

A Perturbation Theory for Hamilton's Principal Function: Applications to Boundary Value Problems

by

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To my family

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ABSTRACT

A Perturbation Theory for Hamilton's Principal Function: Applications to
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This thesis introduces an analytical perturbation theory for Hamilton's principal function and Hamilton's characteristic function. Based on Hamilton's principle and the research carried out by Sir William Rowan Hamilton, a perturbation theory is developed to analytically solve two-point boundary value problems. The principal function is shown to solve the two-point boundary value problem through simple differentiation and elimination. The characteristic function is related to the principal function through a Legendre transformation, and can also be used to solve two-point boundary value problems. In order to obtain the solution to the perturbed two-point boundary value problem the knowledge of the nominal solution is sufficient. The perturbation theory is applied to the two body problem to study the perturbed dynamics in the vicinity of the Hohmann transfer. It is found that the perturbation can actually offer a lower cost two-impulse transfer to the target orbit than the Hohmann transfer. The numerical error analysis of the perturbation theory is shown for different orders of calculation.

Coupling Hamilton's principal and characteristic functions yields an analytical perturbation theory for the initial value problem, where the state of the perturbed system can be accurately obtained. The perturbation theory is applied to the restricted three-body problem, where the system is viewed as a two-body problem perturbed by the presence of a third body. It is shown that the first order theory can be sufficient to solve the problem, which is expressed in terms of Delaunay elements. The solution to the initial value problem is applied to derive a Keplerian periapsis map that can be used for low-energy space mission design problems.

CHAPTER I

Introduction

1.1 Celestial Mechanics and Mathematics

Humans have been fascinated by celestial phenomena since the dawn of times. Most civilizations have traditional tales and legends that explain the existence and role of celestial bodies such as the Sun, Moon and star constellations. Even if these stories and explanations are seldom accurate from a physical perspective, they have played a central role in cultural developments worldwide. Celestial bodies were observed in awe and their behavior was spiritually worshipped. It has been proposed the burial sites such as the Stonehenge in Wiltshire County, England, were aligned with the lunar lines, and were able to take into account the precession of the Earth around its rotational axis (1). The Cromlech of Eteneta is a burial site on the slopes of Adarremendi, Basque Country, that is the first thing that the sun shines on during winter solstice(2).

Ancient Babylonians were among the first to study the motion and behavior of celestial bodies, and actually record them. They realized of the periodic nature of orbits and recorded data in clay tablets, which required knowledge of angular distances (3). Egyptians used a three-hundred and sixty five day calendar and aligned their pyramids towards the North Star. They also developed basic trigonometric functions

to correlate the dimensions of the pyramids to sunrise and sunset angles(4).

Ancient Greeks developed astronomy as a branch of mathematics, and provided several advances in the field. Their work is seen as pivotal today, as they strived for the physical explanation of celestial phenomena, and in fact, the word "planet" comes from the greek word for wanderer. Eudoxus of Cnidus erroneously conjectured an Earth centered universe that prevailed for several centuries in the western world. Aristarchus of Samos proposed a heliocentric model of the Universe, and calculated the sizes of the Earth, Moon and Sun, although these values were incorrect. However, his heliocentric model was not well received and a geocentric model became standard instead. Claudius Ptolemy's research in the geocentric Universe became one of the most influential works in astronomy for centuries to come (5).

Nicolaus Copernicus' published *De revolutionibus orbium coelestium* in 1543, where he proposed a heliocentric solar system that revolutionized the field of astronomy. Although it was well received at first, it later encountered criticism and opposition and the geocentric model prevailed as conventional wisdom until early 17th century. It took several years and restless research by Galileo Galilei and Johannes Kepler to establish the heliocentric Copernican theory (6).

Kepler discovered the three laws of planetary motion that bear his name, and was able to describe the motion of planets around the Sun (7). Although these laws don't take into account the mass of the bodies and tidal perturbations that arise from other celestial bodies, Kepler's work laid the foundation for the present day research and development in the field of astrodynamics and celestial mechanics. In order to solve some of the problems encountered by Kepler, John Napier developed logarithms to solve burdensome equations in a relatively fast manner. Exponential notation made

hand calculations much easier to compute, especially for problems where perturbations were present (8).

Isaac Newton's work led to the further understanding of the motion of bodies in space by developing Kepler's ideas. Newton's law of universal gravitation determines that every body of mass interacts with each other. His work established basic equations of motion for the two-body and three-body problems. He developed infinitesimal calculus to help obtain the oblate spheroid shape of the Earth and took into account the effect of the Moon's gravitational attraction (9).

Joseph Louis Lagrange reformulated Newtonian mechanics by applying the conservation laws for momentum and energy, resulting in what is known as Lagrangian Mechanics. This formulations allow simpler methods to obtain the equations of motion and to analyze a system. Lagrange was able to use this new framework to formulate the motion of a body of negligible mass around two massive bodies, commonly known as the restricted three-body problem. While working on this problem Lagrange encountered what are known as Lagrangian points , which are equilibrium points in the restricted circular three-body problem.

Sir William Rowan Hamilton continued Lagrange's work and reformulated a general framework for classical mechanics, namely Hamiltonian mechanics. His research led to the discovery of Hamilton's principal function, an essential function in the wave theory of light, and Hamilton's principle, which allows to obtain the equations of motion of a system and explicitly showing the relationship between coordinates and momenta. He applied this principle to the motion of the Moon and several other problems in optics and classical mechanics.

Hamilton's proposed framework is convenient for solving several problems in celestial mechanics and draws advantages over other methods. He also developed a technique to solve the initial value problem that arises from Hamilton's principle and principal function, aimed at solving as precisely as possible the trajectories of planets, moon and asteroids in the solar system.

The fascination and desire to understand celestial phenomena and astrodynamics has led to many important advances in mathematics and physics. Throughout history mathematical advances have been closely related to the development of celestial mechanics, as many of the aforementioned were experts in both mathematics and astronomy (as well as other areas of study), and developed mathematical methods to help solve problems in celestial mechanics.

1.2 Two-Point Boundary Value Problems

Solutions to two-point boundary value problems are of significant importance in the field of astrodynamics and have been subject to extensive research over the years. Obtaining the trajectory of a comet, determining the required initial velocity of a missile to hit its target, knowing the optimal velocity of a probe to perform a fly-by, determining the orbital motion of the planets, and several other situations require one to solve boundary value problems. However, the lack of procedures that automatically converge to the desired solution remains as a fundamental difficulty in solving these problems. Usually, solutions involve open ended iterative methods that often have no guaranteed convergence and require a good initial guess. Examples of these methods include the method of homotopy, multiple shooting combined with Newton's iteration, and a variety of other techniques (10).

The multiple shooting method is a numerical procedure to solve boundary value problems that requires the nominal solution as the initial guess. However, the rate of convergence slows down as the system becomes more nonlinear. Newton's iteration method obtains the solution of a function by converging on the solution through multiple successive approximations. However, the function has to satisfy several requirements and some initial guesses might cause the solution to diverge (10).

The method of homotopy is used to analytically solve partial differential equations and nonlinear ordinary differential equations. It is used in conjunction with Newton's method to analytically solve a function, and its performance is tied to the chosen homotopy function (11). This can be a problem for non-trivial functions. Ideally, analytical solutions are desired to solve two-point boundary value problems, even if solutions involve integrating a set of equations by quadratures.

A further challenge for many problems with known solutions is when implicit equations must be solved iteratively to satisfy the two-point boundary value problem. In the two-body problem this situation occurs when finding a solution to Lambert's problem, which involves iterating the semimajor axis of the transfer orbit (12), (13).

Johann Heinrich Lambert developed a solution for the Keplerian two-point boundary value problem, finding a minimum energy orbit between two coordinates given a specific transfer time. However, this technique does not take into account common perturbations like mass distribution and tidal effects. Nevertheless, it provided a crucial step in solving these type of problems and the nominal solution can be used by a variety of numerical algorithms to solve more complex problems.

Even if Lambert's problem can be used to solve the two-point boundary value

problem, specific two-point boundary value problems can be solved with different methods. A special kind of two-point boundary value problem is the Hohmann transfer, developed by Walter Hohmann, which is the minimum fuel two-burn orbit transfer. This transfer is restricted to a 180 degree transfer, so even if often times this is desirable, it does not always offer the best solution. Bi-elliptic transfers can offer a less expensive alternative to Hohmann transfers, but the transfer time is much longer, and therefore is more suitable for missions where transfer time is not important.

Guibout and Scheeres (2002) outlined a novel approach for solving two-point boundary value problems for Hamiltonian dynamical systems using generating functions for the canonical transformation (14). They also suggested using *Hamilton's principal function* to solve the two-point boundary value problem for the two-body problem. This function is derived directly from Hamilton's principle and yields solutions to the equations of motion through simple differentiations and eliminations (15). Although the generating function of the canonical transformation and Hamilton's principal function have different physical significance, they are intimately related. Hamilton's principal function allows the initial and final endpoints and times to be variable, that is, they are free to change without affecting the structure of the function. Generating functions on the other hand, have invariable initial conditions, viewed as constants of motion, only final endpoints being variable.

1.3 Initial Value Problems

The initial value problem has also been extensively studied for solving problems in astrodynamics, as it allows to obtain the trajectory of a body in space over time. The n-body problem originally formulated by Newton falls under this category, as given some initial position and velocity vectors the trajectory is solved. A more com-

mon problem is the restricted three-body problem, where an object of negligible mass orbits two massive bodies that are in orbit around each other, and is an accurate representation of many planet-moon, star-planet, or binary asteroid systems.

Numerous methods and techniques have been developed to solve the initial value problem, where the challenge lies in solving the equations of motion for a set of initial conditions. Obtaining solutions can be straightforward by simply using a numerical integration and propagation technique, however, choosing the right technique for each specific problem can improve results.

Leonhard Euler developed a first-order integration method to solve simple problems, but it can be numerically unstable and the error can be significant, and it is not suitable for difficult equations. For most systems the Runge-Kutta method is widely used to integrate the equations of motion, however, error can build up over long periods of time. Variational integrators alternatively can be used for Hamiltonian dynamical systems, as they preserve the symplectic structure of the system (16).

Solutions to the initial value problem commonly involve numerical methods. These methods can accurately obtain a solution given some initial conditions and equations of motion. However, numerical results alone become very limited as no qualitative understanding is gained. Analytical solutions on the other hand give insight on how the system evolves over time and allow better understanding of its structure. Furthermore, analytical results yield more accurate results for highly complex problems that numerical methods would not.

Traditionally the restricted three-body problem is solved by numerically integrating the equations of motion, which are well known. Analytical analysis can be carried

out in the system to obtain Lagrange or equilibrium points and zero-velocity curves, but the solution to the trajectory has to be solved numerically. Ross and Scheeres (2007), and Grover and Ross (2009) developed an analytical framework to solve initial value problems, and developed it for the planar restricted three-body problem using the Picard iteration method (17), (18).

However, their proposed method is partially based on heuristics and simplifies the problem by ignoring the changing periapse due to the tidal effects of a third body. The ensuing results are not numerically accurate but they do capture the overall behavior of dynamics. Therefore their method is more suited for qualitative analysis and is a useful tool for low-energy spacecraft mission design. Another drawback of their method is that it can only be applied to the planar motion. While this solves many problems of interest, it is more desirable to have a full planar and non-planar motion methods.

1.4 Perturbations

The solution to the Keplerian two-body problem is well known, the equations of motion are integrable and there are six constants of motion that fully describe the system. However, the only acting force is the gravitational pull of the primary mass on the orbiting object, as commonly encountered forces such as irregular mass distribution, third body tidal effects, and atmospheric drag are neglected. Although the two-body problem provides a good approximation for problems with small perturbations and short time durations, the solution deviates when perturbing forces are large or time durations are long. For perturbed environments special perturbation techniques are employed to obtain solutions that vary from analytical to numerical methods (19).

Trajectories in space are usually perturbed from their nominal Keplerian orbits and this has also been focus of research for centuries. Accurately modeling perturbation forces is of crucial importance in many astrodynamics problems, and several methods and models have been developed to solve a variety of perturbed environments.

Newton's law of gravitation allows for gravitational effects from an infinite number of bodies. Hence, a system with multiple bodies and their gravitational effects can be modeled with accuracy. However, this model does not take into account the mass distribution of the bodies nor other perturbing forces, which can have a significant effect on the equations of motion. Obtaining the perturbing force vector can be very difficult using Newton's force balance method, and some systems are impossible to analyze.

Pierre Simon Laplace developed the potential function, which is a scalar function, that allows to obtain the perturbing force vector by simple differentiation (20). This in turn allows to numerically integrate and propagate the equations of motion of the perturbed system as long as the perturbing potential is known.

Adrien Marie Legendre discovered the Legendre functions which are a family of solutions to Laplace's equation and help solve problems of gravitational potential distribution for spheres. This allows to map the gravitational distribution of celestial bodies and is particularly useful to obtain an accurate perturbation potential close to such celestial body.

Lagrange developed the variation of parameters to obtain the Lagrange planetary

equations, which solve the perturbed secular and periodic terms of the orbit elements (21). The orbit elements are expressed as time differentials and have singularity problems. Gauss also developed a variation of parameters approach, which is applicable to non-conservative forces as the equations are expressed in terms of the perturbing accelerations (22).

For the nominal two-body two-point boundary value problem, Lambert's problem is solved in order to obtain the desired solutions (although Kepler's equation must be solved numerically). Therefore, the solution to these types of problems is well established and generally straightforward. Using analytical expressions would be useful not only to obtain a closed form solution for the perturbed Lambert's problem, but to better understand the underlying structure of the system. Furthermore, this possibility would allow for further development of mission design problems, such as low energy trajectories. Therefore, obtaining a family of solutions in the vicinity of the desired target can be very useful and important information. These results can help to better understand the dynamics of the system, even when perturbing forces are present.

When a Hamiltonian dynamical system is altered due to the presence of perturbing forces, the nominal solution no longer accurately solves the problem, and one must usually resort to numerical methods. As stated, conventional solutions to two-point boundary value problems involve numerical integration schemes and open-ended iterative solutions that do not necessarily converge. This type of solution focuses on finding an optimal solution around the vicinity of the nominal solution and can be accurately obtained. However, these numerical solutions do not portray an accurate description of the system and its behavior, they focus on finding the most satisfying numerical solution. While in practice this might be desirable, it does not obtain a

detailed analysis of the vicinity of the desired solution, which can be used to analyze the system.

Hence, it might be practical and desirable to have a method that can analytically obtain solutions to the perturbed problem. Several popular methods exist to solve these kind of problems, however, they are either only suited for specific situations or they are mathematically complex and burdensome. Kozai's method for example involves an averaging technique to solve Lagrange's planetary equations, but it only takes into account a first few zonal harmonics (23). Similarly Brower's method only takes into account zonal harmonic effects when calculating the perturbed orbit elements.

Von Zeipel's method can be used to reduce the Hamiltonian function through canonical transformations into a form independent of periodic perturbed coordinates (24). The reduced Hamiltonian function depends only on the constants of motion of the perturbed system. However, this method can be complex and very cumbersome.

1.4.1 Hamilton's work and Perturbed Dynamics

Sir William Rowan Hamilton developed a framework for classical dynamics that is a reformulation of Lagrangian and Newtonian mechanics. This framework is widely used and served as a base for other advancements in celestial mechanics and classical dynamics. However, there were several aspects of his work that did not receive much attention and have been "forgotten", and these works have been fundamental in the development of the perturbation theories and boundary value problem solutions presented in this dissertation.

Hamilton's principal function and Hamilton's characteristic function have not commonly been used and have been seemed as irrelevant and replaced by the generating function of the canonical transformation. However, these two functions can be used to analytically solve boundary value problems, as they are functions of endpoint states and times, and can be solved by simple partial differentiation.

Hamilton developed a perturbation theory to solve the initial value problem. His findings were also based on the use of Hamilton's principal function and used a similar approach to the one developed in the dissertation. However, he based his development in several assumptions and is only valid for small perturbations. Furthermore, it only solves the problem to the first order, providing an approximate but not exact solution. The method developed in the following chapters is not based on any assumptions and provides an accurate solution to the perturbed system.

1.5 Original Contributions

In the dissertation, an analytical perturbation theory to solve the two-point boundary value problem and the initial value problem is developed. The proposed method arises from previous work done by Sir William Hamilton and his study of Hamiltonian dynamics. Hamilton made several contributions to classical dynamics and successfully applied them to optics and celestial mechanics. He also developed two functions that contain the full dynamics of a Hamiltonian dynamical system that are known as Hamilton's principal function and Hamilton's characteristic function. However, these functions have not found much practical consideration historically, but instead they have been used for theoretical applications

Because of its structure, it is shown that Hamilton's principal function can be

applied to solve two-point boundary value problems. The characteristic function and the principal function share the same structure, therefore the characteristic function can also be used to solve two-point boundary value problems.

A perturbation theory is developed to analytically obtain Hamilton's principal function of a perturbed Hamiltonian dynamical system. Since Hamilton's principal function can be used to solve two-point boundary value problems, this theory allows to analytically solve two-point boundary value problems, where given the initial and final generalized coordinates and time, the endpoint generalized momenta are solved.

Similarly, another perturbation theory is developed to solve Hamilton's characteristic function for a perturbed Hamiltonian system, where given the initial and final generalized momenta and times, the endpoint generalized coordinates are obtained.

Two-point boundary value problems can be reversed to solve the initial value problem. Using the principal and characteristic functions the solution for an initial value problem can be obtained. Through the perturbation method to solve two-point boundary value problems, the perturbed initial value problem can also be solved analytically.

In the nominal two-body problem Lambert's problem is solved to obtain a solution for the two-point boundary value problem. The proposed perturbation theory is applied to analytically solve the perturbed two-point boundary value problem. This not only provides an accurate solution but also the possibility to obtain a family of solutions in the vicinity of the nominal solution, which exposes the full dynamics of the perturbed system.

The perturbation theory is also applied to analytically solve the initial value problem for the two-body problem and the restricted three-body problem. The restricted three-body problem can be viewed as a rotating two-body problem that is perturbed by the presence of the third body. Therefore, an analytical solution for the restricted three-body problem can be obtained, which is required to solve by quadratures at each order.

Ross and Scheeres found applications in low-energy spacecraft mission design problems for analytical solutions in the restricted three-body problem. However, their method is based on heuristics and best suited for qualitative analysis, and is restricted to planar motion. The proposed method in this dissertation is based on fundamental Hamiltonian dynamics and the derivation is consistent and thorough. Accurate numerical results can be obtained for the restricted three-body problem and this can be applied to accurately solve low-energy spacecraft mission design problems. Furthermore, the method developed in this dissertation is not restricted to planar motion and spans all $2n$ dimensions of phase space.

The research presented in this thesis has produced several conference and journal papers, including those that are under the review process or to be submitted in the near future:

Hamilton's Principal Function for the Two-Body Problem Solution by Penagari-cano and Scheeres presented at the 2006 AAS/AIAA Space Flight Mechanics Conference in Tampa, Florida, showed that the principal function can be used to solve two-point boundary value problems and introduced the analytical form of the principal function for the two-body problem as a function of the endpoint coordinates and times (25).

A Perturbation Theory for Hamilton's Principal Function: Applications to Two-Point Boundary Value Problems by Penagaricano and Scheeres was presented at the 2007 AAS/AIAA Space Flight Mechanics Conference in Sedona, Arizona, introduced a perturbation theory to solve perturbed two-point boundary values using Hamilton's principal function, and showed its validity through numerical simulations (26).

A Symplectic Keplerian Map for Perturbed Two-Body Dynamics by Penagaricano and Scheeres was presented at the 2008 AIAA/AAS Astrodynamics Specialists Conference in Honolulu, Hawaii, introduced a perturbation theory to analytically solve the perturbed initial value problem based on Hamilton's principal and characteristic functions (27).

A Perturbation Theory by Penagaricano and Scheeres was presented at the IAF International Astronautical Congress in Glasgow, Scotland, and was later published in *Acta Astronautica*. The paper introduced the perturbations theories for the two-point boundary value problems, and the initial value problem, establishing a clear link between them (28).

Additionally, there is one paper under review and another one ready to be submitted. The former introduces the perturbation theory for the two-point boundary value problem, and develops it in a thorough manner. It also shows how to apply the theory to solve the perturbed Hohmann transfer and the perturbed Lambert's problem. The latter paper develops the initial value problem and applies it to solving the restricted three-body problem for the out-of-plane motion.

CHAPTER II

Hamiltonian Dynamics

In the first half of the 19th century, Sir William Rowan Hamilton introduced a general framework for classical mechanics. This framework was derived from the previous work of Joseph Louis Lagrange in the development of Lagrangian mechanics at the end of the 18th century. The Hamiltonian dynamics framework is yet another reformulation of Newtonian mechanics, and while the Lagrangian framework introduces the conservation of energy and momentum into the system, Hamiltonian dynamics defines the momenta as an independent parameter in the equations of motion.

Both the Lagrangian and Hamiltonian mechanics have advantages and shortcomings, and each are suited better for different systems, although the frameworks are interchangeable. Most common celestial mechanics system such as the two-body and the three-body problems are better suited to be analyzed using the Hamiltonian framework.

2.1 Lagrangian Mechanics

Lagrangian mechanics is a reformulation of Newtonian mechanics and takes into account the laws of conservation of energy and conservation of momentum. Coordinates that are independent of each other are used in order to avoid constraints

between coordinates and to simplify the system. These coordinates are called generalized coordinates and their time derivatives are known as generalized momenta.

The Lagrangian, L , is a basic function introduced by Lagrange and defines the dynamics of the system. The Lagrangian function of an n -dimensional system is expressed in terms of generalized coordinates \vec{q} , generalized velocities $\dot{\vec{q}}$ and time t , and is defined in terms of the kinetic and potential energies:

$$L(\vec{q}, \dot{\vec{q}}, t) = T(\dot{\vec{q}}) - V(\vec{q}, t), \quad (2.1)$$

where T is the kinetic energy and V is the potential energy of the system. The equations of motion of the system are obtained by satisfying the Euler-Lagrange equation:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0. \quad (2.2)$$

2.2 Hamiltonian Dynamical Systems

Hamiltonian mechanics is a further development of Lagrangian mechanics and therefore another reformulation of Newtonian mechanics, and was introduced by Sir William Rowan Hamilton in the 18th century. The equations of motion in a Hamiltonian dynamical system are of first order and define a $2n$ dimensional phase space of generalized coordinates and generalized momenta.

2.2.1 Phase Space

Phase space is a mathematical space where all the states of a dynamical system are encompassed. Each state of the system is represented by a unique point in phase space. For a Hamiltonian dynamical system the phase space is represented by generalized coordinates \vec{q} , and generalized momenta \vec{p} , which are canonical variables and independent of each other. Therefore the phase space is of dimension $2n$ for an n dimensional system.

The extended phase space contains time as an independent variable, therefore the triplet (\vec{q}, \vec{p}, t) represents such space. The extended phase space is of dimension $2n+1$ for an n dimensional system.

The phase space and extended phase space are important in the study of Hamiltonian dynamics since they both represent a set of canonical variables for a system. Such canonical variables can be obtained through canonical transformations. A Lagrangian dynamical system can be transformed into a Hamiltonian dynamical system by means of Legendre transformations, thus eliminating the generalized velocities and replacing them with generalized momenta, which are canonical variables.

2.2.2 Principle of Least Action

Leonhard Euler described the term action as the effort nature has to do to preserve the structure of the system, implying that natural phenomena occur such that this effort is stationary, and adopts the easiest configuration for the dynamics of the system. The action integral of the system can be represented mathematically by the equation

$$A = \int_{t_0}^{t_1} L(\vec{q}, \dot{\vec{q}}) dt. \quad (2.3)$$

2.2.3 Hamilton's Equations

The Hamiltonian function, H , contains the dynamics of the system and is obtained through a Legendre transformation of the Lagrangian. This transformation allows to express the Hamiltonian in terms of generalized coordinates and generalized momenta:

$$H(\vec{q}, \vec{p}, t) = \dot{\vec{q}} \cdot \vec{p} - L(\vec{q}, \dot{\vec{q}}, t), \quad (2.4)$$

where the generalized momenta \vec{p} are defined as

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{q}}}. \quad (2.5)$$

A system is a Hamiltonian dynamical system if there exists a Hamiltonian function, $H(\vec{q}, \vec{p}, t)$, such that it satisfies Hamilton's equations of motion. The Hamiltonian can be obtained from a variation of the Lagrangian:

$$\delta L = \frac{\partial L}{\partial \vec{q}} \delta \vec{q} + \frac{\partial L}{\partial \dot{\vec{q}}} \delta \dot{\vec{q}} + \frac{\partial L}{\partial t} \delta t. \quad (2.6)$$

Taking the definition of generalized momenta in Eq. 2.5, the Euler-Lagrange equation in Eq. 2.2 can be rewritten as:

$$\frac{d}{dt}\vec{p} - \frac{\partial L}{\partial q_i} = 0, \quad (2.7)$$

and rearranging Eq. 2.7, the time differential of the momenta can be written as:

$$\dot{\vec{p}} = \frac{\partial L}{\partial q_i}, \quad (2.8)$$

which can be substituted in Eq. 2.6, and the variation in the Lagrangian becomes

$$\delta L = \dot{\vec{p}}\delta\vec{q} + \vec{p}\delta\dot{\vec{q}} + \frac{\partial L}{\partial t}\delta t. \quad (2.9)$$

The terms inside Eq. 2.9 can be manipulated by simple chain rule differentiation:

$$\delta(\vec{p}\dot{\vec{q}}) = \vec{p}\delta\dot{\vec{q}} + \dot{\vec{q}}\delta\vec{p}, \quad (2.10)$$

and Eq. 2.9 can be written as

$$\delta L = \dot{\vec{p}}\delta\vec{q} + \delta(\vec{p}\dot{\vec{q}}) - \dot{\vec{q}}\delta\vec{p} + \frac{\partial L}{\partial t}\delta t. \quad (2.11)$$

Rearranging and using simple algebra, Eq.2.11 becomes

$$\delta \left(L - \vec{p}\vec{q} \right) = \dot{\vec{p}}\delta\vec{q} - \vec{q}\delta\vec{p} + \frac{\partial L}{\partial t}\delta t. \quad (2.12)$$

The left hand side of Eq. 2.12 is the variation of the Hamiltonian in Eq. 2.4, hence,

$$-\delta H = \dot{\vec{p}}\delta\vec{q} - \vec{q}\delta\vec{p} + \frac{\partial L}{\partial t}\delta t = - \left[\frac{\partial H}{\partial \vec{q}}\delta\vec{q} + \frac{\partial H}{\partial \vec{p}}\delta\vec{p} + \frac{\partial H}{\partial t}\delta t \right]. \quad (2.13)$$

Equating terms in Eq. 2.13 yields Hamilton's equations of motion:

$$\dot{\vec{q}} = \frac{\partial H}{\partial \vec{p}}, \quad \dot{\vec{p}} = -\frac{\partial H}{\partial \vec{q}}, \quad (2.14)$$

and,

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \quad (2.15)$$

2.2.4 Hamilton's Principle

Sir William Hamilton formulated the principle that carries his name, and argued that the solution to a Hamiltonian dynamical system exists if the action integral in Eq. 2.3 is stationary between two specified endpoints in phase space: it satisfies $\delta A = 0$ (29). It can be shown that the condition for $\delta A = 0$ is equivalent to the path in phase space satisfying Eq. 2.14:

$$\delta A = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial \vec{q}} \delta \vec{q} + \frac{\partial L}{\partial \dot{\vec{q}}} \delta \dot{\vec{q}} \right) dt. \quad (2.16)$$

The second term in the integrand can be manipulated by the relationship given by the chain rule:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\vec{q}}} \delta \vec{q} \right) = \frac{\partial L}{\partial \dot{\vec{q}}} \delta \ddot{\vec{q}} + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\vec{q}}} \right) \delta \dot{\vec{q}}. \quad (2.17)$$

Substituting the above equation in Eq. 2.16 yields

$$\delta A = \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial \vec{q}} \delta \vec{q} + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\vec{q}}} \right) \delta \dot{\vec{q}} \right] dt + \frac{\partial L}{\partial \dot{\vec{q}}} \delta \vec{q} \Big|_{t_0}^{t_1}. \quad (2.18)$$

Since the endpoints are specified, it follows that $\delta \vec{q}_0 = \delta \vec{q}_1 = 0$ and Eq. 2.18 becomes

$$\delta A = \int_{t_0}^{t_1} \left[\left(\frac{\partial L}{\partial \vec{q}} + \frac{d}{dt} \frac{\partial L}{\partial \dot{\vec{q}}} \right) \delta \vec{q} \right] dt = 0, \quad (2.19)$$

and Eq. 2.19 is satisfied if

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0, \quad (2.20)$$

which is the Euler-Lagrange equation given in Eq. 2.2. As stated, the solution to the Euler-Lagrange equations correspond to the equations of motion of the system, therefore Hamilton's principle of stationary action states that the equations of motion of a dynamical system will be such that the action integral is stationary $\delta A = 0$ (29).

