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ON MULTIDIMENSIONAL INTEGRAL EQUATIONS OF VOLTERRA TYPE

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I. INTRODUCTION

Multidimensional integral equations of Volterra type have been studied extensively in the past [see N. V. Kasatkina [5], W. Walter [10], Arthur Wouk [11], for partial surveys], and were used to solve boundary value problems—a typical example of such an application is the classical Darboux problem:

$$z_{xy}(x,y) = f(x,y,z,z_x,z_y),(x,y)\epsilon[o,h]x[o,k],$$

$$z(x,o) = \varphi(x);\ z(o,y) = \psi(y);\ \varphi(o) = \psi(o),$$

which corresponds to the integral equation

$$z(x,y) = \varphi(x) + \psi(y) - \varphi(o) + \int_o^x\int_o^y f(\alpha,\beta,z,z_x,z_y)d\alpha d\beta \qquad (1.1)$$

Several forms have been proposed for multidimensional integral equations of Volterra type. One such form has been studied by W. Walter [10] namely,

$$u_\nu(x) = g_\nu(x) + \int_{H_\nu(x)} k_\nu(x,\xi,u\ (\xi))(d\xi)_{p_\nu} \qquad (1.2)$$

$$u = (u_1,u_2,\ldots u_n),\ \nu = 1,2\ldots n,\ x\epsilon B \subset E^m.$$

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where $H_\nu(x) \subset B(x) = \{\xi \in B \mid \xi_i \leq x_i,\ i = 1,2,\ldots n\}$. Precisely, $H_\nu(x)$ is assumed by Walter to be contained in a p_ν-dimensional hyperplane, $1 \leq p_\nu \leq m$, parallel to the coordinate axes—i.e., in a translate of the linear manifold generated by p_ν of the basis vectors $(1,0,0\ldots0)$, $(0,1,0\ldots0)$, $\ldots$ $(0,0,\ldots0,1)$ in E^m. W. Walter gave theorems for the existence of continuous solutions for such systems.

One is now directed naturally to equations where the set H_ν need not be in a p_ν-dimensional hyperplane (with nonzero measure). In other words, one may consider equations where H_ν may consist of several sets, each of which belongs to a hyperplane of different dimension. An example of such an equation would be

$$\varphi(x,y) = f(x,y) + \int_0^x A(x,y,s)\varphi(s,y)ds + \\ + \int_0^y B(x,y,t)\varphi(x,t)dt + \int_0^x\int_0^y C(x,y,s,t)\varphi(s,t)dsdt \qquad (1.3)$$

which is not of the form (1.2) [see T. H. Gronwall [3]].

N. V. Kasatkina [5] has studied the following more general integral equation for local uniqueness of continuous solutions:

$$x(t) = \sum_{\substack{1 \leq i_1 \leq \cdots \leq i_k \leq n \\ 1 \leq k \leq n}} \int_{a_{i_1}}^{t_{i_1}} \cdots \int_{a_{i_k}}^{t_{i_k}} K_{i_1 \ldots i_k}(t\ s_{i_1} \ldots i_k,$$

$$x(t_1, t_2 \ldots t_{i_1 - 1}, s_{i_1}, t_{i_1 + 1}, \ldots t_{i_k - 1}, s_{i_k}, \ldots . t_n) ds_{i_1} \ldots ds_{i_k} + f\ (t) \quad (1.4)$$

where $t = (t_1, t_2 \ldots t_n)$, $s_{i_1 \ldots i_k} = (s_{i_1}, s_{i_2} \ldots s_{i_k})$. His method involves differential inequalities and the results hold locally.

It is the purpose of this paper to study multidimensional nonlinear integral equations of Volterra type of the same general form as (1.4), but with unknowns in spaces L_p, $1 \leq p \leq \infty$, and not necessarily in C, as is usual. Besides, we require less demanding hypotheses on the integrands, as motivated by relevant applications which will be mentioned below. In this situation we establish global existence and uniqueness theorems as well as continuous dependence on initial values, for such integral equations.

The canonical form, emphasized in the present discussion, seems to be the most general since it is found that many equations considered in the past can be put into this form. The theorems we obtain are global—and not local. Further, an analogy with the usual theory of differential equations is maintained as far as possible.

The proofs in the present paper are based on fixed point theorems--precisely, on an extension of Banach's contraction theorem. Indeed, the equations under consideration—being of Volterra type—give rise to an operator T which is not necessarily a contraction by itself; but suitable powers of T are so. It is seen that still, there exists a fixed point for T, by a remark of F. F. Bonsall [1]. This fact allows us to relax the hypotheses.

Essentially, the same argument applies to the space C of continuous functions, as well as to the spaces L_p of the p^{th} summable functions, for $1 \leq p \leq \infty$, and in each of all these cases, we assume a different set of hypotheses. If the equations are assumed to be linear, with analytic coefficients, then existence of analytic solutions is obtained by applying the argument to a space of analytic functions.

The results of this paper are used in—and the present paper has been motivated by—problems of optimal control monitored by nonlinear system (1.1) (and corresponding Darboux data), particularly since the controls are known to be only measurable, and hence the integrands—which contain the controls—may be assumed, at best, to be in some L_p-space, $1 \leq p \leq \infty$. Besides, the corresponding Pontryagin-type multipliers are known to satisfy linear Volterra-type integral equations of the same form (1.4), but again with integrands in an L_p-space.

In §2 we develop suitable notations, in §3 we summarize basic statements, in particular Bonsall's remark, in §4 we discuss the problem under consideration in class C, in §5 we show that analogous results and essentially same proofs using Bonsall's remark hold in L_p, $1 \leq p \leq \infty$; in §6 we consider the linear case and show the existence of analytic solutions, in harmony with classical results. Applications to control problems will be given elsewhere. The special case of the classical Darboux problem (1.1) is discussed in the appendix.

2. NOTATIONS

Let E^n denote the n-dimensional Euclidean space. Let $G = \{t \epsilon E^n | t = (t_1, t_2, \dots t_n),\ a_i \leq t_i \leq a_i + h_i,\ i = 1,2 \dots n\}$. Let α denote the multiindex $(\alpha_1, \alpha_2, \dots \alpha_n)$ with arbitrary nonnegative integers α_i, $i = 1,2\dots n$. As usual, $|\alpha| = \alpha_1 + \alpha_2 + \dots \alpha_n$, and $\alpha! = \alpha_1! \alpha_2! \dots \alpha_n!$. Given any α, let β_j denote the index of the j^{th} nonzero element of the sequence $(\alpha_1, \alpha_2 \dots \alpha_n)$. Let us consider the multiindex $\beta = (\beta_1, \beta_2, \dots \beta_k)$; k being some integer with $1 \leq k \leq n$, determined by α. We shall say that β corresponds to α. Thus, for example, (2,5) corresponds to (0,1,0,0,2); (1,3,6,7) corresponds to (1,0,1,0,0,1,1,), etc. For $t \epsilon E^n$, let t_β denote $(t_{\beta_1}, t_{\beta_2} \dots t_{\beta_k})$ and let $t'_\beta = (t_1, t_2 \dots t_{\beta_1 - 1}, t_{\beta_1 + 1}, \dots t_{\beta_k - 1}, t_{\beta_k + 1}, \dots, t_n)$. In particular let $t'_i = (t_1, t_2 \dots t_{i-1}, t_{i+1}, \dots t_n)$. Let $G_\beta = \{s_\beta \epsilon E^k | s_\beta = (s_{\beta_1}, s_{\beta_2}, \dots s_{\beta_k}), a_i \leq s_i \leq a_i + h_i;\ i = \beta_1, \beta_2 \dots \beta_k\}$. Let $\pi(\theta) = \pi(\theta_1, \theta_2 \dots \theta_n) = \sum_{1 \leq |\alpha| \leq N} \lambda_\alpha \theta^\alpha$ denote a polynomial of degree N in θ with no constant term. Here θ^α denotes $\theta_1^{\alpha_1} \theta_2^{\alpha_2} \dots \theta_n^{\alpha_n}$.

