbf{Q}}\backslash \mathrm{St}_{x,\mathbf{A}}} \langle \boldsymbol{\varphi}^\bullet(h^{-1} \cdot x), \mathbf{f}_{1,2}^\bullet(h_t\hat{w}h)\rangle \, d\hat{w} \, d^1h.
\end{aligned}$$

5.1.7 Archimedean pairing.

We can simplify $\mathbf{a}^T(t)$ one step further before having to pin down $\boldsymbol{\varphi}^\bullet$ and $\mathbf{f}_{1,2}^\bullet$ exactly by a choice of S-basis. Observe that since $\langle \bullet, \bullet \rangle$ is a H_∞-invariant pairing as in §3.3.5, we have

$$\begin{aligned}
\langle \boldsymbol{\varphi}^\bullet(h^{-1} \cdot x), \mathbf{f}_{1,2}^\bullet(h_t\hat{w}h)\rangle &= \langle \sigma_{1,2}^\bullet(h_\infty^{-1}) \cdot \boldsymbol{\varphi}^\bullet(h_f^{-1} \cdot x), \check{\sigma}_{1,2}^\bullet(h_\infty)^{-1} \cdot \mathbf{f}_{1,2}^\bullet(h_t\hat{w}h)\rangle \\
&= \langle \boldsymbol{\varphi}^\bullet(h_f^{-1} \cdot x), \mathbf{f}_{1,2}^\bullet(h_t\hat{w}h_f)\rangle.
\end{aligned}$$

This gives

$$\mathbf{a}^T(t) = \frac{1}{\mathrm{vol}(\mathrm{St}_{x,\infty})} \cdot \int_{\mathrm{St}_{x,f}\backslash H_f} \int_{\mathrm{St}_{x,\mathbf{Q}}\backslash \mathrm{St}_{x,\mathbf{A}}} \langle \boldsymbol{\varphi}^\bullet(h_f^{-1} \cdot x), \mathbf{f}_{1,2}^\bullet(h_t\hat{w}h_f)\rangle \, d\hat{w} \, d^1h_f. \quad (1.7.3)$$

5.2 Specializing to good choices

We fix a good choice of S-basis $(\delta_x, \jmath)$ for D with respect to the fixed choice of $x \in \mathscr{X} \cap \mathbf{X}^T_{\mathbf{Q}}$. We then evaluate $\mathbf{a}^T(t)$ by specializing (1.7.3) to a specific Bruhat–Schwartz function $\boldsymbol{\varphi}^{\delta_x,\jmath}$ and a particular member $\mathbf{f}^{\delta_x,\jmath}_{1,2}$ of the compatible set $\{\mathbf{f}^{\bullet}_{1,2}\}$.

5.2.1 The choice of a set S.

The choice of S now has bearing on our discussion. Recall that this choice only affects the local conditions on the particular Eichler order $\mathscr{D}(dNp^r)$ that we choose. Let x be the fixed representative in $\mathscr{X} \cap \mathbf{X}^T_{\mathbf{Q}}$ as above, we have associated with it an element δ_x in $K^{\delta_x} \subset D$. Denote by $\mathscr{O}^x = \mathscr{O}^{\delta_x}$ the order $\mathbf{Z}[\delta_x]$ in K^{δ_x}. The conductor of $\mathscr{O}$ is exactly $\mathrm{n}(\delta_x^{(0)}) = \det(T)$. In view of this, we set

$$S = \{2, p\} \cup \{\text{primes } l \text{ that divide } 4\det(T)\} - \{l \mid d\}.$$

5.2.2 The choice of an S-basis.

Let $(\delta_x, \jmath)$ be an S-basis associated with x as in §4.4.2, and set $\boldsymbol{\varphi}^x = \boldsymbol{\varphi}^{\delta_x,\jmath,\mathscr{D}}_{2k}$ and $\mathbf{f}^x_i = \mathbf{f}^{\delta_x,\jmath,\mathscr{D}}_i$, $i = 1, 2$. Here $\mathscr{D} = \mathscr{D}^{\delta_x,\jmath,S}(r)$ is an Eichler order of level dNp^r associated with $(\delta_x, \jmath)$ in §1.3.4. Let us evaluate (1.7.3) using $\boldsymbol{\varphi}^x$ and $\mathbf{f}^x_i$.

Under these choices,

$$\boldsymbol{\varphi}^x(h_f^{-1} \cdot x) = \boldsymbol{\varphi}^x_f(h_f^{-1} \cdot x) \cdot \boldsymbol{\varphi}^x_\infty(x)$$

and

$$\begin{aligned} \boldsymbol{\varphi}^x_\infty(x) &= e^{-2\pi\cdot(\mathrm{n}(x_1)+\mathrm{n}(x_2))} \cdot (-\mathrm{n}(\delta_x^{(0)}))^{\frac{k}{2}} \cdot \mathbf{t}_0^k \\ &= e^{-2\pi\cdot\mathrm{tr}(T)} \cdot (-\det(T))^{\frac{k}{2}} \cdot \mathbf{t}_0^k \end{aligned}$$

by Prop. 4.4.2. The integral (1.7.3) now becomes

$$\begin{aligned} \mathbf{a}^T(t) = \frac{1}{\mathrm{vol}(\mathrm{St}_{x,\infty})} &\cdot e^{-2\pi\cdot\mathrm{tr}(T)} \cdot (-\det(T))^{\frac{k}{2}} \\ &\cdot \int_{\mathrm{St}_{x,f}\backslash H_f} \boldsymbol{\varphi}^x_f(h_f^{-1} \cdot x) \int_{\mathrm{St}_{x,\mathbf{Q}}\backslash \mathrm{St}_{x,\mathbf{A}}} \langle \mathbf{t}_0^k, \mathbf{f}^x_{1,2}(h_t \hat{w} h_f)\rangle \, d\hat{w} \, d^1 h_f. \end{aligned} \tag{2.2.4}$$

5.2.3 Integrality-preserving isometries.

We turn our attention to $\boldsymbol{\varphi}^x_f$.

By definition, we have that

$$\boldsymbol{\varphi}_f^x(h_f^{-1}\cdot x) = \begin{cases} \frac{1}{\mathrm{vol}(U_f^{\mathscr{D}})} & \text{if } h_f^{-1}\cdot x \in \mathscr{X}_f, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

It follows that the integrand is supported on the subset $\mathrm{St}_{x,f}\backslash E_{x,f}$ of $\mathrm{St}_{x,f}\backslash H_f$, where

$$\begin{aligned} E_{x,f} &= \{h_f \in H_f : h_f^{-1}\cdot x \in \mathscr{X}_f\} \\ &= \{h_f = (\alpha,\beta) : \alpha^{-1}x_i\beta \in \mathscr{D}_f, i = 1,2\}. \end{aligned}$$

The normalizer $U_f^{\mathscr{D}} \subset H_f$ of $\mathscr{D}_f$ acts on $E_{x,f}$. To utilize this observation, we need to assume that $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ have reciprocal eigenvalues under the Atkin–Lehner operators w_l from §2.3.2 for primes $l \mid dNp^r$; in other words, that $a_l(\boldsymbol{\pi}_1)a_l(\boldsymbol{\pi}_2) = 1$. Under this assumption, $\mathbf{f}_{1,2}^x$ is right-invariant under $U_f^{\mathscr{D}}$, and the integrand in (2.2.4) is constant on the orbits

$$\mathrm{St}_{x,f}\backslash \mathrm{St}_{x,f}\cdot h_f \cdot U_f^{\mathscr{D}} \simeq (h_f^{-1}\mathrm{St}_{x,f}h_f \cap U_f^{\mathscr{D}})\backslash U_f^{\mathscr{D}}$$

of this action. We arrive at the following expression

$$\mathbf{a}^T(t) = \frac{e^{-2\pi\cdot\mathrm{tr}(T)}\cdot(-\det(T))^{\frac{k}{2}}}{\mathrm{vol}(\mathrm{St}_{x,\infty})}\cdot\sum_{h_f}\frac{1}{v_h}\int_{\mathrm{St}_{x,\mathbf{Q}}\backslash\mathrm{St}_{x,\mathbf{A}}}\langle\mathbf{t}_0^k, \mathbf{f}_{1,2}^x(h_t\hat{w}h_f)\rangle\, d\hat{w} \qquad (2.3.5)$$

where $v_h = \mathrm{vol}(h_f^{-1}\mathrm{St}_{x,f}h_f \cap U_f^{\mathscr{D}})$ and h_f runs through a set of double-coset representatives for

$$\bar{E}_{x,f} = \mathrm{St}_{x,f}\backslash E_{x,f}/U_f^{\mathscr{D}}$$

which is in fact finite [Yos84, Prop. 1.5].

5.2.4 Averaging over an ideal class group.

For a fixed h_f, the integrand

$$\langle\mathbf{t}_0^k, \mathbf{f}_{1,2}^x(h_t\hat{w}h_f)\rangle$$

as a function of $\hat{w} \in \mathrm{St}_{x,\mathbf{A}}$ is right-invariant under

$$U_h^{\mathrm{St}} = (h_f^{-1}\mathrm{St}_{x,f}h_f \cap U_f^{\mathscr{D}}) \times \mathrm{St}_{x,\infty}.^\dagger$$

†The right-invariance under $\mathrm{St}_{x,\infty}$ follows from the observation that $\mathbf{t}_0^k$ is of weight 0 in $\mathscr{V}_{2k}$, hence is fixed by the action of $\hat{w}_\infty$.

Under the isomorphism $\mathrm{St}_x \simeq \mathbf{Q}^\times \backslash K^{\delta_x,\times}$, we see that

$$\mathrm{St}_{x,\mathbf{Q}} \backslash \mathrm{St}_{x,\mathbf{A}} / U_h^{\mathrm{St}} \simeq K_{\mathbf{Q}}^{\delta_x,\times} \backslash K_{\mathbf{A}}^{\delta_x,\times} / (\mathscr{O}_{h,f}^{x,\times} \cdot K_\infty^{\delta_x,\times}) = \mathrm{Cl}(\mathscr{O}_h^x)$$

for some order $\mathscr{O}_h^{x,\times}$ in K. The integral (2.3.5) simplifies as a double sum:

$$\begin{aligned} \mathbf{a}^T(t) &= \frac{e^{-2\pi\cdot\mathrm{tr}(T)} \cdot (-\det(T))^{\frac{k}{2}}}{\mathrm{vol}(\mathrm{St}_{x,\infty})} \cdot \sum_{h_f} \frac{1}{v_h} \cdot \frac{v_h \cdot \mathrm{vol}(\mathrm{St}_{x,\infty})}{e_h} \cdot \sum_{w \in \mathrm{Cl}(\mathscr{O}_h^x)} \langle \mathbf{t}_0^k, \mathbf{f}_{1,2}^x(h_t \hat{w} h_f) \rangle \\ &= e^{-2\pi\cdot\mathrm{tr}(T)} \cdot (-\det(T))^{\frac{k}{2}} \cdot \sum_{h_f} \frac{1}{e_h} \sum_{w \in \mathrm{Cl}(\mathscr{O}_h^x)} \langle \mathbf{t}_0^k, \mathbf{f}_{1,2}^x(h_t \hat{w} h_f) \rangle \end{aligned} \tag{2.4.6}$$

where $\hat{w} = (\dot{w}, x_1^{-1}\dot{w}x_1) \in \mathrm{St}_{x,f}$ as before, and

$$e_h = \frac{\#(K_{\mathbf{Q}}^{\delta_x,\times} \cap \mathscr{O}_{h,f}^{x,\times} \cdot K_\infty^{\delta_x,\times})}{\#\mathbf{Z}^\times}$$

is half the number of units in $\mathscr{O}_h^x$. Since $K^{\delta,x}$ is purely imaginary, we see that

$$e_h = \begin{cases} 2 & \text{if } \mathscr{O}_h^x \simeq \mathbf{Z}[\sqrt{-1}]; \\ 3 & \text{if } \mathscr{O}_h^x \simeq \mathbf{Z}[\frac{1+\sqrt{-3}}{2}]; \\ 1 & \text{otherwise.} \end{cases}$$

5.2.5 A formula for $a_t(T)$.

By definition of $\mathbf{f}_{1,2}^x$, we have

$$\begin{aligned} \mathbf{f}_{1,2}^x(h_t \hat{w} h_f) &= \mathbf{f}_{1,2}^x((\dot{t}, 1)(\dot{w}, x_1^{-1}\dot{w}x_1)(\alpha, \beta)) \\ &= \mathbf{f}_1^x(\dot{t}\dot{w}\alpha) \cdot \mathbf{f}_2^x(x_1^{-1}\dot{w}x_1\beta). \end{aligned}$$

Plug this into the inner sum in (2.4.6), we get

$$\mathbf{a}^T(t) = e^{-2\pi\cdot\mathrm{tr}(T)} \cdot (-\det(T))^{\frac{k}{2}} \cdot \sum_{h_f \in \bar{E}_{x,f}} \frac{1}{e_h} \sum_{w \in \mathrm{Cl}(\mathscr{O}_h^x)} \langle \mathbf{t}_0^k, \mathbf{f}_1^x(\dot{t}\dot{w}\alpha_h) \rangle \cdot \mathbf{f}_2^x(x_1^{-1}\dot{w}x_1\beta_h). \tag{2.5.7}$$

Let us relate $\mathbf{a}^T(t)$ to the Fourier coefficient $a_{t_f}^T$ of the (shifted) Siegel modular form $\mathscr{Y}_{t_f}$ using the relation (5.2.9), which in this case simplifies to

$$\begin{aligned}\mathbf{a}^T(t_f t_\infty) &= \det(J(t_\infty, \mathbf{i}))^{-k}\lambda'(t_\infty)^k \cdot a_{t_f}(T) \cdot \boldsymbol{\psi}_\infty(T \cdot (t_\infty\langle \mathbf{i}\rangle)) \\ &= \mathrm{n}(t_\infty)^{-k} \cdot \mathrm{n}(t_\infty)^k \cdot a_{t_f}(T) \cdot e^{2\pi\iota\cdot\mathrm{tr}(T\cdot\mathbf{i})} \\ &= a_{t_f}(T) \cdot e^{-2\pi\cdot\mathrm{tr}(T)}.\end{aligned}$$

Thus we arrive at the following formula for $a_{t_f}(T)$:

$$a_{t_f}(T) = (-\det(T))^{\frac{k}{2}} \cdot \sum_{h_f \in \bar{E}_{x,f}} \frac{1}{e_h} \sum_{w \in \mathrm{Cl}(\mathscr{O}_h^x)} \langle \mathbf{t}_0^k, \mathbf{f}_1^x(\dot{t}\dot{w}\alpha_h)\rangle \cdot \mathbf{f}_2^x(x_1^{-1}\dot{w}x_1\beta_h). \qquad (2.5.8)$$

From this formula, we see that rationality of $a_{t_f}(T)$ follows immediately from the algebraicity of $\mathbf{f}_i^x$'s.

