Effective Correspondents to Cardinal Characteristics in Cichoń's Diagram

by

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CHAPTER I

Introduction

1.1 Notes on Notation

The notation in this thesis can be called standard, with the proviso that different disciplines consider different notations to be standard. Before providing a rough outline of the work to follow, we will here remove most ambiguities. We use ω to denote the set of natural numbers: $\{0, 1, 2, \ldots\}$. The Cantor space is the set of infinite binary sequences, and is denoted by ${}^{\omega}2$. Cantor space can also be characterized as the set of functions from ω to $\{0, 1\}$ ($X \in {}^{\omega}2$ is the function which sends n to the value of the nth bit). The Baire space is the set of functions from ω to ω and is denoted by ${}^{\omega}\omega$.

The set of finite binary sequences (in this finite-length case, we may use the word "strings" instead of "sequences") is denoted by ${}^{<\omega}2$, and the set of binary strings of length n is denoted by ${}^{n}2$. For $\sigma \in {}^{<\omega}2$, $|\sigma|$ is the length of σ. We will use $\subseteq$ or $\subset$ to denote the initial segment relation ($\subset$ will be used for the case of a finite binary string which is an initial segment of an infinite binary sequence). For $X \in {}^{\omega}2$ and $n \in \omega$, $X \upharpoonright n$ is the initial segment of X which has length n.

Cantor space can be topologized, with basic open sets determined by elements of ${}^{<\omega}2$. For $\sigma \in {}^{<\omega}2$, let $[\sigma]$ denote the basic open set of $X \in {}^{\omega}2$ such that $\sigma \subset X$.

Also, if $V \subseteq {}^{<\omega}2$, let $[V]$ denote the union of $[\sigma]$ for $\sigma \in V$. We will also use the same notation (with a similar meaning) for partial functions from ω to $\{0,1\}$ with finite domain. If s is such a function, then $[s]$ denotes the set of X which are extensions of s. Under this topology, Cantor space is a metric space. The Lebesgue measure μ is determined by $\mu([\sigma]) = 2^{-n}$.

Some ambiguity cannot be avoided. The main points of which the reader should be aware are:

1. Brackets will also be used in another context. We will use $[\omega]^\omega$ and $[\omega]^{<\omega}$ to denote the set of (respectively) finite and infinite subsets of ω.

2. Members X of ${}^\omega 2$ will be called both "reals" and "oracles." The latter term will be used when we are concerned with the properties of X that relate to relative computability. However, there will be cases where some properties of X involve relative computability, and others do not. In such cases, either term may be used.

3. The symbol $\emptyset$ can mean either the empty subset of ω, or the empty string. This may cause ambiguity because of an identification between subsets of ω and reals which we will use frequently: $A \subseteq \omega$ is identified with the real X such that $X(n) = 1$ iff $n \in A$. Therefore, care is needed in some situations to distinguish between $\emptyset$ the string, and the real associated with $\emptyset$ (the real consisting of only zeroes).

1.2 This Thesis's Thrust

An element of either Cantor space or Baire space is computable if it is determined by an algorithm – for example, $f : \omega \to \omega$ is computable if there is a program which, given n, determines (in finite time) $f(n)$. The idea of computability can be expanded by relativization. An object is computable relative to $B \subseteq \omega$ (B is called an oracle) if it is determined by an algorithm which is allowed to ask questions of the form "Is n in B?" for any n. This gives rise to Turing reducibility $\leq_T$ – i.e., $B \leq_T A$ if the set B is computable relative to A. Turing reducibility (restricted to sets only) is a pre-order whose equivalence classes are called the Turing degrees [41]. The Turing degrees are partially ordered by $\leq_T$ and form an upper semilattice. A significant portion of computability theory is concerned with the basic question "What is the structure of the Turing degrees of sets with interesting properties, or which compute functions with interesting properties?" Of course, this depends on how one defines "interesting."

Set theory is one source for interesting properties. In set theory, one works with a universe of sets satisfying the Zermelo-Fraenkel axioms (ZF), and frequently also the axiom of choice (ZFC). Certain objects in the universe represent mathematical objects – for example, the natural numbers and Cantor space. This thesis is primarily interested in two topics of set theory, and how they relate to computability theory. These topics are forcing and cardinal characteristics of the continuum.

Forcing is a set-theoretic technique which can be used to show certain statements do not follow from ZFC. Forcing was first used by Cohen to construct a universe of ZFC where the Continuum Hypothesis does not hold [12, 13]. Informally, forcing can be thought of as a process that adds new objects to the initial universe, called the ground model. For example, Cohen forcing adds new sets to 2^ω. If enough sets are added, 2^ω can be enlarged without changing any cardinals, ensuring the Continuum Hypothesis can be made false. The added sets have further properties – for example, they are not contained in any meager (i.e. of first Baire category) subset of 2^ω from the ground model.

A forcing is determined by conditions, objects in the ground model which partially describe objects being added by the forcing. In the case of Cohen forcing, the conditions are finite binary strings, and the added set is the union of all strings used. To ensure the set has the desired properties, we make sure to use at least one string each from certain ground model sets of strings. We may try a similar construction in computability theory by replacing "ground model" here with "computable." The resulting sets have properties similar to those of sets added by Cohen forcing. This method has also been used in certain constructions to prove statements – for example, given $A >_T \emptyset$, there exists B not Turing comparable to it. More recently, forcing constructions have yielded interesting results with respect to many areas of research, including effective Ramsey theory [11], almost everywhere domination [10], and algorithmic randomness [33].

A cardinal characteristic is a cardinal which is descriptive of some property of a model of ZFC. Some cardinal characteristics are defined using a triple $\mathfrak{K} := (K_-, K_+, K)$, where K_- ("the challenges") and K_+ ("the answers") are subsets of Cantor space or Baire space, and K ("is met by") is a relation on $K_- \times K_+$. We call such triples "debates." The cardinal associated with the debate $\mathfrak{K}$, denoted $||\mathfrak{K}||$, is the least cardinality of a family of answers needed to meet all challenges. The relationships among ten cardinal characteristics definable via debates are summarized by Cichoń's diagram [17]:

$$\begin{array}{ccccccc}
\mathbf{cov}(\mathcal{N}) & \longleftarrow & \mathbf{non}(\mathcal{M}) & \longleftarrow & \mathbf{cof}(\mathcal{M}) & \longleftarrow & \mathbf{cof}(\mathcal{N}) \\
 & & \downarrow & & \downarrow & & \\
\downarrow & & \mathfrak{b} & \longleftarrow & \mathfrak{d} & & \downarrow \\
 & & \downarrow & & \downarrow & & \\
\mathbf{add}(\mathcal{N}) & \longleftarrow & \mathbf{add}(\mathcal{M}) & \longleftarrow & \mathbf{cov}(\mathcal{M}) & \longleftarrow & \mathbf{non}(\mathcal{N})
\end{array}$$

An arrow $||\mathfrak{K}|| \to ||\mathfrak{L}||$ indicates $||\mathfrak{K}|| \geq ||\mathfrak{L}||$ is a theorem of ZFC. The diagram is complete, in the sense that every such provable inequality is represented by an arrow or a series of arrows [5].

Debates provide standard mechanisms for proving either ZFC $\vdash ||\mathfrak{K}|| \geq ||\mathfrak{L}||$ or ZFC $\nvdash ||\mathfrak{K}|| \geq ||\mathfrak{L}||$. We consider such results to be, respectively, positive and negative. A positive result can be achieved by finding a morphism from $\mathfrak{K}$ to $\mathfrak{L}$: a pair of maps $\phi_+ : K_+ \to L_+$ and $\phi_- : L_- \to K_-$ such that for all $X \in L_-$ and $Y \in K_+$, if $\phi_-(X)KY$, then $XL\phi_+(Y)$. A negative result can (in all cases relevant to this thesis) be achieved by exhibiting a forcing which adds a challenge to L_- not met by any ground model answer in L_+ but does not add a challenge to K_- not met by any ground model answer in K_+.

In my thesis, we develop the notion of a Turing characteristic – a computability-theoretic correspondent to a cardinal of the form $||\mathfrak{K}||$ – to be the set of oracles which compute a challenge not met by any computable answer (denote this by $\mathbf{K}$). As with cardinal characteristics, we associate with Turing characteristics positive results ($\mathbf{K} \supseteq \mathbf{L}$) and negative results ($\mathbf{K} \not\supseteq \mathbf{L}$). In many cases results may be obtained via an effective version of the corresponding set-theoretic proof. Specifically, we may prove $\mathbf{K} \supseteq \mathbf{L}$ by finding a morphism from $\mathfrak{K}$ to $\mathfrak{L}$ such that for all $X \in L_-$ and $Y \in K_+$, $\phi_-(X)$ is computable from X and $\phi_+(Y)$ is computable from Y. If a notion of forcing proves that ZFC $\nvdash ||\mathfrak{K}|| \geq ||\mathfrak{L}||$, and this forcing can be made effective (as in Section 3.4), it provides a witness to $\mathbf{K} \not\supseteq \mathbf{L}$.

In most cases, using the techniques outlined above, we can prove $\mathbf{K} \supseteq \mathbf{L}$ in the cases where ZFC $\vdash ||\mathfrak{K}|| \geq ||\mathfrak{L}||$, and $\mathbf{K} \not\supseteq \mathbf{L}$ in the cases where ZFC $\nvdash ||\mathfrak{K}|| \geq ||\mathfrak{L}||$. This correspondence is not strict; for Turing characteristics derived from cardinal characteristics in Cichoń's diagram, we obtain the diagram

$$\begin{array}{ccccccc}
\mathbf{COVN} & \longleftarrow & \mathbf{NONM} & \longleftarrow & \mathbf{COFM} & \longleftarrow & \mathbf{COFN} \\
 & & \downarrow & & \downarrow & & \\
\downarrow & & \mathbf{B} & \longleftarrow & \mathbf{D} & & \downarrow \\
 & & \| & & \| & & \\
\mathbf{ADDN} & = & \mathbf{ADDM} & \longleftarrow & \mathbf{COVM} & \longleftarrow & \mathbf{NONN}
\end{array}$$

where $\rightarrow$ means $\supseteq$. This diagram, like Cichoń's diagram, is complete.

We can expand this definition of Turing characteristics by enlarging the set of answers considered, and restricting the set of challenges considered. Specifically, if $\mathbf{A}$ and $\mathbf{C}$ are sets of oracles, both are closed downward with respect to $\leq_T$, and $\mathbf{A}$ is countable, we use the set of answers computable from an oracle in $\mathbf{A}$, and challenges computable from an oracle in $\mathbf{C}$. With these added parameters, proofs of negative results may or may not work, but the proofs of positive results can always be carried out. This leads to the diagram

$$\begin{array}{ccccccc}
\mathbf{COVN}^{\mathbf{C}}_{\mathbf{A}} & \longleftarrow & \mathbf{NONM}^{\mathbf{C}}_{\mathbf{A}} & \longleftarrow & \mathbf{COFM}^{\mathbf{C}}_{\mathbf{A}} & \longleftarrow & \mathbf{COFN}^{\mathbf{C}}_{\mathbf{A}} \\
 & & \downarrow & & \downarrow & & \\
\downarrow & & \mathbf{B}^{\mathbf{C}}_{\mathbf{A}} & \longleftarrow & \mathbf{D}^{\mathbf{C}}_{\mathbf{A}} & & \downarrow \\
 & & \downarrow & & \downarrow & & \\
\mathbf{ADDN}^{\mathbf{C}}_{\mathbf{A}} & \longleftarrow & \mathbf{ADDM}^{\mathbf{C}}_{\mathbf{A}} & \longleftarrow & \mathbf{COVM}^{\mathbf{C}}_{\mathbf{A}} & \longleftarrow & \mathbf{NONN}^{\mathbf{C}}_{\mathbf{A}}
\end{array}$$

which is not known to be complete.

Some Turing characteristics studied in this paper impact the computability-theoretic field of algorithmic randomness; specifically, those Turing characteristics which involve $\mathcal{N}$, the ideal of measure zero subsets of Cantor space. A real may be said to be random if it does not have any "exceptional properties." Exceptional properties are represented by measure zero sets (also called null sets). Thus, a real is algorithmically random if it is not in any set which can be demonstrated by an algorithm to have measure zero. This last property of measure zero sets – which we may term as being "effectively null" – is of interest in of itself, and some results of this thesis can be interpreted as results on the relationship between the structure of a null set and the oracles relative to which it is effectively null.

We begin this thesis with chapters II and III, which outline the necessary computability-theoretic and set-theoretic background. Special attention will be paid to the topics of algorithmic randomness, forcing and its role in computability theory, cardinal characteristics of the continuum. In Chapter IV, we provide the full, formal definition of a Turing characteristic, and deal with some preliminary matters needed to study Turing characteristics arising from cardinals in Cichoń's diagram. Chapter V provides positive results, and Chapter VI provides negative results. Finally, in Chapter VII we list open questions arising from the thesis material.

CHAPTER II

Computability Theory Background

In this chapter we provide the background from computability theory on which this thesis builds. Much of this material is basic – in fact, any graduate level instruction in computability theory should cover most if not all the material of Sections 2.1–2.6. The approach taken in this thesis is most similar to the ones found in [38] and Chapters 4–5 of [20], though [49] and [19] are also important works and useful references. The remaining sections cover a sampling of results from the fields of algorithmic randomness and genericity, fields which are linked to the study of Turing characteristics. The former in particular is a rich field, and only a little of what is known is covered here – for more detailed treatments, see [36] or [15].

2.1 Computable and Partial Computable Objects

Let $\mathcal{P}$ denote the class of partial functions from some ω^k ($k \in \omega$) to ω (i.e., the domain is a subset of ω^k, possibly a proper subset). If $\Sigma(n)$ is a statement dependent on $n \in \omega$, let $\mu n[\Sigma(n)]$ denote the least value of n such that $\Sigma(n)$ is true. We use letters $\phi, \psi, \ldots$ to denote members of $\mathcal{P}$, and $f, g, \ldots$ to denote total members of $\mathcal{P}$ (i.e., ones with domain ω^k).

Definition II.1. The class $\mathcal{PC}$ of *partial computable* functions is the intersection of all $\mathcal{Q} \subseteq \mathcal{P}$ such that

1. if $\phi(n_0,\ldots,n_{k-1}) = m$ (for m constant) for all $(n_0,\ldots,n_{k-1}) \in \omega^k$, then $\phi \in \mathcal{Q}$ ($\mathcal{Q}$ contains all constant functions);

2. if for some $0 \le i < k$, $\phi(n_0,\ldots,n_{k-1}) = n_i$ for all $(n_0,\ldots,n_{k-1}) \in \omega^k$, then $\phi \in \mathcal{Q}$ ($\mathcal{Q}$ contains all projection functions);

3. the function $\phi(n) = n+1$ is in $\mathcal{Q}$ ($\mathcal{Q}$ contains the successor function);

4. if $\phi \in \mathcal{Q}$ takes k inputs and $\psi_0,\ldots\psi_{k-1} \in \mathcal{Q}$ each take r inputs, define ζ by

$$\zeta(n_0,\ldots,n_{r-1}) = \phi(\psi_0(n_0,\ldots,n_{r-1}),\ldots,\psi_{k-1}(n_0,\ldots,n_{r-1}))$$

   whenever the right-hand side is defined (otherwise $\zeta(n_0,\ldots,n_{r-1})$ does not exist); then $\zeta \in \mathcal{Q}$ ($\mathcal{Q}$ is closed under composition);

5. If $\phi \in \mathcal{Q}$ takes $k+2$ inputs and $\psi \in \mathcal{Q}$ takes k inputs, define ζ by

$$\zeta(n_0,\ldots,n_{k-1},0) = \psi(n_0,\ldots,n_{k-1})$$

$$\zeta(n_0,\ldots,n_{k-1},m+1) = \phi(n_0,\ldots,n_{k-1},\zeta(n_0,\ldots,n_{k-1},m),m)$$

   whenever the relevant right-hand side is defined (otherwise $\zeta(n_0,\ldots,n_k)$ does not exist); then $\zeta \in \mathcal{Q}$ ($\mathcal{Q}$ is closed under primitive recursion);

6. If $m \in \omega$ and $\phi \in \mathcal{Q}$ takes $k+1$ inputs, define ζ by

$$\zeta(n_0,\ldots n_{k-1}) = \mu n[\phi(n_0,\ldots n_{k-1},n) = m]$$

   whenever such an n exists (otherwise $\zeta(n_0,\ldots n_{k-1})$ does not exist); then $\zeta \in \mathcal{Q}$ ($\mathcal{Q}$ is closed under search).

If $f \in \mathcal{PC}$ is total, then f is *computable*. A set $A \subseteq \omega^k$ is *computable* iff the function χ_A, which maps members of A to 1 and non-members of A to 0, is computable. A relation R on k integer-valued variables is *computable* iff the set of k-tuples for which R holds is computable.

Remark II.2. Some authors use the terms *partial recursive* and *recursive* in place of partial computable and computable.

The intuition behind this definition is that ϕ is partial computable if there is an algorithm that, given n, returns $\phi(n)$, with the amount of time and memory used allowed to be arbitrarily large, as long as it is finite. For a given n, ϕ is allowed to return no answer – $\phi(n)$ is allowed not to exist. In this case, we say $\phi(n)$ *diverges*, and otherwise say $\phi(n)$ *converges* (these will be abbreviated $\phi(n)\uparrow$ and $\phi(n)\downarrow$, respectively). Given this intuition, the above definition makes sense – the "starting" functions in 1. through 3. are all clearly algorithmic, and any reasonable definition of an algorithm allows for composition, definition by recursion, and search.

It is this intuition, rather than the formalism of Definition II.1, that is paramount. Computability theory is guided by Church's Thesis, an informal statement that any possible (i.e., follows the intuition) definition of a computable (or partial computable) function is equivalent to the one above. This statement cannot be proven, but there is ample evidence for it, as all other proposed definitions are provably equivalent to the one above, and further, the proofs of equivalence follow a specific pattern. Sections 1.5-1.6 of Rogers [19] and Section 5.4 of [20] explain this in more detail. In practice, therefore, when determining if a function is (partial) computable, we will rely on a description of an algorithm that computes the function, rather than trying to trace everything back to the formal definition. In most cases, it will be simple to tell if something is or is not an algorithm.

For a quick and simple example of this algorithmic thinking, consider the following line of reasoning. Addition is clearly a recursion of the successor function, and multiplication a recursion of addition, so both are computable. The relation $|$ (divisibility) is computable since $n|m$ iff there exists $k \leq m$ such that $nk = m$. Therefore, the set of primes is computable – p is prime if and only if p is not 0 or 1 and the only $n < p$ such that $n|p$ is 1. All of this can be formalized according to Definition II.1, but this formalization is clearly not necessary.

Since there are countably many partial computable functions, we can enumerate them via the natural numbers. Further, this enumeration may be performed in such a way that information about the eth partial computable function can be extracted computably from e. In the remainder of this section, we sketch a way to develop such an enumeration and present (without proof) some of its useful properties. We begin with a standard method for enumerating pairs and sequences of natural numbers.

Definition II.3. 1. $\langle\cdot,\cdot\rangle : \omega^2 \to \omega$ denotes the bijection given by

$$\langle n_0, n_1\rangle = \frac{(n_0+n_1)(n_0+n_1+1)}{2} + n_1.$$

2. For $k \geq 1$, $\langle\cdot,\ldots,\cdot\rangle_k : \omega^k \to \omega$ denotes the bijection such that $\langle\cdot\rangle_1$ is the identity map, $\langle\cdot,\cdot\rangle_2 = \langle\cdot,\cdot\rangle$, and

$$\langle n_0, n_1, \ldots, n_k\rangle_{k+1} = \langle n_0, \langle n_1, \ldots n_k\rangle_k\rangle.$$

3. $Seq : \bigcup_{k>0} \omega^k \to \omega$ is the bijection given by

$$(n_0, \ldots n_{k-1}) \mapsto \langle k-1, \langle n_0, \ldots, n_{k-1}\rangle_k\rangle.$$

Proposition II.4. *1. $\langle\cdot,\cdot\rangle$ is computable, as are $\langle n_0, n_1\rangle \mapsto n_0$ and $\langle n_0, n_1\rangle \mapsto n_1$*

2. For all $k \geq 1$ and $0 \leq i < k$, $\langle\cdot,\ldots,\cdot\rangle_k$ and $\langle n_0, \ldots, n_{k-1}\rangle_k \mapsto n_i$ are computable.

3. For all k, the restriction of Seq to ω^k is computable. The function mapping n to the length of $Seq^{-1}(n)$ is computable. For all k, the function mapping n to m if m is the kth number in $Seq^{-1}(n)$ if this has length at least $k+1$ (and otherwise reports failure) is computable.

Remark II.5. By this proposition, the properties of partial computable functions are encapsulated by the properties of partial computable functions with one input (we may compose with the proper $\langle \cdot, \ldots, \cdot \rangle_k$ if need be).

Now, we will describe a method of assigning finite sequences of natural numbers to partial computable functions. For each partial function ϕ and any sequence assigned to ϕ, the first two numbers of the sequence correspond to the case from Definition II.1 by which ϕ is given[1] and the number of inputs ϕ takes, respectively. Remaining numbers in the sequence give any additional information about the definition of ϕ in a reasonable way. For example, if ϕ is the function mapping all k-tuples to m, assign to ϕ the sequence $(1, k, m)$. If ϕ takes r inputs and is defined as a composition of $\psi, \zeta_0, \ldots, \zeta_{k-1}$, to which have been assigned the sequences

$$Seq^{-1}(a), Seq^{-1}(b_0), \ldots, Seq^{-1}(b_{k-1}),$$

assign to ϕ the sequence $(4, r, a, b_0, \ldots, b_{k-1})$. Assignments in other cases are defined analogously. If a sequence is not assigned to a function in this way, it is assigned to the empty function ($\phi(n)$ diverges for all n). Note that it is possible for multiple sequences to be assigned to the same partial computable function; in fact, it is not hard to show that for every partial computable ϕ, there are infinitely many sequences assigned to it.

For $e \in \omega$, let Φ_e denote the partial computable function to which $Seq^{-1}(e)$ is assigned. We say that e is a *code*, or *index*, for Φ_e. Note that a given ϕ has infinitely many codes.

We can also use sequences of natural numbers (hence, natural numbers) to code the computation of a partial computable function, including the code of the function, all inputs, the final output, and all intermediate computation information. Contin-

[1] or more correctly, by which an algorithm for ϕ is given

uing our examples from the previous paragraph, if Φ_e is the function mapping all k-tuples to m, the computation code for $\Phi_e(n_0, \ldots, n_{k-1})$ will be $(e, k, n_0, \ldots, n_{k-1}, m)$. If Φ_e is defined as a composition of $\Phi_a, \Phi_{b_0}, \ldots, \Phi_{b_{k-1}}$ where each Φ_{b_i} takes r inputs, then then computation code for $\Phi_e(n_0, \ldots, n_{r-1}) = y$ is

$$(e, r, n_0, \ldots, n_{r-1}, c, d_0, \ldots, d_{k-1}, y)$$

where $c, d_0. \ldots, d_{k-1}$ are the (natural number) computation codes for the computations of $\Phi_a, \Phi_{b_0}, \ldots, \Phi_{b_{k-1}}$ involved in computing $\Phi_e(n_0, \ldots, n_{r-1})$.

We say that $\Phi_e(n_0, \ldots, n_{k-1})$ converges to y in s steps if $\Phi_e(n_0, \ldots, n_{k-1}) = y$ and the natural number computation code for this result is s. Abbreviate by $\Phi_{e,s}(n_0, \ldots, n_{k-1})$ the result of $\Phi_e(n_0, \ldots, n_{k-1})$ considering only computation codes up to s (if a computation code does not exist below s, then $\Phi_{e,s}(n_0, \ldots, n_{k-1}) \uparrow$). By Proposition II.4, we can "unpack" computation codes, and verify they are correct. For this reason, the enumeration Φ_e is universal, in the following respect.

Proposition II.6. *Given e, n, s, we can computably determine if $\Phi_{e,s}(n) \downarrow$*[2]*. The function which maps (e, n) to $\Phi_e(n)$ if Φ_e takes one input, and diverges otherwise, is partial computable.*

Corollary II.7. *The partial function $J(e) := \Phi_e(e)$ is partial computable.*

Remark II.8. By the remark following Proposition II.4, similar results hold if we replace one input with any fixed number of inputs. A useful property of this system is that if $\Phi_{e,s}(n) = y$, then $s \geq \max e, n, y$. The function in the corollary is called the *diagonal function*, and in Section 2.6 we will see that it provides a method for calibrating how non-computable a function or a set is.

[2] This does not mean we can computably determine if $\Phi_e(n) \downarrow$.

This proposition and the framework it represents help us to manage partial computable functions in a way that is convenient and, in a sense, algorithmic. One application of this is the definition of uniform sequences.

Definition II.9. A sequence $\{\phi_n\}_{n\in\omega}$ of partial functions is *uniformly partial computable* iff there is a computable function g such that for all n, $\phi_n = \Phi_{g(n)}$. *Uniformly computable* sequences of functions are defined similarly. A sequence $\{A_n\}_{n\in\omega}$ of subsets of ω is *uniformly computable* iff the corresponding sequence of characteristic functions is uniformly computable.

Remark II.10. Other kinds of uniform sequences will be mentioned without being explicitly defined. In each case, the definition is similar to this one, where the index of the nth element of the sequence is determined computably from n.

Uniform sequences will play an important role in constructions presented in this thesis, although at the present time we are unable to explain exactly why. We can, however, give the reader an idea of what is to come. Suppose we are trying to construct a computable function f using a sequence of partial computable functions ϕ_n. Each ϕ_{n+1} will extend ϕ_n, and f will extend all ϕ_n. In order to compute $f(m)$, we can look for an n such that $\phi_n(m)$ converges, but to be able to do this, we need the sequence of ϕ_n to be uniform. Further, the definition of ϕ_{n+1} may depend on computations involving ϕ_n. In this case, it may not enough to know that ϕ_n is computable; we may need to know an index for ϕ_n, so that we know what that function is.

We close this section by noting that Proposition II.6 can be thought of as saying all the information about an algorithm can be computably extracted from its index. This follows from Proposition II.4 since elements of a sequence can be extracted computably from the sequence. Since elements can also be put into a sequence computably, we might also suppose that the index of an algorithm depends, computably, on the information about that algorithm. Specifically, if the actions of an algorithm depend comptuably on a parameter d, so does the index of the algorithm. This is made precise by the following proposition.

Proposition II.11. *[24] (s-m-n Theorem) For each m and n, there exists a computable function f such that for all $e, x_0, \ldots, x_m, y_0, \ldots, y_n$,*

$$\Phi_e(x_0, \ldots, x_m, y_0, \ldots, y_n) = \Phi_{f(e,x_0,\ldots,x_m)}(y_0, \ldots, y_n).$$

2.2 Coding Finite Mathematical Objects

Finite objects, in general, can be coded using natural numbers in such a way that information about the object can be deduced computably from the code (and the code can be determined computably from sufficient information about the object). Before we continue, we define, for various types of finite objects, enumerations having these properties. This allows us to talk profitably about computable functions on, for example, finite subsets of ω, and show that the usual functions on them are all computable. The proofs that the functions are computable are simple, and left to the reader.

We begin with the rational numbers $\mathbb{Q}$, letting $\langle n, a, b\rangle_3$ code $(-1)^n \frac{a}{b+1}$ for $a, b, n \in \omega$. Note that in contrast to Definition II.3 this coding does not produce a bijection; however, this does not significantly alter anything.

Proposition II.12. *The following are partial computable on* $\mathbf{Q}$.

1. *Addition, subtraction, multiplication, division, and inverse.*

2. *Exponentiation (as a function on $\mathbb{Q} \times \omega$).*

3. *The relations $=$ and $<$.*

Now consider finite subsets of ω. We code $D \in [\omega]^{<\omega}$ with the number $\sum_{k\in D} 2^k$. An empty sum is taken to be 0, so 0 codes $\emptyset$.

Proposition II.13. *The following are partial computable on* $[\omega]^{<\omega}$.

1. max *and* min.

2. $D \mapsto |D|$ *(cardinality of* D*).*

3. $\{(k, D) : k \in D\}$.

4. *Union, intersection, and* $\setminus$ *(relative complement).*

5. $\{D : D$ *is an interval*$\}$.

Enumerating finite subsets of ω allows us to enumerate ${}^{<\omega}2$, the set of finite binary strings, since there is a natural correspondence between finite binary strings ending in 1 and nonempty finite subsets of ω. If D_σ denotes the set corresponding to a string σ in the way, we may code σ with the value of one subtracted from the code for D_σ.

Proposition II.14. *The following are computable on* ${}^{<\omega}2$.

1. $\sigma \mapsto |\sigma|$ *(length of* σ*).*

2. $\{(\sigma, k) : \sigma(k) = 0\}$ *and* $\{(\sigma, k) : \sigma(k) = 1\}$.

3. *Concatenation and the initial segment relation* $\subseteq$.

4. $\{\sigma : \sigma \subset A\}$ *where* $A \in {}^{\omega}2$ *is computable.*

The previous two enumerations immediately give an enumeration of the set of clopen subsets of ${}^{\omega}2$, since $C \subseteq {}^{\omega}2$ is clopen if and only if it is the union of finitely many basic open neighborhoods (each given by a finite binary string). Specifically, if $C = [D]$, we replace each $\sigma \in D$ with its code, and then code C using the code for D.

Proposition II.15. *The following are computable on clopen sets (and binary strings).*

1. *Intersection, union, complement, and* $\setminus$.

2. $\sigma \mapsto [\sigma]$.

3. $\{(\sigma, C) : [\sigma] \subseteq C\}$.

4. *Lebesgue measure* (μ).

Finally, we briefly consider S, the set of partial functions from ω to $\{0,1\}$ with finite domain. Code $s \in S$ with

$$\sum_{s(k)=0} 3^k + 2 \sum_{s(k)=1} 3^k.$$

Proposition II.16. *The following are computable on* S.

1. $s \mapsto dom(s)$.

2. $\{s : dom(s) \text{ is an interval}\}$.

3. $\{(s,t) : s, t \text{ are compatible}\}$.

4. $s \mapsto [s]$.

5. $\{(s,k) : s(k) = 0\}$, $\{(s,k) : s(k) = 1\}$, *and* $\{(s,k) : k \notin dom(s)\}$ *(as subsets of* $S \times \omega$*).*

Here s *and* t *are compatible if they agree on the intersection of their domains.*

2.3 Relative Computability and the Turing Degrees

Clearly, not all partial functions (or even all functions or sets) are partial computable, since there are only countably many computable partial functions. Therefore, the question arises, can we calibrate how uncomputable a function is? That is, given two functions f and g, when can we say that f is "more uncomputable" than g? Clearly this will be the case if g is computable and f is not. Suppose g is uncomputable, but we then modify Definition II.1 to include g "for free?" Then g will be "computable," and if f is still not "computable," then we are justified in saying f is "more uncomputable" than g. This is somewhat formalized below:

Definition II.17. Let $\phi : \omega^k \to \omega$ be a partial function. The *partial computable closure* of ϕ is the set of partial functions obtained by adding a seventh case to Definition II.1. This seventh case states that the partial computable closure of ϕ contains ϕ.

Remark II.18. Note that if ϕ is partial computable, its partial computable closure is consists of just the partial computable functions.

In almost all cases[3], we will only consider the partial computable closures of sets (identified with their characteristic functions, as usual). If ϕ is in the partial computable closure of A, we say ϕ is partial computable relative to A, or ϕ is Turing reducible to A. This is denoted $\phi \leq_T A$. We say $A \equiv_T B$ if $A \leq_T B$ and $B \leq_T A$, and $A|_T B$ if $A \not\leq_T B$ and $B \not\leq_T A$.

Our intuition is that we are modifying the idea of an algorithm by allowing it to ask, at any time, questions of the form "is n in A?" with further action by the algorithm depending on the answer received. The algorithm never "sees" A in any way beyond the questions it asks in performing a computation. For this reason, in this context A is called an *oracle*. We note that if ϕ is partial computable relative to A and if $\phi(k)\downarrow$, this computation only asks finitely many questions of the form "is n in A?" The *use* of the computation is 1 plus the maximum value of n for which such a question is asked (if no questions are asked, the use is 0). We observe that for *this*

[3]We may also consider the partial computable closures of total functions. We will not consider the partial computable closures of partial functions.

computation only, we achieve the same result whether we use as oracle A or σ, where σ is any finite initial segment of A with length at least the use of the computation.

We can modify the framework presented in Section 2.1 to account for oracle computation. To define sequences coding partial functions computable from A, we need only add a code for χ_A. To define sequences coding computations, we proceed as before, except we also have sequences coding the times when an algorithm asks "is n in A?" in which we record the value of n and the result. Denote by Φ_e^A the partial function coded in this way by e (using oracle A), and by $\Phi_{e,s}^\sigma(n)$ the result of the partial function coded by e, considering only computation codes up to s, using σ as an oracle, with the use of the computation not exceeding $|\sigma|$.

Proposition II.19. *The set of* (e, s, n, σ) *such that* $\Phi_{e,s}^\sigma(n)\downarrow$ *is computable. The function that maps* (e, n) *to* $\Phi_e^A(n)$ *(if the latter exists) is uniformly partial computable in* A *(i.e., there is an index* d *such that this function is given by* Φ_d^A *for all* A*).*

Remark II.20. Compare this proposition to Proposition II.6. For each e, the function that maps A to Φ_e^A is called a *Turing functional*.

It is clear that, as a relation on ${}^\omega 2$, $\leq_T$ is reflexive and transitive. Therefore, we can consider the set of equivalence classes of $\equiv_T$. A member of this set is called a *Turing degree*. Naturally, $\leq_T$ imposes a partial ordering $\leq$ on the set $\mathcal{D}$ of Turing degrees, and in fact $(\mathcal{D}, \leq)$ forms an upper semilattice. Specifically, if $\mathbf{a}$ is the degree of A and $\mathbf{b}$ is the degree of B, then the degree of $A \oplus B$ is the least upper bound of $\mathbf{a}$ and $\mathbf{b}$, where

$$A \oplus B = \{2n : n \in A\} \cup \{2n + 1 : n \in B\}.$$

Finally, we note that if $A \subseteq \omega$ is computable (for example, $A = \emptyset$), then although $\{\Phi_e^A\}_{e\in\omega}$ gives exactly the partial computable functions, the indexing of functions is different. Specifically, values of e which involve a call to the oracle produce the empty function when there is no oracle, and may produce something different if we use A as an oracle. In order to present a more unified approach in cases where we deal with functions which are computable and functions which are computable relative to some B, we assume that all computation is done with an oracle. Specifically, from this point forward, if we say something is computable, we are assuming the use of $\emptyset$ as an oracle, and take Φ_e to mean $\Phi_e^{\emptyset}$.

Remark II.21. The overall idea of this section, if not the specific approach, is derived from [41] and [25].

2.4 Descriptive Complexity of Sets

Consider the following question: when is it possible for an algorithm to describe a set $A \subseteq \omega$? This depends on what we mean by the word "describe." An algorithm can be said to describe A if it is an algorithm for χ_A, but there are other possibilities. For example:

Proposition II.22. *For $A \subseteq \omega$, the following are equivalent.*

1. *A is the domain of some partial computable function ϕ.*
2. *There is a computable $R \subseteq \omega$ such that $n \in A$ iff $\exists s \langle s, n\rangle \in R$.*
3. *A is empty or the range of some computable function f.*
4. *A is finite or the range of some injective computable function g.*
5. *The partial function ψ given by $\psi(n) = 1$ if $n \in A$ and $\psi(n) \uparrow$ if $n \notin A$ is partial computable.*

Proof. $1 \Rightarrow 2$: Let $\phi = \Phi_e$, and $\langle s, n\rangle \in R$ iff $\Phi_{e,s}(n)\downarrow$.

2 $\Rightarrow$ 3: If A is nonempty, choose $a \in A$. Let $f(\langle s,n\rangle) = n$ if $\langle s,n\rangle \in R$ and $f(\langle s,n\rangle) = a$ otherwise.

3 $\Rightarrow$ 4: If A is infinite, we may define $g(0) = f(0)$ and $g(n+1) = f$ applied to $\mu k[\forall m \leq n\ f(k) \neq g(m)]$.

4 $\Rightarrow$ 5: Let h be the constant function equal to 1. If A is finite, then A is computable, and ψ is partial computable since $\psi(n) = h(\mu m[n \in A])$. If A is infinite, then ψ is partial computable since $\psi(n) = h(\mu m[g(m) = n])$.

5 $\Rightarrow$ 1: Trivial. $\square$

Definition II.23. If A satisfies any (all) of the above conditions, A is *computably enumerable* (abbreviated A is c.e.). The eth c.e. set, denoted W_e, is the domain of Φ_e.

