McKay's Correspondence for Klein's Quartic Curve

by

Brian S. Jurgelewicz

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
2010

Doctoral Committee:

      Professor Igor Dolgachev, Chair
      Professor Harm Derksen
      Professor Jeffrey C. Lagarias
      Professor Karen E. Smith
      Associate Professor Martin J. Strauss

ACKNOWLEDGEMENTS

TABLE OF CONTENTS

CHAPTER I

Introduction

Our story begins with McKay's observation regarding binary polyhedral groups, i.e. finite subgroups of $\mathrm{SL}(2, \mathbb{C})$. Each binary polyhedral group G comes with a natural, two-dimensional representation V_{nat}. John McKay associates a graph to G as follows. The vertices of the graph are the irreducible, complex representations of G. Two vertices, corresponding to representations ρ_1, ρ_2, are connected by an edge if ρ_1 is isomorphic to a direct summand of the tensor product $\rho_2 \otimes V_{\mathrm{nat}}$, or vice-versa; the condition turns out to be symmetric. McKay's observation is that the graph associated to G is a Dynkin diagram of affine type ADE.

Chapter II discusses the place of McKay's observation within the ADE correspondence. We present the subject from four different viewpoints; namely, representation theory, equivariant sheaves, invariant theory, and resolution of singularities. Like a cubist painting, the multitude of perspectives has the effect of fleshing out McKay's observation and adding significance to it.

The subject of Chapter III is the T_{pqr}-correspondence, which is a generalization of the ADE correspondence. Note that a finite subgroup of $\mathrm{SL}(2, \mathbb{C})$ acts linearly on $\mathbb{C}^2$, so it acts on the set of lines through the origin in $\mathbb{C}^2$, which is $\mathbb{P}^1$. The basic idea of the generalization is simply to replace $\mathbb{P}^1$ with a smooth projective curve Z of genus

$g \geq 2$, equipped with the action of a finite subgroup of automorphisms, $G \subseteq \mathrm{Aut}\, Z$. We assume there are precisely three exceptional orbits on Z with stabilizer subgroups in G of orders (p, q, r). The name T_{pqr}-correspondence refers to the fact that a graph of type T_{pqr} can be associated to the pair (Z, G) in several ways (see Section 3.1).

The analogue of an irreducible representation of a binary polyhedral group in this setting is a G-invariant vector bundle on Z which is stable with respect to the usual notion of slope stability, and of degree zero. A collection of such objects has a generalized McKay graph (see III.11, III.15, and III.17). The analogue of McKay's observation within the T_{pqr}-correspondence is a sort of Riemann-Hilbert problem; it asks whether there exists a collection of such bundles whose generalized McKay graph is the same T_{pqr}-graph (see Conjectures III.12 and III.17).This is where the background material ends and our original work begins.

Chapter IV is the heart of the thesis. We specialize to the case when Z is Klein's quartic curve, and $G = \mathrm{PSL}(2, \mathbb{F}_7)$ is its full automorphism group, of order 168. We examine a certain collection of modules over the ring $A = \mathbb{C}[x, y, z]/(x^7 + y^3 + z^2)$ canonically associated to the pair (Z, G).

The computer algebra program Macaulay2 is used to find presentations of our modules. Using the presentations, we show that at least four of our modules correspond to stable G-invariant bundles on Z of degree zero. This is the main result.

The novelty of the result is not the discovery of the bundles, but the fact that they have a uniform construction. To our knowledge, this was previously unknown.

Chapter IV concludes with a peek at future research. We find several stable bundles in the cases $(p, q, r) = (2, 3, 11)$ and $(p, q, r) = (3, 3, 5)$ as well. This is perhaps the most mysterious part of the thesis. To be sure, it is a direction one ought to pursue in the future continuation of this project.

CHAPTER II

The ADE Correspondence

Our goal in this introductory chapter is to review the ADE correspondence. There are four sections: Polyhedral groups, Equivariant sheaves, Invariant theory, and Resolution of singularities. In each of these sections, one sees the same combinatorial structure; namely, the Dynkin diagram of type ADE. The fact that one can recover the same combinatorial structure in multiple ways is what we mean by 'ADE correspondence'. The rest of the thesis is dedicated to generalizing this happy situation to star-shaped graphs of type T_{pqr}.

2.1 Polyhedral groups

Our starting point for the ADE correspondence is John McKay's remarkable observation regarding binary polyhedral groups. Let us begin by recalling the definitions involved.

The term binary polyhedral group is synonymous with finite subgroup of $\mathrm{SL}(2, \mathbb{C})$. Polyhedral group means finite subgroup of $\mathrm{PSL}(2, \mathbb{C})$. A binary polyhedral group is the preimage of a polyhedral group under the quotient map $\mathrm{SL}(2, \mathbb{C}) \to \mathrm{PSL}(2, \mathbb{C})$. The reason for the name is as follows. Every finite subgroup of $\mathrm{PSL}(2, \mathbb{C})$ is conjugate into the compact subgroup $\mathrm{PSU}(2, \mathbb{C})$. Since $\mathrm{PSU}(2, \mathbb{C})$ is isomorphic to $\mathrm{SO}(3, \mathbb{R})$, a finite subgroup of $\mathrm{PSU}(2, \mathbb{C})$ is either cyclic, dihedral, or it is the symmetry group of

a regular convex polyhedron in $\mathbb{R}^3$. A regular convex polyhedron is called a Platonic solid. There are five Platonic solids: tetrahedron, cube, octahedron, dodecahedron, and icosahedron. A cube and an octahedron are polar polytopes, hence they have isomorphic symmetry groups, as do a dodecahedron and an icosahedron. As the reader is undoubtedly aware, binary polyhedral groups have a rich history (see, for instance, Chapter 1 of [MS02]).

Let G be a binary polyhedral group. The embedding $\rho_{\mathrm{nat}} : G \to \mathrm{SL}(2, \mathbb{C})$ is called the natural representation. Let $K(\mathrm{Rep}\, G)$ be the Grothendieck group of the category of finite dimensional complex representations of G. The McKay pairing $K(\mathrm{Rep}\, G) \times K(\mathrm{Rep}\, G) \to \mathbb{Z}$ is the bilinear pairing defined by the formula

$$(2.1) \qquad \langle \rho_i, \rho_j \rangle = 2 \dim_{\mathbb{C}} \mathrm{Hom}_G(\rho_i, \rho_j) - \dim_{\mathbb{C}} \mathrm{Hom}_G(\rho_{\mathrm{nat}} \otimes \rho_i, \rho_j).$$

Let $\mathcal{B} = \{\rho_1, \ldots, \rho_n\} \subset K(\mathrm{Rep}\, G)$ be the classes of the irreducible representations of G. They form a basis for $K(\mathrm{Rep}\, G)$. Using the character tables of the binary polyhedral groups, McKay observed that the McKay pairing on $K(\mathrm{Rep}\, G)$ with respect to the basis $\mathcal{B}$ is a Cartan matrix of affine type ADE.

The natural representation defines an action of G on $\mathbb{C}^2$. Let $A = \mathbb{C}[x, y]^G$ be the ring of invariants. The variety $V = \mathrm{Spec}\, A$ is called an ADE singularity, Du Val singularity, rational double point, or simple singularity. The open subset $U = V \backslash \{0\}$ is called the punctured quasicone.

The binary polyhedral group $G < \mathrm{SL}(2, \mathbb{C})$ is isomorphic to the fundamental group of the punctured quasicone U. Hence, by the Riemann-Hilbert correspondence, $\mathrm{Rep}\, G$ is equivalent to $\mathrm{Loc}\, U$, the category of local systems on U, i.e. locally free sheaves on U in the classical, analytic topology.

2.2 Equivariant sheaves

McKay's observation regarding the representation theory of a binary polyhedral group G can be explained in terms of the G-equivariant K-theory of $\mathbb{C}^2$. Let us take a moment to review some terminology from the theory of G-sheaves.

Let X be a smooth quasi-projective variety over $\mathbb{C}$, and G a group acting on X. Let coh X be the category of coherent sheaves on X. A coherent sheaf $E \in \text{coh } X$ is said to be G-invariant if there exists an isomorphism $c_g^E : E \to g^*E$ for each $g \in G$. If there is a finite collection $\{c_g^E\}_{g \in G}$ satisfying $c_1^E = 1$ and $c_{hg}^E = g^*(c_h^E) \circ c_g^E$ then E is said to admit a G-linearization. If Y is a smooth subscheme of X, then Y is said to be G-invariant if $\mathcal{O}_X(Y)$ is G-linearized.

The term 'G-sheaf' will always mean G-linearized sheaf. Whenever we encounter G-invariant sheaves which are not G-linearized, we will be careful to point it out.

If E and F are G-sheaves, then G acts on $\text{Hom}_{\text{coh } X}(E, F)$ by the formula $g(\phi) = (c_g^F)^{-1} \circ g^* \phi \circ c_g^E$. In this way G-sheaves form a category $\text{coh}^G X$, with homomorphisms $\text{Hom}_{\text{coh}^G X}(E, F) = \text{Hom}_{\text{coh } X}(E, F)^G$.

By the assumptions that X is quasi-projective and G is finite, every G-sheaf has a G-equivariant injective resolution (see [Gro57] Prop 5.1.2). This allows one to define the derived functors of $\text{Hom}_{\text{coh}^G X}(-, -)$; they are denoted $\text{Ext}^i_{\text{coh}^G X}(-, -)$. It follows from [Gro57] Prop 5.2.3 that the taking invariants functor is exact, i.e. G acts on the space $\text{Ext}^i_{\text{coh } X}(E, F)$, $i > 0$ in such a way that $\text{Ext}^i_{\text{coh}^G X}(E, F) = \text{Ext}^i_{\text{coh } X}(E, F)^G$. Incidentally, the proof uses the fact that the order of G is coprime to the characteristic of the ground field.

Let $K(\text{coh}^G X)$ be the Grothendieck group of $\text{coh}^G X$. Let $K_c(\text{coh}^G X)$ be the subgroup generated by sheaves with compact support. The equivariant Euler char-

acteristic $K(\mathrm{coh}^G X) \times K_c(\mathrm{coh}^G X) \to \mathbb{Z}$ is given by

$$\chi(E, F) = \sum_{n=0}^{\dim X} (-1)^n \dim \mathrm{Ext}^n_{\mathrm{coh}\, X}(E, F)^G.$$

It induces a well-defined, bilinear pairing $\chi(-, -) : K(\mathrm{coh}^G X) \times K_c(\mathrm{coh}^G X) \to \mathbb{Z}$, called the Euler form.

With the Euler form in hand, we now return to the specific situation of $G < \mathrm{SL}(2, \mathbb{C})$. The natural representation ρ_{nat} defines an action of G on the variety $\mathbb{C}^2$. The origin is fixed, and G acts freely away from the origin.

One may regard $\mathbb{C}^2$ as a rank 2 vector bundle over the origin. There is an inclusion $s : \{0\} \hookrightarrow \mathbb{C}^2$ and a projection $\pi : \mathbb{C}^2 \to \{0\}$.

Both s_* and π^* are faithful and exact. Note that the functor π^* is not full, because $\mathrm{Hom}_{\mathrm{coh}\,\mathbb{C}^2}(\mathcal{O}, \mathcal{O})$ is isomorphic to $\mathbb{C}[x, y]$, not $\mathbb{C}$.

Now by Prop 5.4.9 in [CG97], the Euler form between pushforwards may be calculated by the formula:

$$(2.2) \quad \chi(s_*\rho_i, s_*\rho_j) = \mathrm{Hom}_G(\Lambda^2 \rho_{\mathrm{nat}} \otimes \rho_i, \rho_j) - \mathrm{Hom}_G(\rho_{\mathrm{nat}} \otimes \rho_i, \rho_j) + \mathrm{Hom}_G(\rho_i, \rho_j).$$

Since $G < \mathrm{SL}(2, \mathbb{C})$, the representation $\Lambda^2 \rho_{\mathrm{nat}}$ is isomorphic to the trivial representation. Hence we obtain

$$(2.3) \quad\quad\quad \chi(s_*\rho_i, s_*\rho_j) = 2 \mathrm{Hom}(\rho_i, \rho_j) - \mathrm{Hom}(\rho_{\mathrm{nat}} \otimes \rho_i, \rho_j).$$

That is, the Euler form on $K_c(\mathrm{coh}^G \mathbb{C}^2)$ defines a bilinear form on $K(\mathrm{Rep}\,G)$, via the faithful, exact functor s_*, such that it coincides with equation (2.1).

Together, the following two propositions show that the Euler form is a perfect pairing.

Proposition II.1. *For any representation $\rho \in \mathrm{Rep}\,G$, the dual of the skyscraper sheaf $s_*\rho_i$ with respect to the Euler form is the vector bundle $\pi^*\rho_i$.*

Proof. All higher Ext's between $\pi^*\rho_i$ and $s_*\rho_j$ vanish because the support of the sheaf $s_*\rho_j$ is the origin of $\mathbb{C}^2$, which is 0-dimensional. By [CG97] 5.4, one has the adjunction formula for G-sheaves:

$$(2.4) \qquad \operatorname{Hom}(\pi^*\rho_i, s_*\rho_j)^G = \operatorname{Hom}_G(\rho_i, \rho_j).$$

These two facts imply that the $\pi^*\rho_i$'s and $s_*\rho_j$'s are dual, i.e.

$$(2.5) \qquad \chi(\pi^*\rho_i, s_*\rho_j) = \dim_{\mathbb{C}} \operatorname{Hom}_G(\rho_i, \rho_j) = \delta_{ij}.$$

Thus $\pi^*\mathcal{B} := \{\pi^*\rho_1, \dots, \pi^*\rho_n\}$ and $s_*\mathcal{B} := \{s_*\rho_1, \dots, s_*\rho_n\}$ are a set of dual bases for $K(\operatorname{coh}^G \mathbb{C}^2)$ and $K_c(\operatorname{coh}^G \mathbb{C}^2)$, respectively. $\qquad\square$

Proposition II.2. *Every G-linearized coherent sheaf on $\mathbb{C}^2$ has a resolution of length at most 2 by free G-bundles, i.e. there exists an exact sequence of the form*

$$(2.6) \qquad 0 \to \pi^*\rho_2 \to \pi^*\rho_1 \to \pi^*\rho_0 \to \mathcal{F} \to 0,$$

for all $\mathcal{F} \in \operatorname{coh}^G \mathbb{C}^2$, where ρ_0, ρ_1, ρ_2 are representations of G.

Remark II.3. The analogous statement holds for any smooth variety X with G-action satisfying condition ELFG (enough locally free G-linearized sheaves), i.e. for every G-sheaf $\mathcal{F}$ there exists a G-bundle surjecting onto it.

Proof. Let $\mathbb{C}[x, y] * G$ be the skew group ring. Its elements are formal sums $\sum_{g \in G} a_g g$, $a_g \in \mathbb{C}[x, y]$, with multiplication satisfying

$$(2.7) \qquad (ag)(a'g') = a(g(a'))gg'.$$

A $\mathbb{C}[x, y] * G$-module M is a $\mathbb{C}[x, y]$-module which is also a kG-module, such that $g(ax) = g(a)gx$, for $g \in G, a \in A, x \in X$. One has a canonical equivalence between the category of finitely generated $\mathbb{C}[x, y] * G$-modules and the category of G-linearized

coherent sheaves on $\mathbb{C}^2$. Another basic fact is that every projective $\mathbb{C}[x,y] * G$-module is projective (hence free) as a $\mathbb{C}[x,y]$-module (see [Aus86]).

Now we reproduce the rest of the proof from [HS87], for the reader's convenience.

Let $M \in \mathbb{C}[x,y] * G$-mod. There is a G-compatible epimorphism $\epsilon : M \to M/\mathfrak{m}M$, where $\mathfrak{m}$ is the maximal ideal (x,y) of $\mathbb{C}[x,y]$. Since $M/\mathfrak{m}M$ is finitely generated as a $\mathbb{C}[x,y] * G$-module, it is projective as a $\mathbb{C}[G]$-module. Therefore ϵ admits a $\mathbb{C}[G]$-section $j : M/\mathfrak{m}M \to M$.

We let $H := M/\mathfrak{m}M$ and define $\delta : \mathbb{C}[x,y] \otimes_{\mathbb{C}} H \to M$ by $\delta(a \otimes h) = aj(h)$ for all $a \in \mathbb{C}[x,y]$ and $h \in H$. Then δ is a $\mathbb{C}[x,y]$-module epimorphism and compatible with the G-action on $\mathbb{C}[x,y] \otimes_{\mathbb{C}} H$ and M. The statement of the proposition now follows from the Hilbert syzygy theorem. $\qquad\square$

By the above two propositions, the Euler form is a perfect pairing. Recall that the Euler form $\chi(s_* \rho_i, s_* \rho_j)$ between skyscraper sheaves is well-defined. Thus we get a well-defined bilinear pairing $K(\mathrm{coh}^G \mathbb{C}^2) \times K(\mathrm{coh}^G \mathbb{C}^2) \to \mathbb{Z}$. Note that the pairing is defined between any two classes, with no assumption of compact support.

2.3   Invariant theory

Let $A = \mathbb{C}[x,y]^G$ be the ring of invariants, and $V = \mathbb{C}^2/G$ be the ADE singularity. Let $p : \mathbb{C}^2 \to V$ be the projection.

Let $M = p_*(E)^G \in \mathrm{coh}\, V$ be the invariant subsheaf of a G-linearized vector bundle $E \in \mathrm{coh}^G \mathbb{C}^2$. Since G acts freely away from the origin of $\mathbb{C}^2$, M is locally free away from the singular point of V.

Let U be the punctured quasicone and $i : U \subset V$ be the open embedding. Let $\Gamma : \mathrm{coh}\, V \to \mathrm{mod}\, A$ be the canonical equivalence between $\mathrm{coh}\, V$ and the category of finitely generated A-modules.

An A-module M is said to be maximal Cohen-Macaulay if depth $M = \dim M$. In [BD08], Prop 3.12, it is shown that $\Gamma \circ i_* : \operatorname{coh} U \to \ \operatorname{mod} A$ restricts to an equivalence between $\operatorname{vect} U$, the category of vector bundles on U, and $\operatorname{CM}(A)$, the category of (maximal) Cohen-Macaulay A-modules.

Let $\widehat{A} = \mathbb{C}[\![x,y]\!]^G$ be the completion of A. Auslander has shown that the taking-invariants functor $p_*^G : \mathbb{C}[\![x,y]\!]-\operatorname{mod} \to \widehat{A}-\operatorname{mod}$ induces an equivalence between the full subcategory $\operatorname{pro} \mathbb{C}[\![x,y]\!] * G \subset \mathbb{C}[\![x,y]\!] * G-\operatorname{mod}$ consisting of projective modules and the full subcategory $\operatorname{CM}(\widehat{A}) \subset \widehat{A}-\operatorname{mod}$ consisting of Cohen-Macaulay modules:

$$(2.8) \qquad\qquad p_*^G : \operatorname{pro} \mathbb{C}[\![x,y]\!] * G \simeq \operatorname{CM}(\widehat{A}).$$

By composing the pullback $\pi^* : \operatorname{Rep} G \to \mathbb{C}[\![x,y]\!] * G$ with the equivalence p_*^G, we get a faithful, exact, essentially surjective functor

$$(2.9) \qquad\qquad \varphi : \operatorname{Rep} G \to \operatorname{CM}(\widehat{A}).$$

An $\widehat{A}$-module M is indecomposable iff $\operatorname{End}(M)$ is a local ring (see Prop 1.18 in [Yos90]). This implies that $\widehat{A}$-mod satisfies the Krull-Schmitt property, meaning, every module has a unique decomposition into indecomposables.