Proposition 5.2.1. *Suppose that the $\mathbf{f}_i^x$'s are algebraic in the sense of Definition 2.3.3, with field of definition $K(\mathbf{f}_i^x)$, then the Tth Fourier coefficient $a_{t_f}(T)$ of $\mathscr{Y}_{t_f}$ is defined over the compositum $K(\mathbf{f}_1^x) \cdot K(\mathbf{f}_2^x)$.*

Proof. Indeed, first note that $K(\mathbf{f}_i^x)$ contains $\sqrt{-\det(T)}$ for both $i = 1, 2$. In fact, by §5.1.4 together with the definition of $K(\mathbf{f}_i^x)$ from §2.3.4, we see that $K(\mathbf{f}_i^x)$ contains $K_\jmath = K^T(\sqrt{\mathrm{n}(\jmath)})$, where $K^T = K(\sqrt{-\det(T)})$ is the splitting field of m^T from §5.1.4. It remains to check that

$$\langle \mathbf{t}_0^k, \mathbf{f}_1^x(\dot{t}\dot{w}\alpha_h)\rangle \cdot \mathbf{f}_2^x(x_1^{-1}\dot{w}x_1\beta_h)$$

lies in $K(\mathbf{f}_1^x) \cdot K(\mathbf{f}_2^x)$, and this is the case by the definition of algebraic automorphic forms. $\square$

Corollary 5.2.2. *Suppose that the compatible sets of automorphic forms $\{\mathbf{f}_i^x\}$ are algebraic, then the Fourier coefficients $a_{t_f}(T)$ of $\mathscr{Y}_{t_f}$ are all contained in the compositum $K(\{\mathbf{f}_i\}) = K(\{\mathbf{f}_1\}) \cdot K(\{\mathbf{f}_2\})$.*

5.3 Integrality of Fourier coefficients

Assume for the rest of this section that the compatible sets of automorphic forms $\{\mathbf{f}_i^\bullet\}$ are $\mathfrak{p}$-integral. We establish the $\mathfrak{p}$-integrality of the Fourier coefficients $a_{t_f}(T)$.

5.3.1 A criterion for $\mathfrak{p}$-integrality.

Among the terms appearing in the formula (2.5.8) for $a_{t_f}(T)$, $\mathbf{f}_2^x(x_1^{-1}\dot{w}x_1\beta_h)$ is $\mathfrak{p}$-integral by definition, and $\frac{1}{e_h}$ is either $1, \frac{1}{2}$, or $\frac{1}{3}$, hence is a unit in $\mathbf{Z}_p$ for $p \geq 5$;

therefore, if

$$(-\det(T))^{\frac{k}{2}} \cdot \langle \mathbf{t}_0^k, \mathbf{f}_1^x(\dot{t}\dot{w}\alpha_h)\rangle_k \tag{3.1.9}$$

if $\mathfrak{p}$-integral, then so is $a_{t_f}(T)$.

By the definition of a $\mathfrak{p}$-integral compatible set of automorphic forms on D in §2.3.6, we have

$$\mathbf{f}_1^x(\dot{t}\dot{w}\alpha_h) \in \check{\sigma}^x_{2k,\mathrm{wt},p}(\dot{t}_p\dot{w}_p\alpha_p) \cdot \check{\sigma}_p(C_p^x) \cdot \mathscr{M}$$

where $\sigma_{2k,p} = \sigma_p$, $\check{\sigma}^x_{\mathrm{wt},p} = \check{\sigma}^{\delta_x,\jmath}_{2k,\mathrm{wt},p}$ and $C_p^x = C_p^{\delta_x,\jmath}$. Plug the above formula into the pairing in (3.1.9), we find

$$\begin{aligned}\langle \mathbf{t}_0^k, \mathbf{f}_1^x(\dot{t}\dot{w}\alpha_h)\rangle_k &= \langle \mathbf{t}_0^k, \check{\sigma}^x_{\mathrm{wt},p}(\dot{t}_p\dot{w}_p\alpha_p) \cdot \check{\sigma}_p(C_p^x)\cdot \mathbf{v}\rangle \\ &= \langle \sigma_p(C_p^x)^{-1} \cdot \sigma^x_{\mathrm{wt},p}(\dot{t}_p\dot{w}_p\alpha_p)^{-1} \cdot \mathbf{t}_0^k, \mathbf{v}\rangle \\ &= \langle \sigma_p(C_p^x)^{-1} \cdot \sigma^x_{\mathrm{wt},p}(\alpha_p)^{-1} \cdot \mathbf{t}_0^k, \mathbf{v}\rangle\end{aligned}$$

since $\mathbf{t}_0^k$ is fixed by $(\dot{t}_p\dot{w}_p)^{-1} \in K_p^{\delta_x,\times}$. Altogether, we see that if

$$(-\det(T))^{\frac{k}{2}} \cdot \sigma_p(C_p^x)^{-1} \cdot \sigma^x_{\mathrm{wt},p}(\alpha_p)^{-1} \cdot \mathbf{t}_0^k \tag{3.1.10}$$

lies in $\mathscr{O}_{\mathbf{C}_p}[X]_{\deg\le 2k}$, then $a_{t_f}(T)$ lies in $\frac{1}{k!}\mathscr{O}_{\mathbf{C}_p}$. To continue, we need to understand the matrix

$$(-\det(T))^{\frac{k}{2}} \cdot (C_p^x)^{-1} \cdot \epsilon^x_{\mathrm{wt},p}(\alpha_p)^{-1} = (-\det(T))^{\frac{k}{2}} \cdot \epsilon^x_{\mathrm{ar},p}(\alpha_p)^{-1} \cdot (C_p^x)^{-1}$$

in $\mathrm{GL}_2(\mathbf{C}_p)$ (where for simplicity we set $\epsilon^x_{\mathrm{wt},p} = \epsilon^{\delta_x,\jmath}_{\mathrm{wt},p}$ and $\epsilon^x_{\mathrm{ar},p} = \epsilon^{\delta_x,\jmath}_{\mathrm{ar},p}$). Let us begin by studying the double-coset space $\bar{E}_{x,f}$.

5.3.2 An auxiliary double-coset space.

Instead of studying $\bar{E}_{x,f}$ directly, let us consider the related classes of double-cosets $\bar{E}_{\tilde{x},f}$ where $\tilde{x} \in \mathbf{X}^T_{\mathbf{Q}}$ is defined by

$$\tilde{x}(\mathbf{w}_1) = 1 \qquad \text{and} \qquad \tilde{x}(\mathbf{w}_2) = x_2\bar{x}_1.$$

Unlike $\bar{E}_{x,f}$, $\bar{E}_{\tilde{x},f}$ has the property that every element $h \in \bar{E}_{\tilde{x},f}$ can be represented by (α,α) for some $\alpha \in K_f^{\delta_x,\times}\backslash D_f^\times/\mathscr{D}_f^\times$. Indeed, $E_{\tilde{x},f}$ is by definition the set of elements $h = (\alpha,\beta) \in H_f$ such that

$$\alpha^{-1}\tilde{x}_i\beta \in \mathscr{D}_f$$

for $i = 1, 2$. In particular, we see that $\alpha^{-1}\beta \in \mathscr{D}_f$ is an element of norm 1. It follows that

$$(\alpha, \beta) = (\alpha, \alpha \cdot \alpha^{-1}\beta) = (\alpha, \alpha) \cdot (1, \alpha^{-1}\beta);$$

since $(1, \alpha^{-1}\beta) \in U_f^{\mathscr{D}}$, we conclude that $(\alpha, \beta) = (\alpha, \alpha)$ as elements in $\bar{E}_{\tilde{x},f}$.

5.3.3 Representatives for $\bar{E}_{\tilde{x},f}$.

Using the factorization $\bar{E}_{\tilde{x},f} \simeq \prod_l \bar{E}_{\tilde{x},l}$ into local components, let us consider each $\bar{E}_{\tilde{x},l}$ individually.

As a first reduction, we have the following general proposition which applies to $\bar{E}_{x,l}$ and not just $\bar{E}_{\tilde{x},l}$.

Proposition 5.3.1 (Yoshida). *For a prime $l \neq 2$, if l does not divide $4\det(T_x) = 4ac - b^2$, then $\bar{E}_{x,f} = \{1\}$.*

Proof. We elaborate on the proof given for [Yos84, Prop. 1.3]. Denote by $\mathscr{L}_l^x \subset \mathscr{D}_l$ the image of $\mathscr{L}_l$ under x. Since $l \nmid \det(T_x)$, $\mathscr{L}_l^x$ is a unimodular sub-lattice in $\mathscr{D}_l$, and we have an orthogonal decomposition

$$\mathscr{D}_l = \mathscr{L}_l^x \perp \mathscr{L}_l^{x,\perp}.$$

Given $h \in E_{x,l}$, we have another orthogonal decomposition

$$h^{-1}\mathscr{D}_l = \mathscr{L}_l^x \perp \mathscr{L}_l^{x,\perp,h}$$

where $\mathscr{L}_l^{x,\perp,h} = \{x \in h^{-1}\mathscr{D}_l : (x, y) = 0 \text{ for all } y \in \mathscr{L}_l^x\}$ is the orthogonal complement of $\mathscr{L}_l^x$ in $h^{-1}\mathscr{D}_l$.

By [O'M00, Theorem (92:3)], we have an isometry taking $\mathscr{L}_l^{x,\perp,h}$ to $\mathscr{L}_l^{x,\perp}$. Together with the identity map on $\mathscr{L}_l^x$, we obtain an element $w \in \mathrm{St}_{x,l}$ such that $h^{-1} \cdot w$ is an isometry preserving $\mathscr{D}_l$, hence defines an element in $U_f^{\mathscr{D}}$. □

Next we consider the case that D is ramified at l. In this case $U_l^{\mathscr{D}} = H_l$ since $\mathscr{D}_l$ is the unique maximal order in D_l; therefore $\bar{E}_{x,l} = \{1\}$ as well.

It remains to consider the prime 2 and the primes l such that $l \mid 4\det(T)$ and $l \nmid d$, in other words, the primes in S possibly excluding the special prime p. Let us identify $D_l^\times$ with $\mathrm{GL}_2(\mathbf{Q}_l)$ using the arithmetic embedding $\epsilon_{\mathrm{ar}}^{\delta_x,\jmath}$ defined in §1.3.2 and §1.3.3 depending on whether l is split in K^T or not. By §5.3.2, it suffices to consider the elements (α, α) with α running over a set of representatives for $K_l^{\delta_x,\times} \backslash D_l^\times / \mathscr{D}_l^\times$

and satisfying the condition $\alpha^{-1}x_2\bar{x}_1\alpha \in \mathscr{D}_l$. For convenience, set $r_l = \mathrm{ord}_l(Np^r)$, $\nu = \mathrm{ord}_l(4\det(T_x)) = \mathrm{ord}_l(ac - b^2)$, and ς the conductor of $\mathbf{Z}[\delta_x]$ as in §1.3.6.