Let C (G) denote as usual the space of functions continuous on G, with sup-norm and let $L_p(G)$ be the space of the p-th summable functions for $1 \leq p < \infty$ and $L_\infty(G)$ be the space of all essentially bounded measurable functions on G. For $m \geq 1$ and $X = C(G), L_p(G)$ or $L_\infty(G)$, let $X^m = X x X x \ldots X$ (m times) and for $\varphi = (\varphi_1, \varphi_2 \ldots \varphi_m)$ in X^m let $\|\varphi\| = \sum_{i=1}^{m} \|\varphi_i\|$. We shall denote by $\|\varphi\|_c, \|\varphi\|_p$ and $\|\varphi\|_\infty$ the norms in $[C(G)]^m$, $[L_p(G)]^m$ and $[L_\infty(G)]^m$, respectively.

For i = 1,2...n, let T_i be an operator defined on X^m (with $X = C(G)$, $L_p(G)$ or $L_\infty(G)$] as follows:

$$T_i\varphi(t) = \int_{a_i}^{t_i} \varphi(t_i', s)ds, \quad \varphi \epsilon X^m$$

We define the product T_iT_j as composition, so that $T_i^r\varphi = T_i(T_i^{r-1}\varphi), r \geq 1$ and $T_i^0 \varphi = \varphi; \varphi \epsilon X^m$. By using Hölder's inequality, it is seen that

$$\|T_i^r\varphi\|_p \leq \|\varphi\|_p \, h_i^r \, (r!p^r)^{-1/p} \qquad , \ 1 \leq p < \infty \tag{2.1}$$

and

$$\|T_i^r\varphi\|_\infty \leq \|\varphi\|_\infty \, h_i^r \, (r!)^{-1} \tag{2.2}$$

Also,

$$\|T_i^r\varphi\|_c \leq \|\varphi\|_c \, h_i^r \, (r!)^{-1} \tag{2.3}$$

For any multiindex $\alpha = (\alpha_1, \alpha_2 \ldots \alpha_n)$ as in the beginning of this section, we shall denote by T^α the operator $T_1^{\alpha_1} \ldots T_n^{\alpha_n}$; and given any polynomial $\pi(\theta) = \Sigma\lambda_\alpha\theta^\alpha$ of degree N, we shall denote by $\pi(T)$ the operator $\Sigma\lambda_\alpha T^\alpha \equiv$ $\Sigma \, \lambda_{(\alpha_1, \alpha_2 \ldots \alpha_n)} \, T_1^{\alpha_1} T_2^{\alpha_2} \ldots T_n^{\alpha_n}$.

In this paper we shall consider the following (canonic) form of multi-dimensional integral equations of Volterra type:

$$x(t) = f(t) + (\pi(T)\circ F)(x)(t)$$

$$\equiv f(t) + \sum_{1 \leq |\alpha| \leq N} \lambda_\alpha T^\alpha F_\alpha(t, s_\beta, x(t'_\beta, s_\beta)) \qquad (2.4)$$

where $f(t) \epsilon X^m$ and for each α, with $1 \leq |\alpha| \leq N$, the function $F_\alpha(t, s_\beta, x)$ $(F^1_\alpha, F^2_\alpha \ldots F^m_\alpha)$ is defined on $G x G_\beta x E^m$ and here β is the multi-index corresponding to α, as described earlier. Specific assumptions on f and F_α will be made later.

3. PRELIMINARIES

In the sequel, we shall need a few preliminary statements. They are given below:

If $F: X \to Y$ is a mapping of a metric space (x, ρ) into another metric space (Y, σ) and there is a constant $c > o$ such that $\sigma(Fx_1, Fx_2) \leq c\ \rho(x_1, x_2)$ for all $x_1, x_2 \epsilon X$, then we denote by $\nu(F)$ the number sup $\{\sigma(Fx_1, Fx_2)/\rho(x_1, x_2) : x_1, x_2 \epsilon X, x_1 \neq x_2\}$.

(3.i) (An extension of Banach's contraction mapping theorem) (see F. F. Bonsall [1]): Let $F: X \to X$ be a mapping of a complete metric space (X, ρ) into itself. Let $F^1 = F$ and $F^n = F(F^{n-1})$ for $n > 1$. Let us assume that $\nu(F^n) < +\infty$ for every n and that $\sum_{n=1}^{\infty} \nu(F^n) < \infty$. Then F has a unique fixed point $x_o \epsilon X$.

(3.ii) (F. F. Bonsall, [1]): If $F: X \to X$ is any continuous map of a complete metric space (X, ρ) into itself such that for some integer $N \geq 1$, F^N is a contraction on X, then F has a unique fixed point x_o in X.

(3.iii) Let $G = \{t \in E^n | t = (t_1, t_2 \ldots t_n), a_i \leq t_i \leq a_i + h_i, i = 1,2\ldots n\}$ as in §2 and let $G_i = \{s \in E^1 : a_i \leq s \leq a_i + h_i\}$ $i = 1,2\ldots n$. Let $\varphi = \varphi(t,s)$ be a real valued function defined on GxG_i such that

(a) $|\varphi(t,s)| \leq cm(s)$ for all $(t,s) \in GxG_i$ where c is a constant and m(s) is integrable on G_i.

(b) φ is continuous in t for each fixed $s \in G_i$.

(c) φ is measurable in s for each fixed $t \in G$.

Then the function $T_i\varphi(t) = \int_{a_i}^{t_i} \varphi(t,s)ds$ is continuous on G.

Following the notation of §2, if T_i, $i = 1,2\ldots n$ are the operators defined there, if $\alpha = (\alpha_1, \alpha_2 \ldots \alpha_n)$ is any multiindex, $T^\alpha = T_1^{\alpha_1} \ldots T_n^{\alpha_n}$ and β is the multiindex corresponding to α, then $T^\alpha \varphi(t, s_\beta) \in [C(G)]^m$, provided

(a) $|\varphi(t,s_\beta)| \leq cm(s_\beta)$ for all $(t,s_\beta) \in GxG_\beta$.

(b) φ is continuous in t for each fixed $s_\beta \in G_\beta$.

(c) φ is measurable in s_β for each fixed $t \in G$.

(3.iv) Let $\alpha = (\alpha_1, \alpha_2 \ldots \alpha_n)$ be any multiindex and let $\beta = (\beta_1, \beta_2 \ldots \beta_k)$ correspond to α as in §2. Let $\varphi(t,s_\beta)$ be any real valued function on GxG_β which is continuous in t_β for almost all $(t'_\beta, s_\beta) \in G$ and belongs to $L_1(G)$ for each fixed $t_\beta \in G_\beta$. Let there exist a constant $B \geq 0$ and a function $m(t'_\beta, s_\beta)$ in $L_1(G)$ such that for $(t,s_\beta) \in GxG_\beta$ we have $|\varphi(t,s_\beta)| \leq Bm(t'_\beta, s_\beta)$. Then the function $\Phi(t)$ defined by $\Phi(t) = T^\alpha \varphi(t,s_\beta)$, is measurable on G.

Let B_1 be a measurable subset of E^n and let $g_i(x,y), i = 1,2\ldots m$ be real-valued measurable functions on $B_1 x E^r$. Let p_1 and p_2 be two real numbers with $1 \leq p_1, p_2 < \infty$. Let us consider the following condition:

(H) There exist m functions $a_i(x), i = 1,2\ldots m$, in $L_{p_2}(B_1)$ and a constant $b \geq 0$ such that for each $i = 1,2\ldots m$, we have

$$|g_i(x,y)| \leq a_i(x) + b|y|^{p_1/p_2}. \qquad (3.1)$$

(3.v) (see M. A. Krasnoselskii, [6]) Let B_1 be a measurable subset of E^n and let $g_i(x,y), i = 1,2\ldots m$ be real valued functions on $B_1 x E^r$ such that for each i, $g_i(x,y)$ is continuous on E^r with respect to y for almost all x in B_1, and measurable in x for each fixed y in E^r. Let $Jz = (J_1 z, J_2 z, \ldots J_m z)$ with $J_i z(x) = g_i(x,z(x))$. Then Jz is measurable whenever z is measurable. Furthermore, the operator J maps $[L_{p_1}(B_1)]^r$ into $[L_{p_2}(B_1)]^m$ if and only if condition (H) holds.