The intuition behind Definition II.23 is that A is c.e. if there is an algorithm which puts numbers into A. Such an algorithm is not allowed to take numbers out of A once they are put in, or decide that some n will never be put into A. An algorithm of this type is called an *enumeration* of A, and can be represented by a sequence $\{A_s\}_{s\in\omega}$ such that $\bigcup_{s\in\omega} A_s = A$ and $A_s \subseteq A_{s+1}$ for all s. The enumeration is computable if the set of (s,n) such that $n \in A_s$ is computable. Observe that each condition in Proposition II.22 immediately gives an enumeration of A – for example, if $A = W_e$, it is enumerated by $A_s = W_{e,s}$, where $n \in W_{e,s}$ iff $\Phi_{e,s}(n)\downarrow$.

Clearly, if A is computable, then A is c.e. Further, the two concepts do seem similar in many ways. For example, it is easy to show A is computable iff A is empty or the range of a nondecreasing computable function f iff A is finite or the range of an increasing computable function g (compare to 3 and 4 of Proposition II.22). Also, we have a simple characterization of computable sets in terms of c.e. sets.

Proposition II.24. *For $A \subseteq \omega$, A is computable iff A and $\bar{A}$ are c.e.*

Proof. The forward implication is trivial. Suppose A and $\bar{A}$ are c.e. via condition 2 of Proposition II.22, as witnessed by R and Q, respectively. Let $f(n) = \mu s[\langle s, n\rangle \in R \cup Q]$. Clearly f is partial computable, and since every n is in A or $\bar{A}$, f is total. Now note $n \in A$ iff $\langle h(n), n\rangle \in R$. □

However, it is not the case that every c.e. set is computable.

Proposition II.25. *Let $K = \{e : \Phi_e(e)\downarrow\}$. K is a c.e. set that is not computable.*

Proof. Note K is c.e. since it is the domain of the diagonal function. If K is computable, by Proposition II.24, $\bar{K} = W_e$ for some e. But then $e \in \bar{K}$ iff $e \in \mathrm{dom}(\Phi_e)$ iff $e \in K$, a contradiction. □

The c.e. set K defined in the above proposition plays an important role. It is complete, in the sense that every c.e. set is not "less computable" than K.

Proposition II.26. *If A is c.e., then $A \leq_T K$.*

Proof. Assume $A = W_e$ and let $\psi(n, x) = \mu s[n \in W_{e,s}]$. By Proposition II.11, there is a computable g such that for all n and x, $\psi(n, x) = \Phi_{g(n)}(x)$. It suffices to note that $n \in A$ iff $g(n) \in K$. □

For any oracle A, the above can be relativized to produce a definition for A-c.e. sets (by relativize, we mean replace all instances of "computable" with "computable relative to A"). We denote by W_e^A the eth A-c.e. set (the domain of Φ_e^A), and A' the set $\{e : \Phi_e^A(e)\downarrow\}$ (note that $K = \emptyset'$). As in the computable case, A' is complete with respect to A-c.e. sets. We denote by $A^{(n)}$ the result of applying $'$ to A n times. The operator $'$ (called the jump operator) behaves nicely relative to $\leq_T$.

Proposition II.27. *If $A \leq_T B$, then $A' \leq_T B'$.*

Proof. A' is A-c.e. $\Rightarrow A'$ is B-c.e. $\Rightarrow A' \leq_T B'$. □

Remark II.28. For this reason, we may view the jump operator as an operator on Turing degrees (i.e., it is well-defined).

Definition II.29. A is *low$_n$* if $A^{(n)} \leq_T \emptyset^{(n)}$, and A is *high$_n$* if $A^{(n)} \geq_T \emptyset^{(n+1)}$. In the case $n = 1$, we say simply A is *low* or A is *high*.

The intuition is that an A which is low$_n$ is close to $\emptyset$ in terms of computability (i.e., behaves like $\emptyset$ with respect to the jump operator), and an A which is high$_n$ is close to $\emptyset'$. This intuition is especially relevant when we restrict this definition to sets which are computable relative to $\emptyset'$, so that the sets that are low$_n$ or high$_n$ are the ones that take the lowest or highest possible value after n iterations of the jump.

We now turn our attention to an extension of the notion of c.e. sets, based on condition 2 of Proposition II.22. We make the following definition inductively on n.

Definition II.30. Let $A \subseteq \omega$.

1. A is Σ^0_0 iff A is computable.

2. A is Π^0_n iff $\bar{A}$ is Σ^0_n.

3. A is Σ^0_{n+1} iff there is a Π^0_n set B such that $m \in A$ iff $\exists s \ \langle s, m \rangle \in B$.

4. A is Δ^0_n iff A is Σ^0_n and Π^0_n.

5. A is *arithmetical* iff A is Σ^0_n or Π^0_n for some n.

Remark II.31. Note that Π^0_0 and Δ^0_0 are equivalent to computable, and Σ^0_1 is equivalent to c.e. These definitions relativized to A are denoted $\Sigma^0_n[A]$, $\Pi^0_n[A]$, and $\Delta^0_n[A]$.

The arithmetical sets, though they can be far from computable, are also in a sense computably describable. From the definitions, it is clear that A is arithmetical iff it is obtained by applying a finite number of quantifiers to a computable relation R. If we ignore any bounded quantifiers, the number of blocks of consecutive occurrences of $\exists$ or consecutive occurrences of $\forall$ is the n such that A is Σ^0_n (if the first unbounded quantifier is $\exists$) or Π^0_n (if the first unbounded quantifier is $\forall$). Two consequences are immediate. First, for each n, we can enumerate the Σ^0_n and Π^0_n sets effectively (by indexing the sequence of quantifiers and R). Second, since we can always put in a dummy quantifier (i.e., a quantifier for a variable that is not involved in R), for all n, "A is Π^0_n" and "A is Σ^0_n" each imply "A is Δ^0_{n+1}." These implications fail to reverse for any $n > 0$, as do the implications "A is Δ^0_n"$\Rightarrow$"A is Σ^0_n" and "A is Δ^0_n"$\Rightarrow$"A is Π^0_n." This is easily deduced from the propositions below.

Definition II.32. A Σ^0_n set A is Σ^0_n T-complete iff for all Σ^0_n B, $B \leq_T A$.

Proposition II.33. *[25] For all n, A is Σ^0_{n+1} iff A is c.e. relative to $\emptyset^{(n)}$, and $\emptyset^{(n+1)}$ is Σ^0_{n+1} T-complete.*

Proof. The proof will proceed by induction on n. For $n = 0$, this has already been established (the second statement by Proposition II.26). Now assume both statements for n. Let A be Σ^0_{n+2}, as witnessed by a Π^0_{n+1} B. By the inductive assumption, $\emptyset^{(n+1)} \geq_T \bar{B} \geq_T B$, so A is c.e. relative to $\emptyset^{(n+1)}$ (see condition 2 of Proposition II.22). The second statement is now Proposition II.26 relativized to $\emptyset^{(n+1)}$. □

Corollary II.34. *[25] For all n, A is Δ^0_{n+1} iff $A \leq_T \emptyset^{(n)}$.*

Proof. By a relativization of Proposition II.24, $A \leq_T \emptyset^{(n)}$ iff A and $\bar{A}$ are c.e. relative to $\emptyset^{(n)}$ iff A and $\bar{A}$ are Σ^0_{n+1}. □

For this reason, we may refer to all functions $\leq_T \emptyset^{(n)}$ as Δ^0_{n+1} – certainly each such function is $\equiv_T$ a Δ^0_{n+1} set. The Δ^0_2 functions are computably describable in an especially nice way.

Proposition II.35. *[48] For all f, f is Δ^0_2 iff there is a computable $g : \omega^2 \to \omega$ such that for all n, $f(n) = \lim_s g(n, s)$.*

Proof. For the forward implication, let $f = \Phi_e^{\emptyset'}$, and let $\emptyset'_s$ be a computable enumeration of $\emptyset'$. We may let

$$g(n, s) = \Phi_{e,s}^{\emptyset'_s \upharpoonright s}(n)$$

if the latter converges, and 0 otherwise. For the reverse implication, let A be the set of $\langle n, s\rangle$ such that $\forall t \geq s\ g(n, t) = g(n, s)$. Since A is Π^0_1, it is computable from $\emptyset'$. To compute $f(n)$, search for the least s such that $\langle n, s\rangle \in A$, and compute $g(n, s)$. □

Remark II.36. Equivalently, f is Δ^0_2 iff there is a sequence of functions g_s uniformly computable such that $f = \lim_s g_s$. This statement also holds for sets – i.e., A is Δ^0_2 iff there is a sequence of sets B_s uniformly computable such that $A = \lim_s B_s$.

We now close this section with a proof of a theorem due to Martin [28] closely related to the work of this thesis, using many of the ideas from this section.

Definition II.37. A function f *dominates* a function g iff for all but finitely many n, $f(n) \geq g(n)$. This is abbreviated $f \geq^* g$.

Proposition II.38. *[28] A is high iff A computes a function f which dominates every computable function.*

Proof.

Lemma II.39. *The set Tot* $:= \{e : \Phi_e$ *is total.*$\}$ *is* Π^0_2 *T-complete.*

Proof. First, Tot is Π^0_2 since $e \in$ Tot iff $\forall n \exists s\ n \in W_{e,s}$. Now suppose A is Π^0_2, and fix R computable so that $n \in A$ iff $\forall s \exists t\ \langle n, s, t\rangle \in R$. Then ψ partial computable, where $\psi(n,s) = \mu t[\langle n,s,t\rangle \in R]$. By Proposition II.11, there is a computable f such that $\Phi_{f(n)}(s) = \psi(n,s)$. We now observe $n \in A$ iff $f(n) \in \text{Tot}$. $\square$

Recall $\emptyset''$ is Σ^0_2 T-complete. Easily, it follows that $\overline{\emptyset''}$ is Π^0_2 T-complete, so $\emptyset'' \equiv_T \bar{\emptyset}'' \equiv_T \text{Tot}$. Therefore, by the relativizations of Corollary II.34 and Proposition II.35, A is high iff Tot is $\lim_s B_s$ with the B_s uniformly A-computable.

Now suppose such a sequence B_s exists. Define an A-computable $h(e,n)$ by first finding the least $t \geq n$ such that either $\Phi_{e,t}(n)\downarrow$ or $e \notin B_t$. In the former case, let $h(e,n) = \Phi_{e,t}(n)$, and in the latter case, let $h(e,n) = 0$. Note that if Φ_e is total, then for all but finitely many n, $h(e,n) = \Phi_e(n)$. Hence, $f(n) := \max_{e\leq n} h(e,n)$ dominates all computable functions.

Conversely, suppose $f \leq_T A$ dominates all computable functions. For Φ_e total, let $g_e(n) = \mu s[\Phi_{e,s}(n)\downarrow]$. Note g_e is computable – hence, so is $\tilde{g}_e(n) := \max_{m\leq n} g_e(m)$. Since every $\tilde{g}_e$ is dominated by f, Tot is the limit of B_s where $e \in B_s$ iff for all $n \leq s$ $\Phi_{e,f(s)}(n)\downarrow$. $\square$

2.5 Descriptive Complexity of Classes

We can apply a process similar to the one from the discussion following Definition II.30 to subsets of ${}^\omega 2$ to define arithmetical classes.

Definition II.40. 1. $P \subseteq {}^\omega 2$ is Σ^0_n iff for some computable R,

$$A \in P \iff \exists x_1 \forall x_2 \dots Q x_n (x_1, x_2, \dots, A \upharpoonright x_n) \in R,$$

where $Q = \exists$ if n is odd and $Q = \forall$ if n is even.

2. $P \subseteq {}^\omega 2$ is Π^0_n iff for some computable R,

$$A \in P \iff \forall x_1 \exists x_2 \dots Q x_n (x_1, x_2, \dots, A \upharpoonright x_n) \in R,$$

where $Q = \forall$ if n is odd and $Q = \exists$ if n is even.

3. $P \subseteq {}^{\omega}2$ is an *arithmetical class* iff it is Σ^0_n or Π^0_n for some n.

We will be mostly concerned with Σ^0_1 classes and Π^0_1 classes. The former are also called *c.e. open sets*, because of the following result which essentially says we may view such a set V as the result of a computable process which adds ("enumerates") basic open sets into V.

Definition II.41. A *tree* is a $T \subseteq {}^{<\omega}2$ such that $\sigma \in T$ and $\tau \subseteq \sigma$ imply $\tau \in T$. The *paths* through T are the $A \in {}^{\omega}2$ such that for all n, $A \upharpoonright n \in T$.

Proposition II.42. *$P \subseteq {}^{\omega}2$ is Σ^0_1 iff there is a computable $W \subseteq {}^{<\omega}2$ such that $P = [W]$ iff there is a c.e. $V \subseteq {}^{<\omega}2$ such that $P = [V]$. $P \subseteq {}^{\omega}2$ is Π^0_1 iff there is a computable tree T such that P is the set of paths through T iff there is a co-c.e. tree S such that P is the set of paths through S.*

Proof. The second statement clearly follows from the first, since $\bar{P}$ is Π^0_1 iff P is Σ^0_1, and if $P = [W]$, then $\bar{P}$ is the set of paths through T, where $\sigma \in T$ if it does not extend any $\tau \in W$. We also note that the first equivalence (Σ^0_1 iff equal to $[W]$ for some computable W) is simply a restatement of the definition (though it represents a different perspective that is sometimes useful).

The only case we need to consider, then, is when $P = [V]$ for a c.e. V. In this case, we need to show that $P = [W]$ for a computable W. If V_s is a computable enumeration of V, we may take W to be the set of σ such that some initial segment of σ is in $V_{|\sigma|}$. □

Remark II.43. The equivalence P is Π^0_1 iff it is the set of branches of a computable tree T bears some further remarks. By König's lemma, if T is infinite, P is nonempty. However, König's lemma does not hold computably – that is, P is not guaranteed to have a computable member. In future sections, we will see multiple natural examples of nonempty Π^0_1 classes without computable members.

The Π^0_1 classes will play an especially important role in the material which follows, since there are many cases where, for some property $\mathcal{P}$, there is a Π^0_1 class consisting of only reals with property $\mathcal{P}$. For a quick example, if ϕ is $\{0,1\}$-valued and partial computable, the set of reals extending ϕ is Π^0_1.

Because of Proposition II.42, we may effectively enumerate the Π^0_1 classes (i.e., e codes the Π^0_1 class that is the complement of $[W_e]$). Let P_e denote the Π^0_1 class coded by e, and $P_e[s]$ the complement of $[W_{e,s}]$.

Proposition II.44. *The set $\{e : P_e = \emptyset\}$ is computable relative to $\emptyset'$.*

Proof. Note that for all s, $P_e[s+1] \subseteq P_e[s]$, so the $P_e[s]$ form a nested sequence of closed subsets of a compact space (${}^\omega 2$). Therefore, P_e is empty iff some $P_e[s]$ is empty. Since $W_{e,s}$ is finite for all s, it follows that $P_e[s]$ is in fact uniformly clopen. Therefore, we may computably (from s) determine if $P_e[s]$ is empty. It follows that we may determine computably relative to $\emptyset'$ if any $P_e[s]$ is empty. □

In the cases where we want to know if an oracle A computes a real with property $\mathcal{P}$, the following notion will be useful.

Definition II.45. $\mathcal{A} \subseteq {}^\omega 2$ is a *basis* for Π^0_1 classes iff for all nonempty Π^0_1 P, there exists $A \in \mathcal{A} \cap P$.

Example II.46. It is easy to show that the Δ^0_2 reals are a basis for Π^0_1 classes. If P is nonempty, then an $X \in P$ can be constructed computably in $\emptyset'$ as follows: we fix a computable T such that P is the set of paths through T, and start with $X \upharpoonright 0 = \emptyset$. Given $X \upharpoonright n = \sigma$, we determine computably in $\emptyset'$ if there is $k > n$ such that every extension of $\sigma 0$ of length k is not in T. If so, let $X \upharpoonright n+1 = \sigma 1$, and if not, let it be $\sigma 0$. In fact, the X constructed in this way is $\equiv_T$ to a c.e. set, so the set of such reals is a basis for Π^0_1 classes.

Proposition II.47. *[22] The set $\{A : A \text{ is low}\}$ is a basis for Π^0_1 classes.*

Proof. Let P be a nonempty Π^0_1 class. We will construct a sequence of nonempty Π^0_1 classes P^e satisfying $P^0 = P$ and $P^{e+1} \subseteq P^e$, so that their intersection is nonempty. We will ensure that any member of this intersection is low by using P^{e+1} to decide $\Phi^A_e(e)$ for all $A \in P^{e+1}$.

Given P^e, consider $\tilde{P}^e = P^e \cap \{X : \Phi^X_e(e)\uparrow\}$. Since $\Phi^X_e(e)\uparrow$ iff for all s $\Phi^{X\upharpoonright s}_{e,s}(e)\uparrow$, an index for $\tilde{P}^e$ can be found computably from an index from P^e. The oracle $\emptyset'$ can determine if $\tilde{P}^e$ is empty – if so, let $P^{e+1} = P^e$ and otherwise let $P^{e+1} = \tilde{P}^e$. For $X \in \bigcap_e P^e$, $\Phi^X_e(e)\downarrow$ iff the first case occurred at step e, so X is low. □

We close this section by remarking that there is no "best" choice for a basis for Π^0_1 classes. Specifically, the intersection of all bases is the set of computable reals, which is not a basis (as witnessed by, say, the set of X such that for all e, s, $X(e) \neq \Phi_{e,s}(e)$). We will establish this in the next section, where we will exhibit a basis with no Δ^0_2 members.

2.6 Properties Which Imply Non-Computability

One of the important early questions in computability theory was Post's problem [41], which asked if there was a c.e. A such that $\emptyset <_T A <_T \emptyset'$ – that is, a c.e. Turing degree other than the minimum and the maximum. While this problem (and the further study of c.e. degrees) does not concern this thesis, the following two definitions made in an attempt to solve Post's problem do concern this thesis.

Definition II.48. [41] Let $A \subseteq \omega$ be infinite.

1. A is *immune* iff A does not have an infinite c.e. subset.

2. A is *hyperimmune* iff for any computable sequence $\{D_i\}_{i\in\omega}$ of nonempty pairwise disjoint finite subsets of ω, there exists i such that $A \cap D_i = \emptyset$.

Proposition II.49. *Every hyperimmune set is immune. No immune set is computable.*

Proof. The second statement is trivial since any computable set is c.e. For the first statement, for any infinite c.e. set that is enumerated by an injective computable f, use $D_i = \{f(i)\}$. □

Remark II.50. Although it is not vital to this thesis, we will pause to briefly explain the role of these definitions in solving Post's problem. A c.e. set A is simple (hypersimple) if its complement is immune (hyperimmune). The hope behind these definitions was that a simple/hypersimple set, while being non-computable, would also be prevented from having too much computational power due to the "thinness" of its complement. This hope was supported by known properties of $\emptyset'$. However, it is possible for a simple (or even hypersimple) set to compute $\emptyset'$. Nevertheless, it is possible to construct a low simple set, which provides one solution to Post's problem.

There is a characterization of the hyperimmune reals similar to that of the high oracles from Proposition II.38.

Definition II.51. For an infinite $A \subseteq \omega$, p_A is the increasing function that enumerates A.

Proposition II.52. *[30] A is hyperimmune iff p_A is not dominated by any computable function.*

Proof. $\Rightarrow$: Suppose p_A is dominated by a computable g. Without loss of generality, we may assume g is increasing (otherwise, replace g with $n \mapsto \sum_{m\leq n}(g(m)+1)$).

Let D_i be given by

$$D_0 = [0, g(0)]$$

$$D_{i+1} = [1 + \max D_i, g(1 + \max D_i)].$$

Observe the sequence of D_i is computable and the sets are pairwise disjoint. Also, for all but finitely many n, $n \leq p_A(n) \leq g(n)$ (the former inequality is true always), so for all but finitely many i, $A \cap D_i \neq \emptyset$. If we replace the D_i with $\tilde{D}_i := D_{i+k}$ for a sufficiently large k, the $\tilde{D}_i$ witness A is not hyperimmune.

$\Leftarrow$: Suppose A is not hyperimmune, as witnessed by a sequence of D_i. Let $g(0) = \max D_0$, and $g(n+1) = \max D_m$, where m is minimal such that D_m has no member $\leq g(n)$. Note that for all n, the interval $(g(n), g(n+1)]$ contains some D_i, and so contains some member of A. As also $g(0) \geq p_A(0)$ (since D_0 contains some member of A), it follows that g dominates p_A. □

Remark II.53. Clearly, for all A, $p_A \equiv_T A$. Therefore, B computes a hyperimmune real iff B computes a function that is not dominated by any computable function (by the trick used on g in the $\Rightarrow$ portion of the above proof, if B computes a function not dominated by any computable function, we may assume the function is increasing). For this reason, the B without this property are sometimes called *computably dominated*. We will instead use the more traditional term *hyperimmune-free*.

Hyperimmune reals have more impact on the material of this thesis than do immune reals, and have notable interactions with the concepts from the previous two sections, as witnessed by the following propositions.

Proposition II.54. *[35] If A is non-computable and Δ^0_2, then A computes a hyperimmune real.*

Proof. Let B_s be a uniform sequence of computable sets such that $\lim_s B_s = A$. The function given by $f(s) = \mu t > s[B_t \upharpoonright s = A \upharpoonright s]$ is computable in A (in fact, $f \equiv_T A$). Suppose g dominates f. Then for sufficiently large s, there is a $t \in (s, g(s)]$ such that $B_t \upharpoonright s = A \upharpoonright s$. Also, for any fixed n, for sufficiently large s, $B_t(n) = A(n)$ for all $t > s$. Consider the algorithm (computable in g) that, given n, finds $s > n$ such that $B_t(n)$ is constant for $t \in (s, g(s)]$ and returns this constant value. For sufficiently large n, this constant value must be $A(n)$, so the algorithm (up to a finite modification) gives A. It follows that $A \leq_T g$. As A is not computable, g cannot be computable. □

Remark II.55. The actual result being proven here is that if A is non-computable and Δ^0_2, then A computes a function f such that $g \geq^* f$ implies $g \geq_T f$. Such an f is called a *self-modulus*. The oracles which compute a self-modulus will be a topic of discussion in Chapter VII.

Proposition II.56. *[22] The set of hyperimmune-free reals is a basis for Π^0_1 classes.*

Proof. Let P be a nonempty Π^0_1 class. As in the proof of Proposition II.47, we will build a nested sequence of nonempty Π^0_1 classes with $P^0 = P$. Given P^e, determine if there exists n such that

$$\tilde{P}^{e,n} := P^e \cap \{X : \Phi^X_e(n) \uparrow\} \neq \emptyset.$$

Note this is a Π^0_1 class since X is in the latter set iff for all s, $\Phi_{e,s}(n) \uparrow$. If such an n exists, let $P^{e+1} = \tilde{P}^{e,n}$ for the least such n (in this case, Φ^X_e is not total for any $X \in P^{e+1}$), and if not, let $P^{e+1} = P^e$ (in this case, Φ^X is total for all $X \in P^{e+1}$).

Now, let $X \in \bigcap_e P^e$, and suppose Φ^X_e is total (so $P^e = P^{e+1}$ and Φ^Y_e is total for all $Y \in P^{e+1}$). Let T be a computable tree such that P^{e+1} is the set of branches of T. Temporarily fix n. Note that the set of $\sigma \in T$ such that $\Phi^\sigma_{e,|\sigma|}(n) \uparrow$ is a computable subtree of T. Since its set of branches is empty (any branch would be a $Y \in P^{e+1}$ with Φ_e^Y not total), this subtree is finite. That is, there exists m such that for all $\sigma \in T$ of length m, $\Phi_{e,m}^{\sigma}(n)\downarrow$. The function f that, given n, finds the least such m is clearly computable. We now note that Φ_e^X is dominated by the computable function that maps n to the maximum of $\Phi_{e,f(n)}^{\sigma}(n)$ for $\sigma \in T$ of length $f(n)$. □

Corollary II.57. *The intersection of all bases for Π_1^0 classes is the set of computable reals.*

Proof. If X is in every basis for Π_1^0 classes, then X is low (Proposition II.47) and therefore Δ_2^0 ($X \leq_T X' \leq_T \emptyset'$), and X is hyperimmune-free (the preceding proposition). Therefore, by Proposition II.54, X is computable. □

Corollary II.58. *There is no $\subseteq$-minimal basis for Π_1^0 classes.*

We now turn our attention away from immunity variants and consider another pair of related properties which will figure prominently in this thesis. As with immunity and hyperimmunity, they imply non-computability in a specific way, one related to the diagonal function $J(e) = \Phi_e(e)$.

Definition II.59. A real A is *diagonally non-computable* (or *DNC*) iff it computes a function f such that $f(e) \neq J(e)$ for all e where the latter is defined. If f is additionally $\{0,1\}$-valued, A is *PA*.

In this definition, A computes a function (or a set) whose value on input e is the witness that the function (set) is not given by Φ_e. The properties of DNC reals which are of interest to us involve algorithmic randomness and genericity, and therefore discussion of them will be postponed to the next section. The PA reals, on the other hand, are closely related to Π_1^0 classes in the following way:

Proposition II.60. *[22, 8] Let $A \subseteq \omega$. The following are equivalent.*

1. *A is PA.*

2. *For any $\{0,1\}$-valued partial computable function ϕ, A computes a total extension of ϕ.*

3. *The set of $B \leq_T A$ is a basis for Π^0_1 classes.*

Proof. $1 \Rightarrow 2$: Let $f \leq_T A$ witness A is PA. By Proposition II.11, if ϕ is partial computable, there is a computable g such that for all n and m, $\Phi_{g(n)}(m) = \phi(n)$. For all n in the domain of ϕ, $f(g(n)) \neq \phi(n)$, so $n \mapsto 1 - f(g(n))$ extends ϕ.

$2 \Rightarrow 3$: Let P be a nonempty Π^0_1 class, and T a computable tree such that P is the set of branches through T. For $i \in \{0,1\}$, let $\phi(\sigma) = i$ if there exists s such that σi has extensions in T of length s and $\sigma^\frown(1-i)$ does not (otherwise $\phi(\sigma)$ is undefined). By König's lemma, if $[\sigma] \cap P$ is nonempty and $[\sigma i] \cap P$ is empty, $\phi(\sigma) = 1 - i$. Let $f \leq_T A$ be a $\{0,1\}$-valued total extension of ϕ. We may construct $B \leq_T A$ in P by letting $B(n) = f(B \restriction n)$.

$3 \Rightarrow 1$: The set of Y such that $Y(e) \neq J(e)$ for all e is a Π^0_1 class. □

Remark II.61. Note that the third argument above immediately supplies a low PA real and a hyperimmune-free PA real.

2.7 Algorithmic Randomness

The field of algorithmic randomness has been of intense interest to computability theorists in recent years, and in this thesis we only touch on a small portion of it. We refer the reader to the texts by Nies [36] and Downey and Hirschfeldt [15] for a more detailed treatment.

There are three approaches one can make to answering the question "what makes a real random?"

- The real should not have any exceptional properties. Here, an "exceptional property" is represented by being in a measure zero subset of ${}^{\omega}2$ (according to Lebesgue measure). In this sense, a real is algorithmically random if it is not in any null set that is effectively presented.
- The real should not be easy to compress. In other words, there should be no way to describe, using relatively little information, relatively long initial segments of the real. Here, a description is represented by a function which takes (usually shorter) finite binary strings to (usually longer) finite binary strings. In this sense, a real is algorithmically random if no such function which is effectively presented consistently takes short strings to long initial segments of the real.
- The real should be unpredictable. If one bets on the next bit to appear in the real, one should not expect to win consistently. Here, betting is represented by a function from finite binary strings to positive real numbers that describes the winnings of such a gamble according to a fixed betting process. In this sense, a real is algorithmically random if no such function which is effectively presented consistently produces success (in the form of increased winnings) along the initial segments of the real.

There are therefore many ways to define an algorithmically random real, since we can take any of the above approaches, and also within each approach we have different options for what constitutes being "effectively presented." We will only concern ourselves with the first two approaches (no exceptional properties, incompressibility), and only with two notions of algorithmically random which may be defined from these approaches. The material in this section will also differ from the usual treatment of algorithmic randomness in that we will place greater emphasis on effectively presented null sets, since these sets will be of independent interest later.

The fundamental way to represent a null set N is with a sequence of open sets V_n such that for all n, $N \subseteq V_n$, and $\mu(V_n) \to 0$ as $n \to \infty$. The following definition shows two ways we can make this definition in some sense computable.

Definition II.62. [29, 46]

1. A *Martin-Löf test* is a sequence of uniformly c.e. open sets V_n such that for all n, $\mu(V_n) \leq 2^{-n}$. A *Schnorr test* is a Martin-Löf test $\{V_n\}_{n\in\omega}$ such that for all n, $\mu(V_n) = 2^{-n}$.

2. A null set $N \subseteq {}^{\omega}2$ is *Martin-Löf null* (*Schnorr null*) iff for some Martin-Löf (Schnorr) test $\{V_n\}_{n\in\omega}$, $N \subseteq V_n$ for all n.

3. A real X is *Martin-Löf random* (*Schnorr random*) iff it is not contained in any Martin-Löf (Schnorr) null set.

Remark II.63. Without loss of generality, we may assume the open sets of a Martin-Löf test are nested, since we can replace V_n with $\bigcap_{m\leq n} V_m$. Of course, no computable X is Martin-Löf or Schnorr random, as witnessed by $\{V_n\} = [X \upharpoonright n]$.

Note that $\{V_n\}_{n\in\omega}$ is uniformly c.e. open iff the set of $\langle n, \sigma\rangle$ such that $[\sigma] \subseteq V_n$ is c.e. Therefore, we can enumerate all sequences of uniformly c.e. open sets. This further allows us to effectively list all Martin-Löf tests. That is, if V_n (in some uniform sequence) would attain measure greater than 2^{-n}, halt all enumeration of basic open neighborhoods into V_n. Let $\{V^e_n\}$ denote the eth test in this listing. From this we can construct a *universal Martin-Löf test* – a Martin-Löf test $\{U_n\}_{n\in\omega}$ such that N is Martin-Löf null iff $N \subseteq U_n$ for all n. Specifically, U_n will be the union of all V^e_{n+e+1}. By way of contrast, there is no universal Schnorr test.

The existence of a universal Martin-Löf test has some interesting implications for the Martin-Löf random reals (and, by extension, the Schnorr random reals, since Martin-Löf random implies Schnorr random). Since X is Martin-Löf random iff it is outside of the intersection of the U_n iff it is outside one of them, for all n $P_n := {}^{\omega}2 \backslash U_n$ is a Π^0_1 class of measure at least $1 - 2^{-n}$ consisting of only Martin-Löf random reals. Therefore, there are Martin-Löf random reals that are low and ones that are hyperimmune-free. Also, every Π^0_1 class P with positive measure contains a Martin-Löf random real (if $2^{-n} < \mu(P)$, $P_n \cap P$ has positive measure and cannot be empty).

Both notions of an effectively null set can be expressed in multiple ways.

Definition II.64. A real number r (i.e., $r \in \mathbb{R}$) is a *computable real number* iff there is a computable sequence $\{q_n\}_{n\in\omega}$ of rational numbers such that for all n, $0 \leq r - q_n \leq 2^{-n}$. A sequence of real numbers $\{r_k\}_{k\in\omega}$ is *uniformly computable* iff the corresponding sequence of rational sequences is uniformly computable.

Proposition II.65. *[50, 47, 46]*

1. *For all $N \subseteq {}^{\omega}2$, the following are equivalent.*

   (a) N is Martin-Löf null.

   (b) There is a computable sequence of $\sigma_n \in {}^{<\omega}2$ such that

$$\sum_{n\in\omega} 2^{-|\sigma_n|} < \infty$$

   and

$$N \subseteq \bigcap_{n\in\omega} \bigcup_{m\geq n} [\sigma_m].$$

2. For all $N \subseteq {}^{\omega}2$*, the following are equivalent.*

(a) N *is Schnorr null.*

(b) There is a sequence of uniformly c.e. open sets V_n *such that* $\{\mu(V_n)\}_{n\in\omega}$ *is a decreasing uniformly computable sequence of real numbers,* $\lim_{n\to\infty}\mu(V_n) = 0$*, and* $N \subseteq V_n$ *for all* n*.*

(c) There is a computable sequence of $\sigma_n \in {}^{<\omega}2$ *such that*

$$\sum_{n\in\omega} 2^{-|\sigma_n|}$$

is a finite computable real number and

$$N \subseteq \bigcap_{n\in\omega}\bigcup_{m\geq n}[\sigma_m].$$

(d) There is a computable sequence of clopen sets $E_n \subseteq {}^{\omega}2$ *such that* $\mu(E_n) \leq 2^{-n}$ *for all* n*, and*

$$N \subseteq \bigcap_{n\in\omega}\bigcup_{m\geq n} E_m.$$

Proof. 1. (a)⇒(b): Assume N is Martin-Löf null as witnessed by a test $\{V_n\}_{n\in\omega}$. Fix $W_n \subseteq {}^{<\omega}2$ uniformly c.e. such that $V_n = [W_n]$ for all n. Let $\sigma_{\langle m,k\rangle}$ be the mth finite binary string enumerated into W_k. The sequence of σ_n is computable, and

$$\sum_{n\in\omega} 2^{-|\sigma_n|} = \sum_{n\in\omega}\mu(V_n) \leq 2.$$

(b)⇒(a): Let σ_n be a sequence of strings as described. By removing finitely many strings, we may assume

$$\sum_{n\in\omega} 2^{-|\sigma_n|} \leq 1.$$

Let W_n be the set of τ such that $\sigma_m \subseteq \tau$ for at least 2^n many values of m. Clearly, if $V_n = [W_n]$, then the V_n form a sequence of uniformly c.e. open sets.

Express V_n as the disjoint union of (possibly infinitely many) basic open sets $[\tau_k]$. Then

$$\mu(V_n) = \sum_k 2^{-|\tau_k|} \leq 2^{-n} \sum_{m,k:\sigma_m \subseteq \tau_k} 2^{-|\tau_k|} \leq 2^{-n} \sum_m 2^{-|\sigma_m|} \leq 2^{-n}.$$

Note in the second inequality, we take advantage of the disjointness of the $[\tau_k]$, implying that for a fixed m,

$$\sum_{k:\sigma_m \subseteq \tau_k} 2^{-|\tau_k|} \leq 2^{-|\sigma_m|}.$$

The proof is finished by observing that if X extends infinitely many σ_m, then for every n there is a k such that $X \upharpoonright k$ extends at least 2^n many σ_m. Hence, such an X is in every V_n.

2. (a)⇒(b): Trivial, since any computable sequence of rational numbers is also a uniformly computable sequence of real numbers.

   (b)⇒(c): Suppose we are given a sequence of V_n as stated in (b). We may move to a subsequence $\tilde{V}_n$, which satisfies the necessary properties of V_n and also $\mu(\tilde{V}_n) \leq 2^{-n}$. Specifically, if $\{q_{n,k}\}_{k\in\omega}$ approximate $\mu(V_n)$ as in Definition II.64, let $\tilde{V}_n$ be V_m, where m is minimal such that $q_{m,n+2} \leq 2^{-n-1}$. The proof now proceeds as with (a)⇒(b) of the previous item (as it is easy to verify that the sum of a sequence of uniformly computable real numbers, if finite, is a computable real number).

   (c)⇒(d): Let the sequence of σ_n be given. Let r be the sum of the $2^{-|\sigma_n|}$. Since r is computable, there is a computable function f such that for all n,

$$r - \sum_{m \leq f(n)} 2^{-|\sigma_m|} \leq 2^{-n}.$$

   Specifically, wait for the sum to become $\geq q_n$, where the sequence of q_n witnesses r is computable. Let E_n be the union of $[\sigma_m]$ for $f(n) < m \leq f(n+1)$.

(d)⇒(a): Given the sequence of E_n as described, we obtain the V_n as uniform unions of computable sequences of clopen sets $V_{n,s}$. Let $V_{n,0}$ be empty, and given $V_{n,s}$, let $V_{n,s+1} = V_{n,s} \cup E_{n+s+1} \cup C$, where C is clopen and has minimal index such that this union has measure $2^{-n} - 2^{-n-s-1}$ (by induction on s, such a C always exists since $V_{n,s}$ has measure $2^{-n} - 2^{-n-s}$ and E_{n+s+1} contributes measure at most 2^{-n-s-1}). □

Using the above proposition we can easily prove the following converse to every positive measure Π^0_1 class containing a Martin-Löf random (hence Schnorr random) real.

Proposition II.66. *If a Π^0_1 class P contains a Schnorr random real X, then $\mu(P) > 0$.*

Proof. Let $P = {}^\omega 2 \setminus [W]$, where $W \subseteq {}^{<\omega}2$ is c.e. Let W_s be a computable enumeration of W such that W_s is finite for all s, and let $P_s = {}^\omega 2 \setminus [W_s]$. Then P_s is a computable sequence of clopen sets. In particular, the sequence P_s is uniformly c.e. open and $\mu(P_s)$ is computable uniformly in s. Therefore, if $\mu(P) = 0$, $\lim_s \mu(P_s) = 0$ so by Proposition II.65 $\bigcap_s P_s = P$ is Schnorr null, a contradiction. □

The interaction between Martin-Löf random and Schnorr random (as concepts) is closely related to highness.