5.3.4 The case that l splits in K^T.

By §1.3.6, the union of

$$\left\{A(n,s) = \begin{bmatrix} 1 & l^n \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ sl & 1 \end{bmatrix} : n \le 0, 0 \le s < l^{r_l - 1}\right\}$$

and

$$\left\{B(n,t) = \begin{bmatrix} 1 & l^n \\ & 1 \end{bmatrix}\begin{bmatrix} t & 1 \\ -1 & \end{bmatrix} : n \le 0, 0 \le t < l^{r_l}\right\}$$

is a set of double-coset representatives for $K_l^{T,\times}\backslash \mathrm{GL}_2(\mathbf{Q}_l)/I_l(r)^\times \simeq K_l^{\delta_x,\times}\backslash D_l^\times/\mathscr{D}_l^\times$. Now $\delta_x = x_2\bar{x}_1$ is identified with the matrix

$$\begin{bmatrix} \delta_x & \\ & \bar{\delta}_x \end{bmatrix} = \begin{bmatrix} \frac{-b+\sqrt{\Delta_x}}{2} & \\ & \frac{-b-\sqrt{\Delta_x}}{2} \end{bmatrix}$$

under the arithmetic embedding $\epsilon_{\mathrm{ar}}^{\delta_x, \jmath}$. Here $\Delta_x = b^2 - 4ac$.

After some matrix multiplications, we get:

$$A(n,s)^{-1}\begin{bmatrix} \delta_x & \\ & \bar{\delta}_x \end{bmatrix}A(n,s) = \begin{bmatrix} \delta_x + sl^{n+1}\sqrt{\Delta_x} & l^n\sqrt{\Delta_x} \\ -sl\sqrt{\Delta_x}(1 + sl^{n+1}) & \bar{\delta}_x - sl^{n+1}\sqrt{\Delta_x} \end{bmatrix}.$$

For the product to be in $I_l(r_l)$, we must have

- $n + \frac{\nu}{2} \ge 0$ and
- $\mathrm{ord}_l(sl\sqrt{\Delta_x}(1 + sl^{n+1})) \ge r_l$.

Similarly

$$B(n,t)^{-1}\begin{bmatrix} \delta_x & \\ & \bar{\delta}_x \end{bmatrix}B(n,t) = \begin{bmatrix} \bar{\delta}_x & \\ (t - l^n)\sqrt{\Delta_x} & \delta_x \end{bmatrix}.$$

This translates to the conditions

- $n + \frac{\nu}{2} \ge 0$ and
- $\mathrm{ord}_l(t - l^n) + \frac{\nu}{2} \ge r_l$.

Set $R_{\tilde{x},l}$ to be the inverse image under $\epsilon_{\mathrm{ar}}^{\delta_x,\jmath}$ of the set

$$\{(A(n,s),A(n,s)) : 0 \geq n \geq -\frac{\nu}{2}\} \cup \{(B(n,t),B(n,t)) : 0 \geq n \geq -\frac{\nu}{2}\};$$

then every element in $\bar{E}_{\tilde{x},f}$ can be represented by some $(\alpha,\alpha) \in R_{\tilde{x},l}$.

5.3.5 The case that l does not split in K^T.

Again by §1.3.6, the union of

$$\left\{ C(n,s) = \begin{bmatrix} l^n & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ sl & 1 \end{bmatrix} : n \geq -\nu, 0 \leq s < l^{r_l - 1} \right\}$$

and

$$\left\{ D(n,t) = \begin{bmatrix} l^n & \\ & 1 \end{bmatrix} \begin{bmatrix} t & 1 \\ -1 & \end{bmatrix} : n \geq -\nu, 0 \leq t < l^{r_l} \right\}$$

is a set of double-coset representatives for $K_l^{T,\times} \backslash \mathrm{GL}_2(\mathbf{Q}_l)/I_l(r)^\times \simeq K_l^{\delta_x,\times} \backslash D_l^\times / \mathscr{D}_l^\times$. Now $\delta_x = x_2 \bar{x}_1$ is identified with the matrix

$$\begin{bmatrix} \frac{-b}{2} & 1 \\ \frac{\Delta_x}{4} & \frac{-b}{2} \end{bmatrix} = \begin{bmatrix} \frac{-b}{2} & \\ & \frac{-b}{2} \end{bmatrix} + \begin{bmatrix} & 1 \\ \frac{\Delta_x}{4} & \end{bmatrix}$$

under the arithmetic embedding $\epsilon_{\mathrm{ar}}^{\delta_x,\jmath}$. In particular, we must have that $\mathrm{ord}_l(\Delta_x) \geq r_l$.

After some matrix multiplications, we get:

$$C(n,s)^{-1} \begin{bmatrix} & 1 \\ \frac{\Delta_x}{4} & \end{bmatrix} C(n,s) = \begin{bmatrix} sl^{1-n} & l^{-n} \\ l^n \frac{\Delta}{4} - s^2 l^{2-n} & -sl^{1-n} \end{bmatrix}.$$

For the product to be in $I_l(r_l)$, we must have

- $n \leq 0$ and
- $\mathrm{ord}_l(l^n \frac{\Delta}{4} - s^2 l^{2-n}) \geq r_l$.

Similarly

$$D(n,t)^{-1} \begin{bmatrix} & 1 \\ \frac{\Delta_x}{4} & \end{bmatrix} D(n,t) = \begin{bmatrix} -tl^n \frac{\Delta_x}{4} & -l^n \frac{\Delta_x}{4} \\ t^2 l^n \frac{\Delta_x}{4} - l^{-n} & tl^n \frac{\Delta_x}{4} \end{bmatrix}.$$

This translates to the conditions

- $n \leq 0$ and

- $\mathrm{ord}_l(t^2 l^n \frac{\Delta_x}{4} - l^{-n})$.

Set $R_{\tilde{x},l}$ to be the inverse image under $\epsilon_{\mathrm{ar}}^{\delta_x,\jmath}$ of the set

$$\{(C(n,s),C(n,s)) : 0 \geq n \geq -\nu\} \cup \{(D(n,t),D(n,t)) : 0 \geq n \geq -\nu\};$$

then every element in $\bar{E}_{\tilde{x},f}$ can be represented by some $(\alpha,\alpha) \in R_{\tilde{x},l}$.

5.3.6 Recovering $\bar{E}_{x,f}$.

We can recover the original double-coset space $\bar{E}_{x,f}$ as in proof of [Yos84, Theorem 4.3]. Denote by $\mathscr{D}_f^{\mathrm{n}=\mathrm{n}(x_1)}$ the set of elements in $\mathscr{D}_f$ of norm equal $\mathrm{n}(x_1)$. For each $y \in \mathscr{D}_f^{\mathrm{n}=\mathrm{n}(x_1)}$, set

$$E^y_{\tilde{x},f} = \{(\alpha,\beta) \in E_{\tilde{x},f} : \alpha^{-1}x_2\bar{x}_1\beta \in \mathscr{D}_f y\} \quad \text{and} \quad E^{\bar{y}}_{x,f} = \{(\alpha,\beta) \in E_{x,f} : \alpha^{-1}x_1\beta \in \mathscr{D}_f \bar{y}\}.$$

(Here $\bar{y}$ is the image of y under the main involution on D.) Since an element (α,β) of $E_{\tilde{x},f}$ (resp. of $E_{x,f}$) belongs to $E^{\beta^{-1}x_1\beta}_{\tilde{x},f}$ (resp. to $E^{\beta^{-1}x_1\beta}_{x,f}$), we have that

$$E_{\tilde{x},f} = \bigcup_{y \in \mathscr{D}_f^{\mathrm{n}=\mathrm{n}(x_1)}} E^y_{\tilde{x},f} \quad \text{and} \quad E_{x,f} = \bigcup_{y \in \mathscr{D}_f^{\mathrm{n}=\mathrm{n}(x_1)}} E^{\bar{y}}_{x,f}.$$

The map $c_y : E^y_{\tilde{x},f} \to E^{\bar{y}}_{x,f}$ given by $(\alpha,\beta) \mapsto (\alpha, x_1^{-1}\beta\bar{y})$ is a bijection with inverse $c^y : E^y_{x,f} \to E^y_{\tilde{x},f}$ given by $(\alpha,\beta) \mapsto (\alpha, \bar{x}_1^{-1}\beta y)$. Although the map c_y is not compatible with the right action by $U_f^{\mathscr{D}}$ in general, we still have the following lemma.

Lemma 5.3.2. *Every element of $\bar{E}_{x,f}$ can be represented by*

$$c_y(\alpha u, \alpha u) = (\alpha u, x_1^{-1}\alpha u \bar{y})$$

for some $(\alpha,\alpha) \in R_{\tilde{x},f}$, $y \in \mathscr{D}_f^{\mathrm{n}=\mathrm{n}(x_1)}$, and $u \in \mathscr{D}_f^\times$.

Proof. Given $h \in E_{x,f}$, choose an $y \in \mathscr{D}_f^{\mathrm{n}=\mathrm{n}(x_1)}$ so that $h \in E^{\bar{y}}_{x,f}$. Since c_y is a bijection, $h = c_y(\beta,\gamma)$ for some $(\beta,\gamma) \in E^y_{\tilde{x},f}$. Now we have

$$(\beta,\gamma) = (\dot{w},\dot{w})(\alpha,\alpha)(u_1,u_2)$$

for some $(\dot{w},\dot{w}) \in \mathrm{St}_{\tilde{x},f}$ and $(u_1,u_2) \in U_f^{\mathscr{D}}$. Apply the map c_y to the right-hand side, we get

$$c_y((\dot{w},\dot{w})(\alpha,\alpha)(u_1,u_2)) = (\dot{w}\alpha u_1, x_1^{-1}\dot{w}\alpha u_2\bar{y})$$
$$= (\dot{w}, x_1^{-1}\dot{w}x_1)(\alpha u_2, x_1^{-1}\alpha u_2\bar{y})(u_2^{-1}u_1, 1).$$

The lemma follows by observing that $(\dot{w}, x_1^{-1}\dot{w}x_1) \in \mathrm{St}_{x,f}$ and $(u_2u_1^{-1}, 1) \in U_f^{\mathscr{D}}$. $\square$

5.3.7 Integrality when p splits.

We check that the vector (3.1.10) from §5.3.1 indeed lies in $\mathscr{O}_{\mathbf{C}_p}[X]_{\deg\leq 2k}$. Since p splits in K^T,

$$C_p^x = \begin{bmatrix} \frac{1}{\sqrt{\mathrm{n}(\jmath)}} & \\ & 1 \end{bmatrix}$$

fixes $\mathbf{t}_0^k$; thus we have

$$(-\det(T))^{\frac{k}{2}} \cdot \sigma_p(C_p^x)^{-1} \cdot \sigma_{\mathrm{wt},p}^x(\alpha_p)^{-1} \cdot \mathbf{t}_0^k = (-\det(T))^{\frac{k}{2}} \cdot \sigma_{\mathrm{ar},p}^x(\alpha_p)^{-1} \cdot \sigma_p(C_p^x)^{-1} \cdot \mathbf{t}_0^k$$
$$= (-\det(T))^{\frac{k}{2}} \cdot \sigma_{\mathrm{ar},p}^x(\alpha_p)^{-1} \cdot \mathbf{t}_0^k,$$

where we set $\sigma_{\mathrm{ar},p}^x = \sigma_{2k,\mathrm{ar},p}^{\delta_x,\jmath}$ to reduce the amount of notations. By Lemma 5.3.2, we can represent α_p by either

$$(\epsilon_{\mathrm{ar},p}^x)^{-1}(A(n,s)) \cdot u \quad \text{or} \quad (\epsilon_{\mathrm{ar},p}^x)^{-1}(B(n,t)) \cdot u$$

for some $u \in \mathscr{D}_f^\times$ and $0 \geq n \geq -\frac{\nu}{2}$. In either case we see that $\sigma_{\mathrm{ar},p}^x(\alpha_p)^{-1} = \sigma_{\mathrm{ar},p}^x(M)^{-1} \cdot \sigma_{\mathrm{ar},p}^x(u_l(n))^{-1}$ where $u_l(n) = \begin{bmatrix} 1 & l^n \\ & 1 \end{bmatrix}$ and $M \in \mathrm{GL}_2(\mathbf{Z}_p)$. We have

$$(-\det(T))^{\frac{k}{2}} \cdot \sigma_{\mathrm{ar},p}^x(\alpha_p)^{-1}\mathbf{t}_0^k = (-\det(T))^{\frac{k}{2}} \cdot \sigma_{\mathrm{ar},p}^x(M)^{-1} \cdot \sigma_{\mathrm{ar},p}^x(u_l(n))^{-1}\mathbf{t}_0^k$$
$$= \sigma_{\mathrm{ar},p}(M)^{-1} \cdot (-\det(T))^{\frac{k}{2}} \cdot (-l^nX + 1)^k$$
$$= \sigma_{\mathrm{ar},p}(M)^{-1} \cdot \mu^{\frac{k}{2}} \cdot (-l^{\nu/2} \cdot l^nX + l^{\nu/2})^k$$

for some unit $\mu \in \mathscr{O}_{\mathbf{C}_p}^\times$. Since $n + \nu/2 \geq 0$, we see that the image vector indeed lies in $\mathscr{O}_{\mathbf{C}_p}[X]_{\deg\leq 2k}$.