2.2.5 Symplectic Structure of Hamiltonian Dynamical Systems

Hamilton's equations of motion given in Eq. 2.14 can be expressed in a more compact form:

$$\dot{X} = J \frac{\partial H}{\partial X}, \quad (2.21)$$

where X is the state vector consisting of generalized coordinates and momenta, and J is the block of identity matrices:

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}. \quad (2.22)$$

The system is symplectic if the Jacobian of the system, $M = \frac{\partial x_1}{\partial x_0}$, satisfies the following equation (30):

$$M^T J M = J. \quad (2.23)$$

Due to its form, a Hamiltonian dynamical system has a symplectic structure (30).

2.2.6 Generating Functions

Generating functions transform a set of canonical variables (\vec{q}, \vec{p}, t) with a Hamiltonian $H(\vec{q}, \vec{p}, t)$ into another set of canonical variables (\vec{Q}, \vec{P}, t) with a Hamiltonian $K(\vec{Q}, \vec{P}, t)$, via canonical transformations preserving the symplectic structure of the Hamiltonian dynamical system. There are four types of generating functions, and a total of 4^n different generating functions, equally distributed among the four types.

The first type of generating function, $F_1 = F_1(\vec{q}, \vec{Q}, t)$ depends on the new and old set of generalized coordinates and satisfies the following conditions:

$$\vec{p} = \frac{\partial F_1}{\partial \vec{q}}, \quad \vec{P} = -\frac{\partial F_1}{\partial \vec{Q}}, \quad (2.24)$$

$$K = H + \frac{\partial F_1}{\partial t}. \quad (2.25)$$

The second type of generating function, $F_2(\vec{q}, \vec{P}, t) = F_1 + \vec{Q} \cdot \vec{P}$ depends on the

old generalized coordinates and new generalized momenta and satisfies the following conditions:

$$\vec{p} = \frac{\partial F_2}{\partial \vec{q}}, \quad \vec{Q} = \frac{\partial F_2}{\partial \vec{P}}, \quad (2.26)$$

$$K = H + \frac{\partial F_2}{\partial t}. \quad (2.27)$$

The third type of generating function, $F_3(\vec{p}, \vec{Q}, t) = F_1 - \vec{q} \cdot \vec{p}$ depends on the old generalized momenta and the new generalized coordinates and satisfies the following conditions:

$$\vec{q} = -\frac{\partial F_3}{\partial \vec{p}}, \quad \vec{P} = -\frac{\partial F_3}{\partial \vec{Q}}, \quad (2.28)$$

$$K = H + \frac{\partial F_3}{\partial t}. \quad (2.29)$$

The fourth type of generating function, $F_4(\vec{p}, \vec{P}, t) = F_1 - \vec{q} \cdot \vec{p} + \vec{Q} \cdot \vec{P}$ depends on the old and the new generalized momenta and satisfies the following conditions:

$$\vec{q} = -\frac{\partial F_4}{\partial \vec{p}}, \quad \vec{Q} = \frac{\partial F_4}{\partial \vec{P}}, \quad (2.30)$$

$$K = H + \frac{\partial F_4}{\partial t}. \quad (2.31)$$

2.2.7 Hamilton-Jacobi Equations

The Hamilton-Jacobi equations can be viewed as yet another reformulation of Newtonian mechanics. The solution to these equations allows to obtain the generating function of the system, which in turn can be used to obtain the equations of motion through simple differentiation.

For the first type of generating function, $F_1(\vec{q}, \vec{Q}, t)$, the corresponding Hamilton-Jacobi equation is

$$H\left(\vec{q}, \frac{\partial F_1}{\partial \vec{q}}, t\right) + \frac{\partial F_1}{\partial t} = 0. \quad (2.32)$$

For the second type of generating function, $F_2(\vec{q}, \vec{P}, t)$, the corresponding Hamilton-Jacobi equation is

$$H\left(\vec{q}, \frac{\partial F_2}{\partial \vec{q}}, t\right) + \frac{\partial F_2}{\partial t} = 0. \quad (2.33)$$

For the third type of generating function, $F_3(\vec{p}, \vec{Q}, t)$, the corresponding Hamilton-Jacobi equation is

$$H\left(-\frac{\partial F_3}{\partial p}, p, t\right) + \frac{\partial F_3}{\partial t} = 0. \quad (2.34)$$

For the fourth type of generating function, $F_4(\vec{p}, \vec{P}, t)$, the corresponding Hamilton-Jacobi equation is

$$H\left(-\frac{\partial F_4}{\partial p}, p, t\right) + \frac{\partial F_4}{\partial t} = 0. \quad (2.35)$$

2.3 Hamilton's Principal Function

Sir William Rowan Hamilton first formulated the existence of a fundamental function that fully describes the behavior of a Hamiltonian dynamical system in phase space (31), (15). This function is known as Hamilton's principal function and is intimately related to the generating function of the canonical transformation. The Hamiltonian of a dynamical system has a principal function associated with it, and the equations of motion can be obtained by simple differentiation of Hamilton's principal function (32). The principal function is built from the action integral between two specified points in phase space:

$$A = \int_{t_0}^{t_1} L(\vec{q}, \dot{\vec{q}}, t) dt. \quad (2.36)$$

A solution for a Hamiltonian dynamical system exists if the action integral in Eq.

2.36 is stationary: it satisfies $\delta A = 0$, and the corresponding path in phase space that satisfies Hamiltonian equations of motion in Eq. 2.14 (29) . Note that performing this variation also requires $\delta \vec{q} = 0$ and $\delta t = 0$, which means that

$$\delta \vec{q}(t_0) = \delta \vec{q}(t_1) = 0. \quad (2.37)$$

Hamilton's principal function is defined as the action integral evaluated along a trajectory in phase space that yields a stationary value of A ($\delta A = 0$) between two specified endpoints. Using Eq. 2.4 the action integral can be expressed in the following form:

$$W(\vec{q}_0, \vec{q}_1, t_0, t_1) = \int_{t_0}^{t_1} [\vec{p} \cdot \dot{\vec{q}} - H(\vec{q}, \vec{p})] dt, \quad (2.38)$$

where it is assumed that the generalized velocities are related to the generalized coordinates and generalized momenta, $\dot{\vec{q}} = \dot{\vec{q}}(\vec{q}, \vec{p})$. By definition, the action integral evaluated along a path that connects two specified point in phase space is equal to Hamilton's principal function, $W(\vec{q}_0, \vec{q}_1, t_0, t_1)$, where \vec{q}_0 and \vec{q}_1 are the generalized coordinates at times t_0 and t_1 respectively. These two sets of generalized coordinates are connected by a path in phase space that makes the action integral stationary. Note that Hamilton's principal function is defined by the endpoint generalized coordinates and times. In order to derive the governing equations for W , consider a variation of the action integral's that yields

$$\delta W = [\vec{p} \cdot \delta \vec{q} - H(\vec{q}, \vec{p}) \delta t]_{t_0}^{t_1}, \quad (2.39)$$

$$\delta W = \vec{p}_1 \cdot \delta \vec{q}_1 - H(\vec{q}_1, \vec{p}_1) \delta t_1 - \vec{p}_0 \cdot \delta \vec{q}_0 + H(\vec{q}_0, \vec{p}_0) \delta t_0. \quad (2.40)$$

On the other hand, by simply taking the variation of $W(\vec{q}_0, \vec{q}_1, t_0, t_1)$ the following equation can be obtained:

$$\delta W = \frac{\partial W}{\partial \vec{q}_0} \cdot \delta \vec{q}_0 + \frac{\partial W}{\partial t_0} \delta t_0 + \frac{\partial W}{\partial \vec{q}_1} \cdot \delta \vec{q}_1 + \frac{\partial W}{\partial t_1} \delta t_1. \quad (2.41)$$

Eqs. 2.40 and 2.41 can then be equated to obtain following relationships:

$$\vec{p}_0 = -\frac{\partial W}{\partial \vec{q}_0}, \quad (2.42)$$

$$\vec{p}_1 = \frac{\partial W}{\partial \vec{q}_1}, \quad (2.43)$$

and,

$$-\frac{\partial W}{\partial t_0} + H(\vec{q}_0, \vec{p}_0, t_0) = 0 \quad (2.44)$$

$$\frac{\partial W}{\partial t_1} + H(\vec{q}_1, \vec{p}_1, t_1) = 0.$$

The expressions in Eqs. 2.42, 2.43 are the two boundary conditions that the principal function must satisfy, while Eq. 2.44 shows the pair of partial differential equations that must hold true for W . Substituting for the momenta in the Hamiltonian, Eq. 2.44 becomes:

$$-\frac{\partial W}{\partial t_0} + H(\vec{q}_0, -\frac{\partial W}{\partial \vec{q}_0}, t_0) = 0 \tag{2.45}$$

$$\frac{\partial W}{\partial t_1} + H(\vec{q}_1, \frac{\partial W}{\partial \vec{q}_1}, t_1) = 0.$$

Therefore, it can be said that Hamilton's principal function transforms the initial state of the system $(\vec{q}_0, \vec{p}_0, t_0)$ into a final state $(\vec{q}_1, \vec{p}_1, t_1)$.

2.3.1 Energy Surface of Hamilton's Principal Function

Hamilton's principal function transforms the state of the system \vec{q}_0, \vec{p}_0 at a time t_0 into some later state \vec{q}_1, \vec{p}_1 at time t_1 . In addition, for a conservative system the canonical variables \vec{q}, \vec{p} are restricted to the energy surface $E = H$, and cannot leave that surface during the transformation. That is, if \vec{q}_0, \vec{q}_1 and $t_1 - t_0$ are unchanged then the energy, a direct function of these variables, is unchanged. This implies that the gradient of W with respect to the energy is zero (33):

$$\frac{\partial W}{\partial E} = 0. \tag{2.46}$$

Hamilton's principal function involves moving endpoints and an energy surface, therefore it generates a transformation that maps a point on the energy surface $E = H$ of the extended phase space to another point on the same surface. Since there exists a constraint on how Hamilton's principal function changes as a function of \vec{q}_0 and \vec{q}_1 , the following condition holds (33):

$$\left| \frac{\partial^2 W}{\partial q_{0i} \partial q_{1j}} \right| = 0. \quad (2.47)$$

2.4 Hamilton's Characteristic Function

In his study of dynamics, Hamilton proved the existence of another fundamental function that contains the solution to a dynamical system (31). This function is known as Hamilton's characteristic function and is directly related to Hamilton's principal function:

$$Q(\vec{p}_0, \vec{p}_1, t_0, t_1) = W(\vec{q}_0, \vec{q}_1, t_0, t_1) + \vec{p}_0 \vec{q}_0 - \vec{p}_1 \vec{q}_1. \quad (2.48)$$

Hamilton's characteristic function, Q , can be obtained using a Legendre transformation to change the dependent canonical variables from (\vec{q}_0, \vec{q}_1) to (\vec{p}_0, \vec{p}_1) . As with the principal function, the conditions on the characteristic function can be obtained by calculus of variations. Let the characteristic function undergo a variation:

$$\delta Q = \delta W + \vec{p}_0 \delta \vec{q}_0 - \vec{p}_1 \delta \vec{q}_1 + \vec{q}_0 \delta \vec{p}_0 - \vec{q}_1 \delta \vec{p}_1, \quad (2.49)$$

where

$$\delta W = \vec{p}_1 \delta \vec{q}_1 - \vec{p}_0 \delta \vec{q}_0 - H(\vec{q}_1, \vec{p}_1) \delta t_1 + H(\vec{q}_0, \vec{p}_0) \delta t_0. \quad (2.50)$$

Therefore, plugging the variation of the principal function in Eq. 2.50 in Eq. 2.49 yields

$$\delta Q = \vec{q}_0 \delta \vec{p}_0 - \vec{q}_1 \delta \vec{p}_1 - H(\vec{q}_1, \vec{p}_1) \delta t_1 + H(\vec{q}_0, \vec{p}_0) \delta t_0. \quad (2.51)$$

The variation of the characteristic function along the trajectory in phase space that yields a stationary action integral is

$$\delta Q = \frac{\partial Q}{\partial \vec{p}_0} \delta \vec{p}_0 + \frac{\partial Q}{\partial \vec{p}_1} \delta \vec{p}_1 + \frac{\partial Q}{\partial t_0} \delta t_0 + \frac{\partial Q}{\partial t_1} \delta t_1. \quad (2.52)$$

Therefore, equating Eq. 2.52 and Eq. 2.51 the two boundary conditions and partial differential equations that the characteristic function must satisfy are

$$\vec{q}_0 = \frac{\partial Q}{\partial \vec{p}_0}, \quad \vec{q}_1 = -\frac{\partial Q}{\partial \vec{p}_1}, \quad (2.53)$$

and

$$\frac{\partial Q}{\partial t_0} - H\left(\frac{\partial Q}{\partial \vec{p}_0}, \vec{p}_0, t_0\right) = 0, \quad \frac{\partial Q}{\partial t_1} + H\left(-\frac{\partial Q}{\partial \vec{p}_1}, \vec{p}_1, t_1\right) = 0. \quad (2.54)$$

2.5 Analogy between Principal, Characteristic, and Generating Functions

Hamilton's principal function, Hamilton's characteristic function, and the generating functions for the canonical transformation are intimately related to one another (34). The characteristic function arises from a Legendre transformation of the principal function (31). This is analogous to the first and fourth type of generating functions, which differ by a Legendre transformation.

The transformation from the principal function to the characteristic function changes the independent variables of the fundamental function from the endpoint generalized coordinates to the endpoint generalized momenta. Similarly, the first and fourth type of generating functions differ by a Legendre function that changes the function from generalized coordinates to generalized momenta.

However, there is a fundamental difference between Hamilton's principal and characteristic functions and the generating functions. Both the principal function and the characteristic function describe the system from one endpoint to another in phase space. Since both functions satisfy the equations of motion of the Hamiltonian dynamical system, the endpoints are dynamic, and can be change without affecting the structure of the functions. In essence, Hamilton's principal function and characteristic function transform the system from a point in phase space to another point that satisfies the same equations of motion and Hamiltonian.

The generating function of the canonical transformation transforms the system from a set of canonical variables to another set of canonical variables that might have different equations of motion and a different Hamiltonian. The transformation is also

time dependent.

The analogy between the generating function and principal and characteristic functions is that they all transform between two sets of canonical variables. However, the principal and characteristic functions transform one point in phase space to another under the same Hamiltonian, while the generating function transforms between two set of different variables that have different Hamiltonians. Furthermore, the principal and characteristic functions involve dynamic endpoints that can go forwards or backwards in time, while the generating function only goes forward in time.

CHAPTER III

Perturbation Theory for Hamilton's Principal Function

Hamilton developed a perturbation theory for a general Hamiltonian dynamical system (35). However, Hamilton's perturbation method is developed for first order solutions and makes assumptions about the system while not stated in general terms. Consider the generalized coordinates and momenta of an n-dimensional Hamiltonian dynamical system:

$$\begin{aligned}\vec{q} &= \sum_{i=1}^n q_i \delta q_i, \\ \vec{p} &= \sum_{i=1}^n p_i \delta p_i,\end{aligned}\tag{3.1}$$

where δq_i and δp_i are the i^{th} direction components of \vec{q} and \vec{p} respectively. Following the Einstein convention, the δq_i 's and δp_i 's are dropped. The state vector is defined as $\vec{x} = [\vec{q} \ \vec{p}]$. Let the i^{th} component of the state vector be defined as $X_i = (q_i, p_i)$.

3.1 Perturbation Theory for Hamilton's Principal Function

An unperturbed system with a Hamiltonian function $H^{(0)}(x_i, t)$ has a solution that is provided by Hamilton's principal function $W^{(0)}(q_{0i}, q_{1i}, t_0, t_1)$ (15). A perturbing force that preserves the Hamiltonian structure of the system results in a dynamical system with a Hamiltonian $H(x_i, t) = H^{(0)}(x_i, t) + \epsilon H^{(1)}(x_i, t)$, where ϵ is a small parameter (35).

Due to basic existence theorems the system $H(x_i, t) = H^{(0)}(x_i, t) + \epsilon H^{(1)}(x_i, t)$ has a solution defined by a principal function $W(q_{0i}, q_{1i}, t_0, t_1)$. A Taylor series expansion of this solution assuming ϵ can be arbitrarily small, allows the principal function to take the form

$$W = W^{(0)} + \epsilon W^{(1)} + \epsilon^2 W^{(2)} + \dots + \epsilon^n W^{(n)} + \dots, \quad (3.2)$$

where $W^{(\alpha)}$ is the α order element of the principal function corresponding to the perturbed system. By definition the full principal function satisfies the boundary conditions

$$p_{0i} = -\frac{\partial W}{\partial q_{0i}}, \quad p_{1i} = \frac{\partial W}{\partial q_{1i}}. \quad (3.3)$$

Applying this to the perturbed W the perturbed momenta can be defined as

$$p_i = p_i^{(0)} + \epsilon p_i^{(1)} + \epsilon^2 p_i^{(2)} + \dots + \epsilon^n p_i^{(n)} + \dots, \quad (3.4)$$

balancing orders of ϵ the momenta and principal function are related at each order:

$$p_{0i}^{(\alpha)} = -\frac{\partial W^{(\alpha)}}{\partial q_{0i}}, \quad p_{1i}^{(\alpha)} = -\frac{\partial W^{(\alpha)}}{\partial q_{1i}}, \quad (3.5)$$

where $p_i^{(\alpha)}$ is the α order element of the momenta. The Hamiltonian of the system is a function of $x_i = (q_i, p_i)$ and can be expressed as the following:

$$\begin{aligned} H(x_i, t) &= H^{(0)}(x_i, t) + \epsilon H^{(1)}(x_i, t), \\ &= H^{(0)}\left(q_i, p_i^{(0)} + \Delta p_i, t\right) + \epsilon H^{(1)}\left(q_i, p_i^{(0)} + \Delta p_i, t\right), \end{aligned} \quad (3.6)$$

where $\Delta p_i = \sum_{\alpha=1}^N \epsilon^\alpha p_i^{(\alpha)}$. Therefore the Hamiltonian of the system can be expanded in a Taylor series about the principal function of the unperturbed system. Using the Einstein Summation Convention this expansion is expressed as

$$\begin{aligned} H(x_i, t) &= H^{(0)}(x_i^{(0)}, t) + \sum_{\alpha=1}^N \frac{1}{\alpha!} \frac{\partial^\alpha H^{(0)}}{\partial p_j \partial p_k \partial p_l \dots} \Big|_{x_m^{(0)}} \Delta p_j \Delta p_k \Delta p_l \dots, \\ &+ \epsilon H^{(1)}(x_i^{(0)}, t) + \epsilon \sum_{\alpha=1}^N \frac{1}{\alpha!} \frac{\partial^\alpha H^{(1)}}{\partial p_j \partial p_k \partial p_l \dots} \Big|_{x_m^{(0)}} \Delta p_j \Delta p_k \Delta p_l \dots, \end{aligned} \quad (3.7)$$

The expression in Eq. 3.7 can be collected in powers of ϵ :

$$\begin{aligned}
H(x_i, t) = & H^{(0)}(x_i^{(0)}(t), t) + \epsilon \left[\frac{\partial H^{(0)}}{\partial p_j} \Big|_{x_m^{(0)}} p_j^{(1)} + H^{(1)}(x_i^{(0)}(t), t) \right] + \epsilon^2 \left[\frac{\partial H^{(0)}}{\partial p_j} \Big|_{x_m^{(0)}} p_j^{(2)} + \right. \\
& + \frac{1}{2!} \frac{\partial^2 H^{(0)}}{\partial p_j \partial p_k} \Big|_{x_m^{(0)}} p_j^{(1)} p_k^{(1)} + \left. \frac{\partial H^{(1)}}{\partial p_j} \Big|_{x_m^{(0)}} p_j^{(1)} \right] + \epsilon^3 \left[\frac{\partial H^{(0)}}{\partial p_j} \Big|_{x_m^{(0)}} p_j^{(3)} + \frac{\partial^2 H^{(0)}}{\partial p_j \partial p_k} \Big|_{x_m^{(0)}} p_j^{(1)} p_k^{(2)} + \right. \\
& + \frac{1}{6} \frac{\partial^3 H^{(0)}}{\partial p_j \partial p_k \partial p_l} \Big|_{x_m^{(0)}} p_j^{(1)} p_k^{(1)} p_l^{(1)} + \frac{\partial H^{(1)}}{\partial p_j} \Big|_{x_m^{(0)}} p_j^{(2)} + \left. \frac{1}{2} \frac{\partial^2 H^{(1)}}{\partial p_j \partial p_k} \Big|_{x_m^{(0)}} p_j^{(2)} p_k^{(2)} \right] + \\
& + \epsilon^4 \left[\frac{\partial H^{(0)}}{\partial p_j} \Big|_{x_m^{(0)}} p_j^{(4)} + \frac{\partial^2 H^{(0)}}{\partial p_j \partial p_k} \Big|_{x_m^{(0)}} p_j^{(1)} p_k^{(3)} + \frac{1}{2} \frac{\partial^2 H^{(0)}}{\partial p_j \partial p_k} \Big|_{x_m^{(0)}} p_j^{(1)} p_k^{(1)} + \right. \\
& + \frac{1}{2} \frac{\partial^3 H^{(0)}}{\partial p_j \partial p_k \partial p_l} \Big|_{x_m^{(0)}} p_j^{(1)} p_k^{(1)} p_l^{(2)} + \frac{1}{24} \frac{\partial^4 H^{(0)}}{\partial p_j \partial p_k \partial p_l \partial p_m} \Big|_{x_m^{(0)}} p_j^{(1)} p_k^{(1)} p_l^{(1)} p_m^{(1)} + \frac{\partial H^{(1)}}{\partial p_j} \Big|_{x_m^{(0)}} p_j^{(3)} + \\
& \left. \frac{\partial^2 H^{(1)}}{\partial p_j \partial p_k} \Big|_{x_m^{(0)}} p_j^{(1)} p_k^{(2)} + \frac{1}{6} \frac{\partial^3 H^{(0)}}{\partial p_j \partial p_k \partial p_l} \Big|_{x_m^{(0)}} p_j^{(1)} p_k^{(1)} p_l^{(1)} \right] + \dots \\
& \dots + \epsilon^n \left[\frac{\partial H^{(0)}}{\partial p_j} \Big|_{x_m^{(0)}} p_j^{(n)} + \frac{\partial H^{(1)}}{\partial p_j} \Big|_{x_m^{(0)}} p_j^{(n-1)} + \dots \right] + \dots
\end{aligned} \tag{3.8}$$

The action integral of the system is now

$$\begin{aligned}
W^{(0)} + \epsilon W^{(1)} + \dots + \epsilon^n W^{(n)} + \dots = & \int_{t_0}^{t_1} \left(p_i^{(0)} + \epsilon p_i^{(1)} + \dots + \epsilon^n p_i^{(n)} + \dots \right) dq_i \\
& - \int_{t_0}^{t_1} H \left(q_i, p_i^{(0)} + \epsilon p_i^{(1)} + \dots + \epsilon^n p_i^{(n)} + \dots, t \right) dt,
\end{aligned} \tag{3.9}$$

where the Hamiltonian is as expressed by Eq. 3.8. Taking the variation in end points and time of Eq. 3.9 under the assumption that the trajectory satisfies the Hamiltonian

equations of motion for the perturbed system leads to

$$\begin{aligned} \delta W = & - \left(p_{0i}^{(0)} + \dots + \epsilon^n p_{0i}^{(n)} + \dots \right) \delta q_{0i} + \left(p_{1i}^{(0)} + \dots + \epsilon^n p_{1i}^{(n)} + \dots \right) \delta q_{1i} \\ & + H \left(q_{0i}, p_{0i}^{(0)} + \dots + \epsilon^n p_{0i}^{(n)} + \dots, t \right) \delta t_0 - H \left(q_{1i}, p_{1i}^{(0)} + \dots + \epsilon^n p_{1i}^{(n)} + \dots, t \right) \delta t_1. \end{aligned} \quad (3.10)$$