Proposition II.67. *[37] If X is not high, X is Schnorr random iff X is Martin-Löf random.*

Proof. The reverse implication is always true. Assume X is not high and not Martin-Löf random. For each n, $X \in U_n$, where $\{U_n\}$ is the universal Martin-Löf test. Let $U_{n,s}$ denote a uniform enumeration of the U_n. For every enumeration A_s of a c.e. set A, we can assume A_s is finite for all s. Correspondingly, we can assume $U_{n,s}$

is clopen (i.e., the union of finitely many basic open neighborhoods) for all n, s. The function f mapping n to the least s such that $X \in U_{n,s}$ is computable in X. Since X is not high, by Proposition II.38, there is a computable g such that for infinitely many n, $g(n) > f(n)$. Hence, for infinitely many n, $X \in U_{n,g(n)}$. Since $\{U_{n,g(n)}\}_{n\in\omega}$ is a computable sequence of clopen sets and $\mu(U_{n,g(n)}) \leq \mu(U_n) \leq 2^{-n}$, by Proposition II.65, X is not Schnorr random. □

In contrast to the above, when restricted to the high oracles, Martin-Löf random and Schnorr random are very different concepts. For example, it is a theorem that if A is high, there exists $X \equiv_T A$ such that X is Schnorr random and not Martin-Löf random. In particular, every high oracle computes a Schnorr random, including every high c.e. oracle. But a c.e. oracle can only compute a Martin-Löf random real if it computes $\emptyset'$. We do not prove these statements here, since the parts of them most relevant to the thesis will be implied by later results.

Another way to calibrate the difference between Martin-Löf random and Schnorr random is via initial segment complexity. This corresponds to the second approach to randomness mentioned above, the approach of incompressibility.

Definition II.68. 1. A *machine* is a partial computable function $M : {}^{<\omega}2 \to {}^{<\omega}2$. A machine M is *prefix-free* if for all $\sigma, \tau \in \mathrm{dom}(M)$, $\sigma \not\subsetneq \tau$. A machine M is *computable* iff $\mu([\mathrm{dom}(M)])$ is a computable real number.

2. For M a machine and σ a finite binary string, the *complexity* of σ relative to M, denoted $K_M(\sigma)$, is the minimum length of τ such that $M(\tau) = \sigma$ (if no such τ exists, $K_M(\sigma) = \infty$).

Since partial computable functions can be effectively enumerated, so can machines. In fact, prefix-free machines can be effectively enumerated – simply enumerate the domain of M_e, and if at any point we would enumerate an extension of a σ already in the domain, stop enumerating the domain of M_e, and alter the definition of M_e to only converge for those σ already in the domain. From this, we can define a universal prefix-free machine, and from it, a minimal complexity function.

Definition II.69. 1. Let M_e denote the effective enumeration of prefix-free machines. The *universal prefix-free machine*, denoted $\mathbb{U}$, is the machine that maps $0^e1\sigma$ to $M_e(\sigma)$ (and otherwise is not defined).

2. The *Kolmogorov complexity* of $\sigma \in {}^{<\omega}2$, denoted $K(\sigma)$, is the complexity of σ relative to $\mathbb{U}$.

Remark II.70. Kolmogorov complexity is minimal among complexity functions up to an additive constant – that is, for each prefix-free M, there is a constant d such that for all σ, $K(\sigma) \leq K_M(\sigma) + d$ (specifically, if $M = M_e$, we may use $d = e + 1$). We abbreviate this relationship by $K(\sigma) \leq^+ K_M(\sigma)$; in general, if $\mathcal{P}$ and $\mathcal{Q}$ are expressions, the notation $\mathcal{P} \leq^+ \mathcal{Q}$ indicates there is a constant d such that for all possible values of variables occurring in $\mathcal{P}$ and $\mathcal{Q}$, $\mathcal{P} \leq \mathcal{Q} + d$

A real X is compressible by a machine M if for many values of n, $K_M(X \upharpoonright n)$ is significantly less than n (M provides short descriptions for long initial segments of X). If any machine compresses X, then $\mathbb{U}$ does as well. Both Martin-Löf random reals and Schnorr random reals can be characterized as the reals which cannot be compressed by certain machines.

Proposition II.71. *[47, 14] For a real X, X is Martin-Löf random iff $K(X \upharpoonright n) \geq^+ n$, and X is Schnorr random iff for all computable prefix-free M, $K_M(X \upharpoonright n) \geq^+ n$.*

Remark II.72. In other words, Martin-Löf random reals are those that can't be compressed by any prefix-free machine, and Schnorr random reals are those that can't be compressed by computable prefix-free machines.

Proof. For any prefix-free machine M and $b \in \omega$, let R_b^M denote the set of strings σ such that $K_M(\sigma) \leq |\sigma| - b$. We observe that $K_M(X \upharpoonright n) \geq^+ n$ iff for some b, $X \notin [R_b^M]$. We will actually prove the stronger statement that N is Martin-Löf null iff for some prefix-free M, $N \subseteq \bigcap_b [R_b^M]$, and N is Schnorr null iff this is the case for some computable M. We divide the proof into two lemmas.

Lemma II.73. *For all prefix-free M, R_b^M is uniformly c.e., and $\mu([R_b^M]) \leq 2^{-b}$ for all b. Additionally, if M is computable, this measure is uniformly computable.*

Proof. That R_b^M is uniformly c.e. is easily seen from $\tau \in R_b^M$ iff there exists $\langle \tau, s \rangle$ such that $M_s(\tau) \downarrow = \sigma$ and $|\tau| \leq |\sigma| - b$.

To bound the measure of $[R_b^M]$, express this open set as the disjoint union of basic open sets $[\sigma_k]$. For each k, there is τ_k such that $M(\tau_k) = \sigma_k$ and $|\tau_k| \leq |\sigma_k| - b$. Then

$$\mu([R_b^M]) = \sum_k 2^{-|\sigma_k|} \leq 2^{-b} \sum_k 2^{-|\tau_k|} = 2^{-b} \mu\left(\bigcup_k [\tau_k]\right) \leq 2^{-b} \mu(\mathrm{dom}(M)) \leq 2^{-b}.$$

Note the second equality is due to M being prefix-free, so the $[\tau_k]$ are disjoint.

Now suppose M is computable. It follows that there is a computable f such that $\mu([\mathrm{dom}(M_{f(n)})])$ is within 2^{-n} of $\mu([\mathrm{dom}(M)])$. Let $R_{b,n}^M$ denote the set of those σ in R_b^M which are witnessed by a $\tau \in \mathrm{dom}(M_{f(n)})$. The measure of $[R_{b,n}^M]$ is rational and computable given n, uniformly in b, and by a straightforward variation of the argument above, is within 2^{-n-b} of $\mu([R_b^M])$. Hence, $\mu([R_b^M])$ is uniformly computable. □

Lemma II.74. *If $\{V_m\}_{m \in \omega}$ is a Martin-Löf test, there is a prefix-free machine M such that for all $m \geq 1$, $V_{2m} \subseteq [R_m^M]$. Further, if $\mu(V_m) = 2^{-m}$ for all m, M is computable.*

Proof. Express, uniformly in m, each V_{2m} as the disjoint union of basic open sets $[\sigma_{m,k}]$. Without loss of generality, we may assume that for all m and k, $|\sigma_{m,k}| \leq |\sigma_{m,k+1}|$ by, if need be, replacing $\sigma_{m,k+1}$ with the sequence of all extensions of $\sigma_{m,k+1}$ of length $|\sigma_{m,k}|$, removing any which equal $\sigma_{m,l}$ for $l \leq k$.

The domain of M will be divided into disjoint pieces, with the mth piece (for $m \geq 1$) consisting of strings which extend $0^{m-1}1$. The definition of M on the mth piece of the domain will ensure that $V_{2m} \subseteq [R_m^M]$ by mapping strings of length $m+s$ to $\sigma_{m,k}$ of length $2m+s$ (note that each $\sigma_{m,k}$ has length at least $2m$). M will be undefined on strings which do not have any 1's. We will describe the definition of M restricted to strings in the mth piece – this description will be uniform in m. The description will be done in stages s.

In the description below, let $\tau + n$ denote the string whose index is n plus the index of τ. Let $f(n)$ be the number of k such that $|\sigma_{m,k}| = n$, and $g(n)$ the minimal k such that $|\sigma_{m,k}| \geq n$. All three of these functions are computable (the latter two uniformly in m). All work is restricted to those τ which extend $0^{m-1}1$. Let $M_0(\tau)\uparrow$ for all τ. Given M_s, we define M_{s+1} as follows:

1. If $|\tau| \neq m+s$, $M_{s+1}(\tau)\downarrow \iff M_s(\tau)\downarrow$.
2. If $f(2m+s) = 0$, $M_{s+1}(\tau)\uparrow$ for all τ of length $m+s$.
3. If $f(2m+s) \neq 0$, let τ' have minimal index such that for all $\nu \subseteq \tau'$, $M_s(\nu)\uparrow$.
4. For $i < f(2m+s)$, let $M_{s+1}(\tau'+i) = \sigma_{m,g(2m+s)+i}$. For all other τ of length $m+s$, $M_{s+1}(\tau)\uparrow$.

Note that for all s, $\sum_{k<g(2m+s)} 2^{-|\sigma_{m,k}|} \leq 2^{-2m} - f(2m+s)2^{-2m-s}$, so the domain of M_s (on the mth piece) has measure $2^m \sum_{k<g(2m+s)} 2^{-|\sigma_{m,k}|} \leq 2^{-m} - f(2m+s)2^{-m-s}$. Therefore, there are at most $2^s - f(2m+s)$ many τ of length $m+s$ such that $M_s(\nu)\downarrow$

for some $\nu \subseteq \tau$ – that is, there are at least $f(2m+s)$ many τ for which this is not the case. This implies the above construction works – if $f(2m+s) \neq 0$, τ' exists, and for all $i < f(2m+s)$, $\tau' + i$ has length $m+s$.

Clearly the M constructed is prefix-free, and all $\sigma_{m,k}$ are in R_m^M. Further, the mth piece of the domain of M has measure $2^m\mu(V_{2m})$. If $\mu(V_{2m}) = 2^{-2m}$ for all m, then dom(M) has measure 1, which is certainly a computable real number. □

□

Remark II.75. The construction in Lemma II.74 makes the implicit assumption that there are infinitely many $\sigma_{m,k}$ for all m (else, f and g may not be computable, since we do not know if there will be another $\sigma_{m,k}$). However, the functions f and g are not strictly necessary – they merely serve as bookkeeping. The true nature of the construction is that whenever we see a string of length $2m+s$ in the list of $\sigma_{m,k}$, we find a string of length $m+s$ to be mapped to it by M. We do not need f and g to be computable to show this construction works.

Kolmogorov complexity can also be used to characterize being a DNC real.

Proposition II.76. *[23] A is DNC iff there is $f : \omega \to {}^{<\omega}2$ computable relative to A such that for all n, $K(f(n)) \geq n$.*

Proof. Let σ_i denote the string coded by $i \in \omega$.

$\Rightarrow$: Let r be an index such that Φ_r^A is total and $\Phi_r^A(e) \neq J(e)$ for all e. By Proposition II.11, there is a computable h such that

$$\Phi_{h(m)}(x) = \Phi_r^{\mathbb{U}(\sigma_m)}(x)$$

for all m, x. Let $f(n)$ equal $A \upharpoonright u$, where u is the maximum of the uses for computations $\Phi_r^A(h(m))$ for m such that $|\sigma_m| < n$. Suppose that $K(f(n)) < n$ for some n.

There is an m such that $|\sigma_m| < n$ and $\mathbb{U}(\sigma_m) = \sigma_{f(n)}$. Therefore,

$$\Phi_r^A(h(m)) = \Phi_r^{f(n)}(h(m)) = \Phi_r^{\mathbb{U}(\sigma_m)}(h(m)) = \Phi_{h(m)}(h(m)),$$

a contradiction. Hence, $K(f(n)) \geq n$ for all n, as desired.

$\Leftarrow$: Consider the prefix-free machine M which, given $0^k 1\sigma_e$, where $|\sigma_e| = k$, outputs $\sigma_{J(e)}$. For all e such that $J(e)\downarrow$, $K_M(\sigma_{J(e)}) \leq 2\log e + d$ for some constant d. Hence, $K(\sigma_{J(e)}) \leq 2\log e + d'$ for some constant d' (independent of e). Now, if $f \leq_T A$ satisfies $K(f(n)) \geq n$ for all n, then for sufficiently large n, $K(f(n)) > K(\sigma_{J(e)})$, so $f(n) \neq \sigma_{J(e)}$. Hence, if $\tilde{f}$ is the function mapping n to the index of $f(n)$, a finite modification of $\tilde{f}$ shows A is DNC. □

Corollary II.77. *If X is Martin-Löf random, X is DNC. If X is Schnorr random, X is high or DNC.*

Proof. The first statement follows from Proposition II.71, which implies the existence of a d such that $K(X \upharpoonright (n+d)) \geq n$ for all n. The second statement now follows from Proposition II.67. □

Note that the second part of the proof of Proposition II.76 does not depend on J, but works for any partial computable ϕ. Immediately, we obtain the fact that if A is DNC, then A computes a function f such that for any partial computable ϕ, $f(n) \neq \phi(n)$ for all but finitely many n (i.e., $\tilde{f}$ for an f such that $K(f(n)) \geq n$ for all n). The oracles with this property are clearly just the DNC ones. However, if we relax this property to just computable g (instead of partial computable ϕ), we also obtain the high oracles.

Proposition II.78. *[23] For any oracle A, the following are equivalent.*

1. *A is high or there is $f \leq_T A$ such that for all partial computable ϕ, $f(n) \neq \phi(n)$ for all but finitely many n.*

2. A is high or DNC.

3. There is $f \leq_T A$ such that for all computable g, $f(n) \neq g(n)$ for all but finitely many n.

Proof. $1 \Rightarrow 2$: Either A is high or a finite modification of f witnesses A is DNC (via $\phi = J$).

$2 \Rightarrow 3$: The argument preceding the proposition proves this if A is DNC. If A is high, any $f \leq_T A$ which dominates every computable function must have this property (else, f would fail to dominate $g + 1$).

$3 \Rightarrow 1$: Suppose $f \leq_T A$ has the described property and A is not high. Assume $\phi = \Phi_e$ and for infinitely many n, $f(n) = \Phi_e(n)$. Let $h(n)$ equal the minimal s such that $f(m) = \Phi_{e,s}(m)$ for $n + 1$ values of $m < s$. Clearly $h \leq_T A$, so there is $\tilde{h}$ computable such that $\tilde{h}(n) \geq h(n)$ for infinitely many n. Without loss of generality, assume $\tilde{h}$ is increasing. Let $g(n) = \Phi_{e,\tilde{h}(n)}(n)$ whenever the latter computation converges, and 0 otherwise. Note that if $\tilde{h}(n) \geq h(n)$, then for some $m \geq n$, we have $f(m) = \Phi_{e,h(n)}(m) = \Phi_{e,h(m)}(m) = g(m)$. As there are infinitely many such n, there are infinitely many such m, a contradiction. Hence, for any partial computable ϕ, $f(n) \geq \phi(n)$ for all but finitely many n. □

Corollary II.79. *If X is Schnorr random, there is $f \leq_T X$ such that for all computable g, $f(n) \neq g(n)$ for all but finitely many n.*

We introduce another concept related to randomness which will be useful in later chapters.

Definition II.80. For a real X, the *effective Hausdorff dimension* of X, denoted $\dim(X)$, is

$$\liminf_{n \to \infty} \frac{K(X \upharpoonright n)}{n}.$$

We note some simple facts about effective Hausdorff dimension before continuing. The first requires an upper bound on $K(\sigma)$.

Proposition II.81. $K(\sigma) \leq^+ 2\log|\sigma| + |\sigma|$.

Proof. Let τ_i be the string indexed by i, and M the prefix-free machine such that $M(\nu\sigma) = \sigma$ if $\tau_{|\sigma|} = \mathbb{U}(\nu)$ and otherwise diverges. Also, let N be the prefix-free machine that maps $0^{|\tau_i|}1\tau_i$ to τ_i for every i and otherwise diverges. For all i, $K(\tau_i) \leq^+ K_N(\tau_i) \leq^+ 2|\tau_i| \leq^+ 2\log i$. From M we obtain

$$K(\sigma) \leq^+ K_M(\sigma) \leq^+ K(\tau_{|\sigma|}) + |\sigma| \leq^+ 2\log|\sigma| + |\sigma|. \qquad \square$$

Using $X \upharpoonright n = \sigma$ and taking the limit as $n \to \infty$, we obtain $\dim(X) \leq 1$ for all X. Clearly, $0 \leq \dim(X)$ for all X. If $\dim(X) > 0$, then X is DNC – if $\dim(X) > 1/k$ for $k \in \omega$, then for sufficiently large n, $K(X \upharpoonright kn) \geq n$. By Proposition II.71, if X is Martin-Löf random, X has maximal dimension, as $\lim_n (n-d)/n = 1$. However, if X is Schnorr random, X can have minimal dimension, since X can fail to be DNC.

The notions of randomness and effectively null above can be relativized. In this situation, it has proved interesting to ask when relativization does not change anything. For example, are there non-computable A such that N is Schnorr null relative to A iff N is Schnorr null, and if so, how can they be characterized? This question can also be asked with "Martin-Löf" instead of "Schnorr," or with randomness instead of nullity, but it is the stated question that is of most interest to this thesis. This notion was studied by Terwijn and Zambella [54].

Definition II.82.

1. An *order* is a computable h which is nondecreasing and unbounded.

2. A *trace* with bound h (where h is an order) is a function $T : \omega \to [\omega]^{<\omega}$ such

that for all n, $|T(n)| \leq h(n)$. A trace T *traces* a function f if for all but finitely many n, $f(n) \in T(n)$.

3. An oracle A is *computably traceable* iff for some (any) order h, for every $f \leq_T A$ there is a computable trace T with bound h such that T traces f.

4. An oracle A is *low for Schnorr tests* iff every Schnorr null set relative to A is Schnorr null.

Remark II.83. It is proven by Terwijn and Zambella (and also by work done later in this thesis) that if A is computably traceable via any order, it is computably traceable via all orders; hence the lack of distinction between "any" and "some" in the definition.

Proposition II.84. *[54] A is low for Schnorr tests iff A is computably traceable.*

We omit a proof of this proposition since it is implied by work done later in the thesis.

2.8 Algorithmic Genericity

Algorithmic genericity is a concept that is in some ways similar to algorithmic randomness, and there are some notable interactions between the two, though the former is less developed. If a real being algorithmically random can be expressed as not being in certain sets which are small with respect to measure, a real being algorithmically generic can be expressed as not being in certain sets which are small with respect to category. Here, "small" may mean nowhere dense (N is nowhere dense if ${}^{\omega}2 \backslash N$ has dense interior) or meager (covered by the union of countably many nowhere dense sets). This similarity between randomness and genericity will be of much interest to us.

Definition II.85. [7]

A set $V \subseteq {}^{<\omega}2$ is *dense* iff for every σ, there exists $\tau \in V$ which extends σ.

A real X is n-generic ($n \geq 1$) iff for every Σ^0_n $V \subseteq {}^{<\omega}2$, there is $\sigma \subset X$ such that either $\sigma \in V$ or $\forall \tau \supseteq \sigma$, $\tau \notin V$.

A real X is weakly n-generic ($n \geq 1$) iff for every dense Σ^0_n $V \subseteq {}^{<\omega}2$, there is $\sigma \subset X$ such that $\sigma \in V$ (equivalently, $X \in [V]$).

Remark II.86. In light of the discourse above, note that the definition of weakly n-generic is one that says X must be in certain open dense sets. No computable X is n-generic or weakly n-generic, at witnessed by $V = \{\sigma : \sigma \not\subseteq X\}$.

Note that in the definition of n-generic, if V is dense, then there cannot exist a σ without an extension in V. It follows that n-generic implies weakly n-generic for all n. Also, if V is Σ^0_n, the set of σ with no extension in V is Π^0_n, so the union of this set with V is a dense Σ^0_{n+1} set. It follows that weakly $(n+1)$-generic implies n-generic for all n. We will focus on reals which are weakly 1-generic, 1-generic, and 2-generic.

Algorithmic genericity is a property that tends to be orthogonal to the properties we have already discussed. We give here some examples.

Proposition II.87. *[9] No 1-generic real is DNC.*

Proof. Suppose that for some index d and 1-generic real X, $\Phi^X_d(e) \neq J(e)$ for all e (with the former function total). Let

$$V = \{\sigma : \exists s, e \ \Phi^\sigma_{d,s}(e) \downarrow = \Phi_{e,s}(e) \downarrow\}$$

By assumption, no $\sigma \subset X$ can be in V, so fix $\sigma \subset X$ such that $\tau \notin V$ for all τ extending σ. Define a computable g so that $g(d)$ is $\Phi^\tau_{e,|\tau|}(d)$, where τ is the extension of σ with minimal index such that this computation converges (since Φ^X_d is total, such a τ must exist). But if $g = \Phi_d$, then for some τ and $s \geq |\tau|$, $\Phi^{\tau}_{e,s}(d) = g(d) = \Phi_{d,s}(d)$, a contradiction. □

Corollary II.88. *No 1-generic real computes a Martin-Löf random real.*

Proposition II.89. *No 2-generic real is high.*

Proof. Suppose that d is an index such that for a 2-generic X, Φ^X_e is total. We will construct a computable g such that Φ^X_e does not dominate g. Let

$$V = \{\sigma : \exists n \forall \nu \supseteq \sigma \ \Phi^{\nu}_{e,|\nu|}(n) \uparrow\}.$$

The totality of Φ^X_e implies that no initial segment of X is in V. By 2-genericity, fix $\sigma \subset X$ such that for all τ extending σ and all n, there is $\nu \supseteq \tau$ such that $\Phi^{\nu}_{e,|\nu|}(n) \downarrow$.

We let τ_n be an enumeration of the strings extending σ (i.e., τ_n is σ concatenated with the string indexed by n). To define g, let m be minimal such that $\nu := \tau_m$ extends τ_n and $\Phi^{\nu}_{e,|\nu|}(n) \downarrow$, and let $g(n)$ be 1 plus the output of this computation. Now consider

$$V_m := \{\sigma' : \exists s, n \geq m \ \Phi^{\sigma'}_{e,s}(n) \downarrow < g(n)\}$$

By the definition of g, for every m and $\sigma' \subset X$, σ' is an initial segment of some τ_n for $n \geq m$, and hence also an initial segment of some ν such that $\Phi^{\nu}_{e,|\nu|}(n) \downarrow < g(n)$. This ν is in V_m. Thus, by genericity, for all m some $\sigma' \subset X$ is in V_m. But this implies that for all m there is $n \geq m$ such that $\Phi^X_e(n) < g(n)$ – that is, Φ^X_e doesn't dominate g, as desired. □

Corollary II.90. *No 2-generic real computes a Schnorr random real.*

In contrast, weak 1-genericity dovetails perfectly with hyperimmunity, due to the following weakening of a theorem of Kurtz.

Proposition II.91. *[27] A computes a weakly 1-generic real iff A computes a hyperimmune set.*

Proof. $\Rightarrow$: We show that if X is weakly 1-generic, then X is hyperimmune (when viewed as a subset of ω). For any $m \in \omega$ and computable f, the set of Y such that $p_Y(n) \geq f(n)$ for some $n \geq m$ is c.e. open and dense (specifically, given σ, $[\sigma(0^r1)^{|\sigma|+n+1}] \subseteq$ this set, where $r = f(|\sigma| + n) + 1$). It follows that X is in every such set, so p_X is not dominated by any computable function.

$\Leftarrow$: Let $g \leq_T A$ not be dominated by any computable function, and without loss of generality, assume g is nondecreasing. We construct a weakly 1-generic $X \leq_T A$ as the union of an A-uniform sequence $\{\sigma_n\}_{n\in\omega}$ such that $\sigma_0 = \emptyset$ and $\sigma_{n+1} \supseteq \sigma_n$. View W_e, the eth c.e. set, as a subset of ${}^{<\omega}2$. We aim to fulfill the requirements[4]

$$R_e : W_e \text{ is not dense or } X \in [W_e]$$

We keep track of a list of for which e has R_e been satisfied (this list starts empty). Given σ_n, search for the minimal $e \leq n$ such that R_e is unsatisfied and there exists τ such that $|\tau| \leq g(n)$, $\tau \supseteq \sigma_n$, and $\tau \in W_{e,g(n)}$ (say "R_e needs attention"). If no such e exists, $\sigma_{n+1} = \sigma_n$. Otherwise, for the least e such that R_e needs attention, fix τ with minimal length satisfying the above statements. If $|\tau| \leq n+1$, let $\sigma_{n+1} = \tau$ and add e to the "satisfied" list, and otherwise, let $\sigma_{n+1} = \sigma_n$.

We now show that if W_e is dense, there is n such that $\sigma_{n+1} \in W_e$. This will prove $X := \bigcup_n \sigma_n$ is weakly 1-generic (and that X is a real and not a finite binary segment, since for any k, W_e can consist of the strings of length at least k). There is m such that by stage m of the construction, for all $d < e$ either R_d is satisfied or R_d will never be satisfied. We note that the function

$$f(n) = \mu s[\forall \sigma \in {}^n2\ \exists \tau \supseteq \sigma\ |\tau| \leq s \wedge \tau \in W_{e,s}]$$

[4]This "requirement" framework is common in computability theory proofs.

is computable, and hence there is $m' \geq m$ such that $g(m') \geq f(m')$. At this stage, R_e needs attention (since $|\sigma_{m'}| \leq m'$). For $n \geq m'$, if R_e is not yet satisfied and still needs attention, either $\sigma_{n+1} = \sigma_n$, so R_e still needs attention, or the choice of σ_{n+1} satisfies R_e (it cannot be chosen to satisfy R_d for $d < e$ by assumption, or satisfy R_d for $d > e$ since such a d would not be minimal such that R_d needs attention). If n is large enough to be greater than the length of a τ witnessing that R_e needs attention, then the latter will be the case. This has to happen eventually, so there is n such that $\sigma_{n+1} \in W_e$, as needed. □

We end this chapter with one more concept related to genericity which will be relevant to the thesis work.

Definition II.92. An oracle A is *low for weak 1-genericity* iff every weakly 1-generic X is weakly 1-generic relative to A.

These oracles are nicely characterized by the following theorem of Stephan and Yu. However, their proof involves concepts which will not directly influence the remainder of the thesis. We do not wish to provide exposition for these concepts, and so present their result without proof.

Proposition II.93. *[52] For any A, the following are equivalent.*

1. *A is low for weak 1-genericity.*

2. *Every dense c.e. open set relative to A has a dense c.e. open subset.*

3. *A is hyperimmune-free and not DNC.*

CHAPTER III

Set Theory Background

In this section, we provide some exposition to certain areas of set theory in order for the reader to understand the motivation of the thesis work, and the sources of the proofs contained herein. As this is a thesis in computability theory, and not set theory, most of this material is covered in significantly less detail than the material of the previous chapter. We refer the reader to [21], [16], or Chapter 6 of [20] for a more thorough treatment. Also, [6] and [4] are good sources for additional information on cardinal characteristics of the continuum, and how they are affected by forcing.

3.1 Fundamentals of Set Theory

In set theory, we consider sets to encompass all mathematical objects, with $\in$ a binary relation on sets (i.e., the elements of a set are themselves sets). The collection of all sets is called the *universe*. We emphasize that not all collections of sets are themselves sets. If this were the case, there would be a set A consisting of all sets x such that $x \notin x$. Since $A \in A \iff A \notin A$, we have a contradiction, known as Russell's paradox. We call a collection of sets a *class*. A class is a *proper class* if it is not a set (the universe itself is one example).

We assume the universe satisfies the axiom system ZF. Axioms of ZF fall into two types. The first type describe the behavior of sets. For example, a set is determined exactly by its elements. For another example, $\in$ is a well-founded relation on sets (that is, there is no infinite sequence x_i such that $x_{i+1} \in x_i$ for all i). The second type describe the existence of certain kind of sets. For example, if x is a set, so is $\mathcal{P}(x)$ (defined formally as the class of y such that for all z, $z \in y \Rightarrow z \in x$). For another example, if u is a set and F is an operation mapping sets y to the class of sets z such that $\phi(y, z)$ holds (where ϕ is a formula of first-order logic, i.e., built from $\in$, propositional connectives, and quantifiers $\exists$ and $\forall$ ranging over sets), then the class of those $F(y)$ for $y \in u$ which are sets is itself a set. In other words, if we apply the same operation to every member of a set, we get another set.

Set theory is of interest in part because nearly all of mathematics can be represented in set theory. In some cases, this can be done directly. For example, a topological space is just a set T and a set $\mathcal{O} \subseteq \mathcal{P}(T)$ such that $\mathcal{O}$ satisfies the axioms assumed for open sets. We can then formalize statements about topological spaces entirely in terms of sets (though to prove some of the theorems, we may need stronger axioms than those in ZF).

For a second example which is simpler (though less direct), we may define the ordered pair (x, y) to be the set $\{\{x\}, \{x, y\}\}$ (that is, in ZF, it is provable that this is a set whenever x and y are sets). A relation is then a set of ordered pairs, and a relation f is a function if for all x, y, z, if $(x, y), (x, z) \in f$, then $y = z$ ($(x, y) \in f$ meaning $y = f(x)$). The domain of f is the set of x such that $(x, y) \in f$ for some y, and similarly for the range of f. The set of functions from x to y is denoted ${}^{x}y$ (hence our use of the notation ${}^{\omega}2$ and ${}^{\omega}\omega$). Continuing, we can make formal definitions for concepts involving functions, and prove basic facts in ZF (e.g., if f is injective, there exists g such that $g(f(x)) = x$ for all x in the domain of f). We note that for functions f and g, g extends f iff $f \subseteq g$ (this explains our choice of notation for denoting initial segments of finite binary strings, which can be viewed as functions).

For a third example, the natural numbers can be represented by sets in the following way:

$$0 = \emptyset$$

$$1 = \{0\}$$

$$2 = \{0, 1\}$$

$$3 = \{0, 1, 2\}$$

This representation has several notable features. Each $n \in \omega$ is also a subset of ω. For n, m in ω, $n \in m$ is equivalent to $n < m$, and implies $n \subset m$. The successor of n is $n \cup \{n\}$. From this representation, we can define addition and multiplication as set functions, and begin to formulate statements in number theory. Going further, we can represent rational numbers as equivalence classes of pairs of elements of ω, real numbers as equivalence classes of Cauchy sequences of rational numbers, etc.

The above framework can also be extended to cover the ordinal numbers. Formally, a set is *transitive* if every element is also a subset. A set is an *ordinal* if it is a transitive set whose elements are also transitive. Each ordinal is an initial segment of the ordinals, as α contains all $\beta < \alpha$. We note that the proper class of all ordinals is well-ordered by $\in$ (i.e., it is a well-founded linear order), so each ordinal can be well-ordered within the universe. That is, for each ordinal α, there is a *set* of ordered pairs (β, γ) with $\beta, \gamma \in \alpha$ such that if $\beta \leq \gamma$ iff the ordered pair (β, γ) is in the set, then $\leq$ is a well-order. This set will consist of (β, γ) such that $\beta \subseteq \gamma \in \alpha$.

Note that if α is an ordinal, so is $\alpha \cup \{\alpha\}$. We call this the *successor of* α (denoted $\alpha + 1$). An ordinal is a *successor ordinal* if it is the successor some α, and a *limit ordinal* if this is not the case. For example, ω is a limit ordinal. To define an operation F recursively on the ordinals, it suffices to define $F(0)$, $F(\alpha + 1)$ given α, and for limit ordinals α, $F(\alpha)$ given $F(\beta)$ for $\beta < \alpha$. We will use this framework in the next section to introduce the standard notation for cardinals.

3.2 Choice, Cardinals, and the Continuum

A fundamental assumption most mathematicians make is the axiom of choice. One way to state this in the language of set theory is that for any set x, there is a function f such that $x \subseteq \mathrm{dom}(f)$ and for all $y \in x$, if $y \neq \emptyset$, $f(y) \in y$. In other words, given any set of choices to make, we can (within the universe) make those choices. One application of this axiom comes from the usual proof that there is a subset of $\mathbb{R}$ which is not Lebesgue measurable. At one point, the proof considers a set of equivalence classes of real numbers, and by the axiom of choice, assumes that there is a set of real numbers which contains exactly one number from each equivalence class.

In ZF, the axiom of choice can be described in terms of ordinals.

Theorem III.1. *(ZF) The following are equivalent.*

1. *The axiom of choice.*

2. *For all x, there exists a relation r which well-orders x.*

3. *For all x, there exists a bijective function f from x to an ordinal α.*

We pause to note that although the axiom of choice seems a reasonable mathematical assumption to make, it is not a consequence of ZF (i.e., assuming there is a model of ZF, there is a model of ZF where the axiom of choice fails). Therefore, for some purposes, the axiom system ZFC (ZF plus the axiom of choice) is a better model for mathematical thought. The remainder of this section will focus on ZFC and its models.

An ordinal κ is a *cardinal* if there do not exist f and $\beta < \kappa$ such that $f : \kappa \to \beta$ is a bijection. Note that in ZFC, for all x, the set of ordinals which can be mapped bijectively to x is nonempty, and thus has a minimal element κ (we denote this by $|x| = \kappa$). Clearly, κ is a cardinal. This gives us a way to interpret the cardinals as giving the possible sizes of sets, in the sense that there is a bijection $f : x \to y$ iff $|x| = |y|$.

Given an x, let $Ha(x)$ be the set of ordinals α such that there is an injection $f : \alpha \to x$. Note that $Ha(x)$ is an initial segment of the ordinals, so it is an ordinal. Given that, it is clear that $Ha(x)$ is a cardinal for all x, and if κ is a cardinal, $Ha(\kappa)$ is the least cardinal bigger than κ (for example, $Ha(\omega)$ is the smallest uncountable cardinal). Using this, we "enumerate" the infinite cardinals using ordinals:

$$\aleph_0 = \omega$$
$$\aleph_{\alpha+1} = Ha(\aleph_\alpha)$$
$$\aleph_\alpha = \bigcup_{\beta<\alpha} \aleph_\beta \text{ for limit } \alpha^1.$$

We now wish to turn our attention to the continuum. The use of "the" here is somewhat misleading, as there are multiple sets that can be referred to as the continuum: for instance $\mathbb{R}$, $[0,1]$, $[\omega]^\omega$, $\mathcal{P}(\omega)$, ${}^\omega 2$, and ${}^\omega\omega$. However, they are similar enough that we can call each "the continuum" (we have already seen this to some extent via our identification of $\mathcal{P}(\omega)$ with ${}^\omega 2$ in the previous chapter). They all have the same cardinality, for example. Further, any pair of these sets (viewed as topological spaces) become homeomorphic after removing a countable set from each. In particular, there are maps between any pair that are "almost bijective" and preserve properties like nowhere dense and meager. In each case where there is a natural measure (such that the measure of the whole space is bounded by 1), these maps can also be made to preserve measure, so they preserve the property of being null. This justifies our practice of calling each the continuum.

For cardinals κ and λ, we let κ^λ denote the cardinality of ${}^\lambda\kappa$. Therefore, $2^{\aleph_0}$ is the cardinality of the continuum (sometimes this is also denoted $\mathfrak{c}$). It is simple to show that $\aleph_0 < 2^{\aleph_0}$, and it was conjectured by Cantor that the continuum was the smallest set bigger than ω – in modern terms, that $\aleph_1 = 2^{\aleph_0}$. This conjecture is known as the continuum hypothesis, abbreviated CH. It has been shown that CH is independent of ZFC; in particular, there are models of ZFC where CH fails. We will indicate (without full rigorous detail) in the next section how this was proven.

3.3 Forcing

Cohen [12, 13] established that ZFC does not prove CH using forcing, the name given to a general method for constructing, given a model V of ZFC, another model $V[G]$ of ZFC, by adding new sets to V. By using this process, we may change the truth value of certain statements in V. For example, if V is a model of ZFC+CH (such a model can be constructed by methods other than forcing), and $V[G]$ is a model obtained via a forcing that adds $\aleph_2$ reals to V without changing any cardinals, then $V[G]$ satisfies ZFC+$(2^{\aleph_0} = \aleph_2)$. However, we should be careful in making such statements, since our intuition was that the universe already contained "all" sets. So to carry out forcing, we (roughly speaking) assume we work within the "real" universe, and begin with a model V of ZFC (which may be a class or a set in the real universe). The sets we add to V, although they were not in V, are in the real universe, and therefore, we can describe and study them. This outline, and the remainder of the material in this section, should be read remembering that we are not attempting to rigorously define forcing, but rather, describe its methods and properties to such a degree that we may understand how they apply to computability theory (and in particular to the subject of this thesis).