Since $\widehat{A}$-mod is Krull-Schmitt, one can define the AR quiver of $\operatorname{CM}(\widehat{A})$. Its vertices are the indecomposable objects; and the number of arrows between two indecomposables M_i and M_j is the dimension of the space of irreducible morphisms $\operatorname{Irr}(M_i, M_j) \subset \operatorname{Hom}(M_i, M_j)$, where $\operatorname{Irr}(M_i, M_j)$ is the quotient of the space of non-invertible morphisms $\varphi : M_i \to M_j$ by the subspace of ones which can be written as a product $\varphi = \varphi_1 \circ \varphi_2$.

Theorem II.4. *[Aus86]The AR quiver of* $\operatorname{CM}(\widehat{A})$ *coincides with the quiver associated to the McKay pairing of* G.

Theorem II.5. *[AR87, KST07] The AR quiver of* $\mathrm{CM}(\widehat{A})$ *is a Dynkin diagram of affine type ADE.*

Together, these two facts constitute an independent explanation of McKay's observation. In other words, McKay's observation is 'subsumed' into the theory of CM modules.

Remark II.6. For an irreducible representation ρ, a presentation of the module $\varphi(\rho)$ is known. From the explicit presentation, one can see that $\varphi(\rho)$ is given as the cokernel of a matrix over A, not merely $\widehat{A}$. That is, φ factors through the canonical map $\mathrm{CM}(A) \to \mathrm{CM}(\widehat{A})$. Recall that $\mathrm{Rep}\,G$ is equivalent to the category of local systems on U, and $\mathrm{CM}(A)$ is equivalent to $\mathrm{vect}\,U$. Thus we regard φ as a canonical way of associating an (algebraic) vector bundle on U to a local system on U. In Chapter III, we will see a map that is analogous to φ, using the Narasimhan-Seshadri theorem, and in Chapter IV we will study its image.

2.4   Resolution of Singularities

Let X be a smooth, quasi-projective, complex variety, with a finite subgroup of automorphisms $G < \mathrm{Aut}(X)$. The G-Hilbert scheme G-Hilb X is the closure of the set of regular G-orbits in the Hilbert scheme $\mathrm{Hilb}^{[n]} X$ of 0-dimensional subschemes of length $n = |G|$. It comes with a tautological family $X \xleftarrow{\ p\ } \mathcal{Z} \xrightarrow{\ q\ } \text{G-Hilb}\,X$. A reference for these facts about G-Hilbert schemes is [Blu06].

For an abelian category $\mathcal{A}$, let $D^b(\mathcal{A})$ be its derived category (in the sense of [Del77], pages 262-311). The functor $\Phi : D^b(\mathrm{coh}\,\text{G-Hilb}\,X) \to D^b(\mathrm{coh}^G X)$ given by the formula $\Phi(E) = Rp_*(Lq^*(E))$ is called the *Fourier-Mukai transform,* with kernel $\mathcal{O}_{\mathcal{Z}}$. Here Rp_* and Lq^* are the right and left derived functors, respectively.

The following theorem sparked a lot of excitement in the field because it holds for

three-dimensional varieties.

Theorem II.7. *[BKR01] Suppose X is a complex symplectic variety of dimension ≤ 3, and G acts by symplectic automorphims. Then $Y := $ G-Hilb X is a crepant resolution of the categorical quotient X/G and the Fourier-Mukai transform $\Phi :$ $D^b(\mathrm{coh}\, Y) \rightarrow D^b(\mathrm{coh}^G X)$ is an equivalence of triangulated categories.* $\qquad\square$

One specializes to the case $X = \mathbb{C}^2$. Let $V = \mathrm{Spec}\, A$, where $A = \mathbb{C}[x,y]^G$ is the ring of invariants. Let $Y \rightarrow V$ be a minimal resolution of V. Y is a smooth variety, birational to V, with trivial canonical bundle $\omega_Y \simeq \mathcal{O}$. The reduced exceptional divisor $\mathcal{E} \subset Y$ is an ADE configuration of rational curves $\{E_i, i = 2, \ldots, n\}$. The fundamental cycle, Z, is an effective divisor, supported on $\mathcal{E}$, such that if one adjoins Z to the set $\{E_i, i = 2, \ldots, n\}$, then the configuration is a Dynkin diagram of affine type ADE.

Kapranov and Vasserot examine the functor in the opposite direction.

Theorem II.8. *[KV00] Let $\Psi : D^b(\mathrm{coh}^G \mathbb{C}^2) \rightarrow D^b(\mathrm{coh}\, Y)$ be the Fourier-Mukai transform defined by $\Psi(E) = Rq_*(Lp^*(E))^G$. Then Ψ is an equivalence of categories.*

Moreover, let $i : E_i \rightarrow Y$, $i = 2, \ldots, n$ be the reduced irreducible components of the exceptional curve $\mathcal{E}$. Let Z be the fundamental cycle. Let $s_i = s_*\rho_i, i = 1, \ldots, n$ be the irreducible 0-dimensional G-invariant sheaves supported at the origin, where $s : \{0\} \rightarrow \mathbb{C}^2$ is the inclusion, and $\rho_1, \ldots, \rho_n$ are the irreducible representations of G, with ρ_1 the trivial representation.

Then $\Psi(s_1)$ is quasi-isomorphic to the torsion sheaf $\mathcal{O}_Z$, and $\Psi(s_i)$ is quasi-isomorphic to the torsion sheaf $i_*\mathcal{O}_{E_i}(-1)[1]$ for $i = 2, \ldots, n$. $\qquad\square$

Here (-1) denotes the Serre twist for $\mathcal{O}_{\mathbb{P}^1}$, and $[1]$ denotes the shift functor.

Since an equivalence induces an isometry on the level of K-groups, one has equal-

ities

(2.10) $\chi(i_*\mathcal{O}_{E_i}(-1)[1], i_*\mathcal{O}_{E_j}(-1)[1]) = \chi(s_i, s_j)$ for $i = 2, \ldots, n,\ j = 2, \ldots, n$.

This is yet another interpretation of the McKay pairing. It shows that the Euler characteristic between G-sheaves on $\mathbb{C}^2$ can be interpreted as an Euler characteristic between sheaves on the minimal resolution, Y, and vice-versa. In this sense, McKay's observation is subsumed into intersection theory on Y.

Let us elaborate a bit on the previous sentence. Y is a partial compactification of U. The boundary divisor $\mathcal{E}$ defines a bilinear form on the subgroup $K_{\mathcal{E}}(\mathrm{coh}\,Y) \subset K(\mathrm{coh}\,Y)$ generated by sheaves supported on $\mathcal{E}$. The Euler form $K(\mathrm{coh}\,Y) \times K_{\mathcal{E}}(\mathrm{coh}\,Y) \to \mathbb{Z}$ is a perfect pairing (it is possible to show this directly, but it also this follows from the BKR theorem and our discussion of G-sheaves on $\mathbb{C}^2$). We get a well-defined pairing $K(\mathrm{coh}\,Y) \times K(\mathrm{coh}\,Y) \to \mathbb{Z}$, which agrees with the McKay pairing on $K(\mathrm{Rep}\,G)$.

From an even higher vantage point: One starts with a local system on U. Then one associates to it an algebraic vector bundle on U, by means of a partial compactification of U. The compactification induces a bilinear pairing between vector bundles on U; therefore, it induces a bilinear pairing between local systems on U. This is the explanation of the McKay pairing. Equivalently, one can say that the McKay pairing on $\mathrm{Loc}\,U$, with respect to the natural representation V_{nat}, detects the combinatorial structure of the partial compactification of U.

CHAPTER III

The T_{pqr} Correspondence

The T_{pqr} correspondence is a generalization of the ADE correspondence to star-shaped graphs of type T_{pqr}, that is, graphs with three arms emanating from a central vertex, of lengths p, q, r. For example, here is a picture of T_{237}.
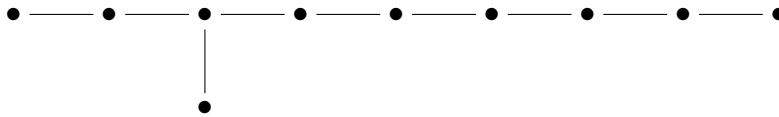
As in the ADE case, the word correspondence refers to the fact that the T_{pqr} graph can be recovered in several ways.

3.1 Analogue of McKay's observation

This section begins with a discussion of equivariant sheaves, derived categories of singularities, and resolution of singularities, just as in Chapter II. Then we formulate a conjecture which, if true, would be analogous to McKay's observation.

3.1.1 G-sheaves on curves

Let Z be a smooth projective curve over $\mathbb{C}$ of genus $g \geq 2$, with a finite subgroup of automorphisms $G \subseteq \operatorname{Aut}(Z)$. Let us assume the quotient Z/G is isomorphic to $\mathbb{P}^1$, and the quotient map $Z \to Z/G$ is ramified over three points, with ramification

indices (e_1, e_2, e_3).

The reduced preimages of the ramification points are G-invariant divisors D_1, D_2, D_3 on Z. The divisors D_1, D_2, D_3 are called *ground forms.*

Let $\mathrm{WDiv}^G(Z)$ be the group of G-invariant divisors modulo G-invariant rational functions. The ground forms generate $\mathrm{WDiv}^G(Z)$, and they satisfy the relations:

$$(3.1) \qquad\qquad e_1 D_1 = e_2 D_2 = e_3 D_3.$$

The common value is the class of a regular orbit D_{reg}. $\mathrm{WDiv}^G(Z)$ is isomorphic to the group of G-linearized line bundles, modulo G-invariant isomorphisms, i.e. we have an isomorphism

$$(3.2) \qquad\qquad \mathrm{WDiv}^G(Z) \simeq \mathrm{Pic}(G; Z).$$

Our reference for all of these facts concerning ground forms is [Dol99].

$\mathrm{Pic}(G; Z)$ is a rank 1 abelian group. One can show, see [Dol09], that the torsion part is isomorphic to $\mathbb{Z}/c_1\mathbb{Z} \oplus \mathbb{Z}/c_2\mathbb{Z}$, where

$$(3.3) \qquad\qquad c_1 = \gcd(e_1, e_2, e_3), c_2 = \gcd(e_1 e_2, e_2 e_3, e_3 e_1).$$

The Riemann-Hurwitz formula is

$$(3.4) \qquad\qquad K = -2D_{\mathrm{reg}} + \sum (e_i - 1)D_i$$

$$= D_{\mathrm{reg}} - \sum D_i.$$

If one prefers, the pair (Z, G) defines an orbifold curve.

Definition III.1. A complex orbifold curve is a smooth, separated, irreducible, DM stack, $\mathfrak{X}$, of dimension 1, and finite type over $\mathbb{C}$, with trivial generic stabilizer.

Following [Kre09], a compact orbifold curve $\mathfrak{X}$, with underlying coarse moduli space X and orders of ramification $e_1, \ldots, e_r$ is spherical, respectively Euclidean, respectively hyperbolic, when the quantity

$$2 - 2g - \sum_{i=1}^{r} \frac{e_i - 1}{e_i}$$

is positive, respectively zero, respectively negative, where g is the genus of X.

Let $\mathcal{O}(D) \in \mathrm{coh}^G Z$ be a G-linearized line bundle on Z. We may write D uniquely in the form

$$(3.5) \qquad D = cD_{\mathrm{reg}} + \sum_{i=1}^{r} c_i D_i, \text{ where } c \in \mathbb{Z}, \text{ and } 0 \le c_i < e_i.$$

Then, for any $n \in \mathbb{Z}$, there is an isomorphism

$$(3.6) \qquad H^0(Z, \mathcal{O}(nD))^G \simeq H^0(\mathbb{P}^1, \mathcal{O}(nc + \sum_{i=1}^{r} \lfloor n\frac{c_i}{e_i} \rfloor)),$$

where $\lfloor q \rfloor$ denotes the round-down integral part of a rational number $q \in \mathbb{Q}$. This follows from the equivalence between G-sheaves on Z and parabolic sheaves on $Z/G = \mathbb{P}^1$ (see [Bis97]).

Example III.2. Let $D \in \{D_1, D_2, D_3\} \subset \mathrm{Div}^G(Z)$ be one of the ground forms. Then the space $\mathrm{H}^0(Z, \mathcal{O}(D))^G$ is one-dimensional, because $\mathrm{H}^0(Z, \mathcal{O}(D))^G$ is isomorphic to $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(\lfloor \frac{1}{e} \rfloor))$. Similarly, we get

$$(3.7) \qquad \mathrm{Hom}(\mathcal{O}(jD_i), \mathcal{O}((kD_i))^G \simeq \mathbb{C}; \ \ 0 \le j < k \le e_i - 1$$

and

$$(3.8) \quad \mathrm{Hom}(\mathcal{O}(jD_i), \mathcal{O}(kD_\ell))^G = 0; \text{ for } i \ne \ell; j = 1, \ldots, e_i - 1; k = 1, \ldots, e_\ell - 1.$$

Recall the following fact from [BD08]:

Lemma III.3. *Let A be an additive category, L an object of A and $\Gamma = \mathrm{End}_A(L)$ its endomorphism ring. Then the functor $\mathrm{Hom}_A(L, -) : A \to \mod \Gamma$ induces an equivalence of categories*

$$\mathrm{Hom}_A(L, -) : \mathrm{add}(L) \to \mathrm{pro}(\Gamma),$$

where $\mathrm{pro}(\Gamma)$ is the category of finitely generated projective right Γ-modules, and $\mathrm{add}\, L$ is the full subcategory consisting of finite direct sums of direct summands of L.

Let L be the following G-linearized vector bundle on Z:

(3.9)
$$L = \mathcal{O} \oplus_{i=1}^{3} \bigoplus_{j=1}^{e_i - 1} \mathcal{O}(jD_i).$$

The above lemma implies that $\mathrm{add}\, L \subset \mathrm{coh}^G Z$ is equivalent to $\mathrm{End}(L) - \mathrm{mod}$. From equations (3.7) and (3.8), one sees that $\mathrm{End}(L)$ is isomorphic to the path algebra of the following quiver Q. Hence $\mathrm{End}(L) - \mathrm{mod}$ is equivalent to $\mathrm{Rep}\, Q$, the category of finite-dimensional representations of Q.

(3.10)
$$Q = \begin{array}{c} \mathcal{O}(D_1) \longrightarrow \cdots \longrightarrow \mathcal{O}((e_1 - 1)D_1) \\ \\ \mathcal{O} \longrightarrow \mathcal{O}(D_2) \longrightarrow \cdots \longrightarrow \mathcal{O}((e_2 - 1)D_2) \\ \\ \mathcal{O}(D_3) \longrightarrow \cdots \longrightarrow \mathcal{O}((e_3 - 1)D_3) \end{array}$$

In particular, we have a fully faithful functor $\varphi : \mathrm{Rep}\, Q \subset \mathrm{coh}^G Z$. We would like to suggest that this is the first part of the T_{pqr}-correspondence.

Remark III.4. φ is not essentially surjective because $\mathcal{O}(D_{\mathrm{reg}})$ is not contained in $\mathrm{add}\, L$ (see [GL87]).

3.1.2   Ground forms

Next, let $A = \oplus_{n \geq 0} \mathrm{H}^0(Z, \omega^n)^G$ be the canonical ring of invariants of the pair (Z, G). The surface $V = \mathrm{Spec}\, A$ is called a canonical singularity, or Fuchsian singu-

larity, and A is called a Fuchsian ring [Loo84].

All the localizations at maximal ideals are regular except for one: $A_{>0}$. It will always be referred to as the origin, or the singular point. As in the ADE case, the complement $U = \operatorname{Spec} A \backslash \{A_{>0}\}$ is called the *punctured quasi-cone.*

Definition III.5. The Poincaré ring is the total ring of invariants:

$$A^{\mathrm{ab}} := \oplus_{L \in \mathrm{Pic}(G;Z)} \mathrm{H}^0(Z, L)^G.$$

It is graded by the abelian group $\mathrm{Pic}(G; Z)$.

Proposition III.6. *The Poincaré ring has a uniform presentation:*

$$A^{\mathrm{ab}} = \mathbb{C}[x, y, z]/(x^{e_1} + y^{e_2} + z^{e_3}).$$

The functions x, y, and z are also called ground forms. □

Remark III.7. The ring $\mathbb{C}[x, y, z]/(x^{e_1} + y^{e_2} + z^{e_3})$ appears on page 237 of Poincaré's article on Fuchsian functions [Poi82]. The brilliant idea behind ground forms is that there is a relation between invariant functions and invariant orbits. For any G-linearized very ample line bundle L_0, one has a G-invariant embedding of Z in the projective space $\mathbb{P}(\mathrm{H}^0(L_0)) \simeq \operatorname{Proj} A_{L_0}$, where $A_{L_0} = \oplus_{n \geq 0} \mathrm{H}^0(Z, L_0{}^n)$.

$$Z \hookrightarrow \operatorname{Proj} A_{L_0}.$$

This implies that the ring of invariants $A_{L_0}^G$ is contained in A^{ab} as a subspace. Thus one makes the startling observation that all the invariant functions, for all the G-invariant embeddings of Z into a projective space, can, in principle, be written as polynomials in the ground forms x, y, z.

Remark III.8. The notation A^{ab} is used because the complement $U^{\mathrm{ab}} = \operatorname{Spec} A^{\mathrm{ab}} \backslash (A^{\mathrm{ab}})_{>0}$ is the maximal abelian, finite, unramified cover of U. It will not be crucial for us,

but there is an exact sequence (see [Dol09])

$$1 \to \mathbb{Z}/k\mathbb{Z} \to \operatorname{Gal}(U^{\mathrm{ab}}/U) \to \operatorname{Pic}(G;Z)_{\mathrm{tors}} \to 1.$$

Let grmod A^{ab} be the category of modules graded with respect to the abelian group $\operatorname{Pic}(G;Z)$. Let qgr A^{ab} be the quotient by the abelian subcategory of torsion modules. The homomorphisms in qgr A^{ab} are given by

(3.11) $$\operatorname{Hom}_{\mathrm{qgr}}(M,N) = \varinjlim \operatorname{Hom}_{\mathrm{grmod}}(M',N),$$

where the limit is taken over all submodules $M' \subset M$ such that the quotient M/M' is finite dimensional.

The following proposition is the analogue of Serre's theorem.

Proposition III.9. *(see [GL87]) Let $A^{\mathrm{ab}} = \oplus_{L \in \operatorname{Pic}(G;Z)} \operatorname{H}^0(Z,L)^G = \mathbb{C}[x,y,z]/(x^{e_1} + y^{e_2} + z^{e_3})$ be the Poincaré ring of (Z,G). Then there exists an equivalence of categories*

(3.12) $$\varphi : \mathrm{qgr}\, A^{\mathrm{ab}} \to \mathrm{coh}^G Z.$$

Moreover, for $L \in \operatorname{Pic}(G;Z)$, let $A^{\mathrm{ab}}(L)$ be the Serre twist of A^{ab}, defined by the formula $A^{\mathrm{ab}}(L)_N = A^{\mathrm{ab}}_{L \otimes N}$, where $N \in \operatorname{Pic}(G;Z)$. Then the equivalence φ can be chosen such that $\varphi(A(L))$ is isomorphic to L, simultaneously for all $L \in \operatorname{Pic}(G;Z)$.

$\square$

Similarly, the category qgr A is equivalent to $\mathrm{coh}^{\mathbb{C}^*} U$, and $\mathrm{coh}^{\mathbb{C}^*} U$ is equivalent to $\mathrm{coh}^G Z$. The latter equivalence may be conveniently expressed as an isomorphism of stacks:

(3.13) $$[[W/\mathbb{C}^*]/G] \simeq [[W/G]/\mathbb{C}^*],$$

where W is the punctured cone over some very ample G-linearized line bundle on Z, e.g. the canonical bundle, in the case when Z is not hyperelliptic.