However, it also known that the variation of the principal function can be expressed as:

$$\delta W = \frac{\partial W}{\partial q_0} \delta q_0 + \frac{\partial W}{\partial t_0} \delta t_0 + \frac{\partial W}{\partial q_1} \delta q_1 + \frac{\partial W}{\partial t_1} \delta t_1. \quad (3.11)$$

Equating the δt_0 and δt_1 in Eqs. 3.10, 3.11 leads to the following expressions:

$$\frac{\partial W^{(0)}}{\partial t_0} + \dots + \epsilon^n \frac{\partial W^{(n)}}{\partial t_0} + \dots - H(q_{0i}, p_{0i}^{(0)} + \dots + \epsilon^n p_{0i}^{(n)} + \dots, t_0) = 0, \quad (3.12)$$

$$\frac{\partial W^{(0)}}{\partial t_1} + \dots + \epsilon^n \frac{\partial W^{(n)}}{\partial t_1} + \dots + H(q_{1i}, p_{1i}^{(0)} + \dots + \epsilon^n p_{1i}^{(n)} + \dots, t_1) = 0,$$

where by definition the momenta at each order is:

$$\begin{aligned}
p_{0i}^{(0)} &= -\frac{\partial W^{(0)}}{\partial q_{0i}}, & p_{1i}^{(0)} &= \frac{\partial W^{(0)}}{\partial q_{1i}}, \\
&\dots & &\dots \\
&&& (3.13) \\
p_{0i}^{(n)} &= -\frac{\partial W^{(n)}}{\partial q_{0i}}, & p_{1i}^{(n)} &= \frac{\partial W^{(n)}}{\partial q_{1i}}, \\
&\dots & &\dots
\end{aligned}$$

Using the expanded Hamiltonian in Eq. 3.8, Eq. 3.12 can be rewritten as

$$\begin{aligned}
&\frac{\partial W^{(0)}}{\partial t_0} - H^{(0)}(x_{0i}^{(0)}, t) + \epsilon \left[\frac{\partial W^{(1)}}{\partial t_0} - \frac{\partial H^{(0)}}{\partial p_{0j}} \Big|_{x_{0m}^{(0)}} p_{0j}^{(1)} - H^{(0)}(x_{0i}^{(0)}, t) \right] \\
&+ \epsilon^2 \left[\frac{\partial W^{(2)}}{\partial t_0} - \frac{\partial H^{(0)}}{\partial p_{0j}} \Big|_{x_{0m}^{(0)}} p_{0j}^{(2)} - \frac{1}{2!} \frac{\partial^2 H^{(0)}}{\partial p_{0j} \partial p_{0k}} \Big|_{x_{0m}^{(0)}} p_{0j}^{(1)} p_{0k}^{(1)} - \frac{\partial H^{(1)}}{\partial q_{0j}} \Big|_{x_{0m}^{(0)}} p_{0j}^{(1)} \right] + \dots = 0,
\end{aligned}$$

and, (3.14)

$$\begin{aligned}
&\frac{\partial W^{(0)}}{\partial t_1} + H^{(0)}(x_{1i}^{(0)}, t) + \epsilon \left[\frac{\partial W^{(1)}}{\partial t_1} + \frac{\partial H^{(0)}}{\partial p_{1j}} \Big|_{x_{1m}^{(0)}} p_{1j}^{(1)} + H^{(1)}(x_{1i}^{(0)}, t) \right] \\
&+ \epsilon^2 \left[\frac{\partial W^{(2)}}{\partial t_1} + \frac{\partial H^{(0)}}{\partial p_{1j}} \Big|_{x_{1m}^{(0)}} p_{1j}^{(2)} + \frac{1}{2!} \frac{\partial^2 H^{(0)}}{\partial p_{1j} \partial p_{1k}} \Big|_{x_{1m}^{(0)} p_{1j}^{(1)}} p_{1k}^{(1)} + \frac{\partial H^{(1)}}{\partial p_{1j}} \Big|_{x_{1m}^{(0)}} p_{1j}^{(1)} \right] + \dots = 0.
\end{aligned}$$

From the unperturbed system it is known that the nominal solution $W^{(0)}$ satisfies

$$\frac{\partial W^{(0)}}{\partial t_0} - H^{(0)}(x_{0i}^{(0)}, t) = 0, \quad \frac{\partial W^{(0)}}{\partial t_1} + H^{(0)}(x_{1i}^{(0)}, t) = 0. \quad (3.15)$$

Hence, Eq. 3.14 yields the following expressions for increasing orders of ϵ :

$$\begin{aligned} \epsilon^1 : \quad & \frac{\partial W^{(1)}}{\partial t_0} - \frac{\partial H^{(0)}}{\partial p_{0j}} \Big|_{x_{0m}^{(0)}} p_{0j}^{(1)} - H^{(1)}(x_{0i}^{(0)}, t) = 0, \\ & \frac{\partial W^{(1)}}{\partial t_1} + \frac{\partial H^{(0)}}{\partial p_{1j}} \Big|_{x_{1m}^{(0)}} p_{1j}^{(1)} + H^{(1)}(x_{1i}^{(0)}, t) = 0, \\ \epsilon^2 : \quad & \frac{\partial W^{(2)}}{\partial t_0} - \frac{\partial H^{(0)}}{\partial p_{0j}} \Big|_{x_{0m}^{(0)}} p_{0j}^{(2)} - \frac{1}{2!} \frac{\partial^2 H^{(0)}}{\partial p_{0j} \partial p_{0k}} \Big|_{x_{0m}^{(0)}} p_{0j}^{(1)} p_{0k}^{(1)} - \frac{\partial H^{(1)}}{\partial p_{0j}} \Big|_{x_{0m}^{(0)}} p_{0j}^{(1)} = 0, \\ & \frac{\partial W^{(2)}}{\partial t_1} + \frac{\partial H^{(0)}}{\partial p_{1j}} \Big|_{x_{1m}^{(0)}} p_{1j}^{(2)} + \frac{1}{2!} \frac{\partial^2 H^{(0)}}{\partial p_{1j} \partial p_{1k}} \Big|_{x_{1m}^{(0)}} p_{1j}^{(1)} p_{1k}^{(1)} + \frac{\partial H^{(1)}}{\partial p_{1j}} \Big|_{x_{1m}^{(0)}} p_{1j}^{(1)} = 0, \\ & \dots \\ & \dots \end{aligned} \quad (3.16)$$

Following Hamilton, note that the total derivative of $W^{(\alpha)}$ with respect to t_0 and t_1 , can be expressed as the following (36):

$$\frac{dW^{(\alpha)}}{dt_0} = \frac{\partial W^{(\alpha)}}{\partial t_0} + \frac{\partial W^{(\alpha)}}{\partial q_{0i}} \frac{dq_{0i}}{dt_0}, \quad (3.17)$$

$$\frac{dW^{(\alpha)}}{dt_1} = \frac{\partial W^{(\alpha)}}{\partial t_1} + \frac{\partial W^{(\alpha)}}{\partial q_{1i}} \frac{dq_{1i}}{dt_1}.$$

Where q_{0i} and q_{1i} lie along the nominal trajectory. By definition $\frac{dq_{0i}}{dt_0} = \frac{\partial H}{\partial p_{0i}} \Big|_{x_{0m}^{(0)}}$ and $\frac{dq_{1i}}{dt_1} = \frac{\partial H}{\partial p_{1i}} \Big|_{x_{1m}^{(0)}}$ since the expansion is about this nominal solution. Also from Eq. 3.5 $\frac{\partial W^{(\alpha)}}{\partial q_{0i}} = -p_{0i}^{(\alpha)}$ and $\frac{\partial W^{(\alpha)}}{\partial q_{1i}} = p_{1i}^{(\alpha)}$, therefore Eq. 3.17 becomes

$$\frac{dW^{(\alpha)}}{dt_0} = \frac{\partial W^{(\alpha)}}{\partial t_0} - p_{0i}^{(\alpha)} \frac{\partial H}{\partial p_{0i}} \Big|_{x_{0m}^{(0)}}, \quad (3.18)$$

$$\frac{dW^{(\alpha)}}{dt_1} = \frac{\partial W^{(\alpha)}}{\partial t_1} + p_{1i}^{(\alpha)} \frac{\partial H}{\partial p_{1i}} \Big|_{x_{1m}^{(0)}},$$

where $\frac{\partial H}{\partial p_{0i}} \Big|_{x_{0m}^{(0)}}$ is a function of $(q_{0i}, p_{0i}^{(0)})$ and $\frac{\partial H}{\partial p_{1i}} \Big|_{x_{0m}^{(0)}}$ is a function of $(q_{1i}, p_{1i}^{(0)})$. Therefore, by substituting the right hand side of Eq. 3.18 into Eq. 3.16 total time derivative expressions for the $W^{(\alpha)}$'s are

$$\begin{aligned}
\epsilon^1 : \quad & \frac{dW^{(1)}}{dt_0} - H^{(1)}(x_{0i}^{(0)}, t) = 0, \\
& \frac{dW^{(1)}}{dt_1} + H^{(1)}(x_{1i}^{(0)}, t) = 0, \\
\epsilon^2 : \quad & \frac{dW^{(2)}}{dt_0} - \frac{1}{2!} \frac{\partial^2 H^{(0)}}{\partial p_{0j} \partial p_{0k}} \Big|_{x_{0m}^{(0)}} p_{0j}^{(1)} p_{0k}^{(1)} - \frac{\partial H^{(1)}}{\partial p_{0j}} \Big|_{x_{0m}^{(0)}} p_{0j}^{(1)} = 0, \\
& \frac{dW^{(2)}}{dt_1} + \frac{1}{2!} \frac{\partial^2 H^{(0)}}{\partial p_{1j} \partial p_{1k}} \Big|_{x_{1m}^{(0)}} p_{1j}^{(1)} p_{1k}^{(1)} + \frac{\partial H^{(1)}}{\partial p_{1j}} \Big|_{x_{1m}^{(0)}} p_{1j}^{(1)} = 0, \\
& \dots \\
& \dots \\
\epsilon^n : \quad & \frac{dW^{(n)}}{dt_0} = F_0^{(n)}(x_{0i}^{(0)}, W^{(0)}, W^{(1)}, \dots, W^{(n-1)}), \\
& \frac{dW^{(n)}}{dt_1} = F_1^{(n)}(x_{1i}^{(0)}, W^{(0)}, W^{(1)}, \dots, W^{(n-1)}), \\
& \dots \\
& \dots
\end{aligned} \tag{3.19}$$

As it can be seen, at each order $W^{(\alpha)}$ apparently needs to satisfy two equations simultaneously. Hence:

$$\begin{aligned}
\epsilon^1 : \quad W^{(1)} &= \int_{t_1}^{t_0} H^{(1)}(x_i^{(0)}(t), t) dt, \\
W^{(1)} &= - \int_{t_0}^{t_1} H^{(1)}(x_i^{(0)}(t), t) dt, \\
\epsilon^2 : \quad W^{(2)} &= \int_{t_1}^{t_0} \left[\frac{1}{2} \frac{\partial^2 H^{(0)}}{\partial p_j \partial p_k} \Big|_{x_m^{(0)}} p_j^{(1)} p_k^{(1)} + \frac{\partial H^{(1)}}{\partial p_j} \Big|_{x_m^{(0)}} p_j^{(1)} \right] dt, \\
W^{(2)} &= - \int_{t_0}^{t_1} \left[\frac{1}{2} \frac{\partial^2 H^{(0)}}{\partial p_j \partial p_k} \Big|_{x_m^{(0)}} p_j^{(1)} p_k^{(1)} + \frac{\partial H^{(1)}}{\partial p_j} \Big|_{x_m^{(0)}} p_j^{(1)} \right] dt, \\
&\dots \\
&\dots \\
\epsilon^n : \quad W^{(n)} &= - \int_{t_1}^{t_0} F^{(n)}(x_i^{(0)}(t), t, W^{(0)}, \dots, W^{(n-1)}) dt, \\
W^{(n)} &= \int_{t_0}^{t_1} F^{(n)}(x_i^{(0)}(t), t, W^{(0)}, \dots, W^{(n-1)}) dt, \\
&\dots \\
&\dots
\end{aligned} \tag{3.20}$$

However, these two simultaneous equations are one and the same, as can be seen by reversing the limits of integration. Hence, at each order $W^{(\alpha)}$ is defined by only one equation:

$$\begin{aligned}
\epsilon^1 : \quad W^{(1)} &= - \int_{t_0}^{t_1} H^{(1)}(x_i^{(0)}(t), t) dt, \\
\epsilon^2 : \quad W^{(2)} &= - \int_{t_0}^{t_1} \left[\frac{1}{2} \frac{\partial^2 H^{(0)}}{\partial p_j \partial p_k} \Big|_{x_m^{(0)}} p_j^{(1)} p_k^{(1)} + \frac{\partial H^{(1)}}{\partial p_j} \Big|_{x_m^{(0)}} p_j^{(1)} \right] dt, \\
&\dots \\
&\dots \\
\epsilon^n : \quad W^{(n)} &= \int_{t_0}^{t_1} F^{(n)}(x_i^{(0)}(t), t, W^{(0)}, \dots, W^{(n-1)}) dt.
\end{aligned} \tag{3.21}$$

These equations allow to recursively solve for the principal function of the full system once the nominal solution $W^{(0)}$ is known, since at each order $W^{(\alpha)}$ is a function of $W^{(0)}, W^{(1)}, \dots, W^{(\alpha-1)}$. In order to solve the two-point boundary value problem, the partial derivatives of W with respect to q_{0i} and q_{1i} need to be taken. However this requires the knowledge of an analytical expression for state vector of the nominal system which is not always possible.

3.2 Perturbation Theory for Hamilton's Characteristic Function

In his study of dynamics, Hamilton proved the existence of another fundamental function that contains the solution to a dynamical system (31). This function is known as Hamilton's characteristic function and is directly related to Hamilton's principal function:

$$Q(p_{0i}, p_{1i}, t_0, t_1) = W(q_{0i}, q_{1i}, t_0, t_1) + p_{0i}q_{0i} - p_{1i}q_{1i}. \quad (3.22)$$

A Taylor series expansion of this solution assuming ϵ can be arbitrarily small, allows the characteristic function take the form

$$Q = Q^{(0)} + \epsilon Q^{(1)} + \epsilon^2 Q^{(2)} + \dots + \epsilon^n Q^{(n)} + \dots, \quad (3.23)$$

where $Q^{(\alpha)}$ is the α order element of the principal function corresponding to the perturbed system. By definition the full principal function satisfies the boundary conditions

$$q_{0i} = \frac{\partial Q}{\partial p_{0i}} \quad q_{1i} = -\frac{\partial Q}{\partial p_{1i}}. \quad (3.24)$$

Applying this to the perturbed Q the generalized coordinates can be defined as

$$q_i = q_i^{(0)} + \epsilon q_i^{(1)} + \epsilon^2 q_i^{(2)} + \dots + \epsilon^n q_i^{(n)} + \dots, \quad (3.25)$$

balancing orders of ϵ leads to

$$q_{0i}^{(\alpha)} = \frac{\partial Q^{(\alpha)}}{\partial p_{0i}}, \quad q_{1i}^{(\alpha)} = -\frac{\partial Q^{(\alpha)}}{\partial p_{1i}}, \quad (3.26)$$

where $q_i^{(\alpha)}$ is the α order element of the momenta. The Hamiltonian of the system is a function of $x_i = (q_i, p_i)$ and can be expressed as the following:

$$\begin{aligned}
 H(x_i, t) &= H^{(0)}(x_i, t) + \epsilon H^{(1)}(x_i, t), \\
 &= H^{(0)}\left(q_i^{(0)} + \Delta q_i, p_i, t\right) + \epsilon H^{(1)}\left(q_i^{(0)} + \Delta q_i, p_i, t\right),
 \end{aligned}
 \tag{3.27}$$

where the change in the generalized coordinates is $\Delta q_i = \sum_{\alpha=1}^N \epsilon^\alpha q_i^{(\alpha)}$. Therefore the Hamiltonian of the system can be expanded in a Taylor series about the principal function of the unperturbed system. Using the Einstein Summation Convention this expansion is expressed as

$$\begin{aligned}
 H(x_i, t) &= H^{(0)}(x_i^{(0)}, t) + \sum_{\alpha=1}^N \frac{1}{\alpha!} \frac{\partial^\alpha H^{(0)}}{\partial q_j \partial q_k \partial q_l \dots} \Big|_{x_m^{(0)}} \Delta q_j \Delta q_k \Delta q_l \dots, \\
 &+ \epsilon H^{(1)}(x_i^{(0)}, t) + \epsilon \sum_{\alpha=1}^N \frac{1}{\alpha!} \frac{\partial^\alpha H^{(1)}}{\partial q_j \partial q_k \partial q_l \dots} \Big|_{x_m^{(0)}} \Delta q_j \Delta q_k \Delta q_l \dots
 \end{aligned}
 \tag{3.28}$$

The expression in Eq. 3.28 can be collected in powers of ϵ :

$$\begin{aligned}
H(x_i, t) = & H^{(0)}(x_i^{(0)}(t), t) + \epsilon \left[\frac{\partial H^{(0)}}{\partial q_j} \Big|_{x_m^{(0)}} q_j^{(1)} + H^{(1)}(x_i^{(0)}(t), t) \right] + \epsilon^2 \left[\frac{\partial H^{(0)}}{\partial q_j} \Big|_{x_m^{(0)}} q_j^{(2)} + \right. \\
& + \frac{1}{2!} \frac{\partial^2 H^{(0)}}{\partial q_j \partial q_k} \Big|_{x_m^{(0)}} q_j^{(1)} q_k^{(1)} + \left. \frac{\partial H^{(1)}}{\partial q_j} \Big|_{x_m^{(0)}} q_j^{(1)} \right] + \epsilon^3 \left[\frac{\partial H^{(0)}}{\partial q_j} \Big|_{x_m^{(0)}} q_j^{(3)} + \frac{\partial^2 H^{(0)}}{\partial q_j \partial q_k} \Big|_{x_m^{(0)}} q_j^{(1)} q_k^{(2)} + \right. \\
& + \frac{1}{6} \frac{\partial^3 H^{(0)}}{\partial q_j \partial q_k \partial q_l} \Big|_{x_m^{(0)}} q_j^{(1)} q_k^{(1)} q_l^{(1)} + \frac{\partial H^{(1)}}{\partial q_j} \Big|_{x_m^{(0)}} q_j^{(2)} + \frac{1}{2} \frac{\partial^2 H^{(1)}}{\partial q_j \partial q_k} \Big|_{x_m^{(0)}} q_j^{(2)} q_k^{(2)} \left. \right] + \\
& + \epsilon^4 \left[\frac{\partial H^{(0)}}{\partial q_j} \Big|_{x_m^{(0)}} q_j^{(4)} + \frac{\partial^2 H^{(0)}}{\partial q_j \partial q_k} \Big|_{x_m^{(0)}} q_j^{(1)} q_k^{(3)} + \frac{1}{2} \frac{\partial^2 H^{(0)}}{\partial q_j \partial q_k} \Big|_{x_m^{(0)}} q_j^{(1)} q_k^{(1)} + \right. \\
& + \frac{1}{2} \frac{\partial^3 H^{(0)}}{\partial q_j \partial q_k \partial q_l} \Big|_{x_m^{(0)}} q_j^{(1)} q_k^{(1)} q_l^{(2)} + \frac{1}{24} \frac{\partial^4 H^{(0)}}{\partial q_j \partial q_k \partial q_l \partial q_m} \Big|_{x_m^{(0)}} q_j^{(1)} q_k^{(1)} q_l^{(1)} q_m^{(1)} + \frac{\partial H^{(1)}}{\partial q_j} \Big|_{x_m^{(0)}} q_j^{(3)} + \\
& \left. \frac{\partial^2 H^{(1)}}{\partial q_j \partial q_k} \Big|_{x_m^{(0)}} q_j^{(1)} q_k^{(2)} + \frac{1}{6} \frac{\partial^3 H^{(0)}}{\partial q_j \partial q_k \partial q_l} \Big|_{x_m^{(0)}} q_j^{(1)} q_k^{(1)} q_l^{(1)} \right] + \dots \\
& \dots + \epsilon^n \left[\frac{\partial H^{(0)}}{\partial q_j} \Big|_{x_m^{(0)}} q_j^{(n)} + \frac{\partial H^{(1)}}{\partial q_j} \Big|_{x_m^{(0)}} q_j^{(n-1)} + \dots \right] + \dots
\end{aligned} \tag{3.29}$$

The action integral of the system can be expressed as

$$\begin{aligned}
Q^{(0)} + \epsilon Q^{(1)} + \dots + \epsilon^n Q^{(n)} + \dots = & \int_{t_0}^{t_1} p_i d \left(q_i + \epsilon q_i^{(1)} + \dots + \epsilon^n q_i^{(n)} \right) \\
& - \int_{t_0}^{t_1} H \left(q_i^{(0)} + q_i^{(1)} + \dots + \epsilon^n q_i^{(n)} + \dots, p_i, t \right) dt + p_{0i} q_{0i} - p_{1i} q_{1i},
\end{aligned} \tag{3.30}$$

where the Hamiltonian is as expressed by Eq. 3.29. Taking the variation in end points and time of Eq. 3.9 under the assumption that the trajectory satisfies the Hamiltonian equations of motion for the perturbed system:

$$\begin{aligned}
\delta Q = & -p_{0i}\delta\left(q_{0i}^{(0)} + \dots + \epsilon^n q_{0i}^{(n)} + \dots\right) + p_{1i}\delta\left(q_{1i}^{(0)} + \dots + \epsilon^n q_{1i}^{(n)} + \dots\right) \\
& + H\left(q_{0i}^{(0)} + \dots + \epsilon^n q_{0i}^{(n)} + \dots, p_{0i}, t_0\right)\delta t_0 - H\left(q_{1i}^{(0)} + \dots + \epsilon^n q_{1i}^{(n)} + \dots, p_{1i}, t_1\right)\delta t_1 \\
& + p_{0i}\delta\left(q_{0i}^{(0)} + \dots + \epsilon^n q_{0i}^{(n)} + \dots\right) - p_{1i}\delta\left(q_{1i}^{(0)} + \dots + \epsilon^n q_{1i}^{(n)} + \dots\right) \\
& + \left(q_{0i}^{(0)} + \epsilon q_{0i}^{(1)} + \dots + \epsilon^n q_{0i}^{(n)} + \dots\right)\delta p_{0i} - \left(q_{1i}^{(0)} + \epsilon q_{1i}^{(1)} + \dots + \epsilon^n q_{1i}^{(n)} + \dots\right)\delta p_{1i}.
\end{aligned} \tag{3.31}$$

However, the variation of the characteristic function can also be expressed as:

$$\delta Q = \frac{\partial Q}{\partial p_0}\delta p_0 + \frac{\partial Q}{\partial t_0}\delta t_0 + \frac{\partial Q}{\partial p_1}\delta p_1 + \frac{\partial Q}{\partial t_1}\delta t_1. \tag{3.32}$$

Equating the δt_0 and δt_1 in Eqs. 3.31, 3.32 terms leads to the following expressions:

$$\frac{\partial Q^{(0)}}{\partial t_0} + \dots + \epsilon^n \frac{\partial Q^{(n)}}{\partial t_0} + \dots - H\left(q_{0i}^{(0)} + \dots + \epsilon^n q_{0i}^{(n)} + \dots, p_{0i}, t_0\right) = 0, \tag{3.33}$$

$$\frac{\partial Q^{(0)}}{\partial t_1} + \dots + \epsilon^n \frac{\partial Q^{(n)}}{\partial t_1} + \dots + H\left(q_{1i}^{(0)} + \dots + \epsilon^n q_{1i}^{(n)} + \dots, p_{1i}, t_1\right) = 0,$$

where by definition the generalized coordinate is

$$\begin{aligned}
q_{0i}^{(0)} &= \frac{\partial Q^{(0)}}{\partial p_{0i}}, & q_{1i}^{(0)} &= -\frac{\partial Q^{(0)}}{\partial p_{1i}}, \\
&\dots & &\dots \\
q_{0i}^{(n)} &= \frac{\partial Q^{(n)}}{\partial p_{0i}}, & q_{1i}^{(n)} &= -\frac{\partial Q^{(n)}}{\partial p_{1i}}, \\
&\dots & &\dots
\end{aligned} \tag{3.34}$$

Using the expanded Hamiltonian in Eq. 3.29, Eq. 3.33 can be rewritten as

$$\begin{aligned}
&\frac{\partial Q^{(0)}}{\partial t_0} - H^{(0)}(x_{0i}^{(0)}, t) + \epsilon \left[\frac{\partial Q^{(1)}}{\partial t_0} - \frac{\partial H^{(0)}}{\partial q_{0j}} \Big|_{x_{0m}^{(0)}} q_{0j}^{(1)} - H^{(0)}(x_{0i}^{(0)}, t) \right] \\
&+ \epsilon^2 \left[\frac{\partial Q^{(2)}}{\partial t_0} - \frac{\partial H^{(0)}}{\partial q_{0j}} \Big|_{x_{0m}^{(0)}} q_{0j}^{(2)} - \frac{1}{2!} \frac{\partial^2 H^{(0)}}{\partial q_{0j} \partial q_{0k}} \Big|_{x_{0m}^{(0)}} q_{0j}^{(1)} q_{0k}^{(1)} - \frac{\partial H^{(1)}}{\partial p_{0j}} \Big|_{x_{0m}^{(0)}} q_{0j}^{(1)} \right] + \dots = 0,
\end{aligned}$$

and (3.35)

$$\begin{aligned}
&\frac{\partial Q^{(0)}}{\partial t_1} + H^{(0)}(x_{1i}^{(0)}, t) + \epsilon \left[\frac{\partial Q^{(1)}}{\partial t_1} + \frac{\partial H^{(0)}}{\partial q_{1j}} \Big|_{x_{1m}^{(0)}} q_{1j}^{(1)} + H^{(1)}(x_{1i}^{(0)}, t) \right] \\
&+ \epsilon^2 \left[\frac{\partial Q^{(2)}}{\partial t_1} + \frac{\partial H^{(0)}}{\partial q_{1j}} \Big|_{x_{1m}^{(0)}} q_{1j}^{(2)} + \frac{1}{2!} \frac{\partial^2 H^{(0)}}{\partial q_{1j} \partial q_{1k}} \Big|_{x_{1m}^{(0)}} q_{1j}^{(1)} q_{1k}^{(1)} + \frac{\partial H^{(1)}}{\partial p_{1j}} \Big|_{x_{1m}^{(0)}} q_{1j}^{(1)} \right] + \dots = 0.
\end{aligned}$$

From the unperturbed system the nominal solution $Q^{(0)}$ satisfies

$$\frac{\partial Q^{(0)}}{\partial t_0} - H^{(0)}(x_{0i}^{(0)}, t) = 0, \quad \frac{\partial Q^{(0)}}{\partial t_1} + H^{(0)}(x_{1i}^{(0)}, t) = 0. \quad (3.36)$$

Hence, Eq. 3.35 yields the following expressions for increasing orders of ϵ :

$$\begin{aligned} \epsilon^1 : \quad & \frac{\partial Q^{(1)}}{\partial t_0} - \frac{\partial H^{(0)}}{\partial q_{0j}} \Big|_{x_{0m}^{(0)}} q_{0j}^{(1)} - H^{(1)}(x_{0i}^{(0)}, t) = 0, \\ & \frac{\partial Q^{(1)}}{\partial t_1} + \frac{\partial H^{(0)}}{\partial q_{1j}} \Big|_{x_{1m}^{(0)}} p_{1j}^{(1)} + H^{(1)}(x_{1i}^{(0)}, t) = 0, \\ \epsilon^2 : \quad & \frac{\partial Q^{(2)}}{\partial t_0} - \frac{\partial H^{(0)}}{\partial q_{0j}} \Big|_{x_{0m}^{(0)}} q_{0j}^{(2)} - \frac{1}{2!} \frac{\partial^2 H^{(0)}}{\partial q_{0j} \partial q_{0k}} \Big|_{x_{0m}^{(0)}} q_{0j}^{(1)} q_{0k}^{(1)} - \frac{\partial H^{(1)}}{\partial p_{0j}} \Big|_{x_{0m}^{(0)}} q_{0j}^{(1)} = 0, \\ & \frac{\partial Q^{(2)}}{\partial t_1} + \frac{\partial H^{(0)}}{\partial q_{1j}} \Big|_{x_{1m}^{(0)}} q_{1j}^{(2)} + \frac{1}{2!} \frac{\partial^2 H^{(0)}}{\partial q_{1j} \partial q_{1k}} \Big|_{x_{1m}^{(0)}} q_{1j}^{(1)} q_{1k}^{(1)} + \frac{\partial H^{(1)}}{\partial p_{1j}} \Big|_{x_{1m}^{(0)}} q_{1j}^{(1)} = 0, \end{aligned} \quad (3.37)$$

...