Suppose we have such a V, which we will call the *ground model*. A *notion of forcing* is a partial order $(P, \leq)$ in V – that is, P and $\leq$ are both elements of V such that $\leq$ partially orders P. We call the elements of P *conditions*. A set $D \subseteq P$ in V is *dense* if for every $p \in P$ there is $q \leq p$ such that $q \in D$. A *generic filter* is a $G \subset P$ (not necessarily in V!) such that

1. For all $p, q \in G$, there is $r \leq p, q$ in G.

2. For all $p \in G$, if $p \leq q$, $q \in G$.

3. For all dense D in V, there is $p \in G \cap D$.

We now sketch the construction of $V[G]$, given G a generic filter. We associate a set of *names* to $(P, \leq)$. We will not concern ourselves with the exact structure of these names. The basic idea is that a name a is an element of V which contains instructions for how to interpret it based on which $p \in P$ are in G. Then the elements of $V[G]$ will be names interpreted via G. For the particular way that we define the names, each element x of V has a name that is interpreted as x regardless of which p are in G, so that $V \subseteq V[G]$ (in the real universe). Also, there will be a name whose interpretation will be G, so that this containment is strict[2].

From the details of the construction, it is possible to show that $V[G]$ is a model of ZFC. We also have the useful concept of a condition *forcing* a statement. Let $\phi(x_1, \ldots, x_k)$ be a formula of set theory, $a_1, \ldots, a_k$ a list of names, and p a condition. Then p forces $\phi(a_1, \ldots, a_k)$ if whenever G is a generic filter containing p, $\phi(a_1, \ldots, a_k)$ holds in $V[G]$. Although this is a statement involving truth in $V[G]$, it can repre-

[2]We will not concern ourselves with the question of whether or not G exists, even in the real universe. With enough work, such a G can always be found for the notions of forcing involved in this thesis.

sented by a formula in V (i.e., V can tell if p forces $\phi(a_1, \ldots, a_k)$). Also, $\phi(a_1, \ldots, a_k)$ is true in $V[G]$ iff there is $p \in G$ which forces $\phi(a_1, \ldots, a_k)$.

To see an example of this, we consider Cohen forcing. The notion of forcing is $({}^{<\omega}2, \supseteq)$. Let G be a generic filter for this notion of forcing, and let $X = \bigcup_{p \in G} p$ as an element of $V[G]$. Note that X is a partial function from ω to 2 (particularly, for $n \in \text{dom}(X)$, $p(n)$ and $q(n)$ cannot have different values for $p, q \in G$). Actually, $X \in {}^{\omega}2$ – for any n, the set of p such that $n \in \text{dom}(p)$ is dense in ${}^{<\omega}2$ (in the ground model). Further, if $Y \in {}^{\omega}2$ is in the ground model, $X \neq Y$, since the set of p such that $p(n) \neq Y(n)$ for some n is also dense. Therefore, this forcing adds a new real to ${}^{\omega}2$. If we repeat this $\aleph_2$ times,[3] in the resulting model there are $\aleph_2$ reals. Of course, when we say $\aleph_2$ in this context, we mean $\aleph_2$ as interpreted in V. However, this forcing does not change the values of any cardinals. Therefore, we have produced a model of ZFC where $2^{\aleph_0} = \aleph_2$.

We note that in this example, we have not explicitly used the idea of a condition forcing a statement. This idea is present nonetheless, as it is necessary to show $V[G]$ is a model of ZFC, and has the same cardinals as V. In the next section, we will show another example of a statement true in the $V[G]$ obtained by Cohen forcing which illustrates explicitly the usefulness of forcing.

3.4 Forcing in Computability Theory

Although primarily a set theoretic method, forcing can also be adapted to prove results in computability theory. In fact, forcing in computability theory technically predates forcing in set theory (see the Kleene-Post construction of incomparable Turing degrees [25], or Spector's construction of a minimal Turing degree [51]), although it was not formalized until its use in set theory.

[3]We need to take care doing this, but such care is beyond the scope of this thesis

Forcing in computability theory can be based on any notion of forcing which adds new reals to the ground model. The conditions are assumed to be computable, or at least computably described, as are the dense sets of conditions. The computable reals take the place of the ground model, and the reals computable relative to some oracle A take the place of $V[G]$. The Turing programs Φ_e^A take the place of names. We no longer have general theorems about conditions forcing statements, but in certain cases we can recapture results if the forcing of a statement can be verified sufficiently computably. In this section, we will indicate specific proofs involving computable forcing which will illustrate these points more clearly.

Forcing proofs in computability theory fall into two categories: constructive and nonconstructive (this mirrors the situation in set theory, where a proof involving a forcing extension may involve constructing it, or, knowing the construction is possible, producing a nonconstructive proof of certain properties it has). In the constructive proofs, we think of conditions as approximations of some A we are constructing. Given a condition p at some stage of the construction, we find $p' \leq p$ which forces some statement about A (i.e., we can verify that for all $\tilde{A}$ which p' approximates, the statement is true for $\tilde{A}$). To guarantee we can do this, we need to verify that the set of conditions which can force the desired statement is dense. Applying this process ω many times, we obtain an A with desired properties (which may depend on the overall relative computability of the construction – i.e., what oracle B is needed to always find the p' given p). Note that we could describe such a proof as finding a set of conditions G that meets sufficiently many dense sets – finding a G that is sufficiently close to the definition of a generic filter.

For an example of this, we point the reader to the proof of the Low Basis Theorem (Proposition II.47). There, our conditions were Π_1^0 classes (i.e., computable trees) with $Q \leq P$ if $Q \subseteq P$. We were interested in classes P which could force either $\Phi_e^A(e)\downarrow$ or $\Phi_e^A(e)\uparrow$ (i.e., this is true for all $A \in P$). We showed that for any e, the set of such P was dense and constructed a set of conditions which met all of them (for any P^e, we could define a $P^{e+1} \subseteq P^e$ which forced one of the two statements). The A desired was then just the intersection of all the conditions obtained. That A was low depended on $\emptyset'$ being sufficient to compute a sequence of indices for the sequence of Π_1^0 classes (we note that the sequence of indices being incomputable is analogous to a ground model V not containing a generic filter G).

Another example that is somewhat more in a "set theoretic style" is the proof of the Hyperimmune-free Basis Theorem (Proposition II.56). As with the Low Basis Theorem, we force with Π_1^0 classes, except now, we want to force some $\phi \leq_T A$ to be total or not total. Just as any element of a forcing extension is given by a name, such a ϕ must be given by Φ_e^A for some e. Therefore, it suffices to show that given e and a Π_1^0 class P, we can find a Π_1^0 class $P' \subseteq P$ such that either P' forces Φ_e^A to be total or P' forces Φ_e^A to not be total. In the former case, using the representation of P' as the paths through a computable tree, we can show Φ_e^A is dominated by a computable function.

This last portion of the argument points the way to a consideration of nonconstructive forcing proofs. For a notion of forcing $(P, <)$ and a member x of the generic extension $V[G]$, any statement about x must be forced by a $p \in P$. We can then exploit the fact that $p \in V$ to prove that x must have certain properties relative to the ground model. Take, for example, $(P, <)$ to be some notion of forcing, and consider $f \in {}^\omega\omega$ an extension $V[G]$. There is a name a for f, and a condition p such that p forces "a is a name for a member f of ${}^\omega\omega$." Further, for each n, we can find $p_n < p$ such that p_n forces "$f(n)$ is defined." Perhaps by examining p_n from the ground model, we can determine an upper bound for the value of $f(n)$ (i.e., an upper bound for the m such that "$f(n) = m$" can be forced). If so, we can define a ground model function g which dominates f, proving that every function in the extension is dominated by a ground model function. The arguments are similar to the ones employed to show that if P is a Π^0_1 class forcing "Φ^A_e is total," then there is a computable function dominating Φ^A_e.

Nonconstructive forcing proofs in computability theory work in much the same way, although sometimes more care is needed. Given a real with "sufficient genericity" relative to a computable version of a notion of forcing, any statement about it is forced by a condition. By computably manipulating this condition, we can show the real has certain properties relative to the computable reals. The main difference is that, say, we may need to define a function computably, which is a more restrictive notion than defining a function in a model of ZFC.

We take as an example the proof that no 2-generic real is high (Proposition II.89). This is the computable version of the argument that if $V[G]$ is a generic extension for the Cohen forcing, and $f \in {}^\omega\omega$ is in this extension, then f does not dominate all ground model functions. We compare and contrast the salient features of these proofs in the table below.

Given	$f \leq_T A$	$f \in {}^\omega\omega$ in $V[G]$
there is	an e with $f = \Phi_e^A$.	a name a for f.
Since	A is 2-generic,	$V[G]$ is a generic extension,
there is a condition	$\sigma \subset A$	$\sigma \in G$
which forces the statement "$f \in {}^\omega\omega$." Enumerate the $\tau \supseteq \sigma$.		
Construct a g which is	computable	in the ground model
by finding, for each τ_n, a $\nu_n \supseteq \tau_n$ such that		
for some m, ν_n forces	$\Phi_e^A(n) \downarrow = m$	$f(n) = m$
and let $g(n)$ be $1 + m$. There is no		
$\tau \supseteq \sigma$ such that	$\tau \subset A$	$\tau \in G$
and m such that τ forces	$\Phi_e^A(n) \geq g(n)$	$f(n) \geq g(n)$
for all $n \geq m$. Hence, the opposite must be forced, and it follows that		
g is a	computable	ground model
function not dominated by f.		

The computability theoretic proof is very nearly a direct translation of the set theoretic proof. There are two major differences. First, we do not have access to general forcing theorems, so we had to make specific arguments based on 2-genericity to say that certain statements were forced. Second, we had to make sure that g was defined computably, not just defined. For these reasons, not all set theoretic proofs of this kind can be translated. Many, however, can be if sufficient care is taken. We will see additional proofs of this kind (as well as constructive forcing proofs) later in the thesis.

3.5 Cardinal Characteristics of the Continuum

Before we can have any discussion of cardinal characteristics of the continuum, we must first acknowledge the difficulty that there is no formal definition for this term. Informally, a cardinal characteristic is a cardinal (whose value may vary between different models of ZFC) which describes some property of the continuum. In (somewhat) more detail, let $\mathcal{O}$ be some property of infinite cardinals such that $\neg\mathcal{O}(\aleph_0)$, $\mathcal{O}(\mathfrak{c})$, and if $\mathfrak{c} \geq \kappa \geq \lambda \geq \aleph_0$, then $\mathcal{O}(\lambda) \Rightarrow \mathcal{O}(\kappa)$. Then the minimal cardinal κ such that $\mathcal{O}(\kappa)$ can be thought of as marking the boundary between cardinals that "behave like $\aleph_0$" from those that "behave like $\mathfrak{c}$" (with respect to the property $\mathcal{O}$). For a more concrete idea, consider the following examples.

Example III.2.

- $\mathcal{O}_1(\kappa)$ iff there is an unbounded family $\mathcal{B} \subseteq {}^\omega\omega$ such that $|\mathcal{B}| \leq \kappa$. $\mathcal{B}$ is unbounded if there is no $f \in {}^\omega\omega$ that dominates every $g \in \mathcal{B}$.

- $\mathcal{O}_2(\kappa)$ iff there is a family $\mathcal{F}$ of null subsets of ${}^\omega 2$ whose union is not null such that $|\mathcal{F}| \leq \kappa$.

- $\mathcal{O}_3(\kappa)$ iff there is a splitting family $\mathcal{S} \subseteq [\omega]^\omega$ such that $|\mathcal{S}| \leq \kappa$. $\mathcal{S}$ is splitting if for every $R \in [\omega]^\omega$, there is $S \in \mathcal{S}$ such that $|R \cap S| = |R \backslash S| = \aleph_0$ (S splits R).

- $\mathcal{O}_4(\kappa)$ iff there is a family $\mathcal{X} \subseteq [\omega]^\omega$ such that $\{Y : X \subseteq Y \text{ for some } X \in \mathcal{X}\}$ is an ultrafilter and $|\mathcal{X}| \leq \kappa$. $\mathcal{U} \subseteq \mathcal{P}(\omega)$ is an ultrafilter if it is closed under supersets and finite intersections, and for all $A \subseteq \omega$, $A \in \mathcal{U}$ or $\bar{A} \in \mathcal{U}$.

It is easy to verify that for $1 \leq i \leq 4$, $\neg\mathcal{O}_i(\aleph_0)$ and $\mathcal{O}_i(\mathfrak{c})$. Let $\mathfrak{b}$, $\mathbf{add}(\mathcal{N})$[4], $\mathfrak{s}$, and $\mathfrak{u}$ denote the least κ such that (respectively) $\mathcal{O}_1(\kappa)$, $\mathcal{O}_2(\kappa)$, $\mathcal{O}_3(\kappa)$, and $\mathcal{O}_4(\kappa)$. The cardinals $\mathfrak{b}$, $\mathbf{add}(\mathcal{N})$, $\mathfrak{s}$, and $\mathfrak{u}$ are examples of cardinal characteristics of the continuum.

[4] The $\mathcal{N}$ denotes the ideal of null subsets of ${}^\omega 2$. The reason for using this notation will become clearer shortly.

Of course, $\aleph_1 \leq \mathfrak{b}, \mathbf{add}(\mathcal{N}), \mathfrak{s}, \mathfrak{u} \leq \mathfrak{c}$. Studying what other inequalities between these cardinals are true under various assumptions has been an active area of research. For example, it can be proven that $\mathfrak{b} \geq \mathbf{add}(\mathcal{N})$ is a consequence of ZFC. Note that ZFC is the proper axiom system to use, and not, say, ZFC+CH. In the latter system, the question at hand is trivial, since all cardinal characteristics of the continuum equal $\mathfrak{c} = \aleph_1$ (however, as will be explained later, the mere fact of equality between two such cardinals, even in a model of ZFC+CH, does not necessarily properly describe the relationship between the two cardinals).

Although there is no universal formal definition for cardinal characteristics of the continuum, all such cardinals studied in this thesis can be defined in a certain way. These cardinals include the cardinals $\mathfrak{s}, \mathfrak{b}$, and $\mathbf{add}(\mathcal{N})$ from Example III.2, but not $\mathfrak{u}$. We note that these cardinals predate the formal unifying definition (some by a great deal). Nevertheless, we find the definition we give to be preferable.[5] The formalization was first achieved by Vojtáš [57], though our notation and terminology derives mainly from the handbook chapter by Blass [6].

Definition III.3. A *debate* is a triple $\mathfrak{K} := (K_-, K_+, K)$, where K_+ and K_- are sets and K is a relation with domain K_- and range K_+. The *norm of* $\mathfrak{K}$, denoted $||\mathfrak{K}||$, is the smallest cardinality of any subset $\mathcal{Y}$ of K_+ such that for every $X \in K_-$, there exists $Y \in \mathcal{Y}$ such that XKY. The *dual to* $\mathfrak{K}$, denoted $\mathfrak{K}^\perp$, is the triple $(K_+, K_-, \hat{K})$, where $Y\hat{K}X$ iff $\neg(XKY)$.

Remark III.4. Informally, we say that K_- is the set of "challenges," K_+ the set of "answers," and if XKY we say that X "is met by" Y. Therefore, $||\mathfrak{K}||$ is the minimum cardinality of a family of answers needed to meet all challenges. We use the convention of denoting a debate and its norm by matching uppercase and lowercase

[5] At least, for our purposes.

letters if possible (so $||\mathfrak{K}|| = \mathfrak{k}$). Although the definition in its generality does not state this, the sets K_+ and K_- should have something to do with the continuum in order for $\mathfrak{K}$ to be of use to us.

Before we provide the examples of interest, we will need some notation.

Notation III.5.
- $f \leq^* g$ denotes f is dominated by g.
- $\mathcal{N}$ denotes the set of null subsets of ${}^\omega 2$.
- $\mathcal{M}$ denotes the set of meager subsets of ${}^\omega 2$.

Example III.6. Consider Example III.2. It is easy to see that $\mathfrak{b}$, $\mathbf{add}(\mathcal{N})$, and $\mathfrak{s}$ are the norms of the debates

$$\mathfrak{B} := ({}^\omega\omega, {}^\omega\omega, \not\geq^*)$$

$$\mathbf{Add}(\mathcal{N}) := (\mathcal{N}, \mathcal{N}, \not\supseteq)$$

$$\mathfrak{S} := ([\omega]^\omega, [\omega]^\omega, \text{is split by})$$

Example III.7. The additional cardinal characteristics of the continuum which we will study can be defined as the norms of the following debates.

- $\mathfrak{D} := \mathfrak{B}^\perp = ({}^\omega\omega, {}^\omega\omega, \leq^*)$.
- $\mathfrak{R} := \mathfrak{R}^\perp = ([\omega]^\omega, [\omega]^\omega, \text{does not split})$.
- Let $\mathcal{J}$ be either $\mathcal{N}$ or $\mathcal{M}$.
  - $\mathbf{Add}(\mathcal{J}) := (\mathcal{J}, \mathcal{J}, \not\supseteq)$.
  - $\mathbf{Cof}(\mathcal{J}) := \mathbf{Add}(\mathcal{J})^\perp = (\mathcal{J}, \mathcal{J}, \subseteq)$.
  - $\mathbf{Cov}(\mathcal{J}) := ({}^\omega 2, \mathcal{J}, \in)$.
  - $\mathbf{Non}(\mathcal{J}) := \mathbf{Cov}(\mathcal{J})^\perp = (\mathcal{J}, {}^\omega 2, \not\ni)$.

In the interest of clarity, it is useful to explicitly write out the definitions of the corresponding cardinals without using debates (as is usually done in the literature).

- $\mathfrak{d}$ is the minimal cardinality of a dominating family $\mathcal{D} \subseteq {}^\omega\omega$. $\mathcal{D}$ is dominating if every $g \in {}^\omega\omega$, is dominated by some $f \in \mathcal{D}$.

- $\mathfrak{r}$ is the minimal cardinality of an unsplittable family $\mathcal{R} \subseteq [\omega]^\omega$. $\mathcal{R}$ is unsplittable if there is no $S \in [\omega]^\omega$ which splits every $R \in \mathcal{R}$ (i.e., for each S, R has finite intersection with either S or its complement).

- $\mathbf{add}(\mathcal{M})$ is the minimal cardinality of a family of meager sets whose union is not meager.

- $\mathbf{cof}(\mathcal{N})$ ($\mathbf{cof}(\mathcal{M})$) is the minimal cardinality of a family $\mathcal{F}$ of null (meager) sets such that every null (meager) set is the subset of some $F \in \mathcal{F}$ (such an $\mathcal{F}$ is a base for the ideal).

- $\mathbf{cov}(\mathcal{N})$ ($\mathbf{cov}(\mathcal{M})$) is the minimal cardinality of a family of null (meager) sets whose union is ${}^\omega 2$.

- $\mathbf{non}(\mathcal{N})$ ($\mathbf{non}(\mathcal{M})$) is the minimal cardinality of a subset of ${}^\omega 2$ that is not null (meager).

Of the cardinals defined above, our primary focus will be on the ten $\mathfrak{b}$, $\mathfrak{d}$, and for $\mathcal{J} = \mathcal{N}$ or $\mathcal{M}$, $\mathbf{add}(\mathcal{J})$, $\mathbf{cof}(\mathcal{J})$, $\mathbf{cov}(\mathcal{J})$, and $\mathbf{non}(\mathcal{J})$. The inequalities between these cardinals that are provable in ZFC, established over the course of fifty years by Rothberger [43, 44], Truss [55], Miller [32], Bartoszynski [2], and Raisonnier and Stern [42], are summarized by what is known as Cichoń's diagram [17].

Proposition III.8. *Let $\mathfrak{l} \to \mathfrak{k}$ denote ZFC proves $\mathfrak{k} \leq \mathfrak{l}$. Then:*

$$\begin{array}{ccccccc}
\mathbf{cov}(\mathcal{N}) & \longleftarrow & \mathbf{non}(\mathcal{M}) & \longleftarrow & \mathbf{cof}(\mathcal{M}) & \longleftarrow & \mathbf{cof}(\mathcal{N}) \\
 & & \downarrow & & \downarrow & & \\
\downarrow & & \mathfrak{b} & \longleftarrow & \mathfrak{d} & & \downarrow \\
 & & \downarrow & & \downarrow & & \\
\mathbf{add}(\mathcal{N}) & \longleftarrow & \mathbf{add}(\mathcal{M}) & \longleftarrow & \mathbf{cov}(\mathcal{M}) & \longleftarrow & \mathbf{non}(\mathcal{N})
\end{array}$$

This diagram is complete, in the sense that any inequality between two of these cardinals that is provable in ZFC is represented by an arrow or a series of arrows (announced in [5]; a summary of the relevant consistency results can be found in [4]).

Remark III.9. We note that this diagram is highly symmetrical with respect to dualization. For each $\mathfrak{K}$, if we replace $||\mathfrak{K}||$ with $||\mathfrak{K}^{\perp}||$ and reverse all arrows, we obtain a 180° rotation of Cichoń's diagram. It is also a theorem of ZFC that $\mathfrak{s} \leq \mathbf{non}(\mathcal{M}), \mathfrak{d}, \mathbf{non}(\mathcal{M})$ and $\mathfrak{r} \geq \mathbf{cov}(\mathcal{N}), \mathfrak{b}, \mathbf{cov}(\mathcal{M})$.

We consider Cichoń's diagram to consist of two kinds of results: positive results (ZFC proves $\mathfrak{k} \leq \mathfrak{l}$) and negative results (ZFC does not prove $\mathfrak{k} \leq \mathfrak{l}$). Both kinds of results can be proven using general methods based on debates. We begin by considering positive results.

Definition III.10. A *morphism* ϕ from one debate $\mathfrak{K} = (K_-, K_+, K)$ to another $\mathfrak{L} = (L_-, L_+, L)$ is a pair of functions $\phi_- : L_- \to K_-$ and $\phi_+ : K_+ \to L_+$ such that for all $X \in L_-$ and $Y \in K_+$, if $\phi_-(X)KY$, then $XL\phi_+(Y)$.

Remark III.11. Some authors use instead generalized Galois-Tukey connections, for which the maps and the implication travel in the opposite direction (that is, one needs maps from $K_- \to L_-$ and $L_+ \to K_+$, etc.). Note that any morphism from $\mathfrak{K} \to \mathfrak{L}$ gives a morphism from $\mathfrak{L}^{\perp} \to \mathfrak{K}^{\perp}$ – the maps ϕ_+ and ϕ_- merely switch places.

Proposition III.12. *(ZFC) If there is a morphism* $\phi : \mathfrak{K} \to \mathfrak{L}$ *then* $||\mathfrak{K}|| \geq ||\mathfrak{L}||$ *and* $||\mathfrak{L}^\perp|| \geq ||\mathfrak{K}^\perp||$.

Proof. Suppose we have such a ϕ. Fix an arbitrary $\mathcal{Y} \subseteq K_+$ with $|\mathcal{Y}| < ||\mathfrak{L}||$. Then $|\phi_+(\mathcal{Y})| \leq |\mathcal{Y}| < ||\mathfrak{L}||$, so there exists $X \in L_-$ such that for all $Y \in \mathcal{Y}$, $\neg X L \phi_+(Y)$. Consequently, for all $Y \in \mathcal{Y}$, $\neg \phi_-(X) K Y$. It thus impossible to have $||\mathfrak{K}|| < ||\mathfrak{L}||$. The second statement follows from the existence of a "dual" morphism from $\mathfrak{L}^\perp$ to $\mathfrak{K}^\perp$. □

All positive results from Cichoń's diagram can be proven by establishing the existence of the proper morphisms. In fact, the framework of debates and morphisms was created to describe recurring patterns in the original proofs of these results. Specifically, the original proofs of such results followed the pattern of the proof above, but with specific cardinals and specific constructions in place of ϕ_+ and ϕ_-. In this respect, morphisms between debates can be said to describe the relationships between cardinal characteristics of the continuum.

However, situations can arise in which general morphisms between debates are unsatisfactory in terms of describing relationships between cardinal characteristics. For example, by a result of Yiparaki [58], in a model V of ZFC+CH, there exist $\phi_+ : {}^\omega\omega \to {}^\omega 2$ and $\phi_- : \mathcal{M} \to {}^\omega\omega$ such that $\phi_+(f) \in M$ implies $\phi_-(M) \geq^* f$. In other words, there exists in V a morphism from $\mathfrak{B}$ to $\mathbf{Non}(\mathcal{M})$ witnessing that, in V, $\mathfrak{b} \geq \mathbf{non}(\mathcal{M})$ (in fact both equal $\aleph_1$), though the existence of such a morphism is not guaranteed by ZFC alone. But these maps (as guaranteed by Yiparaki's result) are highly nonconstructive: $\phi_-(M)$ and $\phi_+(f)$ need not be related to M and f in any meaningful way, so the mere existence of these maps does not impart any real information about the relationship between $\mathfrak{b}$ and $\mathbf{non}(\mathcal{M})$ in V. Therefore, it is worth asking when $\mathfrak{K} \to \mathfrak{L}$ is achievable with maps ϕ_- and ϕ_+ that are sufficiently "nice."

For example, suppose that instead of working with $\mathcal{N}$ and $\mathcal{M}$ directly, we work with bases $\{N_f\}_{f\in{}^\omega\omega}$ and $\{M_f\}_{f\in{}^\omega\omega}$, and view morphisms as working on indices. In other words, a morphism from $\mathbf{Non}(\mathcal{M})$ to $\mathfrak{B}$ now consists of maps $\phi_+ : {}^\omega 2 \to {}^\omega\omega$ and $\phi_- : {}^\omega\omega \to {}^\omega\omega$ such that $f \geq^* \phi_+(X)$ implies $X \in M_{\phi_-(f)}$). We note that these changes do not affect the values of any cardinal characteristics or the existence of any morphisms. Then we may ask when $\mathfrak{K} \to \mathfrak{L}$ can be achieved with ϕ_- and ϕ_+ both Borel. This is the case for all $\mathfrak{K}, \mathfrak{L}$ such that $\mathfrak{k}, \mathfrak{l}$ are in Cichoń's diagram and $\mathfrak{l} \leq \mathfrak{k}$ is a theorem of ZFC (to put this more concisely, all inequalities in Cichoń's diagram are realized by Borel morphisms). Pawlikowski and Recław improved this by showing most inequalities are realized with continuous morphisms. Specifically:

Proposition III.13. *[40] Let an unmarked arrow from $\mathfrak{K}$ to $\mathfrak{L}$ denote that there exists a continuous morphism from $\mathfrak{K}$ to $\mathfrak{L}$. If the arrow is labelled with a B, there exists only a Borel morphism.*

$$\begin{array}{ccccccc}
\mathbf{Cov}(\mathcal{N}) & \longleftarrow & \mathbf{Non}(\mathcal{M}) & \longleftarrow & \mathbf{Cof}(\mathcal{M}) & \longleftarrow & \mathbf{Cof}(\mathcal{N}) \\
 & & \downarrow{\scriptstyle B} & & \downarrow & & \\
\downarrow & & \mathfrak{B} & \longleftarrow & \mathfrak{D} & & \downarrow \\
 & & \downarrow & & \downarrow{\scriptstyle B} & & \\
\mathbf{Add}(\mathcal{N}) & \longleftarrow & \mathbf{Add}(\mathcal{M}) & \longleftarrow & \mathbf{Cov}(\mathcal{M}) & \longleftarrow & \mathbf{Non}(\mathcal{N})
\end{array}$$

To understand the proofs of negative results, we must understand first the interaction between cardinal characteristics of the continuum and forcing. If $\mathfrak{k}$ is the norm of $\mathfrak{K}$, we may use forcing to produce a model where $\mathfrak{k}$ is large as follows. We begin with a model of ZFC+CH, and iterate a forcing which adds to the ground model a challenge not met by any ground model answer. If we iterate such a forcing $\aleph_2$ times,[6] $\mathfrak{k}$ is increased to $\aleph_2$, since any family of answers with smaller cardinality

[6] Again, care needs to be taken with the iteration, and again, this is beyond the scope of this thesis.

must be contained in the model obtained after the αth iteration, for some $\alpha < \aleph_2$. The challenge added in the $(\alpha+1)$th iteration shows such a family does not suffice to meet all challenges.

If $\mathfrak{l} \leq \mathfrak{k}$ is witnessed by a sufficiently well-behaved morphism $\phi : \mathfrak{K} \to \mathfrak{L}$, then any forcing which adds a challenge to L_- not met by any ground model answer in L_+ will also add a challenge to K_-[7] not met by any ground model answer in K_+. Intuitively, this makes sense, since it implies that a suitable iterated forcing that increases $\mathfrak{l}$ will also increase $\mathfrak{k}$. All of the proofs of positive results for Cichoń's diagram can be restated as proofs of such relationships between forcings. For example, the proof that $\mathbf{cov}(\mathcal{N}) \leq \mathbf{non}(\mathcal{M})$ holds in ZFC can also be used to show that any forcing which adds a real not in any ground model null set also adds a meager set covering all ground model reals.

Therefore, if we want to construct a model where $\mathfrak{l} > \mathfrak{k}$, we want to find a forcing which adds a challenge to L_- not met by any ground model answer in L_+ *without* also adding a challenge to K_- not met by any ground model answer in K_+. All of the negative results from Cichoń's diagram can be proved using such forcings. Of course, simply finding such a forcing is not by itself enough. An iterated forcing construction needs to be carried out, so there are additional properties which the forcing (and the iteration) need to satisfy. However, we have enough information to state the following fact (which is nevertheless a gross oversimplification) which will be a key motivating factor in the study of Turing characteristics (to be defined in the following chapter).

Fact III.14. *For cardinal characteristics of the continuum $\mathfrak{k}, \mathfrak{l}$ from Cichoń's diagram, ZFC proves $\mathfrak{l} \leq \mathfrak{k}$ iff any forcing which adds a challenge to L_- not met by any*

[7]specifically, ϕ_- applied to the challenge added to L_-

ground model answer in L_+ *will also add a challenge to* K_- *not met by any ground model answer in* K_+*.*

CHAPTER IV

Turing Characteristics

The goal of this thesis is twofold. First, make a reasonable definition for effective correspondents to the cardinals in Cichoń's diagram (and similarly defined cardinals, such as $\mathfrak{r}$ and $\mathfrak{s}$). Second, investigate the extent to which relationships among these correspondents reflect relationships among cardinals. In this chapter, we define the correspondents as classes of oracles with sufficient computing power. These classes of oracles are closely related to some of the topics outlined in Chapter 2.

4.1 Definitions

A slightly informal definition which encapsulates our idea of a correspondent to a cardinal characteristic defined via a debate is the following:

Definition IV.1. Let $\mathfrak{K} = (K_-, K_+, K)$ be a debate such that $||\mathfrak{K}|| > \aleph_0$, and let $\mathbf{C}$ and $\mathbf{A}$ be sets of oracles closed downward with respect to $\leq_T$ with $\mathbf{A}$ countable. The Turing norm of $\mathfrak{K}$ relative to $\mathbf{A}$ in $\mathbf{C}$ consists of the oracles $A \in \mathbf{C}$ which "compute" an $X \in K_-$ such that for all $Y \in K_+$ with "computable" relative to some oracle in $\mathbf{A}$, $\neg(XKY)$.

Remark IV.2. There are two intuitions behind this definition. The first is that the set of answers computable from an oracle in $\mathbf{A}$ do not suffice to meet all challenges (there are only countably many such answers). The canonical case is where **A** consists of the computable oracles (so we are looking at computable answers). The Turing norm of $\mathfrak{K}$ relative to **A** (in some set **C** of oracles) consists of oracles which are sufficiently complex to compute a witness to the inadequacy of these answers. This intuition is sound as long as $||\mathfrak{K}|| > \aleph_0$, which is the case for all the debates we considered in Section 3.5.

The second intuition is that we have seen before the basic idea of a challenge not met by any "weak" answer. Recall that $||\mathfrak{K}||$ is increased by a suitable iteration of a forcing which adds to the ground model a challenge not met by any ground model answer. From Section 3.4, the computable analogue to a real being added by a forcing is an oracle computing that real, with the computable objects standing in for the ground model. Therefore, this definition can be viewed as the computable version of this forcing, plus its relativizations.

This definition, however, currently lacks rigor because we do not know when an oracle computes a challenge or response. If the debate is $\mathfrak{B}$ or $\mathfrak{S}$ (for example), this doesn't matter – every challenge or answer is either a function from ω to itself or a subset of ω. But how does one define the complexity of, say, a null subset of ${}^{\omega}2$?

To resolve this difficulty, we use the same strategy as was used for finding Borel or continuous morphisms. First, we replace $\mathcal{N}$ and $\mathcal{M}$ (which have cardinality $2^{\mathfrak{c}}$) with a $\mathfrak{c}$-sized base for each. Parameterize each base with members of ${}^{\omega}\omega$, and replace each debate involving $\mathcal{N}$ and $\mathcal{M}$ with a corresponding debate on the codes for members in the correct base. We make this rigorous with the following definitions.

Definition IV.3. Let $\mathfrak{K} = (K_-, K_+, K)$ be a debate with K_- and K_+ parameterized by elements of ${}^{\omega}\omega$. Then $\tilde{\mathfrak{K}}$ denotes the debate $(\tilde{K}_-, \tilde{K}_+, \tilde{K})$ where $\tilde{K}_\pm$ is the set of codes for elements of $K_\pm$ and if $\tilde{X}$, $\tilde{Y}$ code X, Y, then $\tilde{X}\tilde{K}\tilde{Y}$ iff XKY.

Remark IV.4. Naturally, different coding schemes will produce different debates. From a set theoretic perspective, there is no real difference, since for any $\tilde{\mathfrak{K}}$, there are morphisms from $\mathfrak{K}$ to $\tilde{\mathfrak{K}}$, and vice versa. However, different choices of codes will affect how the material to follow proceeds. We will forego any discussion of which codes to choose (and why) until a later section.

Now we rephrase our definition from earlier:

Definition IV.5. Let $\mathfrak{K} = (K_-, K_+, K)$ be a debate such that $K_\pm \subseteq {}^\omega\omega$ and $||\mathfrak{K}|| > \aleph_0$, and let $\mathbf{C}$ and $\mathbf{A}$ be sets of oracles closed downward with respect to $\leq_T$ with $\mathbf{A}$ countable. The *Turing norm of* $\mathfrak{K}$ *relative to* $\mathbf{A}$ *in* $\mathbf{C}$ (denoted by $\langle\mathfrak{K}\rangle^{\mathbf{C}}_{\mathbf{A}}$) consists of the oracles $A \in \mathbf{C}$ which compute an $X \in K_-$ such that for all $Y \in K_+$ computable relative to some oracle in $\mathbf{A}$, $\neg(XKY)$.

Remark IV.6. We refer to classes of degrees arising from Turing norms as Turing characteristics. In the definition above, if $\mathbf{A}$ is omitted, it is understood to be the set of computable oracles, and if $\mathbf{C}$ is omitted, it is understood to be ${}^\omega 2$. Note that $\langle\mathfrak{K}\rangle^{\mathbf{C}}_{\mathbf{A}} = \mathbf{C} \cap \langle\mathfrak{K}\rangle_{\mathbf{A}}$. For simplicity we denote a debate and its Turing norm by matching Fraktur and boldface letters if possible (so $\langle\mathfrak{K}\rangle^{\mathbf{C}}_{\mathbf{A}} = \mathbf{K}^{\mathbf{C}}_{\mathbf{A}}$).

Using this definition, we can describe the 12 Turing characteristics this paper will study (with the default choices for $\mathbf{A}$ and $\mathbf{C}$).

- $A \in \mathbf{B} = \langle\mathfrak{B}\rangle$ iff A computes a function dominating all computable functions.
- $A \in \mathbf{D} = \langle\mathfrak{D}\rangle$ iff A computes a function not dominated by any computable function.
- $A \in \mathbf{ADDN} = \langle\widetilde{\mathbf{Add}(\mathcal{N})}\rangle$ iff A computes a code for a null set containing all computably coded null sets.

- $A \in \mathbf{COFN} = \langle \widetilde{\mathbf{Cof}(\mathcal{N})} \rangle$ iff A computes a code for a null set not contained in any computably coded null set.

- $A \in \mathbf{COVN} = \langle \widetilde{\mathbf{Cov}(\mathcal{N})} \rangle$ iff A computes a real not contained in any computably coded null set.

- $A \in \mathbf{NONN} = \langle \widetilde{\mathbf{Non}(\mathcal{N})} \rangle$ iff A computes a code for a null set containing all computable reals.

- $A \in \mathbf{ADDM} = \langle \widetilde{\mathbf{Add}(\mathcal{M})} \rangle$ iff A computes a code for a meager set containing all computably coded meager sets.

- $A \in \mathbf{COFM} = \langle \widetilde{\mathbf{Cof}(\mathcal{M})} \rangle$ iff A computes a code for a meager set not contained in any computably coded meager set.