5.3.8 Integrality when p does not split.

Again, we check that the vector (3.1.10) from §5.3.1 indeed lies in $\mathscr{O}_{\mathbf{C}_p}[X]_{\deg\leq 2k}$. Since p does not split in K^T, we have

$$C_p^x = \begin{bmatrix} \frac{\sqrt{\Delta_x}}{2} & 1 \\ \frac{\sqrt{-\Delta_x}}{2} & -\sqrt{-1} \end{bmatrix}$$

and

$$\begin{aligned}\sigma_p(C_p^x)^{-1}\cdot\mathbf{t}_0^k = \sigma_p\left(\begin{bmatrix} \frac{1}{\sqrt{\Delta_x}} & \frac{1}{\sqrt{-\Delta_x}} \\ \frac{1}{2} & \frac{\sqrt{-1}}{2} \end{bmatrix}\right)\cdot\mathbf{t}_0^k &= \left(-\frac{1}{\sqrt{-\Delta_x}}\right)^{-k}\left(\frac{X}{\sqrt{\Delta_x}}+\frac{1}{2}\right)^k\left(\frac{X}{\sqrt{-\Delta_x}}+\frac{\sqrt{-1}}{2}\right)^k \\ &= (-1)^k\left(\frac{X}{\sqrt{\Delta_x}}+\frac{1}{2}\right)^k\left(X-\frac{\sqrt{\Delta_x}}{2}\right)^k.\end{aligned}$$

From this we find that

$$\begin{aligned}&(-\det(T))^{\frac{k}{2}}\cdot\sigma_p(C_p^x)^{-1}\cdot\sigma^x_{\mathrm{wt},p}(\alpha_p)^{-1}\cdot\mathbf{t}_0^k \\ &=(-\det(T))^{\frac{k}{2}}\cdot\sigma^x_{\mathrm{ar},p}(\alpha_p)^{-1}\cdot\sigma_p(C_p^x)^{-1}\cdot\mathbf{t}_0^k \\ &=(-\det(T))^{\frac{k}{2}}\cdot\sigma^x_{\mathrm{ar},p}(\alpha_p)^{-1}\cdot(-1)^k\left(\frac{X}{\sqrt{\Delta_x}}+\frac{1}{2}\right)^k\left(X-\frac{\sqrt{\Delta_x}}{2}\right)^k \\ &=\sigma^x_{\mathrm{ar},p}(\alpha_p)^{-1}\cdot(-1)^k\left(\frac{X}{2}+\frac{\sqrt{\Delta_x}}{4}\right)^k\left(X-\frac{\sqrt{\Delta_x}}{2}\right)^k,\end{aligned}$$

since $\det(T) = -\frac{\Delta_x}{4}$.

By Lemma 5.3.2, we can represent α_p by either

$$(\epsilon^x_{\mathrm{ar},p})^{-1}(C(n,s))\cdot u \quad \text{or} \quad (\epsilon^x_{\mathrm{ar},p})^{-1}(D(n,t))\cdot u$$

for some $u\in\mathscr{D}_f^\times$ and $0\geq n\geq -\nu$. In either case we see that

$$\sigma^x_{\mathrm{ar},p}(\alpha_p)^{-1} = \sigma^x_{\mathrm{ar},p}(M)^{-1}\cdot\sigma^x_{\mathrm{ar},p}(d_l(n))^{-1}$$

where $d_l(n) = \begin{bmatrix} l^n & \\ & 1 \end{bmatrix}$ and some $M \in \mathrm{GL}_2(\mathbf{Z}_p)$. We have

$$\begin{aligned} &\sigma^x_{\mathrm{ar},p}(\alpha_p)^{-1} \cdot (-1)^k \left(\frac{X}{2} + \frac{\sqrt{\Delta_x}}{4}\right)^k \left(X - \frac{\sqrt{\Delta_x}}{2}\right)^k \\ = &\left(-\frac{1}{2}\right)^k \cdot \sigma^x_{\mathrm{ar},p}(M)^{-1} \cdot \sigma^x_{\mathrm{ar},p}(d_l(n))^{-1} \cdot \left(X^2 - \frac{\Delta_x}{4}\right)^k \\ = &\left(-\frac{1}{2}\right)^k \cdot \sigma_{\mathrm{ar},p}(M)^{-1} \cdot \left(l^{-n}X^2 - l^n\frac{\Delta_x}{4}\right) \end{aligned}$$

which lies in $\mathscr{O}_{\mathbf{C}_p}[X]_{\deg \le 2k}$ since $n \le 0$ and $n + \nu \ge 0$.

5.3.9 The theorem.

We synthesize the above discussion into the following theorem.

Theorem 5.3.3. *Suppose as before that the compatible sets of automorphic forms $\{\mathbf{f}_i^\bullet\}$ are both $\mathfrak{p}$-integral, then the Fourier coefficients $a_{t_f}(T)$ of $\mathscr{Y}_{t_f}$ all lie in $\bar{\mathbf{Q}} \cap \frac{1}{k!}\mathscr{O}_{\mathbf{C}_p}$.*

Proof. Recall the formula (2.5.8) reduced the question to checking that

$$(-\det(T))^{\frac{k}{2}} \cdot \langle \mathbf{t}_0^k, \mathbf{f}_1^x(\dot{t}\dot{w}\alpha_h)\rangle_k$$

lies in $\mathscr{O}_{\mathbf{C}_p}$. We then reduced this to showing that

$$(-\det(T))^{\frac{k}{2}} \cdot \sigma_p(C_p^x)^{-1} \cdot \sigma^x_{\mathrm{wt},p}(\alpha_p)^{-1} \cdot \mathbf{t}_0^k$$

lies in $\mathscr{O}_{\mathbf{C}_p}[X]_{\deg \le 2k}$. We have just verified this criterion in §5.3.4 and §5.3.8 depending on whether p is split in K^T or not. □

Chapter 6. Non-Vanishing

6.1 Bessel model of the Yoshida lift

We compute the Bessel model for a collection of Fourier coefficients. We make the additional assumption that the central character ε_i of $\boldsymbol{\pi}_i$ is trivial.

6.1.1 A collection of good indices.

We choose a collection of indices whose Bessel models are of arithmetic interest. Let $\Delta > 3$ be a square-free odd integer satisfying the following conditions:

- $\Delta \equiv -1 \pmod 4$,
- $(\Delta, dNp) = 1$, and
- $\left(\frac{q}{-\Delta}\right) = \begin{cases} 1 & \text{if } q \mid Np, \text{ and} \\ -1 & \text{if } q \mid d. \end{cases}$

Set $m^{\Delta}(X) = X^2 - X + \frac{1+\Delta}{4}$. Fix a root δ of m^{Δ} in D as in §1.1.3. Note such an δ exists by the above conditions. Let $\dot{K} = K^{\delta} = \mathbf{Q}(\delta)$ be the imaginary sub-field of D determined by δ. Let l be a prime inert in $\dot{K}$ and not dividing $2dNp$. For each integer $n \geq 0$, we have the following data:

- $\delta_n = l^n \delta \in \dot{K} \subset D$;
- $x_n \in \mathbf{X}$ given by $x_n(\mathbf{w}_1) = 1$ and $x_n(\mathbf{w}_2) = l^n \delta$;
- $T_n = T_{x_n} = \begin{bmatrix} 1 & \frac{l^n}{2} \\ \frac{l^n}{2} & l^{2n} \frac{1+\Delta}{4} \end{bmatrix}.$

Set $S = \{2, l, p\} \cup \{\text{prime divisors of } \Delta\}$, and fix an element $\jmath \in \dot{K}^{\perp}$ so that $(\delta_n, \jmath)$ is an S-basis.

6.1.2 Eichler orders and conjugation.

Let $\epsilon^n_{\mathrm{ar},l} = \epsilon^{\delta_n,\jmath}_{\mathrm{ar},l}$ be the arithmetic embedding at l attached to the S-basis $(\delta_n, \jmath)$ as in §1.3.3. It determines a local Eichler order $\mathscr{D}^n_l = (\epsilon^n_{\mathrm{ar},l})^{-1}(\mathrm{M}_2(\mathbf{Z}_l))$, which in turn can be spliced with other local Eichler orders $\mathscr{D}_q$ to give an Eichler order $\mathscr{D}^n = \mathscr{D}^{\delta_n,\jmath,S}$ of level dNp^r in D. As such, we have that $\mathscr{D}^n_q = \mathscr{D}^m_q$ for all $m, n \geq 0$ and $q \neq l$. Note that the intersection $\dot{\mathscr{O}}^n_f = \dot{K}_f \cap \mathscr{D}^n_f$ is exactly $\mathbf{Z}[\delta^{(0)}_n]$ in the notations from §1.3.3, and we have $\mathbf{Z}[\delta^{(0)}_n] = \mathbf{Z}[\delta_n]$ is the order of conductor l^n in $\dot{K}$.

Let us compare $\mathscr{D}^n_l$ and $\mathscr{D}^m_l$. Set $\alpha^{\delta_n}_i = (\epsilon^n_{\mathrm{ar},l})^{-1}(\mathrm{diag}[l^i, 1])$, we claim that

$$\alpha^{\delta_n}_i \mathscr{D}^n_f (\alpha^{\delta_n}_i)^{-1} = \mathscr{D}^{n+i}_f$$

for $0 \geq i \geq -n$. Indeed, we have

$$\epsilon^n_{\mathrm{ar},l} \circ \mathrm{Ad}(\alpha^{\delta_n}_i)^{-1}(\delta^{(0)}_{n+i}) = \begin{bmatrix} l^{-i} & \\ & 1 \end{bmatrix} \cdot l^i \begin{bmatrix} & 1 \\ \frac{l^{2n}\Delta}{4} & \end{bmatrix} \cdot \begin{bmatrix} l^i & \\ & 1 \end{bmatrix} = \begin{bmatrix} & 1 \\ \frac{l^{2(n+i)}\Delta}{4} & \end{bmatrix} = \epsilon^{n+i}_{\mathrm{ar},l}(\delta^{(0)}_{n+i})$$

and $\epsilon^n_{\mathrm{ar},l} \circ \mathrm{Ad}(\alpha^{\delta_n}_i)^{-1}(\jmath) = \epsilon^n_{\mathrm{ar},l}(\jmath)$; hence

$$\epsilon^n_{\mathrm{ar},l} \circ \mathrm{Ad}(\alpha^{\delta_n}_i)^{-1} = \epsilon^{n+i}_{\mathrm{ar},l}$$

and the claim follows since $\mathscr{D}^m_l = (\epsilon^m_{\mathrm{ar},l})^{-1}(\mathrm{M}_2(\mathbf{Z}_l))$.

Remark 6.1.1. Since $\epsilon^n_{\mathrm{ar},l}(\alpha^{\delta_n}_i) = \mathrm{diag}[l^i, 1]$ commutes with all diagonal matrices, it follows from the above discussion that

$$(\epsilon^n_{\mathrm{ar},l})^{-1}(\mathrm{diag}[t_1, t_2]) = (\epsilon^m_{\mathrm{ar},l})^{-1}(\mathrm{diag}[t_1, t_2])$$

for all $m, n \in \mathbf{Z}_{\geq 0}$. This in particular applies to $\alpha^{\delta_n}_i$ itself, which we now simply denote by α_i. It is an element in D_l independent of n.

6.1.3 The Fourier functional for T_n.

Let $K = K^{T_n}$ be the imaginary quadratic field determined by T_n, and let $K^\times$ be the corresponding torus in GSp_4 as in §3.5.4. We have an isomorphism

$$K \simeq \dot{K} \qquad \text{given by} \qquad l^n \frac{1 + \sqrt{-\Delta}}{2} \mapsto \delta_n.$$

We emphasize that K, $\dot{K}$, and the weight embeddings $\epsilon^{\delta_n,\jmath}_{\mathrm{wt}} = \epsilon^{\delta_m,\jmath}_{\mathrm{wt}}$ are all independent of n; whereas the arithmetic embeddings $\epsilon^n_{\mathrm{ar},l}$ and the corresponding Eichler orders

$\mathscr{D}^n$ do depend on n. Denote by $\dot{t}$ the image of $t \in K$ under this isomorphism. As we observed in §5.1.5, this isomorphism intertwines the actions by $K^\times \subset \mathrm{GSp}_4$ and $\dot{K}^\times \subset D^\times$ on x_n:

$$(\dot{t}, 1) \cdot x_n = x_n \cdot t.$$

Let us evaluate $\mathbf{a}^{T_n}(t)$ for $t \in K_{\mathbf{A}}^\times$. Under the above choices made on T_n, the formula (2.5.7) from §5.2.5 simplifies* to

$$\mathbf{a}^{T_n}(t) = e^{-2\pi \cdot \mathrm{tr}(T_n)} \cdot (-l^{2n}\frac{\Delta}{4})^{\frac{k}{2}} \sum_{i=-n}^{0} \sum_{w} \langle \mathbf{t}_0^k, \mathbf{f}_1^{x_n}(\dot{t}\dot{w}\alpha_i) \rangle \cdot \mathbf{f}_2^{x_n}(\dot{w}\alpha_i).$$

Recall here $\mathbf{f}_i^{x_n}$ is an automorphic form associated with $(\delta_n, \jmath)$ and $\mathscr{D}^n$ in §2.3.1 for $i = 1, 2$; and $w = w(i)$ runs over a set of representatives for the ideal class group

$$\mathrm{Cl}(\mathscr{O}_{n+i}) = K_{\mathbf{Q}}^\times K_\infty^\times \backslash K_{\mathbf{A}}^\times / \mathscr{O}_{n+i,f}^\times$$

(since $\mathscr{O}_{n+i,f}$ is the inverse image of $\alpha_i \mathscr{D}_f^n \alpha_i^{-1} \cap \dot{K}_f$ under the isomorphism $K \simeq \dot{K}$ above).