...

As with the principal function, the total derivative of $Q^{(\alpha)}$ with respect to t_0 and t_1 can be expressed as the following:

$$\frac{dQ^{(\alpha)}}{dt_0} = \frac{\partial Q^{(\alpha)}}{\partial t_0} + \frac{\partial Q^{(\alpha)}}{\partial p_{0i}} \frac{dp_{0i}}{dt_0}, \quad (3.38)$$

$$\frac{dQ^{(\alpha)}}{dt_1} = \frac{\partial Q^{(\alpha)}}{\partial t_1} + \frac{\partial Q^{(\alpha)}}{\partial p_{1i}} \frac{dp_{1i}}{dt_1}.$$

where p_{0i} and p_{1i} lie along the nominal trajectory. By definition, the time differentials of the momenta are $\frac{dp_{0i}}{dt_0} = -\frac{\partial H}{\partial q_{0i}} \Big|_{x_{0m}^{(0)}}$, and $\frac{dp_{1i}}{dt_1} = -\frac{\partial H}{\partial q_{1i}} \Big|_{x_{1m}^{(0)}}$, since the expansion is about the nominal solution. Also from Eq. 3.26 the boundary conditions of the characteristic function are $\frac{\partial Q^{(\alpha)}}{\partial p_{0i}} = q_{0i}^{(\alpha)}$ and $\frac{\partial Q^{(\alpha)}}{\partial p_{1i}} = -q_{1i}^{(\alpha)}$, therefore Eq. 3.38 becomes

$$\frac{dQ^{(\alpha)}}{dt_0} = \frac{\partial Q^{(\alpha)}}{\partial t_0} - q_{0i}^{(\alpha)} \frac{\partial H}{\partial q_{0i}} \Big|_{x_{0m}^{(0)}}, \quad (3.39)$$

$$\frac{dQ^{(\alpha)}}{dt_1} = \frac{\partial Q^{(\alpha)}}{\partial t_1} + q_{1i}^{(\alpha)} \frac{\partial H}{\partial q_{1i}} \Big|_{x_{1m}^{(0)}},$$

where $\frac{\partial H}{\partial p_{0i}} \Big|_{x_{0m}^{(0)}}$ is a function of $(q_{0i}, p_{0i}^{(0)})$ and $\frac{\partial H}{\partial p_{1i}} \Big|_{x_{0m}^{(0)}}$ is a function of $(q_{1i}, p_{1i}^{(0)})$. Therefore, by substituting the right hand side of Eq. 3.39 into Eq. 3.37 the total time derivative expressions for the $Q^{(\alpha)}$'s are obtained:

$$\begin{aligned}
\epsilon^1 : \quad & \frac{dQ^{(1)}}{dt_0} - H^{(1)}(x_{0i}^{(0)}, t) = 0 \\
& \frac{dQ^{(1)}}{dt_1} + H^{(1)}(x_{1i}^{(0)}, t) = 0 \\
\epsilon^2 : \quad & \frac{dQ^{(2)}}{dt_0} - \frac{1}{2!} \frac{\partial^2 H^{(0)}}{\partial q_{0j} \partial q_{0k}} \Big|_{x_{0m}^{(0)}} q_{0j}^{(1)} q_{0k}^{(1)} - \frac{\partial H^{(1)}}{\partial p_{0j}} \Big|_{x_{0m}^{(0)}} q_{0j}^{(1)} = 0 \\
& \frac{dQ^{(2)}}{dt_1} + \frac{1}{2!} \frac{\partial^2 H^{(0)}}{\partial q_{1j} \partial q_{1k}} \Big|_{x_{1m}^{(0)}} q_{1j}^{(1)} q_{1k}^{(1)} + \frac{\partial H^{(1)}}{\partial p_{1j}} \Big|_{x_{1m}^{(0)}} q_{1j}^{(1)} = 0 \\
& \dots \\
& \dots \\
\epsilon^n : \quad & \frac{dQ^{(n)}}{dt_0} = F_0^{(n)}(x_{0i}^{(0)}, Q^{(0)}, Q^{(1)}, \dots, Q^{(n-1)}) \\
& \frac{dQ^{(n)}}{dt_1} = F_1^{(n)}(x_{1i}^{(0)}, Q^{(0)}, Q^{(1)}, \dots, Q^{(n-1)}) \\
& \dots \\
& \dots
\end{aligned} \tag{3.40}$$

As it can be seen, at each order $Q^{(\alpha)}$ apparently needs to satisfy two equations simultaneously. Hence the characteristic function at each order is

$$\begin{aligned}
\epsilon^1 : \quad Q^{(1)} &= \int_{t_1}^{t_0} H^{(1)}(x_i^{(0)}(t), t) dt \\
Q^{(1)} &= - \int_{t_0}^{t_1} H^{(1)}(x_i^{(0)}(t), t) dt, \\
\epsilon^2 : \quad Q^{(2)} &= \int_{t_1}^{t_0} \left[\frac{1}{2} \frac{\partial^2 H^{(0)}}{\partial q_j \partial q_k} \Big|_{x_m^{(0)}} q_j^{(1)} q_k^{(1)} + \frac{\partial H^{(1)}}{\partial p_j} \Big|_{x_m^{(0)}} q_j^{(1)} \right] dt, \\
Q^{(2)} &= - \int_{t_0}^{t_1} \left[\frac{1}{2} \frac{\partial^2 H^{(0)}}{\partial q_j \partial q_k} \Big|_{x_m^{(0)}} q_j^{(1)} q_k^{(1)} + \frac{\partial H^{(1)}}{\partial p_j} \Big|_{x_m^{(0)}} q_j^{(1)} \right] dt, \\
&\dots \\
&\dots \\
\epsilon^n : \quad Q^{(n)} &= - \int_{t_1}^{t_0} F^{(n)}(x_i^{(0)}(t), t, Q^{(0)}, \dots, Q^{(n-1)}) dt, \\
Q^{(n)} &= \int_{t_0}^{t_1} F^{(n)}(x_i^{(0)}(t), t, Q^{(0)}, \dots, Q^{(n-1)}) dt, \\
&\dots \\
&\dots
\end{aligned} \tag{3.41}$$

However, these two simultaneous equations are one and the same, as can be seen by reversing the limits of integration. Hence, at each order $Q^{(\alpha)}$ is defined by only one equation:

$$\begin{aligned}
\epsilon^1 : \quad Q^{(1)} &= - \int_{t_0}^{t_1} H^{(1)}(x_i^{(0)}(t), t) dt, \\
\epsilon^2 : \quad Q^{(2)} &= - \int_{t_0}^{t_1} \left[\frac{1}{2} \frac{\partial^2 H^{(0)}}{\partial q_j \partial q_k} \Big|_{x_m^{(0)}} q_j^{(1)} q_k^{(1)} + \frac{\partial H^{(1)}}{\partial p_j} \Big|_{x_m^{(0)}} q_j^{(1)} \right] dt, \\
&\dots \\
&\dots \\
\epsilon^n : \quad Q^{(n)} &= \int_{t_0}^{t_1} G^{(n)}(x_i^{(0)}(t), t, Q^{(0)}, \dots, Q^{(n-1)}) dt.
\end{aligned} \tag{3.42}$$

In order to solve the two-point boundary value problem, the partial derivatives of Q with respect to p_{0i} and p_{1i} need to be taken. However this requires the knowledge of an analytical expression for state vector of the nominal system which is not always possible.

CHAPTER IV

Perturbation Theory for the Two-Point Boundary Value Problem

The two-point boundary value problem connects two points in phase space, (q_{0i}, p_{0i}, t_0) and (q_{1i}, p_{1i}, t_1) . For a Hamiltonian dynamical system, there are four kinds of boundary value problems. The first kind of two-point boundary value problem involves obtaining solutions for the endpoint momenta given the endpoint coordinates and times. The second and third kind involve solving one endpoint coordinate and momentum given the other endpoint coordinate and momentum. The fourth kind of two-point boundary value problem involves solving for the endpoint generalized coordinates given the endpoint momenta and times.

4.1 Two-Point Boundary Value Problem Solutions for Hamilton's Principal Function

In order to solve the first kind of two-point boundary value problems, Hamilton's principal function can be used (37). Given the generalized coordinates $q_{0i}(t_0)$ and $q_{1i}(t_1)$, the solution for the required momenta $p_{0i}(t_0)$ and $p_{1i}(t_1)$ that connects the two points needs to be obtained by taking its partial derivative of the principal function

with respect to the endpoint generalized coordinates:

$$p_{0i} = -\frac{\partial W}{\partial q_{0i}}(q_{0i}, q_{1i}, t_0, t_1), \quad p_{1i} = \frac{\partial W}{\partial q_{1i}}(q_{0i}, q_{1i}, t_0, t_1). \quad (4.1)$$

At each order, the principal function and the generalized momenta are related by the following:

$$p_{0i}^{(\alpha)} = -\frac{\partial W^{(\alpha)}}{\partial q_{0i}}, \quad p_{1i}^{(\alpha)} = \frac{\partial W^{(\alpha)}}{\partial q_{1i}}. \quad (4.2)$$

Thus if W is computed to the N^{th} order, an approximate solution to the two-point boundary value problem can be obtained to that order using $W = \sum_{\alpha=1}^N \epsilon^\alpha W^{(\alpha)}$ to find $p_i = \sum_{\alpha=1}^N \epsilon^\alpha p_i^{(\alpha)}$. Hence, at each order the perturbed momenta is expressed as

$$\begin{aligned}
\epsilon^1 : \quad p_{0i}^{(1)} &= -\frac{\partial W^{(1)}}{\partial q_{0i}} = \frac{\partial}{\partial q_{0i}} \int_{t_0}^{t_1} H^{(1)}(x_i^{(0)}(t), t) dt, \\
p_{1i}^{(1)} &= \frac{\partial W^{(1)}}{\partial q_{1i}} = -\frac{\partial}{\partial q_{1i}} \int_{t_0}^{t_1} H^{(1)}(x_i^{(0)}(t), t) dt, \\
\epsilon^2 : \quad p_{0i}^{(2)} &= -\frac{\partial W^{(2)}}{\partial q_{0i}} = \frac{\partial}{\partial q_{0i}} \int_{t_0}^{t_1} \left[\frac{1}{2} \frac{\partial^2 H^{(0)}}{\partial p_j \partial p_k} \Big|_{\vec{x}^{(0)}} p_j^{(1)} p_k^{(1)} + \frac{\partial H^{(1)}}{\partial p_j} \Big|_{\vec{x}^{(0)}} p_j^{(1)} \right] dt, \\
p_{1i}^{(2)} &= \frac{\partial W^{(2)}}{\partial q_{1i}} = -\frac{\partial}{\partial q_{1i}} \int_{t_0}^{t_1} \left[\frac{1}{2} \frac{\partial^2 H^{(0)}}{\partial p_j \partial p_k} \Big|_{\vec{x}^{(0)}} p_j^{(1)} p_k^{(1)} + \frac{\partial H^{(1)}}{\partial p_j} \Big|_{\vec{x}^{(0)}} p_j^{(1)} \right] dt, \\
&\dots \\
&\dots \\
\epsilon^n : \quad p_{0i}^{(n)} &= -\frac{\partial W^{(n)}}{\partial q_{0i}} = -\frac{\partial}{\partial q_{0i}} \int_{t_0}^{t_1} F^{(n)}(x_i^{(0)}(t), t, W^{(0)}, \dots, W^{(n-1)}) dt, \\
p_{1i}^{(n)} &= \frac{\partial W^{(n)}}{\partial q_{1i}} = \frac{\partial}{\partial q_{1i}} \int_{t_0}^{t_1} F^{(n)}(x_i^{(0)}(t), t, W^{(0)}, \dots, W^{(n-1)}) dt, \\
&\dots \\
&\dots
\end{aligned} \tag{4.3}$$

The general form Eq. 4.3 can be expressed as

$$p_{0i} = - \int_{t_0}^{t_1} \frac{\partial F^{(\alpha)}}{\partial q_{0i}}, \quad (4.4)$$

$$p_{1i} = \int_{t_0}^{t_1} \frac{\partial F^{(\alpha)}}{\partial q_{1i}},$$

where the function F is defined as

$$\frac{\partial F^{(\alpha)}}{\partial q_{0i}} = \frac{\partial F^{(\alpha)}}{\partial x_j^{(0)}} \frac{\partial_b x_j^{(0)}}{\partial_b q_{0i}}, \quad (4.5)$$

$$\frac{\partial F^{(\alpha)}}{\partial q_{1i}} = \frac{\partial F^{(\alpha)}}{\partial x_j^{(0)}} \frac{\partial_b x_j^{(0)}}{\partial_b q_{1i}}, \quad (4.6)$$

since F can be expressed as $F^{(\alpha)} = F^{(\alpha)} \left(t, x_j^{(0)}(t, x_{0i}, t_0) \right)$.

Fig. 4.1 illustrates the problem in the presence of a perturbation. In order to take the variation of $x^{(0)}$ with respect to q_0 , there are constraints that have to be taken into account. For this problem, the two-point boundary value problem, the endpoint generalized coordinates are fixed, even if the path in phase space of the dynamics differs from the nominal. Therefore, any variation of q_1 , $\delta q_1 = 0$, and this fact imposes a constraint on the dynamics of the system. Under these circumstances $x^{(0)}$ is a function of the constraint and the proper chain rule needs to be applied.

The subscript b in the partial derivative corresponds to the constraint case that solves the boundary value problem using the nominal solution. This partial derivative consists of direct and indirect parts. The direct part comes from the state transition

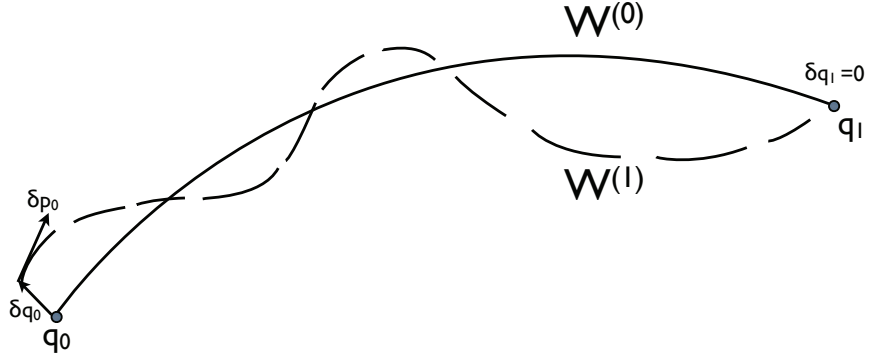


Figure 4.1: Illustration of the constraint endpoint

matrix, while the indirect part is the necessary variation to ensure that $\delta q(t_1) = 0$.

4.1.1 Constrained Partial Derivative for Hamilton's Principal Function

In order to solve the aforementioned constrained partial derivative, consider the state vector of the nominal solution at some arbitrary time $\tau \in [t_0, t_1]$ expressed as

$$x_j^{(0)}(\tau) = x_j^{(0)}(\tau, q_\alpha^{(0)}(t), p_\alpha^{(0)}(t), t). \quad (4.7)$$

The state vector at time τ is dependent on the state at a time $t \in [t_1, \tau]$. A variation to the state vector at time t yields

$$x_j(t) = x_j^{(0)}(t) + \delta x_j(t). \quad (4.8)$$

The state vector at time τ due to variation in the state at t is therefore

$$x_j^{(0)}(\tau, q_\alpha^{(0)}(t) + \delta q_\alpha(t), p_\alpha^{(0)}(t) + \delta p_\alpha(t), t) = x_j(\tau) + \delta x_j(\tau). \quad (4.9)$$

The variation on generalized coordinates at time τ is

$$\begin{aligned} q_j(\tau)^{(0)} + \delta q_j(\tau) &= q_j(\tau)^{(0)} + \Phi_{j\alpha} \delta q_\alpha(t) + \Phi_{j(n+\alpha)} \delta p_\alpha(t) + \\ &+ \frac{1}{2!} [\Phi_{j\alpha\beta} \delta q_\alpha(t) \delta q_\beta(t) + \Phi_{j(n+\alpha)\beta} \delta p_\alpha(t) \delta q_\beta(t) + \\ &+ \Phi_{j\alpha(n+\beta)} \delta q_\alpha(t) \delta p_\beta(t) + \Phi_{j(n+\alpha)(n+\beta)} \delta p_\alpha(t) \delta p_\beta(t)] + \dots, \end{aligned} \quad (4.10)$$

where $\Phi_{ij}(\tau, t)$ is the state transition matrix and $\Phi_{ijkl\dots}(\tau, t)$ are the higher order partials of the state transition matrix or state transition tensors. Note that these tensors can be solved for from the nominal integrable solution. The dimension of the state vector is $2n$.

For this type of two-point boundary value problem, $q_j(t_1) = q_{1j}$ is fixed and the

variation $\delta q_j(\tau)$ vanishes when $\tau = t_1$, hence, $\delta q_{1j} = 0$. This condition imposes an n -dimensional constraint on the system, allowing to obtain the solution for the other n variables.

Assuming that the momenta can be expressed as a function of coordinates only, $p_i(t) = f(q_\gamma, q_\psi, \dots)$, allows to express the variation of the momenta as a power series of the following form:

$$\delta p_i(t) = C_{i\gamma} \delta q_\gamma(t) + C_{i\gamma\psi} \delta q_\gamma(t) \delta q_\psi(t) + \dots, \quad (4.11)$$

where the C 's are coefficient tensors to be determined by boundary conditions. Plugging Eq. 4.25 into Eq. 4.10 and combining terms, following result is obtained:

$$\begin{aligned} \delta q_j(\tau) = & [\Phi_{j\gamma} + \Phi_{j(n+\alpha)} C_{\alpha\gamma}] \delta q_\gamma(t) + \frac{1}{2} [\Phi_{j\gamma\psi} + \Phi_{j(n+\alpha)\psi} C_{\alpha\gamma} + \\ & + \Phi_{j\gamma(n+\beta)} C_{\beta\psi} + \Phi_{j(n+\alpha)(n+\beta)} C_{\alpha\gamma} C_{\beta\delta} + \\ & + 2\Phi_{j(n+\alpha)} C_{\alpha\gamma\psi}] \delta q_\gamma(t) \delta q_\psi(t) + \dots \end{aligned} \quad (4.12)$$

Therefore $\delta q_j(t_1) = 0$ and using this boundary condition the coefficient tensors in Eq. 4.12 can be solved at each order:

$$\begin{aligned}
C_{\alpha\gamma}(t_1, t) &= -\Phi_{j(n+\alpha)}^{-1}(t_1, t)\Phi_{j\gamma}(t_1, t) \\
C_{\alpha\gamma\psi}(t_1, t) &= -\frac{1}{2}\Phi_{j(n+\alpha)}^{-1}(t_1, t) \left[\Phi_{j\gamma\psi}(t_1, t) + 2\Phi_{j(n+\alpha)\psi}(t_1, t)C_{\alpha\gamma} + \right. \\
&\quad \left. + \Phi_{j(n+\alpha)(n+\beta)}(t_1, t)C_{\alpha\gamma}(t_1, t)C_{\beta\psi}(t_1, t) \right]
\end{aligned} \tag{4.13}$$

...

The tensors are functions of the state transition matrix and state transition tensors from t to t_1 . In order to solve these it is necessary for unique inverses to exist, which are not always guaranteed. If these tensors are singular they correspond to the existence of multiple solutions to the nominal two-point boundary value problem. Discussions on how to solve the problem in the presence of singularities were explored by Guibout (38). Hence, the constrained partial derivatives take the form

$$\begin{aligned}
\frac{\partial_b x_j^{(0)}}{\partial_b q_{0\gamma}}(\tau, t) &= \Phi_{j\gamma}(\tau, t) + \Phi_{j(n+\alpha)}(\tau, t)C_{\alpha\gamma}(t_1, t), \\
\frac{\partial_b^2 x_j^{(0)}}{\partial_b q_{0\gamma}\partial_b q_{0\psi}}(\tau, t) &= \frac{1}{2} \left[\Phi_{j\gamma\psi}(\tau, t) + \Phi_{j(n+\alpha)\psi}(\tau, t)C_{\alpha\gamma}(t_1, t) + \Phi_{j\gamma(n+\beta)}(\tau, t)C_{\beta\psi}(t_1, t) + \right. \\
&\quad \left. \Phi_{j(n+\alpha)(n+\beta)}(\tau, t)C_{\alpha\gamma}(t_1, t)C_{\beta\delta}(t_1, t) + 2\Phi_{j(n+\alpha)}(\tau, t)C_{\alpha\gamma\psi}(t_1, t) \right],
\end{aligned} \tag{4.14}$$

...

The solution at each order can be obtained by the knowledge of the nominal solution $\vec{x}^{(0)}$ since $W^{(n)} = W^{(n)}(W^{(0)})$. Thus the expressions in Eq. 4.3 can be solved using quadratures. From a practical standpoint, the state transition matrix and state transition tensors can be obtained numerically. Park and Scheeres 2005 derive a scheme to obtain numerical expressions for these tensors (39).

4.2 Two-Point Boundary Value Problem Solutions for Hamilton's Characteristic Function

To solve the fourth kind of two-point boundary value problems, Hamilton's characteristic function can be used. Given the generalized momenta $p_{0i}(t_0)$ and $p_{1i}(t_1)$, the solution for the required generalized coordinates $q_{0i}(t_0)$ and $q_{1i}(t_1)$ that connects the two points needs to be obtained by taking its partial derivative of the principal function with respect to the endpoint generalized momenta:

$$q_{0i} = \frac{\partial Q}{\partial p_{0i}}(p_{0i}, p_{1i}, t_0, t_1), \quad q_{1i} = -\frac{\partial Q}{\partial p_{1i}}(p_{0i}, p_{1i}, t_0, t_1). \quad (4.15)$$

At each order, the characteristic function and the generalized coordinates are related by the following equations

$$q_{0i}^{(\alpha)} = \frac{\partial Q^{(\alpha)}}{\partial p_{0i}}, \quad q_{1i}^{(\alpha)} = -\frac{\partial Q^{(\alpha)}}{\partial p_{1i}}. \quad (4.16)$$

Thus if Q is computed to order N an approximate solution to the two-point

boundary value problem can be obtained to that order using $Q = \sum_{\alpha=1}^N \epsilon^\alpha Q^{(\alpha)}$ to find $q_i = \sum_{\alpha=1}^N \epsilon^\alpha q_i^{(\alpha)}$. Therefore the perturbed coordinates at each order are:

$$\epsilon^1 : \quad q_{0i}^{(1)} = \frac{\partial Q^{(1)}}{\partial p_{0i}} = \frac{\partial}{\partial p_{0i}} \int_{t_0}^{t_1} H^{(1)}(x_i^{(0)}(t), t) dt,$$

$$q_{1i}^{(1)} = -\frac{\partial Q^{(1)}}{\partial p_{1i}} = -\frac{\partial}{\partial p_{1i}} \int_{t_0}^{t_1} H^{(1)}(x_i^{(0)}(t), t) dt,$$

$$\epsilon^2 : \quad q_{0i}^{(2)} = \frac{\partial Q^{(2)}}{\partial p_{0i}} = \frac{\partial}{\partial p_{0i}} \int_{t_0}^{t_1} \left[\frac{1}{2} \frac{\partial^2 H^{(0)}}{\partial q_j \partial q_k} \Big|_{\vec{x}^{(0)}} q_j^{(1)} q_k^{(1)} + \frac{\partial H^{(1)}}{\partial q_j} \Big|_{\vec{x}^{(0)}} q_j^{(1)} \right] dt,$$

$$q_{1i}^{(2)} = -\frac{\partial Q^{(2)}}{\partial p_{1i}} = -\frac{\partial}{\partial p_{1i}} \int_{t_0}^{t_1} \left[\frac{1}{2} \frac{\partial^2 H^{(0)}}{\partial q_j \partial q_k} \Big|_{\vec{x}^{(0)}} q_j^{(1)} q_k^{(1)} + \frac{\partial H^{(1)}}{\partial p_j} \Big|_{\vec{x}^{(0)}} q_j^{(1)} \right] dt,$$

...