- $A \in \mathbf{COVM} = \langle \widetilde{\mathbf{Cov}(\mathcal{M})} \rangle$ iff A computes a real not contained in any computably coded meager set.

- $A \in \mathbf{NONM} = \langle \widetilde{\mathbf{Non}(\mathcal{M})} \rangle$ iff A computes a code for a meager set containing all computable reals.

- $A \in \mathbf{R} = \langle \mathfrak{R} \rangle$ iff A computes a set splitting all infinite computable sets.

- $A \in \mathbf{S} = \langle \mathfrak{S} \rangle$ iff A computes an infinite set not split by any computable set.

Now that we have a definition for Turing characteristics, we must determine which relationships among them we wish to study. This issue is resolved by considering Fact III.14 in light of our intuition concerning forcings and Definition IV.5. The computable version of the second statement from Fact III.14 is that any A which computes a challenge in L_- which is not met by any computable answer in L_+ also computes a challenge in K_- which is not met by any computable answer in K_+. In other words, $\mathbf{L} \subseteq \mathbf{K}$. Therefore, we might hope that for $\mathfrak{k}, \mathfrak{l}$ in Cichoń's diagram, ZFC proves $\mathfrak{l} \leq \mathfrak{k}$ iff $\mathbf{L} \subseteq \mathbf{K}$, or perhaps iff $\mathbf{L}^{\mathbf{C}}_{\mathbf{A}} \subseteq \mathbf{K}^{\mathbf{C}}_{\mathbf{A}}$ for all $\mathbf{A}, \mathbf{C}$. We will show that for both potential equivalences, the forward implication holds. For the former potential equivalence, the reverse implication does not hold, and for the second, it is still an open question. Specifically:

Theorem IV.7. *In both the diagrams to follow, for* $\mathbf{P}, \mathbf{Q} \subseteq {}^{\omega}2$*, let* $\mathbf{P} \rightarrow \mathbf{Q}$ *denote* $\mathbf{Q} \subseteq \mathbf{P}$ *(= denotes equality as usual). Then:*

$$\begin{array}{ccccccc}
\mathbf{COVN} & \longleftarrow & \mathbf{NONM} & \longleftarrow & \mathbf{COFM} & \longleftarrow & \mathbf{COFN} \\
 & & \downarrow & & \downarrow & & \\
\downarrow & & \mathbf{B} & \longleftarrow & \mathbf{D} & & \downarrow \\
 & & \| & & \| & & \\
\mathbf{ADDN} & = & \mathbf{ADDM} & \longleftarrow & \mathbf{COVM} & \longleftarrow & \mathbf{NONN}
\end{array}$$

This diagram is complete.

Also, for all $\mathbf{A}, \mathbf{C} \subseteq {}^{\omega}2$ *closed downward with respect to* $\leq_T$ *and* $\mathbf{A}$ *countable,*

$$\begin{array}{ccccccc}
\mathbf{COVN}^{\mathbf{C}}_{\mathbf{A}} & \longleftarrow & \mathbf{NONM}^{\mathbf{C}}_{\mathbf{A}} & \longleftarrow & \mathbf{COFM}^{\mathbf{C}}_{\mathbf{A}} & \longleftarrow & \mathbf{COFN}^{\mathbf{C}}_{\mathbf{A}} \\
 & & \downarrow & & \downarrow & & \\
\downarrow & & \mathbf{B}^{\mathbf{C}}_{\mathbf{A}} & \longleftarrow & \mathbf{D}^{\mathbf{C}}_{\mathbf{A}} & & \downarrow \\
 & & \downarrow & & \downarrow & & \\
\mathbf{ADDN}^{\mathbf{C}}_{\mathbf{A}} & \longleftarrow & \mathbf{ADDM}^{\mathbf{C}}_{\mathbf{A}} & \longleftarrow & \mathbf{COVM}^{\mathbf{C}}_{\mathbf{A}} & \longleftarrow & \mathbf{NONN}^{\mathbf{C}}_{\mathbf{A}}
\end{array}$$

Additionally, for all such $\mathbf{A}$ *and* $\mathbf{C}$*,*

$$\mathbf{S}^{\mathbf{C}}_{\mathbf{A}} \subseteq \mathbf{NONM}^{\mathbf{C}}_{\mathbf{A}}, \mathbf{D}^{\mathbf{C}}_{\mathbf{A}}, \mathbf{NONN}^{\mathbf{C}}_{\mathbf{A}}$$

$$\mathbf{R}^{\mathbf{C}}_{\mathbf{A}} \supseteq \mathbf{COVM}^{\mathbf{C}}_{\mathbf{A}}, \mathbf{B}^{\mathbf{C}}_{\mathbf{A}}, \mathbf{COVN}^{\mathbf{C}}_{\mathbf{A}}$$

Remark IV.8. The second diagram above is not known to be complete.

As was the case with cardinal characteristics, we make a distinction between positive results (e.g., $\mathbf{L} \subseteq \mathbf{K}$) and negative results (e.g., $\mathbf{L} \nsubseteq \mathbf{K}$). The former results are needed to establish both diagrams hold, while the latter results are needed to establish the completeness of the first diagram. In order to prove positive results, we make effective the notion of a morphism, and to prove negative results, we use forcing proofs. Before we turn to either matter, we must attend to some preliminary matters.

4.2 Coding of Objects

In order to go further, we must decide how we are going to use ${}^{\omega}\omega$ to code bases for $\mathcal{N}$ and $\mathcal{M}$. The choice generally does not matter in set theory, but in computability theory, our choice of codes can drastically alter the results. For example, let σ_i denote the binary string coded by i, and C_j the clopen subset of ${}^{\omega}2$ coded by j (in the manner described in Section 2.2). Now if $N \subseteq {}^{\omega}2$,

$$\mu(N) = 0 \Leftrightarrow \exists f \in {}^{\omega}\omega \left(\sum_{n \in \omega} 2^{-|\sigma_{f(n)}|} < \infty \wedge A \subseteq \bigcap_{n} \bigcup_{m \geq n} [\sigma_{f(m)}] \right)$$
$$\Leftrightarrow \exists g \in {}^{\omega}\omega \left(\forall n[\mu(C_{g(n)}) \leq 2^{-n}] \wedge A \subseteq \bigcap_{n} \bigcup_{m \geq n} C_{g(m)} \right).$$

These facts present us with two ways to code null sets. Either we use f to code the null set of the first kind if $\sum_n 2^{-|\sigma_{f(n)}|}$ is finite or we use g to code the null set of the second kind if for all n, $\mu(C_{g(n)}) \leq 2^{-n}$.

Suppose we use the former system. Let N_f denote the null set coded by f. Then by Proposition II.65, for all N and A, $N \subseteq N_f$ for some $f \leq_T A$ iff N is Martin-Löf null relative to A. Because there is a universal Martin-Löf null set, there is a computable $\tilde{f}$ such that $N_f \subseteq N_{\tilde{f}}$ for all computable f. In our parlance, this would imply every computable oracle is in **ADDN**. But we expect that every oracle in **ADDN** is also in **B** – that is, computes a function dominating all computable functions. As this is impossible for a computable oracle to do, this coding is unsuitable for our purposes.

In proofs of $\mathfrak{b} \geq \mathbf{add}(\mathcal{N})$ which use this coding one needs to find, for any f, a function h such that for all k

$$\sum_{n \geq h(k)} 2^{-|\sigma_{f(n)}|} < 2^{-k}.$$

In set theory, this is not an issue – the function exists in whatever model we're working in, so we can use it. In computability theory, however, we have to ask if such an h can be found computably in f. The answer, generally, is no. The second coding above contains this information naturally, since we explicitly require $\mu(C_{g(n)}) \leq 2^{-n}$ for all n[1].

Therefore, there is a difference between the information content of these two coding systems. We will later see that the second coding allows for proofs of the results indicated in Theorem IV.7, so it is the coding we will use. In the absence of a clear, overarching reason to choose one coding over another, we will continue to select the codings which produce the desired results. Let us now make the formal definition:

Definition IV.9. $\tilde{\mathcal{N}}$ denotes the set of $f \in {}^{\omega}\omega$ such that $\mu(C_{f(n)}) \leq 2^{-n}$ for all n. For $f \in \tilde{\mathcal{N}}$,

$$N_f = \bigcap_n \bigcup_{m \geq n} C_{f(m)}.$$

For $f, g \in \tilde{\mathcal{N}}$ and $X \in {}^{\omega}2$, $f \mathrel{\tilde{\subseteq}} g$ iff $N_f \subseteq N_g$ and $X \mathrel{\tilde{\in}} f$ iff $X \in N_f$.

Remark IV.10. Note that for all N and A, by Proposition II.65, $N \subseteq N_f$ for some $f \leq_T A$ iff N is Schnorr null relative to A. We will avoid the notation $\tilde{\subseteq}$ and $\tilde{\in}$ whenever possible, although we defined this notation (in order to be in line with Definitions IV.3 and IV.5).

Our parametrization of $\mathcal{M}$ requires some further preparation.

[1]This, and not our choice of basic open sets vs. clopen sets, is the significant difference between the codings

Definition IV.11. 1. An *interval partition* is a partition of ω into intervals. For I an interval partition, let I^n denote the nth interval in the partition. For $f \in {}^\omega\omega$ such that f is increasing and $f(0) = 0$ (denote the set of such f by ${}^\omega\omega\uparrow$), I_f denotes the interval partition such that $f(n)$ is the left endpoint of I_f^n.

2. A *chopped real* is a pair (X, I) where $X \in {}^\omega 2$ and I is an interval partition. A real $Y \in {}^\omega 2$ *matches* (X, I) iff for infinitely many n, $Y \upharpoonright I^n = X \upharpoonright I^n$. Denote the set of Y matching (X, I) by $\mathrm{Match}(X, I)$.

One can prove that sets of the form ${}^\omega 2 \backslash \mathrm{Match}(X, I)$ constitute a base for the ideal $\mathcal{M}$. We can therefore code meager sets by coding chopped reals. Two options immediately present themselves. We could code (X, I) with $X \oplus f$, where f codes I. Or we could code (X, I) with the function mapping n to an index for $X \upharpoonright I^n$. Each is more convenient than the other for certain purposes, though they are equivalent, as shown below. Recall that M is meager iff it is covered by the union of countably many nowhere dense sets, and M is nowhere dense iff it is the complement of a dense open set iff for every σ, there is $\tau \supseteq \sigma$ such that $[\tau]$ is disjoint from M.

Proposition IV.12. *For any $M \subseteq {}^\omega 2$ and $A \in {}^\omega 2$, the following are equivalent.*

1. *There is a sequence M_n of nowhere dense sets and $g : \omega \times {}^{<\omega}2 \to {}^\omega 2$ computable relative to A such that $M \subseteq \bigcup_n M_n$ and for all n, σ, $g(n, \sigma)$ extends σ and $[g(n, \sigma)]$ is disjoint from M_n.*

2. *There is a sequence M_n of nowhere dense sets and $g : \omega \times {}^{<\omega}2 \to {}^\omega 2$ computable relative to A such that $M \subseteq \bigcup_n M_n$, $M_n \subseteq M_{n+1}$ for all n, and for all n, σ, $g(n, \sigma)$ extends σ and $[g(n, \sigma)]$ is disjoint from M_n.*

3. *There are $X, f \leq_T A$ such that M is disjoint from $Match(X, I_f)$.*

4. *There is* $h \leq_T A$*,* $X \in {}^\omega 2$*, and* I *an interval partition such that* M *is disjoint from Match*(X, I) *and for all* n*,* $h(n)$ *is an index for* $X \upharpoonright I^n$*.*

5. M *is disjoint from the intersection of an* A*-uniform sequence* $\{V_n\}_{n\in\omega}$ *of* A*-c.e. dense open sets.*

Proof. (based on an argument in Section 5 of [6], in turn based on ideas from [53])

$1 \Rightarrow 2$: Let M_n and g be given as in 1. We let $\tilde{M}_n = \bigcup_{k\leq n} M_n$, and $\tilde{g}$ be the function defined by

$$\tilde{g}(0, \sigma) = g(0, \sigma)$$

$$\tilde{g}(n + 1, \sigma) = g(n + 1, \tilde{g}(n, \sigma)).$$

Clearly $\tilde{g}$ and the sequence of $\tilde{M}_n$ satisfy the properties specified by 2 (note $\tilde{g} \leq_T g \leq_T A$).

$2 \Rightarrow 3$: Let M_n and g be given as in 2. We will construct (X, I_f) such that for each n, no $Y \in M_n$ agrees with X on I^n_f. Then any Y which matches (X, I) will be out of infinitely many M_n, and thus out of all of them by monotonicity.

Assume $f(n)$ has been defined, and $X(k)$ for all $k < m := f(n)$. Note that $\{\sigma_{2^m+i}\}_{i<2^m}$ is exactly the set of binary strings of length m. Let $\tau_0 = \emptyset$, and for $i < 2^m$, define τ_{i+1} such that $\sigma_{2^m+i}\tau_{i+1} = g(n, \sigma_{2^m+i}\tau_i)$. Now let $f(n+1) = m+|\tau_{2^m}|$ and for $m + k < f(n + 1)$, $X(m + k) = \tau_{2^m}(k)$.

Now suppose Y agrees with X on I^n. Let m be as above, and fix i such that $\sigma_{2^m+i} = Y \upharpoonright m$. Then $Y \in [\sigma_{2^m+i}\tau_{i+1}]$, implying $Y \notin M_n$. Thus M is disjoint from Match(X, I_f). Also, $X, f \leq_T g \leq_T A$.

$3 \Rightarrow 4$: If X and f are given as in 3, we may let

$$h(n) = \sum_{\substack{f(n)\leq m<f(n+1) \\ X(m)=0}} 3^m + 2 \cdot \sum_{\substack{f(n)\leq m<f(n+1) \\ X(m)=1}} 3^m.$$

Note $h \leq_T X \oplus f \leq_T A$.

$4 \Rightarrow 5$: Suppose h is as given in 4. Then $V_n := \bigcup_{m \geq n}[h(m)]$ is A-uniformly A-c.e. open. Also, each V_n is dense; given σ, for sufficiently large m σ and $h(m)$ have disjoint domains, so there is a Y extending both. We may assume $m \geq n$, so $Y \in [\sigma] \cap V_n$. Since $\bigcap_n V_n = \mathrm{Match}(X, I)$, M is disjoint from this intersection.

$5 \Rightarrow 1$: If $\{V_n\}_{n\in\omega}$ is as given in 5, we may let $M_n = {}^{\omega}2 \backslash V_n$. Let $\{W_n\}_{n\in\omega}$ be an A-uniform sequence of A-c.e. subsets of ${}^{<\omega}2$ such that $V_n = [W_n]$ for all n. To define g, given n and σ, find the least $\langle m, s\rangle$ such that $\sigma_m \in W_{n,s}$ and either $\sigma_m \subseteq \sigma$ or $\sigma \subseteq \sigma_m$. In the former case, let $g(n, \sigma) = \sigma$, and in the latter case, let $g(n, \sigma) = \sigma_m$. □

Remark IV.13. We note that in 1. and 2. above, it is a stronger condition to require that the set of (τ, n) such that $[\tau] \cap M_n = \emptyset$ is computable relative to A (if this is the case, we let $g(n, \sigma)$ be the first such τ we find for n). Although we do not need this stronger condition, in some cases it will be possible to supply it, and furthermore convenient to do so.

For simplicity of notation, we code meager sets by functions h as in item 4 of the preceding proposition. However, it is understood that we may just as well use $X \oplus f$ for a code whenever that is more convenient. Recall from Section 2.2 the set S of partial functions from ω to $\{0, 1\}$ with finite domain. Let s_i denote the member of S coded by i.

Definition IV.14. $\tilde{\mathcal{M}}$ denotes the set of $f \in {}^{\omega}\omega$ such that:

- For all n, $\mathrm{dom}(s_{f(n)})$ is a nonempty interval.
- $\min \mathrm{dom}(s_{f(0)}) = 0$.
- For all n, $\min \mathrm{dom}(s_{f(n+1)}) = 1 + \max \mathrm{dom}(s_{f(n)})$.

For $f \in \tilde{\mathcal{M}}$,

$$M_f = {}^{\omega}2 \setminus \bigcap_n \bigcup_{m \geq n} [f(m)].$$

For $f, g \in \tilde{\mathcal{M}}$ and $X \in {}^{\omega}2$, $f \mathrel{\tilde{\subseteq}} g$ iff $M_f \subseteq M_g$ and $X \mathrel{\tilde{\in}} f$ iff $X \in M_f$.

Remark IV.15. As before, we will avoid the notation $\tilde{\subseteq}$ and $\tilde{\in}$ as much as possible.

4.3 Immediately Seen Equivalents From Computability Theory

For many $\mathfrak{k}$ in Cichoń's diagram, the corresponding Turing characteristic $\mathbf{K}$ is a set of oracles that has already been mentioned in Chapter II. In several cases, this is seen immediately from the definitions, with little to no further work. In this section, we quickly point out these cases. We do stress that in this section, we are only dealing with the case where $\mathbf{C} = {}^{\omega}2$ and $\mathbf{A}$ is the set of computable reals. Varying $\mathbf{C}$ does not produce anything interesting, since $\mathbf{K}^{\mathbf{C}} = \mathbf{C} \cap \mathbf{K}$. Varying $\mathbf{A}$ is not desirable since not everything in this section relativizes. For example:

Proposition IV.16. *$A \in \mathbf{B}$ iff A is high, and $A \in \mathbf{D}$ iff A computes a hyperimmune set.*

Remark IV.17. This follows from Propositions II.38 and II.52. The first part does not relativize well – in particular, if $\mathbf{A} = \{X : X \leq_T B\}$, where B is not computable, it is not necessarily the case that $A \in \mathbf{B_A}$ iff $A' \geq B''$. The proof of Proposition II.38 relied on an f dominating all computable functions being able to *compute* all such functions. Since an f dominating every function computable relative to B is not guaranteed to compute B, the proof may not work in this case. Trying to relativize to more complicated sets $\mathbf{A}$ clearly just makes this problem worse. This is why, in this section, we only present results for $\mathbf{A}$ the set of computable reals.

Proposition IV.18. *$A \in \mathbf{COVN}$ iff A computes a Schnorr random real, and $A \in \mathbf{COFN}$ iff A is not low for Schnorr tests.*

Remark IV.19. This follows from our choice of codes $\tilde{\mathcal{N}}$ and Proposition II.65.

Proposition IV.20. *$A \in \mathbf{COVM}$ iff A computes a weakly 1-generic real.*

Proof. By Proposition IV.12, $A \in \mathbf{COVM}$ iff there is $X \leq_T A$ which is in every uniform intersection of dense c.e. open sets. If X is in every dense c.e. open set, it is certainly in any intersection of such sets (uniform or not). Conversely, any dense c.e. open V is the intersection of a uniform sequence of dense c.e. open sets (just use $V_n = V$ for all n), so if X is in any such intersection, it is in every dense c.e. open set. □

We also briefly consider $\mathbf{R}$ and $\mathbf{S}$, which are also equivalent to sets of oracles already considered in computability theory. These sets of oracles involve concepts not discussed in Chapter II, since they do not have a significant impact on this thesis (we are mainly concerned with the properties of $\mathbf{R}_{\mathbf{A}}^{\mathbf{C}}$ and $\mathbf{S}_{\mathbf{A}}^{\mathbf{C}}$ for arbitrary $\mathbf{A}$ and $\mathbf{C}$).

Definition IV.21. Let $X \in \mathcal{P}(\omega)$. X is *r-cohesive* iff for every infinite computable set A, $X \subseteq^* A$ or $X \subseteq^* \bar{A}$ (i.e., no computable set splits X). X is *bi-immune* iff X and $\bar{X}$ are immune.

Proposition IV.22. *$A \in \mathbf{S}$ iff A computes an r-cohesive set.*

Remark IV.23. Immediate from the definition.

Proposition IV.24. *$A \in \mathbf{R}$ iff A computes a bi-immune set.*

Proof. It suffices to show X is bi-immune iff neither X nor $\bar{X}$ has an infinite computable subset – if A is not split by X, then there is a finite $F \subset A$ such that $A \backslash F$ is an infinite computable subset of either X or $\bar{X}$. The forward implication is trivial since every computable set is c.e. For the reverse implication, it suffices to show that any infinite c.e. set A has an infinite computable subset. By Proposition II.22, let A be the range of f, an injective computable function. Let B be the set

$$\{f(n) : f(n) > f(m) \text{ for all } m < n\}.$$

Clearly B is an infinite subset of A, and it is computable since $k \in B$ iff $k = f(n)$, where n is minimal such that $f(n) \geq k$. $\square$

CHAPTER V

Positive Results

In this chapter, we mainly devote our attention to proving, for $\mathfrak{K}$ and $\mathfrak{L}$ such that $\mathfrak{l} \leq \mathfrak{k}$ is an inequality in Cichoń's diagram, statements of the form: for all $\mathbf{A}$ and $\mathbf{C}$, $\mathbf{L}_{\mathbf{A}}^{\mathbf{C}} \subseteq \mathbf{K}_{\mathbf{A}}^{\mathbf{C}}$. To do so, we make the corresponding set theoretic arguments sufficiently effective by demonstrating a type of morphism from $\mathfrak{K}$ to $\mathfrak{L}$ which we will call an *effective morphism*. As we are simply looking for morphisms with additional properties, we can find suitable morphisms already in the set theoretic literature. We also note that the goal of finding effective morphisms is similar to Pawlikowski and Recław's goal of finding continuous morphisms, and we derive considerable inspiration from their work [40]. In addition to proving a version of Cichoń's diagram for Turing characteristics, we also use effective morphisms to prove results about $\mathbf{S}_{\mathbf{A}}^{\mathbf{C}}$ and $\mathbf{R}_{\mathbf{A}}^{\mathbf{C}}$.

However, effective morphisms do not tell the whole story, at least not when we consider a specific $\mathbf{A}$ and $\mathbf{C}$. For example, we saw in Section 4.3 that

$$\begin{aligned} A \in \mathbf{D} &\iff A \text{ computes a hyperimmune set} \\ &\iff A \text{ computes a weakly 1-generic real} \\ &\iff A \in \mathbf{COVM}. \end{aligned}$$

In particular, $\mathbf{D} \subseteq \mathbf{COVM}$. But this cannot be witnessed by an effective morphism from $\mathbf{Cov}(\mathcal{M})$ to $\mathfrak{D}$ since such a morphism would imply $\mathfrak{d} \leq \mathbf{cov}(\mathcal{M})$, which is not true in all models of ZFC. We shall see a similar situation holds with $\mathbf{B}$. Both anomalies can be explained by modifying the definition of an effective morphism to produce the notion of an *effective semi-morphism*. Further, the proof of Proposition II.91 can be phrased in terms of effective semi-morphisms, so there is some evidence that this notion can be used to explain completely the behavior of Turing characteristics when particular choices are made for $\mathbf{C}$ and $\mathbf{A}$.

Remark V.1. In this chapter and the next, many proofs will be modifications of set-theoretic proofs. We will cite the source of the original argument whenever one may be reasonably assigned.

5.1 Effective Morphisms

We now define the type of morphism to be used in conjunction with Turing characteristics:

Definition V.2. Let $\mathfrak{K} = (K_-, K_+, K)$ and $\mathfrak{L} = (L_-, L_+, L)$ be debates with $K_\pm$ and $L_\pm \subseteq {}^\omega\omega$. A morphism $\phi : \mathfrak{K} \to \mathfrak{L}$ is *effective* iff for all $X \in L_-$ and $Y \in K_+$, $\phi_-(X) \leq_T X$ and $\phi_+(Y) \leq_T Y$.

Theorem V.3. *Let $\phi : \mathfrak{K} \to \mathfrak{L}$ be an effective morphism and suppose $\mathbf{C}$ and $\mathbf{A}$ are closed downward with respect to $\leq_T$ with $\mathbf{A}$ countable. Then $\langle \mathfrak{L} \rangle^{\mathbf{C}}_{\mathbf{A}} \subseteq \langle \mathfrak{K} \rangle^{\mathbf{C}}_{\mathbf{A}}$ and $\langle \mathfrak{K}^\perp \rangle^{\mathbf{C}}_{\mathbf{A}} \subseteq \langle \mathfrak{L}^\perp \rangle^{\mathbf{C}}_{\mathbf{A}}$.*

Proof. Fix $A \in \langle \mathfrak{L} \rangle^{\mathbf{C}}_{\mathbf{A}}$, and $X \leq_T A$ in L_- not met by any answer in L_+ computable from an oracle in $\mathbf{A}$. Since ϕ is effective, $\phi_-(X) \leq_T A$, and for any $Y \in K_+$ computable from an oracle in $\mathbf{A}$, $\phi_+(Y)$ is also computable from an oracle in $\mathbf{A}$. For any such Y, $\neg X L \phi_+(Y)$ so $\neg \phi_-(X) K Y$. It follows that $\phi_-(X)$ witnesses that

$A \in \langle \mathfrak{K} \rangle^{\mathbf{C}}_{\mathbf{A}}$, as needed. The second result follows from the dual morphism, which is clearly also effective. $\square$

Now we can translate many proofs of positive results from set theory to computability theory. For example, denote the space of interval partitions by IP. An interval partition I is said to dominate J if for almost all n, there exists a k such that $J^k \subseteq I^n$. Code interval partitions by members of ${}^\omega\omega\uparrow$ as detailed in Definition IV.11. Note I_f dominates I_g iff for almost all n there exists k such that $f(n) \leq g(k)$ and $g(k+1) \leq f(n+1)$. If f and g satisfy this condition, we say f *interval dominates* g (we use this instead of "$\widetilde{\text{dominates}}$" in order to avoid confusion with "dominates").

Define the debate $\mathfrak{D}' = (IP, IP, \text{is dominated by})$, and observe that the corresponding $\tilde{\mathfrak{D}}'$ is $({}^\omega\omega\uparrow, {}^\omega\omega\uparrow, \text{is interval dominated by})$. One can use morphisms to show that $\mathfrak{d} = ||\mathfrak{D}'||$, and in fact the morphisms used (when applied to codes) are effective:

Theorem V.4. *There exist effective morphisms $\phi : \mathfrak{D} \to \tilde{\mathfrak{D}}'$ and $\psi : \tilde{\mathfrak{D}}' \to \mathfrak{D}$.*

Proof. (based on an argument presented in [6], possibly folklore)

First, we need $\phi_+ : {}^\omega\omega \to {}^\omega\omega\uparrow$ and $\phi_- : {}^\omega\omega\uparrow \to {}^\omega\omega$ such that if $\phi_-(f)$ is dominated by g, then $\phi_+(g)$ interval dominates f. Let

$$\phi_-(f)(n) = f(n+1),$$

$$\phi_+(g)(0) = 0, \text{ and}$$

$$\phi_+(g)(n+1) = \max\{\phi_+(g)(n)+1\} \cup \{g(m) : m \leq \phi_+(g)(n)\}.$$

Clearly $\phi_-(f) \leq_T f$ and $\phi_+(g) \leq_T g$, so it remains to prove these maps form a morphism.

Suppose g dominates $\phi_-(f)$. Note that for all n, $\phi_+(g)(n) \leq f(\phi_+(g)(n))$, and for sufficiently large n,

$$\begin{aligned} f(\phi_+(g)(n)+1) &= \phi_-(f)(\phi_+(g)(n)) \\ &\leq g(\phi_+(g)(n)) \\ &\leq \phi_+(g)(n+1) \end{aligned}$$

That is, if $k = \phi_+(g)(n)$, for large enough n $\phi_+(g)(n) \leq f(k)$ and $f(k+1) \leq \phi_+(g)(n+1)$, so $\phi_+(g)$ interval dominates f. Thus ϕ is a morphism, as needed.

For the morphism in the other direction, let $\psi_+ = \phi_-$ and $\psi_- = \phi_+$ from above. Note that for any f, g, n, and k for which $n \leq g(n) \leq \psi_-(f)(k)$ and $\psi_-(f)(k+1) \leq g(n+1) = \psi_+(g)(n)$, we have $f(n) \leq \psi_-(f)(k+1) \leq \psi_+(g)(n)$. It follows that if g interval dominates $\psi_-(f)$ then $\psi_+(g)$ dominates f, as needed. □

Corollary V.5. *For all* $\mathbf{A}$ *and* $\mathbf{C}$ *which are closed downward with respect to* $\leq_T$ *and with* $\mathbf{A}$ *countable,* $\mathbf{D}^{\mathbf{C}}_{\mathbf{A}} = \langle \tilde{\mathfrak{D}}' \rangle^{\mathbf{C}}_{\mathbf{A}}$ *and* $\mathbf{B}^{\mathbf{C}}_{\mathbf{A}} = \langle \tilde{\mathfrak{D}}'^{\perp} \rangle^{\mathbf{C}}_{\mathbf{A}}$.

Remark V.6. For future results of this kind, we will say simply "For all $\mathbf{A}$ and $\mathbf{C}$," with the understanding that the additional conditions are implicitly assumed.

Therefore, if we seek effective morphisms to or from $\mathfrak{D}$ or $\mathfrak{B}$, we may as well seek effective morphisms to or from $\tilde{\mathfrak{D}}'$ or $\tilde{\mathfrak{D}}'^{\perp}$, respectively. This will prove convenient, as some morphisms will be easier to demonstrate if we use interval partitions.

For ease of reference, we now restate the diagram whose proof will be the major work of this chapter.

Theorem V.7. *For all* $\mathbf{A}, \mathbf{C}$,

$$\begin{array}{ccccccc}
\mathbf{COVN}^{\mathbf{C}}_{\mathbf{A}} & \longleftarrow & \mathbf{NONM}^{\mathbf{C}}_{\mathbf{A}} & \longleftarrow & \mathbf{COFM}^{\mathbf{C}}_{\mathbf{A}} & \longleftarrow & \mathbf{COFN}^{\mathbf{C}}_{\mathbf{A}} \\
 & & \downarrow & & \downarrow & & \\
\downarrow & & \mathbf{B}^{\mathbf{C}}_{\mathbf{A}} & \longleftarrow & \mathbf{D}^{\mathbf{C}}_{\mathbf{A}} & & \downarrow \\
 & & \downarrow & & \downarrow & & \\
\mathbf{ADDN}^{\mathbf{C}}_{\mathbf{A}} & \longleftarrow & \mathbf{ADDM}^{\mathbf{C}}_{\mathbf{A}} & \longleftarrow & \mathbf{COVM}^{\mathbf{C}}_{\mathbf{A}} & \longleftarrow & \mathbf{NONN}^{\mathbf{C}}_{\mathbf{A}}
\end{array}$$

where $\leftarrow$ *means* $\subseteq$*.*

The morphism necessary for one arrow in the diagram can be supplied immediately, and trivially.

Theorem V.8. *There is an effective morphism* $\phi : \mathfrak{D} \to \mathfrak{B}$*.*

Proof. We seek $\phi_- : {}^\omega\omega \to {}^\omega\omega$ and $\phi_+ : {}^\omega\omega \to {}^\omega\omega$ such that $\phi_-(g) \leq^* f$ implies $g \not\geq^* \phi_+(f)$. We may simply let ϕ_- be the identity, and $\phi_+(f)(n) = f(n) + 1$. □

Corollary V.9. *For all* $\mathbf{A}, \mathbf{C}$*,* $\mathbf{B}^{\mathbf{C}}_{\mathbf{A}} \subseteq \mathbf{D}^{\mathbf{C}}_{\mathbf{A}}$*.*

5.2 Meager Sets and Chopped Reals

Because of their combinatorial nature, chopped reals are a convenient way to represent meager sets. One manifestation of this is the way that they afford an easy description of when one meager set contains another:

Proposition V.10. *Match*$(X, I) \subseteq$ *Match*(Y, J) *iff for almost all* n*, there exists an* m *such that* $J^m \subseteq I^n$ *and* $X \restriction J^m = Y \restriction J^m$*.*

Proof. The reverse implication is clear – if the latter condition holds and Z matches (X, I), then for almost all n such that $X \restriction I^n = Z \restriction I^n$, we have for some m $Z \restriction J^m = X \restriction J^m = Y \restriction J^m$. There will be infinitely many such m, so Z matches (Y, J).

Conversely, assume there exists A an infinite set of n such that there is no m with $J^m \subseteq I^n$ and $X \restriction J^m = Y \restriction J^m$. Without loss of generality, we may assume either all $n \in A$ are even or all $n \in A$ are odd. Define a real Z by $Z(k) = X(k)$ if $k \in I^n$ and $n \in A$, and $1 - Y(k)$ otherwise. Note that Z matches (X, I). Also, if $Z \restriction J^m = Y \restriction J^m$, then J^m must be covered by the union of some I^n for $n \in A$.

Since A contains no consecutive members, J^m must be covered by a single I^n – a contradiction. It follows that Z does not match (Y, J). □

We say that (X, I) *engulfs* (Y, J) if the conditions of Proposition V.10 hold. Two corollaries to Proposition V.10 should be readily apparent. Suppose $X, Y \in {}^\omega 2$, $I, J \in IP$, and $f, g \in \widetilde{mathcalM}$ (see Definition IV.14).

Corollary V.11. *If $Match(X, I) = {}^\omega 2 \backslash M_f$ and $Match(Y, J) = {}^\omega 2 \backslash M_g$, then (X, I) engulfs (Y, J) iff $M_g \subseteq M_f$.*

Corollary V.12. *If (X, I) engulfs (Y, J), then I dominates J.*

Theorem V.13. *There exists an effective morphism $\phi : \widetilde{\mathbf{Cof}(\mathcal{M})} \to \tilde{\mathfrak{D}}'$.*

Proof. We require $\phi_- : {}^\omega\omega \uparrow\to \tilde{\mathcal{M}}$ and $\phi_+ : \tilde{\mathcal{M}} \to {}^\omega\omega$ such that $M_{\phi_-(g)} \subseteq M_f$ implies g is interval dominated by $\phi_+(f)$.

Fix a computable Y, and let $\phi_-(g)(n)$ be a code for $Y \upharpoonright I^n_g$. Given f, let $\phi_+(f) = \min \mathrm{dom}(s_{f(n)})$ – in other words, $\phi_+(f)$ is the function such that for all n, $f(n)$ is a code for $X \upharpoonright I^n_{\phi_+(f)}$. By the corollaries above, if $M_{\phi_-(g)} \subseteq M_f$, then f is engulfed by $\phi_-(g)$, so $\phi_+(f)$ is interval dominated by g. Therefore these maps form a morphism. By Proposition IV.12, it is an effective morphism. □

Corollary V.14. *For all* $\mathbf{C}$ *and* $\mathbf{A}$, $\mathbf{D}^{\mathbf{C}}_{\mathbf{A}} \subseteq \mathbf{COFM}^{\mathbf{C}}_{\mathbf{A}}$ *and* $\mathbf{B}^{\mathbf{C}}_{\mathbf{A}} \supseteq \mathbf{ADDM}^{\mathbf{C}}_{\mathbf{A}}$

There are two more morphisms to be found before we need to include $\mathcal{N}$. They are:

- $\widetilde{\mathbf{Cof}(\mathcal{M})} \to \widetilde{\mathbf{Non}(\mathcal{M})}$.
- $\mathfrak{D} \to \widetilde{\mathbf{Cov}(\mathcal{M})}$.

Along with duality, these will prove this much of the diagram of Theorem V.7:

$$\begin{array}{ccc}
\mathbf{NONM_A^C} & \longleftarrow & \mathbf{COFM_A^C} \\
\downarrow & & \downarrow \\
\mathbf{B_A^C} & \longleftarrow & \mathbf{D_A^C} \\
\downarrow & & \downarrow \\
\mathbf{ADDM_A^C} & \longleftarrow & \mathbf{COVM_A^C}
\end{array}$$

The first morphism is easily supplied. In general, when looking for a morphism $\phi : \mathbf{Cof}(\mathcal{I}) \to \mathbf{Non}(\mathcal{I})$ (where $\mathcal{I}$ is a nontrivial ideal of subsets of X), we let ϕ_- be the identity, and for $I \in \mathcal{I}$, $\phi_+(I) = X$, where $X \notin I$ – so that if $J \subseteq I$, $X \notin J$. So for meager (or measure zero) sets, the goal is to find, recursively in a code, some real which is not in the coded object. For meager sets, this is simple, since $X \in \mathrm{Match}(X, I)$ for all I. The result requires no further proof:

Theorem V.15. *There exists an effective morphism* $\phi : \widetilde{\mathbf{Cof}(\mathcal{M})} \to \widetilde{\mathbf{Non}(\mathcal{M})}$.

Corollary V.16. *For all* $\mathbf{C}$ *and* $\mathbf{A}$, $\mathbf{NONM_A^C} \subseteq \mathbf{COFM_A^C}$ *and* $\mathbf{ADDM_A^C} \subseteq \mathbf{COVM_A^C}$.

The second morphism listed above relies on the observation that for any $f \in {}^\omega\omega$, the set $\{g : g \leq^* f\}$ is a meager subset of ${}^\omega\omega$.