6.1.4 The Bessel model for T_n.

Given a Hecke character $\chi\colon K_{\mathbf{Q}}^\times \backslash K_{\mathbf{A}}^\times \to \mathbf{C}^1$ which is trivial on $\mathbf{A}^\times$, we have

$$\begin{aligned} B_{\mathbf{Y}}^{T_n,\chi}(1) &= \int_{\mathbf{A}^\times K_{\mathbf{Q}}^\times \backslash K_{\mathbf{A}}^\times} \bar{\chi}(t) \cdot \mathbf{a}^T(t)\, d^\times t \\ &= e^{-2\pi \cdot \mathrm{tr}(T_n)} \cdot (-l^{2n}\frac{\Delta}{4})^{\frac{k}{2}} \cdot \sum_{i=-n}^{0} \sum_{w} \mathbf{f}_2^{x_n}(\dot{w}\alpha_i) \\ &\qquad \cdot \int_{\mathbf{A}^\times K_{\mathbf{Q}}^\times \backslash K_{\mathbf{A}}^\times} \bar{\chi}(t) \cdot \langle \mathbf{t}_0^k, \mathbf{f}_1^{x_n}(\dot{t}\dot{w}\alpha_i) \rangle\, d^\times t. \end{aligned}$$

Let us untangle the integral by substituting t with tw^{-1}; we get

$$\begin{aligned} B_{\mathbf{Y}}^{T_n,\chi}(1) = e^{-2\pi \cdot \mathrm{tr}(T_n)} \cdot (-l^{2n}\frac{\Delta}{4})^{\frac{k}{2}} \sum_{i=-n}^{0} \sum_{w} \chi(w) \cdot \mathbf{f}_2^{x_n}(\dot{w}\alpha_i) \\ \cdot \int_{\mathbf{A}^\times K_{\mathbf{Q}}^\times \backslash K_{\mathbf{A}}^\times} \bar{\chi}(t) \cdot \langle \mathbf{t}_0^k, \mathbf{f}_1^{x_n}(\dot{t}\alpha_i) \rangle\, d^\times t. \end{aligned} \tag{1.4.1}$$

*Since l, Δ, and dNp^r are mutually co-prime, and l is inert in K^{T_n}, a set of double-coset representatives for $\bar{E}_{x,f} \simeq \bar{E}_{x,l}$ is given by $\{(\alpha_i, \alpha_i) : 0 \geq i \geq -n\}$. Also, we see that $e_i = 1$ for $i = -n, \dots, 1$ since $\Delta > 3$.

6.1.5 Specializing to primitive characters.

Denote by $\mathscr{O}_n = \mathbf{Z}[l^n \frac{1+\sqrt{-\Delta}}{2}]$ the order of conductor l^n in K. It is mapped to the order $\dot{\mathscr{O}}^n$ of $\dot{K}$ under the isomorphism described in §6.1.3. We now take $\chi = \chi_n$ to be a primitive character of conductor l^n, i.e., χ is trivial on the image of $\mathscr{O}_{n,f}^\times$ but not on the image of $\mathscr{O}_{n+i,f}^\times$ for any $i < 0$.

We claim in this case all the terms in (1.4.1) vanish except for $i = 0$. By primitivity, we have $\chi(z_i) \neq 1$ for some $z_i \in \mathscr{O}_{n+i,l}^\times$ if $i < 0$. Denote by $\dot{z}_i$ the corresponding element in $\dot{\mathscr{O}}_l^{n+i,\times} \subset \mathscr{D}_l^{n+i,\times}$. By §6.1.2, we see that $\alpha_i^{-1}\dot{z}_i\alpha_i$ lies in $\mathscr{D}_l^{n,\times}$. As a result, we find that

$$\begin{aligned}\int_{\mathbf{A}^\times K_{n,\mathbf{Q}}^\times \backslash K_{n,\mathbf{A}}^\times} \bar{\chi}(t)\cdot\langle \mathbf{t}_0^k, \mathbf{f}_1^{x_n}(\dot{t}\alpha_i)\rangle\, d^\times t &= \int_{\mathbf{A}^\times K_{n,\mathbf{Q}}^\times \backslash K_{n,\mathbf{A}}^\times} \bar{\chi}(t)\cdot\langle \mathbf{t}_0^k, \mathbf{f}_1^{x_n}(\dot{t}\alpha_i\cdot\alpha_i^{-1}\dot{z}_i\alpha_i)\rangle_k\, dt \\ &= \int_{\mathbf{A}^\times K_{n,\mathbf{Q}}^\times \backslash K_{n,\mathbf{A}}^\times} \bar{\chi}(t\cdot z_i^{-1})\cdot\langle \mathbf{t}_0^k, \mathbf{f}_1^{x_n}(\dot{t}\alpha_i)\rangle_k\, dt \\ &= \chi(z_i)\cdot\int_{\mathbf{A}^\times K_{n,\mathbf{Q}}^\times \backslash K_{n,\mathbf{A}}^\times} \bar{\chi}(t)\cdot\langle \mathbf{t}_0^k, \mathbf{f}_1^{x_n}(\dot{t}\alpha_i)\rangle_k\, dt\end{aligned}$$

which implies both sides must be zero.

6.1.6 A linear combination of Fourier coefficients.

For a primitive character $\chi = \chi_n$ of conductor l^n, we see that (1.4.1) simplifies to

$$\begin{aligned}B_{\mathbf{Y}}^{T_n,\chi}(1) = e^{-2\pi\cdot\mathrm{tr}(T_n)}\cdot(-l^{2n}\frac{\Delta}{4})^{\frac{k}{2}} \\ \cdot\sum_{w\in\mathrm{Cl}(\mathscr{O}_n)}\chi(w)\cdot\mathbf{f}_2^{x_n}(\dot{w})\cdot\int_{\mathbf{A}^\times K_{\mathbf{Q}}^\times\backslash K_{\mathbf{A}}^\times}\bar{\chi}(t)\cdot\langle\mathbf{t}_0^k,\mathbf{f}_1^{x_n}(\dot{t})\rangle\, d^\times t\end{aligned}$$

since $e_\chi = 1$ and $U_f^\chi = \mathscr{O}_{n,f}^\times$. The analysis from §3.5.6 with $\mathbf{a}^T(t)$ replaced by $\langle\mathbf{t}_0^k,\mathbf{f}_1^{x_n}(\dot{t})\rangle$ gives that

$$\int_{\mathbf{A}^\times K_{\mathbf{Q}}^\times\backslash K_{\mathbf{A}}^\times}\bar{\chi}(t)\cdot\langle\mathbf{t}_0^k,\mathbf{f}_1^{x_n}(\dot{t})\rangle\, d^\times t = 2\pi\cdot\mathrm{vol}(\mathscr{O}_{n,f}^\times)\sum_{t\in\mathrm{Cl}(\mathscr{O}_n)}\bar{\chi}(t)\cdot\langle\mathbf{t}_0^k,\mathbf{f}_1^{x_n}(\dot{t})\rangle.$$

Finally, substituting $B_{\mathbf{Y}}^{T_n,\chi}(1)$ with the linear combination of translated Fourier coefficients from (5.6.12) in §3.5.6, we get after cancellation

$$\begin{aligned}&\sum_{i=1}^{h_n}\bar{\chi}_n(t_i)\cdot a_{t_i}(T_n) = \\ &\quad(-l^{2n}\frac{\Delta}{4})^{\frac{k}{2}}\left(\sum_{w\in\mathrm{Cl}(\mathscr{O}_n)}\chi(w)\cdot\mathbf{f}_2^{x_n}(\dot{w})\right)\cdot\left(\sum_{t\in\mathrm{Cl}(\mathscr{O}_n)}\bar{\chi}(t)\cdot\langle\mathbf{t}_0^k,\mathbf{f}_1^{x_n}(\dot{t})\rangle\right)\end{aligned} \tag{1.6.2}$$

where $h_n = \#\mathrm{Cl}(\mathscr{O}_n)$.

6.2 Equidistribution and non-vanishing

We show that for $n \gg 0$ and some $t_i \in \mathrm{Cl}(\mathscr{O}_n)$, $a_{t_i}(T_n)$ is non-zero. Furthermore, in favorable situations, we show that $a_{t_i}(T_n)$ is in fact non-zero modulo $\mathfrak{p}$ assuming Artin's conjecture on primitive roots. We assume that the automorphic forms $\mathbf{f}_i^{x_n}$ are $\mathfrak{p}$-integral throughout this section. Also denote by $K(\mathbf{f}_i^{x_n})$ the field of definition of $\mathbf{f}_i^{x_n}$ and denote by $K(\mathbf{f}_1^{x_n}, \mathbf{f}_2^{x_n})$ their compositum. Moreover, we assume that $p > k$, so the $a_{t_i}(T_n)$ in fact lies in $K(\mathbf{f}_1^{x_n}, \mathbf{f}_2^{x_n}) \cap \mathscr{O}_{\mathbf{C}_p}$ by Theorem 5.3.3 and Proposition 5.2.1.

6.2.1 Character sums.

If $\sum_{i=1}^{h_n} \bar{\chi}_n(t_i) \cdot a_{t_i}(T_n)$ is not zero modulo $\mathfrak{p}$ for some primitive χ_n of conductor l^n, then $a_{t_i}(T_n)$ is also not zero modulo $\mathfrak{p}$ for some i. By (1.6.2), the non-vanishing of $a_{t_i}(T_n)$ becomes a question of the non-vanishing modulo $\mathfrak{p}$ of the two character sums

$$\sum_{w \in \mathrm{Cl}(\mathscr{O}_n)} \chi(w) \cdot \mathbf{f}_2^{x_n}(\dot{w}) \qquad \text{and} \qquad \sum_{t \in \mathrm{Cl}(\mathscr{O}_n)} \bar{\chi}(t) \cdot \langle \mathbf{t}_0^k, \mathbf{f}_1^{x_n}(\dot{t}) \rangle. \tag{2.1.3}$$

Since the first sum is but a special case of the second sum, let us study

$$\sum_{t \in \mathrm{Cl}(\mathscr{O}_n)} \bar{\chi}(t) \cdot \langle \mathbf{t}_0^k, \mathbf{f}^{x_n}(\dot{t}) \rangle = \bar{\chi}(s) \sum_{t \in \mathrm{Cl}(\mathscr{O}_n)} \bar{\chi}(t) \cdot \langle \mathbf{t}_0^k, \mathbf{f}^{x_n}(\dot{t}\dot{s}) \rangle$$

for a generic $\mathbf{f}^{x_n}$ and some $s \in K_f^\times$ using the methods of [Vat03] and [CV07]. Specifically, we aim to address the question of whether

$$\sum_{t \in \mathrm{Cl}(\mathscr{O}_n)} \bar{\chi}(t) \cdot \langle \mathbf{t}_0^k, \mathbf{f}^{x_n}(\dot{t}\dot{s}) \rangle \tag{2.1.4}$$

is non-zero modulo $\mathfrak{p}$.†

6.2.2 Galois groups.

We begin by describing a special instance of [CV07, Proposition 5.6] that applies to our context. Fix a choice of the reciprocity map

$$\mathrm{rec}_K \colon \overline{K_{\mathbf{Q}}^\times} \backslash K_f^\times \simeq \mathrm{Gal}(K^{\mathrm{ab}}/K),$$

†Note since χ is a ring class character, it is trivial on $\mathbf{A}^\times$, which in turn forces us to work with automorphic forms with trivial central character.

and define K_n to be the finite abelian extension of K corresponding to the open subgroup given by the image of $U_n = \mathscr{O}_{n,f}^\times$ in $\overline{K_{\mathbf{Q}}^\times}\backslash K_f^\times$. We identify $\mathrm{Cl}(\mathscr{O}_n)$ with $G_n = \mathrm{Gal}(K_n/K)$ via the reciprocity map, and set $G_\infty = \varprojlim G_n$. We note that the reciprocity map induces a topological isomorphism [CV07, Lemma 2.1]

$$K_{\mathbf{Q}}^\times\backslash K_f^\times/U_\infty \simeq G_\infty$$

where $U_\infty = \bigcap_n U_n$.