(4.17)

...

$$\epsilon^n : \quad q_{0i}^{(n)} = \frac{\partial Q^{(n)}}{\partial p_{0i}} = -\frac{\partial}{\partial p_{0i}} \int_{t_0}^{t_1} G^{(n)}(x_i^{(0)}(t), t, Q^{(0)}, \dots, Q^{(n-1)}) dt,$$

$$q_{1i}^{(n)} = -\frac{\partial Q^{(n)}}{\partial p_{1i}} = \frac{\partial}{\partial p_{1i}} \int_{t_0}^{t_1} G^{(n)}(x_i^{(0)}(t), t, Q^{(0)}, \dots, Q^{(n-1)}) dt,$$

...

...

The general form of Eq. 4.17 is

$$q_{0i} = - \int_{t_0}^{t_1} \frac{\partial G^{(\alpha)}}{\partial p_{0i}}, \quad (4.18)$$

$$q_{1i} = \int_{t_0}^{t_1} \frac{\partial G^{(\alpha)}}{\partial p_{1i}},$$

where the function G with respect to momenta is

$$\frac{\partial G^{(\alpha)}}{\partial p_{0i}} = \frac{\partial G^{(\alpha)}}{\partial x_j^{(0)}} \frac{\partial_b x_j^{(0)}}{\partial_b p_{0i}}, \quad (4.19)$$

$$\frac{\partial G^{(\alpha)}}{\partial p_{1i}} = \frac{\partial G^{(\alpha)}}{\partial x_j^{(0)}} \frac{\partial_b x_j^{(0)}}{\partial_b p_{1i}}, \quad (4.20)$$

since $G^{(\alpha)} = G^{(\alpha)}(t, x_j^{(0)}(t, x_{0i}, t_0))$. In order to take the variation of $x_j^{(0)}$ with respect to p_0 , there are constraints that need to be taken into account. Since the two-point boundary value problem is being solved, the endpoint generalized coordinates are fixed, even if the path in phase space of the dynamics differs from the nominal. Therefore, any variation of p_{1j} , $\delta p_{1j} = 0$, and this fact imposes a constraint on the dynamics of the system. Under these circumstances $x_j^{(0)}$ is a function of the constraint and the proper chain rule has to be applied.

The subscript b in the partial derivative corresponds to the constraint case that solves the boundary value problem using the nominal solution. This partial derivative

consists of direct and indirect parts. The direct part comes from the state transition matrix, while the indirect part is the necessary variation to ensure that $\delta p_j(t_1) = 0$.

4.2.1 Constrained Partial Derivative for Hamilton's Characteristic Function

The derivation of the constrained partial derivative for Hamilton's characteristic function follows along the same lines as for the principal function. Recall that the state vector of the nominal solution at some arbitrary time $\tau \in [t_0, t_1]$ can be expressed as

$$x_j^{(0)}(\tau) = x_j^{(0)}(\tau, q_\alpha^{(0)}(t), p_\alpha^{(0)}(t), t). \quad (4.21)$$

As with the principal function the state vector $x_j^{(0)}(\tau)$ is dependent on the state at a time $t \in [t_1, \tau]$. A variation to the state vector at time t yields

$$x_j(t) = x_j^{(0)}(t) + \delta x_j(t). \quad (4.22)$$

The state vector at time τ due to variation in the state at t is therefore

$$x_j^{(0)}(\tau, q_\alpha^{(0)}(t) + \delta q_\alpha(t), p_\alpha^{(0)}(t) + \delta p_\alpha(t), t) = x_j(\tau) + \delta x_j(\tau). \quad (4.23)$$

The variation on generalized momenta at time τ is

$$\begin{aligned}
p_j(\tau)^{(0)} + \delta p_j(\tau) &= p_j(\tau)^{(0)} + \Phi_{j(n+\alpha)} \delta p_\alpha(t) + \Phi_{j\alpha} \delta q_\alpha(t) + \\
&+ \frac{1}{2!} \left[\Phi_{j(n+\alpha)(n+\beta)} \delta p_\alpha(t) \delta p_\beta(t) + \Phi_{j\alpha(n+\beta)} \delta q_\alpha(t) \delta p_\beta(t) + \right. \\
&\left. + \Phi_{j(n+\alpha)\beta} \delta p_\alpha(t) \delta q_\beta(t) + \Phi_{j\alpha\beta} \delta q_\alpha(t) \delta q_\beta(t) \right] + \dots,
\end{aligned} \tag{4.24}$$

where $\Phi_{ij}(\tau, t)$ is the state transition matrix and $\Phi_{ijkl\dots}(\tau, t)$ corresponds to the state transition tensors. For this type of two-point boundary value problem the momenta at t_1 , $p_j(t_1) = p_{1j}$, is fixed and the variation $\delta p_j(\tau)$ vanishes when $\tau = t_1$, hence, $\delta p_{1j} = 0$ imposing a constraint on the system. The generalized coordinates can be expressed as a function of generalized momenta only, $q_i(t) = f(p_\gamma, p_\psi, \dots)$, and the variation of the generalized coordinates can be expressed as a power series:

$$\delta q_i(t) = K_{i\gamma} \delta p_\gamma(t) + K_{i\gamma\psi} \delta p_\gamma(t) \delta p_\psi(t) + \dots \tag{4.25}$$

where the K 's are coefficient tensors to be determined by boundary conditions. Combining Eq. 4.25 and Eq. 4.24 leads to the general expression for the variation of the generalized momenta:

$$\begin{aligned}
\delta p_j(\tau) = & [\Phi_{j\gamma} + \Phi_{j\alpha} K_{\alpha\gamma}] \delta p_\gamma(t) + \frac{1}{2} [\Phi_{j(n+\gamma)(n+\psi)} + \Phi_{j\alpha(n+\psi)} K_{\alpha\gamma} + \\
& + \Phi_{j(n+\gamma)\beta} K_{\beta\psi} + \Phi_{j\alpha\beta} K_{\alpha\gamma} K_{\beta\delta} + \\
& + 2\Phi_{j\alpha} K_{\alpha\gamma\psi}] \delta p_\gamma(t) \delta p_\psi(t) + \dots
\end{aligned} \tag{4.26}$$

Since $\delta p_j(t_1) = 0$, and using this boundary condition, the coefficient tensors, K 's in Eq. 4.26 can be solved at each order:

$$\begin{aligned}
K_{\alpha\gamma}(t_1, t) &= -\Phi_{j\alpha}^{-1}(t_1, t) \Phi_{j(n+\gamma)}(t_1, t), \\
K_{\alpha\gamma\psi}(t_1, t) &= -\frac{1}{2} \Phi_{j\alpha}^{-1}(t_1, t) [\Phi_{j(n+\gamma)(n+\psi)}(t_1, t) + 2\Phi_{j\alpha(n+\psi)}(t_1, t) K_{\alpha\gamma} + \\
& + \Phi_{j\alpha\beta}(t_1, t) K_{\alpha\gamma}(t_1, t) K_{\beta\psi}(t_1, t)] \\
&\dots
\end{aligned} \tag{4.27}$$

The tensors are functions of the state transition matrix and state transition tensors from t to t_1 . Hence, the constrained partial derivatives take the form

$$\frac{\partial_b x_j^{(0)}}{\partial_b p_{0\gamma}}(\tau, t) = \Phi_{j(n+\gamma)}(\tau, t) + \Phi_{j\alpha}(\tau, t)K_{\alpha\gamma}(t_1, t),$$

$$\frac{\partial_b^2 x_j^{(0)}}{\partial_b p_{0\gamma} \partial_b p_{0\psi}}(\tau, t) = \frac{1}{2} [\Phi_{j(n+\gamma)(n+\psi)}(\tau, t) + \Phi_{j\alpha\psi}(\tau, t)K_{\alpha\gamma}(t_1, t) + \Phi_{j\gamma\beta}(\tau, t)K_{\beta\psi}(t_1, t) +$$

$$\Phi_{j\alpha\beta}(\tau, t)K_{\alpha\gamma}(t_1, t)K_{\beta\delta}(t_1, t) + 2\Phi_{j\alpha}(\tau, t)K_{\alpha\gamma\psi}(t_1, t)],$$

...

(4.28)

As with principal function, the solution at each order can be obtained with the knowledge of the nominal solution $\vec{x}^{(0)}$ since $Q^{(n)} = Q^{(n)}(Q^{(0)})$. Hence, the expressions in Eq. 4.17 can be solved using quadratures.

4.3 Numerical Error Analysis of Perturbation Theory for The Two-Point Boundary Value Problem

Carrying out a double integral by means of quadratures inevitably leads to numerical errors in calculations. In order to obtain highly accurate results in a perturbed system, it is important to understand how these errors alter the solutions that are obtained numerically. The calculations for the integrals involved in the perturbed solutions are carried out by simple quadratures, or using the midpoint rule. An integrand $f(t)$ integrated from t_0 to t_1 can be expressed as:

$$\int_{t_0}^{t_1} F(t)dt = \sum_{i=1}^N \left[\Delta F_i \frac{t_1 - t_0}{N} \right] + \frac{(t_1 - t_0)^3}{24N} \frac{d^2 F}{dt^2} \Big|_{t=\xi}, \quad (4.29)$$

where ξ lies on the interval defined by the limits of integration $[t_0, t_1]$. Assuming that there is a difference Δ between the calculated first-order theory result and the true solution, the true solution can be expressed as

$$p_0^{(1)*} = p_0^{(1)} + \Delta p_0^{(1)}, \quad (4.30)$$

where $p_0^{(1)}$ is the numerically obtained first order solution, $\Delta p_0^{(1)}$ is the difference between the calculated and true solutions, $p_0^{(1)*}$ is the true first-order solution:

$$p_0^{(1)*} = \int_{t_0}^{t_1} dp_0^{(1)*}. \quad (4.31)$$

The first order calculated solution is:

$$p_0^{(1)} = \sum_{i=1}^N p_{0i}^{(1)}. \quad (4.32)$$

In Eq. 4.29, the parameter $\xi \in [t_0, t_1]$, hence, $\xi = t_0$ results in the following expression for the numerical error $\delta p_0^{(1)}$ substituting $p_0^{(1)*}$ and $p_0^{(1)}$ by the integral in Eq. 4.29:

$$\delta p_0^{(1)} = \frac{(t_1 - t_0)^3}{24N^2} \frac{d^2}{dt^2} \left[-\frac{\partial H^{(1)}}{\partial q_0} \right] \Bigg|_{t=t_0} = \frac{(t_1 - t_0)^3}{24N^2} \frac{d}{dt} \left[-\frac{\partial^2 H^{(1)}}{\partial q_0^2} \frac{dq_0}{dt_0} \right] \Bigg|_{t=t_0}. \quad (4.33)$$

From the Hamiltonian dynamics theory it is clear that $\frac{dq_0}{dt_0} = -p_0$, and Eq. 4.33 becomes

$$\Delta p_0^{(1)} = -\frac{(t_1 - t_0)^3}{24N^2} \frac{d}{dt} \left[\frac{\partial^2 H^{(1)}}{\partial q_0^2} p_0 \right] \Bigg|_{t=t_0}. \quad (4.34)$$

Therefore expanding Eq. 4.34 yields

$$\Delta p_0^{(1)} = \frac{(t_1 - t_0)^3}{24N^2} \left[\frac{\partial^3 H^{(1)}}{\partial q_0^3} (p_0)^2 + \frac{\partial^2 H^{(1)}}{\partial q_0^2} \frac{\partial p_0}{\partial q_0} p_0 \right]. \quad (4.35)$$

The result in Eq. 4.35 implies that the first order error is dependent on the time of flight, the integration time-step, the perturbing function and the nominal initial velocity of the orbiting particle. For a specified two-point boundary value problem, the numerical error is a function of the chosen step size for that particular numerical integration:

$$\Delta p_0^{(1)} \propto \frac{1}{N^2}. \quad (4.36)$$

Similarly for the second-order, the actual solution can be expressed as an integral:

$$p_0^{(2)*} = \int_{t_0}^{t_1} dp_0^{(2)*}, \quad (4.37)$$

and the numerically obtained solution is

$$p_0^{(2)} = \sum_{i=1}^N p_{0i}^{(2)}. \quad (4.38)$$

The numerical second-order error is can be obtained substituting $p_0^{(2)*}$ and $p_0^{(2)}$ by the integral in Eq. 4.29:

$$\Delta p_0^{(2)} = -\frac{(t_1 - t_0)^3}{24N^2} \frac{d^2}{dt^2} \left[\frac{\partial p^{(1)*}}{\partial q_0} p^{(1)*} \right] \Bigg|_{t=t_0}. \quad (4.39)$$

Eq. 4.39 can be further expanded by plugging the expression for $p_0^{(1)*}$ in Eq. 4.31:

$$\Delta p_0^{(2)} = -\frac{(t_1 - t_0)^3}{24N^2} \frac{d^2}{dt^2} \left[\left(-\int_t^{t_1} \frac{\partial^2 H^{(1)}}{\partial q_0^2} dt \right) \left(-\int_t^{t_1} \frac{\partial H^{(1)}}{\partial q_0} dt \right) \right] \Bigg|_{t=t_0}. \quad (4.40)$$

Eq. 4.40 can be further expanded:

$$\Delta p_0^{(2)} = -\frac{(t_1 - t_0)^3}{24N^2} \frac{d}{dt} \left[-\frac{\partial^2 H^{(1)}}{\partial q_0^2} p_0^{(1)*} - \frac{\partial p_0^{(1)*}}{\partial q_0} \frac{\partial H^{(1)}}{\partial q_0} \right] \Big|_{t=t_0}. \quad (4.41)$$

Eq. 4.41 takes the following form:

$$\Delta p_0^{(2)} = -\frac{(t_1 - t_0)^3}{24N^2} \left[\left(\frac{\partial^3 H^{(1)}}{\partial q_0^3} p_0^{(1)*} + \frac{\partial p_0^{(1)*}}{\partial q_0} \frac{\partial^2 H^{(1)}}{\partial q_0^2} \right) p_0 + 2 \frac{\partial^2 H^{(1)}}{\partial q_0^2} \frac{\partial H^{(1)}}{\partial q_0} \right]. \quad (4.42)$$

For the second-order solution, it is also assumed that there is a difference Δ between the calculated second-order theory solution and the true solution, hence, the true solution can be expressed as:

$$p_0^{(2)*} = p_0^{(2)} + \Delta p_0^{(2)} = \sum_{i=1}^N p_{0i}^{(2)} + \Delta p_0^{(2)}. \quad (4.43)$$

Where $p_{0i}^{(2)}$ is the numerically computed solution that depends on the numerically computed first order solution $p_i^{(1)}$. Therefore Eq. 4.38 can be expressed as:

$$\sum_{i=1}^N p_{0i}^{(2)}(p_i^{(1)}) = \sum_{i=1}^N p_i^{(2)}(p_i^{(1)*} - \Delta p_i^{(1)}) = \sum_{i=1}^N p_{0i}^{(2)}(p_i^{(1)*}) - \sum_{i=1}^N \frac{\partial p_i^{(2)}}{\partial p_i^{(1)*}} \Delta p_i^{(1)} - \sum_{i=1}^N \frac{\partial^2 p_0^{(2)}}{\partial (p_i^{(1)*})^2} (\Delta p_i^{(1)})^2, \quad (4.44)$$

where each $p_{0i}^{(2)}$ can be expressed as:

$$p_{0i}^{(2)} = \frac{\partial \left(p_i^{(1)*} - \Delta p_i^{(1)} \right)}{\partial q_0} \left(p_i^{(1)*} - \Delta p_i^{(1)} \right). \quad (4.45)$$

Eq. 4.45 can be further expanded:

$$p_{0i}^{(2)} = \frac{\partial p_i^{(1)*}}{\partial q_0} p_i^{(1)*} - \frac{\partial p_i^{(1)*}}{\partial q_0} \Delta p_i^{(1)*} - \frac{\partial \Delta p_i^{(1)}}{\partial q_0} p_i^{(1)*} + \frac{\partial \Delta p_i^{(1)}}{\partial q_0} \Delta p_i^{(1)}. \quad (4.46)$$

Therefore, differentiating Eq. 4.46 with respect to $p_i^{(1)*}$ yields

$$\frac{\partial p_{0i}^{(2)}}{\partial p_i^{(1)*}} = \frac{\partial p_i^{(1)*}}{\partial q_0} = \int_{t_0}^{t_1} \frac{\partial^2 H^{(1)}}{\partial q_0^2} dt. \quad (4.47)$$

Plugging Eq. 4.47 into 4.42 leads to the final expression for $\Delta p_0^{(2)}$ that can be analyzed:

$$\Delta p_0^{(2)} = p_0^{(2)*} - \sum_{i=1}^N \Delta p_{0i}^{(2)}(p_i^{(1)*}) = \delta p_0^{(2)} - \sum_{i=1}^N \frac{\partial p_i^{(1)*}}{\partial q_0} \delta p_i^{(1)}. \quad (4.48)$$

Investigating Eq. 4.48, the second order error depends on the numerical error from the integration routine, which as with the first order is proportional to $\frac{1}{N^2}$. However, there is additional numerical error caused by the first order integration numerical error, which is carried over. Therefore it is necessary to obtain the first order numer-

ical solution as accurately as possible in order to obtain an improved second-order solution, otherwise, a second-order solution could lead to a larger error than the first-order solution, defeating the purpose of higher order solutions.

4.4 Analytical Example: 1-Dimensional Particle Dynamics

To illustrate the described perturbation theory, one-dimensional particle motion is an appropriate example. A particle moves along a straight line with constant velocity, and the set of initial conditions (x_0, v_0, t_0) and final conditions (x_1, v_1, t_1) , represent the endpoint generalized coordinate and momentum. The obvious choice is for the position to be the generalized coordinate and the velocity to be the generalized momentum. The Hamiltonian of the system and the endpoint positions and velocities are:

$$\begin{aligned}
 H &= \frac{1}{2}v^2 \\
 v_0 &= \frac{x_1 - x_0}{t_1 - t_0}, \\
 v_1 &= \frac{x_1 - x_0}{t_1 - t_0}, \\
 x_1 &= x_0 + v_0(t_1 - t_0), \\
 x_0 &= x_1 - v_1(t_1 - t_0).
 \end{aligned}
 \tag{4.49}$$

The principal function of the nominal system described by Eq. 4.49 is,

$$W^{(0)} = \frac{1}{2} \frac{(x_1 - x_0)^2}{t_1 - t_0}. \quad (4.50)$$

If the nominal system is altered by a constant acceleration that results in the Hamiltonian $H = \frac{1}{2}v^2 - \epsilon ax$, the new system and its solution can be described by the following set of equations:

$$\begin{aligned} H &= H^{(0)} + \epsilon H^{(1)} = \frac{1}{2}v^2 - \epsilon ax, \\ v_0 &= v_0^{(0)} + \epsilon v_0^{(1)} = \frac{x_1 - x_0}{t_1 - t_0} - \epsilon \frac{1}{2}a(t_1 - t_0), \\ v_1 &= v_1^{(0)} + \epsilon v_1^{(1)} = \frac{x_1 - x_0}{t_1 - t_0} + \epsilon \frac{1}{2}a(t_1 - t_0), \\ x_1 &= x_0 + v_0^{(0)}(t_1 - t_0) + \epsilon \frac{1}{2}a(t_1 - t_0)^2, \\ x_0 &= x_1 - v_1^{(0)}(t_1 - t_0) - \epsilon \frac{1}{2}a(t_1 - t_0)^2. \end{aligned} \quad (4.51)$$

The principal function for the solution of the perturbed system is

$$W = \frac{1}{2} \frac{(x_1 - x_0)^2}{t_1 - t_0} + \epsilon \frac{1}{2} (x_0 + x_1)(t_1 - t_0) - \epsilon^2 \frac{1}{24} a^2 (t_1 - t_0)^3. \quad (4.52)$$

It can be shown that Eq. 4.52 is the principal function given for the solution of the system by satisfying the required boundary conditions:

$$-\frac{\partial W}{\partial x_0} = v_0 = \frac{x_1 - x_0}{t_1 - t_0} - \epsilon \frac{1}{2} a (t_1 - t_0), \quad (4.53)$$

$$\frac{\partial W}{\partial x_1} = v_1 = \frac{x_1 - x_0}{t_1 - t_0} + \epsilon \frac{1}{2} a (t_1 - t_0).$$

Application of the perturbation theory leads to the following relationships:

$$W^{(1)} = \int_{t_0}^{t_1} a x(t) dt, \quad (4.54)$$

$$W^{(2)} = -\frac{1}{8} \int_{t_0}^{t_1} a^2 (t - t_0)^2 dt,$$

where the position coordinate of the unperturbed system as a function of time is $x(t) = x_0 + \frac{x_1 - x_0}{t_1 - t_0} (t - t_0)$, and the perturbing acceleration a is constant. Therefore, integrating the equations in Eq. 4.54 yields the following expressions for $W^{(1)}$ and $W^{(2)}$:

$$W^{(1)} = \frac{1}{2} a (x_0 + x_1) (t_1 - t_0), \quad (4.55)$$

$$W^{(2)} = -\frac{1}{24} a^2 (t_1 - t_0)^3,$$

and $W^{(0)} + \epsilon W^{(1)} + \epsilon^2 W^{(2)} = W$. To solve the two-point boundary value problem it suffices to differentiate W with respect to x_0 and x_1 .

In this example it is possible to find an analytical solution for the principal function and then differentiate it to solve the two-point boundary value problem:

$$v_0^{(1)} = -a \int_{t_0}^{t_1} \frac{\partial x}{\partial x_0} dt = -a \int_{t_0}^{t_1} \left[1 - \frac{t-t_0}{t_1-t_0} \right] dt = -\frac{1}{2}a(t_1 - t_0),$$

(4.56)

$$v_1^{(1)} = a \int_{t_0}^{t_1} \frac{\partial x}{\partial x_1} dt = a \int_{t_0}^{t_1} \left[\frac{t-t_0}{t_1-t_0} \right] dt = \frac{1}{2}a(t_1 - t_0).$$

Note that $\frac{\partial W^{(2)}}{\partial x_0} = \frac{\partial W^{(2)}}{\partial x_1} = 0$.

CHAPTER V

Perturbation Theory for the Initial Value Problem

Hamilton's principal function, $W(q_{0i}, q_{1i}, t_0, t_1)$ can be used to solve the two-point boundary value problem given the initial and final generalized coordinates and times. Similarly, given the initial and final generalized momenta and times, a different kind of two-point boundary value problem can be solved by using Hamilton's characteristic function $Q(p_{0i}, p_{1i}, t_0, t_1)$.

5.1 Perturbation Theory for the Initial Value Problem

In the previous chapter a perturbation theory was developed to solve two-point boundary value problems for Hamilton's principal and characteristic functions. The α order term for both perturbation theories are

$$W^{(\alpha)} = \int_{t_0}^{t_1} F^{(\alpha)}(x_i^{(0)}(t), t, W^{(0)}, \dots, W^{(\alpha-1)}) dt, \quad (5.1)$$

$$Q^{(\alpha)} = \int_{t_0}^{t_1} G^{(\alpha)}(x_i^{(0)}(t), t, Q^{(0)}, \dots, Q^{(\alpha-1)}) dt. \quad (5.2)$$

For the initial value problem the initial endpoint generalized coordinates and momenta are fixed, while the final endpoints are perturbed. From the two-point boundary value problem, the final perturbed coordinates and momenta can be approximated to the α order by the following:

$$p_{1i}^{(\alpha)} = \int_{t_0}^{t_1} \frac{\partial F^{(\alpha)}}{\partial q_{1i}} dt, \quad q_{1i}^\alpha = - \int_{t_0}^{t_1} \frac{\partial G^{(\alpha)}}{\partial p_{1i}} dt. \quad (5.3)$$

When solving the two-point boundary value problems the final endpoint is fixed, creating a constraint on the system. The solution to the constraint can be obtained by using the fact that the endpoint is fixed at time t_1 . If the constraint of the state at the final time is removed, the term $\frac{\partial x_j}{\partial x_{1i}}$ should be the backwards state transition matrix $\Phi(\tau, t_1)$, therefore the partials of F and G with respect to the coordinates and momenta at t_1 are

$$\frac{\partial F^{(\alpha)}}{\partial q_{1i}} = \frac{\partial F^{(\alpha)}}{\partial x_j^{(0)}} \Phi_{xq}(\tau, t_1), \quad (5.4)$$

$$\frac{\partial G^{(\alpha)}}{\partial p_{1i}} = \frac{\partial G^{(\alpha)}}{\partial x_j^{(0)}} \Phi_{xp}(\tau, t_1), \quad (5.5)$$

where the state transition matrix is defined as

$$\Phi(\tau, t_1) = \begin{bmatrix} \Phi_{qq} & \Phi_{qp} \\ \Phi_{pq} & \Phi_{pp} \end{bmatrix}, \quad (5.6)$$

and the terms Φ_{xq} and Φ_{xp} are

$$\Phi_{xq} = \begin{bmatrix} \Phi_{qq} \\ \Phi_{pq} \end{bmatrix} \quad \Phi_{xp} = \begin{bmatrix} \Phi_{qp} \\ \Phi_{pp} \end{bmatrix}, \quad (5.7)$$

Matrix identities allow to write the state transition matrix as

$$\Phi(\tau, t_1) = \Phi^{-1}(t_1, \tau) = -J\Phi^T J. \quad (5.8)$$

Hence, the α order term of the perturbed state is

$$\begin{bmatrix} q_{1i}^{(\alpha)} & p_{1i}^{(\alpha)} \end{bmatrix} = \int_{t_0}^{t_1} \begin{bmatrix} -G_x^{(\alpha)} & F_x^{(\alpha)} \end{bmatrix} \begin{bmatrix} \Phi_{qp} & \Phi_{pp} \\ \Phi_{qq} & \Phi_{pq} \end{bmatrix} \Big|_{(t_1, \tau)} d\tau. \quad (5.9)$$

5.2 First-Order Perturbation Theory

The first order perturbed terms of the principal and characteristic function are

$$W^{(1)} = - \int_{t_0}^{t_1} H^{(1)}(x_i^{(0)}, \tau) d\tau, \quad Q^{(1)} = - \int_{t_0}^{t_1} H^{(1)}(x_i^{(0)}, \tau) d\tau. \quad (5.10)$$

Therefore, the correction terms to solve the different first-order two-point bound-

ary value problems are

$$q_{1i}^{(1)} = -\frac{\partial Q^{(1)}}{\partial p_{1i}} = \int_{t_0}^{t_1} \frac{\partial H^{(1)}}{\partial x_j} \frac{\partial x_j}{\partial p_{1i}} d\tau, \quad p_{1i}^{(1)} = \frac{\partial W^{(1)}}{\partial q_{1i}} = -\int_{t_0}^{t_1} \frac{\partial H^{(1)}}{\partial x_j} \frac{\partial x_j}{\partial q_{1i}} d\tau. \quad (5.11)$$

Combining the two terms in Eq. 5.11 the first-order perturbed state can be obtained:

$$\begin{bmatrix} -p_{1i}^{(1)} \\ q_{1i}^{(1)} \end{bmatrix} J = \int_{t_0}^{t_1} H_x^{(1)} \frac{\partial x_j}{\partial x_{1i}} J d\tau. \quad (5.12)$$

By manipulating the above equation by the state transition matrix properties, the first order term of the perturbed state is

$$\begin{bmatrix} q_{1i}^{(1)} \\ p_{1i}^{(1)} \end{bmatrix} = \int_{t_0}^{t_1} J \begin{bmatrix} H_q^{(1)} & H_p^{(1)} \end{bmatrix} \begin{bmatrix} \Phi_{pp}^T & -\Phi_{qp}^T \\ -\Phi_{pq}^T & \Phi_{qq}^T \end{bmatrix} \Big|_{(t_1, \tau)} d\tau. \quad (5.13)$$

Therefore the first order perturbed state at time t_1 can be expressed as:

$$x_{1i}^{(1)} = \int_{t_0}^{t_1} \begin{bmatrix} -H_q^{(1)} & H_p^{(1)} \end{bmatrix} \begin{bmatrix} \Phi_{qp}^T & \Phi_{pp}^T \\ \Phi_{qq}^T & \Phi_{pq}^T \end{bmatrix} d\tau. \quad (5.14)$$

Grouping the term leads to the following expression for the first order state vector:

$$x_{1i}^{(1)} = \int_{t_0}^{t_1} [\Phi(t_1, \tau) JH_x^{(1)}]^T d\tau. \quad (5.15)$$

The first order solution to the perturbed state is then:

$$x_{1i} = \Phi(t_1, t_0) \bar{x}_0^{(0)} + \epsilon \left[\int_{t_0}^{t_1} N(t_1, \tau) d\tau \right], \quad (5.16)$$

where $N(t_1, \tau) = [\Phi(t_1, \tau) JH_x^{(1)}]^T$. The integration must be carried out over the time defined by the nominal system, and along the nominal path in phase space. Hence, in order to calculate the perturbed state of the system the knowledge of the nominal system and the perturbed form of the Hamiltonian is needed.