Following the terminology of R. C. Gunning and H. Rossi [4], a complex valued function f defined on an open subset $B \subset C^n$ (the n-dimensional complex vector space) is called "holomorphic" in B if each point $w \epsilon B$ has an open neighborhood $U, w \epsilon U \subset B$, such that the function f has a power series expansion $f(z) = \sum_{\nu_1,\nu_2\ldots\nu_n = o}^{\infty} a_{\nu_1\nu_2\ldots\nu_n}(z_1 - w_1)^{\nu_1}\ldots(z_n - w_n)^{\nu_n}$ which converges absolutely for all $z \epsilon U$. A function f is said to be holomorphic on a closed set $D \subset C^n$, if f is holomorphic on an open set containing D. For functions of several real variables, the same definition above holds, with the word "analytic" being used instead of "holomorphic." The set of all functions holomorphic on D will be denoted by O(D).

If D denotes the rectangle $\{\zeta \epsilon C^n \,|\, |\zeta_i| \leq H_i, i = 1,2\ldots n\} \subset C^n$, then we can define a norm on O(D) as:

$$\|x\| = \sum_{|\alpha| = o} |a_\alpha| H^\alpha \quad \text{where} \quad x = \sum_{|\alpha| = o}^{\infty} a_\alpha \zeta \epsilon O(D)$$

(3.vi) The set O(D) with the above norm is a Banach space.

4. CONTINUOUS SOLUTIONS

In this section we shall discuss the existence and uniqueness of continuous solutions (as well as the dependence of solutions on the "initial" values) of the canonical system:

$$x(t) = f(t) + \sum_{1 \leq |\alpha| \leq N} \lambda_\alpha T^\alpha F_\alpha(t, s_\beta, x(t'_\beta, s_\beta)) \tag{2.4}$$

Theorem 1 below shows the existence of continuous solutions of (2.4) under the assumption that the F_α are merely continuous in x, and not necessarily linear. In this situation, the modulus of continuity is required to be "small" and suitable bounds are given for the λ_α.

If all F_α are known to be Lipschitzian in x, the condition of "small modulus of continuity" can be removed due to the fact that the equation (2.4) is of Volterra type, and the λ_α are then arbitrary. Besides, the solutions are unique in this case. Precise formulations are found in theorem 2. If F_α are linear in x, with analytic coefficients A_α, then solutions can be found which are analytic—not merely continuous. This is in harmony with classical results. This case is studied in §6.

In order to state a theorem on the existence of continuous solutions of the integral equation (2.4), we shall need the following set of hypotheses:

(H_1): Let f(t) be a given element of $[C(G)]^m$, and let $M_1 > 0$ be such that $|f(t)| \leq M_1$ for $t \in G$. Let $M_2 > M_1$ and S_{M_2} denote $\{\zeta | \zeta \in E^m, |\zeta| \leq M_2\}$.

(H_2): Let $F_\alpha(t,s_\beta,x) = (F_\alpha^1,F_\alpha^2 \ldots F_\alpha^m)$ be functions defined on $GxG_\beta xS_{M_2}$ where $\beta = (\beta_1,\beta_2,\ldots \beta_{d_\alpha})$ corresponds to α, as described in §2. Let

(a) $F_\alpha(t,s_\beta,x)$ be measurable in s_β for fixed (t,x);

(b) there exist functions $k_{1\alpha}(s_\beta)$ integrable on G_β such that on $G \times G_\beta \times S_{M_2}$ we have $|F_\alpha(t,s_\beta,x)| \leq k_{1\alpha}(s_\beta)$.

(H_3): There exist monotone nondecreasing continuous functions $w_{i\alpha}$ with $w_{i\alpha}(0) = 0$, $i = 1,2$ and functions $\tilde{k}_{2\alpha}(s_\beta)$ integrable on G_β such that for $(t^1,x_1),(t^2,x_2)$ in GxS_{M_2} and $s_\beta \epsilon G_\beta$, we have

$$|F_\alpha(t^1,s_\beta,x_1) - F_\alpha(t^2,s_\beta,x_2)| \leq \tilde{k}_{2\alpha}(s_\beta)[w_{1\alpha}(|t^1-t^2|) + w_{2\alpha}(|x_1-x_2|)] \quad (4.1)$$

It is to be noted that if we take $H_\alpha(t,s_\beta,x) = (t_\beta - s_\beta)^{\beta-1}((\beta-1)!)^{-1} F_\alpha(t,s_\beta,x)$, then we have $T_\beta H_\alpha = T_{\beta_1}.T_{\beta_2}\ldots T_{\beta_{d\alpha}}(H_\alpha) = T^\alpha F_\alpha$. Furthermore, as a consequence of the condition (H_3) above, there exist functions $k_{2\alpha}(s_\beta)$ integrable in G_β such that

$$\begin{aligned} &|H_\alpha(t^1,s_\beta,x_1) - H_\alpha(t^2,s_\beta,x^2)| \leq \\ &\leq k_{2\alpha}(s_\beta)[w_{1\alpha}(|t^1-t^2|) + w_{2\alpha}(|x_1-x_2|)] \end{aligned} \quad (4.2)$$

for $(t^1,x_1),(t^2,x_2)$ in GxS_{M_2} and $s_\beta \epsilon F_\beta$. This is what we shall need in the sequel.

Let D_{β_i} denote the product of intervals $[a_j,a_j + h_j]$ for $j = 1,2\ldots i-1,i+1,\ldots d_\alpha$ where d_α is the number of nonzero elements in $\alpha = (\alpha_1,\alpha_2\ldots\alpha_n)$. Let us consider the functions $K_{i_\alpha}(t_{\beta_i})$ defined on $[a_{\beta_i},a_{\beta_i} + h_{\beta_i}]$ as follows:

$$K_{i_\alpha}(t_{\beta_i}) = \int (t_\beta - s_\beta)^{\beta-1} (\beta-1)!)^{-1} k_{1\alpha}(s_\beta) ds_\beta \qquad (4.2)$$

where the integration is performed over the set $[a_{\beta_i}, t_{\beta_i}] \times D_{\beta_i}$ and $k_{1\alpha}(s_\beta)$ are the functions found in (H_2). Since $k_{1\alpha}(s_\beta)$ are L-integrable in G_β, it follows that the functions K_{i_α} are continuous in $[a_{\beta_i}, a_{\beta_i} + h_{\beta_i}]$. Also, by (H_1), the function f is continuous on G. Thus, there exist monotone nondecreasing functions $\nu(.), \sigma_{i_\alpha}(.)$ such that $\nu(0) = 0, \sigma_{i_\alpha}(0) = 0$, and

$$|f(t^1) - f(t^2)| \leq \nu(|t^1 - t^2|) \text{ and}$$

$$|K_{i_\alpha}(t^1_{\beta_i}) - K_{i_\alpha}(t^2_{\beta_i})| \leq \sigma_{i_\alpha}(|t^1_{\beta_i} - t^2_{\beta_i}|). \qquad (4.3)$$

Let $k_{i_\alpha} = \int_{G_\beta} k_{i_\alpha}(s_\beta) ds_\beta, i = 1,2;\ 1 \leq |\alpha| \leq N.$

(H_4): Let there exist real numbers λ_α such that

$$M_1 + \sum_\alpha |\lambda_\alpha| k_\alpha h_\beta^{\beta-1} ((\beta-1)!^{-1} \leq M_2$$

(H_5): Let there exist monotone nondecreasing, continuous functions $\eta(.)$ vanishing at zero such that

$$\eta(\theta) \geq \zeta(\theta) + \sum_\alpha |\lambda_\alpha| k_{2\alpha} w_{2\alpha}(\eta(\theta))$$

where

$$\zeta(\theta) = \nu(\theta) + \sum_\alpha |\lambda_\alpha| [\sum_i \sigma_{i\alpha}(|\theta_{\beta_i}|) + k_2 w_{1\alpha}(\theta)]$$

and

$$\theta = (\theta_1, \theta_2 \ldots \theta_n) \text{ with } \theta_i \epsilon [0, h_i],\ i = 1,2\ldots n \qquad (4.4)$$

<u>Theorem 1</u>: Let the above hypotheses H_1-H_5 hold. Let K denote $\{x \epsilon [C(G)]^m \big| |x(t)| \leq M_2$ and $|x(t^1) - x(t^2)| \leq \eta(|t^1 - t^2|)$ for $t^1, t^2 \epsilon G\}$. Let τ be an operator defined on $[C(G)]^m$ by

$$\tau x(t) = f(t) + \sum_{1 \leq |\alpha| \leq N} \lambda_\alpha T^\alpha F_\alpha(t, s_\beta, x(t'_\beta, s_\beta)).$$

Then, there exists at least one $x \epsilon K$ with $x = \tau x$.