Denote by G_t the torsion subgroup of G_∞; the quotient group $G_w = G_\infty/G_t$ is free of rank 1 over $\mathbf{Z}_l$, and $K_{l^\infty} = K^{G_t}$ is the anti-cyclotomic $\mathbf{Z}_l$-extension of K. We note that for $n \gg 0$, $G_t \to G_n$ is an injection and $G_n \simeq G_t \times G_{w,n}$ where $G_{w,n} \simeq G_n/G_t$ is a cyclic l-group given by the projection of G_w onto G_n.

6.2.3 The genus subgroup.

For each prime $q \mid \Delta$, denote by $\sigma_q \in G_\infty$ the Frobenius element for the unique prime $\mathfrak{q} = \mathfrak{q}_K \subset K$ lying over q. Note that $\mathrm{rec}_K(\varpi_q) = \sigma_q$ for a choice of uniformizer ϖ_q in $K_{\mathfrak{q}}$, and that $\sigma_q^2 = 1$. Denote by $\tilde{G}$ the subgroup generated by all the σ_q's. We see that $\tilde{G} \simeq \prod_{q|\Delta} \mu_2$ as groups. Let

$$R = \{1, \tau_2, \cdots, \tau_m\}$$

be a fixed set of representatives for the quotient $G_t/\tilde{G}$.

6.2.4 CM points.

Let $\mathscr{R}$ be a fixed order in D, we *do not* require $\mathscr{R}$ to be an Eichler order. Set

$$CM_{\mathscr{R}} = K_{\mathbf{Q}}^\times\backslash D_f^\times/\mathscr{R}_f^\times,$$

we shall refer to it as the set of *CM points* attached to $\mathscr{R}$. The group

$$\overline{K_{\mathbf{Q}}^\times}\backslash K_f^\times \simeq \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

acts on $CM_{\mathscr{R}}$ by left multiplication. Given a CM point $x = [\beta] \in CM_{\mathscr{R}}$, we define the *conductor* of x to be the conductor of the stabilizer of x, $\mathscr{O}(x)_f^\times = \dot{K}_f^\times \cap \beta \cdot \mathscr{R}_f^\times \cdot \beta^{-1}$. We denote by $CM_{\mathscr{R}}(l^n)$ the set of CM points of conductor l^n and $CM_{\mathscr{R}}(l^\infty) = \bigcup_n CM_{\mathscr{R}}(l^n)$. Note that the action by $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ on $CM_{\mathscr{R}}(l^\infty)$ factors through G_∞.

6.2.5 An equidistribution result.

Set

$$M_{\mathscr{R}} = D_{\mathbf{Q}}^{\times} \backslash D_f^{\times} / \mathscr{R}_f^{\times}.$$

Since D is definite, $M_{\mathscr{R}}$ is finite [Vig80, Ch. III, Théorème 5.4]. Let R be the set of representatives of G_t/G_1 fixed above. We have a simultaneous reduction map

$$CM_{\mathscr{R}}(l^n) \xrightarrow{\text{RED}} \prod_{\tau \in R} M_{\mathscr{R}} \qquad x \mapsto (\tau \cdot x)_\tau = (x, \tau_2 \cdot x, \cdots, \tau_m \cdot x).$$

Proposition 6.2.1 ([CV07, Prop. 5.6]). *Suppose that* $\mathrm{n}(\mathscr{R}_f^{\times}) = \mathbf{Z}_f^{\times}$*, then for all but finitely many* $x \in CM_{\mathscr{R}}(l^\infty)$*, we have*

$$\text{RED}(G_\infty \cdot x) = \prod_{\tau \in R} M_{\mathscr{R}}.$$

Proof. Since the base field is $F = \mathbf{Q}$, and $H = \mathscr{R}_f^{\times}$ is assumed to satisfy $\mathrm{n}(\mathscr{R}_f^{\times}) = \mathbf{Z}_f^{\times}$, we see that the set $N_H = \mathbf{Q}_{>0}^{\times} \backslash \mathbf{A}_f^{\times} / \mathrm{n}(\mathscr{R}_f^{\times})$ in [CV07, §5] has only one element. The proposition is then almost word-for-word [CV07, Prop. 5.6]. □

6.2.6 Averaging over characters.

Following [CV07], we massage the sum (2.1.4) in order to apply the above key proposition.

For a group G, denote by $\hat{G}$ the group of characters on G. For $n \gg 0$, $G_n \simeq G_t \times G_{w,n}$ and we can factor $\chi \in \hat{G}_n$ into $\chi = \chi_t \cdot \chi_w$ with $\chi_t \in \hat{G}_t$ and $\chi_w \in \hat{G}_{w,n}$. To show that (2.1.4) is non-zero modulo $\mathfrak{p}$, it suffices to show that there exists* some primitive character $\chi_t \in \hat{G}_t$ such that the average over the primitive characters of $\hat{G}_{w,n}$

$$\sum_{\substack{\chi_w \in \hat{G}_{w,n} \\ \chi_w \text{ prim.}}} \sum_{t \in G_n} (\bar{\chi}_w \cdot \bar{\chi}_t)(t) \cdot \langle \mathbf{t}_0^k, \mathbf{f}^{x_n}(\dot{t}\dot{s}) \rangle = \sum_{t \in G_t} \bar{\chi}_t(t) \sum_{w \in G_{w,n}} \sum_{\chi_w \text{ prim.}} \bar{\chi}_w(w) \cdot \langle \mathbf{t}_0^k, \mathbf{f}^{x_n}(\dot{t}\dot{s}\dot{w}) \rangle.$$

is non-zero modulo $\mathfrak{p}$.

*In fact, it turns out that for *any* such character, not necessarily primitive, we can show the sum is non-zero for some primitive χ_w.

We can evaluate the two inner sums,

$$\begin{aligned}
&\sum_{w\in G_{w,n}}\sum_{\chi_w \text{ prim.}} \bar{\chi}_w(w)\cdot\langle \mathbf{t}_0^k, \mathbf{f}^{x_n}(\dot{t}\dot{s}\dot{w})\rangle \\
&= \sum_{w\in G_{w,n}}\left(\sum_{\chi_w\in\hat{G}_{w,n}} \bar{\chi}_w(w) - \sum_{\chi'_w\in\hat{G}_{w,n-1}} \bar{\chi}'_w(w)\right)\cdot\langle \mathbf{t}_0^k, \mathbf{f}^{x_n}(\dot{t}\dot{s}\dot{w})\rangle \\
&= |G_w(n)|\cdot\langle \mathbf{t}_0^k, \mathbf{f}^{x_n}(\dot{t}\dot{s})\rangle - \sum_{\sigma\in Z_n}|G_w(n-1)|\cdot\langle \mathbf{t}_0^k, \mathbf{f}^{x_n}(\dot{t}\dot{s}\sigma)\rangle \\
&= |G_w(n)|\cdot\langle \mathbf{t}_0^k, ((1-l^{-1}\cdot \mathrm{tr}_n)\cdot\mathbf{f}^{x_n})(\dot{t}\dot{s})\rangle
\end{aligned}$$

where $Z_n = \ker(G_{w,n}\to G_{w,n-1})$ and $(\mathrm{tr}_n\cdot\mathbf{f}^{x_n})(\dot{t}\dot{s}) = \sum_{\sigma\in Z_n}\mathbf{f}^{x_n}(\dot{t}\dot{s}\cdot\sigma)$.

Now $|G_w(n)| = l^m$ for some $m\le n-1$ is a $\mathfrak{p}$-unit, thus our problem reduces to showing that

$$\langle \mathbf{t}_0^k, \sum_{t\in G_t}\bar{\chi}_t(t)\cdot((1-l^{-1}\cdot\mathrm{tr}_n)\cdot\mathbf{f}^{x_n})(\dot{t}\dot{s})\rangle \not\equiv 0 \pmod{\mathfrak{p}}. \tag{2.6.5}$$

6.2.7 Averaging over the genus subgroup.

Let us average over the genus subgroup $\tilde{G}$. For simplicity, we further assume that Δ is a prime, so that $\tilde{G}$ is a group of order 2 generated by σ_Δ. We get

$$\sum_{t\in G_t}\bar{\chi}_t(t)\cdot((1-l^{-1}\cdot\mathrm{tr}_n)\cdot\mathbf{f}^{x_n})(\dot{t}\dot{s}) = \sum_{\tau\in R}\bar{\chi}_t(\tau)\cdot((1-l^{-1}\cdot\mathrm{tr}_n)\cdot(\mathbf{f}^{x_n}+\bar{\chi}_t(\sigma_\Delta)\cdot\sigma_\Delta\cdot\mathbf{f}^{x_n}))(\dot{\tau}\dot{s})$$

where $(\sigma_\Delta\cdot\mathbf{f}^{x_n})(\beta) = \mathbf{f}^{x_n}(\beta\dot{\sigma}_\Delta)$ for $\beta\in D^\times_{\mathbf{A}}$.

Set

$$\tilde{\mathbf{f}}^{x_n} = \mathbf{f}^{x_n} + \bar{\chi}_t(\sigma_\Delta)\cdot\sigma_\Delta\cdot\mathbf{f}^{x_n},$$

it is an automorphic form on $D^\times$ of level $\tilde{\mathscr{D}}_f^{n,\times} = \mathscr{D}_f^{n,\times}\cap\sigma_\Delta\mathscr{D}_f^{n,\times}\sigma_\Delta^{-1}$ in the sense that it satisfies the transformation law (3.1.1) from §2.3.1 but with $\mathscr{D}(r)_f^\times$ replaced by $\tilde{\mathscr{D}}_f^{n,\times}$.

Suppose that the automorphic form $\mathbf{f}^{x_n}$ is non-Eisenstein at $\mathfrak{p}$ in the sense of §2.3.8. That is, $\langle \mathbf{t}_0^k, \mathbf{f}^{x_n}(\beta)\rangle$ is not constant modulo $\mathfrak{p}$ as a function in $\beta\in D_f^\times$. We claim that $\tilde{\mathbf{f}}^{x_n}$ is then also non-Eisenstein at $\mathfrak{p}$. Suppose otherwise that $\langle \mathbf{t}_0^k, \tilde{\mathbf{f}}^{x_n}(\beta)\rangle \equiv \xi \pmod{\mathfrak{p}}$ for a constant $\xi\in\mathscr{O}_{\mathbf{C}_p}$ and all $\beta\in D_f^\times$. It follows that

$$\langle \mathbf{t}_0^k, \mathbf{f}^{x_n}(\beta)\rangle \equiv \langle \mathbf{t}_0^k, (\bar{\chi}_t(\sigma_\Delta)\cdot\sigma_\Delta\cdot\mathbf{f}^{x_n})(\beta)\rangle + \xi \pmod{\mathfrak{p}}.$$

Now the right-hand side is right-invariant under $\sigma_\Delta\mathscr{D}_f^{n,\times}\sigma_\Delta^{-1}$, hence $\langle \mathbf{t}_0^k, \mathbf{f}^{x_n}\rangle$ modulo $\mathfrak{p}$ is right-invariant under $D^1_\Delta\cdot\mathscr{D}_f^{n,\times} = \langle\mathscr{D}_f^{n,\times}, \sigma_\Delta\mathscr{D}_f^{n,\times}\sigma_\Delta^{-1}\rangle$ [CV07, Proof of Lemma 5.9].

Since D is split at Δ, by strong approximation and the fact that $\mathbf{Q}$ has class number 1, we get

$$D_f^\times = D_{\mathbf{Q}}^\times \cdot D_\Delta^1 \cdot \mathscr{D}_f^{n,\times}.$$

This implies $\langle \mathbf{t}_0^k, \mathbf{f}^{x_n} \rangle$ is constant modulo $\mathfrak{p}$, contradicting our assumption.

Remark 6.2.1. More generally, without assuming $\mathfrak{p}$-integality, we have that $\langle \mathbf{t}_0^k, \tilde{\mathbf{f}}^{x_n} \rangle$ is not constant as a function on $D_f^\times$. Indeed, since $\boldsymbol{\pi}$ is cuspidal, $\langle \mathbf{t}_0^k, \mathbf{f}^{x_n}(\beta) \rangle$ is not constant on $D_f^\times$, then the same argument applies.

6.2.8 Distribution relation.

In order to apply Proposition 6.2.1, we need to first make explicit the dependence on n in (2.6.5). Set $\epsilon_{\mathrm{ar},l} = \epsilon^0_{\mathrm{ar},l}$, $\mathscr{D}_l = \mathscr{D}_l^0$, $\mathbf{f} = \mathbf{f}^{x_0}$, and $\dot{\mathscr{O}}_l = \dot{\mathscr{O}}_l^0$. By §6.1.2, we have that $\mathscr{D}_l^n = \alpha_{-n}^{-1} \mathscr{D}_l \alpha_{-n}$. It follows that

$$\mathbf{f}^{x_n} = \alpha_{-n} \cdot \mathbf{f}$$

as functions on $D_{\mathbf{A}}^\times$ since they only differ in the choice of the Eichler order. Similarly

$$\tilde{\mathbf{f}}^{x_n} = \alpha_{-n} \cdot \tilde{\mathbf{f}}$$

since $\alpha_{-n} \in D_l^\times$ and $\sigma_\Delta \in D_\Delta^\times$ commute.[†]

It remains to show that

$$\langle \mathbf{t}_0^k, \sum_{\tau \in R} \bar{\chi}_t(\tau) \cdot ((1 - l^{-1} \cdot \mathrm{tr}_n) \cdot \alpha_{-n} \cdot \tilde{\mathbf{f}})(\dot{\tau}\dot{s}) \rangle \not\equiv 0 \pmod{\mathfrak{p}}. \tag{2.8.6}$$

For this, we need to understand $(1 - l^{-1} \cdot \mathrm{tr}_n) \cdot \alpha_{-n} \cdot (\tilde{\mathbf{f}})$. This is leads to the following lemma.