The morphisms in $\mathrm{coh}^{\mathbb{C}^*} U$ are $\mathbb{C}^*$-equivariant; so, for instance, $\mathrm{coh}^{\mathbb{C}^*} U$ is not a full subcategory of $\mathrm{coh}\, U$. Thus, we are led to consider $T_+ \subset \mathrm{coh}\, U$, the subcategory whose are objects are $\mathbb{C}^*$-linearized sheaves, and whose morphisms are given by

$$(3.14) \qquad \mathrm{Hom}_{\mathrm{coh}\, T_+}(E, F) = \mathrm{Hom}_{\mathrm{coh}\, U}(E, F)_{\geq 0}.$$

Here, $\mathrm{Hom}_{\mathrm{coh}\, U}(E, F)_{\geq 0}$ is the positively graded submodule of the graded A-module $\mathrm{Hom}_{\mathrm{coh}\, U}(E, F)$.

Geometrically, T_+ is equivalent to the full subcategory $\pi_+^* \mathrm{coh}^G Z \subset \mathrm{coh}^G T$, where $\pi_+ : T \to Z$ is the tangent bundle of Z. Indeed, homorphisms between pullbacks are given by the following formula:

$$(3.15) \qquad \mathrm{Hom}_{\mathrm{coh}^G T}(\pi_+^* E, \pi_+^* F) = \oplus_{n \geq 0} \mathrm{Hom}_{\mathrm{coh}^G Z}(E, F(nK))^G.$$

Let us fix an equivalence $\mathrm{coh}^G Z \equiv \mathrm{qgr}\, A$. Let $\Omega_+ : \mathrm{qgr}\, A \to \mathrm{grmod}\, A$ be the functor given by

$$(3.16) \qquad \Omega_+(E) = \oplus_{n \geq 0} \mathrm{H}^0(Z, E(nK))^G.$$

Geometrically, Ω_+ is the same as pulling back to the tangent bundle of Z, and then taking global sections:

$$(3.17) \qquad \Omega_+(E) = \mathrm{H}^0(T, \pi_+^* E)^G.$$

Ω_- and T_- are defined similarly; Ω_- is the same as pulling back to the cotangent bundle and then taking global sections.

Now, Orlov [Orl09] shows that the derived functor $R\Omega_+$ exists and is fully faithful:

$$R\Omega_+ : D^b(\mathrm{qgr}\, A) \subset D^b(\mathrm{grmod}\, A).$$

Moreover, by examining the image of $R\Omega_+$, he obtains a fully faithful functor

$$(3.18) \qquad \varphi : D^b(\mathrm{qgr}\, A) \to D^b(\mathrm{grmod}\, A)/\mathrm{Perf}.$$

Here, Perf is the full triangulated subcategory generated by bounded complexes of locally free sheaves, and the quotient $q : D^b(\mathrm{grmod}\, A) \to D^b(\mathrm{grmod}\, A)/\mathrm{Perf}$ is taken in the sense of localization. The category $D^b(\mathrm{coh}\, V)/\mathrm{Perf}$ is defined for any variety V; it is a fundamental object of study.

Recall that we had a fully faithful functor $\mathrm{Rep}\, Q \subset \mathrm{coh}^G Z$. The composition of fully faithful functors is fully faithful. Thus, there is a fully faithful functor

$$(3.19) \qquad\qquad \mathrm{Rep}\, Q \subset D^b(\mathrm{grmod}\, A)/\mathrm{Perf}\,.$$

We would like to suggest this is the second part of the T_{pqr}-correspondence.

3.1.3 The K-trivial smooth surface at infinity

To physicists, $D^b(\mathrm{grmod}\, A)/\mathrm{Perf}$ is known as the category of B-branes. String theory predicts that the category of B-branes should not change under crepant resolution [Asp07, BD96, Rei02]. The variety V does not admit a crepant resolution, but there is a K-trivial, smooth surface, Y, which is birational to V.

Y is the G-Hilbert scheme of the cotangent bundle $\pi : T^* \to Z$, where the G-action on T^* is the canonical one, extending the action of G on Z.

The local coordinate ring in an affine neighborhood a point x on the zero-section $s \subset T^*$ is of the form $\mathbb{C}[x, y]$, where y is the coordinate along the fiber. Sometimes one writes $y = \partial_x$, but we avoid this because $\mathbb{C}[x, y]$ is a commutative ring; there are no relations between x and y.

Suppose $x \in s$ is a fixed point. Then the stabilizer subgroup $H = \mathrm{Stab}(x) < G$ is a cyclic group, say $H = \mathbb{Z}/m\mathbb{Z}$. Let ξ be a fixed primitive m-th root of unity.

Since we are on the cotangent bundle, as opposed to the tangent bundle, the action of G on the ring $\mathbb{C}[x, y]$ factors through an action of H given by the formula $(x, y) \mapsto (\xi x, \xi^{-1} y)$.

Thus the surface T^*/G has 3 singular points along the curve $Z/G \simeq \mathbb{P}^1$. These singular points are of type $A_{e_i-1}, i = 1, 2, 3$.

Since T^*/G has three singular points of type A_{e_i-1}, the reduced pullback on Y of the line $Z/G \subset T^*/G$ is a T_{e_1,e_2,e_3}-configuration of rational curves, $\mathcal{E} = \cup_{i,j} E_{ij} \subset Y$.

(3.20)

$$
\mathcal{E} = \quad E_{0,0}
\begin{array}{c}
E_{1,1} \text{ ------ } \cdots \text{ ------ } E_{1,e_1-1} \\
\\
E_{2,1} \text{ ------ } \cdots \text{ ------ } E_{2,e_2-1} \\
\\
E_{3,1} \text{ ------ } \cdots \text{ ------ } E_{3,e_3-1}
\end{array}
$$

Since G-acts symplectically on T^*, we are in the situation of the BKR theorem. Let $Y = \text{G-Hilb}\, T^*$, and let $\mathcal{Z}$ be the tautological family, with projections $T^* \xleftarrow{\; p \;} \mathcal{Z} \xrightarrow{\; q \;} Y$.

By the BKR theorem, the Fourier-Mukai transform $\Phi : D^b(\text{coh}\, Y) \to D^b(\text{coh}^G T^*)$ with kernel $\mathcal{O}_{\mathcal{Z}}$ is an equivalence of categories.

Since Y is K-trivial, e.g. by the BKR theorem, all of the irreducible components of the T_{e_1,e_2,e_3}-configuration $\mathcal{E}$ have self-intersection -2, including the central vertex.

Remark III.10. There is an alternate construction of the surface Y as an open subset of a rational surface. Unlike the cotangent bundle construction discussed above, the alternate construction relies on the fact that there are precisely three ramification indices.

3.1.4   Conjecture

The three main points of the chapter thus far can be summarized as follows: Starting from the punctured quasicone U, one encounters the T_{pqr}-diagram in three ways. First, one examines the category of $\mathbb{C}^*$-linearized coherent sheaves on U, and finds that there is a fully faithful functor $\text{Rep}\, Q \subset \text{coh}^{\mathbb{C}^*} U$, where Q is the

quiver in equation (3.10). Second, one considers the derived category of singularities, $D^b(\text{grmod } A)/\text{Perf}$, and finds a fully faithful functor $\text{Rep } Q \subset \text{coh}^{\mathbb{C}^*} U \subset D^b(\text{grmod } A)/\text{Perf}$. Third, one has the open embedding $U \subset Y$, such that the complement $\mathcal{E} = Y \backslash U$ is a T_{pqr}-configuration of rational curves.

Admittedly, one can imagine an even tighter analogy with the ADE case. However, the point we would like to make is that there is a very glaring omission: One expects to be able to recover the T_{pqr}-diagram from $\text{Loc } U$, the category of local systems on U.

Let Π be the fundamental group of U. By the Riemann-Hilbert correspondence, $\text{Rep } \Pi$ is equivalent to $\text{Loc } U$.

The group Π comes with a natural embedding $\Pi < \widetilde{\text{SU}(1,1)}$, where $\widetilde{\text{SU}(1,1)}$ is the universal cover of the topological group $\text{SU}(1,1)$ [BPR03]. Π is a cocompact, discrete subgroup of $\widetilde{\text{SU}(1,1)}$. Composing with the covering homomorphism $\widetilde{\text{SU}(1,1)} \to \text{SU}(1,1)$ yields a natural representation $\rho_{\text{nat}} : \Pi \to \text{SU}(1,1)$.

Conjecture III.11. *Let Π be a cocompact discrete subgroup of $\widetilde{\text{SU}(1,1)}$ of signature $(0; p, q, r)$. Let $\rho_{\text{nat}} : \Pi \to \text{SU}(1,1)$ be the natural representation. Then there exists a finite collection of complex representations $\rho_1, \ldots, \rho_n \in \text{Rep } \Pi$ such that the McKay pairing*

$$(3.21) \qquad (\rho_i, \rho_j) = 2 \dim_{\mathbb{C}} \text{Hom}_{\Pi}(\rho_i, \rho_j) - \dim_{\mathbb{C}} \text{Hom}_{\Pi}(\rho_j, \rho_i \otimes \rho_{\text{nat}})$$

on the set $\{\rho_1, \ldots, \rho_n\}$ is a Cartan matrix of type T_{pqr}.

One can of course make stronger conjectures by imposing additional conditions on the representations.

An irreducible representation of Π is said to be *absolutely irreducible* if it remains irreducible when restricted to any subgroup of finite index.

Conjecture III.12. *One may choose the representations appearing in Conjecture III.11 to be unitary and absolutely irreducible.*

3.2 The Narasimhan-Seshadri theorem

Recall that in the ADE correspondence, there is a procedure that associates an indecomposable CM A-module to an irreducible local system on U. The idea is to generalize that procedure to the Fuchsian case, then describe which CM A-modules arise from absolutely irreducible local systems. One of our main tools for doing this is the Narasimhan-Seshadri (NS) theorem.

3.2.1 Classical case

We begin with the statement of the original NS theorem.

Theorem III.13. *[NS64] Let Z be a smooth projective curve over $\mathbb{C}$. Let $\pi_1(Z)$ be the fundamental group of Z, acting on the hyperbolic plane $\mathbb{D}$ by means of the natural representation $\rho_{\mathrm{nat}} : \pi_1(Z) \to \mathrm{SU}(1,1)$. Then the map $\mathrm{Rep}\,\pi_1(Z) \to \mathrm{coh}\,Z$ given by $E \mapsto (\mathbb{D} \times_{\mathbb{C}} E)^{\pi_1(Z)}$ establishes a natural bijective correspondence between (isomorphism classes of) representations of $\pi_1(Z)$, and vector bundles on Z of degree zero. Under this correspondence, unitary representations correspond to semistable bundles, and irreducible unitary representations correspond to stable bundles.*

More generally, there is a natural correspondence between stable bundles of rank n and irreducible unitary representations of $\pi_1^{(n)}(Z)$, where $\pi_1^{(n)}(Z)$ is a discrete group, acting effectively and properly on a simply connected Riemann surface Y with quotient Z such that $Y \to Z$ is ramified over a single point $z \in Z$, with order of ramification n. $\qquad\square$

The degree zero condition is very interesting. Recall that in the category coh Z,

there is a non-split extension $\text{Ext}^1(\mathcal{O}(-K), \mathcal{O}) \simeq \text{H}^0(\mathcal{O})^\vee \simeq \mathbb{C}$ called the Euler sequence:

$$(3.22) \qquad\qquad 0 \to \mathcal{O} \to V_{\text{nat}} \to \mathcal{O}(-K) \to 0.$$

In [Ati57], it is shown that a vector bundle E on Z is of degree zero if and only if the Euler sequence splits after tensoring with E.[1]

Stable bundles generate the Grothendieck group of coh Z in finitely many steps. Let us explain what this means. First, all objects of the derived category of Z are quasi-isomorphic to finite length complexes of coherent sheaves. Second, since Z is smooth, all coherent sheaves have a finite length resolution by vector bundles. Third, it is a standard fact (see [LP97]) that every vector bundle admits a flag such that the indecomposable components of the associated graded bundle are semistable. Fourth, every semistable bundle admits a flag such that the indecomposable components of the associated graded bundle are stable. Putting all four pieces of information together, one sees that stable bundles generate the Grothendieck group of coh Z.

It is also known that every vector bundle admits a full flag, i.e. a filtration by line bundles; hence, the image of Pic Z under the natural inclusion Pic $Z \subset K(\text{coh } Z)$ is a spanning set for $K(\text{coh } Z)$.

Note: Line bundles on Z of degree zero correspond to one-dimensional irreducible unitary representations of $\pi_1(Z)$, i.e. characters of $\pi_1(Z)$.

3.2.2 Equivariant case

In this section we discuss the equivariant version of the Narasimhan-Seshadri theorem, following [Dol99, Dol09]. Another good reference for this material is [BPR03].

---

[1]There is a very interesting cohomological characterization of semistable bundles as well: Let E be a coherent sheaf on Z. Then E is a semi-stable vector bundle if and only if there is a sheaf $0 \neq F \in \text{coh } Z$ such that $\text{H}^0(E \otimes F) = \text{H}^1(E \otimes F) = 0$ (see [HP05]).

Let $\Gamma \subset \mathrm{SU}(1,1)$ be the orientation preserving subgroup of the reflection group of a hyperbolic triangle with angles $(\pi/p, \pi/q, \pi/r)$. Γ is an index 2 subgroup of the reflection group, for the product of reflections in all three sides does not preserve the orientation of the triangle. The triple (p, q, r) is called the signature of Γ. Sometimes one also writes $(0; p, q, r)$ to signify that the genus of the curve $\mathbb{D}/\Gamma \simeq \mathbb{P}^1$ is zero.

Let $\pi_1(Z)$ be a surface subgroup of Γ, equal to the fundamental group of a Riemann surface $Z := \overline{\mathbb{D}}/\pi_1(Z)$, where $\overline{\mathbb{D}} \simeq \mathbb{H} \cup \mathbb{Q} \cup \infty$ is the unit disk model of the compactification of the upper half plane obtained by adjoining $\mathbb{Q}$ along the real line, and adjoining a point at infinity.

Let $G = \Gamma/\pi_1(Z)$ be the quotient of Γ by $\pi_1(Z)$. G is a finite group acting faithfully on Z by automorphisms.

Let $g \geq 2$ be the genus of Z. Let m be a positive integer dividing $2g - 2$. Let mG be a central extension of G by $\mathbb{Z}/m\mathbb{Z}$ such that the subgroup $\mathbb{Z}/m\mathbb{Z}$ acts trivially on Z

$$(3.23) \qquad 1 \to \mathbb{Z}/m\mathbb{Z} \to mG \to G \to 1.$$

Let $m\Gamma \subset \mathrm{SU}(1,1)$ be the double extension of $\pi_1(Z)$ corresponding to the (unfaithful) action of mG on Z. As in [Dol09], $m\Gamma$ is the image of the class of mG under the inflation map $\mathrm{H}^2(G, \mathbb{Z}/m\mathbb{Z}) \to \mathrm{H}^2(\Gamma, \mathbb{Z}/m\mathbb{Z})$. We have an exact sequence

$$(3.24) \qquad 1 \to \mathbb{Z}/m\mathbb{Z} \to m\Gamma \to \Gamma \to 1.$$

Let Π be the preimage of Γ in the universal cover of $\mathrm{SU}(1,1)$. The group Π is an infinite cyclic extension of Γ.

$$(3.25) \qquad 1 \to \mathbb{Z} \to \Pi \to \Gamma \to 1.$$

This is the analogue of the extension $1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{SL}(2, \mathbb{C}) \to \mathrm{PSL}(2, \mathbb{C}) \to 1$. The group Π is analogous to a binary polyhedral group.

Π is also an infinite cyclic extension of $m\Gamma$:

$$(3.26) \qquad\qquad 1 \to m\mathbb{Z} \to \Pi \to m\Gamma \to 1.$$

Representations of Π which factor through some such $m\Gamma$ are said to be of level m (more precisely: 'level at most m', since we allow for the possibility of factoring through some smaller $n\Gamma$, $n < m$.)

Recall that absolutely irreducible representations of Π are representations which are a) irreducible and b) their restriction to any subgroup of finite index remains irreducible.

A G-linearized bundle E is said to be G-stable if every G-linearized subbundle $F \subset E$ has smaller slope, i.e. $\deg F / \operatorname{rank} F < \deg E / \operatorname{rank} E$. In general, G-stability is a weaker condition than stability, because in G-stability, one only considers G-linearized subbundles.

Proposition III.14. *(Equivariant NS, Part 1) The natural map* $\operatorname{Rep}\Gamma \to \operatorname{coh}^G Z$ *given by* $E \mapsto (\mathbb{D} \times_{\mathbb{C}} E)^{\pi_1(Z)}$ *establishes a bijection between (isomorphism classes of) representations of* Γ, *and G-linearized bundles on Z of degree zero. Under this correspondence, unitary representations correspond to semistable bundles, and irreducible, unitary representations correspond to G-stable bundles.*

By the classical NS theorem, unitary representation of Γ such that their restriction to $\pi_1(Z)$ remains irreducible define stable bundles on Z. This is the idea behind the following, stronger version of the equivariant NS theorem.

Proposition III.15. *(Equivariant NS, Part 2) There is a natural bijection between irreducible, unitary representations of $m\Gamma$ and mG-stable mG-linearized degree zero bundles on Z. Under this correspondence, unitary representations of $m\Gamma$ which are*

absolutely irreducible as representations of Π *correspond to* mG-*linearized degree zero bundles on* Z *which are stable in the usual sense.* $\square$

For instance, G-linearized line bundles of degree zero correspond to irreducible characters of Γ.

Let us now specialize to the case when Z is Klein's quartic curve, and $G = \mathrm{PSL}(2, \mathbb{F}_7)$. In this case, the group of G-invariant line bundles is the same as the group of $\mathrm{SL}(2, \mathbb{F}_7)$-linearized line bundles. We write $\mathrm{SL}(2, \mathbb{F}_7) = 2G$, because there is an extension $1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{SL}(2, \mathbb{F}_2) \to \mathrm{PSL}(2, \mathbb{F}_7) \to 1$.

Specializing to the $(2, 3, 7)$-case, one has the following proposition.

Proposition III.16. *Let* Z *be Klein's quartic curve, and* $G = \mathrm{PSL}(2, \mathbb{F}_7)$. *Let* Π *be the fundamental group of the punctured quasicone* U *canonically associated to* (Z, G). *Then there is a natural bijective correspondence between (isomorphism classes of) G-invariant, stable vector bundles on* Z *of degree zero, and absolutely irreducible, unitary representations of* Π. $\square$

To summarize: Starting with an absolutely irreducible, unitary local system on the punctured quasicone U, the equivariant Narasimhan-Seshadri theorem associates to it an (algebraic) vector bundle on U, linearized with respect to a potentially unfaithful $\mathbb{C}^*$-action on U, such that the corresponding bundle on $[U/\mathbb{C}^*]$ is stable in a very strong sense.

There are many questions one can ask at this point. For instance, the vector bundle on U comes with a canonical extension to Y, the smooth compactification of U at infinity. By Deligne's theory (see [Bry96]), such an extension corresponds to a boundary divisor on Y. The conjecture in [Dol09] asks whether there exist finitely many such representations, such that the Chern classes of the corresponding

vector bundles on Y are dual to the reduced, irreducible components of the boundary divisor $\mathcal{E} \subset Y$.

Another natural question goes as follows. Let $\underline{\mathrm{CM}}(A)$ be the stable category, whose objects are CM A-modules, and whose morphisms are given by $\mathrm{Hom}_{\underline{\mathrm{CM}}}(M, N) = \mathrm{Hom}_A(M, N)/\mathcal{P}$, where $\mathcal{P}$ is the subspace of morphisms factoring through a projective A-module.

We have just seen that the NS theorem associates a vector bundle on U to a unitary local system on U. Composing with the natural functor $\mathrm{CM}(A) \to \underline{\mathrm{CM}}(A)$, we get a map

$$(3.27) \qquad\qquad \varphi : \mathrm{Rep}_{\mathrm{unitary}} \Pi \to \underline{\mathrm{CM}}(A).$$

We want to describe the image of φ. Let us say a CM A-module is *absolutely irreducible* if it is of the form $\varphi(\rho)$, where ρ is an absolutely irreducible unitary representation.