5.2.1 Symplectic Structure of the Perturbation Theory

In order for the Hamiltonian structure of the perturbed system to be preserved, the perturbation theory must conserve the symplectic structure of the dynamical system. The Hamiltonian form of the equations of the motion of the perturbed system is

$$\dot{x} = J \frac{\partial H}{\partial x}, \quad (5.17)$$

where the perturbed state and Hamiltonian are expanded by a power series $x = x^{(0)} + \epsilon x^{(1)}$, $H(x, t) = H^{(0)}(x^{(0)}, t) + \epsilon H^{(1)}(x, t)$, and the J is the block matrix:

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}. \quad (5.18)$$

For the nominal system, the dynamics of the system satisfy the equations of motion of the following form:

$$\dot{x}^{(0)} = J \frac{\partial H^{(0)}}{\partial x}, \quad (5.19)$$

which have a symplectic structure and from which the nominal solution $x^{(0)}(t; x_0)$ can be obtained. The equations of motion of the full perturbed system can therefore be approximated to the first order by

$$\dot{x}(t; x_0, \epsilon) = \dot{x}^{(0)} + \epsilon \dot{x}^{(1)} = J \frac{\partial H^{(0)}}{\partial x} \Big|_{x^{(0)}} + \epsilon \left[J \frac{\partial H^{(0)}}{\partial x} \Big|_{x^{(0)}} \right]. \quad (5.20)$$

The first-order solution to the initial value problem is be of the form

$$x_1 = \Phi(t_1, t_0)x_0 + \epsilon \int_{t_0}^{t_1} \Phi(t_1, \tau) J H_x^{(0)} d\tau, \quad (5.21)$$

where $H_x^{(0)} = \frac{\partial H^{(0)}}{\partial x} \Big|_{x^{(0)}}$. The system is symplectic if the Jacobian of the system $M = \frac{\partial x_1}{\partial x_0}$ satisfies the following:

$$M^T J M = J. \quad (5.22)$$

For the initial value problem, the state at the initial time is the same for the nominal solution and the perturbed solution, $x_0 = x_0^{(0)}$. Therefore the first-order solution can be rewritten as

$$x_1 = \Phi(t_1, t_0) \left[x_0^{(0)} + \epsilon \int_{t_0}^{t_1} \Phi(t_0, \tau) J H_x^{(0)} d\tau \right]. \quad (5.23)$$

The Jacobian of the perturbed system then becomes

$$M = \frac{\partial x_1}{\partial x_0} = \Phi(t_1, t_0) \left[I + \epsilon \int_{t_0}^{t_1} \Phi^{-1}(\tau, t_0) J H_{xx}^{(0)} \Phi(\tau, t_0) d\tau \right]. \quad (5.24)$$

Therefore the symplectic condition for the state transition matrix can be checked:

$$\begin{aligned} M^T J M = & \\ & \left[I + \epsilon \int \Phi^T(\tau, t_0) H_{xx}^{(0)} \Phi(\tau, t_0) \right] \Phi^T(t_1, t_0) J \Phi(t_1, t_0) \left[I + \epsilon \int_{t_0}^{t_1} \Phi^T(\tau, t_0) H_{xx}^{(0)} J^T \Phi^{-T}(\tau, t_0) \right]. \end{aligned} \quad (5.25)$$

Since the unperturbed system is a Hamiltonian dynamical system, it is symplectic, $\Phi^T(t_1, t_0) J \Phi(t_1, t_0) = J$ and Eq. 5.25 becomes

$$M^T J M = \left[I + \epsilon \int \Phi^T(\tau, t_0) H_{xx}^{(0)} \Phi(\tau, t_0) \right] J \left[I + \epsilon \int_{t_0}^{t_1} \Phi^T(\tau, t_0) H_{xx}^{(0)} J^T \Phi^{-T}(\tau, t_0) \right]. \quad (5.26)$$

For an approximation of a perturbed system to the first order, the system must be symplectic to the first order and the following condition must be satisfied:

$$\int_{t_0}^{t_1} \Phi^T(\tau, t_0) H_{xx}^{(0)} J^T \Phi^{-T}(\tau, t_0) J d\tau + \int_{t_0}^{t_1} J \Phi^{-1}(\tau, t_0) J H_{xx}^{(0)} \Phi(\tau, t_0) d\tau = 0. \quad (5.27)$$

The following relationships can be obtained from $M^T J M = J$:

$$J^T \Phi^{(-T)}(\tau, t_0) J = \Phi(\tau, t_0), \quad (5.28)$$

$$-\Phi = J \Phi^{-1} J.$$

Therefore Eq. 5.28 becomes

$$\int_{t_0}^{t_1} \Phi^T(\tau, t_0) H_{xx}^{(0)} \Phi(\tau, t_0) d\tau - \int_{t_0}^{t_1} \Phi^T(\tau, t_0) H_{xx}^{(0)} \Phi(\tau, t_0) d\tau = 0. \quad (5.29)$$

Therefore, the system is symplectic to the first order.

CHAPTER VI

Perturbed Rotating Two-Body Problem

6.1 The Two-Body Problem

The two-body problem determines the dynamics of two point masses, m_0 and m_1 , interacting with each other. The combined gravitational parameter of these two-bodies is $\mu = G(m_0 + m_1)$, where $G = 6.67 \times 10^{-11} N(m/kg)^2$ is the universal gravitational constant and the two bodies follow the equations of motion

$$\ddot{\vec{r}}_{m_0} = -\frac{\mu}{m_0} \frac{\vec{r}_{m_0} - \vec{r}_{m_1}}{|\vec{r}_{m_0} - \vec{r}_{m_1}|^3} \quad \ddot{\vec{r}}_{m_1} = -\frac{\mu}{m_1} \frac{\vec{r}_{m_1} - \vec{r}_{m_0}}{|\vec{r}_{m_1} - \vec{r}_{m_0}|^3}, \quad (6.1)$$

where \vec{r}_{m_0} is the position vector of m_0 and \vec{r}_{m_1} is the position vector of m_1 . Both position vectors are relative to the origin of the coordinate system.

6.2 Keplerian Two-Body Problem

Kepler's problem is a special case of the two-body problem, where one of the bodies is assumed to have negligible mass ($m_0 = 0$) and the other body is a point mass at the center of mass of the system. The massive body is therefore stationary

and the particle orbits the central body. The equations of motion of the orbiting particle with respect to the central body are

$$\ddot{\vec{r}} = -\mu_1 r^{-3} \vec{r}, \quad (6.2)$$

where $\mu_1 = Gm_1$ is the mass parameter of the main body and \vec{r} is the position vector of the particle with respect to the center of mass. The equations of motion can be expressed in terms of the force potential function U , which is convenient to analyze perturbations on the system. For the Keplerian two-body problem, the equations of motion become a simple expression

$$\ddot{\vec{r}} = -\frac{\partial U}{\partial \vec{r}}, \quad (6.3)$$

where $U = \frac{\mu_1}{r}$ is the force potential function for the Keplerian two-body problem, which is a Hamiltonian dynamical system, where the Hamiltonian is defined as

$$H = \frac{1}{2} p_i^2 - \frac{\mu}{\sqrt{(q_j^2)}}, \quad (6.4)$$

where the generalized coordinates and momenta (q_i, p_i) are the relative position coordinates and momenta of the particle (\vec{r}, \vec{v}) respectively. The Keplerian two-body problem is commonly viewed as a nominal solution to most systems encountered in celestial mechanics. However, it only takes into account the gravitational effect of the central body as a point mass, and other effects such as tidal effects, mass dis-

tribution, and aerodynamic drag are neglected. Nevertheless, for most systems the Keplerian two-body problem offers the best approximation, and perturbed systems can be modeled by adding higher order terms.

6.3 Delaunay Elements

In the two-body problem, it is convenient to express the trajectory of the orbiting particle in terms of the orbit elements, as they give a better qualitative understanding of the system than the position and momentum coordinates. Delaunay elements are a set for canonical variables of the two-body problem, and can be expressed in terms of the classical orbit elements $(a, e, i, \Omega, \omega, M)$ (40):

$$l_d = M, \quad L_d = \sqrt{\mu a}, \quad (6.5)$$

$$g_d = \omega, \quad G_d = L_d \sqrt{1 - e^2}, \quad (6.6)$$

$$h_d = \Omega, \quad H_d = G_d \cos i, \quad (6.7)$$

where a is the semimajor axis of the orbit, e is the eccentricity, i is the inclination, Ω is the right ascension of the ascending node, ω is the argument of periapse, and M is the mean anomaly. The Delaunay elements can be divided in two subsets, where l_d, g_d, h_d are the generalized coordinates and L_d, G_d, H_d are the generalized momenta. The position vector of the orbiting particle can be expressed as

$$\vec{r} = r \begin{bmatrix} \cos(g_d + f) \cos h_d - \sin(g_d + f) \sin h_d \cos i \\ \cos(g_d + f) \sin h_d + \sin(g_d + f) \cos h_d \cos i \\ \sin(g_d + f) \sin i \end{bmatrix}, \quad (6.8)$$

where f is the true anomaly of the particle orbit and the distance from the center of mass to the orbiting particle r is

$$r = \frac{a(1 - e^2)}{1 + e \cos f}. \quad (6.9)$$

The true anomaly, f , can be obtained through the following relationships with the eccentric anomaly, E :

$$\tan \frac{f}{2} = \sqrt{\frac{1 + e}{1 - e}} \tan \frac{E}{2}, \quad (6.10)$$

and the mean anomaly of the orbit is

$$M = E - e \sin E. \quad (6.11)$$

6.4 Poincare Elements

The set of Delaunay elements for the two-body problem become singular for circular orbits, where the eccentricity $e = 0$. The Poincare elements for the two-body problem are related to the Delaunay elements by (40)

$$\begin{aligned}
 l_p &= l_d + g_d + h_d, & L_p &= L_d, \\
 g_p &= \sqrt{2(L_d - G_d)} \cos(g_d + h_d), & G_p &= \sqrt{2(L_d - G_d)} \sin(g_d + h_d), \\
 h_p &= \sqrt{-2G_d(\cos i - 1)} \cos h_d, & H_p &= \sqrt{-2G_d(\cos i - 1)} \sin h_d.
 \end{aligned} \tag{6.12}$$

6.5 The Restricted Three-Body Problem

Isaac Newton first studied the influence of gravitational attraction between numerous bodies in order to study a system that included the gravitational effects of the Earth and the Moon on another body. In this problem the three bodies are assumed to be point masses and only first-order gravitational effects are taken into account.

The restricted three-body problem assumes that one of the masses is negligible and does not affect the motion of the other two. The two massive bodies are in an orbit around their center of mass and a third particle is in orbit around the two point masses.

The restricted circular three-body problem further assumes that the two massive bodies are in a circular orbit around their center of mass but the particle is in an elliptical orbit around the bodies. The equations of motion of the particle can be

expressed relative to the barycenter or one of the bodies of interest.

The most common form of the equations of motion are expressed in the synodic frame, which is a rotating frame where the origin is in the barycenter of the system:

$$\begin{aligned}\ddot{x} - 2\dot{y} - x &= -\frac{(1-\epsilon)(x-\epsilon)}{r_1^3} - \frac{\epsilon(x+1-\epsilon)}{r_2^3}, \\ \ddot{y} + 2\dot{x} - y &= -\frac{(1-\epsilon)y}{r_1^3} - \frac{\epsilon y}{r_2^3}, \\ \ddot{z} &= -\frac{(1-\epsilon)z}{r_1^3} - \frac{-\epsilon z}{r_2^3}\end{aligned}\quad . \quad (6.13)$$

The parameters in Eq. 6.13 are normalized by the distance between the primary and the secondary and their orbit period around the barycenter.

Another interesting frame of reference is the inertially fixed frame in one of the massive bodies. This frame offers the possibility to study the motion of the particle with respect to one of the bodies while the other body has a smaller effect on the particle. The equations of motion of the particle with respect to the primary in the inertial frame are

$$\ddot{\vec{r}} = -\mu_1 \frac{\vec{r}}{r^3} + \mu_2 \left[\frac{\vec{r}_2 - \vec{r} - \vec{r}_1}{|\vec{r}_2 - \vec{r} - \vec{r}_1|^3} - \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|^3} \right]. \quad (6.14)$$

where \vec{r} is the position vector of the particle relative to the first body, \vec{r}_1 is the position vector of the first body with respect to the barycenter, and \vec{r}_2 is the position vector of the second body with respect to the barycenter. The first term of Eq. 6.14 is known as the direct part, as it represents the effect of the central body on the particle. The second term is the indirect part as it represents the effect between the

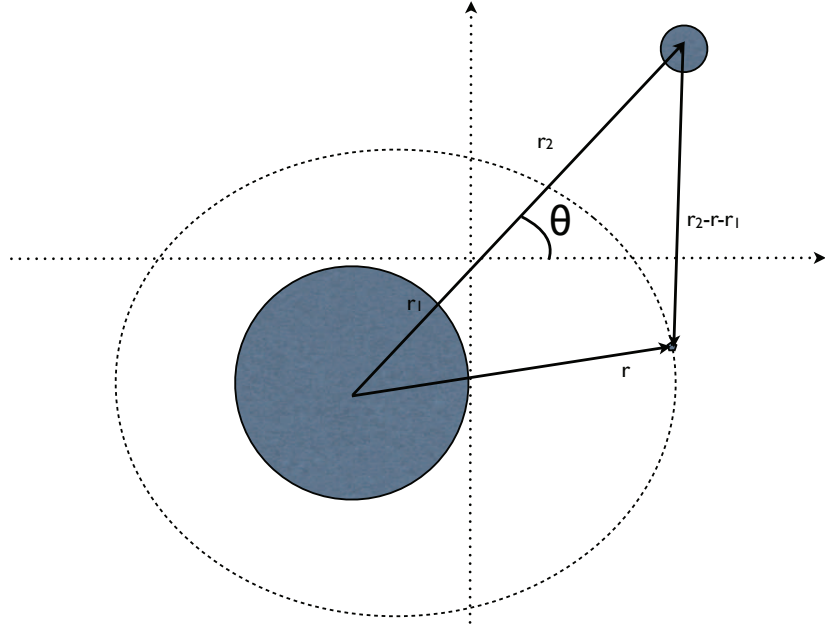


Figure 6.1: Illustration of the restricted three-body problem

central and secondary bodies and the effect between the secondary and the particle.

6.5.1 The Jacobi Integral for the Restricted Circular Three-Body Problem

There are ten known integrals for the restricted three-body problem, and eighteen are needed to completely and analytically solve the problem. Therefore, the equations of motion are solved through numerical integration. The Jacobi constant is a widely used constant of motion for the restricted three-body problem, which only exists in the synodic frame. The Jacobi integral is important for stability analysis and finding equilibrium points of the system. The Jacobi integral or constant for the restricted three-body problem is commonly expressed as the position and momentum variables

$$C_J = |\vec{v}|^2 - (x^2 + y^2) - 2 \left[\frac{\mu_1}{|\vec{r}|} + \frac{\mu_2}{|\vec{r}_1 + \vec{r} - \vec{r}_2|} \right], \quad (6.15)$$

where μ_1 and μ_2 are the gravitational parameters of the two bodies, \vec{v} is the momentum of the particle in the synodic frame, (x, y) are the cartesian coordinates of the particle in the plane of the orbit of the bodies around the barycenter, \vec{r} is the position vector from the first primary to the particle, and \vec{r}_1 and \vec{r}_2 are the position vectors of the bodies from the barycenter.

6.6 The Perturbed Rotating Two-Body Problem

The restricted circular three-body problem can be expressed as a perturbed rotating two-body problem, where a particle of negligible mass orbits a central point mass and a perturbation arises from the gravitational pull of the third body, which is rotating along with the central body around the barycenter of the perturbed system. The equations of motion for the nominal (Keplerian) two-body problem are

$$\ddot{\vec{r}} = -\mu_1 \frac{\vec{r}}{r^3}, \quad (6.16)$$

where μ_1 is the gravitational parameter of the central body and \vec{r} is the position vector of the particle with respect to the central body. The Hamiltonian of the Keplerian system is

$$H = \frac{1}{2} |\vec{v}|^2 - \frac{\mu}{|\vec{r}|}. \quad (6.17)$$

To introduce the third body into the system, it is expressed as a perturbing force that changes the Hamiltonian of the two-body system with the following first-order perturbing term:

$$H^{(1)} = \mu_2 \left[\frac{1}{|\vec{r}_2 - \vec{r} - \vec{r}_1|} - \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|^3} \cdot \vec{r} \right]. \quad (6.18)$$

Therefore, the acceleration on the particle due to the perturbation of the third body is

$$\ddot{\vec{a}} = \mu_2 \frac{\vec{r}_2 - \vec{r} - \vec{r}_1}{|\vec{r}_2 - \vec{r} - \vec{r}_1|^3} - \mu_2 \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|^3}. \quad (6.19)$$

The first term of Eq. 6.19 is the acceleration of the particle from the third body and the second term is the acceleration between the two massive bodies.

6.6.1 The Jacobi Integral for the Perturbed Rotating Two-Body Problem

Eq. 6.15 gives the Jacobi constant for the restricted three-body problem:

$$C_J = |\vec{v}|^2 - (x^2 + y^2) - 2 \left[\frac{\mu_1}{|\vec{r}|} + \frac{\mu_2}{|\vec{r}_1 + \vec{r} - \vec{r}_2|} \right]. \quad (6.20)$$

However, for the perturbed rotating two-body problem is convenient to express the Jacobi integral as a separate function of the nominal and perturbed system. Delaunay elements allow this to happen and the Jacobi integral can be expressed as:

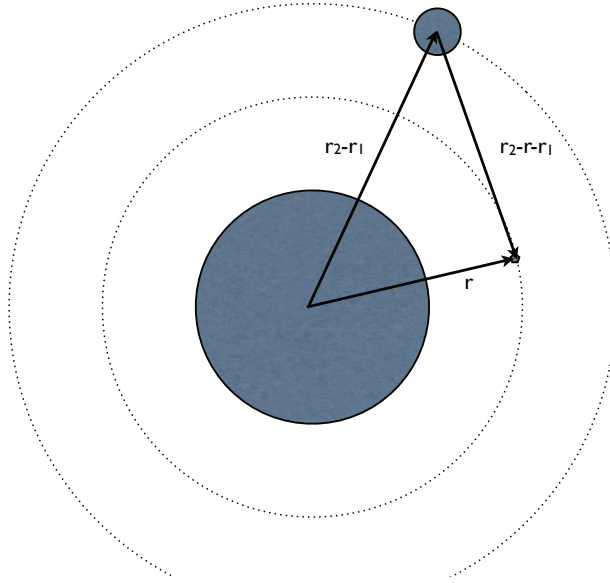


Figure 6.2: Illustration of the rotating perturbed two-body problem

$$C_J = -\frac{\mu_1^2}{2L^2} - \dot{\phi}H + \epsilon H^{(1)}(l, g, h, L, G, H), \quad (6.21)$$

where ϕ is the angular rate of the orbit of the two massive bodies and R is the perturbing part of the Hamiltonian function, and is used in order to not to confuse it with the angular momentum Delaunay element H . To show that the Jacobi integral is constant consider the time rate of change:

$$\dot{C}_J = \frac{\mu^2}{L^3} \dot{L} - \dot{\phi} \dot{H} + \epsilon \dot{H}^{(1)}(l, g, h, L, G, H). \quad (6.22)$$

Eq. 6.22 can be expanded with the appropriate chain rule

$$\dot{C}_J = \frac{\mu^2}{L^3} \dot{L} - \dot{\phi} \dot{H} + \epsilon \left[\frac{\partial R}{\partial l} \dot{l} + \frac{\partial R}{\partial g} \dot{g} + \frac{\partial R}{\partial h} \dot{h} + \frac{\partial R}{\partial L} \dot{L} + \frac{\partial R}{\partial G} \dot{G} + \frac{\partial R}{\partial H} \dot{H} \right]. \quad (6.23)$$

The Delaunay elements X_D are related to the Jacobi integral with the following relationship

$$\dot{X}_D = J \frac{\partial C_J}{\partial X_D}, \quad (6.24)$$

where J is the block matrix of the identity matrix. The equations of motion for the Delaunay elements is therefore

$$\begin{aligned} \dot{l} &= \frac{\partial C_J}{\partial L} = \frac{\mu^2}{L^3} + \epsilon \frac{\partial R}{\partial L}, & \dot{L} &= -\frac{\partial C_J}{\partial l} = -\epsilon \frac{\partial R}{\partial l}, \\ \dot{g} &= \frac{\partial C_J}{\partial G} = \epsilon \frac{\partial R}{\partial G}, & \dot{G} &= -\frac{\partial C_J}{\partial g} = -\epsilon \frac{\partial R}{\partial g}, \\ \dot{h} &= \frac{\partial C_J}{\partial H} = -\dot{\phi} + \epsilon \frac{\partial R}{\partial H}, & \dot{H} &= -\frac{\partial C_J}{\partial h} = -\epsilon \frac{\partial R}{\partial h}. \end{aligned} \quad (6.25)$$

Therefore Eq. 6.23 can be written as

$$\begin{aligned} \dot{C}_J &= \epsilon \left[-\frac{\mu^2}{L^3} \frac{\partial R}{\partial l} + \dot{\phi} \frac{\partial R}{\partial h} + \frac{\partial R}{\partial l} \frac{\mu^2}{L^3} - \frac{\partial R}{\partial h} \dot{\phi} \right] + \\ &+ \epsilon^2 \left[\frac{\partial R}{\partial l} \frac{\partial R}{\partial L} + \frac{\partial R}{\partial g} \frac{\partial R}{\partial G} + \frac{\partial R}{\partial h} \frac{\partial R}{\partial H} - \frac{\partial R}{\partial L} \frac{\partial R}{\partial l} - \frac{\partial R}{\partial G} \frac{\partial R}{\partial g} - \frac{\partial R}{\partial H} \frac{\partial R}{\partial h} \right] = 0. \end{aligned} \quad (6.26)$$

Therefore the Jacobian for the perturbed two-body problem is constant.

6.6.2 The Perturbed Circular Restricted Three-Body Problem

This section describes how the restricted circular three-body problem can be viewed as a perturbed rotating two-body problem, where the tidal effects from the third body's gravity is modeled as a perturbing force. If yet another perturbation is added to the three body system that preserves the Hamiltonian structure of the system, the problem still remains a perturbed two-body problem and can be modeled as such. Other common perturbations include higher order gravitational forces, solar radiation pressure, and the gravitational forces of other bodies.

The perturbed two-body problem where two or more perturbing forces are present, and where one of the perturbations is due to the gravitational effect of a third body, can be modeled as a perturbed three-body problem. If several more bodies are present, the perturbed two-body problem can be modeled as a restricted multiple body problem (restricted three-body problem, restricted four-body problem...), where the extra bodies act as perturbing forces.

To model the perturbed restricted three-body problem, the Keplerian orbit is still the nominal system. The equations of motion of the nominal system are therefore:

$$\ddot{\vec{r}} = -\mu_1 \frac{\vec{r}}{r^3}, \quad (6.27)$$

where \vec{r} is the position vector from the central body of the two-body problem to the orbiting particle, and μ_1 is the gravitational parameter of the body. The Hamiltonian of the Keplerian system is the same as the one given by Eq. 6.17

$$H = \frac{1}{2}|\vec{v}|^2 - \frac{\mu}{|\vec{r}|}. \quad (6.28)$$

The third body is introduced into the system as given by Eq 6.18, and the additional perturbing potential can be added. Therefore, the perturbing potential for the system is:

$$H^{(1)} = \mu_2 \left[\frac{1}{|\vec{r}_2 - \vec{r} - \vec{r}_1|} - \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|^3} \cdot \vec{r} \right] + U_p(\vec{r}), \quad (6.29)$$

where $U_p(\vec{r})$ is the potential that perturbs the restricted three-body problem. Therefore, the acceleration on the particle due to the presence of the third body and the perturbation to the restricted three-body problem is

$$\ddot{\vec{a}} = \mu_2 \frac{\vec{r}_2 - \vec{r} - \vec{r}_1}{|\vec{r}_2 - \vec{r} - \vec{r}_1|^3} - \mu_2 \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|^3} + \frac{\partial U_p}{\partial \vec{r}}. \quad (6.30)$$

The first term of Eq. 6.30 is the acceleration of the particle from the third body, the second term is the acceleration between the two massive bodies, and the third term is the perturbation acceleration to the restricted three-body problem.