Lemma 6.2.2 ([CV07, Corollary 6.6])**.**

$$\mathrm{tr}_n \cdot \alpha_{-n} = T_l \cdot \alpha_{-n} \cdot \alpha_{-1} - l \cdot \alpha_{-n} \cdot \alpha_{-2} \tag{2.8.7}$$

as elements in $\mathbf{Z}[D_l^\times / \mathscr{D}_l^\times]$.

[†]Indeed, we have

$$\begin{aligned}
\tilde{\mathbf{f}}^{x_n}(\tau) &= \mathbf{f}(\dot{\tau}\alpha_{-n}) + \bar{\chi}_t(\sigma_\Delta) \cdot \mathbf{f}(\dot{\tau}\dot{\sigma}_\Delta \alpha_{-n}) \\
&= (\alpha_{-n} \cdot \mathbf{f})(\dot{\tau}) + (\bar{\chi}_t(\sigma_\Delta) \cdot \sigma_\Delta \cdot \mathbf{f})(\dot{\tau}\alpha_{-n}) \\
&= (\alpha_{-n} \cdot (\mathbf{f} + \bar{\chi}_t(\sigma_\Delta) \cdot \sigma_\Delta \cdot \mathbf{f}))(\dot{\tau}) \\
&= \alpha_{-n} \cdot \tilde{\mathbf{f}}(\dot{\tau}).
\end{aligned}$$

Before going into the proof, let us first explain what the equality says.

6.2.9 Lattices in $\dot{K}_l$.

The map

$$x \mapsto \mathscr{I}_x = \epsilon_{\mathrm{ar},l}(x) \cdot \dot{\mathscr{O}}_l$$

defines a bijection between $D_l^\times/\mathscr{D}_l^\times$ and the set of $\mathbf{Z}_l$-lattices in $\dot{K}_l$, which we denote by $\mathbf{I}$. For convenience, we shall treat $D_l^\times/\mathscr{D}_l^\times$ synonymously when there is no danger of confusion. We now introduce some terminologies useful for the ensuing discussion.

Given $\mathscr{I}$, we have that $\mathscr{I} = \epsilon_{\mathrm{ar},l}(t) \cdot \dot{\mathscr{O}}_l^n$ for some integer $n \geq 0$ and $t \in \dot{K}_l^\times$ [Iha67, §Prop. 1]. We shall refer to n as the *level* of the lattice $\mathscr{I}$.

As in [CV07, §6.2], the *lower neighbors* of $\mathscr{I} \in \mathbf{I}$ are the lattices $\mathscr{I}^\downarrow$such that $\mathscr{I}^\downarrow \subset \mathscr{I}$ and $\mathscr{I}/\mathscr{I}^\downarrow \simeq \mathbf{Z}/l\mathbf{Z}$. In particular, the lower neighbors of $\dot{\mathscr{O}}_l = \mathbf{Z} \oplus \mathbf{Z}\delta^{(0)}$ are the lattices $\dot{\mathscr{O}}_i^\downarrow = \mathbf{Z}(l\delta^{(0)} + i) \oplus \mathbf{Z}$ for $i = 0, \dots, l-1$ and $\dot{\mathscr{O}}_l^\downarrow = \mathbf{Z}\delta^{(0)} \oplus \mathbf{Z}l$. They correspond to the elements

$$\mu_i = (\epsilon_{\mathrm{ar},l})^{-1}\left(\begin{bmatrix} l & i \\ & 1 \end{bmatrix}\right) \text{ for } i = 0, \dots, l-1, \text{ and } \mu_l = (\epsilon_{\mathrm{ar},l})^{-1}\left(\begin{bmatrix} 1 & \\ & l \end{bmatrix}\right)$$

in $D_l^\times/\mathscr{D}_l^\times$. From this we conclude that the neighbors of $\mathscr{I}_x = \epsilon_{\mathrm{ar},l}(x)$ are exactly the lattices $\mathscr{I}_{x,i}^\downarrow = \epsilon_{\mathrm{ar},l}(x) \cdot \dot{\mathscr{O}}_i^\downarrow$ for $i = 0, \dots, l$ and they correspond to the elements $x \cdot \mu_i \in D_l^\times/\mathscr{D}_l^\times$. We note that if $\mathscr{I}_x$ has level $n > 1$, then $\mathscr{I}_{x,i}^\downarrow$ all have level $n+1$ except for $\mathscr{I}_{x,l}^\downarrow$, which has level $n-1$.

Lastly, the *upper predecessor* of a lattice $\mathscr{I}_x \in \mathbf{I}$ of level $n \geq 1$ is the unique lattice $\mathscr{I}_x^\uparrow = \dot{\mathscr{O}}_l^{n-1} \cdot \mathscr{I}_x$ of level $n-1$ which has $\mathscr{I}_x$ as a lower neighbor. It corresponds to the element $x\alpha_{-1} \in D_l^\times/\mathscr{D}_l^\times$ where $\alpha_{-1} = (\epsilon_{\mathrm{ar},l})^{-1}(\mathrm{diag}[l^{-1}, 1])$ as in §6.1.2.

6.2.10 Trace map on lattices.

For $n \gg 0$, $Z_n = \ker(G_{w,n} \to G_{w,n-1}) \simeq \dot{\mathscr{O}}_l^{n-1,\times}/\dot{\mathscr{O}}_l^{n,\times}$. Using this identification, we define a trace map on $x \in D_l^\times/\mathscr{D}_l^\times$ of level n (i.e., $\mathscr{I}_x$ has level n) taking value in $\mathbf{Z}[D_l^\times/\mathscr{D}_l^\times]$ by

$$\mathrm{tr}_n \cdot x = \sum_{\sigma \in Z_n} \sigma \cdot x.$$

It is compatible with the trace map on $\mathbf{f}$ (and $\tilde{\mathbf{f}}$) in the sense that $(\mathrm{tr}_n \cdot x) \cdot \mathbf{f} = \mathrm{tr}_n \cdot (x \cdot \mathbf{f})$ as functions on $D_{\mathbf{A}}^\times$.

In more geometric terms, if we write $\mathscr{I}_x = \epsilon_{\mathrm{ar},l}(t_x)\cdot\dot{\mathscr{O}}_l^n$, then

$$\begin{aligned}\sum_{\sigma\in Z_n}\sigma\cdot\mathscr{I}_x &= \sum_{\sigma\in Z_n}\sigma\cdot\epsilon_{\mathrm{ar},l}(t_x)\cdot\dot{\mathscr{O}}_l^n\\ &= \sum_{\sigma\in Z_n}\epsilon_{\mathrm{ar},l}(t_x)\cdot\sigma\cdot\dot{\mathscr{O}}_l^n.\end{aligned}$$

Now as σ runs over Z_n, $\sigma\cdot\dot{\mathscr{O}}_l^n$ runs over the lattices in $\dot{\mathscr{O}}_l^{n-1}$ that are of level n and index l, which are exactly the lower neighbors $\dot{\mathscr{O}}_i^{n,\downarrow}$, $i=0,\ldots,l-1$ of level n; thus

$$\mathrm{tr}_n\cdot\mathscr{I}_x = \sum_{i=0}^{l-1}(\mathscr{I}_x^{\uparrow})_i^{\downarrow}, \tag{2.10.8}$$

which corresponds $\sum_{i=0}^{l-1}x\cdot\alpha_{-1}\cdot\mu_i$ in $\mathbf{Z}[D_l^\times/\mathscr{D}_l^\times]$.

6.2.11 Hecke action on lattices.

We also have a Hecke action on $\mathbf{Z}[D_l^\times/\mathscr{D}_l^\times]\simeq\mathbf{Z}[\mathbf{I}]$ defined as follows. Recall we have the abstract Hecke element $\mathscr{D}_l^\times\varpi_l\mathscr{D}_l^\times = \{\alpha\in D_l^\times : \mathrm{n}(\alpha)=l\}$ where ϖ_l is any element in $D_l^\times$ of reduced norm l. The Hecke action is then

$$T_l\cdot x = \sum_i x\cdot\varpi_{i,l}$$

with respect to any decomposition $\mathscr{D}_l^\times\varpi_l\mathscr{D}_l^\times = \bigcup_i\varpi_{i,l}\cdot\mathscr{D}_l^\times$. A comparison with the Hecke action on automorphic forms described in §2.3.2 shows that $(T_l\cdot x)\cdot\mathbf{f} = \omega_l(l)\cdot x\cdot(T_l\cdot\mathbf{f})$.* Now we have a natural choice for the elements $\varpi_{i,l}$, namely, μ_i for $i=0,\ldots,l$. It follows that $T_l\cdot x = \sum_i x\cdot\mu_i$. In terms of lattices, we see that $T_l\cdot\mathscr{I}_x = \sum_{i=0}^{l}\mathscr{I}_{x,i}^{\downarrow}$ is the just the sum of the lower neighbors of $\mathscr{I}_x$.

Proof of Lemma 6.2.2. First observe that α_{-n} is of level n. By the preceding discus-

*Indeed,

$$\begin{aligned}((T_l\cdot x)\cdot\mathbf{f})(y) &= \sum_i\mathbf{f}(y\cdot x\cdot\varpi_{i,l})\\ &= \omega_l(l)\cdot(T_l\cdot\mathbf{f})(y\cdot x)\\ &= \omega_l(l)\cdot(x\cdot(T_l\cdot\mathbf{f}))(y).\end{aligned}$$

sion, we have:

$$\begin{aligned}
\mathrm{tr}_n \cdot \alpha_{-n} &= \sum_{i=0}^{l-1} \alpha_{-(n-1)}^{-1} \cdot \alpha_{-1} \cdot \mu_i \\
&= T_l \cdot (\alpha_{-n} \cdot \alpha_{-1}) - \alpha_{-n} \cdot \alpha_{-1} \cdot \mu_l \\
&= T_l \cdot \alpha_{-n} \cdot \alpha_{-1} - l \cdot \alpha_{-n} \cdot \alpha_{-2}.
\end{aligned}$$

□

6.2.12 The upshot.

By Lemma 6.2.2 which we just proved, we have

$$\begin{aligned}
&((1 - l^{-1} \cdot \mathrm{tr}_n) \cdot \alpha_{-n} \cdot \tilde{\mathbf{f}})(\dot{\tau}) \\
= &(\alpha_{-n} \cdot \tilde{\mathbf{f}})(\dot{\tau}\dot{s}) - l^{-1} \cdot ((T_l \cdot \alpha_{-n} \cdot \alpha_{-1}) \cdot \tilde{\mathbf{f}})(\dot{\tau}\dot{s}) + (\alpha_{-n} \cdot \alpha_{-2} \cdot \tilde{\mathbf{f}})(\dot{\tau}\dot{s}) \\
= &\tilde{\mathbf{f}}(\dot{\tau}\dot{s} \cdot \alpha_{-n}) - l^{-1} \cdot (\alpha_{-n} \cdot \alpha_{-1} \cdot T_l \cdot \tilde{\mathbf{f}})(\dot{\tau}\dot{s}) + (\alpha_{-2} \cdot \tilde{\mathbf{f}})(\dot{\tau}\dot{s} \cdot \alpha_{-n}) \\
= &\tilde{\mathbf{f}}(\dot{\tau}\dot{s} \cdot \alpha_{-n}) - l^{-1} \cdot \omega_l(l) \cdot a_l \cdot (\alpha_{-1} \cdot \tilde{\mathbf{f}})(\dot{\tau}\dot{s} \cdot \alpha_{-n}) + (\alpha_{-2} \cdot \tilde{\mathbf{f}})(\dot{\tau}\dot{s} \cdot \alpha_{-n})
\end{aligned}$$

where $a_l = a_l(\mathbf{f})$ is the eigenvalue of $\mathbf{f}$ under T_l. Set

$$\mathbf{f}^* = \tilde{\mathbf{f}} - l^{-1} \cdot \omega_l(l) \cdot a_l \cdot \alpha_{-1} \cdot \tilde{\mathbf{f}} + \alpha_{-2} \cdot \tilde{\mathbf{f}},$$

it is an automorphic form of level $\mathscr{D}_f^{*,\times} = \tilde{\mathscr{D}}_f^\times \cap \alpha_{-1}\tilde{\mathscr{D}}_f^\times \alpha_{-1}^{-1} \cap \alpha_{-2}\tilde{\mathscr{D}}_f^\times \alpha_{-2}^{-1}$, again in the sense of (3.1.1) from §2.3.1. Arguing as in §6.2.7, we see that if $\langle \mathbf{t}_0^k, \mathbf{f}^* \rangle$ is constant modulo $\mathfrak{p}$, then $\langle \mathbf{t}_0^k, \tilde{\mathbf{f}} \rangle$ is constant on $D_{\mathbf{Q}}^\times \cdot D_l^1 \cdot \tilde{\mathscr{D}}_f^\times = D_f^\times$, which contradicts the fact that $\tilde{\mathbf{f}}$ is non-Eisenstein at $\mathfrak{p}$; therefore $\mathbf{f}^*$ is also non-Eisenstein at $\mathfrak{p}$.