Conjecture III.17. *There exists a finite collection of absolutely irreducible CM A-modules whose incidence graph with respect to the Euler characteristic pairing on $\underline{\mathrm{CM}}(A)$ is a star-shaped graph of type T_{pqr}.*

Remark III.18. The Euler pairing on $\underline{\mathrm{CM}}(A)$ is finite. In fact, $\underline{\mathrm{CM}}(A)$ is a Calabi-Yau category of dimension 1 (see [BIKR08]). It is not known whether the Euler form on $\underline{\mathrm{CM}}(A)$ restricted to the image of φ agrees with the McKay pairing on $(\mathrm{Rep}\,\Pi, \rho_{\mathrm{nat}})$.

3.3   The partial sum of twists of a G-invariant divisor

In this section, we will discuss a technique for producing CM A-modules which seem like they might be absolutely irreducible, i.e. they are candidates. Several of

our candidate CM A-modules will turn out to be absolutely irreducible, in Chapter IV.

The idea is to start with coherent sheaves on the K-trivial surface, $Y := \text{G-Hilb}\, T^*$. In analogy with the ADE case, we consider torsion sheaves of the form $\mathcal{O}_{E_i}$, where E_i is an irreducible component of the boundary curve $\mathcal{E}$. We may take their second syzygies on Y:

$$(3.28) \qquad 0 \to P_2 \to P_1 \to P_0 \to \mathcal{O}_{E_i} \to 0 \in \text{coh}\, Y.$$

In an affine open subset containing the arm of the boundary curve $\mathcal{E}$ to which E_i belongs, this minimal resolution restricts to a resolution of $\mathcal{O}_{E_i}$ which may not be minimal, but this is Ok.

Let $U = Y \backslash \mathcal{E}$ be the complement of the boundary curve. The P's restrict to non-trivial vector bundles on U.

We ask whether the vector bundle $P_2|_U$ admits a connection such that the sheaf of horizontal sections is a local system on U corresponding to an irreducible unitary representation of $\Pi = \pi_1(U)$.

Observe that $\Omega_+(\mathcal{O}(D))$ cannot be a torsion A-module, by equation (3.6) on page 15 (recall that Ω_+ is defined on page 19). Also note that torsion sheaves supported on the zero section of the tangent bundle are not of the form $\pi_+^* \mathcal{F}$ for any $\mathcal{F} \in \text{coh}\, Z$. On the other hand, $\Omega_-(\mathcal{O}(D))$ is a torsion A-module, by equation (3.6). Torsion sheaves supported on the zero section of the cotangent bundle are not of the form $\pi_-^* \mathcal{F}$ for any $\mathcal{F} \in \text{coh}\, Z$.

Such observations lead us to consider A-modules of the form $\Omega_+(\mathcal{O}(D))$. These are the modules we will be studying Chapter IV. Let us restart and precisely formulate this idea.

Let Z be a smooth projective curve and let D be an effective G-invariant divisor on Z of degree d. Let $C_{\geq d} \subset \mathbb{Z}^n$ be the positive cone given by

$$(3.29) \qquad C_{\geq d} = \mathbb{Z}_{\geq 0}\{\mathcal{O}(E) \in \mathrm{Pic}(G; Z)| \deg E - \deg D \geq 0\}.$$

For a fixed D, let $\mathcal{I}$ be the ideal of the Poincaré ring A^{ab} given by

$$(3.30) \qquad \mathcal{I} = \{f \in A^{\mathrm{ab}}| \deg(f) \in C_{\geq d}\},$$

where $\deg(f)$ refers to the $\mathrm{Pic}(G; Z)$-grading on A^{ab}.

$\mathcal{I}$ is indeed an ideal. Since A^{ab} is noetherian, $\mathcal{I}$ is finitely generated.

In the 237 case, A is equal to A^{ab}, so $\mathcal{I}$ is the same as pulling back to the tangent bundle of Z and then taking G-invariant global sections:

$$(3.31) \qquad \mathcal{I} = \Omega_+(D) = \mathrm{H}^0(T, \pi_+^* \mathcal{O}(D))^G \in \mathrm{grmod}\, A.$$

Returning to the general (p, q, r)-case, let $\mathrm{syz}^2(\mathcal{I})$ be the second syzygy of $\mathcal{I}$:

$$(3.32) \qquad 0 \to \mathrm{syz}^2(\mathcal{I}) \to F_1 \to F_0 \to \mathcal{I} \to 0.$$

Here P_1 and P_0 are free graded A^{ab}-modules, and this sequence is $\mathrm{Pic}(G; Z)$-graded.

Now composing with the fundamental sequence $0 \to \mathcal{I} \to A^{\mathrm{ab}} \to A^{\mathrm{ab}}/\mathcal{I} \to 0$, we obtain an exact sequence:

$$(3.33) \qquad 0 \to \mathrm{syz}^2(\mathcal{I}) \to F_1 \to F_0 \to A^{\mathrm{ab}} \to A^{\mathrm{ab}}/\mathcal{I} \to 0.$$

Let $j : U^{\mathrm{ab}} \to V^{\mathrm{ab}}$ be the inclusion of the punctured quasi-cone. Applying j^* we get an exact sequence of vector bundles on U:

$$(3.34) \qquad 0 \to j^* \mathrm{syz}^2(\mathcal{I}) \to j^* F_1 \to j^* F_0 \to \mathcal{O} \to 0.$$

We are asking whether $j^* \mathrm{syz}^2(\mathcal{I})$ admits a connection such that the sheaf of horizontal sections corresponds to an irreducible unitary representation of $[\Pi, \Pi] = \pi_1(U^{\mathrm{ab}})$.

Applying the exact functor $\pi : \mathrm{grmod}\, A \to \mathrm{qgr}\, A$ to equation (3.33), we get an exact sequence of G-linearized vector bundles on Z:

$$(3.35) \qquad 0 \to \pi\, \mathrm{syz}^2(\mathcal{I}) \to \pi F_1 \to \pi F_0 \to \pi A \to 0.$$

The simplest possible example is the natural action of the group $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{C}[x,y]$. In that case, $A = \mathbb{C}[x,y]^{\mathbb{Z}/2\mathbb{Z}}$ is isomorphic to the ring $\mathbb{C}[u,v,w]/(u^2 + v^2 + w^2)$, with grading $\deg(u,v,w) = (1,1,1)$. Let $\mathfrak{m}$ be the maximal ideal $A_{>0}$. One can find a presentation of $\mathfrak{m}$.

$$A_{>0} = \mathrm{coker} \begin{pmatrix} v & w & 0 & u \\ -u & 0 & w & v \\ 0 & -u & -v & w \end{pmatrix}.$$

One can also find a presentation of the kernel of the above matrix, either by hand or using a computer.

$$\mathrm{syz}^2 A_{>0} = \mathrm{coker} \begin{pmatrix} -w & v & -u & 0 \\ v & w & 0 & -u \\ -u & 0 & w & -v \\ 0 & u & v & w \end{pmatrix}.$$

The map $A^4 \to A$ given by the vector $(w, -v, u, 0)$ defines an exact sequence

$$(3.36) \qquad 0 \to A(-1) \to \mathrm{syz}^2 A_{>0} \to A_{>0} \to 0$$

Let $k = A/A_{>0}$ be the skyscraper sheaf of the singular point. Composing with the fundamental sequence $0 \to A_{>0} \to A \to k \to 0$ we get an exact sequence:

$$(3.37) \qquad 0 \to A(-1) \to \mathrm{syz}^2 A_{>0} \to A \to k \to 0.$$

Applying Serre's theorem, we have an exact sequence of vector bundles on $\mathbb{P}^1$:

$$(3.38) \qquad 0 \to \mathcal{O}(-2) \to \pi\, \mathrm{syz}^2 A_{>0} \to \mathcal{O} \to 0.$$

Tensoring with $\mathcal{O}(2)$ yields the Euler sequence:

$$(3.39) \qquad\qquad 0 \to \mathcal{O} \to \pi \operatorname{syz}^2 A_{>0}(2) \to \mathcal{O}(2) \to 0.$$

In short, the natural representation V_{nat} of $G = \mathbb{Z}/2\mathbb{Z}$ can be obtained as the second syzygy of $A_{>0}$, up to a twist.

We will carry out a similar procedure in the case $(p, q, r) = (2, 3, 7)$.

CHAPTER IV

The Case $(p, q, r) = (2, 3, 7)$

We now specialize to the case when the signature of the triangle group $\Gamma <$ SU$(1, 1)$ is $(0; 2, 3, 7)$.

4.1 Klein's quartic

For the pair (Z, G), one may take

$$(4.1) \qquad Z = \operatorname{Proj} \mathbb{C}[X, Y, Z]/(X^3Y + Y^3Z + Z^3X),$$

and $G = \operatorname{PSL}(2, \mathbb{F}_7)$. The curve Z is lovingly known as Klein's quartic. A great deal of things are known about Klein's quartic [Lev99], therefore it is a good testing ground.

One important property is $\operatorname{Pic}(G; Z) \simeq \mathbb{Z}$. This is actually true whenever the weights (p, q, r) are coprime, by equation (3.3). Moreover, in the 237 case, the canonical bundle ω generates $\operatorname{Pic}(G; Z)$. This is not always true, even when the weights are coprime. For example, in the case $(2, 3, 11)$, the class of ω is divisible by 5 (see [Dol99]).

Let $A = A^{\mathrm{ab}} = \mathbb{C}[X, Y, Z]/(X^3Y + Y^3Z + Z^3X)^G$ be the ring of invariants of the canonical ring of Z. We have a canonical isomorphism

$$(4.2) \qquad A = \oplus_{n \geq 0} \operatorname{H}^0(\mathcal{O}(nK))^G = \mathbb{C}[x, y, z]/(x^7 + y^3 + z^2).$$

This is a very natural $\mathbb{Z}$-grading on A. As the relation must be homogeneous, one sees that the weights of the variables are $\deg(x, y, z) = (6, 14, 21)$. Using the relations $e_1 D_1 = e_2 D_2 = e_3 D_3$ and the Riemann-Hurwitz formula, one can easily see there are relations $D_1 = 6K, D_2 = 14K, D_3 = 21K$. This is a consistency check. The ground forms x, y, z are generators of $H^0(D_1)^G, H^0(D_2)^G$, and $H^0(D_3)^G$, respectively.

Let $2G = \mathrm{SL}(2, \mathbb{F}_7)$ and $G = \mathrm{PSL}(2, \mathbb{F}_7)$. Let λ be the $2G$-linearized line bundle on Z which generates the G-invariant Picard group. Let $R = \mathbb{C}[X, Y, Z]/(X^3 Y + Y^3 Z + Z^3 X)$ be the canonical ring of Z, where X, Y, Z are generators of V_+, the three dimensional irrep representation of G, which is equal to the positive eigenspace of the seven-dimensional irrep V of $2G$.

Remark IV.1. With the grading $(12, 28, 42)$, qgr A is equivalent to the category of $\mathrm{SL}(2, \mathbb{F}_7)$-linearized bundles on Z. Since every $\mathrm{PSL}(2, \mathbb{F}_7)$-invariant sheaf admits an $\mathrm{SL}(2, \mathbb{F}_7)$-linearization, qgr A is equivalent to the category of $\mathrm{PSL}(2, \mathbb{F}_7)$-invariant sheaves on Z. However, for consistency, we will always stick with the grading $(6, 14, 21)$.

Let us familiarize ourselves with the notation by performing some elementary, yet interesting calculations with the tensor product structure of the group $K(\mathrm{coh}^G Z)$. By simple arithmetic we get

$$(4.3) \quad [6K]^3[14K] + [14K]^3[-21K] + [-21K]^3[6K] = [32K] + [21K] + [-57K].$$

Applying the group isomorphism $\varphi : K(\mathrm{coh}^G Z) \simeq \mathbb{Z} \oplus \mathrm{Pic}(G; Z)$, we obtain

$$(4.4) \quad \varphi([32K] + [21K] + [-57K]) = (3[\mathcal{O}], [(32 + 21 - 57)K]) = (3[\mathcal{O}], [-4K]).$$

Since $[6K][14K][-21K] = [(6 + 14 - 21)K] = [-K]$, we have the relation $[-4K] = ([6K][14K][-21K])^4$.

4.2 Computations with Macaulay2

The computer algebra program Macaulay2 can be used to compute syzygies of finitely generated, graded A-modules. This is the new part of the thesis, so we carefully explain the procedure again.

$A_{\geq d}$ is by definition the ideal generated by all elements $f \in A$ such that $\deg(f) \geq d$.

For any $n \in \mathbb{Z}$, the graded A-module $A(n)$ is defined by $A(n)_i = A_{n+i}$. For example, $A(-6)$ is isomorphic to the ideal $(x) \subset A$.

Observe that $A_{>d}$ is isomorphic to $A_{>d+1}$, as a graded A-module, for $d \in \{6, 12, 18, 24, 30, 36, 14, 28, 21, 42\}$; i.e. A_{d+1} is zero for these d's.

The second syzygy of $A_{\geq d}$ fits into an exact sequence:

$$(4.5) \qquad 0 \to \operatorname{syz}^2 A_{\geq d} \to F_1 \to F_0 \to A_{\geq d} \to 0.$$

Here the F's are free graded A-modules of the form

$$(4.6) \qquad F_1 = \oplus_i A(-b_{1,i}), F_0 = \oplus_j A(-b_{0,j}).$$

The b's are strictly positive integers, because the generators of the ideal $A_{\geq d}$ are in positive degrees, and the resolution is assumed to be minimal.

Combining with the fundamental sequence $0 \to A_{\geq d} \to A \to A/A_{\geq d} \to 0$, we get an exact sequence:

$$(4.7) \qquad 0 \to \operatorname{syz}^2 A_{\geq d} \to F_1 \to F_0 \to A \to A/A_{\geq d} \to 0.$$

Applying the exact functor $\pi : \operatorname{grmod} A \to \operatorname{qgr} A$ kills the torsion module; whence, we have an exact sequence:

$$(4.8) \qquad 0 \to \pi \operatorname{syz}^2 A_{\geq d} \to \pi F_1 \to \pi F_0 \to \pi A \to 0.$$

Let $\varphi : \operatorname{qgr} A \to \operatorname{coh}^G Z$ be the equivalence of categories such that $\varphi(A(n)) = \mathcal{O}(nK)$.

Applying φ, we arrive at an exact sequence of G-bundles on Z:

$$(4.9) \qquad 0 \to \varphi\pi \operatorname{syz}^2 A_{\geq d} \to \varphi\pi F_1 \to \varphi\pi F_0 \to \varphi\pi A \to 0.$$

In particular, on the level of the K-group,

Proposition IV.2. *The class of $\varphi\pi \operatorname{syz}^2 A_{\geq d}$ in $K(\operatorname{coh}^G Z)$ is given by*

$$(4.10) \qquad [\varphi\pi \operatorname{syz}^2 A_{\geq d}] = \sum_i [\mathcal{O}(-b_{(1,i)}K)] - \sum_j [\mathcal{O}(-b_{(0,j)}K)] + [\mathcal{O}],$$

where all the b's are positive.

Remark IV.3. The Grothendieck group $K(\operatorname{coh}^G Z)$ does not capture all the information regarding stability, i.e. there can two bundles with the same class in the K-group one of which is stable and the one not. We will address this point in Section 4.4.

The following very simple program finds the generators of A up to degree 84, then truncates that to get generators of the ideals $A_{[d,84]}$ and $A_{[d+1,84]}$. Here $A_{[d,84]}$ is the ideal generated by all elements $f \in A$ such that $d \leq \deg f \leq 84$.

Program: geqIdeals

start

A= ZZ/5857[x,y,z,Degrees=>{6,14,21}]/(x^7+y^3+z^2)

d = 7

I = mingens ideal for i from d to 84 list basis(i,A)

– going to find 0 -> M -> F1 -> F0 -> I -> 0

F0 = (degrees I)_1

output: (12, 14, 21)

syz I;

F1 = degrees source syz I

output: (26, 33, 35, 42)

M = presentation ker syz I

output: $M = \text{coker} \begin{pmatrix} z & y^2 & x^5 & 0 \\ y & -z & 0 & x^5 \\ x^2 & 0 & -z & -y^2 \\ 0 & -x^2 & y & -z \end{pmatrix}$

RankM = #F1 - #F0 +1

output: RankM = 4 - 3 +1 = 2.

The results of running program geqIdeals for $d \in \{7, 13, 19, 25, 31, 37, 15, 29, 22, 43\}$ are displayed below.

The ideals are:

d	$A_{[d,84]}$
7	(x^2, y, z)
13	(y, x^3, z)
19	(xy, z, x^4, y^2)
25	(x^2y, xz, y^2, x^5, yz)
31	$(x^3y, x^2z, xy^2, yz, x^6, z^2)$
37	$(x^4y, x^3z, x^2y^2, xyz, z^2, y^3, y^2z)$
15	(x^3, xy, z, y^2)
29	$(x^5, x^3y, x^2z, xy^2, yz, z^2)$
22	(x^4, x^2y, xz, y^2, yz)
43	$(x^5y, x^4z, x^3y^2, x^2yz, xz^2, xy^3, y^2z, yz^2)$

The Betti numbers are:

d	$(F0, F1)$
7	$((12, 14, 21), (26, 33, 35, 42))$
13	$((14, 18, 21), (32, 35, 39, 42))$
19	$((20, 21, 24, 28), (34, 38, 41, 42, 45, 49))$
25	$((26, 27, 28, 30, 35), (40, 41, 44, 47, 48, 49, 51, 56))$
31	$((32, 33, 34, 35, 36, 42), (46, 47, 48, 50, 53, 54, 55, 56, 57, 63))$
37	$((38, 39, 40, 41, 42, 42, 49), (52, 53, 54, 55, 56, 59, 60, 61, 62, 63, 63, 70))$
15	$((18, 20, 21, 28), (32, 34, 39, 41, 42, 49))$
29	$((30, 32, 33, 34, 35, 42), (44, 46, 47, 48, 51, 53, 54, 55, 56, 63))$
22	$((24, 26, 27, 28, 35), (38, 40, 41, 45, 47, 48, 49, 56))$
43	$((44, 45, 46, 47, 48, 48, 49, 56), (58, 59, 60, 61, 62, 62, 65, 66, 67, 68, 69, 69, 70, 77))$