Theorem V.17. *There exists an effective morphism* $\phi : \mathfrak{D} \to \widetilde{\mathbf{Cov}(\mathcal{M})}$

Proof. (based on an argument presented in [6], perhaps originally from [44])

We seek $\phi_- : {}^\omega 2 \to {}^\omega\omega$ and $\phi_+ : {}^\omega\omega \to \tilde{\mathcal{M}}$ such that $\phi_-(X) \leq^* f$ implies $X \in M_{\phi_+(f)}$ – that is, for all but finitely many n, X doesn't extend the function coded by $\phi_+(f)(n)$.

First, for $h \in {}^\omega\omega$, denote by Y_h the real such that $h(n)$ is the number of 0's between the nth and $(n+1)$th 1 ($h(0)$ is the number of 0's at the start of Y_h). Note that for all h, $Y_h \equiv_T h$ – in particular, p_{Y_h} is the function $n \mapsto \sum_{i \leq n}(h(n) + 1)$.

Now, define $\phi_-(X)$ to be X if it contains finitely many 1's, and otherwise the (unique) g such that $X = Y_g$. For $f \in {}^\omega\omega$, let

$$M = \{Y_g : g \leq^* f\} \cup \{Y : \forall^\infty m\ Y(m) = 0\}.$$

Note that if $\phi_-(X) \leq^* f$, either X contains finitely many 1's or it equals Y_g for some $g \leq^* f$ – thus, it is in M. Also, M is meager, as it is covered by the nowhere dense sets

$$M_n = \{Y_g : \forall m \geq n\ g(m) \leq f(m)\} \cup \{Y : \exists^{\leq n} m\ Y(m) = 1\}.$$

By Proposition IV.12, it remains to observe that we can answer (computably in f) if $[\tau] \cap M_n = \emptyset$ – this will only be the case if τ has at least $n+1$ 1's and for some $m \geq n$ (but less than the number of 1's in τ) the number of 0's between the mth and $(m+1)$th 1 is more than $f(m)$. □

Remark V.18. This is the only effective morphism in this paper that cannot be made uniformly so, in the sense that there are constants e_+ and e_- such that $\phi_+(f) = \Phi^f_{e_+}$ and $\phi_-(X) = \Phi^X_{e_-}$. Further, no such morphism could exist – such a morphism would be continuous, and it is possible to prove there is no continuous morphism from $\mathfrak{D}$ to $\widetilde{\mathbf{Cov}(\mathcal{M})}$. It is unclear if there are any consequences to this fact, but it is interesting nonetheless. We will discuss it somewhat further in Chapter VII.

Corollary V.19. *For all* $\mathbf{C}$ *and* $\mathbf{A}$, $\mathbf{COVM}^{\mathbf{C}}_{\mathbf{A}} \subseteq \mathbf{D}^{\mathbf{C}}_{\mathbf{A}}$ *and* $\mathbf{B}^{\mathbf{C}}_{\mathbf{A}} \subseteq \mathbf{NONM}^{\mathbf{C}}_{\mathbf{A}}$.

5.3 Null Sets

In this section, we describe two simple effective morphisms:

- $\widetilde{\mathbf{Cof}(\mathcal{N})} \to \widetilde{\mathbf{Non}(\mathcal{N})}$.
- $\widetilde{\mathbf{Non}(\mathcal{M})} \to \widetilde{\mathbf{Cov}(\mathcal{N})}$.

With duality and the results of the previous section, this establishes:

$$\begin{array}{ccccccc}
\mathbf{COVN^C_A} & \longleftarrow & \mathbf{NONM^C_A} & \longleftarrow & \mathbf{COFM^C_A} & & \mathbf{COFN^C_A} \\
 & & \big\downarrow & & \big\downarrow & & \\
\big\downarrow & & \mathbf{B^C_A} & \longleftarrow & \mathbf{D^C_A} & & \big\downarrow \\
 & & \big\downarrow & & \big\downarrow & & \\
\mathbf{ADDN^C_A} & & \mathbf{ADDM^C_A} & \longleftarrow & \mathbf{COVM^C_A} & \longleftarrow & \mathbf{NONN^C_A}
\end{array}$$

The remaining necessary morphism, from $\widetilde{\mathbf{Cof}(\mathcal{N})}$ to $\widetilde{\mathbf{Cof}(\mathcal{M})}$ is considerably more complicated, and will require some additional preparation.

To exhibit the first listed morphism, recall from the discussion prior to Theorem V.15 that it suffices to prove:

Proposition V.20. *For every $f \in \tilde{\mathcal{N}}$ (recall Definition IV.9), there exists $X \notin N_f$ such that $X \leq_T f$.*

Proof. (based on a proof presented in [40], possibly folklore)

Letting C_i denote the clopen set coded by i, we need to construct an X that is out of all but finitely many of the $C_{f(n)}$. We will in fact construct an X outside all $C_{f(n)}$ for $n \geq 4$ (note these $C_{f(n)}$ have total measure at most 1/8). Let $U_n = \bigcup_{4 \leq m \leq 2n+5} C_{f(m)}$, and observe that $\mu(U_{n+1} \backslash U_n) \leq 2^{-2n-6} + 2^{-2n-7} < 2^{-2n-5}$. We will construct X so: assume $X(m)$ has been defined for $m < n$. Choose $X(n)$ such that $\mu([X|_{n+1}] \cap U_n)$ is minimal – in case of a tie, let $X(n) = 0$. We can determine $X(n)$ computably relative to f since the function mapping n to an index for U_n (as a clopen set) is computable in f.

Lemma V.21. *For all n, $\mu([X|_{n+1}] \cap U_n) < 2^{-n-2} - 2^{-2n-3}$.*

Proof. Proof by induction. For $n = 0$, note

$$\mu([X|_1] \cap U_0) \leq 1/2\mu(U_0) < 1/2(1/8) < 1/4 - 1/8.$$

Now suppose the claim is true for n. Then

$$\begin{aligned}\mu([X|_{n+1}]\cap U_{n+1}) &\le \mu([X|_{n+1}]\cap U_n)+\mu(U_{n+1}\backslash U_n)\\ &< 2^{-n-2}-2^{-2n-3}+2^{-2n-5}\\ &= 2^{-n-2}-3\cdot 2^{-2n-5}\\ &< 2^{-n-2}-2\cdot 2^{-2n-5}\\ &= 2^{-n-2}-2^{-2n-4}\end{aligned}$$

By construction, $\mu([X|_{n+2}]\cap U_{n+1}) \le \frac{1}{2}\mu([X|_{n+1}]\cap U_{n+1})$, so this finishes the induction step and the proof. □

Now suppose X is in some $C_{f(m)}$ and fix k such that $[X|_k]\subseteq C_{f(m)}$. Then for any n such that $2n+5\ge m$ and $n+1\ge k$, $[X|_{n+1}]\subseteq U_n$, which contradicts the above lemma. □

As before, this is enough to prove:

Theorem V.22. *There exists an effective morphism* $\phi : \widetilde{\mathbf{Cof}(\mathcal{N})} \to \widetilde{\mathbf{Non}(\mathcal{N})}$.

Corollary V.23. *For all* $\mathbf{C}$ *and* $\mathbf{A}$, $\mathbf{NONN}^{\mathbf{C}}_{\mathbf{A}} \subseteq \mathbf{COFN}^{\mathbf{C}}_{\mathbf{A}}$ *and* $\mathbf{ADDN}^{\mathbf{C}}_{\mathbf{A}} \subseteq \mathbf{COVN}^{\mathbf{C}}_{\mathbf{A}}$.

Theorem V.24. *There exists an effective morphism* $\phi : \widetilde{\mathbf{Non}(\mathcal{M})} \to \widetilde{\mathbf{Cov}(\mathcal{N})}$.

Proof. (based on an argument in [43])

We seek $\phi_+ : {}^{\omega}2 \to \tilde{\mathcal{N}}$ and $\phi_- : {}^{\omega}2 \to \tilde{\mathcal{M}}$ such that $X \notin N_{\phi_-(Y)}$ implies $Y \in M_{\phi_+(X)}$.

Let $h(n)=n(n+1)$. Note that h is strictly increasing with $h(0)=0$, and further, $|I^n_h| > n$ – thus, for any real Z, $\mathrm{Match}(Z, I_h)$ is a comeager set of measure 0 (because the set of reals matching Z on I^n_h has measure less than 2^{-n}). Let $\phi_+(X)$ code $X \restriction I^n_h$ and $\phi_-(Y)$ code the clopen set $[Y \restriction I^n_h]$. Since h is computable, these satisfy the effectivity requirement. Also, if $X \notin N_{\phi_-(Y)}$, then for almost all n, $X \upharpoonright I_h^n \neq Y \upharpoonright I_h^n$. Thus, $Y \in {}^\omega 2 \setminus \mathrm{Match}(X, I_h)$, as needed. □

Corollary V.25. *For all* $\mathbf{C}$ *and* $\mathbf{A}$, $\mathbf{COVN}^{\mathbf{C}}_{\mathbf{A}} \subseteq \mathbf{NONM}^{\mathbf{C}}_{\mathbf{A}}$ *and* $\mathbf{COVM}^{\mathbf{C}}_{\mathbf{A}} \subseteq \mathbf{NONN}^{\mathbf{C}}_{\mathbf{A}}$.

5.4 Traces and Null Sets

In order to complete Theorem V.7, it remains only to exhibit an effective morphism from $\widetilde{\mathbf{Cof}(\mathcal{N})}$ to $\widetilde{\mathbf{Cof}(\mathcal{M})}$. A direct morphism is difficult to demonstrate. It will be easier to go through an intermediate debate involving traces (recall Definition II.82).

Definition V.26. Let h be an order. Denote by $\mathcal{T}_h$ the set of traces with bound h, and by $\mathfrak{TR}_h$ the debate $({}^\omega\omega, \mathcal{T}_h, \text{traces})$.

Remark V.27. We omit the use of ˜ in this definition, since we may identify $[\omega]^{<\omega}$ with ω. It is worth noting that in set theory (from whence we derive much of the work in this and the next section), traces are referred to as *slaloms*.

While different orders technically produce different debates, for the purposes of this thesis, all such debates are equivalent.

Proposition V.28. *If* f *and* g *are orders, then there is an effective morphism* $\phi : \mathfrak{TR}_g \to \mathfrak{TR}_f$.

Proof. Now we require $\phi_+ : \mathcal{T}_g \to \mathcal{T}_f$ and $\phi_- : {}^\omega\omega \to {}^\omega\omega$ such that if T traces $\phi_-(h)$, then $\phi_+(T)$ traces h. Let $x_n = \mu x(f(x) \geq g(n))$. Since f and g are orders, so is $n \mapsto x_n$. Let $k_n = x_{n+1} - x_n$.

We can think of our construction as having 2 sides – an f side ($\phi_+(T)$ and h) and a g side (T and $\phi_-(h)$). Objects on the f side will be considered k_n outputs at a time, and on the g side one output at a time. The k_n outputs for an object on the f

side will code the single output of the corresponding object on the g side using the coordinate function $\langle\rangle_{k_n}$.

Explicitly, we define $\phi_-(h)(n) = \langle h(x_n), h(x_n+1), \ldots, h(x_{n+1}-1)\rangle_{k_n}$ if $k_n > 0$ and 0 otherwise. To define ϕ_+, suppose $x_n \leq m < x_{n+1}$. For each $\langle y_0, y_1, \ldots, y_{k_n-1}\rangle_{k_n} \in T(n)$, put y_i into $\phi_+(T)(m)$, where $i = m - x_n$. If no such n exists for m (i.e., $m < x_0$) let $\phi_+(T)(m) = \emptyset$. Note that according to this definition, if $\phi_+(T)(m)$ is nonempty,

$$|\phi_+(T)(m)| \leq |T(n)| \leq g(n) \leq f(x_n) \leq f(m)$$

so $\phi_+(T) \in \mathcal{T}_f$. We have $\phi_-(h) \leq_T h$ and $\phi_+(T) \leq_T T$, since $\langle y_0, y_1, \ldots, y_{k_n-1}\rangle_{k_n} \mapsto y_i$ is computable uniformly in k_n and i for $0 \leq i < k_n$, and $(y_0, y_1, \ldots, y_{k_n-1}) \mapsto \langle y_0, y_1, \ldots, y_{k_n-1}\rangle_{k_n}$ is computable uniformly in k_n. Also, by definition of ϕ_- and ϕ_+, if $\phi_-(h)(n) \in T(n)$, then for all m such that $x_n < m \leq x_{n+1}$, $h(m) \in \phi_+(T)(m)$. It follows that if T traces $\phi_-(h)$ then $\phi_+(T)$ traces h, as needed. □

Remark V.29. The morphism in this proof is cumbersome to explain, but it follows from a technique which we will use repeatedly with traces. We can think of it in terms of one map in the morphism trying to undo the other. If ϕ_- turns k_n numbers into one using the coordinate map, ϕ_+ assumes it works on a set whose elements are already in that form (i.e., coding k_n coordinates). So ϕ_+ undoes the coordinate map on each element in the set. However, this method only works for morphisms *from* something of the form $\mathfrak{TR}_f$.

It follows that the cardinal and Turing characteristics associated to $\mathfrak{TR}_f$ are unaffected by the choice of f. In the material to follow, it will be convenient to use $f(n) = 2^n$. Since this is the only bound we will use in this thesis, we will from this point forward omit the subscript f.

The primary usefulness of traces in set theory is the equivalence $\mathbf{cof}(\mathcal{N}) = ||\mathfrak{TR}||$. As we shall see below, this equivalence transfers to Turing characteristics, which will prove a key intermediate step in showing that $\mathbf{COFM}_{\mathbf{A}}^{\mathbf{C}} \subseteq \mathbf{COFN}_{\mathbf{A}}^{\mathbf{C}}$ for all $\mathbf{A}, \mathbf{C}$. However, the equivalence (and its proof) are also of independent interest. Taking the default values for $\mathbf{A}, \mathbf{C}$, we get $\mathbf{COFN} = \mathbf{TR}$. As mentioned earlier (Proposition IV.18), **COFN** is the set of oracles not low for Schnorr tests, and from the definition, it is clear that **TR** is the set of oracles not computably traceable. Thus, the work below gives a proof of Proposition II.84 which is slightly different than the one originally given by Terwijn and Zambella. This is our reason for not giving a proof with the original citation (in Section 2.7) of this fact.

In all the work to follow, C_i denotes the clopen set coded by i, as usual.

Theorem V.30. *There is an effective morphism* $\phi : \mathfrak{TR} \to \widetilde{\mathbf{Cof}(\mathcal{N})}$

Proof. (based on an argument from [40]

We seek maps $\phi_+ : \mathcal{T} \to \tilde{\mathcal{N}}$, $\phi_- : \tilde{\mathcal{N}} \to {}^\omega\omega$ such that if T traces $\phi_-(f)$, then $N_f \subseteq N_{\phi_+(T)}$. We may use

$$\phi_-(f)(n) = \text{an index for } (C_{f(2n+1)} \cup C_{f(2n+2)})$$

$$\phi_+(T)(n) = \text{an index for } \left(\bigcup\{C_i : i \in T(n) \wedge \mu(C_i) \leq 2^{-2n}\}\right)$$

Note that the second union has measure at most $2^n \cdot 2^{-2n} = 2^{-n}$, so $\phi_+(T) \in \tilde{\mathcal{N}}$. Suppose $\forall^\infty n\ \phi_-(f)(n) \in T(n)$ and $X \in N_f$. Then there exist infinitely many n such that $X \in C_{f(2n+1)} \cup C_{f(2n+2)}$. Since $C_{f(2n+1)} \cup C_{f(2n+2)}$ has measure not more than 2^{-2n}, for almost all n $C_{f(2n+1)} \cup C_{f(2n+2)} \subseteq C_{\phi_+(T)(n)}$. It follows that $X \in N_{\phi_+(T)}$, as needed.

Since unions of clopen sets are computable, $\phi_-(f) \leq_T f$ and $\phi_+(T) \leq_T T$, and the morphism is effective. □

Remark V.31. The logic behind this morphism runs thus: in order to construct the measure 0 subset from T, we would like to assume every member of T codes a clopen set (of unspecified measure), and take their union to give another clopen set. But this union must have measure at most 2^{-n}. Since we might be taking the union of 2^n sets, each one must have measure at most 2^{-2n}. So in order to make sure enough of the $C_{f(k)}$ are included in the unions, we need to take them two at a time – otherwise, their measures will not diminish fast enough. Note that we again use the idea of assuming that elements in a trace are of the type created by ϕ_-.

Theorem V.32. *There is an effective morphism* $\phi : \widetilde{\mathbf{Cof}(\mathcal{N})} \to \mathfrak{TR}$.

Proof. (based on an argument from citeparam)

We seek maps $\phi_+ : \tilde{\mathcal{N}} \to \mathcal{T}$, $\phi_- : {}^{\omega}\omega \to \tilde{\mathcal{N}}$ such that $N_{\phi_-(g)} \subseteq N_f$ implies $\phi_+(f)$ traces g.

First we define ϕ_+. Temporarily fix $f \in \tilde{\mathcal{N}}$. For notational convenience, we let $E_n := C_{f(n)}$. Define a sequence of clopen sets F_n computably in f by the procedure:

$$F_0 = \emptyset$$

$$F^*_{n+1} = F_n \cup E_{2n+4} \cup E_{2n+5}$$

$$F_{n+1} = \begin{cases} F^*_{n+1} \cup C_{n+1} & \text{if } \mu(C_{n+1} \backslash F^*_{n+1}) < 2^{-2n-2} \\ F^*_{n+1} & \text{otherwise.} \end{cases}$$

Let $F = \bigcup_n F_n$. The goal is to add almost all the E_n to F, as well as any clopen set that does not increase the measure by too much. The use of E_n guarantees $N_f \subseteq F$, and the control over the measure of F makes it possible to determine which clopen sets are subsets of F.

Lemma V.33. *For all n:*

1. $\mu(F \backslash F_n) < 2^{-2n-1}$.

2. $C_n \nsubseteq F_n$ *implies* $\mu(C_n \backslash F) > 2^{-2n-1}$. *Hence,* $C_n \subseteq F$ *iff* $C_n \subseteq F_n$*, so* $\{n : C_n \subseteq F\}$ *is computable in* f*.*

Proof. 1. By construction, for all m,

$$\mu(F_{m+1} \backslash F_m) < 2^{-2m-4} + 2^{-2m-5} + 2^{-2m-2} = 11 \cdot 2^{-2m-5}.$$

Summing over all $m \geq n$, we have

$$\mu(F \backslash F_n) < 44/3 \cdot 2^{-2n-5} < 2^{-2n-1}.$$

2. If $C_n \nsubseteq F_n$, then $F_n = F_n^*$, so $\mu(C_n \backslash F_n) \geq 2^{-2n}$. As $\mu(C_n \backslash F) \geq \mu(C_n \backslash F_n) - \mu(F \backslash F_n)$, the previous result proves this claim. □

Now we define a computable array of clopen sets G_n^i which are measure independent and satisfy $\mu(G_n^i) = 2^{-n}$ for all n, i. Let

$$B_n^i = \{\langle m_1, m_2 \rangle | m_1 + 1 = n \wedge i \cdot n \leq m_2 < (i+1) \cdot n\}.$$

The B_n^i are pairwise disjoint, and $|B_n^i| = n$ for all n, i. Take G_n^i to be the set of reals X such that $X(k) = 0$ for all $k \in B_n^i$. Note that if $D \subset 2^{<\omega}$ is finite and $|\sigma| \leq i \cdot n$ for all $\sigma \in D$, then $[D]$ and G_n^i are measure independent.

Let $R(n, k)$ be the set $\{i | G_n^i \cap (C_k \backslash F) = \emptyset\}$ (or, equivalently, the set $\{i | (G_n^i \cap C_k) \backslash F = \emptyset\}$). By Lemma V.33, the relation $i \in R(n, k)$ is computable in f, and if $C_k \subseteq F$, $R(n, k) = \omega$. Otherwise, the following lemmas prove useful:

Lemma V.34. *There exists a function* h *computable in* f *such that if* $C_k \nsubseteq F$*,* $\max R(n, k) \leq h(n, k)$*.*

Proof. The value of h for (n,k) where $C_k \subseteq F$ is irrelevant, so assume otherwise. Let m be minimal such that $2m+1 \geq n+2k+1$, D a finite set of binary strings such that $[D] = C_k \backslash F_m$, and $h(n,k)$ the minimal j such that $|\sigma| \leq j \cdot n$ for all $\sigma \in D$. For $i \geq h(n,k)$,

$$\begin{aligned}\mu(G_i^n \cap (C_k \backslash F_m)) &= 2^{-n} \cdot \mu(C_k \backslash F_m)\\ &\geq 2^{-n} \cdot \mu(C_k \backslash F)\\ &> 2^{-n} \cdot 2^{-2k-1} \geq 2^{-2m-1}.\end{aligned}$$

Since $\mu(F \backslash F_m) < 2^{-2m-1}$, $(G_n^i \cap C_k) \backslash F$ cannot be empty. □

Lemma V.35. *If $C_k \nsubseteq F$, then for almost all n, $|R(n,k)| < 2^{n-k-1}$.*

Proof. Note that if $i \in R(n,k)$, then $C_k \backslash F \subseteq 2^\omega \backslash G_n^i$. Therefore,

$$\begin{aligned}0 < 2^{-2k-1} &< \mu(C_k \backslash F)\\ &\leq \mu\left(\bigcap_n \bigcap_{i \in R(n,k)} (2^\omega \backslash G_n^i)\right)\\ &= \prod_n \prod_{i \in R(n,k)} \mu(2^\omega \backslash G_n^i) = \prod_n (1-2^{-n})^{|R(n,k)|}.\end{aligned}$$

Note we are using the previous lemma, which implies $R(n,k)$ is finite, and the measure independence of the G_n^i. The nonzero bound on the product implies $\liminf_n (1 - 2^{-n})^{|R(n,k)|} = 1$. On the other hand, if there are infinitely many n such that $|R(n,k)| \geq 2^{n-k-1}$, then

$$\liminf_n (1-2^{-n})^{|R(n,k)|} \leq \lim_{n\to\infty} (1-2^{-n})^{2^{n-k-1}} = e^{-2^{-k-1}},$$

a contradiction. □

For $k < n$, let $\tilde{R}(n,k)$ be the set containing the 2^{n-k-1} least i in $R(n,k)$ if $|R(n,k)| > 2^{n-k-1}$, and $R(n,k)$ otherwise. Finally, let $\phi_+(f)(n)$ be the union of all $\tilde{R}(n,k)$ for $k < n$ (so that $|\phi_+(f)(n)| \leq 2^n$). By Lemma V.34, $\phi_+(f) \leq_T f$ (since we know when to stop looking for members of $R(n,k)$), and by Lemma V.35, if $C_k \nsubseteq F$, then for almost all n, $R(n,k) \subseteq \phi_+(f)(n)$.

The definition of ϕ_- is much simpler – we let $\phi_-(g)(n)$ be a clopen set index for $G_n^{g(n)}$ (it is clear that $\phi_-(g) \leq_T g$). We now suppose that for some f, $N_g \subseteq N_f \subseteq F$ (the F defined in the course of determining $\phi_+(f)$). We work in the space ${}^\omega 2\backslash F$, using the topology inherited as a subspace of ${}^\omega 2$. Since this is a closed subset of a Baire space (a space where any countable intersection of open dense sets is nonempty), ${}^\omega 2\backslash F$ is a Baire space as well. For $N \subseteq {}^\omega 2$, let $\vec{N}$ denote $N\backslash F$. Then we have

$$\bigcap_{n\in\omega}\bigcup_{m\geq n} \vec{G_m^{g(m)}} = \vec{\emptyset}.$$

Therefore, for some m, $\bigcup_{m\geq n} \vec{G_n^{g(n)}}$ is not dense. That is, there is a k such that $\vec{C_k} \neq \vec{\emptyset}$ and for all $n \geq m$, $\vec{G_n^{g(n)}} \cap \vec{C_k} = \vec{\emptyset}$, so $g(n) \in R(n,k)$ for almost all n. As $C_k \nsubseteq F$, $g(n) \in \phi_+(f)(n)$ for almost all n, as needed. □

Corollary V.36. *For all* $\mathbf{A}, \mathbf{C}$, $\mathbf{COFN}_{\mathbf{A}}^{\mathbf{C}} = \mathbf{TR}_{\mathbf{A}}^{\mathbf{C}}$ *and* $\mathbf{ADDN}_{\mathbf{A}}^{\mathbf{C}} = \langle \mathfrak{TR}^\perp \rangle_{\mathbf{A}}^{\mathbf{C}}$.

5.5 Traces and Meager Sets

In this section, we finish the proof of Theorem V.7 by relating $\mathfrak{TR}$ to $\mathbf{Cof}(\mathcal{M})$. Recall the proof of Theorem V.30. We can think of the proof in this way: given a trace T (with bound 2^n), imagine each $T(n)$ as a set of codes for clopen sets used in the definition of up to 2^n measure zero sets. To construct a superset of these increasingly many measure zero sets, use the unions of clopen sets coded by elements of $T(n)$. We do this because a measure zero set is expressed as the set of reals in infinitely many of some sequence of clopen sets. The rest of the proof was bookkeeping to ensure the resulting superset also had measure 0.

With meager sets, we will use a similar strategy. The main change is that we express the meager set as the set of reals out of almost all of some sequence of clopen sets. Specifically, if s_i denotes the member of S coded by i (recall S is the set of partial functions from ω to $\{0,1\}$ with finite domain), then M_f is the set of reals out of almost all of the sets of the form $[s_{f(n)}]$. So now, given a trace T, imagining each $T(n)$ as a set of codes for clopen sets used in the definition of some meager sets, we will intersect the clopen sets coded instead of taking their union. But in doing so, we need to be more careful, since we don't want the intersections to become empty. The following fact will prove useful in avoiding these empty intersections.

Proposition V.37. *For all $r \in \omega$, if $\{u_{p,q}\}_{0 \leq p,q \leq r}$ is an array of elements of S such that for all q, the domains of $u_{p,q}$ are pairwise disjoint intervals, then there exists $t \in S$ such that for all q, there exists p with $u_{p,q} \subseteq t$.*

Proof. Proof by induction on r. The case $r = 0$ is trivial. Assume the proposition holds for r, and let $\{u_{p,q}\}_{0 \leq p,q \leq r+1}$ be an array with the described properties. For each q, denote by $\tilde{u}_q$ the $u_{p,q}$ with $\min \operatorname{dom}(u_{p,q})$ maximal, and let q' be the q with $\min \operatorname{dom}(\tilde{u}_q)$ maximal. Apply the induction hypothesis to

$$\{u_{p,q} | q \neq q' \wedge u_{p,q} \neq \tilde{u}_q\},$$

obtaining $\tilde{t}$. Note that for all $u_{p,q}$ in the set above, $\max \operatorname{dom}(u_{p,q}) < \min \operatorname{dom}(\tilde{u}_{q'})$, so we may assume $\max \operatorname{dom}(\tilde{t}) < \min \operatorname{dom}(\tilde{u}_{q'})$. Clearly, there exists $t \supseteq \tilde{t} \cup \tilde{u}_{q'}$. □

Remark V.38. Without loss of generality, we may assume t has domain an interval. Since we can check the necessary conditions on t computably, we can find such a t computably (i.e., search for a t with minimal index).

Theorem V.39. *There exists an effective morphism $\phi : \mathfrak{TR} \to \widetilde{\mathbf{Cof}(\mathcal{M})}$.*

Proof. (based on an argument from [39])

We seek $\phi_+ : \mathcal{T} \to \tilde{\mathcal{M}}$ and $\phi_- : \tilde{\mathcal{M}} \to {}^\omega\omega$ such that if T traces $\phi_-(g)$, then $M_g \subseteq M_{\phi_+(T)}$.

To define $\phi_-(g)$, first note that, using the codings from Section 2.2 of S and $[\omega]^{<\omega}$ by members of ω, we may also code $[S]^{<\omega}$ using members of ω. Let $\phi_-(g)(n)$ be an index for $\{s_{g(m)}\}_{n\leq m<n+2^n}$.

Given T, view the elements of $T(n)$ as indices for elements of $[S]^{<\omega}$, and let $\tilde{T}(n)$ consist of only the elements of $T(n)$ which satisfy the condition of corresponding to subsets of S of cardinality 2^n whose elements have domains that are pairwise disjoint *and* disjoint from $\{0, 1, \ldots, n-1\}$. This condition is computable, so $\tilde{T} \leq_T T$.

Now fix n and let $s_{p,q}$ denote the pth member of the qth element of $\tilde{T}(n)$. Apply Proposition V.37 (with $r = 2^n$) to this array, obtaining (computably in $\tilde{T}$) the existence of a $t \in S$ such that for all q there exists a p with $s_{p,q} \subseteq t$ (note that for some $q < r$, $s_{p,q}$ may not be defined, but this clearly does not alter the existence of t). Without loss of generality, we may also assume there exists such a t with $\min \operatorname{dom}(t) = n$. Denote this t by t_n.

Now, let M_n be the set of X for which $t_m \nsubseteq X$ for all $m \geq n$. Note that $\bigcup_n M_n$ is meager, as witnessed by $h(n, \sigma) = \sigma 0^e t_k$, where $k = \max(|\sigma|, n)$ and $e = k - |\sigma|$. Clearly h is computable in $\tilde{T}$, and hence in T, so by Proposition IV.12 there is $\phi_+(T) \leq_T T$ such that $\bigcup M_n \subseteq M_{\phi_+(T)}$.

Assume that T traces $\phi_-(g)$. Note that if $\phi_-(g)(n) \in T(n)$, then $\phi_-(g)(n) \in \tilde{T}(n)$. It follows that for all but finitely many n, there is an $m \in [n, n+2^n)$ such that $s_{g(m)} \subseteq t_n$. Suppose $X \in M_g$, and fix n such that for all $m \geq n$, $s_{g(m)} \not\subset X$. It follows that for all $m \geq n$, $t_m \not\subset X$, so $X \in M_{\phi_+(T)}$. Therefore $M_g \subseteq M_{\phi_+(T)}$, proving ϕ is a morphism. $\square$

Corollary V.40. *For all* $\mathbf{C}$ *and* $\mathbf{A}$, $\mathbf{ADDN}^{\mathbf{C}}_{\mathbf{A}} \subseteq \mathbf{ADDM}^{\mathbf{C}}_{\mathbf{A}}$ *and* $\mathbf{COFN}^{\mathbf{C}}_{\mathbf{M}} \supseteq \mathbf{COFM}^{\mathbf{C}}_{\mathbf{A}}$.

Proof. By the preceding theorem and Theorem V.32. □

This finishes the proof of Theorem V.7.

5.6 Splitting and Unsplittable Families

Although they will not play a role in the thesis beyond this chapter, the Turing characteristics $\mathbf{R}_{\mathbf{A}}^{\mathbf{C}}$ and $\mathbf{S}_{\mathbf{A}}^{\mathbf{C}}$ interact notably with the Turing characteristics we have been studying. These interactions are, as usual, proven by making the set theory proofs effective via finding effective morphisms. In this section, we demonstrate the existence of the following effective morphisms:

- $\mathfrak{R} \to \tilde{\mathfrak{D}}'^{\perp}$,
- $\mathfrak{R} \to \widetilde{\mathbf{Cov}(\mathcal{N})}$, and
- $\mathfrak{R} \to \widetilde{\mathbf{Cov}(\mathcal{M})}$.

All proofs in this section are based on arguments presented in [6] which may be folklore.

Theorem V.41. *There exists an effective morphism* $\phi : \mathfrak{R} \to \tilde{\mathfrak{D}}'^{\perp}$.

Proof. We require $\phi_+ : [\omega]^\omega \to {}^\omega\omega \uparrow$ and $\phi_- : {}^\omega\omega \uparrow \to [\omega]^\omega$ such that if f interval dominates $\phi_+(A)$, then $\phi_-(f)$ splits A. Let $\phi_+(A) = p_{A\cup\{0\}}$, and $m \in \phi_-(f)$ iff the least n such that $f(n) > m$ is odd (that is, $\phi_-(f)$ is $\bigcup_n I_f^{2n}$). Clearly ϕ_+ and ϕ_- satisfy the effectiveness conditions. Now suppose f interval dominates $\phi_+(A)$. Then for almost all n, there exists a k such that $f(n) \leq p_A(k) < f(n+1)$, so $p_A(k) \in I_f^n$. In particular, for almost all n, $A \cap I_f^{2n}$ and $A \cap I_f^{2n+1}$ are nonempty. It follows that $\phi_-(f)$ splits A, as needed. □

Theorem V.42. *There exists an effective morphism* $\phi : \mathfrak{R} \to \widetilde{\mathbf{Cov}(\mathcal{M})}$.

Proof. We seek $\phi_- : {}^{\omega}2 \to [\omega]^{\omega}$ and $\phi_+ : [\omega]^{\omega} \to \tilde{\mathcal{M}}$ such that $\phi_-(X)$ not splitting A implies $X \in M_{\phi_+(A)}$.

Consider the set U_A of reals which do not split A. This set is meager, since it is covered by the nowhere dense sets

$$U_{A,n} = \{X : A \backslash \{m : m < n\} \subseteq X \text{ or } \bar{X}\}$$

Note that $[\tau] \cap U_{A,n} = \emptyset$ iff there exist $m_0, m_1 \in A$ such that $|\tau| > m_i \geq n$ and $\tau(m_i) = i$ for $i = 0, 1$. This is clearly recursive in A, so by Proposition IV.12 there is $\phi_+(A) \leq_T A$ such that $U_A \subseteq M_{\phi_+(A)}$. Using the identity map for ϕ_- completes the proof. □

Theorem V.43. *There exists an effective morphism* $\phi : \mathfrak{R} \to \widetilde{\mathbf{Cov}(\mathcal{N})}$.

Proof. We seek $\phi_- : {}^{\omega}2 \to [\omega]^{\omega}$ and $\phi_+ : [\omega]^{\omega} \to \tilde{\mathcal{N}}$ such that $\phi_-(x)$ not splitting A implies x is in the null set coded by $\phi_+(A)$. As in Theorem V.42, ϕ_- will be the identity, and $\phi_+(A)$ will code a null set covering U_A (which is null as well as meager).

First, we define recursive functions f and g such that $f(\langle m, k \rangle)$ (respectively, $g(\langle m, k \rangle)$) is the finite string $\sigma_m 0^{\langle m,k \rangle + 1}$ ($\sigma_m 1^{\langle m,k \rangle + 1}$), where σ_m is the string coded by m. Note that any finite (cofinite) real can be expressed as a finite string followed by ω many 0's (1's), and thus is in infinitely many of the neighborhoods $[f(n)]$ ($[g(n)]$).

For $Y \in {}^{\omega}2$, define $A \circ Y$ to be the subset of A such that $A \circ Y(p_A(n)) = Y(n)$, and for $\sigma \in 2^{<\omega}$ define $A \circ \sigma$ to be the finite function with domain a subset of A such that $A \circ \sigma(p_A(n)) = \sigma(n)$ if the latter is defined and $A \circ \sigma(p_A(n))$ is otherwise undefined. Then X fails to split A iff $X \cap A = A \circ Y$ for some finite or cofinite Y. It follows that if X does not split A, then for infinitely many n, X extends either

$A \circ f(n)$ or $A \circ g(n)$. That is, we can take $\phi_+(A)(n)$ to be the (clopen set) index of $[A \circ \sigma_{f(n)}] \cup [A \circ \sigma_{g(n)}]$ (which has measure at most 2^{-n}). Note that this can be determined computably relative to A, using p_A to find finite function indices for $A \circ f(n)$ and $A \circ g(n)$. □

Corollary V.44. *For all* $\mathbf{A}$ *and* $\mathbf{C}$*,*

$$\mathbf{S}^{\mathbf{C}}_{\mathbf{A}} \subseteq \mathbf{NONM}^{\mathbf{C}}_{\mathbf{A}}, \mathbf{D}^{\mathbf{C}}_{\mathbf{A}}, \mathbf{NONN}^{\mathbf{C}}_{\mathbf{A}}$$

$$\mathbf{R}^{\mathbf{C}}_{\mathbf{A}} \supseteq \mathbf{COVM}^{\mathbf{C}}_{\mathbf{A}}, \mathbf{B}^{\mathbf{C}}_{\mathbf{A}}, \mathbf{COVN}^{\mathbf{C}}_{\mathbf{A}}$$

5.7 B, D, and Effective Semi-Morphisms

At this point, it is worth taking some time to reflect on how the results of Sections 5.1–5.6 fit in with what is already known in computability theory. Specifically, we look at the case where $\mathbf{A}$ is the set of computable oracles, and $\mathbf{C}$ is ${}^{\omega}2$. In most cases that we may talk about, we get confirmations of known (and not difficult) results. For example, any high oracle computes a hyperimmune set ($\mathbf{B} \subseteq \mathbf{D}$), any weakly 1-generic real computes a hyperimmune set ($\mathbf{COVM} \subseteq \mathbf{D}$), and any weakly 1-generic real computes a bi-immune set ($\mathbf{COVM} \subseteq \mathbf{R}$).