Remark 6.2.2. Again, without assuming $\mathfrak{p}$-integality, the same argument shows that $\langle \mathbf{t}_0^k, \mathbf{f}^* \rangle$ is not constant on $D_f^\times$ (c.f. Remark 6.2.1).

The upshot of the preceding discussion is that (2.8.6) now becomes

$$\langle \mathbf{t}_0^k, \sum_{\tau \in R} \bar{\chi}_t(\tau) \cdot \mathbf{f}^*(\dot{\tau}\dot{s} \cdot \alpha_{-n}) \rangle \not\equiv 0 \pmod{\mathfrak{p}}, \tag{2.12.9}$$

and this puts us in a position to apply Proposition 6.2.1.

6.2.13 Conductor of some CM points.

Let $\mathscr{D}^*$ be the Eichler order of level $dNp^r l^2\Delta$ defined locally by

$$\mathscr{D}_q^* = \begin{cases} \mathscr{D}_\Delta^0 \cap \sigma_\Delta \mathscr{D}_\Delta^0 \sigma_\Delta^{-1} & \text{if } q = \Delta, \\ \mathscr{D}_l^0 \cap \alpha_{-2} \mathscr{D}_l^0 \alpha_{-2}^{-1} & \text{if } q = l, \text{ and,} \\ \mathscr{D}_q^0 & \text{otherwise.} \end{cases}$$

Note that $\mathscr{D}_f^{*,\times}$ agrees with the compact open subgroup of $D_f^\times$ defined in the previous section.

The last detail to check before applying Proposition 6.2.1 is the conductor of α_{-n} as a CM point in $CM_{\mathscr{D}^*}$. We claim it is l^{n+2}. Indeed, since $\alpha_{-n} \in D_l^\times$, and $\dot{\sigma}_\Delta \in \dot{K}_\Delta^\times$ commutes with $\dot{K}_\Delta^\times$, we have that

$$\dot{K}_q^\times \cap \mathscr{D}_q^{*,\times} = \dot{K}_q^\times \cap \mathscr{D}_q^{0,\times} = \dot{\mathscr{O}}_{K,q}^\times$$

for all $q \neq l$. At the prime l, we have that

$$\epsilon_{\mathrm{ar},j}(\alpha_{-n}\mathscr{D}_l^{*,\times}\alpha_{-n}^{-1}) = \begin{bmatrix} l^{-n} & \\ & 1 \end{bmatrix} \cdot I_l(2) \cdot \begin{bmatrix} l^n & \\ & 1 \end{bmatrix} = \left\{ \begin{bmatrix} a & l^{-n}b \\ l^{n+2}c & d \end{bmatrix} \in \mathrm{M}_2(\mathbf{Q}_p) : a,b,c,d \in \mathbf{Z}_p \right\}$$

where $I_l(2)$ is the order in $\mathrm{M}_2(\mathbf{Z}_l)$ from §1.3.1. It then follows from the definition of $\epsilon_{\mathrm{ar},j}$ that

$$\dot{K}_l^\times \cap \alpha_{-n}\mathscr{D}_l^{*,\times}\alpha_{-n}^{-1} = \dot{\mathscr{O}}_l^{n+2,\times}.$$

6.2.14 Non-vanishing of a character sum.

Now we are ready to prove the following proposition.

Proposition 6.2.3. *For any character $\chi_t \in \hat{G}_t$ and all $n \gg 0$, there exists a primitive character $\chi_w \in \hat{G}_{w,n}$ such that*

$$\sum_{t \in \mathrm{Cl}(\mathscr{O}_n)} (\bar{\chi}_t \bar{\chi}_w)(t) \cdot \langle \mathbf{t}_0^k, \mathbf{f}^{x_n}(\dot{t}) \rangle$$

is not zero modulo $\mathfrak{p}$.

Proof. The discussion up to now has reduced the problem to showing that

$$\sum_{\tau \in R} \bar{\chi}_t(\tau) \cdot \langle \mathbf{t}_0^k, \mathbf{f}^*(\dot{\tau}\dot{s} \cdot \alpha_{-n}) \rangle \not\equiv 0 \pmod{\mathfrak{p}}$$

for some $s \in K_f^\times$.

Now $\langle \mathbf{t}_0^k, \mathbf{f}^* \rangle$ defines a $\frac{1}{k!}\mathscr{O}_{\mathbf{C}_p}$-valued function on $CM_{\mathscr{D}^*}$. Furthermore, since $\mathbf{f}^*$ is non-Eisenstein at $\mathfrak{p}$, we can find two CM points y, z in $CM_{\mathscr{D}^*}$ such that

$$\langle \mathbf{t}_0^k, \mathbf{f}^*(y) \rangle \not\equiv \langle \mathbf{t}_0^k, \mathbf{f}^*(z) \rangle \pmod{\mathfrak{p}}.$$

By Proposition 6.2.1, for $n \gg 0$, we can find s_1 and s_2 in $\dot{K}_f^\times$ such that $\tau_i \cdot s_1 \cdot \alpha_{-n} = \tau_i \cdot s_2 \cdot \alpha_{-n}$ for $i = 2, \ldots, m$, but

$$s_1 \cdot \alpha_{-n} = y \qquad \text{and} \qquad s_2 \cdot \alpha_{-n} = z.$$

Consequently,

$$\sum_{\tau \in R} \bar{\chi}_t(\tau) \cdot \langle \mathbf{t}_0^k, \mathbf{f}^*(\dot{\tau}\dot{s}_1 \cdot \alpha_{-n}) \rangle \not\equiv \sum_{\tau \in R} \bar{\chi}_t(\tau) \cdot \langle \mathbf{t}_0^k, \mathbf{f}^*(\dot{\tau}\dot{s}_2 \cdot \alpha_{-n}) \rangle \pmod{\mathfrak{p}}$$

and so one of them is non-zero modulo $\mathfrak{p}$. □

More generally, apply the same argument and using Remarks 6.2.1 and 6.2.2, we have the following proposition along the lines of [CV07, Theorem 1.13]

Proposition 6.2.4. *For any character $\chi_t \in \hat{G}_t$ and all $n \gg 0$, there exists a primitive character $\chi_w \in \hat{G}_{w,n}$ such that*

$$\sum_{t \in \mathrm{Cl}(\mathscr{O}_n)} (\bar{\chi}_t \bar{\chi}_w)(t) \cdot \langle \mathbf{t}_0^k, \mathbf{f}^{x_n}(\dot{t}) \rangle$$

is not zero.

6.2.15 Simultaneous non-vanishing.

To establish that $a_{t_i}(T_n)$ is non-zero (modulo $\mathfrak{p}$) for $n \gg 0$ and some $t_i \in \mathrm{Cl}(\mathscr{O}_n)$, we need to prove the simultaneous non-vanishing (modulo $\mathfrak{p}$) of the two character sums in (2.1.3). For this, we modify the argument for [Vat03, Corollary 4.2] and exploit the fact that $G_{w,n}$ is cyclic of l-power order, and so its primitive characters are permuted by a certain Galois action. Unfortunately, for non-vanishing modulo $\mathfrak{p}$, we also need to assume the following version of Artin's conjecture on primitive roots if we want to apply this argument for the decomposition group at $\mathfrak{p}$.

Let us first state the general non-vanishing theorem. Let $K(\mathbf{f}_1^{x_n}, \mathbf{f}_2^{x_n}) = K(\mathbf{f}_1^{x_n}) \cdot K(\mathbf{f}_2^{x_n})$ be the compositum of the fields of definition for the $\mathfrak{p}$-integral automorphic forms $\mathbf{f}_i^{x_n}$, $i = 1, 2$. Denote by $K' = K(\mathbf{f}_1^{x_n}, \mathbf{f}_2^{x_n}) \cdot \mathbf{Q}(\chi_t)$ the number field generated the values of χ_t over $K(\mathbf{f}_1^{x_n}, \mathbf{f}_2^{x_n})$.

Theorem 6.2.5. *Suppose l is a prime that is unramified in the extension $K'/\mathbf{Q}$. For this l, and $n \in \mathbf{Z}_{\geq 0}$, let $\mathscr{O}_n$ and T_n be as defined in §6.1.5 and §6.1.1 respectively. Then for all $n \gg 0$, there exists some $t_i \in \mathrm{Cl}(\mathscr{O}_n)$ such that the Fourier coefficient $a_{t_i}(T_n)$ of $\mathscr{Y}_{t_i}$ is not zero.*

Proof. Indeed, since l is unramified in K', and the prime l is totally ramified in $\mathbf{Q}(\zeta_{l^n})/\mathbf{Q}$, we see that $K' \cap \mathbf{Q}(\zeta_{l^n}) = \mathbf{Q}$ for all $n \geq 0$. The Galois group $\Gamma_n = \mathrm{Gal}(K'(\zeta_{l^n}))/K')$ acts transitively on the primitive l^n-th roots of unity, and hence on the primitive characters in $\hat{G}_{w,n}$. The theorem then follows from Proposition 6.2.3 since we can use Γ_n to permute the primitive characters in $\hat{G}_{w,n}$ while leaving $\bar{\chi}_t(t)$, $\langle \mathbf{t}_0^k, \mathbf{f}^{x_n}(\dot{t})\rangle$ fixed. This shows that

$$\sum_{t \in \mathrm{Cl}(\mathscr{O}_n)} (\bar{\chi}_t \bar{\chi}_w)(t) \cdot \langle \mathbf{t}_0^k, \mathbf{f}^{x_n}(\dot{t})\rangle$$

is not zero for *all* primitive characters $\chi_w \in \hat{G}_{w,n}$ and $n \gg 0$. Consequently, we see that

$$\sum_{i=1}^{h_n} \bar{\chi}_n(t_i) \cdot a_{t_i}(T_n)$$

is not zero for all $n \gg 0$; therefore one of $a_{t_i}(T_n)$ must be non-zero also. □

6.2.16 A conditional non-vanishing modulo $\mathfrak{p}$ result.

For non-vanishing modulo $\mathfrak{p}$, we need to assume the following conjecture of Artin:

Conjecture 6.2.6 (Artin). *Given any prime p, there exist infinitely many primes l such that p is a primitive root modulo l. In other words, p is a generator in $(\mathbf{Z}/l\mathbf{Z})^\times$.*

Remark 6.2.3. This conjecture is a theorem of Hooley [Hoo67] assuming yet another conjecture, namely the General Riemann Hypothesis. Also, Heath-Brown [HB86] has shown that this conjecture holds unconditionally for all but possibly two primes p.

Under Conjecture 6.2.6, we again choose l so that the number field K' as in Theorem 6.2.5 is linear disjoint from $\mathbf{Q}(\zeta_{l^\infty})$, and moreover, that l is totally inert in the decomposition group $D_\mathfrak{p} \subset \mathrm{Gal}(\bar{\mathbf{Q}}/K')$. Then as in the proof of Theorem 6.2.5, $D_\mathfrak{p}$ acts transitively on the primitive characters in $\hat{G}_{w,n}$, while leaving $\bar{\chi}_t(t)$, $\langle \mathbf{t}_0^k, \mathbf{f}^{x_n}(\dot{t})\rangle$, and the p-adic valuation of $\sum_{t \in \mathrm{Cl}(\mathscr{O}_n)} (\bar{\chi}_t \bar{\chi}_w)(t) \cdot \langle \mathbf{t}_0^k, \mathbf{f}^{x_n}(\dot{t})\rangle$ fixed. This implies this character sum in question is non-zero modulo $\mathfrak{p}$ for all primitive characters in $\hat{G}_{w,n}$

when $n \gg 0$, from which we see that $\sum_{i=1}^{h_n} \bar{\chi}_n(t_i) \cdot a_{t_i}(T_n)$ is not zero modulo $\mathfrak{p}$ for all $n \gg 0$. We have proved the following theorem.

Theorem 6.2.7. *Assume that $p > k$ and that $\mathbf{f}_i^{x_n}$ are non-Eisenstein at $\mathfrak{p}$ for all $n \in \mathbf{Z}_{\geq 0}$. Assuming in addition that Conjecture 6.2.6 holds. Let l, $\mathscr{O}_n$, and T_n be as above, then for all $n \gg 0$, there exists some $t_i \in \mathrm{Cl}(\mathscr{O}_n)$ such that the Fourier coefficient $a_{t_i}(T_n)$ of $\mathscr{Y}_{t_i}$ is not zero modulo $\mathfrak{p}$. In other words, $a_{t_i}(T_n) \in \mathscr{O}_{K',(\mathfrak{p}_{K'})}^{\times}$ if $p > k$.*

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