The matrices are:

$$\operatorname{syz}^2 A_{[7,84]} = \operatorname{coker} \begin{pmatrix} z & y^2 & x^5 & 0 \\ y & -z & 0 & x^5 \\ x^2 & 0 & -z & -y^2 \\ 0 & -x^2 & y & -z \end{pmatrix}$$

$$\operatorname{syz}^2 A_{[13,84]} = \operatorname{coker} \begin{pmatrix} z & x^4 & y^2 & 0 \\ x^3 & -z & 0 & y^2 \\ y & 0 & -z & -x^4 \\ 0 & -y & x^3 & -z \end{pmatrix}$$

$$\operatorname{syz}^2 A_{[19,84]} = \operatorname{coker} \begin{pmatrix} z & 0 & y^2 & 0 & x^3y & x^6 \\ 0 & z & x^4 & 0 & -y^2 & -x^3y \\ y & x^3 & -z & 0 & 0 & 0 \\ 0 & 0 & xy & z & x^4 & -y^2 \\ 0 & -y & 0 & x^3 & -z & 0 \\ x & 0 & 0 & -y & 0 & -z \end{pmatrix}$$

$$\operatorname{syz}^2 A_{[25,84]} = \operatorname{coker} \begin{pmatrix} z & 0 & 0 & y^2 & 0 & x^5 & x^3y & 0 \\ 0 & z & x^4 & 0 & y^2 & 0 & 0 & 0 \\ 0 & x^3 & -z & 0 & 0 & 0 & -y^2 & 0 \\ y & 0 & -x^3 & -z & 0 & 0 & 0 & x^5 \\ 0 & y & 0 & 0 & -z & 0 & x^4 & 0 \\ x^2 & 0 & 0 & 0 & xy & -z & 0 & -y^2 \\ 0 & 0 & y & 0 & -x^3 & 0 & -z & 0 \\ 0 & -x & 0 & x^2 & 0 & -y & 0 & z \end{pmatrix}$$

$$\mathrm{syz}^2\, A_{[31,84]} = \mathrm{coker}
\begin{pmatrix}
z & 0 & 0 & 0 & y^2 & 0 & x^5 & 0 & x^3y & 0 \\
0 & z & 0 & x^4 & 0 & y^2 & 0 & 0 & 0 & 0 \\
0 & 0 & z & 0 & x^2y & 0 & -y^2 & 0 & x^5 & 0 \\
0 & x^3 & 0 & -z & 0 & 0 & 0 & 0 & -y^2 & xy^2 \\
y & 0 & 0 & -x^3 & -z & 0 & 0 & x^4 & 0 & 0 \\
0 & y & 0 & 0 & 0 & -z & 0 & 0 & x^4 & -x^5 \\
x^2 & 0 & -y & 0 & 0 & 0 & -z & 0 & 0 & 0 \\
0 & -x^2 & 0 & 0 & x^3 & 0 & -xy & z & 0 & -y^2 \\
0 & 0 & x^2 & y & 0 & 0 & 0 & -xy & -z & 0 \\
0 & 0 & -x & 0 & 0 & -x^2 & 0 & y & 0 & z
\end{pmatrix}$$

$$\mathrm{syz}^2\, A_{[37,84]} = \mathrm{coker}
\begin{pmatrix}
z & 0 & 0 & 0 & 0 & y^2 & 0 & x^5 & 0 & x^3y & 0 & 0 \\
0 & z & 0 & 0 & x^4 & 0 & y^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & z & 0 & 0 & x^2y & 0 & -y^2 & 0 & x^5 & 0 & 0 \\
x^3 & 0 & 0 & z & 0 & 0 & x^2y & 0 & y^2 & 0 & 0 & x^6 \\
0 & x^3 & 0 & 0 & -z & 0 & 0 & 0 & 0 & 0 & y^2 & 0 \\
y & 0 & 0 & 0 & -x^3 & -z & 0 & 0 & -x^4 & 0 & 0 & x^3y \\
0 & y & 0 & 0 & 0 & 0 & -z & 0 & 0 & 0 & -x^4 & 0 \\
x^2 & 0 & -y & 0 & 0 & 0 & 0 & -z & 0 & 0 & 0 & 0 \\
0 & -x^2 & 0 & y & 0 & x^3 & 0 & 0 & -z & 0 & 0 & 0 \\
0 & 0 & x^2 & 0 & y & 0 & 0 & 0 & xy & -z & 0 & -y^2 \\
0 & 0 & 0 & 0 & y & 0 & -x^3 & 0 & 0 & 0 & z & 0 \\
0 & 0 & 0 & x & 0 & 0 & 0 & x^2 & 0 & y & y & -z
\end{pmatrix}$$

$$\operatorname{syz}^2 A_{[15,84]} = \operatorname{coker} \begin{pmatrix} z & 0 & y^2 & x^5 & 0 & x^4y \\ 0 & z & x^2y & -y^2 & 0 & x^6 \\ y & 0 & -z & 0 & x^4 & 0 \\ x^2 & -y & 0 & -z & 0 & 0 \\ 0 & 0 & -x^3 & xy & -z & y^2 \\ 0 & x & 0 & 0 & -y & -z \end{pmatrix}$$

$$\operatorname{syz}^2 A_{[29,84]} = \operatorname{coker} \begin{pmatrix} z & 0 & 0 & 0 & y^2 & x^5 & 0 & x^3y & 0 & 0 \\ 0 & z & 0 & 0 & x^2y & -y^2 & 0 & x^5 & 0 & 0 \\ x^3 & 0 & z & 0 & 0 & 0 & y^2 & 0 & x^5 & 0 \\ 0 & 0 & 0 & z & x^4 & -x^2y & 0 & -y^2 & 0 & 0 \\ y & 0 & 0 & x^3 & -z & 0 & 0 & 0 & 0 & 0 \\ x^2 & -y & 0 & 0 & 0 & -z & 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 & x^3 & 0 & -z & 0 & 0 & x^5 \\ 0 & x^2 & 0 & -y & 0 & 0 & 0 & -z & 0 & 0 \\ 0 & 0 & x^2 & 0 & 0 & x^3 & 0 & xy & -z & -y^2 \\ 0 & 0 & 0 & -x & 0 & 0 & -x^2 & 0 & y & -z \end{pmatrix}$$

$$\operatorname{syz}^2 A_{[22,84]} = \operatorname{coker} \begin{pmatrix} z & 0 & x^4 & y^2 & 0 & 0 & 0 & 0 \\ 0 & z & 0 & x^2y & y^2 & 0 & x^5 & 0 \\ x^3 & 0 & -z & 0 & 0 & y^2 & 0 & 0 \\ y & 0 & 0 & -z & 0 & -x^4 & 0 & 0 \\ -x^2 & y & 0 & 0 & -z & 0 & 0 & x^5 \\ 0 & 0 & -y & x^3 & 0 & -z & 0 & 0 \\ 0 & x^2 & 0 & 0 & 0 & xy & -z & -y^2 \\ 0 & 0 & -x & 0 & -x^2 & 0 & y & -z \end{pmatrix}$$

$$\mathrm{syz}^2\, A_{[43,84]} =$$

$$\mathrm{coker}\begin{pmatrix}
z & 0 & 0 & 0 & 0 & 0 & y^2 & 0 & x^5 & 0 & 0 & x^3y & 0 & 0 \\
0 & z & 0 & 0 & x^4 & 0 & 0 & y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & z & 0 & 0 & 0 & x^2y & 0 & -y^2 & 0 & 0 & x^5 & 0 & 0 \\
x^3 & 0 & 0 & z & 0 & 0 & 0 & x^2y & 0 & y^2 & 0 & 0 & x^5 & 0 \\
0 & x^3 & 0 & 0 & -z & 0 & 0 & 0 & 0 & 0 & y^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & z & x^4 & 0 & -x^2y & 0 & 0 & -y^2 & 0 & 0 \\
y & 0 & 0 & 0 & 0 & x^3 & -z & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & y & 0 & 0 & 0 & 0 & 0 & -z & 0 & 0 & -x^4 & 0 & 0 & 0 \\
x^2 & 0 & -y & 0 & 0 & 0 & 0 & 0 & -z & 0 & 0 & 0 & 0 & 0 \\
0 & -x^2 & 0 & y & 0 & 0 & x^3 & 0 & 0 & -z & 0 & 0 & 0 & x^5 \\
0 & 0 & 0 & 0 & -y & 0 & 0 & x^3 & 0 & 0 & -z & 0 & 0 & 0 \\
0 & 0 & x^2 & 0 & 0 & -y & 0 & 0 & 0 & 0 & 0 & -z & 0 & 0 \\
0 & 0 & 0 & x^2 & 0 & 0 & 0 & 0 & x^3 & 0 & xy & xy & -z & -y^2 \\
0 & 0 & 0 & 0 & -x & -x & 0 & 0 & 0 & -x^2 & 0 & 0 & y & -z
\end{pmatrix}$$

The ranks of these vector bundles are:

$$d \quad r(d) = \operatorname{rank} \operatorname{syz}^2 A_{[d,84]} = \#F0 - \#F1 + 1$$

d	$r(d)$
7	2
13	2
19	3
25	4
31	5
37	6
15	3
29	5
22	4
43	7

We display the ranks in a Dynkin diagram according to the following ordering:

$$r(15) \text{---} r(29) \text{---} r(43) \text{---} r(37) \text{---} r(31) \text{---} r(25) \text{---} r(19) \text{---} r(13) \text{---} r(7)$$

$$r(22)$$

(where $r(22)$ connects to $r(43)$)

Ranks:

$$3 \text{---} 5 \text{---} 7 \text{---} 6 \text{---} 5 \text{---} 4 \text{---} 3 \text{---} 2 \text{---} 2$$

$$4$$

(where 4 connects to 7)

We also run the program geqIdeals for the list $\{6, 12, 18, 24, 30, 36, 14, 28, 21, 42\}$.

This time we only report the ideals and the ranks:

d	$A_{[d,84]}$	rank $\mathrm{syz}^2 A_{[d,84]}$
6	(x, y, z)	2
12	(x^2, y, z)	2
18	(x^3, xy, z, y^2)	3
24	(x^4, x^2y, xz, y^2, yz)	4
30	$(x^5, x^3y, x^2z, xy^2, yz, z^2)$	5
36	$(x^6, x^4y, x^3z, x^2y^2, xyz, z^2, y^2z)$	6
14	(y, x^3, z)	2
28	$(y^2, x^5, x^3y, x^2z, yz)$	4
21	(z, x^4, x^2y, y^2)	3
42	$(z^2, y^3, x^5y, x^4z, x^3y^2, x^2yz, y^2z)$	6

$$2 \rule[0.5ex]{1.5em}{0.4pt} 4 \rule[0.5ex]{1.5em}{0.4pt} 6 \rule[0.5ex]{1.5em}{0.4pt} 6 \rule[0.5ex]{1.5em}{0.4pt} 5 \rule[0.5ex]{1.5em}{0.4pt} 4 \rule[0.5ex]{1.5em}{0.4pt} 3 \rule[0.5ex]{1.5em}{0.4pt} 2 \rule[0.5ex]{1.5em}{0.4pt} 2$$

$$\underset{3}{\overset{}{|}}$$
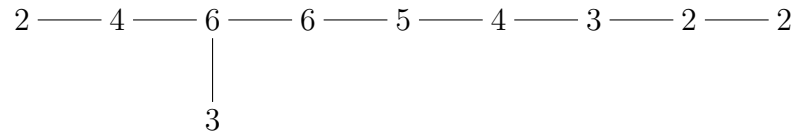
4.3 Data analysis

There is a lot of data to comment on! Therefore let us begin with the rank two bundles $\mathrm{syz}^2 A_{[7,84]}$ and $\mathrm{syz}^2 A_{[13,84]}$.

Proposition IV.4. *The following matrix defines a degree zero G-invariant stable bundle E on Klein's quartic curve, fitting into an exact sequence*

$$(4.11) \qquad\qquad 0 \to \mathcal{O}(-5\lambda) \to E \to \mathcal{O}(5\lambda) \to 0,$$

where λ is the unique G-invariant line bundle class satisfying $\lambda^2 \simeq \omega$.

$$(4.12) \qquad \mathrm{syz}^2 A_{[7,84]} = \mathrm{coker} \begin{pmatrix} z & y^2 & x^5 & 0 \\ y & -z & 0 & x^5 \\ x^2 & 0 & -z & -y^2 \\ 0 & -x^2 & y & -z \end{pmatrix}.$$

Proof. This matrix defines a homogeneous map $A(-21) \oplus A(-28) \oplus A(-30) \oplus A(-37) \to A(0) \oplus A(-7) \oplus A(-9) \oplus A(-16)$. Let $s_1, \ldots, s_4$ be the canonical generators of the cokernel of this map. The columns of the matrix are homogeneous relations among $s_1, \ldots, s_4$. To aid the reader in verifying these statements, we write the degrees in a matrix all by themselves:

$$(4.13) \qquad\qquad \begin{pmatrix} 21 & 28 & 30 & 0 \\ 14 & 21 & 0 & 30 \\ 12 & 0 & 21 & 28 \\ 0 & 12 & 14 & 21 \end{pmatrix}$$

The choice of the grading is not unique. We now shift by an overall factor, to obtain $(6, 13, 15, 22)$ for the target and $(27, 34, 36, 43)$ for the source:

$$(4.14) \quad A(-27) \oplus A(-34) \oplus A(-36) \oplus A(-43) \to A(-6) \oplus A(-13) \oplus A(-15) \oplus A(-22).$$

Increasing the degrees of the module by 6 corresponds to tensoring with $\mathcal{O}(-6K)$ on the curve Z.

The map $\varphi : \mathrm{syz}^2 A_{[7,84]} \to A(-1)$ given by

$$(4.15) \qquad (s_1, s_2, s_3, s_4) \mapsto (0, -x^2, y, z)$$

is well-defined, because the relations go to zero. It is also homogeneous (e.g. s_2 and x^2 are both in degree 13). Note: we are regarding $A(-1)$ as an A-module, not a ring.

The image of φ is the submodule $(-x^2, y, z) \subset A(-1)$. The cokernel of φ is the torsion module $T := \mathbb{C}1 \oplus \mathbb{C}x$, with degrees $1, 7$. The kernel of φ is the submodule of $\mathrm{syz}^2 A_{[7,84]}$ generated by s_1; it is a free module isomorphic to $A(-6)$. Thus we have an exact sequence

$$(4.16) \qquad 0 \to A(-6) \to \mathrm{syz}^2 A_{[7,84]} \to A(-1) \to T \to 0.$$

Applying $\pi : \mathrm{grmod}\, A \to \mathrm{qgr}\, A$ kills the torsion module T.

$$(4.17) \qquad 0 \to \mathcal{O}(-6K) \to \pi\, \mathrm{syz}^2 A_{[7,84]} \to \mathcal{O}(-K) \to 0.$$

One of the special properties of $(2,3,7)$ is that there exists a unique, G-invariant (but not G-linearized) theta characteristic, i.e. a line bundle class λ such that $\lambda^2 \simeq \omega$.

In terms of λ, the sequence reads

$$(4.18) \qquad 0 \to \mathcal{O}(-12\lambda) \to \pi\, \mathrm{syz}^2 A_{[7,84]} \to \mathcal{O}(-2\lambda) \to 0.$$

Tensoring with 7λ, we get an exact sequence

$$(4.19) \qquad 0 \to \mathcal{O}(-5\lambda) \to \pi\, \mathrm{syz}^2 A_{[7,84]}(7\lambda) \to \mathcal{O}(5\lambda) \to 0.$$

The degree of the central term $E := \pi\, \mathrm{syz}^2 A_{[7,84]}(7\lambda)$ is zero.

As for E being a stable bundle, recall that a bundle is said to be stable if $\deg E'/\operatorname{rank} E' < \deg E/\operatorname{rank} E$ for every subbundle $E' \subset E$.

Suppose E is not stable. Then there is a maximal destabilizing line bundle, which is unique [Ses82] Prop 2, hence it must be G-invariant, so it is of the form λ^a, for some $a \geq 0$. Applying $\operatorname{Hom}(\lambda^a, -)^G$, one has a long exact sequence in G-invariant cohomology:

$$(4.20) \quad 0 \to \operatorname{H}^0(\lambda^a, \lambda^{-5})^G \to \operatorname{H}^0(\lambda^a, E)^G \to \operatorname{H}^0(\lambda^a, \lambda^5)^G \to \operatorname{H}^1(\lambda^a, \lambda^{-5})^G \to \ldots$$

The first term $\operatorname{H}^0(\lambda^a, \lambda^{-5})$ is zero because the degree of λ^{-5-a} is less than zero. So if there was a destabilizing subbundle, then $\operatorname{H}^0(\lambda^a, \lambda^5)^G$ would be nonzero for some $a \geq 0$.

The only a's to check are $0, \ldots, 5$. For a even, the space $\operatorname{H}^0(\lambda^{5-a})^G$ is empty, because λ (hence λ^{odd}) is $\mathrm{SL}(3, \mathbb{F}_7)$-linearized but not $\mathrm{PSL}(2, \mathbb{F}_7)$-linearized, i.e. the line bundle λ is not represented by a G-invariant divisor.

For $a = 1, 3$, we get $\operatorname{H}^0(K)^G = A_1 = 0$ and $\operatorname{H}^0(2K)^G = A_2 = 0$, respectively, since $\lambda^2 = K$.

As for $a = 5$, we have $\dim_{\mathbb{C}} \operatorname{H}^1(\lambda^5, \lambda^{-5})^G = \dim_{\mathbb{C}} \operatorname{H}^0(6K)^G$ by Serre duality. Since $\operatorname{H}^0(6K)^G = A_6$ is non-zero (spanned by x), the extension $0 \to \lambda^{-5} \to E \to \lambda^5 \to 0$ does not split. This implies that the map $\varphi : \operatorname{H}^0(\lambda^5, \lambda^5)^G \to \operatorname{H}^1(\lambda^5, \lambda^{-5})^G$ from the long exact sequence is nonzero, because, by definition, it is cupping with the class of the extension. Thus $\operatorname{H}^0(\lambda^{-5}, E)^G$ is zero.

We have arrived at a contradiction; therefore, a destabilizing subbundle cannot exist.

One may also consult [Dol99], where it is shown that the bundle $E_{(-5,5)}$ is stable using different techniques. $\qquad\square$

Let us further study the nonsplit G-invariant sequence

$$(4.21) \qquad 0 \to \mathcal{O} \to E \to \mathcal{O}(5K) \to 0.$$

Tensoring with $2K$ we get

$$(4.22) \qquad 0 \to \mathcal{O}(2K) \to E(2K) \to \mathcal{O}(7K) \to 0.$$

Now let us pull this sequence back to the tangent bundle $\pi_+ : T \to Z$. Equivalently, we pull it back to the subcategory $T_+ \subset \operatorname{coh} U$. Since π_+^* is exact, we get an exact sequence:

$$(4.23) \qquad 0 \to \pi_+^* \mathcal{O}(2K) \to \pi_+^* E(2K) \to \pi_+^* \mathcal{O}(7K) \to 0.$$

Now we take the long exact sequence in G-invariant cohomology:

$$(4.24) \qquad 0 \to \mathrm{H}^0(\pi_+^* \mathcal{O}(2K))^G \to \mathrm{H}^0(\pi_+^* E(2K))^G \to \mathrm{H}^0(\pi_+^* \mathcal{O}(7K))^G \to$$

$$\mathrm{H}^1(\pi_+^* \mathcal{O}(2K))^G \to \mathrm{H}^1(\pi_+^* E(2K))^G \to \mathrm{H}^1(\pi_+^* \mathcal{O}(7K))^G \to 0.$$

To compute cohomology on T, recall that we use the formula:

$$(4.25) \qquad \mathrm{H}^k(\pi_+^* \mathcal{F}) = \oplus_{n \geq 0} \mathrm{H}^k(\mathcal{F}(nK))^G.$$

Furthermore, recall that a version of Serre duality holds in the category $\operatorname{coh}^G Z$:

$$(4.26) \qquad \mathrm{H}^1(\mathcal{F})^G \simeq (\mathrm{H}^0(\mathcal{F}(K))^\vee)^G \simeq (\mathrm{H}^0(\mathcal{F}(K))^G)^\vee.$$

Indeed, the first isomorphism follows from Serre duality in $\operatorname{coh} Z$, and the second isomorphism follows from the fact that a finite dimensional representation of G is trivial if and only if its dual representation is trivial.

The term $\mathrm{H}^0(\pi_+^* \mathcal{O}(2K))^G$ is canonically isomorphic to $A_{>0}(2)$. The term $\mathrm{H}^0(\pi_+^* \mathcal{O}(7K))^G$ is canonically isomorphic to $A_{\geq 7}(7)$. The term $\mathrm{H}^1(\pi_+^* \mathcal{O}(2K)))^G$ is zero, by Serre duality. Hence we get a short exact sequence in $\operatorname{grmod} A$:

$$(4.27) \qquad 0 \to A_{>0}(2) \to \mathrm{H}^0(\pi_+^* E(2K))^G \to A_{\geq 7}(7) \to 0.$$

Recall that one has the Euler sequence:

$$(4.28) \qquad 0 \to \mathcal{O} \to V_{\mathrm{nat}} \to \mathcal{O}(-K) \to 0,$$

corresponding to the generator of $\mathrm{H}^1(K)^G \simeq (\mathrm{H}^0(\mathcal{O})^G)^\vee = A_0^\vee$. Caution: For an A-module M, the module $M^\vee := \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ is not to be confused with $M^* := \mathrm{Hom}_A(M, A)$.