Proof: The set K defined above is a nonempty compact convex subset of the normed space $[C(G)]^m$. Let us prove that τ maps K into K and that τ is continuous. For $x \in K$, t^1, $t^2 \in G$ with $t^1_i \leq t^2_i$, $i = 1,2\ldots n$ we have

$$\left| x(t^1) - x(t^2) \right| < \left| f(t^1) - f(t^2) \right| +$$

$$+ \sum_\alpha \left| \lambda_\alpha [T^\alpha F_\alpha(t^1,s_\beta,x(t^1,s_\beta)) - T^\alpha F_\alpha(t^2,s_\beta,x(t^2,s_\beta))] \right|$$

$$\leq \nu(|t^1-t^2|) + \sum_\alpha \left| \lambda_\alpha [T_\beta H_\alpha(t^1,s_\beta,x(t^1,s_\beta)) - T_\beta H_\alpha(t^2,s_\beta,x(t^2,s_\beta))] \right|$$

$$\leq \nu(|t^1-t^2|) + \sum_\alpha \left| \lambda_\alpha \sum_{i=1}^{d\alpha} \int_{E_i} ((\beta-1)!(t_\beta - s_\beta))^{-1} k_{1\alpha}(s_\beta) ds_\beta \right.$$

$$\left. + \int_{G_\beta} k_{2\alpha}(s_\beta)\, w_{1\alpha}(|t^1-t^2|) + w_{2\alpha}(|x(t^1,s_\beta) - x(t^2,s_\beta)|))\}] \right|$$

$$\leq \nu(|t^1-t^2|) + \sum_\alpha |\lambda_\alpha| \; [\Sigma \sigma_{i_\alpha}(|t^1_{\beta_i} - t^2_{\beta_i}|) + k_{2_\alpha}(w_{1_\alpha}(|t^1-t^2|)$$

$$+ w_{2\alpha} \circ \eta(|t^1-t^2|))] \leq \eta(|t^1-t^2|)$$

by (H_5). (Here, as in (H_3), E_i denotes $[t^1_{\beta_i}, t^2_{\beta_i}] \times D_{\beta_i}$). This shows in particular that τx is continuous on G. Also, for $x \in K$,

$$|\tau x(t)| \leq |f(t)| + \sum_{\alpha} \left|\lambda_\alpha T^\alpha F_\alpha(t, s_\beta, x(t'_\beta, s_\beta))\right|$$

$$\leq M_1 + \sum_{\alpha}|\lambda_\alpha| \left|T_\beta H_\alpha(t, s_\beta, x(t'_\beta, s_\beta))\right|$$

$$M_1 + \sum_{\alpha}|\lambda_\alpha| h_\beta^{\beta-1}((\beta-1)!)^{-1} k_{1_\alpha} \leq M_2.$$

Thus, τ maps K into K.

Now, let x_1 and x_2 be any two elements of K.

Then,

$$|\tau x_1(t) - \tau x_2(t)| \leq \sum_{\alpha}|\lambda_\alpha [T^\alpha F_\alpha(t, s_\beta, x_1) -$$

$$- T^\alpha F_\alpha(t, s_\beta, x_2)]| \leq \sum_{\alpha}|\lambda_\alpha| k_{2_\alpha} w_{2_\alpha}(\|x_1 - x_2\|)$$

where

$$\|x_1 - x_2\| = \sup \{|x_1(t) - x_2(t)| : t \in G\}.$$

Hence for

$$x_1, x_2 \in K, \|\tau x_1 - \tau x_2\| \leq \sum_{\alpha}|\lambda_\alpha| k_{2_\alpha} w_{2_\alpha}(\|x_1 - x_2\|) \tag{4.5}$$

This shows τ maps K continuously into it itself.

Now, by Schauder's fixed point theorem there exists at least one $x \in K$ with $\tau x = x$. This concludes the proof of theorem 1. [Note: Details of above calculations for $n = 2$, may be found in [8]].

Remark: Concerning the hypothesis (H_5) of above theorem, let us consider the case where F_α are Lipschitzian with constants A_α in x. Then $w_{2_\alpha}(\nu) = A_\alpha \nu$ (see (4.1)). If

$$\sum_{\alpha}|A_\alpha \lambda_\alpha h^\alpha| < 1$$

then

$$\eta(\theta) = \zeta(\theta) + \sum_{\alpha}|\lambda_{\alpha}|A_{\alpha}h^{\alpha}\eta(\theta)$$

yields

$$\eta(\theta) = (1-\sum_{\alpha}|A_{\alpha}\lambda_{\alpha}h^{\alpha}|)^{-1}\zeta(\theta)$$

and hence by choosing this function as η in (H_5), it follows by theorem 1 that there exists at least one $x \epsilon K$ satisfying (2.4). Further, in this case, i.e., if $\sum_{\alpha}|A_{\alpha}\lambda_{\alpha}h^{\alpha}| < 1$ then the solutions of (2.4) are unique. Indeed, if x_1 and x_2 are solutions then $x_1 = \tau x_1$ and $x_2 = \tau x_2$. Now with $w_{2\alpha}(\nu) = A_{\alpha}\nu$, the inequality (4.5) will reduce to

$$\begin{aligned}\|x_1-x_2\| = \|\tau x_1-\tau x_2\| &\leq \sum_{\alpha}|A_{\alpha}\lambda_{\alpha}h^{\alpha}|\,\|x_1-x_2\| \\ &< \|x_1-x_2\|\end{aligned}$$

which is impossible if $x_1 \neq x_2$.

This proves uniqueness.

On the other hand, if F_{α} are known to be Lipschitzian, then by a completely different argument one can prove the existence and uniqueness of continuous solutions of (2.4)—without the further condition $\sum_{\alpha}|A_{\alpha}\lambda_{\alpha}h^{\alpha}| < 1$. Precise formulations follow. We shall omit the proof here, since it is the same as for L_p-solutions which will be discussed in the next section.

Theorem 2: Hypotheses: For each α, $1 \leq |\alpha| \leq N$, let $F_{\alpha}(t,s_{\beta},x) = (F^1_{\alpha},F^2_{\alpha}\dots F^m_{\alpha})$ be defined on $GxG_{\beta}xE^m$ where β corresponds to α. Let F_{α} be continuous in (t_{β},t'_{β}) and be measurable in s_{β}. Let there exist constants $M_{\alpha} \geq 0$ and functions $A_{\alpha}(s_{\beta})$ in $L_1(G_{\beta})$ such that for all $(t,s)\epsilon GxG_{\beta}$ and x_1,x_2 in E^m, we have

$$|F_\alpha(t,s,x_1)-F_\alpha(t,s,x_2)| \leq M_\alpha|x_1-x_2|$$

and $|F_\alpha(t,s,o)| \leq A_\alpha(t'_\beta, s_\beta)$. Given a sequence of real numbers $\{\lambda_\alpha | 1 \leq |\alpha| \leq N\}$ and a positive integer r, let δ_r denote $\Sigma\mu_\alpha h^\alpha(\alpha!)^{-1}$ where summation extends over $r \leq |\alpha| \leq rN$ and μ_α is the coefficient of θ^α in the binomial expansion of $(\sum_{1 \leq |\alpha| \leq N} |\lambda_\alpha| M_\alpha \theta^\alpha)^r$. It is seen that $\delta = \sum_{r=o}^{\infty} \delta_r < \infty$. Let $R \geq 1$ be such that $\delta_R < 1$. Let $M > 0$ be any real number such that

$$M > \delta(1-\delta_R)^{-1}(\|f\| + \sum_{1 \leq |\alpha| \leq N} \|\lambda_\alpha A_\alpha\| h^\alpha(\alpha!)^{-1}) \tag{4.6}$$

where f is a given element of $[C(G)]^m$ and $\|.\|$ refers to the supremum norm.