Modeling the perturbing acceleration terms for the perturbed two-body problem as given in Eq. 6.30 yields a perturbed restricted three-body problem, where the perturbing potential is U_p given in Eq. 6.30.

6.7 Orbit Transfers and Impulsive Maneuvers

Orbit transfers are an important aspect of the two-body problem. Isaac Newton studied the different orbits that a celestial body could occupy, but did not envision artificial satellites nor the impulsive nature of certain orbit transfers. The goal of an orbit transfer is to minimize the cost of the maneuver.

6.7.1 The Hohmann Transfer

Walter Hohmann developed an orbit transfer method that yields the fuel optimal two-impulse solution for the two-body problem (41). The transfer takes a particle from a circular orbit to another circular orbit in the same plane, while the transfer angle is always 180 degrees. The impulses are assumed to be instantaneous, which is not feasible in real situations. The semi-major axis of the transfer orbit that takes the particle between the two circular orbits is

$$a_H = \frac{|\vec{r}_0| + |\vec{r}_1|}{2}. \quad (6.31)$$

The transfer time for transfer orbit is half the period of the transfer orbit, since the transfer angle is exactly π :

$$T_H = \pi \sqrt{\frac{a_H^3}{\mu}}. \quad (6.32)$$

The impulsive change in velocities are tangent to the orbit either along the direction of the velocity vector or in the opposite direction. The initial impulse occurs along the velocity vector if increasing the size of the orbit, and opposite to the velocity vector if decreasing. The final impulse is in the same direction as the initial impulse.

$$\begin{aligned} \Delta V_0 &= \sqrt{\frac{2\mu}{|\vec{r}_0|} - \frac{\mu}{a_H}} - \sqrt{\frac{\mu}{|\vec{r}_0|}}, \\ \Delta V_1 &= \sqrt{\frac{\mu}{|\vec{r}_1|}} - \sqrt{\frac{2\mu}{|\vec{r}_1|} - \frac{\mu}{a_H}}. \end{aligned} \quad (6.33)$$

The total cost of the impulses is the sum of the impulsive changes:

$$\Delta V_H = \Delta V_0 + \Delta V_1. \quad (6.34)$$

6.7.2 The Bi-Elliptic Transfer

The Hohmann transfer is the most fuel efficient two impulse transfer, and in most cases the most fuel efficient impulsive transfer. However, there are certain cases in which it is not the most fuel efficient impulsive transfer. The bi-elliptic is the fuel-optimal three impulse maneuver for the two-body problem. The bi-elliptic transfer requires less ΔV for the same initial and final positions if the ratio of final to the initial semimajor axis is greater than 11.94 (41).

The bi-elliptic transfer consists of a transfer between two circular orbits by first going to an intermediate orbit of higher energy, where an intermediate impulse is carried out. Therefore, there are two transfer orbits, one that takes from the initial orbit to the intermediate orbit, and one that takes from the intermediate orbit to the final orbit. The bi-elliptic transfer always consists of two 180 degree transfers, for a total of three impulsive maneuvers.

The semimajor axii of the transfer orbits are

$$\begin{aligned} a_{t0} &= \frac{|\vec{r}_0| + |\vec{r}_1|}{2}, \\ a_{t1} &= \frac{|\vec{r}_1| + |\vec{r}_2|}{2}. \end{aligned} \tag{6.35}$$

where a_{t0} is the semimajor axis of the first transfer orbit and a_{t1} is the semimajor axis of the second transfer orbit. The impulse maneuvers are

$$\begin{aligned} \Delta V_0 &= \sqrt{\frac{2\mu}{|\vec{r}_0|} - \frac{\mu}{a_{t0}}} - \sqrt{\frac{\mu}{|\vec{r}_0|}} \\ \Delta V_1 &= \sqrt{\frac{2\mu}{|\vec{r}_1|} - \frac{\mu}{a_{t1}}} - \sqrt{\frac{2\mu}{|\vec{r}_1|} - \frac{\mu}{a_{t0}}}, \\ \Delta V_2 &= \sqrt{\frac{\mu}{|\vec{r}_1|}} - \sqrt{\frac{2\mu}{|\vec{r}_2|} - \frac{\mu}{a_{t1}}}, \end{aligned} \tag{6.36}$$

The total transfer time is

$$T_t = \pi \sqrt{\frac{a_{t0}^3}{\mu}} + \pi \sqrt{\frac{a_{t1}^3}{\mu}}. \tag{6.37}$$

6.8 Lambert's Problem

Lambert's problem represents the two-point boundary value problem for the two-body problem. Given two position vectors \vec{r}_0 , \vec{r}_1 , and a transfer time $T = t_1 - t_0$, the problem is to find a solution for the orbit that joins these two position vectors in the given time span.

Based on geometric reasoning, Johann Heinrich Lambert formulated the theorem that bears his name, which claims that *the orbital transfer time depends only upon the semimajor axis, the sum of the distances of the initial and final points of the arc from the center of force, and the length of the chord joining these points* (12):

$$\Delta t = \Delta t(a, |\vec{r}_0| + |\vec{r}_1|, |\vec{r}_1 - \vec{r}_0|). \quad (6.38)$$

This theorem was proven by Joseph Louis Lagrange and gave mathematical form with the following equation for elliptic, parabolic and hyperbolic orbits (42):

$$\sqrt{\mu}(t_1 - t_0) = \begin{cases} a^{3/2}[(\alpha - \sin \alpha) - (\beta - \sin \beta)] & E < 0, \\ \sqrt{\frac{2}{9}} \left[(s)^{3/2} - (s - c)^{3/2} \right] & E = 0, \\ (-a)^{3/2}[(\sinh \gamma - \gamma) - (\sinh \delta - \delta)] & E > 0, \end{cases} \quad (6.39)$$

where E is the Keplerian energy of the system, t_0 and t_1 are the initial and final times, $\alpha, \beta, \delta, \gamma$ are related to the transfer orbit angle, c is the chord length between the initial and final position vectors, and s is a parameter that contains the initial and final position vectors:

$$\sin \frac{\alpha}{2} = \sqrt{\frac{|\vec{r}_0| + |\vec{r}_1| + |\vec{r}_1 - \vec{r}_0|}{4a}}, \quad (6.40)$$

$$\sin \frac{\beta}{2} = \sqrt{\frac{|\vec{r}_0| + |\vec{r}_1| - |\vec{r}_1 - \vec{r}_0|}{4a}}, \quad (6.41)$$

$$\sinh \frac{\gamma}{2} = \sqrt{\frac{|\vec{r}_0| + |\vec{r}_1| + |\vec{r}_1 - \vec{r}_0|}{-4a}}, \quad (6.42)$$

$$\sinh \frac{\delta}{2} = \sqrt{\frac{|\vec{r}_0| + |\vec{r}_1| - |\vec{r}_1 - \vec{r}_0|}{-4a}}, \quad (6.43)$$

$$s = \frac{|\vec{r}_0| + |\vec{r}_1| + |\vec{r}_1 - \vec{r}_0|}{2}, \quad (6.44)$$

$$c = |\vec{r}_1 - \vec{r}_0|. \quad (6.45)$$

Eq. 6.39 is known as *Lambert's equation* in its elliptic, parabolic and hyperbolic forms, and provides a constraint relating Δt to \vec{r}_0 and \vec{r}_1 , through a . For a given 2PBVP Lambert's equation defines a semimajor axis for the transfer. Solution for this a is known as Lambert's Problem. Many techniques exist to solve Lambert's problem, which include the use of universal variables, f and g series, and other iterative methods. A detailed description of the techniques can be found in Battin (12) and Vallado (13).

6.8.1 Solutions to Lambert's Problem

Battin developed the equations for the endpoint velocities to solve the two-point boundary value problem for the two-body problem. Once the semimajor axis of the orbit that connects the two points in the given time is found, the velocity vectors can be obtained with the following equations (43):

$$\vec{V}_0 = (B - A) \frac{\vec{r}_0}{|\vec{r}_0|} + (A + B) \frac{\vec{r}_1 - \vec{r}_0}{|\vec{r}_1 - \vec{r}_0|}, \quad (6.46)$$

$$\vec{V}_1 = (A + B) \frac{\vec{r}_1 - \vec{r}_0}{|\vec{r}_1 - \vec{r}_0|} - (B - A) \frac{\vec{r}_1}{|\vec{r}_1|}, \quad (6.47)$$

where the terms A and B will vary depending on whether the orbit is elliptic, parabolic, or hyperbolic

$$A = \begin{cases} \sqrt{\frac{\mu}{4a}} \cot \frac{\alpha}{2} & E < 0 \\ \sqrt{\frac{\mu}{2s}} & E = 0 \\ \sqrt{\frac{\mu}{-4a}} \coth \frac{\gamma}{2} & E > 0, \end{cases} \quad (6.48)$$

$$B = \begin{cases} \sqrt{\frac{\mu}{4a}} \cot \frac{\beta}{2} & E < 0, \\ \sqrt{\frac{\mu}{2(s-c)}} & E = 0, \\ \sqrt{\frac{\mu}{-4a}} \coth \frac{\delta}{2} & E > 0. \end{cases} \quad (6.49)$$

CHAPTER VII

Example: Solving the Perturbed Two-Body Two-Point Boundary Value Problem

The two-body problem is a Hamiltonian dynamical system, and it can be analyzed by the tools offered by Hamiltonian dynamics theory. The perturbation theory for Hamilton's principal function described in Chapter *III* can be applied to solve the perturbed two-body two-point boundary value problem.

In a Hamiltonian form, the set of generalized coordinates can be chosen to represent the position vector \vec{r} of the orbiting particle and the generalized momenta to represent the velocity vector \vec{v} . Therefore the Hamiltonian of the Keplerian two-body problem is equal to the Hamiltonian of the system:

$$H = \frac{1}{2}\vec{v} \cdot \vec{v} - \frac{\mu_1}{r} = -\frac{\mu_1}{2a}, \quad (7.1)$$

where μ_1 is the gravitational parameter of the central body and a is the semimajor axis of the particle's orbit.

7.1 Hamilton's Principal Function for the Two-Body Problem

Hamilton's principal function solves the two-body problem by obtaining the end-point momenta through simple differentiation, and since the two-body problem is a Hamiltonian dynamical system, Hamilton's principal function can be used to obtain the solution for the problem for the two-body problem. Hamilton's principle implies that there must exist a principal function relating the initial state at t_0 to the final state at t_1 . The principal function needs to satisfy the following partial differential equations:

$$-\frac{\partial W}{\partial t_0} + H(r_{0i}, -\frac{\partial W}{\partial r_{0i}}, t_0) = 0, \tag{7.2}$$

$$\frac{\partial W}{\partial t_1} + H(r_{1i}, \frac{\partial W}{\partial r_{1i}}, t_1) = 0,$$

where the Hamiltonian H is given by Eq. 7.1. The principal function also needs to satisfy two partial differential equations:

$$v_{0i} = -\frac{\partial W}{\partial r_{0i}}, \tag{7.3}$$

$$v_{1i} = \frac{\partial W}{\partial r_{1i}}. \tag{7.4}$$

If Hamilton's principal function for the two-body problem is known, then the two-point boundary value problem can be solved analytically.

Theorem 1. For a specific energy level E , Hamilton's principal function for the two

body problem is

$$W(r_{0i}, r_{1i}, t_0, t_1) = \begin{cases} \sqrt{\mu a} [(\alpha + \sin \alpha) - (\beta + \sin \beta)] + \frac{\mu}{2a} (t_1 - t_0) & E < 0, \\ \sqrt{8\mu} \left[\sqrt{\frac{s}{2}} - \sqrt{\frac{s-c}{2}} \right] & E = 0, \\ \sqrt{-\mu a} [(\delta + \sinh \delta) - (\gamma + \sinh \gamma)] + \frac{\mu}{-2a} (t_1 - t_0) & E > 0, \end{cases} \quad (7.5)$$

where the angles and parameters are defined as:

$$\sin \frac{\alpha}{2} = \sqrt{\frac{\sqrt{r_{0i}^2} + \sqrt{r_{1i}^2} + \sqrt{(r_{1i} - r_{0i})^2}}{4a}}, \quad (7.6)$$

$$\sin \frac{\beta}{2} = \sqrt{\frac{\sqrt{r_{0i}^2} + \sqrt{r_{1i}^2} - \sqrt{(r_{1i} - r_{0i})^2}}{4a}}, \quad (7.7)$$

$$\sinh \frac{\gamma}{2} = \sqrt{\frac{\sqrt{r_{0i}^2} + \sqrt{r_{1i}^2} + \sqrt{(r_{1i} - r_{0i})^2}}{-4a}}, \quad (7.8)$$

$$\sinh \frac{\delta}{2} = \sqrt{\frac{\sqrt{r_{0i}^2} + \sqrt{r_{1i}^2} - \sqrt{(r_{1i} - r_{0i})^2}}{-4a}}, \quad (7.9)$$

$$s = \frac{\sqrt{r_{0i}^2} + \sqrt{r_{1i}^2} + \sqrt{(r_{1i} - r_{0i})^2}}{2}, \quad (7.10)$$

$$c = \sqrt{(r_{1i} - r_{0i})^2}, \quad (7.11)$$

where the semimajor axis, a , is related to the total energy/Hamiltonian of the system

by:

$$a = -\frac{\mu}{2E}. \quad (7.12)$$

Proof: Hamilton's principal function, W , needs to satisfy the following two partial differential equations:

$$\frac{\partial W}{\partial t_0} = \begin{cases} -\frac{\mu}{2a} & E < 0, \\ 0 & E = 0, \\ \frac{\mu}{2a} & E > 0. \end{cases} \quad (7.13)$$

Differentiating W with respect to t_1 the following expression is obtained:

$$\frac{\partial W}{\partial t_1} = \begin{cases} \frac{\mu}{2a} & E < 0, \\ 0 & E = 0, \\ -\frac{\mu}{2a} & E > 0. \end{cases} \quad (7.14)$$

Therefore Eqs. 7.13 and 7.14 satisfy the two partial differential equations. Recall that the energy equation for the Keplerian two-body problem is

$$E = -\frac{\mu}{2a} = H \left(r_i, \frac{\partial W}{\partial r_i} \right). \quad (7.15)$$

Next, it must be shown that W satisfies the two boundary conditions described in Chapter 2. Differentiating W with respect to q_{0i} the following expression is obtained:

$$\frac{\partial W}{\partial r_{0i}} = \begin{cases} \frac{\partial W}{\partial \alpha} \frac{\partial \alpha}{\partial r_{0i}} + \frac{\partial W}{\partial \beta} \frac{\partial \beta}{\partial r_{0i}} & E < 0, \\ \frac{\partial W}{\partial s} \frac{\partial s}{\partial r_{0i}} + \frac{\partial W}{\partial (s-c)} \frac{\partial (s-c)}{\partial r_{0i}} & E = 0, \\ \frac{\partial W}{\partial \gamma} \frac{\partial \gamma}{\partial r_{0i}} + \frac{\partial W}{\partial \delta} \frac{\partial \delta}{\partial r_{0i}} & E > 0, \end{cases} \quad (7.16)$$

which is equal to the following expression:

$$\frac{\partial W}{\partial r_{0i}} = \begin{cases} \sqrt{\mu a} \left[(1 + \cos \alpha) \frac{\partial \alpha}{\partial r_{0i}} - (1 + \cos \beta) \frac{\partial \beta}{\partial r_{0i}} \right] & E < 0, \\ \sqrt{8\mu} \left[\frac{1}{2} \frac{1}{\sqrt{s}} \frac{\partial s}{\partial r_{0i}} - \frac{1}{2} \frac{1}{\sqrt{s-c}} \frac{\partial (s-c)}{\partial r_{0i}} \right] & E = 0, \\ \sqrt{-\mu a} \left[(1 + \cosh \delta) \frac{\partial \delta}{\partial r_{0i}} - (1 + \cosh \gamma) \frac{\partial \gamma}{\partial r_{0i}} \right] & E < 0. \end{cases} \quad (7.17)$$

The partial differential equations of the angle quantities with respect to the position vectors in Eq. 7.17 are included in Appendix A. The partial of the principal function with respect to r_{0i} can therefore be written as :

$$\frac{\partial W}{\partial r_{0i}} = \begin{cases} \frac{1}{2} \sqrt{\frac{\mu}{a}} \left[\left(\cot \frac{\alpha}{2} - \cot \frac{\beta}{2} \right) \frac{r_{0i}}{\sqrt{r_{0i}^2}} + \left(-\cot \frac{\alpha}{2} - \cot \frac{\beta}{2} \right) \frac{r_{1i} - r_{0i}}{\sqrt{(r_{1i} - r_{0i})^2}} \right] & E < 0, \\ \sqrt{\frac{\mu}{2}} \left[\left(\frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s-c}} \right) \frac{r_{0i}}{\sqrt{r_{0i}^2}} + \left(\frac{1}{\sqrt{s}} + \frac{1}{\sqrt{s-c}} \right) \frac{r_{1i} - r_{0i}}{\sqrt{(r_{1i} - r_{0i})^2}} \right] & E = 0, \\ \frac{1}{2} \sqrt{\frac{\mu}{-a}} \left[\left(\coth \frac{\gamma}{2} - \coth \frac{\delta}{2} \right) \frac{\bar{r}_1}{|\bar{r}_1|} + \left(-\coth \frac{\gamma}{2} - \coth \frac{\delta}{2} \right) \frac{r_{1i} - r_{0i}}{\sqrt{(r_{1i} - r_{0i})^2}} \right] & E < 0. \end{cases} \quad (7.18)$$

Eq. 7.18 can be collected in a compact form as the following expression:

$$\frac{\partial W}{\partial r_{0i}} = (A - B) \frac{r_{0i}}{\sqrt{r_{0i}^2}} - (A + B) \frac{r_{1i} - r_{0i}}{\sqrt{(r_{1i} - r_{0i})^2}}, \quad (7.19)$$

where the A and B parameters in Eq. 7.19 are expressed as:

$$A = \begin{cases} \sqrt{\frac{\mu}{4a}} \cot \frac{\alpha}{2} & E < 0, \\ \sqrt{\frac{\mu}{2s}} & E = 0, \\ \sqrt{\frac{\mu}{-4a}} \coth \frac{\gamma}{2} & E > 0, \end{cases} \quad (7.20)$$

$$B = \begin{cases} \sqrt{\frac{\mu}{4a}} \cot \frac{\beta}{2} & E < 0, \\ \sqrt{\frac{\mu}{2(s-c)}} & E = 0, \\ \sqrt{\frac{\mu}{-4a}} \coth \frac{\delta}{2} & E > 0. \end{cases} \quad (7.21)$$

Similarly, differentiating W with respect to r_{1i} the following relationship is obtained,

$$\frac{\partial W}{\partial r_{1i}} = \begin{cases} \sqrt{\mu a} \left[(1 + \cos \alpha) \frac{\partial \alpha}{\partial r_{1i}} - (1 + \cos \beta) \frac{\partial \beta}{\partial r_{1i}} \right] & E < 0, \\ \sqrt{8\mu} \left[\frac{1}{2} \frac{1}{\sqrt{s}} \frac{\partial s}{\partial r_{1i}} - \frac{1}{2} \frac{1}{\sqrt{s-c}} \frac{\partial (s-c)}{\partial r_{1i}} \right] & E = 0, \\ \sqrt{-\mu a} \left[(1 + \cosh \delta) \frac{\partial \delta}{\partial r_{1i}} - (1 + \cosh \gamma) \frac{\partial \gamma}{\partial r_{1i}} \right] & E > 0. \end{cases} \quad (7.22)$$

Eq. 7.22 can be collected in the following form:

$$\frac{\partial W}{\partial r_{1i}} = (A + B) \frac{r_{1i} - r_{0i}}{\sqrt{(r_{1i} - r_{0i})^2}} - (B - A) \frac{r_{1i}}{\sqrt{r_{1i}^2}}, \quad (7.23)$$

where A and B are given by Eqs. 7.20 and 7.21. To verify that this W indeed satisfies the boundary conditions, Eqs. 7.19 and 7.23 are compared to the solution to the two-point boundary value problem. Richard Battin goes through a detailed procedure by

the use of geometry to obtain the following solution (12), (43):

$$V_{0i} = (B - A) \frac{r_{0i}}{\sqrt{r_{0i}^2}} + (A + B) \frac{r_{1i} - r_{0i}}{\sqrt{(r_{1i} - r_{0i})^2}}, \quad (7.24)$$

$$V_{1i} = (A + B) \frac{r_{1i} - r_{0i}}{\sqrt{(r_{1i} - r_{0i})^2}} - (B - A) \frac{r_{1i}}{\sqrt{r_{1i}^2}}. \quad (7.25)$$

It becomes apparent that Eq. 7.19 is the negative of Eq. 7.24 and Eq. 7.23 is exactly equal to 7.25. From Hamilton's theorems it is known that Eq. 7.19 has to be the negative of the initial velocity, and that Eq. 7.23 has to be equal to the final velocity. Therefore the proposed W function satisfies the required conditions to be the Hamilton's principal function of the two-body problem.

7.1.1 Energy Constraint

The principal function is restricted to the energy surface $E = H$, where H is the Hamiltonian of the system. The degeneracy condition that holds for the principal function is (33)

$$\left| \frac{\partial^2 W}{\partial r_{0i} \partial r_{1j}} \right| = 0. \quad (7.26)$$

Therefore, there exists a zero-valued eigenvalue with corresponding left and right eigenvectors.

Theorem 2 1. The velocity vectors v_{0i} and v_{1i} are the left and right eigenvectors cor-

responding to the nullspace of $\frac{\partial^2 W}{\partial r_{0i} \partial r_{1i}}$.

Proof:

$$\frac{\partial^2 W}{\partial r_{0i} \partial r_{1i}} v_{1i} = \frac{\partial^2 W}{\partial r_{0i} \partial r_{1i}} \frac{\partial W}{\partial r_{1i}}. \quad (7.27)$$

Eq. 7.27 can also be expressed as:

$$\frac{\partial^2 W}{\partial r_{0i} \partial r_{1i}} v_{1i} = \frac{\partial}{\partial r_{0i}} \left[\frac{1}{2} \frac{\partial W}{\partial r_{1i}} \frac{\partial W}{\partial r_{1i}} \right] = \frac{\partial}{\partial r_{0i}} \left[\frac{1}{2} v_{1i}^2 \right]. \quad (7.28)$$

The energy equation for the two-body problem is equal to the Hamiltonian of the system given by 7.1

$$E = \frac{1}{2} v_i^2 - \frac{\mu}{\sqrt{r_i^2}}. \quad (7.29)$$

Therefore Eq. 7.28 can be expressed as

$$\frac{\partial}{\partial r_{0i}} \left[\frac{1}{2} v_{1i}^2 \right] = \frac{\partial}{\partial r_{0i}} \left[E + \frac{\mu}{\sqrt{r_{1i}^2}} \right] = 0, \quad (7.30)$$

as r_{0i} and r_{1i} are independent and the energy is fixed. Thus, the second partial of Hamilton's principal function is

$$\frac{\partial^2 W}{\partial r_{0i} \partial r_{1i}} v_{1i} = 0. \quad (7.31)$$

Similarly, the left eigenvector expression can be written as:

$$v_{0i} \frac{\partial^2 W}{\partial r_{0i} \partial r_{1i}} = \frac{\partial}{\partial r_{1i}} \left[\frac{1}{2} \frac{\partial W}{\partial r_{0i}} \cdot \frac{\partial W}{\partial r_{0i}} \right] = \frac{\partial}{\partial r_{1i}} \left[\frac{1}{2} v_{0i}^2 \right]. \quad (7.32)$$

Substituting the relationship given by Eq. 7.29 in Eq 7.32 gives:

$$\frac{\partial}{\partial r_{1i}} \left[\frac{1}{2} v_{0i}^2 \right] = \frac{\partial}{\partial r_{1i}} \left[E + \frac{\mu}{\sqrt{r_{0i}^2}} \right] = 0, \quad (7.33)$$

and therefore Eq. 7.32 becomes the following:

$$v_{0i} \frac{\partial^2 W}{\partial r_{0i} \partial r_{1i}} = 0. \quad (7.34)$$

7.2 Perturbation Theory for the Two-Body Problem Hamilton's Principal Function

The Hamiltonian of the Keplerian two-body problem is given Eq. 7.1. If a perturbation is present from an integrable potential, the new Hamiltonian can be expressed as:

$$H = \frac{1}{2}v_i^2 - \frac{\mu}{\sqrt{r_i^2}} + H^{(1)}(r_i), \quad (7.35)$$

where $H^{(1)}(r_i)$ is the perturbation term of the Hamiltonian, a function of position vectors only, and is the difference between the perturbed and unperturbed Hamiltonians. Since the Hamiltonian of the system is only quadratic in the momenta, the total principal function will be simplified as $\frac{\partial^\alpha H}{\partial v_i \partial v_j \partial v_k \dots}$ and for $\alpha = 3, 4, \dots, \infty$ it vanishes. Therefore the perturbed principal function is,

$$W^{(1)} = - \int_{t_0}^{t_1} [H^{(1)}(r_i(t)) dt], \quad (7.36)$$

$$W^{(\alpha)} = - \frac{1}{2} \int_{t_0}^{t_1} \left[\sum_{\beta=1}^{\alpha-1} v_i^{(\beta)}(t) v_i^{(\alpha-\beta)}(t) dt \right] \quad \alpha = 2, \dots, n,$$

where the first order perturbed velocity vector is

$$v_i^{(1)}(t) = \int_t^{t_1} \frac{\partial H^{(1)}}{\partial r_j(\tau)} \frac{\partial_b r_j(\tau)}{\partial_b r_i(t)} d\tau, \quad (7.37)$$

and the i^{th} component of the initial velocity, $v_{0i} = -\frac{\partial W}{\partial r_{0i}}$. Therefore the required initial velocity can be obtained in order to solve the two-point boundary value problem for the flight time $t_1 - t_0$:

$$v_{0i}^{(1)} = \int_{t_0}^{t_1} \frac{\partial H^{(1)}}{\partial r_j} \frac{\partial_b r_j}{\partial_b r_{1i}} dt, \tag{7.38}$$

$$v_{0j}^{(\alpha)} = \frac{1}{2} \int_{t_0}^{t_1} \left[\sum_{\beta=1}^{\alpha-1} \left(\frac{\partial v_i^{(\beta)}}{\partial r_{0j}} v_i^{(\alpha-\beta)} + \frac{\partial v_i^{(\alpha-\beta)}}{\partial r_{1j}} v_i^{(\beta)} \right) \right] dt.$$

Therefore, at each order, the correction to the initial momenta of order α depends on the momenta of order $(\alpha - 1)$.