Tensoring with K, then applying π_-^* to the Euler sequence yields an exact sequence:

$$(4.29) \qquad 0 \to \mathrm{H}^1(\pi_-^* \mathcal{O}(K))^G \to \mathrm{H}^1(\pi_-^* V_{\mathrm{nat}}(K))^G \to \mathrm{H}^1(\pi_-^* \mathcal{O})^G \to 0.$$

This yields an exact sequence of graded A-modules:

$$(4.30) \qquad 0 \to A_{>0}(2)^\vee \to \mathrm{H}^1(\pi_-^* V_{\mathrm{nat}}(K))^G \to A_{>0}(1)^\vee \to 0.$$

The functor $\mathrm{Hom}_{\mathbb{C}}(-, \mathbb{C})$ is exact, so we get

$$(4.31) \qquad 0 \to A_{>0}(1) \to (\mathrm{H}^1(\pi_-^* V_{\mathrm{nat}})^G)^\vee \to A_{>0}(2) \to 0.$$

Putting the two short exact sequences (4.27) and (4.31) together, we arrive at a long exact sequence in grmod A:

$$(4.32) \qquad 0 \to A_{>0}(1) \to (\mathrm{H}^1(\pi_-^* V_{\mathrm{nat}})^G)^\vee \to \mathrm{H}^0(\pi_+^* E(2K))^G \to A_{\geq 7}(7) \to 0.$$

Caution: this does not necessarily imply that $\mathrm{syz}^2(A_{\geq 7}(7))$ is isomorphic to $A_{>0}(1)$, because the two central terms may not be projective modules.

Proposition IV.5. *The ideal (y, x^3, z) defines a G-invariant, degree zero stable bundle on Z fitting into an exact sequence $0 \to \mathcal{O}(\lambda^{-11}) \to E \to \mathcal{O}(\lambda^{11}) \to 0$.*

$$(4.33) \qquad \text{syz}^2 A_{[13,84]} = \text{coker} \begin{pmatrix} z & x^4 & y^2 & 0 \\ x^3 & -z & 0 & y^2 \\ y & 0 & -z & -x^4 \\ 0 & -y & x^3 & -z \end{pmatrix}$$

Proof. The grading data of the matrix is $(6, 9, 13, 16)$. The map coker $M \to A(5)$ defined by $(s_1, \ldots, s_4) \mapsto (0, y, x^3, z)$ defines an exact sequence

$$(4.34) \qquad 0 \to \mathcal{O}(-6K) \to E \to \mathcal{O}(5K) \to 0.$$

Tensoring with λ we get $0 \to \mathcal{O}(\lambda^{-11}) \to E \to \mathcal{O}(\lambda^{11}) \to 0$. This is shown to be stable in the same way as above, using the long exact sequence in cohomology, or by consulting the paper [Dol99], where this bundle is also studied. $\qquad \square$

4.4 Filtration by line bundles

By [Dol99], every G-linearized vector bundle $E \in \text{vect}^G Z$ of rank r admits a full flag

$$(4.35) \qquad 0 \subset E_1 \subset E_2 \subset \cdots \subset E_r = E,$$

such that E_{i+1}/E_i is a G-linearized line bundle, for $i = 1, \ldots, r$. This is called a filtration of E by line bundles.

In the previous sections, we found the filtration by hand. However, this becomes impossible as the rank of E increases. We now explain a systematic way of finding a filtration by line bundles for $E = \varphi \pi \, \text{syz}^2 A_{[d,84]}$. Here, and throughout this entire section, A is the ring $\mathbb{C}[x, y, z]/(x^7 + y^3 + z^2)$.

Suppose we have a finitely generated CM A-module M. Let $s_1, \ldots, s_n$ be a set of generators of M, such that the s_i's are distinct. Let r be the rank of M. Necessarily, one has $n \geq r$.

Let $b_1, \ldots, b_n$ be the degrees of the generators $s_1, \ldots, s_n$. Assume that $b_i > 0$ for all $i = 1, \ldots, n$.

Now let $(s_1, \ldots, s_k)$ be the submodule of M generated by $s_1, \ldots, s_k$, for $1 \le k \le n$. We have a flag:

$$(4.36) \qquad s_1 \subset (s_1, s_2) \subset \cdots \subset (s_1, \ldots, s_{r-1}).$$

Let $\pi : \operatorname{grmod} A \to \operatorname{qgr} A$ be the projection, and let $\varphi : \operatorname{qgr} A \to \operatorname{coh}^G Z$ be the equivalence such that $\varphi(A(n)) = \mathcal{O}(nK)$, for all $n \in \mathbb{Z}$.

Lemma IV.6. *The quotient $M/(s_1, \ldots, s_{r-1})$ is of the form $(t_0) \oplus T$, where (t_0) is a principle ideal, and T is a finite dimensional A-module.*

Proof. Since M is CM, $\varphi\pi M$ is locally free. The subsheaf $\varphi\pi(s_1)$ is also locally free. Since s_1 is not of the form as_1' for any $a \in A_{>0}, s_1' \in M$, the inclusion $\varphi\pi s_1 \subset \varphi\pi M$ is saturated, i.e. the quotient $\varphi\pi(M)/\varphi\pi(s_1)$ is locally free, of rank $r - 1$.

Proceeding by induction, one gets that $\varphi\pi M/\varphi\pi(s_1, \ldots, s_{r-1})$ is locally free of rank 1. Hence it corresponds to an A-module of the form $X \oplus T$, where T is finite dimensional, and X is a rank one CM A-module. But the class group of A is trivial, because it coincides with the torsion part of $\operatorname{Pic}(G; Z)$, which is trivial in the case when the weights (p, q, r) are coprime, by equation (3.3); therefore X must be a principle ideal generated by a single element, $t_0 \in A$. $\square$

Let $(s_1, \ldots, s_{r-1}, t_0)$ be the submodule of M obtained by taking the preimage of (t_0) in M. Applying $\pi : \operatorname{grmod} A \to \operatorname{qgr} A$, we obtain an isomorphism

$$(4.37) \qquad \pi M \simeq \pi(s_1, \ldots, s_{r-1}, t_0).$$

Let $E = \varphi\pi(s_1, \ldots, s_{r-1}, t_0)$. Then E is a G-bundle of rank r, and we get a full

flag:

$$(4.38) \qquad 0 \subset E_1 \subset E_2 \subset \cdots \subset E_r = E,$$

such that $E_{i+1}/E_i \simeq \mathcal{O}(-b_i K)$, for $i = 1, \ldots, r - 2$, and $E_r/E_{r-1} \simeq \mathcal{O}(b_t)$, where $b_t \in \mathbb{Z}$ is the degree of the element $t_0 \in A$.

We call $(-b_1, \ldots, -b_{r-1}, b_t)$ a *sequence of exponents* of E.

Recall that the integers $b_1, \ldots, b_{r-1}$ are the degrees of the $r - 1$ distinct elements $s_1, \ldots, s_{r-1}$. To compute b_t, we use the following isomorphism:

$$(4.39) \qquad \mathcal{O}(b_t K) \simeq \det E \bigotimes_{i=1}^{r-1} \mathcal{O}(-b_i K)^{-1}.$$

This equation can be used to compute b_t assuming that the right hand side is known. For our bundles of the form $\varphi \pi \, \mathrm{syz}^2 A_{[d,84]}$, the right hand side is known. Indeed, we have seen how to find their class in the K-group $K(\mathrm{coh}^G Z)$, and there is an isomorphism $K(\mathrm{coh}^G Z) \simeq \mathbb{Z} \oplus \mathrm{Pic}(G; Z)$ given by $[E] \mapsto (\mathrm{rank}\, E, \det E)$.

For example, in the case of the ideal (x^2, y, z), the degree of t_0 can be read off from the Betti numbers:

$$(4.40) \qquad \deg t_0 = 12 + 14 + 21 - (26 + 33 + 35 + 42) = -89.$$

The command

$$(4.41) \qquad \text{degrees target presentation ker syz gens ideal } (x^2, y, z)$$

immediately gives one the degrees of the presentation of $M = \mathrm{syz}^2(x^2, y, z)$:

$$(4.42) \qquad A(-47) \oplus A(-54) \oplus A(-56) \oplus A(-63) \to M \to 0.$$

By the above discussion, we obtain a sequence

$$(4.43) \qquad 0 \to \mathcal{O}(-47K) \to E \to \mathcal{O}(-42K) \to 0.$$

The last term, $\mathcal{O}(42K)$, was chosen to ensure $\det E = \mathcal{O}(-89K)$.

Therefore, $(-47, -42)$ is a sequence of exponents for E.

Now tensoring with $\mathcal{O}(42K)$, we obtain

$$(4.44) \qquad 0 \to \mathcal{O}(-5K) \to E(42K) \to \mathcal{O} \to 0.$$

So $(-5K, 0)$ is a sequence of exponents of $E(42K)$.

Recall that $\lambda = (1/2)K$ generates the G-invariant Picard group of Z.

A necessary and sufficient condition for a G-linearized bundle E to define a G-invariant bundle of degree zero is

$$(4.45) \qquad \deg E(\lambda^k) = 0,$$

for some $k \in \mathbb{Z}$. In terms of a sequence of exponents $(a_1 K, \ldots, a_r K)$, the necessary and sufficient condition becomes:

$$(4.46) \qquad 2kr + \sum_{i=1}^{r} 4a_i = 0,$$

for some $k \in \mathbb{Z}$. Here we are using $\deg \lambda = 2$, and $\deg K = 4$, since the genus of Klein's quartic curve is 3.

In the following table, we calculate the sequences of exponents for the bundles $\varphi\pi \operatorname{syz}^2 A_{[d,84]}(42K)$, and whether or not they can be twisted to become degree zero G-invariant bundles. In the case when it can be untwisted, we also display the level, i.e. whether the degree zero bundle is $\mathrm{PSL}(2, \mathbb{F}_7)$-linearized (= level 1) or strictly $\mathrm{SL}(2, \mathbb{F}_7)$-linearized (= level 2); equivalently, whether the k in equation (4.46) is even

(= level 1) or odd (= level 2).

d	$(a_1, \ldots, a_r)$	twist deg 0?	level
7	$(-5K, 0)$	y	2
13	$(-11K, 0)$	y	2
19	$(-13K, -17K, 0)$	y	1
25	$(-19K, -20K, -23K, 0)$	y	2
31	$(-25K, -26K, -27K, -29K, 0)$	n	—
37	$(-31K, -32K, -33K, -34K, -35K, 0)$	y	2
15	$(-11K, -13K, 0)$	y	1
29	$(-23K, -25K, -26K, -27K, 0)$	n	—
22	$(-17K, -19K, -20K, 0)$	y	1
43	$(-37K, -38K, -39K, -40K, -41K, -41K, 0)$	n	—

d	$(a_1, \ldots, a_r)$	twist deg 0?	level
6	$(K, 0)$	y	2
12	$(-5K, 0)$	y	2
18	$(-11K, -13K, 0)$	y	1
24	$(-17K, -19K, -20K, 0)$	y	1
30	$(-23K, -25K, -26K, -27K, 0)$	n	—
36	$(-29K, -31K, -32K, -33K, -34K, 0)$	y	2
14	$(-11K, 0)$	y	2
28	$(-23K, -25K, -26K, 0)$	y	2
21	$(-17K, -19K, 0)$	y	1
42	$(-35K, -37K, -38K, -39K, -40K, 0)$	y	2

Remark IV.7. A sequence of exponents is not necessarily unique. Also, it is possi-

ble for the sequence of exponents of an unstable bundle to be the same unordered sequence as the sequence of exponents of a stable bundle.

Honesty forces us to emphasize the fact that not all of the above vector bundles on Z can be untwisted to be of degree zero. Stability is even more delicate, as we will now see.

From [Dol99], it is known that there are precisely four rank three stable G-invariant stable bundles of degree zero, with sequence of exponents:

$$(-10\lambda, 0, 10\lambda)$$
$$(-22\lambda, 0, 22\lambda)$$
$$(-2\lambda, -4\lambda, 6\lambda)$$
$$(-6\lambda, 4\lambda, 2\lambda)$$

Switching to K, we get:
$$(-10K, -5K, 0)$$
$$(-22K, -11K, 0)$$
$$(-4K, -5K, 0)$$
$$(-4K, K, 0)$$

Our three candidates are:
$$(-13K, -17K, 0)$$
$$(-11K, -13K, 0)$$
$$(-17K, -19K, 0)$$

The following proposition says that we have found two of the four stable bundles of rank three.

Proposition IV.8. *The vector bundles with sequences of exponents* $(-17K, -19K, 0)$ *and* $(-10K, -5K, 0)$ *are isomorphic upto twisting by a power of K. Also, the vector*

bundles with sequences of exponents $(-11K, -13K, 0)$ *and* $(-4K, K, 0)$ *are isomorphic upto twisting by a power of* K.

Proof. Let $E_{(a_1,\ldots,a_r)}$ denote a bundle with sequence of exponents $(a_1, \ldots, a_r)$.

Tensoring $E_{(-17,-19,0)}$ with $7K$, we get $E_{(-10,-12,7)}$. To show that $E_{(-10,-12,7)}$ is isomorphic to $E_{(-10,-5,0)}$, it suffices to show that $\operatorname{Hom}(\mathcal{O}(-10K), E_{(-10,-5,0)})^G \simeq \mathbb{C}$, and $E_{(-12,7)} \simeq E_{(-5,0)}$.

Applying $\operatorname{Hom}(\mathcal{O}(-5K), -)^G$ to the sequence $0 \to \mathcal{O}(-12K) \to E_{(-12,7)} \to \mathcal{O}(7K) \to 0$ yields

$$(4.47) \qquad 0 \to \operatorname{Hom}(\mathcal{O}(-5K), E_{(-12,7)})^G \to \operatorname{Hom}(\mathcal{O}(-5K), \mathcal{O}(7K)) \to 0,$$

whence $\mathcal{O}(-5K)$ is a subbundle of $E_{(-12,7)}$. Hence $(-5K, 0)$ is a sequence of exponents for $E_{(-12,7)}$. Since $\operatorname{Ext}^1(\mathcal{O}, -5K)^G$ is one-dimensional, the bundles $E_{(-5,0)}$ and $E_{(-12,7)}$ are isomorphic.

As for the claim that $\operatorname{Hom}(\mathcal{O}(-10K), E_{(-10,-5,0)})^G$ is one-dimensional, the long exact sequence in cohomology reads:

$$(4.48)$$
$$0 \to \operatorname{Hom}(\mathcal{O}(-10K), \mathcal{O}(-10K))^G \to \operatorname{Hom}(\mathcal{O}(-10), E_{(-10,-5,0)})^G \to \operatorname{Hom}(\mathcal{O}(-10K), E_{(-5,0)})^G.$$

Hence it suffices to show $\operatorname{Hom}(\mathcal{O}(-10K), E_{(-5,0)})^G = 0$. Taking cohomology we get

$$(4.49)$$
$$0 \to \operatorname{Hom}(\mathcal{O}(-10K), \mathcal{O}(-5K))^G \to \operatorname{Hom}(\mathcal{O}(-10K), E_{(-5,0)})^G \to \operatorname{Hom}(\mathcal{O}(-10K), 0)^G \to 0.$$

Since A_5 and A_{10} are both zero, the central term $\operatorname{Hom}(\mathcal{O}(-10K), \mathcal{O}(-5K))^G$ is zero, as desired.

The argument for $E_{(4,1,0)}$ proceeds in the same fashion. Tensoring $E_{(-11,-13,0)}$ with $\mathcal{O}(7K)$ yields $E_{(-4,-6,7)}$. Applying $\operatorname{Hom}(\mathcal{O}(K), -)^G$ to the short exact sequence

$0 \to \mathcal{O}(-6K) \to E_{(-6,7)} \to \mathcal{O}(7K) \to 0$ yields $0 \to \mathrm{Hom}(\mathcal{O}(K), E_{(-6,7)}) \to$ $\mathrm{H}^0(\mathcal{O}(6K))^G \to 0$, whence K is a subbundle of $E_{(-6,7)}$. Hence $E_{(-6,7)}$ is isomorphic to $E_{(1,0)}$. $\qquad\square$

However, the bundle $E_{(-13,-17,0)}$ is not isomorphic to the stable bundle $E_{(-22,-11,0)}$ upto a twist. Indeed, tensoring $E_{(-13,-17,0)}$ with $\mathcal{O}(-9K)$ yields $E_{(-22,-26,-9)}$. Clearly $E_{(-26,-9)}$ is not isomorphic to $E_{(-11,0)}$, because they have different degrees. One can show that $\mathrm{Hom}(\mathcal{O}(-22K), E_{(-22,-11,0)})^G$ is one-dimensional, using the long exact sequence in cohomology. Thus $E_{(-22,-11,0)}$ cannot be isomorphic to $E_{(-22,-26,-9)}$.

Similarly, the stable bundle $E_{(-13,-17,0)}$ is not isomorphic to $E_{(-4,-5,0)}$ upto a twist. Therefore, one of our bundles; namely, $E_{(-13,-17,0)} = \mathrm{syz}^2 A_{[19,84]}$, is not stable.

For bundles of rank 4 and higher, it becomes increasingly difficult to check for stability. One very helpful fact is that a non-semistable G-invariant bundle admits a G-invariant destabilizing subbundle which is stable. So, in principle, it would be possible to check for semistability provided we know all sequences of exponents of all G-invariant stable bundles. As of this writing, we only know those G-invariant stable bundles of rank 2 and 3 which can be untwisted to have degree zero. At this point, we are forced to postpone the treatment of higher rank bundles to future research.

We would like to conclude this section with an extended remark on the relation with the work of [KST09] on matrix factorizations. We have been working with CM modules, but for computing homomorphisms it makes more sense to work in the stable category $\underline{\mathrm{CM}}(A)$. One reason is that $\underline{\mathrm{CM}}(A)$ has a triangulated structured, and it is Calabi-Yau (see [BIKR08]).

In the case when A is a hypersurface with equation $f(x_0, \ldots, x_n) = 0$, $\underline{\mathrm{CM}}(A)$ is

equivalent to the homological category of matrix factorizations of f (see [KST09])

$$\mathrm{HMF}(f) \simeq \underline{\mathrm{CM}}(A).$$

A matrix factorization over the ring $A^{ab} = \mathbb{C}[x, y, z]/(x^p + y^q + z^r)$ is simply a pair of matrices over $\mathbb{C}[x, y, z]$, whose product is a diagonal matrix with $f(x, y, z) = x^p + y^q + z^r$ along the diagonal. Such a matrix factorization is graded if the matrices define homogeneous maps with respect to the group $\mathrm{Pic}(G; Z)$.

Let's see an example of what a $\mathrm{Pic}(G; Z)$-graded matrix factorization looks like.

Let $(a; b, c, d)$ stand for $aD_{\mathrm{reg}} + bD_1 + cD_3 + dD_4 \in \mathrm{Pic}(G; Z)$, where D_1, D_2, D_3 are the ground forms. The $\mathrm{Pic}(G; Z)$-grading on $A^{ab} = \mathbb{C}[x, y, z]/(x^p + y^q + z^r)$ is given by

$$(4.50) \qquad \deg(x, y, z) = \{(0; 1, 0, 0), (0; 0, 1, 0, 0); (0; 0, 0, 1)\}.$$

Suppose we have a decomposition of (p, q, r), i.e. positive integers (p', q', r') and (p'', q'', r'') such that $(p' + p'', q' + q'', r' + r'') = (p, q, r)$.