Conclusion: Given $f\epsilon[C(G)]^m$, λ_α real, $1 \leq |\alpha| \leq N$, and $M > 0$ satisfying (4.6), there exists a unique $x\epsilon[C(G)]^m$ with $\|x\| \leq M$ such that $\tau x = x$ where, as in Theorem 1,

$$\tau x(t) = f(t) + \sum_{1 \leq |\alpha| \leq N} \lambda_\alpha T^\alpha F_\alpha(t, s_\beta, x(t'_\beta, s_\beta)) \tag{4.7}$$

Further, if f_1 and f_2 are any two elements of $[C(G)]^m$ and if x_1, x_2 are the corresponding solutions of $\tau x = x$, then

$$\|x_1 - x_2\| \leq (1-\delta_R)^{-1}\|f_1 - f_2\| \tag{4.8}$$

Thus, the solutions depend continuously on the "initial" values.

<u>Remark</u>: The inequality (4.8) is readily obtained by repeated application of the following

$$|x_1(t)-x_2(t)| = |f_1(t)-f_2(t) + \Sigma\lambda_\alpha T^\alpha[F_\alpha(t,s_\beta,x_1)$$
$$- F_\alpha(t,s_\beta,x_2)| \leq |f_1(t)-f_2(t)| + \Sigma|\lambda_\alpha|T^\alpha M_\alpha|x_1-x_2|$$

Thus, for each r

$$\|x_1-x_2\| \leq \delta\|f_1-f_2\| + \delta_r\|x_1-x_2\|$$

so that

$$\|x_1-x_2\| \leq \delta(1-\delta_R)^{-1}\|f_1-f_2\|$$

where R is such that $\delta_R < 1$.

5. L_p-SOLUTIONS ($1 \leq p \leq \infty$).

It is of interest to observe that theorem 2 with slight changes yields unique L_p-solution for the equation (2.4). The changes needed are made clear by the following.

Theorem 3: Hypotheses: Let F_α be as before defined on $GxG_\beta xE^m$; where β corresponds to α. Let F_α be continuous in t_β and be measurable (t_β', s_β). Let there exist constants $M_\alpha \geq 0$ and functions $A_\alpha(t_\beta', s_\beta)$ in $[L_p(G)]$, $1 \leq p \leq \infty$, such that for all $(t,s)\epsilon GxG_\beta$ and x_1,x_2 in E^m

$$|F_\alpha(t,s,x_1) - F_\alpha(t,s,x_2)| \leq M_\alpha|x_1 - x_2| \tag{5.1}$$

and

$$|F_\alpha(t,s,o)| \leq A_\alpha(t_\beta^1, s_\beta) \tag{5.2}$$

Given a sequence of real numbers $\{\lambda_\alpha | 1 \leq |\alpha| \leq N\}$ and a positive integer r, let δ_r denote $\sum_{r \leq |\alpha| \leq rN} \mu_\alpha h^\alpha(\alpha! p^\alpha)^{-1/p}$(with p = 1 in case of L_∞) where μ_α is the coefficient of θ^α in the binomial expansion of $(\sum_{1 \leq |\alpha| \leq N} |\lambda_\alpha| M_\alpha\theta^\alpha)^r$. It is seen that $\delta = \sum_{r=1}^{\infty}\delta_r < \infty$. Let $R \geq 1$ be such that $\delta_R < 1$. Let M > 0 be any real number such that

$$M > \delta(1-\delta_R)^{-1}(\|f\| + \sum_{1 \leq |\alpha| \leq N} \|\lambda_\alpha A_\alpha\|_p h^\alpha(\alpha! p^\alpha)^{-1/p}) \tag{5.3}$$

where f is a given element of $[L_p(G)]^m$. (In the case of $L_\infty(G)$, p is taken as 1 in (5.3)).

Conclusion: Given f in $[L_p(G)]^m$, λ_α real, $1 \leq |\alpha| \leq N$ and $M > 0$ satisfying (5.3), there exists a unique $x \in [L_p(G)]^m$ with $\|x\|_p \leq M$ such that $\tau x = x$, where τ is defined as in theorem 2, by

$$\tau x(t) = f(t) + \sum_{1 \leq |\alpha| \leq N} \lambda_\alpha T^\alpha F_\alpha(t, s_\beta, x(t'_\beta, s_\beta)).$$

Further, if f_1 and f_2 are any two elements of $[L_p(G)]^m$, and x_1, x_2 are the corresponding solutions of $\tau x = x$, then $\|x_1 - x_2\|_p \leq (1-\delta_R)^{-1}\delta\|f_1 - f_2\|_p$.

Proof: Let us observe from (5.1) and (5.2) that

$$\begin{aligned} |F_\alpha(t,s,x)| &\leq |F_\alpha(t,s,x) - F_\alpha(t,s,o)| + |F_\alpha(t,s,o)| \\ &\leq M_\alpha|x| + A_\alpha(t'_\beta, s_\beta) \end{aligned} \tag{5.4}$$

As a consequence of the assumption on F_α and the inequalities (5.4), it follows (see 3.v in §3) that $F_\alpha(t, s_\beta, x(t'_\beta, s_\beta))$ for fixed t_β is in $[L_p(G)]^m$. Hence $T^\alpha F_\alpha(t, s_\beta, x(t'_\beta, s_\beta))$ is in $[L_p(G)]^m$ and consequently maps $[L_p(G)]^m$ into itself.

Let us now show that for x_1, x_2 in $[L_p(G)]^m$ and any integer $r \geq 1$, we have

$$\|\tau^r x_1 - \tau^r x_2\|_p \leq \delta_r \|x_1 - x_2\|_p \tag{5.5}$$

Indeed,

$$|\tau x_1(t) - \tau x_2(t)| =$$

$$= |\Sigma\lambda_\alpha T^\alpha F_\alpha(t,s_\beta,x(t'_\beta,s_\beta))-\Sigma\lambda_\alpha T^\alpha F_\alpha(t,s_\beta,x_2(t'_\beta,s_\beta))|$$

$$\leq \Sigma|\lambda_\alpha|T^\alpha M_\alpha|x_1-x_2|(t'_\beta,s_\beta)$$

and

$$|\tau^r x_1(t)-\tau^r x_2(t)| \leq \Sigma|\lambda_\alpha|T^\alpha M_\alpha|\tau^{r-1}x_1-\tau^{r-1}x_2|$$

It follows now by induction that

$$|\tau^r x_1(t)-\tau^r x_2(t)| \leq (\Sigma|\lambda_\alpha|T^\alpha M_\alpha)^r|x_1-x_2|(t'_\beta,s_\beta) .$$

Now, it follows by using the inequalities (2.1) that

$$\|\tau^r x_1 - \tau^r x_2\|_p \leq \delta_r\|x_1 - x_2\|_p .$$

This concludes the proof of the inequality (5.5). As a consequence, it is seen that $\tau = \tau^1$ is continuous on $[L_p(G)]^m$. Further, since $\delta_r \to 0$ as $r \to \infty$, there is an $R \geq 1$ such that $\delta_R < 1$. The corresponding operator τ^R is a contraction on $[L_p(G)]^m$.