Somewhat more interesting are the results concerning **ADDN** and **NONN**. Parsing the definitions,

- **ADDN** is the set of A such that the union of all (unrelativized) Schnorr null sets is Schnorr null relative to A.

- **NONN** is the set of A such that the set of computable reals is Schnorr null relative to A.

The article [45] calls such oracles *Schnorr covering* and *weakly Schnorr covering*, respectively. Intuitively, A is Schnorr covering if it computes a “universal” relativized

Schnorr test (i.e., universal in every respect except that it is not computable). The fact that every such oracle is high ($\mathbf{ADDN} \subseteq \mathbf{B}$) shows how badly there fails to be a universal Schnorr test (see the discussion following Definition II.62). Additionally, any weakly 1-generic real computes a weakly Schnorr covering oracle, which in turn is not computably traceable ($\mathbf{COVM} \supseteq \mathbf{NONN} \subseteq \mathbf{COFN}$), though the last implication is trivial – if the set R of computable reals is Schnorr null relative to A, since R is not Schnorr null, it witnesses A is not low for Schnorr tests.

However, there are gaps in the results from Sections 5.1–5.6 (at least, for our specific choices for $\mathbf{A}$ and $\mathbf{C}$). For example, we have already mentioned that $\mathbf{COVM} = \mathbf{D}$ (by Proposition II.91), even though we do not have a result stating there is an effective morphism from $\widetilde{\mathbf{Cov}(\mathcal{M})}$ to $\mathfrak{D}$. In fact, by the completeness of Cichoń's diagram, there are models of ZFC where such a morphism definitely does not exist. Similarly, we know (though we did not exhibit a proof) that $\mathbf{B} \subseteq \mathbf{COVN}$, and can further prove that $\mathbf{B} \subseteq \mathbf{ADDN}$.

Proposition V.45. *If A is a high oracle, then A computes a trace T which traces all computable functions.*

Proof. Let $f \leq_T A$ dominate all computable functions. If Φ_e is total, then f dominates $n \mapsto \mu s[\Phi_{e,s}(n)\downarrow]$. Therefore, we may let $T(n)$ be the set of m such that $\Phi_{e,f(n)}(n) = m$ for some $e < 2^n$. □

Corollary V.46. $\mathbf{COVN} \subseteq \mathbf{ADDN} = \mathbf{TR} = \mathbf{ADDM} = \mathbf{B}$*. In other words, A is Schnorr covering iff A is high, and if A is high A computes a Schnorr random real.*

We make two observations about this proof. The first observation is that it relies on $\mathbf{A}$ containing only computable oracles – i.e., this argument will not prove $\mathbf{B_A} \subseteq \mathbf{TR_A}$ for any $\mathbf{A}$ which contains some non-computable B. The reason is that to construct T, A would have to be able to compute $\Phi^B_{e,f(n)}(n)$ (in place of $\Phi_{e,s}(n)$), which it cannot necessarily do because $A \geq_T B$ is not guaranteed.

The second observation to make is that while it does not provide an effective morphism, it does provide something that looks very similar to an effective morphism. Given an f, we map it to a trace $T \leq_T f$, and given a computable g, we map it to an $h \leq_T g$ – namely, for an e such that $g = \Phi_e$, we let $h(n) = \mu s[\Phi_{e,s}(n) \downarrow]$. This is done in such a way that if f dominates h, then T traces g. The only reason this is not a morphism is that we only use computable g. If we generalize this procedure which almost gives a morphism, we get the following definition.

Definition V.47. Let $\mathfrak{K} = (K_-, K_+, K)$ and $\mathfrak{L} = (L_-, L_+, L)$ be debates such that $K_\pm, L_\pm \subseteq {}^\omega\omega$. An *effective semi-morphism* $\phi : \mathfrak{K} \to \mathfrak{L}$ is a pair (ϕ_-, ϕ_+) consisting of a function $\phi_- : L_- \to K_-$ and a partial function $\phi_+ :$ from K_+ to L_+ whose domain is the set of computable members of K_+, such that for all $X \in L_-$ and computable $Y \in K_+$, $\phi_-(X) \leq_T X$, $\phi_+(Y)$ is computable, and $\phi_-(X)KY$ implies $XL\phi_+(Y)$.

Theorem V.48. *Let $\phi : \mathfrak{K} \to \mathfrak{L}$ be an effective semi-morphism and suppose $\mathbf{C}$ is closed downward with respect to $\leq_T$. Then $\langle \mathfrak{L} \rangle^{\mathbf{C}} \subseteq \langle \mathfrak{K} \rangle^{\mathbf{C}}$*

Proof. Identical to the proof of Theorem V.3, except with "computable from an oracle in **A**" replaced with "computable." □

Theorem V.49. *There are effective semi-morphisms $\phi : \mathbf{Add}(\mathcal{N}) \to \mathfrak{B}$ and $\psi : \mathbf{Cov}(\mathcal{M}) \to \mathfrak{D}$.*

Proof. This is implicit in the proofs of Propositions II.91 and V.45. We have already discussed how to phrase the proof of Proposition V.45 in terms of finding an effective semi-morphism. The other semi-morphism is more complicated to describe.

Consider the proof of the $\Leftarrow$ implication in Proposition II.91. In order to view this as giving an effective semi-morphism ψ as described above, we need to define $\psi_+(g) \in {}^\omega\omega$ for computable $g \in \tilde{\mathcal{M}}$ and $\psi_-(f) \in {}^\omega 2$ for $f \in {}^\omega\omega$ such that if f dominates $\psi_+(g)$, then $\psi_-(f) \notin M_g$. The proof of Proposition II.91 almost fits this mold, except instead of using $g \in \tilde{\mathcal{M}}$, it uses a c.e. dense $W_e \subseteq {}^{<\omega}2$, and defines from it the function

$$h_e(n) = \mu s[\forall \sigma \in {}^n 2 \exists \tau \supseteq \sigma \ |\tau| \leq s \wedge \tau \in W_{e,s}].$$

Given an f, we construct a real $X = \psi_-(f)$ in such a way that if f is not dominated by h_e, then $X \in [W_e]$. That is, follow the construction given in the proof, *without assuming f is not dominated by any h_e* (if the construction fails to produce a real – that is, for some n, $\sigma_m = \sigma_n$ for all $m \geq n$ – let X be σ_n followed by ω many 0's). The remainder of the proof shows that for all e such that h_e does not dominate f, then $X \in [W_e]$.

The only obstruction, then, is that the proof starts with a c.e. open dense set, rather than a $g \in \tilde{\mathcal{M}}$. We can remove this obstacle by first representing ${}^\omega 2 \backslash M_g$ as $\bigcup_{m \in \omega} [W_{\hat{g}(m)}]$ for some computable $\hat{g}$ such that each $W_{\hat{g}(m)}$ is dense (which we can do by Proposition IV.12). Now let

$$\psi_+(g)(n) = \max_{m \leq n} h_{\hat{g}(m)}(n),$$

which is computable since h_e is uniformly partial computable and total for all e such that W_e is dense. Observe that if f is not dominated by $\psi_+(g)$, then f is not dominated by any $h_{\hat{g}(m)}$. Therefore, $\psi_-(f) \in [W_{\hat{g}(m)}]$ for all m, so $\psi_-(f) \notin M_g$, as needed. □

Of course, Theorem V.49 produces nothing new, since we just reformulated proofs we already had. However, it does suggest that effective morphisms and their variants may completely describe relationships between Turing characteristics. We return to this idea in Chapter VII.

CHAPTER VI

Negative Results

Combining the results from Chapter V, we have proven:

Theorem VI.1. *Let* $\leftarrow$ *denote* $\subseteq$. *Then:*

$$
\begin{array}{ccccccc}
\mathbf{COVN} & \longleftarrow & \mathbf{NONM} & \longleftarrow & \mathbf{COFM} & \longleftarrow & \mathbf{COFN} \\
 & & \downarrow & & \downarrow & & \\
\downarrow & & \mathbf{B} & \longleftarrow & \mathbf{D} & & \downarrow \\
 & & \| & & \| & & \\
\mathbf{ADDN} & = & \mathbf{ADDM} & \longleftarrow & \mathbf{COVM} & \longleftarrow & \mathbf{NONN}
\end{array}
$$

In this chapter, we prove this diagram is complete – if there is no arrow or sequence of arrows leading from $\mathbf{P}$ to $\mathbf{Q}$, then $\mathbf{Q} \not\subseteq \mathbf{P}$.

For some $\mathbf{P}$ and $\mathbf{Q}$, we can already prove this using the equivalences from Section 4.3 and the material from Chapter II. For example, consider any non-computable low real X (for example, a low Martin-Löf random real, which from the discussion following Definition II.62 exists). Since $\emptyset <_T X \leq_T X' \leq_T \emptyset'$, X computes a hyperimmune set (Proposition). But X is not high. Therefore, X witnesses both $\mathbf{D} \not\subseteq \mathbf{B}$ and $\mathbf{COVM} \not\subseteq \mathbf{ADDM}$.

Similarly, let X be a hyperimmune-free Schnorr random real (also guaranteed to exist by the discussion following Definition II.62). Then X witnesses $\mathbf{COVN} \not\subseteq \mathbf{D}$ (and therefore also $\mathbf{COVN} \not\subseteq \mathbf{ADDN}$ and $\mathbf{NONM} \not\subseteq \mathbf{B}$).

In order to go farther, we will need to study **NONM**, **COFM**, and **NONN** more carefully. We will do so in this chapter. In particular, we will be able to establish, by adapting set theoretic arguments, equivalents to **NONM** and **COFM** which appeared in Chapter II (much as we did in Section 4.3). These equivalents will allow us to prove $\mathbf{NONM} \not\subseteq \mathbf{COVN}$ and $\mathbf{COFM} \not\subseteq \mathbf{NONM}, \mathbf{D}$.

Forcing proofs will play an important role in this chapter, which is to be expected since such arguments are used to separate cardinal characteristics. The first forcing argument we present is one that shows $\mathbf{COVM} \not\subseteq \mathbf{NONM}$. Although this particular result will prove redundant, the proof is instructive, and will prepare for later forcing arguments, especially two involving **NONN** which can be used to show $\mathbf{COFN} \not\subseteq \mathbf{NONN}, \mathbf{COFM}$. This will finish the proof of the following theorem.

Theorem VI.2. *The diagram of Theorem VI.1 is complete.*

6.1 Cohen Forcing and Meager Sets

Recall that by the completeness of Cichoń's diagram, is it not a theorem of ZFC that $\mathbf{cov}(\mathcal{M}) \leq \mathbf{non}(\mathcal{M})$. To construct a model of ZFC where $\mathbf{non}(\mathcal{M}) < \mathbf{cov}(\mathcal{M})$, we may iterate a forcing which adds a real not in any ground model meager set without adding a meager set which covers all ground model reals. As it turns out, Cohen forcing (where the conditions are elements of ${}^{<\omega}2$) has this property. As we saw in Section 3.4, algorithmically generic reals are the computable analogue to Cohen generic reals (an idea reinforced by the fact that the weakly 1-generic reals are exactly those not in M_g for any computable $g \in \tilde{\mathcal{M}}$). Therefore, we would suspect that a sufficiently generic real would witness $\mathbf{COVM} \not\subseteq \mathbf{NONM}$, and furthermore, a proof of this fact can be obtained by adapting the proof that the Cohen forcing does not add a meager set covering the ground model reals. Both suspicions are correct.

Notation VI.3. If s_n denotes the finite domain partial function from ω to $\{0,1\}$ which is coded by n, define the functions $Smax$ and $Smin$ by

$$Smax(n) = \max \mathrm{dom}(s_n)$$

$$Smin(n) = \min \mathrm{dom}(s_n).$$

Observe that both functions are computable. Also, let Int $\subseteq \omega$ be the computable set of n for which $\mathrm{dom}(s_n)$ is a nonempty interval.

Theorem VI.4. *If A is a 2-generic real, $A \notin$ **NONM**.*

Proof. This proof will follow a strategy similar to that of Proposition II.89. Assume that A is 2-generic and $\Phi_e^A \in \tilde{\mathcal{M}}$. Then we can show there is a $\sigma \subset A$ which forces "$\Phi_e^A \in \tilde{\mathcal{M}}$." Then we construct a computable real X such that for any $\tau \supseteq \sigma$, there is ν extending τ such that X matches the chopped real (or fragment thereof) given by Φ_e^ν beyond a specified point. That is, we will consider all such τ in turn, and define more of X to ensure that this is the case. It will follow that no $\tau \supseteq \sigma$ can force $X \notin M_{\Phi_e^A}$, and therefore $X \in M_{\Phi_e^A}$.

Now let us provide details. Fix e such that $\Phi_e^A \in \tilde{\mathcal{M}}$. Let

$$\begin{aligned} V = \{\sigma : \exists n \forall \nu \supseteq \sigma\ (&\Phi_{e,|\nu|}^\nu(n)\uparrow \\ &\vee \Phi_{e,|\nu|}^\nu(n) \notin \mathrm{Int} \\ &\vee Smax(\Phi_{e,|\nu|}^\nu(n)) + 1 \neq Smin(\Phi_{e,|\nu|}^\nu(n+1)) \\ &\vee Smin(\Phi_{e,|\nu|}^\nu(0)) \neq 0)\} \end{aligned}$$

No $\sigma \subset A$ can be contained by V. By 2-genericity of A, there is $\sigma \subset A$ such that for all $\tau \supseteq \sigma$, $\tau \notin V$ (i.e., σ forces $\Phi_e^A \in \tilde{\mathcal{M}}$). We will not actually need the full strength of this fact – it suffices to know that for all $\tau \supseteq \sigma$ and $n \in \omega$, there is ν extending τ such that $\Phi_{e,|\nu|}^\nu(n)\downarrow$, and furthermore $Smin(\Phi_{e,|\nu|}^\nu(n)) \geq n$.

We construct a computable $X \notin M_{\Phi_e^A}$ by initial segments X_s. Let τ_n be an enumeration of the strings extending σ (e.g., τ_n is σ concatenated with the string coded by n). Begin with $X_0 = \emptyset$. Given X_s, let r be minimal such that $\nu := \tau_r$ extends τ_s and $\Phi^{\nu}_{e,|\nu|}(|X_s|)\downarrow$, and let u be the finite domain function coded by this computed value. Since $Smin(\Phi^{\nu}_{e,|\nu|}(X_s)) \geq |X_s|$, we may let X_{s+1} be the string with minimal index which extends $X_s \cup u$ (i.e., such strings exist). Let $X = \bigcup_s X_s$.

Now consider

$$V_m := \{\sigma' : \exists t, s \geq m \ \Phi^{\sigma'}_{e,t}(s)\downarrow \wedge s_{\Phi^{\sigma'}_{e,t}(s)} \subset X\}.$$

By construction of X, for every m and $\tau \supseteq \sigma$, there is an extension of τ in V_m (this was guaranteed by the definition of any X_s such that $\tau_s \supseteq \tau$ and $s \geq m$). It follows that it is impossible for there to be a $\sigma' \subset A$ such that $\tau \notin V_m$ for all $\tau \supseteq \sigma'$. So by the 2-genericity of A, for some $\sigma' \subset A$, $\sigma' \in V_m$. It follows that for infinitely many n, X extends $s_{\Phi_e^A(n)}$, so $X \notin M_{\Phi_e^A}$, as claimed. □

We observe that the proof above is not quite sharp. We used the 2-genericity of A twice. For the second use, only 1-genericity was necessary. For the first use, we needed 2-genericity to ensure ν could be defined for every τ extending σ. However, we only needed to be able to do this for τ which are initial segments of A, which is always possible regardless of the genericity of A. However, while we are constructing X, we do not know which τ are actually initial segments of A. Therefore, we need to be able to employ a computable function which tells us, for a given τ, how long to look for a ν before giving up. In the cases where τ actually is an initial segment of A, we only need the function to be large enough to find ν infinitely often. This line of reasoning suggests that we may be able to strengthen the proof of Theorem VI.4 to work for nonhigh 1-generic reals. We can, although some additional care is required.

Theorem VI.5. *If A is a non-high 1-generic real, $A \notin$* **NONM***.*

Proof. Suppose $\Phi_e^A \in \tilde{\mathcal{M}}$. Let σ_m denote the finite binary string coded by m, and let $\phi(\sigma, m, t)$ be the formula

$$\sigma \subseteq \sigma_m \wedge \forall k \leq |\sigma| + 2^{|\sigma|} \; [\Phi_{e,t}^{\sigma_m}(k) \downarrow]$$
$$\wedge \forall k < |\sigma| + 2^{|\sigma|} \; [Smax(\Phi_{e,t}^{\sigma_m}(k)) < Smin(\Phi_{e,t}^{\sigma_m}(k+1))]$$

Define f by $f(n) = \mu\langle m, t\rangle(\phi(A \upharpoonright n, m, t))$. Clearly f is computable relative to A, so there exists an increasing computable g for which $\exists^\infty n \; g(n) \geq f(n)$.

Construct a computable X by initial segments X_s. Let $X_0 = \emptyset$. To construct X_{s+1}, let $n = |X_s|$. For σ of length n say that σ requires attention if there exists $\langle m, t\rangle \leq g(s)$ such that $\phi(\sigma, m, t)$. Also, say that σ allows action if it requires attention and for the least $\langle m, t\rangle$ witnessing this, $Smax(\Phi_{e,t}^{\sigma_m}(n + 2^n)) \leq s$.

If every σ requiring attention also allows action, we can computably extend X_s to a string X_{s+1} of length $s+1$ such that for each such σ, X_{s+1} agrees with the finite function coded by $\Phi_{e,t}^{\sigma_m}(k)$ for some k with $n \leq k \leq n + 2^n$ (using Proposition V.37 since there are at most 2^n σ and for each σ we have 2^n options with disjoint domain). If no σ requires attention or some σ requires attention but doesn't allow action, let $X_{s+1} = X_s$. Note that for all s, $|X_s| \leq s$. Let $X = \bigcup_s X_s$. Now we prove several properties of this construction:

Lemma VI.6. *If $X_v = X_s$ for all $v \in [s, s']$, and σ requires attention at stage $s+1$, then σ requires attention at stage $s'+1$. Further, if σ allows action at stage $s+1$, it allows action at stage $s'+1$.*

Proof. The first statement follows from the monotonicity of g. The second statement follows from the fact that we use the least $\langle m, t\rangle$ in the computation that determines if σ allows action, and this doesn't change from stage to stage. □

Lemma VI.7. *For every stage s, there exist a σ of length $|X_s|$ and $s' \geq s$ such that σ requires attention at stage $s'+1$.*

Proof. Let $v \geq s$ be minimal such that $g(v) \geq f(v)$. If $X_v \neq X_s$, the lemma holds for some σ and some s' between s and v. Otherwise, the lemma holds for $\sigma = A \upharpoonright |X_s|$ and $s' = v$ (note $f(v) \geq f(|X_s|)$ since $v \geq |X_s|$). □

Lemma VI.8. *For every s, there exists $s' \geq s$ such that $X_{s'+1} \neq X_s$. Thus, X is total.*

Proof. By Lemma VI.7, fix $v \geq s$ such that some σ of length $|X_s|$ requires attention at stage $v+1$. Also, for each σ of length $|X_s|$ such that there exists $\langle m, t\rangle$ with $\phi(\sigma, m, t)$, choose the minimal $\langle m, t\rangle$ with this property. Let v' be the maximum of the chosen $\langle m, t\rangle$ and v'' the maximum of v and v'. Either $X_{v''} \neq X_s$, or some σ will require attention at stage $v''+1$ and all σ requiring attention will allow action, so $X_{v''+1} \neq X_{v''}$. □

Lemma VI.9. *If σ requires attention at stage $s+1$, then for some τ extending σ and $k \geq |\sigma|$, $\Phi_e^\tau(k)\downarrow$ and X agrees with $s_{\Phi_e^\tau(k)}$ on its domain.*

Proof. This clearly follows from Lemmas VI.6 and VI.8. □

Now we prove that for all $\sigma \subset A$ and m, there exist τ extending σ and $k \geq m$ such that $\Phi_e^\tau(k)\downarrow$ and X agrees with $s_{\Phi_e^\tau(k)}$ on its domain. Fix any s such that $|X_s| \geq \max(m, |\sigma|)$ and $g(s) \geq f(s)$. Then at stage $s+1$, $A \upharpoonright |X_s|$ requires attention (note $f(s) \geq f(|X_s|)$). The desired property now follows from Lemma VI.9.

Now consider

$$V_m := \{\sigma' : \exists t, k \geq m \ \Phi_{e,t}^{\sigma'}(k)\downarrow \wedge s_{\Phi_{e,t}^{\sigma'}(k)} \subset X\}$$

Note that for each m every $\sigma \subset A$ has an extension τ in V. By the 1-genericity of A, for every m, there exists $\sigma \subset A$ for which $\sigma \in V_m$, so there exists $k \geq m$ such that X agrees with $s_{\Phi_e^A(k)} = s_{\Phi_e^\sigma(k)}$ on its domain. It follows that $X \notin M_{\Phi_e^A}$, as needed. $\square$

6.2 DNC Reals and Meager Sets

Consider the following debate:

Definition VI.10. Two functions $f, g \in {}^\omega\omega$ are *infinitely equal*, denoted $f =^\infty g$, iff for infinitely many n, $f(n) = g(n)$. $\mathfrak{IE}$ is the debate $({}^\omega\omega, {}^\omega\omega, =^\infty)$.

By a theorem of Bartoszyński [3], this debate has norm equal to $\mathbf{non}(\mathcal{M})$. If we can transfer the proof of this fact to computability theory, we would obtain the equivalence $A \in \mathbf{NONM}$ iff $A \in \mathbf{IE}$, which by definition is the case iff A computes a function f such that for all computable g, $f(n) \neq g(n)$ for all but finitely many n. This would be greatly beneficial, since we have already characterized the latter set of oracles as those which are high or DNC (Proposition II.78).

There is a morphism from $\mathbf{Non}(\mathcal{M})$ to $\mathfrak{IE}$, and this does provide an effective morphism from $\widetilde{\mathbf{Non}(\mathcal{M})}$ to $\mathfrak{IE}$. Consider Theorem V.17, which by duality implies the existence of an effective morphism from $\widetilde{\mathbf{Non}(\mathcal{M})}$ to $\mathfrak{B}$. The crux of that theorem's proof is the fact that for any $f \in {}^\omega\omega$, the set of $g \leq^* f$ is a meager subset of ${}^\omega\omega$. It is also the case that the set of g not infinitely equal to f is a meager subset of ${}^\omega\omega$, by almost the same argument. Therefore, a straightforward modification of the proof of Theorem V.17 establishes the required effective morphism from $\widetilde{\mathbf{Non}(\mathcal{M})}$ to $\mathfrak{IE}$. This proves that $\mathbf{IE} \subseteq \mathbf{NONM}$ (in fact $\mathbf{IE}_\mathbf{A}^\mathbf{C} \subseteq \mathbf{NONM}_\mathbf{A}^\mathbf{C}$ for all $\mathbf{A}$ and $\mathbf{C}$).

However, effective morphisms do not help us as much as they did in Chapter V, because the inequality $||\mathfrak{IE}|| \geq \mathbf{non}(\mathcal{M})$ is not proven by exhibiting a morphism from

$\mathfrak{IE}$ to $\mathbf{Non}(\mathcal{M})$[1]. Instead, the usual proof exhibits a morphism to $\mathbf{Non}(\mathcal{M})$ from a more complex debate formed from $\mathfrak{B}$ and $\mathfrak{IE}$. This debate involves functions from the continuum to itself, so there does not appear to be a reasonable way to assign to it a Turing characteristic. Nevertheless, we can still carry out the set theoretic proof with sufficient computability (just without the machinery of morphisms).

Theorem VI.11. NONM $\subseteq$ IE.

Proof. Let A be an oracle in **NONM**, and $f \leq_T A$ a member of $\tilde{M}$ such that M_f contains every computable real. If A is high, then $A \in \mathbf{IE}$ automatically, so assume A is not high. Let $X, g \leq_T A$ be such that for all n, $X \upharpoonright I_g^n$ is the member of S (the set of partial functions from ω to $\{0,1\}$ with finite domain) coded by $f(n)$. By Corollary V.5, since $A \notin \mathbf{B}$, there is a computable h such that I_g does not dominate I_h. In other words, there are infinitely many n such that I_g^n does not contain an interval of I_h. For such n, I_g^n is covered by the union of two consecutive intervals of I_h. Therefore,

$$B := \{m : \exists n \; I_g^n \subseteq I_h^m \cup I_h^{m+1}\}$$

is an infinite set computable relative to A. Define (by recursion on m) a $\hat{B} \leq_T A$ such that $m+1 \in \hat{B}$ iff $m+1 \in B$ and $m \notin \hat{B}$. Note that $\hat{B}$ is still infinite, and the intervals of the form $I_h^m \cup I_h^{m+1}$ for $m \in \hat{B}$ are pairwise disjoint, with each covering an interval of I_g. Now, using the codings of Sections 2.1 and 2.2 to enumerate ${}^{<\omega}S$, define $p : \omega \to \omega$ such that $p(s)$ codes the set of $X \upharpoonright I_h^m \cup I_h^{m+1}$ for the $3s+1$ least values of $m \in \hat{B}$.

Clearly, $p \leq_T A$. We claim that for every computable function q, $p(s) \neq q(s)$ for all but finitely many s. If not, we fix a computable q which is infinitely equal to p, and construct a computable Y which extends infinitely many of the finite functions

[1] In fact it is an open question as to whether or not such a morphism is guaranteed by ZFC to exist.

$X \upharpoonright I_g^n$ (i.e., $Y \notin M_f$), which will be a contradiction. We construct Y in stages s, by defining it on intervals of I_h, beginning with $Y_0 = \emptyset$. The partial real Y_s will be defined on at most $3s$ intervals.

Given Y_s, view $q(s)$ as a code for a member of ${}^{<\omega}S$. If this sequence has length less than $3s+1$, or if any element in the sequence has a domain which is not the union of two consecutive intervals of I_h, or if the elements of the sequence do not have disjoint domains, let $Y_{s+1} = Y_s$. Otherwise (which will be the case infinitely often), let t be the first element in the sequence coded by $q(s)$ which has domain disjoint from the domain of Y_s (this is possible since Y_s is defined on at most $3s$ intervals and the sequence coded by $q(s)$ has at least $3s+1$ elements). Define Y_{s+1} to agree with Y_s and t on their respective domains, and then define Y_{s+1} to be constantly 0 on the first interval of I_h on which it is undefined. Note that Y_{s+1} is defined on three more intervals of I_h than Y_s is, and the action of defining Y_{s+1} on the first empty interval of I_h guarantees that $Y := \bigcup_s Y_s$ is a total real.

For every s such that $p(s) = q(s)$, Y_{s+1} is defined to match $X \upharpoonright I_h^m \cup I_h^{m+1}$. for a new value of m. It follows that Y agrees with X on infinitely many intervals of I_g, as needed. □

Corollary VI.12. *$A \in$ **NONM** iff A is high or DNC.*

Proof. By Theorem VI.11 and the preceding discussion, and Proposition II.78. □

Remark VI.13. This corollary obviates Theorem VI.5, since no 1-generic real is DNC (Proposition II.89).

6.3 Fractional Dimension

Armed with the characterization of **NONM** from the previous section, we are ready to show how to construct an oracle A in **NONM** and not **COVN**. Since every high oracle is in **COVN**, such an A must be DNC and non-high. Also, A must not compute a Martin-Löf random real, since such reals are Schnorr random (and **COVN** is the set of oracles which compute Schnorr random reals). On the other hand, if A is non-high, DNC, and does not compute a Martin-Löf random real, it is in **NONM**$\setminus$**COVN** since every non-high Schnorr random real is Martin-Löf random (Proposition II.67). Our goal is to demonstrate the existence of such an A.

Recall the definition of effective Hausdorff dimension from Section 2.7. Informally, one can think of the dimension of A as a measure of how random A is. This makes sense because every Martin-Löf random real has the maximum dimension, 1 (although not every real of dimension 1 is Martin-Löf random). For another example, if X is Martin-Löf random, $Z(n) = 0$ for all n, and $A = X \oplus Z$ (i.e., the bits of A alternate between 0 and random), then A has dimension 1/2. One reasonable question to ask about dimension is if $\dim(A) > 0$, does A compute a real of higher dimension (or even a real of dimension 1)? That is, if A has some randomness, can we extract more randomness from it computably? Note that in the example $A = X \oplus Z$ above, this is the case, since $X \leq_T A$. However, this is not always true; Miller [33] has provided a construction of a real A which has dimension 1/2 and does not compute a real of higher dimension[2]. Note that such an A is DNC and does not compute a Martin-Löf random real (see discussion following Proposition II.81). Therefore, for our purposes, it suffices to show that this A can be made non-high. Fortunately, the A constructed by Miller is automatically low_2, a fact which we will demonstrate in this section[3].

The full proof of Miller's theorem is somewhat beyond the scope of this thesis. It heavily relies on the properties of the Kolmogorov complexity K, which is of interest

[2]There is nothing special about 1/2. Miller's proof can be easily modified to work for any rational $q \in (0, 1)$.
[3]We stress that the observation that A is low_2 is the sole contribution of this thesis's author.

for this thesis only insofar as it links randomness, diagonal non-computability, and effective dimension. Therefore, in our presentation of Miller's theorem, we will be brief, only presenting enough detail so that it can be seen that A is low$_2$. For the full treatment, we encourage the reader to consult [33].

Before we begin, it should be noted that Miller's proof is a forcing proof, but one that does not have any clear set-theoretic precedent. In that it involves subtrees of ${}^{\omega}2$ which branch "a lot," but not "too much," it is somewhat similar to $PT_{f,g}$, a notion of forcing whose conditions are subtrees of ${}^{\omega}\omega$ which branch "a lot," but not "too much" [5]. This makes sense, since $PT_{f,g}$ is a forcing that can be used to increase $\mathbf{non}(\mathcal{M})$ without increasing $\mathbf{cov}(\mathcal{N})$. However, the forcing used by Miller does not appear to be otherwise connected to $PT_{f,g}$.

Definition VI.14. [33] Let $V \subseteq {}^{<\omega}2$. The *direct weight* of V, denoted $DW(V)$, is

$$\sum_{\sigma \in V} 2^{-|\sigma|/2}.$$

The *weight* of V, denoted $W(V)$, is

$$\inf\{DW(W) : [V] \subseteq [W]\}.$$

The *optimal cover* of V, denoted V^{oc}, is the subset of ${}^{<\omega}2$ such that

- $[V] \subseteq [V^{oc}]$;
- $DW(V^{oc}) = W(V)$;
- $\mu([V^{oc}])$ is minimal among all sets of strings satisfying the first two requirements.

Proposition VI.15. *[33] For all V, V^{oc} exists, and is unique and prefix-free. If V is finite, so is V^{oc}, and the functions mapping V to V^{oc} and $W(V)$ are computable. Further, if V is c.e., V^{oc} is uniformly Δ^0_2 and $[V]$ is uniformly Σ^0_1.*

Definition VI.16. [33] For $\sigma \in {}^{<\omega}2$ and V a c.e. subset of ${}^{<\omega}2$ such that every $\tau \in V$ extends σ and $\sigma \notin V^{oc}$, $P_{\sigma,V}$ is the nonempty Π^0_1 class $[\sigma]\backslash[V^{oc}]$.

Remark VI.17. The set of pairs (σ, V) satisfying the conditions of Definition VI.16 will be our set of forcing conditions. We let $(\sigma, V) \leq (\tau, W)$ if $P_{\sigma,V} \subseteq P_{\tau,W}$. A will be constructed as the unique member of an intersection of Π^0_1 classes corresponding to conditions. In this context, (σ, V) forces a statement about A if that statement is true for all $A \in P_{\sigma,V}$.

Proposition VI.18. *[33] Let $V_0 = \{\sigma : K(\sigma) \leq |\sigma|/2\}$. Then $(\emptyset, {}^{<\omega}2)$ is a condition. Hence, there is a condition which forces dim$(A) \geq 1/2$.*

We now sketch Miller's construction. The construction involves finding (computably in $\emptyset'$) a sequence of conditions $\{(\sigma_s, V_s)\}_{s\in\omega}$ such that for all s, $(\sigma_{s+1}, V_{s+1}) \leq (\sigma_s, V_s)$ and σ_{s+1} properly extends σ_s. Then A is taken to be $\bigcup_s \sigma_s$, or equivalently $\bigcap_s P_{\sigma_s,V_s}$. The construction begins by letting (σ_0, V_0) be the condition from Proposition VI.18.

If we let $s = \langle e, n\rangle$, then (σ_{s+1}, V_{s+1}) is chosen to make sure A fulfills the requirement

$$R_{e,n} : \text{If } \Phi^A_e \text{ is total, then } \exists k > n\ K(\Phi^A_e \restriction k) \leq (1/2 + 2^{-n}) \cdot k.$$

Clearly, for any fixed e, if every $R_{e,n}$ is met, then Φ^A_e either[4] is nontotal or has dimension at most 1/2.

Miller's strategy to meet $R_{e,n}$ is to construct (σ_{s+1}, V_{s+1}) in order to force one of the following statements:

$$\Phi^A_e \text{ is not total.}$$

$$\exists k > n\big([\forall i < k\ \Phi^A_e(i)\downarrow] \wedge\ K(\Phi^A_e \restriction k) \leq (1/2 + 2^{-n}) \cdot k\big).$$

[4]This skips over the fact that Φ^A_e may not be $\{0,1\}$-valued. This, however, is not an issue, since we may computably modify Φ^A_e to output 1 in place of any nonzero output.

Furthermore, given e and n, we can determine computably in $\emptyset'$ which statement is forced – the choice is determined by whether $M(\rho)\downarrow$, where M is a prefix-free machine and ρ is a finite binary segment, both of which are defined at stage $s = \langle e, n\rangle$. Fix e. Clearly, if the first statement is forced for any n, Φ_e^A is not total. Conversely, if the second statement is forced for all n, then for all n, $\Phi_e^A(n)\downarrow$ – i.e., Φ_e^A is total. Let Tot^A be the set of e for which Φ_e^A is total. Then $e \in \mathrm{Tot}^A$ iff for all n the second statement is forced. The latter condition is computable relative to $\emptyset''$, so $A'' \leq_T \mathrm{Tot}^A \leq_T \emptyset''$[5]. Hence:

Theorem VI.19. *There is a low$_2$ oracle A such that dim$(A) = 1/2$ and for all $B \leq_T A$, dim$(B) \leq 1/2$.*

Corollary VI.20. NONM $\not\subseteq$ COVN*.*

6.4 Challenges Below $\emptyset'$

When confronted with a new computability theoretic property of reals, natural first questions to ask are "How does it interact with being c.e., or being Δ_2^0?" Of the incomputable reals, those which are c.e. or Δ_2^0 still feel somewhat tractable, and somehow "realistic." After all, c.e. reals are computably listable (i.e., as subsets of ω), and Δ_2^0 reals are those computable from the halting problem, which tells us which computations will and will not converge. We observe that in this thesis we have some new properties – for A, the property of A being in some Turing characteristic derived from cardinal characteristics appearing in Cichoń's diagram. What can be said about c.e. or Δ_2^0 oracles with such properties? To put it another way, what happens to the diagram of Theorem VI.1 if we let **C** be the set of c.e. oracles or the set of Δ_2^0 oracles? After the results of Section 6.3, we are able to answer both questions.

[5]Note that by the relativization of Lemma II.39, Tot^A is $\Pi_2^0[A]$ T-complete, and $\bar{A}''$ is $\Pi_2^0[A]$ by the relativization of Proposition II.33.

Remark VI.21. Technically, the set of c.e. oracles is not a suitable choice for **C**, since in our original definition, we required **C** to be closed downward with respect to $\leq_T$. However, this requirement was not for technical reasons – that is, all of the results involving morphisms do not rely on **C** being closed downward. The restriction was placed on **C** so that it would correspond more closely to a set theoretic situation. For an oracle in **C**, we should have access to everything that oracle defines (i.e., computes). Technically, however, it does make sense to allow **C** not to be closed downward, although the only time we will do this is for the case where **C** is the set of c.e. oracles.

First, we establish an upper bound for Turing characteristics derived from cardinal characteristics appearing in Cichoń's diagram.

Proposition VI.22. *Let $\mathfrak{K} = (K_-, K_+, K)$ be any debate such that $K_\pm \subseteq {}^\omega\omega$ and $\mathfrak{k}$ is in Cichoń's diagram. For every $X \in K_-$, there exists $Y \in K_+$ such that $Y \leq_T X$ and XKY. Colloquially, every challenge computes an answer which meets it.*

Proof. The proof of this proposition is simple, though tedious because of the number of cases that need to be considered.

- $\mathfrak{B}$ or $\mathfrak{D}$: In these cases, given f, we need to find g such that either g is not dominated by f, or g dominates f. In either case, $g = f + 1$ serves.