Consider the following matrix $\varphi \in M_4(A^{ab})$

$$(4.51) \qquad \varphi = \begin{pmatrix} z' & x' & y' & 0 \\ x'' & z'' & 0 & y' \\ y'' & 0 & z'' & x' \\ 0 & y'' & x'' & z' \end{pmatrix}$$

Let $L^1 = \{0, (-1; p', 0, r'), (-1; 0, q', r'), (-1; p', q', 0)\}$ and
$L^2 = \{(0; 0, 0, r'), (0; p', 0, 0), (0; 0, q', 0), (-1; p', q', r')\}$ be two lists. Let $(A^{ab})^{L^i}$ denote the direct sum $A^{ab}(L_0^i) \oplus \cdots \oplus A^{ab}(L_3^i)$.

The following elementary computations show that φ defines a degree zero homogeneous map $(A^{ab})^{L^2} \xrightarrow{\varphi} (A^{ab})^{L^1}$.

$$\text{column } 1: \qquad (0;0,0,r') + 0 = \qquad (0;0,0,r')$$

$$(0;p'',0,0) + (-\mathbf{1};\mathbf{p'},\mathbf{0},\mathbf{r'}) = \quad (0;0,0,r')$$

$$(0;0,q'',0) + (-\mathbf{1};\mathbf{0},\mathbf{q'},\mathbf{r'}) = \quad (0;0,0,r')$$

$$\text{column } 2: \qquad (0;p',0,0) + 0 = \qquad (0;p',0,0)$$

$$(0;0,0,r'') + (-1;p',0,r') = \quad (0;p',0,0)$$

$$(0;0,q'',0) + (-\mathbf{1};\mathbf{p'},\mathbf{q'},\mathbf{0}) = \quad (0;p',0,0)$$

$$\text{column } 3: \qquad (0;0,q',0) + 0 = \qquad (0;0,q',0)$$

$$(0;0,0,r'') + (-1;0,q',r') = \quad (0;0,q',0)$$

$$(0;p'',0,0) + (-1;p',q',0) = \quad (0;0,q',0)$$

$$\text{column } 4: \quad (0;0,q',0) + (-1;p',0,r') = \quad (-1,p',q',r')$$

$$(0;p',0,0) + (-1;0,q',r') = \quad (-1,p',q',r')$$

$$(0;0,0,r') + (-1;p',q',0) = \quad (-1,p',q',r')$$

One can find by hand another matrix ψ such that the product $\psi\varphi$ is $\mathrm{diag}(f)$, where $f = x^p + y^q + z^r$.

(4.52)

$$\psi\varphi = \begin{pmatrix} z'' & x' & y' & 0 \\ x'' & -z' & 0 & y' \\ -y'' & 0 & z' & x' \\ 0 & -y'' & x'' & -z'' \end{pmatrix} \begin{pmatrix} z' & x' & -y' & 0 \\ x'' & -z'' & 0 & -y' \\ y'' & 0 & z'' & x' \\ 0 & y'' & x'' & -z' \end{pmatrix} = \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & f & 0 \\ 0 & 0 & 0 & f \end{pmatrix}$$

The matrix ψ gives a presentation of the kernel of φ.

All of our presentation matrices for modules of the form $\mathrm{syz}^2 A_{[d,84]}$ define matrix factorizations. Some of our matrix factorizations in the $(2,3,7)$-case first appeared in the paper [KST09]. In fact, that is how this project got started.

The cokernel of their matrix $q_0(V_{10}) = q_1(V_0)$ is equal to $\mathrm{syz}^2 A_{>0}$, and defines a

bundle with sequence of exponents $(0, -K)$, i.e. it corresponds to the central term of the Euler sequence.

Their matrix $q_0(V_{23}) = q_1(V_{23})$ defines a G-bundle on Z with sequence of exponents $(\lambda^{-11}, \lambda^{11})$, so it is equivalent to the stable bundle defined by $\mathrm{syz}^2 A_{[13,84]}$.

The rank 3 bundle defined by their matrix $q_0(V_{12}) = q_1(V_{12})$ has sequence of exponents $(\lambda^{-10}, 0, \lambda^{10})$, so it is equivalent to the stable bundle defined by $\mathrm{syz}^2 A_{[21,84]}$.

It is tempting to speculate about the relation of our bundles with ranks ≥ 4 to those in [KST09]. For instance, our bundle $\mathrm{syz}^2 A_{[43,84]}$ is of rank 7, and they also have a matrix factorization of rank 7, namely $V_{\bar{1}}$. And there is a seven-dimensional irreducible representation of $\mathrm{SL}(2, \mathbb{F}_7)$. But this requires further thought.

4.5 Towards the general (p, q, r)-case

In the previous sections, we focused on the case $(p, q, r) = (2, 3, 7)$. It is desirable to have a uniform treatment for all (p, q, r). The $(2, 3, 11)$-case is similar to the $(2, 3, 7)$-case, but some of the arithmetic coincidences fail to hold. The arithmetic of the $(3, 3, 5)$-case is even more delicate. We will exhibit several G-invariant stable bundles in both the $(2, 3, 11)$ and $(3, 3, 5)$-case.

4.5.1 The case $(p, q, r) = (2, 3, 11)$.

Since $(2, 3, 11)$ are coprime, the Picard group $\mathrm{Pic}(G; Z)$ is isomorphic to $\mathbb{Z}$. The major difference between $(2, 3, 7)$ and $(2, 3, 11)$ is that the class of the canonical bundle ω is divisible by 5 in $\mathrm{Pic}(G; Z)$ in the $(2, 3, 11)$ case. That is, we have

$$(4.53) \qquad\qquad [\omega] \in 5\mathbb{Z}, \text{ for } (2, 3, 11),$$

whereas $[\omega]$ generates $\mathrm{Pic}(G; Z)$ in the $(2, 3, 7)$ case.

Recall, the Poincaré ring $A^{\mathrm{ab}} := \oplus_{L \in \mathrm{Pic}(G; Z)} \mathrm{H}^0(Z, L)^G$ has the following uniform

presentation:

$$(4.54) \qquad\qquad A^{\mathrm{ab}} = \mathbb{C}[x, y, z]/(x^{11} + y^3 + z^2).$$

The fact that $(2, 3, 11)$ are mutually coprime implies that there is an isomorphism $A^{\mathrm{ab}} \simeq A := \oplus_{n \geq 0} \mathrm{H}^0(Z, \mathcal{O}(nK))^G$.
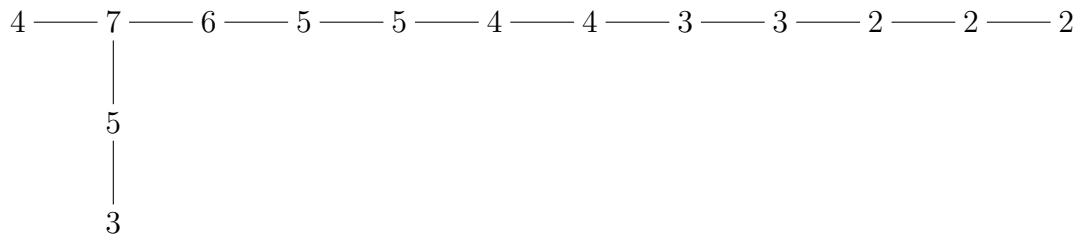
The question is whether the weights of the variables (x, y, z) are $(6, 22, 33)$ or $(6 * 5, 22 * 5, 33 * 5)$. In what follows, we will always use the grading $(6, 22, 33)$. We will never use $(6 * 5, 22 * 5, 33 * 5)$.

Remark IV.9. Incidentally, the isomorphism $A \simeq \oplus_{n \geq 0} \mathrm{H}^0(Z, nK)^G$ is known to exist, but it is not explicit. Klein and Gordon calculated the invariant functions for (2,3,7), but not (2,3,11).

The following tables were obtained by running the program geqIdeals for the ring $A = \mathbb{Z}/5857[x, y, z]/(x^{11} + y^3 + z^2)$, with $\deg(x, y, z) = (6, 22, 33)$. We record the

ranks of the corresponding vector bundles in a Dynkin diagram.

d	$A_{[d,132]}$	rank $\mathrm{syz}^2 A_{[d,132]}$
7	(x^2, y, z)	2
13	(x^3, y, z)	2
19	(y, x^4, z)	2
25	(xy, x^5, z, y^2)	3
31	(z, x^2y, x^6, y^2)	3
37	(xz, x^3y, x^7, y^2, yz)	4
43	$(y^2, x^2z, x^4y, x^8, yz)$	4
49	$(xy^2, x^3z, x^5y, x^9, yz, z^2)$	5
55	$(yz, x^2y^2, x^4z, x^6y, x^{10}, z^2)$	5
61	$(xyz, x^3y^2, x^5z, x^7y, z^2, y^3, y^2z)$	6
23	(x^4, xy, z, y^2)	3
45	$(x^2z, x^4y, x^8, xy^2, yz, z^2)$	5
34	(x^2y, x^6, xz, y^2, yz)	4
67	$(x^2yz, x^4y^2, x^6z, x^8y, xz^2, xy^3, y^2z, yz^2)$	7

$$4 \text{ --- } 7 \text{ --- } 6 \text{ --- } 5 \text{ --- } 5 \text{ --- } 4 \text{ --- } 4 \text{ --- } 3 \text{ --- } 3 \text{ --- } 2 \text{ --- } 2 \text{ --- } 2$$
$$|$$
$$5$$
$$|$$
$$3$$

d	$A_{[d,132]}$	rank $\operatorname{syz}^2 A_{[d,132]}$
6	(x, y, z)	2
12	(x^2, y, z)	2
18	(x^3, y, z)	2
24	(x^4, xy, z, y^2)	3
30	$(x^5, z, x^2 y, y^2)$	3
36	$(x^6, xz, x^3 y, y^2, yz)$	4
42	$(x^7, y^2, x^2 z, x^4 y, yz)$	4
48	$(x^8, xy^2, x^3 z, x^5 y, yz, z^2)$	5
54	$(x^9, yz, x^2 y^2, x^4 z, x^6 y, z^2)$	5
60	$(x^{10}, xyz, x^3 y^2, x^5 z, x^7 y, z^2, y^2 z)$	6
22	(y, x^4, z)	2
44	$(y^2, x^2 z, x^4 y, x^8, yz)$	4
33	$(z, x^2 y, x^6, y^2)$	3
66	$(z^2, y^3, x^2 yz, x^4 y^2, x^6 z, x^8 y, y^2 z)$	6

```
3 ── 6 ── 6 ── 5 ── 5 ── 4 ── 4 ── 3 ── 3 ── 2 ── 2 ── 2
     │
     4
     │
     2
```
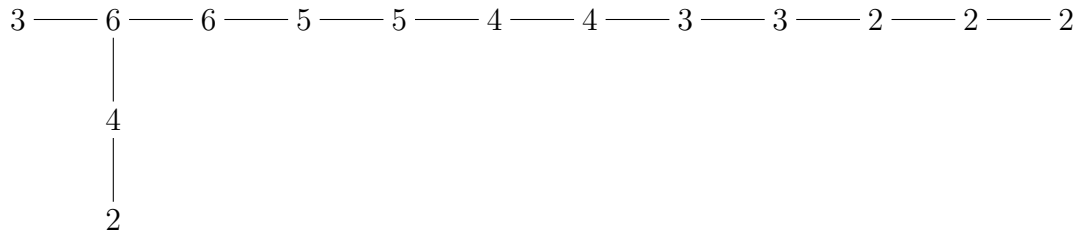
Here are presentations for the rank 2 bundles $\text{syz}^2 A_{[d,132]}$, where $d \in \{6, 7, 13, 19\}$:

$$\left\{ \begin{pmatrix} z & y^2 & x^{10} & 0 \\ y & -z & 0 & x^{10} \\ x & 0 & -z & -y^2 \\ 0 & -x & y & -z \end{pmatrix} \right.$$

$$\begin{pmatrix} z & y^2 & x^9 & 0 \\ y & -z & 0 & x^9 \\ x^2 & 0 & -z & -y^2 \\ 0 & -x^2 & y & -z \end{pmatrix}, \begin{pmatrix} z & y^2 & x^8 & 0 \\ y & -z & 0 & x^8 \\ x^3 & 0 & -z & -y^2 \\ 0 & -x^3 & y & -z \end{pmatrix}, \left. \begin{pmatrix} z & x^7 & y^2 & 0 \\ x^4 & -z & 0 & y^2 \\ y & 0 & -z & -x^7 \\ 0 & -y & x^4 & -z \end{pmatrix} \right\}$$

Here are the sequences of exponents, calculated by the procedure discussed at the end of section 4.4. We also record the level. A bundle E can be untwisted to be of degree zero if there exists an integral solution k to the equation $\deg E(\lambda^k) = 0$. In terms of the sequence of exponents, this reads:

$$(4.55) \qquad \sum_{i=1}^{r} a_i \deg K + rk \deg \lambda = 0$$

$$\sum_{i=1}^{r} 5a_i + rk = 0.$$

When an integral solution k exists, the level is the equivalence class of k modulo 5.

d	$(a_1, \ldots, a_r)$	level
7	$(-1, 0)$	–
13	$(-7, 0)$	–
19	$(-13, 0)$	–
25	$(-17, -19, 0)$	0
31	$(-23, -25, 0)$	0
37	$(-28, -29, -31, 0)$	0
43	$(-34, -35, -37, 0)$	0
49	$(-39, -40, -41, -43, 0)$	–
55	$(-45, -46, -47, -49, 0)$	2
61	$(-50, -51, -52, -53, -55, 0)$	2
23	$(-13, -17, 0)$	0
45	$(-34, -35, -37, -39, 0)$	0
34	$(-23, -25, -28, 0)$	0
67	$(-56, -57, -58, -59, -61, -61, 0)$	–

d	$(a_1, \ldots, a_r)$	level
6	$(5, 0)$	–
12	$(-1, 0)$	–
18	$(-7, 0)$	–
24	$(-13, -17, 0)$	0
30	$(-19, -23, 0)$	0
36	$(-25, -28, -29, 0)$	–
42	$(-31, -34, -35, 0)$	0
48	$(-37, -39, -40, -41, 0)$	2
54	$(-43, -45, -46, -47, 0)$	1
60	$(-49, -50, -51, -52, -53, 0)$	–
22	$(-13, 0)$	–
44	$(-34, -35, -37, 0)$	–
33	$(-23, -25, 0)$	0
66	$(-55, -56, -57, -58, -59, 0)$	–

In particular, we get precisely four rank two bundles. This is the same as the number of absolutely irreducible unitary representations of the Fuchsian group of signature $(0; 2, 3, 11)$ by the formula in [Dol99, Bod94]. The rank 2 bundle corresponding to $d = 6$ is obviously unstable, as expected, because it corresponds to the maximal ideal (x, y, z). It fits into an exact sequence

$$(4.56) \qquad 0 \to \mathcal{O} \to E \to \omega^{-5} \to 0.$$

Meanwhile, the bundle for $d = 7$ fits into an exact sequence

$$(4.57) \qquad 0 \to \mathcal{O} \to E \to \lambda^5 \to 0.$$

This sequence splits because $\dim \mathrm{Ext}^1(\lambda^5, \mathcal{O})^G = \dim \mathrm{H}^0(2K)^G = \dim A_2 = 0$.

We are led to consider the exact sequence:

(4.58)
$$0 \to \mathcal{O} \to E \to \omega^5 \to 0,$$

corresponding to the ground form $x \in (\mathrm{Ext}^1(\omega^5, \mathcal{O})^G)^\vee \simeq \mathrm{H}^0(6K)^G = A_6$. We claim E is stable. As before, we argue by examining the long exact sequence in cohomology:

(4.59)
$$0 \to \mathrm{H}^0(\lambda^{-a})^G \to \mathrm{H}^0(E(-a\lambda))^G \to \mathrm{H}^0(\lambda^{25-a})^G \to \mathrm{H}^1(\lambda^{-a}) \to \cdots$$

The only a's to check are $a = 13, \ldots, 25$. For a not divisible by 5, the space $\mathrm{H}^0(\lambda^a)^G$ is zero because λ is not $\mathrm{PSL}(\mathbb{F}_{11})$-linearized. For $a = 15, 20$, we get $\mathrm{H}^0(2K)^G = 0$ and $\mathrm{H}^0(K)^G = 0$, respectively. For $a = 25$, we need to examine the map $\varphi : \mathrm{H}^0(\mathcal{O})^G \to \mathrm{H}^1(\lambda^{-25})^G$. Both the source and target of φ are one-dimensional. By definition, φ is cupping with a nontrivial extension class; hence φ is an isomorphism. Thus E is stable.

The same chain of arguments shows that the bundle E fitting into the exact sequence $0 \to \mathcal{O} \to E \to \mathcal{O}(11K) \to 0$ is also stable.

4.5.2 The case $(p, q, r) = (3, 3, 5)$.

In the $(3, 3, 5)$-case the ring of invariants is $A = x^3 z + y^3 x + z^2$, and the dual weights are $\deg(x, y, z) = (3, 5, 9)$.

Throughout this section we shall make repeated use of the following proposition.

Proposition IV.10. *All G-invariant line bundles are of the form $\mu^a \otimes T^k, a \in \mathbb{Z}, k \in \{0, 1, 2\}$ where μ satisfies $\mu^4 = \mathcal{O}(K)$ and T is a generator of $\mathrm{Pic}(G; Z)_{\mathrm{tors}} \simeq \mathbb{Z}/3\mathbb{Z}$.*

Proof. Let $2G < \mathrm{SL}(2, \mathbb{C})$ be a binary polyhedral group which is not cyclic of odd order. Let G be the image of $2G$ under the $2 : 1$ map $\mathrm{SL}(2, \mathbb{C}) \to \mathrm{PSL}(2, \mathbb{C})$. Since

$\mathrm{PSL}(2, \mathbb{C})$ is the automorphism group of $\mathbb{P}^1$, G acts on $\mathbb{P}^1$. There is another natural action of G on the weighted projective space $\mathbb{P}(1, 1, 2)$. One extends the action of $2G$ on $\mathbb{C}[x, y]$ to an action of $2G$ on $\mathbb{C}[x, y, z]$. The element $-1 \in 2G$ acts by $(-1, -1, 1)$. In the weighted projective space $\mathbb{P}(1, 1, 2)$, multiplication by scalars (t, t, t^2) is the identity. So the action of $2G$ on $\mathbb{P}(1, 1, 2)$ factors through G.

Similarly, let $F_{d_1}, F_{d_2}, F_{d_3} \in \mathbb{C}[x, y]^{2G}$ be three generators of the ring of invariants for $2G$, of degrees d_1, d_2, d_3, respectively. Then each F defines a G-invariant curve in $\mathbb{P}(1, 1, d_i/2)$ given by the homogeneous equation

$$(4.60) \qquad\qquad z^2 = F_i.$$

This is a hyperelliptic curve Z on which G acts by automorphisms. We claim the $(3, 3, 5)$-case arises in this way, when G is the icosahedral group A_5 of order 60, and F is the ground form of degree 12:

$$(4.61) \qquad\qquad F_{12} = xy((x^2)^5 + 11(xy)^5 - (y^2)^5).$$

The zeroes of F_{12} correspond to the vertices of the icosahedron. For the sake of completeness, the other ground forms found by Klein are

$$F_{20} = -((x^2)^{10} + (y^2)^{10}) + 228((xy)^5(x^2)^5 - (xy)^5(y^2)^5) - 494(xy)^{10},$$

whose zeroes correspond to the 20 faces of the icosahedron, and

$$F_{30} = (x^2)^{15} + (y^2)^{15} + 522((xy)^5(x^2)^{10} - (xy)^5(y^2)^{10}) - 10005((xy)^{10}(x^2)^5 + (xy)^{10}(y^2)^5),$$

whose zeroes correspond to the 30 edges. They satisfy the relation:

$$R := -1728 F_{12}^5 + F_{20}^3 + F_{30}^2 = 0.$$

The group A_5 acts on both Z and $\mathbb{P}^1$, and Z is a double cover of $\mathbb{P}^1$.