As a further consequence of (5.5), we have

$$\|\tau^{i+1}x-\tau^i x\| \leq \delta_i\|\tau x-x\|, x\epsilon[L_p(G)]^m$$

$$i = 1,2...$$

Hence, for $r \geq 1$,

$$\|\tau^r x\| \leq \|x\| + \sum_{i=o}^{r-1}\|\tau^{i+1}x-\tau^i x\|$$

$$\leq \|x\| + \|\tau x-x\|\sum_{i=o}^{r-1}\delta_i$$

For $x = o$, this inequality yields

$$\|\tau^r(0)\| \leq (\sum_{i=o}^{\infty}\delta_i)\|\tau(0)\| = \delta\|\tau(0)\| \qquad (5.6)$$

Hence, by (5.5)

$$\|\tau^r x\| \leq \delta_r\|x\| + \delta\|\tau(0)\|. \qquad (5.7)$$

But, since

$$|\tau(0)|(t) \leq |f(t)| + \Sigma|\lambda_\alpha T^\alpha A_\alpha(t'_\beta, s_\beta)|$$

it follows by the inequality (2.1) that

$$\|\tau(0)\| \leq \|f\| + \sum_{1 \leq |\alpha| \leq N} \|\lambda_\alpha A_\alpha\| h^\alpha(\alpha! p^\alpha)^{-1/p}$$

Thus, by (5.7),

$$\|\tau^r x\| \leq \delta_r \|x\| + \delta(\|f\| + \Sigma\| \lambda_\alpha A_\alpha\| h^\alpha(\alpha! p^\alpha)^{-1/p} \tag{5.8}$$

Let us consider the set

$$X_M = \{x\epsilon[L_p(G)]^m \Big| \|x\|_p \leq M\}$$

This set is mapped into itself by τ^R. Indeed it follows from (5.8) that for $x\epsilon X_M$

$$\|\tau^R x\| \leq M\delta_R + M(1-\delta_R) = M.$$

Now, since τ^R is a contraction on $[L_p(G)]^m$, it is so on X_M. Further, since X_M is a closed subspace of the Banach space $[L_p(G)]^m$, and thus itself is a Banach space, it follows that there is a unique $x\epsilon X_M$ with $\tau^r x = x$. But since τ is a continuous operator on $[L_p(G)]^m$ it follows (see 3.ii in §3) that $\tau x = x$. This concludes the proof of existence and uniqueness.

Let f_1 and f_2 be any two elements of $[L_p(G)]^m$ and let x_1 and x_2 be the corresponding solutions of (2.4). Let $f = |f_1 - f_2|$ and $x = |x_1 - x_2|$. It is seen that

$$x(t) \leq f(t) + (\Sigma|\lambda_\alpha| T^\alpha M_\alpha) x(t).$$

Applying the inequality again, we get

$$x(t) \le f(t) + \Sigma|\lambda_\alpha| T^\alpha M_\alpha (f(t) + (\Sigma|\lambda_\alpha| T^\alpha M_\alpha) x(t))$$
$$= (1 + \Sigma|\lambda_\alpha| T^\alpha M_\alpha) f(t) + (\Sigma|\lambda_\alpha| T^\alpha M_\alpha)^2 . x(t) .$$

In general for $r \ge 1$,

$$x(t) \le [1 + (\Sigma|\lambda_\alpha| T^\alpha M_\alpha) + \ldots + (\Sigma|\lambda_\alpha| T^\alpha M_\alpha)^{r-1}] f(t)$$
$$+ (\Sigma|\lambda_\alpha| T^\alpha M_\alpha)^r . x(t).$$

But then

$$\|x\|_p \le \|f\|_p (1 + \delta_1 + \delta_2 + \ldots + \delta_{r-1}) + \delta_r \|x\|_p .$$

If R is such that $\delta_R < 1$, then

$$\|x\|_p (1-\delta_R) \le \|f\|_p (1 + \delta_1 + \ldots + \delta_{r-1}) \le \delta \|f\|_p$$

thus,

$$\|x\|_p \le \delta (1-\delta_R)^{-1} \|f\|_p$$

i.e.,

$$\|x_1 - x_2\|_p \le \delta (1-\delta_R)^{-1} \|f_1 - f_2\|_p .$$

This concludes the proof of the theorem.

6. GENERAL REMARKS

A. The arguments of the previous sections, when applied to a space of analytic functions yield a unique analytic solution of (2.4) provided the function F_α are assumed linear in x with analytic coefficients. Precise formulations follow:

Theorem 4: Let $G = \{t \in E^n \,\big|\, |t_i| \leq h_i, i = 1,2\ldots n\}$ and let $f(t)$, $A_\alpha(t)$, $1 \leq |\alpha| \leq N$ be functions analytic on an open rectangle $R_1 = \{t \in E^n \,\big|\, |t_i| < H_i', i = 1,2\ldots n\}$ containing G. Then there exists a unique function $x(t)$ analytic on G and satisfying

$$x(t) = f(t) + \sum_{1 \leq |\alpha| \leq N} \lambda_\alpha T^\alpha (A_\alpha x)(t_\beta', s_\beta) \tag{6.1}$$

Proof: Let $R = \{t \in E^n \,\big|\, |t_i| \leq H_i, i=1,2\ldots n\}$ be such that $G \subset R \subset R_1$. Let $D = \{\zeta \in C^n \,\big|\, |\zeta_i| \leq H_i, i=1,2\ldots n\} \subset C^n$, where C denotes as usual, the set of complex numbers. Let $O(D)$ denote the set of all functions holomorphic in D. Then $O(D)$ is a Banach space with the norm given by

$$\|x\| = \sum_{|\alpha|=0}^{\infty} |a_\alpha| H^\alpha \quad \text{for} \quad x = \sum_{|\alpha|=0}^{\infty} a_\alpha \zeta^\alpha \qquad \text{(see 3.vi)}$$

Let $\tilde{f}(\zeta)$ and $\tilde{A}_\alpha(\zeta)$ be natural holomorphic extensions in D, of $f(t)$ and $A_\alpha(t)$ respectively. Let τ be an operator defined on $O(D)$ by

$$\tau x(\zeta) = \tilde{f}(\zeta) + \sum_{1 \leq |\alpha| \leq N} \lambda_\alpha \tilde{T}_\alpha (\tilde{A}_\alpha x)(\zeta_\beta', \xi_\beta)$$

where $\tilde{T}_\alpha = \tilde{T}_1^{\alpha_1} \ldots \tilde{T}_n^{\alpha_n}$ is analogous to T_α; for example,

$$\tilde{T}_i x(\zeta_i', \xi_i) = \int_{-H_i}^{\zeta_i} x(\zeta_i', \xi_i) d\xi_i, i = 1,2\ldots n.$$

It is to be noted that for $x \in O(D)$, the integral defining $\tilde{T}_i$, does not depend on the path.

It is clear that maps $O(D)$ into itself. Further, for any positive integer r and any $x_1, x_2 \in O(D)$, we have

$$\|\tau^r x_1 - \tau^r x_2\| \le \sum_{r \le |\alpha| \le rN} \mu_\alpha H^\alpha (\alpha!)^{-1} \|x_1 - x_2\|$$

where μ_α is the coefficient of θ^α in the binomial expansion of $(\sum_{1 \le |\alpha| \le N} |\lambda_\alpha| A_\alpha \theta^\alpha)^r$—here, $A_\alpha = \mathrm{Sup}\{|A_\alpha(\zeta)| : \zeta \epsilon D\}$. [To obtain the above inequality, we observe that if

$$x_1 - x_2 = \sum_{|\nu|=0} a_\nu \zeta^\nu$$

then

$$|\tilde{T}_\alpha(x_1 - x_2)| \le |\sum_\nu a_\nu \zeta^{\alpha+\nu} \nu!((\alpha + \nu)!)^{-1}$$

$$\le \sum_\nu |a_\nu| H^{\alpha+\nu} \nu! \, ((\alpha + \nu)!)^{-1}$$

$$\le \sum_\nu |a_\nu| H^\nu H^\alpha (\alpha!)^{-1} \quad]$$

It follows that $\|\tau^r x_1 - \tau^r x_2| \le \delta_r \|x_1 - x_2\|$ where $\delta_r = \sum_{r \le |\alpha| \le rN} \mu_\alpha H^\alpha (\alpha!)^{-1}$

Since this is true for any $x_1, x_2 \epsilon O(D)$ and $\sum_{r=1}^{\infty} \delta_r < \infty$—(it is majorised by an exponential function)—and since $O(D)$ is a Banach space, it follows by statement (3.i) that τ has a unique fixed point in $O(D)$. If $\tilde{x}(\zeta)$ denotes this fixed point, and if $x(t)$ denotes the restriction of $\tilde{x}(\zeta)$ for ζ real, it is clear that $x(t)$ is the unique analytic solution of (6.1).