- $\widetilde{\mathbf{Non}(\mathcal{J})}$: Given f in $\tilde{\mathcal{N}}$ or $\tilde{\mathcal{M}}$, there exists $X \leq_T f$ such that X is not in N_f or M_f, respectively. This was proven in Chapter V (Proposition V.20 and the discussion preceding Theorem V.15, respectively).

- $\widetilde{\mathbf{Cov}(\mathcal{J})}$: Given X in ${}^\omega 2$, there exists f in $\tilde{\mathcal{N}}$ or $\tilde{\mathcal{M}}$ such that X is in N_f or M_f, respectively. Let I be the (computable) interval partition such that $|I^n| = n+1$.

If we seek an $f \in \tilde{\mathcal{N}}$, we may use the one which maps n to $[X \upharpoonright I^n]$, and if we seek an $f \in \tilde{\mathcal{M}}$, we may use the one which maps n to $\bar{X} \upharpoonright I^n$.

- $\widetilde{\mathbf{Add}(\mathcal{J})}$: Given f in $\tilde{\mathcal{N}}$ or $\tilde{\mathcal{M}}$, by the previous two cases, there exist $g \leq_T X \leq_T f$ such that, respectively, $X \in N_g \backslash N_f$ (so $N_g \nsubseteq N_f$), or $X \in M_g \backslash M_f$ (so $M_g \nsubseteq M_f$).
- $\widetilde{\mathbf{Cof}(\mathcal{J})}$: Given f, we may simply use f (i.e., N_f as an answer is a superset of N_f as a challenge). □

Corollary VI.23. *Let $\mathfrak{K}$ be a debate satisfying the assumptions of Proposition VI.22. Then for any* $\mathbf{C}$ *and* $\mathbf{A}$, $\mathbf{K}^{\mathbf{C}}_{\mathbf{A}} \subseteq \mathbf{C} \backslash \mathbf{A}$.

For an application of Corollary VI.23, let $\mathbf{C}$ be the set of Δ^0_2 oracles, and let $\mathbf{A}$ be the set of computable oracles. Then by Propositions II.54 and II.91, $\mathbf{COVM}^{\mathbf{C}}_{\mathbf{A}} = \mathbf{C} \backslash \mathbf{A}$. Therefore, for this choice of $\mathbf{C}$ and $\mathbf{A}$, all the Turing characteristics from the right half of Cichoń's diagram equal $\mathbf{C} \backslash \mathbf{A}$.

This observation is almost enough to provide a complete version of the diagram from Theorem V.7 for the case where $\mathbf{C}$ is the set of Δ^0_2 oracles and $\mathbf{A}$ is the (default) set of computable oracles. It remains only to note that the facts $\mathbf{COVN} \nsubseteq \mathbf{B}$, $\mathbf{NONM} \nsubseteq \mathbf{COVN}$, and $\mathbf{COVM} \nsubseteq \mathbf{NONM}$ may all be realized by Δ^0_2 witnesses (i.e., respectively, by a low Martin-Löf random real, the real A given by Miller's construction from Section 6.3, and a low 1-generic real[6]). Therefore:

Theorem VI.24. *Let* $\mathbf{C}$ *be the set of* Δ^0_2 *oracles. Then the following diagram holds*

[6] The existence of such a real has not been provided in this thesis. See [49] for a proof.

and is complete.

$$
\begin{array}{ccccccc}
\mathbf{COVN^C} & \longleftarrow & \mathbf{NONM^C} & \longleftarrow & \mathbf{COFM^C} & = & \mathbf{COFN^C} \\
 & & \downarrow & & \| & & \\
\downarrow & & \mathbf{B^C} & \longleftarrow & \mathbf{D^C} & & \| \\
 & & \| & & \| & & \\
\mathbf{ADDN^C} & = & \mathbf{ADDM^C} & \longleftarrow & \mathbf{COVM^C} & = & \mathbf{NONN^C}
\end{array}
$$

The corresponding diagram for the case where $\mathbf{C}$ is the set of c.e. oracles is even simpler. As before, $\mathbf{COVM^C}$ is $\mathbf{C}\backslash\mathbf{A}$ ($\mathbf{A}$ is the set of computable oracles), so all Turing characteristics on the right side of the diagram are equal. All Turing characteristics on the left side of the diagram are also equal.

Theorem VI.25. *Any c.e. oracle in* **NONM** *is high.*

Proof.

Lemma VI.26. *If $f \in \tilde{\mathcal{M}}$, M_f contains all computable reals, and f_s is a computable approximation to f, then any function h satisfying*

$$(\forall s \geq h(n))(\forall m \leq n) f_s(m) = f(m)$$

dominates all computable functions.

Proof. Assume g is a computable function such that $\exists^\infty n g(n) > h(n)$. Without loss of generality, we may assume g is increasing.

We will construct a recursive X in stages by finite initial segments X_n, with $X = \bigcup_n X_n$, such that $X \notin M_f$, a contradiction. At stage 0, $X_0 = \emptyset$. At stage $n+1$, look for an $m \leq n+1$ such that $r := f_{g(n+1)}(m)$ satisfies[7].

- $Smin(r) \geq |X_n|$ and
- $Smax(r) \leq n$

[7] Recall Notation VI.3 for the meanings of $Smin$ and $Smax$.

If such an m exists, then for the least such m and for all $k \in [|X_n|, n]$, define $X_{n+1}(k)$ to be $s_r(k)$ if it exists and 0 otherwise. If no such m exists, let $X_{n+1} = X_n$. Note that $|X_n| \leq n$ for all n.

Now fix an arbitrary n_0 and $n \geq n_0$ such that $g(n+1) > h(n+1)$. Thus, for all $m \leq n$ and $n' \geq n$, the value of $f_{g(n'+1)}(m)$ is correct (in the sense that it equals $f(m)$). Also,

$$|X_n| \leq n \leq Smin(f(n)) = Smin(f_{g(n+1)}(n)).$$

Therefore, there is an $m \leq n$ such that $Smin(f_{g(n+1)}(m)) \geq |X_n|$, and the least m satisfying this requirement will continue to be the least such m at all stages $n'+1$ for $n' \geq n$. In particular, if $n' = \max\{n, Smax(f_{g(n+1)}(m))\}$, then the construction will guarantee that $X_{n'}$ is extended to be compatible with $s_{f(m)}$.

This establishes that X_n is extended an infinite number of times (thus X is total). Further, an infinite number of these extensions force it to be compatible with some (new) $s_{f(m)}$, so $X \in [s_{f(m)}]$ for infinitely many m. It remains to note that X is computable since the construction is computable. Thus, if g is computable, h dominates g, as claimed. □

Now assume A is c.e., and let $f \leq_T A$ witness $A \in$ **NONM**. Recall that if $f = \Phi_e^A$, we can approximate f computably by letting

$$f_s(n) = \Phi_{e,s}^{A_s \upharpoonright s}(n)$$

where A_s is a fixed computable enumeration of A (if the right-hand side computation diverges let $f_s(n) = 0$). Define $h(n)$ to be the minimum s such that for all $m \leq n$, $\Phi_{e,s}^A(m)\downarrow$, and if u is the maximum use of A in this computation, $A_s \upharpoonright u+1 = A \upharpoonright u+1$. Then $h \leq_T A$ and by the lemma, h dominates all computable functions. Therefore, A is high. □

Remark VI.27. As with Theorem VI.5, this theorem provides nothing new once we have the characterization of **NONM** from the Corollary to Theorem VI.11, since no c.e. oracles which do not compute $\emptyset'$ can be DNC[1].

Since **ADDN** is the set of high oracles, $\mathbf{ADDN}^{\mathbf{C}}$ is the set of high c.e. oracles, and thus equal to $\mathbf{NONM}^{\mathbf{C}}$. Also, $\mathbf{COVM}^{\mathbf{C}}$ is not equal to $\mathbf{NONM}^{\mathbf{C}}$, since the former contains a low oracle. Therefore:

Theorem VI.28. *Let* $\mathbf{C}$ *be the set of c.e. oracles. Then the following diagram holds and is complete.*

$$\begin{array}{ccccccc}
\mathbf{COVN}^{\mathbf{C}} & = & \mathbf{NONM}^{\mathbf{C}} & \longleftarrow & \mathbf{COFM}^{\mathbf{C}} & = & \mathbf{COFN}^{\mathbf{C}} \\
 & & \| & & \| & & \\
\| & & \mathbf{B}^{\mathbf{C}} & \longleftarrow & \mathbf{D}^{\mathbf{C}} & & \| \\
 & & \| & & \| & & \\
\mathbf{ADDN}^{\mathbf{C}} & = & \mathbf{ADDM}^{\mathbf{C}} & \longleftarrow & \mathbf{COVM}^{\mathbf{C}} & = & \mathbf{NONN}^{\mathbf{C}}
\end{array}$$

6.5 Lowness For Genericity and Meager Sets

In addition to the inequalities implied by Cichoń's diagram, it is also true in all models of ZFC that

$$\left.\begin{array}{c}\mathbf{add}(\mathcal{M}) \geq \min\{\mathbf{cov}(\mathcal{M}), \mathfrak{b}\} \\ \mathbf{cof}(\mathcal{M}) \leq \max\{\mathbf{non}(\mathcal{M}), \mathfrak{d}\}\end{array}\right. [31].$$

Both inequalities are proven using a morphism. However, as was the case with the inequality $||\mathfrak{IE}|| \geq \mathbf{non}(\mathcal{M})$ described in Section 6.2, the morphism involves a debate which does not translate to a Turing characteristic; specifically, one which is built from $\mathbf{Non}(\mathcal{M})$ and $\mathfrak{D}$. Therefore, we cannot hope to use an effective morphism to prove any versions of these inequalities for Turing characteristics. Nevertheless, (again, as was the case before) we can modify a set theoretic proof to obtain a result in the case where $\mathbf{A}$ is the set of computable oracles.

Theorem VI.29. **COFM** $\subseteq$ **NONM** $\cup$ **D**.

Proof. (based on an argument from [31])

Fix $A \in$ **COFM** and $f \leq_T A$ in $\tilde{\mathcal{M}}$ such that $M_f \nsubseteq M_g$ for all computable $g \in \tilde{\mathcal{M}}$. If M_f contains every computable real, we are done ($A \in$ **NONM**), so assume X is a computable real such that $X \notin M_f$. By Proposition IV.12, we may fix $Y, h \leq_T A$ such that M_f includes all reals which do not match (Y, I_h), and let $\tilde{h} \leq_T A$ be such that every $I^n_{\tilde{h}}$ contains some I^m_h such that $X \upharpoonright I^m_h = Y \upharpoonright I^m_h$.

We claim $\tilde{h}$ is not interval dominated by any computable $\tilde{g}$ (if this is the case, then by Corollary V.5 $A \in$ **D**). Suppose then that $\tilde{g}$ is computable and interval dominates $\tilde{h}$. Then $(X, I_{\tilde{g}})$ engulfs $(X, I_{\tilde{h}})$, which by definition of $\tilde{h}$ engulfs (Y, I_h). In other words (see Proposition V.10 and the ensuing discussion), if $g \in \tilde{\mathcal{M}}$ is computable and M_g includes all reals which do not match $(X, I_{\tilde{g}})$, then $M_f \subseteq M_g$, a contradiction. □

Corollary VI.30. *A is in* **COFM** *iff A computes a hyperimmune set or is DNC (iff A is not low for weak 1-genericity – see Proposition II.93).*

Remark VI.31. It appears the other three-cardinal inequality does not yield any useful information involving Turing characteristics; that is, the above proof cannot be modified to prove a useful result relating **ADDM**, **B**, and **COVM**.

6.6 Weakly Schnorr Covering

As mentioned in Section 4.3, in [45] the oracles in **NONN** are called "weakly Schnorr covering." Two forcing proofs in that paper (which consists of some of the research for this thesis) shed light on the nature of the weakly Schnorr covering oracles, and allow us to finish proving Theorem VI.2. We recall that it will suffice to show that

1. **NONN** $\nsubseteq$ **COFM**, and

2. **COVN $\not\subseteq$ NONN**.

In other words, (from the equivalences detailed in Sections 4.3 and 6.5)

1. there exists an oracle A which is weakly Schnorr covering and low for weak 1-genericity;

2. there exists a Schnorr random real A which is not weakly Schnorr covering.

We may achieve these by modifying set theory proofs which involve

1. a forcing which adds to the ground model a null set covering all ground model reals *without* adding any new meager sets;

2. a forcing which adds to the ground model a real not in any ground model null set *without* adding a null set covering all ground model reals.

For each of the above cases, there is a simple notion of forcing which has the desired property, which allows for a direct adaptation of the set theory methods to provide proofs in computability theory. For example, let I be the interval partition such that $|I^n| = n+1$ for all n. Consider the notion of forcing whose conditions are partial functions $p : \omega \to \{0,1\}$ whose domains are the unions of coinfinitely many intervals I^n, and $p \leq q$ if $q \subseteq p$. This is a forcing which adds to the ground model a null set covering all ground model reals without adding any new meager sets [18, 4]. A computable modification yields:

Theorem VI.32. *There is an oracle which is weakly Schnorr covering and low for weak 1-genericity. Furthermore, such an oracle is computable from $\emptyset''$.*

Proof. (based on an argument presented in [4], itself based on an argument from [18])

We construct an A with these properties via a forcing construction. Let I be the interval partition such that $|I^n| = n + 1$ for all n. The conditions will be partial computable functions p from ω to $\{0, 1\}$ such that $\text{dom}(p)$ is computable, coinfinite, and the union of some of the intervals I^n. For any partial computable p with domain a union of I^n, let f_p be the function which enumerates, in increasing order, the n such that $I^n \nsubseteq \text{dom}(p)$. Note that p is a condition iff f_p is total and computable.

We will construct A in stages s – if p_s is the condition at stage s, $A = \bigcup_s p_s$. Let σ_i denote the finite binary string coded by i, and let W_e^A be the eth A-c.e. subset of ${}^{<\omega}2$. We assume each W_e^A is closed upward with respect to $\subseteq$, so that $[W_e^A]$ is dense iff W_e^A is dense; we may assume this because we may uniformly adjust the enumeration of W_e^A so that whenever σ is enumerated into W_e^A, so are all extensions of σ. Fix an enumeration X_e of the computable binary reals (which may be done computably in Tot $\equiv_T \emptyset''$). We let p_0 be the empty function, and at stage $s + 1$ we attend to requirement R_s:

$$R_{2e} : A \text{ matches } X_e \text{ on infinitely many } I^n.$$

$$R_{2e+1} : W_e^A \text{ is not dense, or } \exists V \subseteq {}^{<\omega}2 \text{ c.e. such that}$$

$$V \text{ is dense and } V \subseteq W_e^A.$$

Suppose all requirements are satisfied. If C_i denotes the clopen set coded by i, and $g \leq_T A$ is such that $C_{g(n)} = [A \upharpoonright I^n]$, then satisfaction of even requirements implies N_g contains all computable reals, so A is weakly Schnorr covering. The odd requirements guarantee (by Proposition II.93) that A is low for weak 1-genericity.

We now describe the action taken by the construction at stage $s+1$. If $s = 2e$, let p_{s+1} match p_s on its domain, and X_e on $I^{f_p(2n)}$ for all n – otherwise, p_{s+1} is undefined. Clearly this suffices to satisfy R_s and finding an index for p_{s+1} is computable in $\emptyset''$ (this latter point will be important for the action taken for $s = 2e + 1$). It is also straightforward to check that p_{s+1} is actually a condition (i.e., partial computable with computable coinfinite domain). Note that $\mathrm{dom}(p_{s+1})$ includes the least interval not in $\mathrm{dom}(p_s)$, so the actions taken at stages of this form ensure A will be total.

If $s = 2e+1$, we consider the following subroutine which is computable in $\emptyset'$. We construct a sequence of conditions q_i (with $q_{i+1} \supseteq q_i$) and enumerate a set $V \subseteq 2^{<\omega}$ in stages i. The purpose of the subroutine is to find, at stage $i+1$, an extension ν of σ_i such that ν extends some string in W_e^A for all $A \supset q_{i+1}$, and enumerate ν into V. The V we enumerate in the subroutine will play the role of V mentioned in R_{2e+1}. We will want to let $p_{s+1} = \bigcup_i q_i$, so q_{i+1} will be left undefined on $i+1$ designated intervals (so that $\bigcup_i q_i$ will be undefined on infinitely many intervals). If instead we see an opportunity to force W_e^A to be nondense, we will take it and interrupt the subroutine.

Let $q_0 = p_s$ and $V_0 = \emptyset$. Given q_i and V_i, let

$$\Gamma_i = \bigcup_{n \leq i} I^{f_{q_i}(n)}$$

$$\Theta_i = \{\theta : \Gamma_i \to \{0,1\}\}.$$

We intend that Γ_i is the union of the $i+1$ intervals where q_{i+1} will be undefined; Θ_i is the set of possibilities for $A \upharpoonright \Gamma_i$ assuming A extends q_{i+1}. Index the members of Θ_i by $t < 2^{|\Gamma_i|}$. We now work in substages $t \leq 2^{|\Gamma_i|}$, at each substage $t+1$ constructing a condition r_{t+1} which extends r_t and has domain disjoint from Γ_i (let $r_0 = q_i$) and a string ν_{t+1} which extends ν_t (let $\nu_0 = \sigma_i$). The intention is for ν_{t+1} to extend a string in W_e^A for all A extending $r_{t+1} \cup \theta_t$ – that is, substage $t+1$ accounts for the possibility that $A \upharpoonright \Gamma_i$ will be θ_t.

At substage $t+1$, determine[8] if there exist $\tau, \rho \in 2^{<\omega}$ such that τ is compatible with $r_t \cup \theta_t$, $|\tau| = (n+1)(n+2)/2$ for some n, ρ extends ν_t, and $\tau \in W_e^\rho$. If not, let

[8] To do this uniformly in $\emptyset'$, we need an index for r_t. This is why we ensure $\emptyset''$ can provide indices for all conditions used.

$p_{s+1} = r_t \cup \theta_t$ – here, we say the subroutine terminates. If so, for the first such pair $\langle \tau, \rho \rangle$, let $\nu_{t+1} = \rho$ and let r_{t+1} be $r_t \cup \tau \upharpoonright \omega \backslash \Gamma_i$ (note an index for r_{t+1} can be found computably from τ, Γ_i, and an index for r_t).

If the subroutine does not terminate at any substage during stage $i+1$, let $q_{i+1} = \bigcup_{t \leq 2^{|\Gamma_i|}} r_t$ (this is a condition since it finitely extends q_i) and enumerate $\nu := \nu_{2^{|\Gamma_i|}}$ into V. If the subroutine does not terminate during any stage, let $p_{s+1} = \bigcup_i q_i$. We need to establish:

Lemma VI.33. *1. The partial function p_{s+1} is a condition.*

2. V is c.e.

3. An index for p_{s+1} can be found computably in $\emptyset''$.

4. R_{2e+1} is satisfied.

Proof. For each item, we proceed by cases, based on whether or not the subroutine terminated.

1. Note first that the domain of p_{s+1} is the union of some I^n. If the subroutine terminated, $p_{s+1} =^* p_s$ and is therefore a condition. If the subroutine never terminated, it can be carried out computably – each time, we can search for a τ and ρ, knowing we will find one. It follows immediately that p_{s+1} is computable and the sequence q_i is uniform. Note $\Gamma_{i+1} = \Gamma_i \cup I^{f_{q_{i+1}}(i+1)}$ since q_{i+1} is never defined on Γ_i. Thus $f_p(n) = f_{q_{n+1}}(n+1)$ witnesses p is a condition.

2. Either the subroutine terminated and V is finite or the subroutine is computable, giving V a computable enumeration.

3. Since the subroutine is computable in $\emptyset'$, $\emptyset''$ can determine if and when it terminates. From this, we can find an index for p_{s+1} computably.

4. If the subroutine terminated in stage $i+1$, substage $t+1$, then W_e^A contains no extension of ν_t (since A extends $r_t \cup \theta_t$), and the requirement is satisfied. If the subroutine never terminates, then at stage $i+1$, we enumerate an extension of σ_i into V, so V is dense. Also, for some t, A extends $r_{t+1} \cup \theta_t$, so $\nu_{t+1} \in W_e^A$; since $\nu_{2^{|\Gamma_i|}} \in V$ extends ν_{t+1}, it follows that $[V] \subseteq [W_e^A]$ (i.e., every string enumerated into V has an initial segment in W_e^A). □

Since the constructed A satisfies all requirements, it is weakly Schnorr covering and low for weak 1-genericity. □

Corollary VI.34. NONN $\not\subseteq$ COFM.

The second result of this section uses a nonconstructive forcing proof. To understand the motivation behind this proof, we need to recall the original goal: prove **COFN $\not\subseteq$ NONN**. If we are to adapt a set theoretic forcing to prove this, it must be a forcing which adds a new[9] null set to the ground model without adding a null set containing all ground model reals. The simplest forcing which accomplishes this is the random forcing, which adds a real not in any ground model null set (hence our statement that we will prove **COVN $\not\subseteq$ NONN**).

In the random forcing, the conditions can be taken to be positive measure closed sets, with $P \leq Q$ if $P \subseteq Q$. The computable version should be forcing with positive measure Π^0_1 classes, which we have already seen in the proof of Proposition II.56. Therefore, we might guess that a real A which is "sufficiently generic" for the computable version of the random forcing should be hyperimmune-free. Of course, A should also be algorithmically random to some degree. However, we should note that, in contrast to "sufficiently algorithmically generic" reals sufficing for proofs based on the Cohen forcing (see Proposition II.89 and Theorem VI.4), A cannot be

[9] by this we mean a null set not covered by any ground model null set

"sufficiently algorithmically random," since if A is Martin-Löf random relative to $\emptyset'$, A is not hyperimmune-free [37]. Schnorr random turns out to be a correct level of randomness.

Theorem VI.35. *If A is hyperimmune-free and Schnorr random, and $N \subseteq {}^{\omega}2$ is Schnorr null relative to A, then the set of computable reals in N is Schnorr null (unrelativized).*

Remark VI.36. This is a translation of the set theoretic result that if N is a null subset of ${}^{\omega}2$ in the extension created by the random forcing, the set of ground model reals in N is null in the ground model.

Proof. (based on an argument from [26])

We first present some notation, and then two lemmas which serve as weak versions of Fubini's Theorem for null sets, with Schnorr null replacing null. For $M, N \subseteq {}^{\omega}2$, $M \oplus N = \{X \oplus Y : X \in M \wedge Y \in N\}$, and for $N \subseteq 2^{\omega}$ and $Y \in 2^{\omega}$, let $N_Y = \{X : X \oplus Y \in N\}$.

Lemma VI.37. *If N is Schnorr null, so is $\{Y : \mu(N_Y) > 0\}$.*

Proof. Due to van Lambalgen [56]. □

Lemma VI.38. *If P is a Π^0_1 class and e an index such that for all $Y \in P$, $\Phi^Y_e \in \tilde{\mathcal{N}}$, then*

$$e(P) := \{X \oplus Y : Y \in P \wedge X \in N_{\Phi^Y_e}\}$$

is Schnorr null.

Proof. Let $T \subseteq {}^{<\omega}2$ be a computable tree such that P is the set of branches of T, and f a computable function such that for all n and $\tau \in T$ with $|\tau| = f(n)$, $\{e\}^{\tau}_{f(n)}(n)$ converges, and $E^{\tau}_n := C_{\{e\}^{\tau}(n)}$ has measure at most 2^{-n}. Note that f must exist due to compactness (this argument is similar to the one employed to prove Proposition II.44). Let F_n be the set of $X \oplus Y$ such that $\tau := Y \upharpoonright f(n) \in T$ and $X \in E_n^\tau$. Note that F_n is the union of at most $2^{f(n)}$ sets of the form $E_n^\tau \oplus [\tau]$, each of which is clopen and has measure at most $2^{-n-f(n)}$. Thus, F_n is clopen and $\mu(F_n) \leq 2^{-n}$. Also, $e(P) \subseteq \bigcap_n \bigcup_{m \geq n} F_m$, since if $Y \in P$ and $\exists^\infty n\ X \in C_{\Phi_e^Y(n)}$, then $\exists^\infty n\ X \in E_n^{Y \upharpoonright f(n)}$, so $\exists^\infty n\ X \oplus Y \in F_n$. Finally, the sequence of F_n is uniform (each F_n is the union of finitely many clopen sets given uniformly by T and f). □

Now, let A be Schnorr random and hyperimmune-free, and let N be a Schnorr null set relative to A. Fix e such that $\Phi_e^A \in \tilde{\mathcal{N}}$ and $N \subseteq N_{\Phi_e^A}$. Since A is hyperimmune-free, there exists a computable $\tilde{f}$ majorizing[10] $f(n) := \mu s\ \Phi_{e,s}^{A \upharpoonright s}(n) \downarrow$. Then let P be the Π_1^0 class

$$\{Y : \forall n\ \Phi_{e,\tilde{f}(n)}^{Y \upharpoonright \tilde{f}(n)}(n) \downarrow =: m \wedge \mu(C_m) \leq 2^{-n}\}.$$

Clearly $A \in P$, and for all $Y \in P$, $\Phi_e^Y \in \tilde{\mathcal{N}}$. Let $e(P)$ be the Schnorr null set given by Lemma VI.38, and $M = \{X : \mu(e(P)_X) > 0\}$; by Lemma VI.37, M is Schnorr null.

If X is computable and $X \in N$, then there exists a Π_1^0 class $Q \subseteq P$ such that for all $Y \in Q$, $X \in N_{\Phi_e^Y}$. To define Q, let $g(n)$ be the nth value of m such that $X \in C_{\Phi_e^A}(m)$. Then $g \leq_T A$ and thus is majorized by some computable $\tilde{g}$. Let Q be the set of $Y \in P$ such that for all n, X is in $C_{\Phi_e^Y(m)}$ for at least n values of $m \leq \tilde{g}(n)$ ($\tilde{f}$ can be used to calculate the requisite values of $\Phi_e^Y(m)$). Note that $A \in Q$, so Q has positive measure (see Proposition II.66); as $Q \subseteq e(P)_X$, we have $X \in M$. Therefore, M is a Schnorr null set containing every computable real in N. □

Corollary VI.39. *If A is a hyperimmune-free Schnorr random real, A is not weakly*

[10] We say f *majorize* g if $f(n) \geq g(n)$ for all n. Clearly, if f dominates g, then a finite modification of f majorizes g.

Schnorr covering.

Corollary VI.40. COVN $\not\subseteq$ NONN.

This completes the proof of Theorem VI.2.

CHAPTER VII

Questions

In this chapter we lay out some possible directions for future research, some of which are very speculative. We can divide these into two topics.

7.1 Algorithmic Randomness

If we ignore the original motivation of studying effective correspondents to cardinal characteristics, the most striking results of this thesis deal with topics related to algorithmic randomness. Specifically,

1. A is Schnorr covering iff A is high;
2. A computes a hyperimmune set implies A is weakly Schnorr covering implies A is not computably traceable, and neither implication reverses;
3. A being weakly Schnorr covering does not imply, nor is it implied by, A being not low for 1-genericity;
4. If A is hyperimmune-free and Schnorr random, and N is Schnorr null relative to A, then the set of computable reals in N is Schnorr null.

Each of these items offers further questions. We first recall the definition of Schnorr covering: A is Schnorr covering if the union of all (unrelativized) Schnorr null sets is Schnorr null relative to A. But from the perspective of algorithmic randomness, there is no particular reason to consider only Schnorr null in this definition; we may replace either instance of "Schnorr null" in the definition with some other kind of effectively null. For example, we may use Martin-Löf null, covered by a null Π^0_1 class, covered by a null Π^0_2 class, or contained in the success set of a computable martingale (these are the types of effectively null which appear most frequently in the study of algorithmic randomness). What sets of oracles can result from such modifications?

The paper [34] investigates this general question. There are 25 possible definitions, 13 of which are trivial (either produce the set of all oracles or the empty set). Most of the remaining 12 are characterized in terms of previously defined computability-theoretic concepts. The exceptions center on the set $\mathbf{P}$ of oracles A such that there is a martingale computable relative to A which succeeds on every real in the universal Martin-Löf null set. It is known that $\mathbf{P}$ includes all PA oracles, every $A \in \mathbf{P}$ computes a Martin-Löf random real, and $\mathbf{P}$ has measure zero.

Question VII.1. *Does* $\mathbf{P}$ *consist of exactly the PA oracles?*

In contrast to the set of Schnorr covering oracles, the weakly Schnorr covering oracles have not been characterized in terms of previously defined computability-theoretic concepts. One possibility involves the oracles which bound a very strong array for computable reals[1]. A *very strong array* is a computable sequence $\{D_n\}_{n\in\omega}$ of finite subsets of ω such that for all n, $|D_{n+1}| > |D_n|$ and $\bigcup_{n\in\omega} D_n = \omega$. An oracle *A bounds a very strong array for computable reals* if there is a real $X \leq_T A$ and a very strong array $\{D_n\}_{n\in\omega}$ such that for every computable Y, $X \upharpoonright D_n = Y \upharpoonright D_n$ for infinitely many n. Clearly, if A bounds a very strong array for computable reals,

[1] a notion closely related to that of array nonrecursiveness

A is weakly Schnorr covering. We also note that our method for showing an oracle to be weakly Schnorr covering has always been to show it bounds a very strong array for computable reals (for example, the satisfaction of even requirements in Theorem VI.32). This suggests that perhaps, informally speaking, the only way to build a null set containing all computable reals is via a very strong array.

Question VII.2. *Do all weakly Schnorr covering oracles bound a very strong array for computable reals?*

Recall that if A is computably traceable (i.e., not in **COFN**), it is not weakly Schnorr covering (not in **NONM**) and low for weak 1-genericity (not in **COFM**). Does the converse hold? Set-theoretic results suggest the answer is no, since there are forcings which add to the ground model a new null set without adding a null set containing all computable reals or adding a new meager set. However, the forcings which accomplish this are very complex, and therefore unlikely to produce proofs in computability theory. This leaves unsettled the question, which can be rephrased as:

Question VII.3. *If A is not DNC and not weakly Schnorr covering, is A computably traceable?*

Informally, we can regard Theorem VI.35 as saying that a hyperimmune-free Schnorr random real A can only compute a "new" null set by adding non-computable reals to an "existing" null set. Of course, A still does compute "new" null sets; the canonical example is $\{A\}$. Is this, effectively, the only thing A can do? Is it impossible for A to compute "new" null sets by adding non-random reals?

Question VII.4. *If A is hyperimmune-free and Schnorr random, and N is Schnorr null relative to A, is the set*

$$\{X \in N : X \textit{ is not Schnorr random}\}$$

Schnorr null?

Another interesting interpretation of Theorem VI.35 is that it expresses a dichotomy for Schnorr random reals. If A is Schnorr random, then either A computes a hyperimmune set, and the set of all computable reals is Schnorr null relative to A, or A is hyperimmune-free, and any set of computable reals which is Schnorr null relative to A is Schnorr null. There is no middle ground, where A can compute a new null set of computable reals without being able to compute a null set covering all the computable reals. Does this dichotomy extend to all oracles?

Question VII.5. *Do there exist A an oracle and N a set of computable reals such that A is not weakly Schnorr covering, N is Schnorr null relative to A, and N is not Schnorr null?*

7.2 Turing Characteristics

The key open question of this thesis is of course:

Question VII.6. *Is the diagram of Theorem V.7 complete? That is, are the containments implied by this diagram the only ones which hold for all $\mathbf{A}$ and $\mathbf{C}$ which are closed downward with respect to $\leq_T$ and with $\mathbf{A}$ countable?*

This appears to be a difficult problem. In order to prove, for example, that $\mathbf{ADDN_A}$ can be separated from $\mathbf{B_A}$, we may need to carry out at least two constructions: one for a non-computable oracle $A \in \mathbf{A}$, and one dependent on A for an oracle B which will be in $\mathbf{B_A}$ but not $\mathbf{ADDN_A}$. However, in certain cases, there are other, seemingly more tractable, options.

- For separating $\mathbf{ADDN_A}$ from $\mathbf{B_A}$: It is known that if A is an oracle of minimal Turing degree (there exists no non-computable $B <_T A$) and $A \not\leq_T \emptyset'$, then

$\emptyset' \in \mathbf{B_A}$, where $\mathbf{A}$ is the set of oracles computable relative to A. This allows us to try to achieve separation with one construction, which may be a simpler process.

- For separating $\mathbf{COVM_A}$ from $\mathbf{D_A}$: If A is hyperimmune-free and $\mathbf{A}$ is the set of oracles computable relative to A, then it is easy to see that $\mathbf{D_A} = \mathbf{D}$. On the other hand, if A is also DNC, it is not low for weak 1-genericity; there is an X which is weakly 1-generic but not weakly 1-generic relative to A. Since $\mathbf{COVM_A}$ will be the set of oracles which compute a real which is weakly 1-generic relative to A, it may be the case that additionally $\mathbf{COVM_A} \neq \mathbf{COVM}$.

A related question is: to what extent do effective morphisms (and their variants) describe relationships between Turing characteristics?

Question VII.7. *Let $\mathfrak{K}$ and $\mathfrak{L}$ be debates. Is it the case that*

- *there is an effective morphism from $\mathfrak{K}$ to $\mathfrak{L}$ iff $\mathbf{L_A^C} \subseteq \mathbf{K_A^C}$ for all $\mathbf{A}$ and $\mathbf{C}$;*
- *there is an effective semi-morphism from $\mathfrak{K}$ to $\mathfrak{L}$ iff $\mathbf{L} \subseteq \mathbf{K}$?*

Remark VII.8. One of the original goals of this thesis was to find a strong correspondence between relationships among cardinal characteristics and relationships among Turing characteristics, mainly in the hopes of shedding further light on cardinal characteristics using computability-theoretic methods. If the first item above holds, it would further this goal, since we may be able to prove, via some method other than finding an effective morphism, that $\mathbf{L_A^C} \subseteq \mathbf{K_A^C}$ for all $\mathbf{A}$ and $\mathbf{C}$. If this were the case, it would also imply $\mathfrak{l} \leq \mathfrak{k}$.

Cichoń's diagram is complete in a stronger way than the one we have mentioned before. It has been proven that for any assignment of the cardinals in the diagram to either $\aleph_1$ or $\aleph_2$, as long as the assignment respects the inequalities of the diagram and the three-cardinal inequalities mentioned in Section 6.5, then there is a model of ZFC such that $\mathfrak{c} = \aleph_2$ and each cardinal in the diagram equals the cardinal to which it was assigned. Does a similar result hold for the corresponding Turing characteristics? We note that Turing characteristics have a natural maximum ($\mathbf{C}\backslash\mathbf{A}$, as discussed in Section 6.4) and a natural minimum (the empty set). Therefore, we may ask:

Question VII.9. *If we assign Turing characteristics in the diagram of Theorem V.7 to either* $\mathbf{C}\backslash\mathbf{A}$ *or* $\emptyset$*, and this assignment respects the containments of Theorem V.7, can we guarantee that there exist* $\mathbf{C}$ *and* $\mathbf{A}$ *such that each Turing characteristic equals its assignment? Can* $\mathbf{C}$ *and* $\mathbf{A}$ *be chosen so that there are no other options (i.e., each characteristic must be* $\mathbf{C}\backslash\mathbf{A}$ *or* $\emptyset$[2]*)? If not, what other relationships characterize the additional assignments (in the manner of the three-cardinal inequalities)?*

Some of our results have proofs that depend on $\mathbf{A}$ being the set of computable oracles, and it is not known if the results themselves also depend on this. Aside from the results of this kind contained in Theorem VI.1, which we have already addressed above, we can ask the following questions.

Question VII.10. *Do there exist* $\mathbf{A}$ *and* $\mathbf{C}$ *such that* $\mathbf{NONM}_{\mathbf{A}}^{\mathbf{C}} \neq \mathbf{IE}_{\mathbf{A}}^{\mathbf{C}}$*?*

Question VII.11. *Do there exist* $\mathbf{A}$ *and* $\mathbf{C}$ *such that* $\mathbf{COFM}_{\mathbf{A}}^{\mathbf{C}} \neq \mathbf{D}_{\mathbf{A}}^{\mathbf{C}} \cup \mathbf{NONM}_{\mathbf{A}}^{\mathbf{C}}$*?*

A final (vague) question would be: what other cardinal characteristics yield interesting results for Turing characteristics? We have discussed $\mathbf{R}$ and $\mathbf{S}$ to some extent, but it is not completely known how these sets of oracles fit in with those appearing in Theorem VI.1. What about other cardinal characteristics? For cardinal characteristics not defined using debates, can reasonable effective correspondents be defined?

[2]For example, this is the case if $\mathbf{A}$ is the set of computable oracles and $\mathbf{C}$ is the set of oracles computable relative to some oracle of minimal Turing degree.

A good place to start would be the consideration of the similarly defined cardinals $\mathfrak{g}$ and $\mathfrak{h}$. Some variants of $\mathfrak{g}$ studied by Mildenberger [] appear to be likely candidates for effectivization, and similar variants of $\mathfrak{h}$ can be defined.

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