Let $O_5 = \{[x_i, y_i], i = 1, \ldots, 12\}$ be the 12 points on $\mathbb{P}^1$ with stabilizer subgroup in A_5 of order 5, i.e. the 12 vertices of the icosahedron. Then $(x_i, y_i, 0)$ is an A_5-invariant orbit on Z with stabilizer subgroup of order 5.

Let $O_3 = \{[x_i : y_i], i = 1, \ldots, 20\}$ be the 20 points on $\mathbb{P}^1$ with stabilizer subgroup in A_5 of order 3. Let $\pm z_i$ be the solutions to the equation $z_i^2 = F_{12}(x_i, y_i)$. Then $(x_i, y_i, z_i), i = 1, \ldots, 20$ and $(x_i, y_i, -z_i), i = 1, \ldots, 20$ are two different A_5-orbits, on Z, with stabilizer subgroup of order 3.

Let $O_2 = \{[x_i, y_i], i = 1, \ldots, 30\}$ be the 30 points on $\mathbb{P}^1$ with stabilizer subgroup in A_5 of order 2. Their stabilizer subgroup in $2A_5 < \mathrm{SL}(2, \mathbb{C})$ of order 4 is generated by the matrix

$$g_2 := \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$

From the matrix one sees that $x_i = y_i$. Thus the pullback of O_2 consists of points on Z of the form $(1, 1, a)$, where a is a solution to $z^2 = F_{12}(1, 1)$. The action of g_2 on the pullback of O_2 is $g_2(1, 1, a) = (\sqrt{-1}, \sqrt{-1}, a)$. Recall that multiplication by (t, t, t^6) is the identity on $\mathbb{P}(1, 1, 6)$. Thus $(\sqrt{-1}, \sqrt{-1}, a)$ is the same point on Z as $(1, 1, -a)$ (which is not equivalent to $(1, 1, a)$). Thus the pullback of an element of O_2 is not a fixed point on Z.

The above discussion shows that there are three ramification points, of orders $(3, 3, 5)$. The genus of Z is 5, as can be seen from the Riemann-Hurwitz formula, equation (3.4):

$$(4.62) \qquad\qquad 2g - 2 = -2(60) + 4(12) + 2(20) + 2(20).$$

Now, it follows from equation (3.3) that the torsion part of $\mathrm{Pic}(G; Z)$ is $\mathbb{Z}/3\mathbb{Z}$. In terms of the ground forms, $D_1, D_2, D_3 \in \mathrm{Div}^G Z$, the torsion bundle T is represented by the difference $D_1 - D_2$.

The free part of $\mathrm{Pic}(G; Z)$ is rank one (as always happens in the case $Z/G \simeq \mathbb{P}^1$), and it is generated by a square root of K, say λ. A representative for λ is $2D_3 - D_1$ of degree $2(12) - 20 = 4$.

There is no G-invariant torsion, by the same proof as Prop 2.1 of [Dol99]. Hence the line bundle μ generating free part of the G-invariant Picard group is an m-th root of K, for some m divisible by 2.

Z is hyperelliptic, so it has a unique (hence G-invariant) g_2^1. Since the genus of Z is 5, the degree of g_2^1 is 2; hence the only possibility is $\mu = g_2^1$.

We have $\mu^2 = \lambda$ and $\lambda^2 = K$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

First, let us consider the second syzygy of $A_{[4,36]} : E = \mathrm{syz}^2 A_{[4,36]}$. One has the following explicit presentation of E :

$$(4.63) \qquad E = \mathrm{coker} \begin{pmatrix} x^3 + z & 0 & xy^2 & 0 \\ x^2 & z & 0 & xy^2 \\ y & 0 & -z & 0 \\ 0 & y & x^2 & -x^3 - z \end{pmatrix}$$

Let

$$T_1 := \mathrm{coker} \begin{pmatrix} x^3 + z & xy^2 \\ y & -z \end{pmatrix} \in \mathrm{grmod}\, A.$$

Let

$$T_2 := \mathrm{coker} \begin{pmatrix} z & xy^2 \\ y & -x^3 - z \end{pmatrix} \in \mathrm{grmod}\, A.$$

Note that the product of the above 2×2 matrices is a diagonal matrix with $(x^3 z + y^3 x + z^2)$ along the diagonal.

Let (s_1, s_2, s_3, s_4) be the four generators of E. Let (t_1, t_2) be the two generators

of T_2. Let $F : T_2 \to E$ be the map given by

$$(4.64) \qquad F(at_1 + bt_2) = as_2 + bs_4, \text{ for } a, b \in A.$$

Note that F is well defined because the relations go to zero. Also note that F is injective. Hence we have an exact sequence:

$$(4.65) \qquad 0 \to T_2 \to E \to \langle s_1, s_3 \rangle \to 0.$$

One may observe by looking at the relations that $\langle s_1, s_3 \rangle$ is isomorphic to T_1:

$$(4.66) \qquad 0 \to T_2 \to E \to T_1 \to 0.$$

Since we could have reversed the roles of T_1 and T_2, the sequence (4.66) splits.

Next, let us consider the bundle E corresponding to the ground form $x \in H^0(3K)^G$:

$$(4.67) \qquad 0 \to \mathcal{O} \to E \to \mathcal{O}(2K) \to 0.$$

We claim E is stable. To prove this, we apply $\mathrm{Hom}(\mu^a T^k, -)^G$, and examine the long exact sequence in cohomology. The slope of $\mu^a T^k$ is $\deg \mu^a$, because the degree of T is zero. Hence, to show stability, it suffices to show that $H^0(\mu^8 \mu^{-a} T^{-k})^G = 0$ for $a \geq 4$. This is obvious for $a > 8$, so the only a's we have to check are $a = 4, 5, 6, 7, 8$.

To compute G-invariant global sections of $\lambda^n T^k$, one may use the following formula:

$$(4.68) \qquad H^0(Z, \lambda^n T^k)^G = H^0(\mathbb{P}^1, \mathcal{O}(-n + \lfloor (n+k)/3 \rfloor + \lfloor (n-k)/3 \rfloor + \lfloor 2n/5 \rfloor)).$$

For $a = 4$, we have

$$(4.69) \qquad H^0(\mu^4)^G = H^0(K)^G = 0,$$

because the degree one component of the ring A is zero. We also have

$$(4.70) \qquad H^0(Z, \mu^4 T)^G = H^0(\mathbb{P}^1, -2p + \frac{2}{3}p_1 + \frac{2}{3}p_2 + \frac{4}{5}p_3 + \frac{1}{3}p_1 - \frac{1}{3}p_2) = 0$$

and

$$(4.71) \qquad \mathrm{H}^0(Z, \mu^4 T^2)^G = \mathrm{H}^0(\mathbb{P}^1, -2p + \frac{2}{3}p_1 + \frac{2}{3}p_2 + \frac{4}{5}p_3 + \frac{2}{3}p_1 - \frac{2}{3}p_2) = 0.$$

For $a = 5$ and $a = 7$, the line bundle $\mu^{8-a}T^k$ has no G-invariant sections, because odd powers of μ are not G-linearized.

For $a = 6$, $\mu^2 = \lambda = 1/2K$ is G-linearized, i.e. linearly equivalent to a G-invariant divisor, namely λ. We have

$$(4.72) \qquad \mathrm{H}^0(\lambda)^G = \mathrm{H}^0(\mathbb{P}^1, -p + \lfloor \frac{1}{3} \rfloor p_1 + \lfloor \frac{1}{3} \rfloor p_2 + \lfloor \frac{2}{5} \rfloor p_3) = 0$$

and

$$(4.73) \qquad \mathrm{H}^0(Z, \lambda + T)^G = \mathrm{H}^0(\mathbb{P}^1, -p + \lfloor \frac{1}{3} + \frac{1}{3} \rfloor p_1 + \lfloor \frac{1}{3} - \frac{1}{3} \rfloor p_2 + \lfloor \frac{2}{5} \rfloor p_3) = 0$$

and

$$(4.74) \qquad \mathrm{H}^0(Z, \lambda + 2T)^G = \mathrm{H}^0(\mathbb{P}^1, -p + \lfloor \frac{1}{3} + \frac{2}{3} \rfloor p_1 + \lfloor \frac{1}{3} - \frac{2}{3} \rfloor p_2 + \lfloor \frac{2}{5} \rfloor p_3) = 0.$$

For $a = 8$, we have

$$(4.75) \qquad \mathrm{H}^0(Z, T)^G = \mathrm{H}^0(\mathbb{P}^1, \lfloor \frac{1}{3} \rfloor p_1 - \lfloor \frac{1}{3} \rfloor p_2) = 0$$

and

$$(4.76) \qquad \mathrm{H}^0(Z, 2T)^G = \mathrm{H}^0(\mathbb{P}^1, \lfloor \frac{2}{3} \rfloor p_1 - \lfloor \frac{2}{3} \rfloor p_2) = 0.$$

Of course T and $2T$ must not have any G-invariant sections, because they are not linearly equivalent to an effective G-invariant divisor. Lastly, $\mathrm{Hom}(\mu^8, E)^G = 0$ follows from the fact that the map $\varphi : \mathrm{H}^0(\mathcal{O})^G \to \mathrm{H}^1(\mu^8, \mathcal{O})^G$ is injective, which follows from the fact that the extension E does not split, and that $\mathrm{H}^0(\mathcal{O})^G$ is one-dimensional.[1] Therefore, E is stable. Untwisting by $-K$, we get an exact sequence

$$(4.77) \qquad 0 \to \mathcal{O}(-K) \to E(-K) \to \mathcal{O}(K) \to 0,$$

---

[1] When $A_0 \simeq \mathbb{C}$ one says that A is connected.

where the central term $E(-K)$ is degree zero, G-linearized, and stable. Hence $E(-K)$ corresponds to an absolutely irreducible unitary representation of a Fuchsian group $\Gamma < \mathrm{SU}(1,1)$ of signature $(0; 3, 3, 5)$.

Notice that, in the above calculation, if we replace μ with K, then we get

$$(4.78) \qquad \mathrm{H}^0(Z, 4K + T)^G = \mathrm{H}^0(\mathbb{P}^1, -8p + \lfloor \tfrac{8}{3} + \tfrac{1}{3} \rfloor p_1 + \lfloor \tfrac{8}{3} - \tfrac{1}{3} \rfloor p_2 + \tfrac{16}{5} p_3)$$

$$= \mathrm{H}^0(\mathbb{P}^1, -8p + 3p + 2p + 3p) = \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}) \simeq \mathbb{C}.$$

This shows that there is a non-split extension:

$$(4.79) \qquad\qquad 0 \to \mathcal{O} \to E \to 3K + T \to 0.$$

As before, to show that E is stable it suffices to show that $\mathrm{H}^0(\mu^{12-a} T^k)^G = 0$ for $a = 6, \ldots, 11$, $k = 0, 1, 2$. We need only check even values of a, because odd powers of μ are not G-invariant.

For $a = 8, 10$, we have already shown that the space is $\mathrm{H}^0(\mu^{12-a} T^k)^G$ is zero. However, for $a = 6$, we get

$$(4.80) \qquad\qquad \mathrm{H}^0(\mu^6)^G = \mathrm{H}^0(\lambda^3)^G$$

$$= \mathrm{H}^0(\mathbb{P}^1, -3p + \lfloor \tfrac{3}{3} \rfloor p_1 + \lfloor \tfrac{3}{3} \rfloor p_2 + \lfloor \tfrac{6}{5} \rfloor p_3) = \mathrm{H}^0(\mathbb{P}^1, \lfloor \tfrac{1}{5} \rfloor p_3) \simeq \mathbb{C}.$$

That is, 3λ is linearly equivalent to the G-invariant divisor D_3 ($=$ the ground form with stabilizer subgroup $\mathbb{Z}/5\mathbb{Z}$.) So E is not stable. Rather, E is (strictly) semi-stable.

Similarly, the central term E of the nonsplit extension $0 \to \mathcal{O} \to E \to 3K + 2T \to 0$ is strictly semi-stable.

We are led to consider the nonsplit extension $E \in \mathrm{H}^1(\lambda)^G \simeq (\mathrm{H}^0(3\lambda)^G)^\vee$ corresponding to the ground form $D_3 = 3\lambda$.

$$(4.81) \qquad\qquad 0 \to \mathcal{O} \to E \to \mu^2 \to 0.$$

We claim that the G-invariant bundle E is stable. Indeed, arguing as above, we get that $\mathrm{Hom}(\mu T^k, \mu^2)^G = 0$, because μ is not G-invariant. And $\varphi : \mathrm{H}^0(\lambda, \lambda)^G \to \mathrm{Ext}^1(\lambda, \mathcal{O})^G$ is an isomorphism, because φ is defined to be cupping with a nontrivial extension class. Therefore, E is stable. Twisting with μ^{-1}, we get a short exact sequence

$$(4.82) \qquad 0 \to \mu^{-1} \to E(\mu^{-1}) \to \mu \to 0,$$

such that the central term $E(\mu^{-1})$ is G-invariant, degree 0, and stable. Hence $E(\mu^{-1})$ corresponds to an absolutely irreducible, level 2, unitary representation of the fundamental group $\Pi < \widetilde{\mathrm{SU}(1,1)}$ of signature $(3, 3, 5)$.

BIBLIOGRAPHY

BIBLIOGRAPHY

[AR87] Maurice Auslander and Idun Reiten, *Almost split sequences for rational double points*, Trans. Amer. Math. Soc. **302** (1987), no. 1, 87–97.

[Asp07] Paul S. Aspinwall, *Landau-Ginzburg to Calabi-Yau dictionary for D-branes*, J. Math. Phys. **48** (2007), no. 8.

[Ati57] M. F. Atiyah, *Complex analytic connections in fibre bundles*, Trans. Amer. Math. Soc. **85** (1957), 181–207.

[Aus86] Maurice Auslander, *Rational singularities and almost split sequences*, Trans. Amer. Math. Soc. **293** (1986), no. 2, 511–531.

[BD96] Victor V. Batyrev and Dimitrios I. Dais, *Strong McKay correspondence, string-theoretic Hodge numbers and mirror symmetry*, Topology **35** (1996), no. 4, 901–929.

[BD08] Igor Burban and Yuriy Drozd, *Maximal Cohen-Macaulay modules over surface singularities*, Trends in representation theory of algebras and related topics, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008, pp. 101–166.

[BIKR08] Igor Burban, Osamu Iyama, Bernhard Keller, and Idun Reiten, *Cluster tilting for one-dimensional hypersurface singularities*, Adv. Math. **217** (2008), no. 6, 2443–2484.

[Bis97] Indranil Biswas, *Parabolic bundles as orbifold bundles*, Duke Math. J. **88** (1997), no. 2, 305–325.

[BKR01] Tom Bridgeland, Alastair King, and Miles Reid, *The McKay correspondence as an equivalence of derived categories*, J. Amer. Math. Soc. **14** (2001), no. 3, 535–554.

[Blu06] Mark Blume, *Construction of G-Hilbert schemes*, arXiv.org: 0607577, 2006.

[Bod94] Hans U. Boden, *Unitary representations of Brieskorn spheres*, Duke Math. J. **75** (1994), no. 1, 193–220.

[BPR03] Egbert Brieskorn, Anna Pratoussevitch, and Frank Rothenhäusler, *The combinatorial geometry of singularities and Arnold's series E, Z, Q*, Mosc. Math. J. **3** (2003), no. 2, 273–333, 741.

[Bry96] J.-L. Brylinski, *A correspondence dual to McKay's*, arXiv: 9612003, 1996.

[CG97] Neil Chriss and Victor Ginzburg, *Representation theory and complex geometry*, Birkhäuser Boston Inc., Boston, MA, 1997.

[Del77] P. Deligne, *Cohomologie étale*, Lecture Notes in Mathematics, Vol. 569, Springer-Verlag, 1977, Séminaire de Géométrie Algébrique du Bois-Marie SGA 4 1/2, Avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier.

[Dol99] Igor V. Dolgachev, *Invariant stable bundles over modular curves $X(p)$*, Recent progress in algebra (Taejon/Seoul, 1997), Contemp. Math., vol. 224, Amer. Math. Soc., 1999, pp. 65–99.

[Dol09] _____, *McKay's correspondence for cocompact discrete subgroups of* SU$(1, 1)$, Groups and symmetries, CRM Proc. Lecture Notes, vol. 47, Amer. Math. Soc., 2009, pp. 111–133.

[GL87] Werner Geigle and Helmut Lenzing, *A class of weighted projective curves arising in representation theory of finite-dimensional algebras*, Singularities, representation of algebras, and vector bundles (Lambrecht, 1985), Lecture Notes in Math., vol. 1273, Springer, 1987, pp. 265–297.

[Gro57] Alexander Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. (2) **9** (1957), 119–221.

[HP05] Georg Hein and David Ploog, *Fourier-Mukai transforms and stable bundles on elliptic curves*, Beiträge Algebra Geom. **46** (2005), no. 2, 423–434.

[HS87] Jürgen Herzog and Herbert Sanders, *The Grothendieck group of invariant rings and of simple hypersurface singularities*, Singularities, representation of algebras, and vector bundles (Lambrecht, 1985), Lecture Notes in Math., vol. 1273, Springer, 1987, pp. 134–149.

[Kre09] Andrew Kresch, *On the geometry of Deligne-Mumford stacks*, Algebraic geometry—Seattle 2005. Part 1, Proc. Sympos. Pure Math., vol. 80, Amer. Math. Soc., Providence, RI, 2009, pp. 259–271.

[KST07] Hiroshige Kajiura, Kyoji Saito, and Atsushi Takahashi, *Matrix factorization and representations of quivers. II. Type ADE case*, Adv. Math. **211** (2007), no. 1, 327–362.

[KST09] _____, *Triangulated categories of matrix factorizations for regular systems of weights with $\epsilon = -1$*, Adv. Math. **220** (2009), no. 5, 1602–1654.

[KV00] M. Kapranov and E. Vasserot, *Kleinian singularities, derived categories and Hall algebras*, Math. Ann. **316** (2000), no. 3, 565–576.

[Lev99] Silvio Levy (ed.), *The eightfold way*, Mathematical Sciences Research Institute Publications, vol. 35, Cambridge University Press, Cambridge, 1999, The beauty of Klein's quartic curve.

[Loo84] E. Looijenga, *The smoothing components of a triangle singularity. II*, Math. Ann. **269** (1984), no. 3, 357–387.

[LP97] J. Le Potier, *Lectures on vector bundles*, Cambridge Studies in Advanced Mathematics, vol. 54, Cambridge University Press, 1997.

[MS02] Peter McMullen and Egon Schulte, *Abstract regular polytopes*, vol. 92, Cambridge University Press, 2002.

[NS64] M. S. Narasimhan and C. S. Seshadri, *Stable bundles and unitary bundles on a compact Riemann surface*, Proc. Nat. Acad. Sci. U.S.A. **52** (1964), 207–211.

[Orl09] D. Orlov, *Derived categories of coherent sheaves and triangulated categories of singularities*, Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II ed., Progress in Mathematics, vol. 270, Birkhäuser Boston Inc., 2009.

[Poi82] H. Poincaré, *Mémoire sur les fonctions fuchsiennes*, Acta Math. **1** (1882), no. 1, 193–294.

[Rei02] Miles Reid, *La correspondance de McKay*, Astérisque (2002), no. 276, 53–72.

[Ses82] C. S. Seshadri, *Fibrés vectoriels sur les courbes algébriques*, Astérisque, vol. 96, Société Mathématique de France, Paris, 1982, Notes written by J.-M. Drezet from a course at the École Normale Supérieure, June 1980.

[Yos90] Yuji Yoshino, *Cohen-Macaulay modules over Cohen-Macaulay rings*, London Mathematical Society Lecture Note Series, vol. 146, Cambridge University Press, Cambridge, 1990.