B. The canonical form suggested in this paper is very similar to the form studied by Kasatkina [5]. It is to be noted, however, that the notation proposed here simplifies the exposition. Further, it takes care of repeated itegrals also, in a natural way. Of course, a repeated integral can be transformed into a single integral and the author found that it made no difference in the estimates obtained here. The arguments remain the same too.

C. A function $v\epsilon L_p(G)$ will be said to be the generalized partial derivatives of order $\alpha = (\alpha_1 \ldots \alpha_n)$ of a function $u\epsilon L_p(G)$, or $v = D^{\alpha}u$, in the usual sense (C. B. Morrey [7], L. S. Sobolev [9]). We mention here that generalized partial derivatives of order one, have a simple characterization. A function $x_{t_i} \epsilon L_p(G)$ is the generalized partial derivative of $x\epsilon L_p(G)$ with respect to t_i if and only if for almost all closed rectangles $R \subset G$, $R = \{t | c_i \leq t_i \leq d_i, i=1,2\ldots n\}$, we have

$$\int_R x_{t_i}\, dt = \int_{c'_i}^{d'_i} [x(d_i,s)-x(c_i,s)]ds$$

where c'_i and d'_i are defined as usual, by $c = (c_i,c'_i)$ and $d = (d_i,d'_i)$. Here the expression "for almost all rectangles $R = [c,d]$" means that the set of all (c,d) forms a set of measure zero in GxG.

It is not difficult to verify that if $x\epsilon L_p(G)$ then $T_i x$ possesses generalized partial derivative with respect to t_i, $i=1,2\ldots n$. Consequently, if x is an L_p-solution of an equation of the form

$$x(t) = f(t) + T_i \Sigma \lambda_{\alpha} T^{\alpha} F_{\alpha}(t,s,x(t,s))$$

for a given i, i = 1,2...n, and if f possesses generalized partial derivative with respect to t_i, then x also possesses generalized partial derivative with respect to t_i. An example of such a situation is the equivalent of Darboux problem.

$$z(x,y) = \int_0^x \int_0^y F(\alpha,\beta,z(\alpha,\beta))d\alpha d\beta$$

Any L_p-solution of this equation possesses generalized partials with respect to both x and y.

APPENDIX

We shall discuss now the application of our existence theorems to the special case of the classical Darboux problem in a rectangle $G = [a, a + h] \times [b, b + k] \subset E^2$:

$$z^i_{xy} = F_i(x,y,z,z_x,z_y) \;,\; (x,y)\in G \;,$$

$$z^i(x,b) = \varphi^i(x) \;;\; z^i(a,y) = \psi^i(y) \;;\; \varphi^i(a) = \psi^i(b),$$

$$z = (z^1, z^2, \ldots z^n) \;;\; i = 1,2\ldots n \qquad \text{(A.1)}$$

The precise results which will be stated below as corollaries of our theorem 3 of §5 will be applied in the optimal control problem mentioned in the introduction. We shall need the following hypothesis on (A.1) to be able to apply theorem 3, §5 and obtain solutions of (A.1) belonging to a Sobolev class (see [2]).

(H_1): The functions $\varphi(x) = (\varphi^1, \varphi^2, \ldots \varphi^n)$, and $\psi(y) = (\psi^1, \psi^2, \ldots \psi^n)$ are defined and absolutely continuous on $[a, a + h]$ and $[b, b + k]$ respectively. The derivatives φ_x and ψ_y which exist almost everywhere, belong to $L_p([a, a + h])$ and $L_p([b, b + k])$ respectively; here $1 \leq p \leq \infty$. Further $\varphi(a) = \psi(b)$.

(H_2): $F = F(x,y,z,r,t) = (F_1, F_2, \ldots F_n)$ is defined for all $(x,y)\in G$ and $(z,r,t)\in E^{3n}$. For each i, F_i is measurable in (x,y) for fixed $(z,r,t)\in E^{3n}$.

(H_3): There exists a constant $K > 0$ such that for $(x,y)\in G$ and (z_1, r_1, t_1), $(z_2, r_2, t_2)\in E^{3n}$, we have

$$|F(x,y,z_1,r_1,t_1) - F(x,y,z_2,r_2,t_2)| \leq K(|z_1-z_2| + |r_1-r_2| + |t_1-t_2|).$$

(H_4): Let $F(x,y,0,0,0)\epsilon[L_p(G)]^n$.

Remark: One may assume, instad of (H_3) that, there exist constants K_{1j}, K_{2j}, $K_{3j} > 0$ with $j = 1,2,\ldots,n$, such that,

$$|F_i(x,y,z_1,r_1,t_1) - F_i(x,y,z_2,r_2,t_2)| \leq$$

$$\leq \sum_{j=1}^{n} [K_{1j}|z_1^j - z_2^j| + K_{2j}|r_1^j - r_2^j| + K_{3j}|t_1^j - t_2^j|].$$

But with no loss of generality we may set $K_{1j} = K_{2j} = K_{3j} = K^1 > 0$ and $K = nK^1$ so that the above inequality reduces to (H_3).

Let $W_p^1(G)$ denote the Sobolev space of all $z\epsilon L_p(G)$ with first order generalized partial derivatives (see §6) z_x, z_y belonging to $L_p(G)$. Let $\|z\| = \|z\|_p + \|z_x\|_p + \|z_y\|_p$ denote the norm in $[W_p^1(G)]^n$.

Theorem 5: Let the hypotheses H_1-H_4 hold. Then, there exists a unique $z\epsilon[W_p^1(G)]^n$, $1 \leq p \leq \infty$ (same as in (H_1)), such that (i) the generalized partial derivative $z_{xy}^i(x,y)$ exists and equals $F_i(x,y,z,z_x,z_y)$ a.e. in G and (ii) $z(x,b) = \varphi(x)$; $z(a,y) = \psi(y)$.

Further,

$$\|z\| \leq (1-\delta_r)^{-1}\delta[k^{1/p}(\|\varphi_x\|_p + 2^{-1}\|\varphi\|_p) +$$

$$+ h^{1/p}(\|\psi_y\|_p + 2^{-1}\|\psi\|_p) + (h+k)\|s(x,y)\|_p] \qquad \text{(A.2)}$$

where $s(x,y) = F(x,y,0,0,0)$; $\delta = \sum_{n=1}^{\infty}\delta_n$; $\delta_n = [n/2]^{-1}\rho^n$ with $\rho = (2K + 1)$ x $(p^{-1/p})$ $(h^p + k^p)^{1/p}$ if $1 < p < \infty$ and $\rho = (k + 2^{-1})$ $(h + k)$ if $p = 1$ or $p = \infty$.

The number k here is same as in (H_3) and the number R in (A.2) is that positive integer for which $\delta_R < 1$.

If (φ_1, ψ_1) and (φ_2, ψ_2) are any two pairs of functions satisfying (H_1) and if z_1, z_2 are the corresponding solutions of (A.1) then

$$\|z\| \leq (1-\delta_R)^{-1}\delta[k^{1/p}(\|\varphi_x\|_p + 2^{-1}\|\varphi\|_p) + h^{1/p}(\|\psi_y\|_p + 2^{-1}\|\psi\|_p)] \quad (A.3)$$

where

$$z = z_1 - z_2 \; ; \; \varphi = \varphi_1 - \varphi_2 \; ; \; \psi = \psi_1 - \psi_2 \; ;$$

δ's are as above.

Proof: Let us consider the integral equation

$$z(x,y) = \varphi(x) + \psi(y) - \varphi(a) + \int_a^x\int_b^y F(\alpha,\beta,z(\alpha,\beta),z_x(\alpha,\beta),z_y(\alpha,\beta))d\alpha d\beta \quad (A.4)$$

where z_x and z_y are understood as generalized partials of z. Clearly, any solution of (A.4) (which is necessarily continuous on G and hence in $[L_p(G)]^n$) has, indeed, generalized partials z_x, z_y which satisfy the following

$$w_1(x,y) = 2^{-1}(\varphi(x) + \psi(y)) + 2^{-1}\int_a^x w_2(\alpha,y)d\alpha + 2^{-1}\int_b^y w_3(x,\beta)d\beta$$

$$w_2(x,y) = \varphi_x(x) + \int_a^x 0.\, d\alpha + \int_b^y F(x,\beta,w(x,\beta))d\beta$$

$$w_3(x,y) = \psi_y(y) + \int_a^x F(\alpha,y,w(\alpha,y))d\alpha + \int_b^y 0.\, d\beta \quad (A.5)$$

Where $w_1 = z$; $w_2 = z_x$; $w_3 = z_y$; $W = (w_1, w_2, w_3)$ (A.6)

Further, z_{xy} exists and equals $F(x,y,z,z_x,z_y)$ a.e. in G. Thus, every $[W_p^1(G)]^n$ solution of (A.4) (and hence of (A.1) corresponds in a unique manner (as in (A.6)) to a $[L_p(G)]^{3n}$ solution $W = (w_1, w_2, w_3)$ of (A.5). Now, the system of equations (A.5) is exactly in the canonical form (2.4). Since the hypotheses $H_1 - H_4$ guarantee the assumptions in theorem 3, §5, existence of a unique solution $w \in [L_p(G)]^{3n}$ and hence of the corresponding $z \in [W_p^1(G)]^n$ is concluded.

The norm bound (A.2) for the solution z follows from the inequality (5.3) with the observation that in the present case, $\|w\|_p = \|z\|$; $N = 1$; $A_\alpha(x,y) = F(x,y,0,0,0)$; $\lambda_\alpha = 1$; $f = (2^{-1}(\varphi+\psi), \varphi_x, \psi_y)$; $\alpha = (1,0)$ or $(0,1)$. It is to be noted that $\|.\|$ in (5.3) denotes the norm in $L_p(G)$ while $\|\varphi\|$ in (A.2) denotes the norm of $\varphi(x)$ in $L_p([a,a + h])$, and so on.

The same observation leads us from the conclusion of theorem 3, §5 to the inequality (A.3). This concludes the proof of Theorem 5.

Special cases: (i) If F of (H_2) does not depend on r and t, then (A.4) is itself in the canonical form (2.4). In this case, the norm bound for the solution z is given by

$$\|z\| \leq \delta(1 - \delta_R)^{-1}[2^{-1}(k^{1/p}\|\varphi\| + h^{1/p}\|\psi\|) + p^{-1/p}(hk\|s\|_p + k^{1/p}2^{-1}h\|\varphi_x\| + h^{1/p}2^{-1}k\|\psi_y\|)],$$

where

$$\delta_r = (r!)^{-2}(Khk)^r \text{ if } p = 1 \text{ or } p = \infty \text{ and}$$

$$\delta_r = (r!p^r)^{-2/p}(khk)^r \text{ if } 1 < p < \infty \text{ and}$$

$$\delta = \sum_{n=1}^{\infty} \delta_n, \text{ as before.}$$

(ii) Let $U \subset E^m$ and let Γ be any set of measurable functions $u: G \to U$. Let $f = f(x,y,z,r,t,u)$ be defined on $G \times E^{3n} \times U$ and let f be measurable in (x,y), continuous in u, and Lipschitzian (as in (H_3)) in (z,r,t). Let for each

$v\in\Gamma$, the function $f(x,y,0,0,0,u(x,y))$ belong to $[L_p(G)]^n$. For a given $u\in\Gamma$, define

$$F(x,y,z,r,t) = f(x,y,z,r,t,u(x,y)) \tag{A.7}$$

Then F satisfies H_2, H_3, H_4. By applying theorem 5, to this F we obtain a unique solution z of (A.1) corresponding to the data φ,ψ and v (which defines F). The inequality (A.2) gives the norm-bound on z, as before. If v_1 and v_2 are any two elements of Γ, (φ_1,ψ_1) and (φ_2,ψ_2) satisfy (H_1) and if z_i is the solution of (A.1) corresponding to the data (φ_i,ψ_i, v_i), i=1,2 then $z = z_1-z_2$ satisfies the inequality (A.2) with $\varphi = \varphi_1-\varphi_2$; $\psi = \psi_1-\psi_2$ and

$$s(x,y) = f(x,y,z_1,z_{1x},z_{1y},v_1) - f(u,y,z_1,z_{1x},z_{1y},v_2) \tag{A.8}$$

<u>Pointwise estimates</u>: Since any solution z of (A.1) satisfies the integral equation (A.4), it is absolutely continuous in the sense of Tonelli. Hence, there is a set $E\subset G$, with meas $E = 0$ such that for $(x_0,y_0)\in G-E$, we have in view of (H_3),

$$|z_x(x_0,y_0)| \le |\varphi_x(x_0)| + \int_b^{y_0}[s + k(|z| + |z_x| + |z_y|)](x_0,\beta)d\beta \tag{A.9}$$

where $s(x,y)$ denotes $F(x,y,0,0,0)$. Similar inequality holds for $z_y(x_0,y_0)$. Further $z(x,y) = \psi(y) + \int_a^x z_x(\alpha,y)d\alpha = \varphi(x) + \int_b^y z_y(x,\beta)d\beta$. Using these facts along with repeated application of Gronwall's lemma one finds that

$$|z(x,y)| \le 2^{-1}[\|\varphi\|_c + \|\psi\|_c + e^{K(h+k)} \iint_G s(\alpha,\beta)d\alpha d\beta + A_5(Ah + Ak + e^{K(h+k)})]$$

$$|z_x(x,y)| \le \theta_1(x) + AA_5 \text{ and } |z_y(x,y)| \le \theta_2(y) + AA_5 \tag{A.10}$$

where

$$\theta_1(x) = e^{Kk}[|\varphi_x(x)| + Kk|\varphi(x)| + \int_b^{b+k} s(x,\beta)d\beta]$$

$$\theta_2(y) = e^{Kh}[|\psi_y(y)| + Kh|\psi(y)| + \int_a^{a+h} s(\alpha,y)d\alpha]$$

$$A_5 = \|\varphi_x\|_p h^{1/q} + \|\psi_y\|k^{1/q} + (\|\varphi\|_c + \|\psi\|_c)Khk + \iint_G s(\alpha,\beta)d\alpha d\beta$$

and A is a constant depending only on K,h and k. In the above $\|\cdot\|_c$ denotes the supremum norm.

Dependence on data: Let z_i denote, as before, the solution of (A.1), corresponding to the data (φ_i, ψ_i, v_i), $i = 1,2$. (F being given by (A.7).) It is seen then that $z_x = z_{1x} - z_{2x}$ and $z_y = z_{1y} - z_{2y}$ satisfy the inequalities (A.9) and hence pointwise estimates for $z = z_1 - z_2$ and its derivatives are also given by (A.10) where z, φ, ψ and s are understood as follows:

$$z = z_1 - z_2 \ ; \ \varphi = \varphi_1 - \varphi_2 \ ; \ \psi = \psi_1 - \psi_2 \ ;$$

$$s(x,y) = |f(x,y,z_{1x},z_{1y},v_1) - f(x,y,z_1,z_{1x},z_{1y},v_2)|$$

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