7.3 Implementation of Perturbation Theory to Numerical Simulations

The theory described in Chapter IV allows to solve perturbed two-point boundary value problems using Hamilton’s principal function. The perturbation theory yields the required change of the initial velocity in order to hit the intended target. In the two-body problem there are several known analytical solutions to solve two-point boundary value problems. These include the Hohmann and bi-elliptic transfers, as well as Lambert’s problem, which are described in Chapter VI.

In order to successfully implement the perturbation theory, the initial and final position coordinates and times must be defined: $(r_{0i}^{(0)}, r_{1i}^{(0)}, t_1 - t_0)$. Once the parameters are defined the nominal Keplerian two-body problem must be solved and the initial and final velocities obtained along the nominal trajectory in phase space.

If a perturbation is introduced to the system, the nominal solution is no longer valid to solve the two-point boundary value problem, the initial velocity obtained through the nominal system will drive the particle to a point other than the intended

target $r_{i_i}^{(0)}$. Therefore, the initial velocity vector must be corrected in order to hit the desired target.

Assuming that the mathematical form of the perturbing potential is known and that the perturbing force preserves the Hamiltonian structure of the system, the perturbation theory can be applied to solve the perturbed two-body two-point boundary value problem. As seen in Chapter VI, the knowledge of the nominal solution along the mathematical form of the perturbing potential is sufficient to solve the perturbed problem.

The nominal solution can be obtained by integrating the equations of motion using a Runge-Kutta integration function in MATLAB, *ode45*, where initial conditions are given. The time step size of the integration method has to be also defined. The MATLAB *ode45* function will yield the positions and velocities of the nominal system at every time step.

The nominal solution can be used to solve for the the higher order required velocities given in Eq. 7.38. The equations for the corrected velocities must be solved by quadratures and starting with the first order solution, as each order is dependent on the solution of the smaller orders. One must choose the order at which to truncate the perturbation theory, as the higher orders will get computationally more burdensome. The quadratures to solve the integrals can be carried out by functions already offered by MATLAB such as the trapezoidal rule can be coded in MATLAB using the extended Simpson's rule, or Boole's rule.

Once the correction term velocities are calculated using Eq. 7.38, it is added to the nominal velocity such that the corrected initial velocity has the following form:

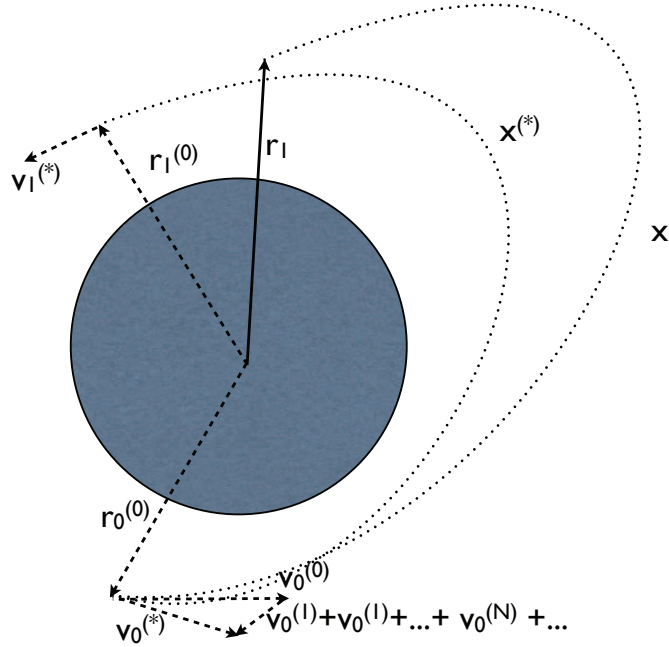


Figure 7.1: Illustration of the corrected two-point boundary value problem

$$v_{0i}^{(*)} = v_{0i}^{(0)} + \sum_{\alpha=1}^N v_{0i}^{(\alpha)}. \quad (7.39)$$

The corrected velocities yield a trajectory $x_i^{(*)}$ that hits the intended target $r_{1i}^{(0)}$ in the desired transfer time $T = t_1 - t_0$. The situation is illustrated in Fig. 7.1, where it is shown that if no correction is made the nominal initial velocity will transfer the particle to a point other than the intended target. The correction to the velocity allow the particle to hit the originally intended target despite the presence of a perturbing force.

7.4 Perturbation Theory for the Hohmann Transfer

For certain missions, the Hohmann transfer is the desired transfer method between two orbits. However, the Hohmann transfer only accurately solves the problem at the nominal level, if there are perturbations presents the particle will deviate from the nominal path. Therefore, using the nominal solution, the initial impulse will drive the particle to a point other than the intended target.

Fig. 7.2 illustrates the Hohmann transfer in the presence of a perturbation. The nominal solutions obtains ΔV_0 and ΔV_1 that takes that particle from the initial orbit r_{0i} to r_{1i} in the given transfer time $T = t_1 - t_0$ through the trajectory $\vec{x}^{(0)}$. The figure also shows what happens when the system is perturbed: the perturbed trajectory \vec{x} starts at the same orbit as the nominal solution, but when the nominal ΔV_0 is applied the perturbed transfer orbit follows a trajectory that in the transfer time $t_1 - t_0$ will hit a point other than the desired target \vec{r}_1 .

The perturbation theory allows to obtain the solution that will let the particle achieve the nominal target $r_1^{(0)}$ despite the presence of a perturbation. The expressions in Eq. 7.38 allow to obtain the correction terms for the initial velocity that will let the particle reach its target. Therefore the required velocity to go from the position at t_0 , \vec{r}_0 , to the position at t_1 , \vec{r}_1 , ΔV_{0i}^* is

$$\Delta V_{0i}^* = \Delta V_{0i}^{(0)} + v_{0i}^{(1)} + v_{0i}^{(2)} + \dots + v_{0i}^{(n)} + \dots \quad (7.40)$$

The solution obtained by the perturbation theory allows for the particle to hit the desired target for the transfer time $T = t_1 - t_0$ point as seen in Fig. 7.3. When

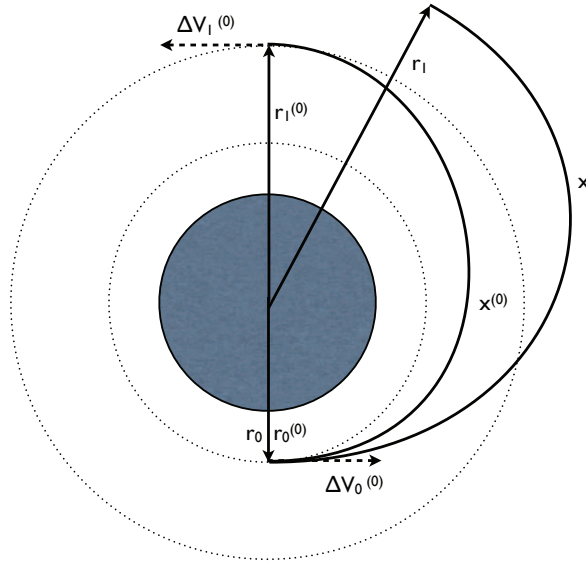


Figure 7.2: Illustration of the nominal and perturbed Hohmann transfer

the particle reaches r_{1i} a second burn is needed to enter the target orbit. However, this final burn will be different from the nominal impulse ΔV_{1i} . This problem can be solved by knowing the momenta at t_1 and subtracting the circular velocity of the particle necessary to orbit in the target orbit:

$$\Delta V_1^{(*)} = \sqrt{\frac{\mu}{\sqrt{r_{1i}^2}}} - v(t_1). \quad (7.41)$$

7.5 Perturbation Theory for Lambert's Problem

As with the Hohmann, the perturbation theory can be applied to Lambert's problem, which solves the two-body two-point boundary value problem. Lambert's problem obtains the orbit that connects two points in space for a transfer time, $T = t_1 - t_0$,

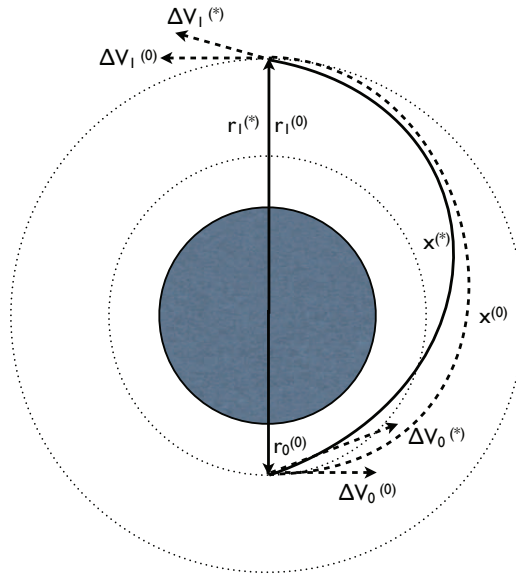


Figure 7.3: Illustration of the corrected Hohmann transfer

and the velocities required to connect those two points are given in Chapter 4. The nominal solution does not take into account perturbations such as oblateness and third-body tidal effects.

The solution to Lambert's problem allows to obtain the positions and velocities at times t_0 and t_1 given a transfer time $T = t_1 - t_0$. When a perturbation is present the initial velocity obtained from solving Lambert's problem will lead the orbiting particle to be at point other than the desired target at t_1 . Fig. 7.4 illustrates the nominal two-body two-point boundary value problem solution and what happens to the trajectory when the orbit is perturbed.

The perturbation theory allows to change the nominal initial velocity in order to hit the desired target in the given transfer time. This situation is illustrated in Fig. 7.5. The nominal system goes from $r_{0i}^{(0)}$ to $r_{1i}^{(0)}$ in time $T = t_1 - t_0$, the initial and final

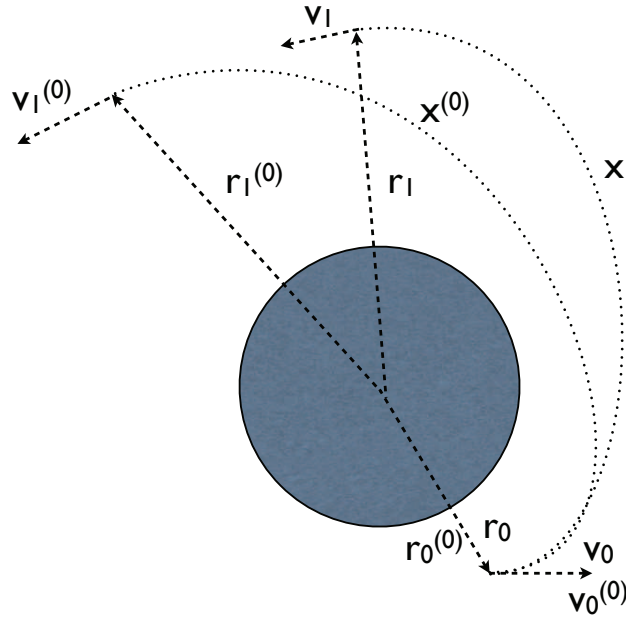


Figure 7.4: Illustration of the nominal and perturbed two-body 2BVP

velocities are $v_{0i}^{(0)}$ and $v_{1i}^{(0)}$ respectively. However in the presence of a perturbation the particle will be at $r_1^{(*)}$ as shown in Fig. 7.4. With the perturbation theory, a correction to the initial velocity can be obtained:

$$v_{0i}^* = v_{0i}^{(0)} + v_{0i}^{(1)} + v_{0i}^{(2)} + \dots + v_{0i}^{(n)} + \dots \quad (7.42)$$

where $v_{0i}^{(1)} + v_{0i}^{(2)} + \dots + v_{0i}^{(n)} + \dots$ are calculated from Eq. 7.38, once the nominal trajectory is obtained.

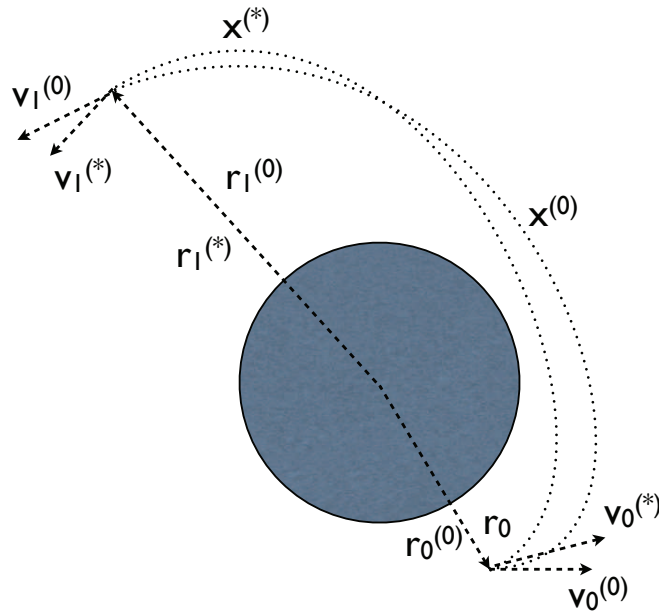


Figure 7.5: Illustration of the nominal and perturbed two-body 2BVP

7.6 Contour Map for the Nominal Two-Point Boundary Value Problem

In order to analyze the perturbed two-body two-point boundary value problem, the perturbation theory for Hamilton's principal function can be applied to obtain the required change in initial velocity necessary to hit the target point, and therefore have a "corrected" two-point boundary value problem. If the goal of the mission is to go from r_{0i} to r_{1i} in a transfer time $T = t_1 - t_0$, one can calculate the nominal trajectory in phase space and use that result to obtain the necessary ΔV_0 to solve the perturbed case.

The theory is useful to solve individual cases of two-point boundary value problems, and might offer fuel savings in certain situations, however not much insight to the problem is gained with such approach. Instead, the behavior of the system in the

vicinity of the solution can be studied by simultaneously solving multiple two-point boundary value problems. This can be done by solving the nominal problem that transfers a particle from r_{0i} to r_{1i} in time $T = t_1 - t_0$ while calculating the total impulse, ΔV , of the transfer. Solutions in the vicinity of the problem can be obtained by slightly deviating the final position r_{1i} and the final time t_1 . This method will allow to obtain a contour map of the total impulse cost of the transfer around the vicinity of the solution.

An example of this method is illustrated by Fig. 7.6. Consider a spacecraft on a circular low-Earth orbit. The desired maneuver is a Hohmann transfer to a circular geocentric orbit, where the transfer angle is $\Theta = 180$ degrees. The goal is to study what happens in the vicinity of the solution at r_{1i} , therefore several Lambert's problems have to be solved simultaneously. In order to do that, the transfer angle and transfer time are slightly deviated by $\Delta\Theta$ and ΔT from the Hohmann solution, as illustrated in Fig. 7.6, while the initial position remains the same for all orbits, $r_{0i}^{(0)}$.

In order to carry out this procedure, the Hohmann transfer needs to be solved first yielding the nominal solution for the transfer. The total transfer impulse, ΔV , and the transfer time, T_H , to go from initial position vector, r_{0i} , to the final position vector, r_{1i} , are obtained. Simultaneously, the transfer is deviated by an angle $\Delta\Theta$ and transfer time ΔT while transferring to the same intended orbit. By solving for the required transfer impulse for several points in the $(\Delta\Theta, \Delta T_H)$ space, where the point $(0, 0)$ corresponds to the Hohmann transfer, a contour map of the vicinity of the Hohmann transfer can be obtained. The contour lines would illustrate the different impulse levels ΔV .

Consider then a spacecraft in a circular low-Earth orbit where the desired ma-

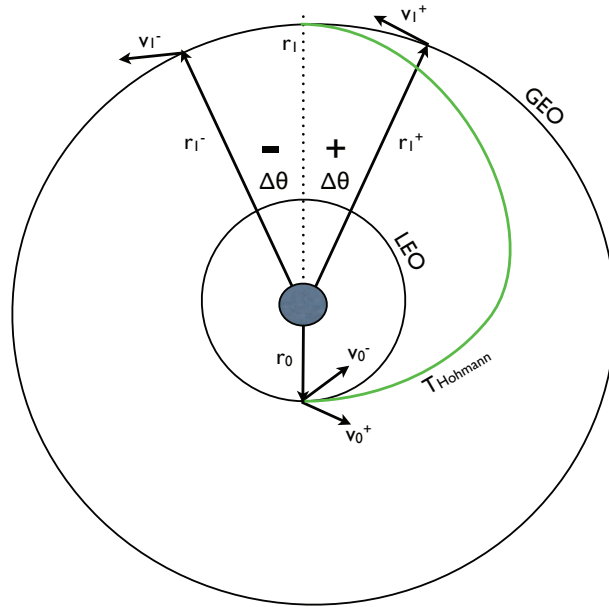


Figure 7.6: Illustration of the vicinity of the Hohmann transfer

neuver is a two-impulse transfer to a circular GEO orbit, one at the beginning of the transfer and one at the end. The distances are normalized by the Earth-Moon semimajor axis $D = 3.844 \times 10^5$ km. Therefore the initial and final semimajor axes of the circular orbits are $a_0 = 0.01743$ and $a_1 = 0.11$ respectively. The normalized radius of the Earth is therefore $R_E = 0.01665$.

The masses are normalized by the total mass of the Earth-Moon system. Therefore the gravitational parameter of the Earth and Moon are $\mu_1 = 0.98785$ and $\mu_2 = 0.01215$ respectively. The mass of the spacecraft is assumed to be negligible.

The time parameter is normalized by the Moon's orbit period around the Earth, $T_M = 2.3606 \times 10^6$ seconds. The Hohmann transfer time from a low-Earth orbit to a geocentric orbit is $T_H = 0.0508$. The total impulse for this transfer is $\Delta V_H = \Delta V_0 + \Delta V_1 = 2.3634 + 1.4294 = 3.7928$

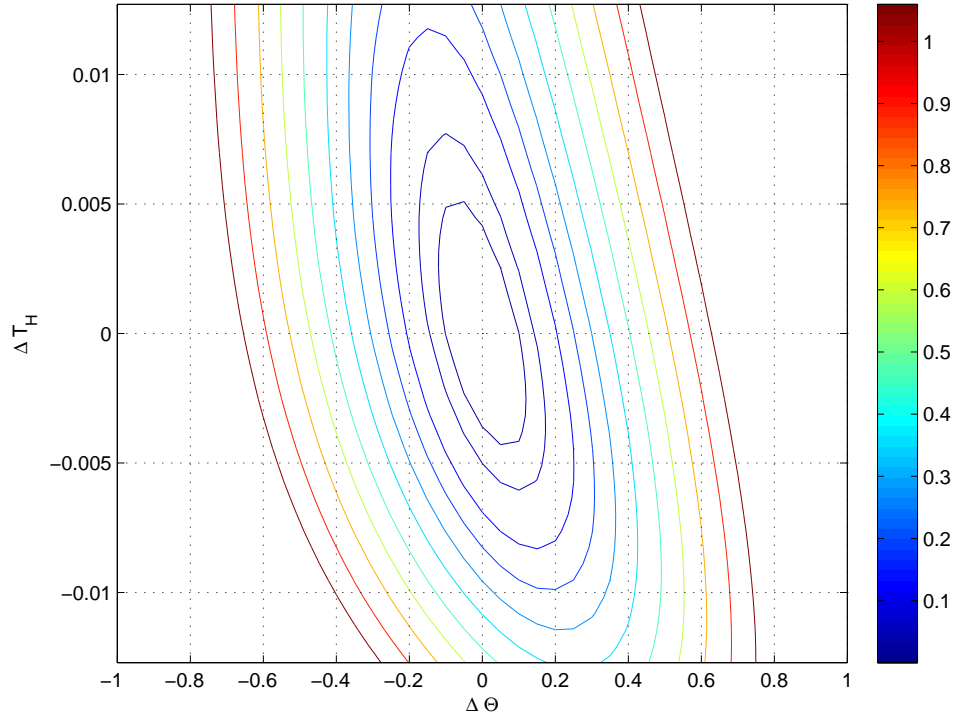


Figure 7.7: ΔV Contour map around the Hohmman Transfer

The equations of motion for the nominal trajectories are solved by using the *ode45* function in MATLAB, which is a Runge-Kutta numerical integrator. The time step of the integration is $\Delta t = 1E - 5$, and the relative tolerance is $1E - 9$.

Figs. 7.7 and 7.8 illustrate these contour maps for the nominal two-body two-point boundary value problem solutions. The values of the contour lines are relative to the Hohmann transfer results, the relative Hohmann transfer impulse being zero.

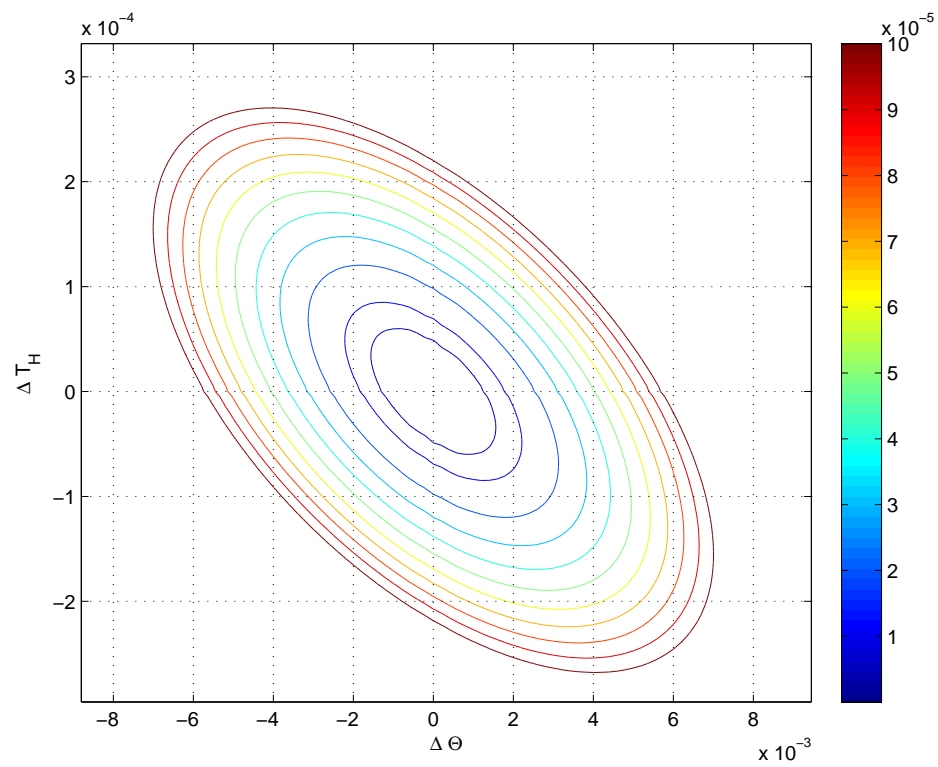


Figure 7.8: Detailed ΔV Contour map around the Hohmann Transfer

7.7 Implementation of Perturbation Theory to Perturbed Contour Map

The goal of the application of the perturbation theory is to create a velocity impulse contour map of the perturbed system, allowing to find the most fuel efficient trajectory. If the perturbation is relatively small, the optimal solution of the perturbed system should be close to the nominal solution in the transfer angle-time space $(\Delta\Theta, \Delta T_H)$. If the perturbation is relatively large, the perturbed optimal solution could be far from the optimal nominal solution in the angle-time space.

As shown by the perturbation theory developed in Chapter *IV* the nominal solution is sufficient to obtain the perturbed solution. The perturbation theory yields the necessary change in the initial nominal velocity in order to hit the desired target. A contour map of the perturbed system allows to obtain the fuel-optimal solution in the transfer angle-time space.

In order to build the contour map, several steps must be taken. First, the ranges of the angle shift $\Delta\Theta$ and transfer time shift ΔT_H , where the origin, $(0, 0)$, belongs to the Hohmann transfer have to be defined for the contour map, as well as the number of points to build the contour map. The initial position is the same for all transfer orbits, which corresponds to the initial Hohmann transfer position, $r_{0i}^{(0)}$.

When the transfer parameters are set, the trajectories can be obtained for the nominal Keplerian orbits. These nominal trajectories can be used for the perturbation theory to obtain the correction velocity terms that will yield the desired target, and the trajectories in phase space of the perturbed system. The correction terms, $\Delta V_{0i}^{(\alpha)}$'s, yield a change in velocity to solve the perturbed two-point boundary value

problem, $\delta V_{0i}^{(*)}$:

$$\Delta V_{0i}^{(*)} = \Delta V_{0i}^{(0)} + \sum_{\alpha=1}^N \Delta V_{0i}^{(\alpha)}. \quad (7.43)$$

The corrected impulse ΔV_{0i}^* yields a trajectory such that the particle will be at the intended $r_{1i}^{(0)}$ despite the presence of a perturbation. The second impulse to enter the target orbit can be obtained from the local circular velocity, V_{1Ci} , and the velocity of the particle when reaching the target point, $V_{1i}^{(*)}$, which is obtained from the perturbation theory:

$$\Delta V_1^* = V_{1C} - V_1^{(*)}, \quad (7.44)$$

where V_{1C} is the local circular velocity of the target orbit. Therefore, the total transfer impulse of the perturbed two-point boundary value problem is:

$$\Delta V^* = \Delta V_0^* + \Delta V_1^* = \Delta V_0^{(0)} + \sum_{\alpha=1}^N \Delta V_0^{(\alpha)} + V_{1C} - V_1^{(*)}. \quad (7.45)$$

These results can be used to build a contour map of the total cost of the transfer as a function of the transfer angle and transfer time. It is important to note that the accuracy of the results will depend on the order to which the theory is solved.

Obtaining solutions for higher orders of the theory is computationally burdensome as higher order state transition tensors are involved. However, for small perturbations first order theory is usually sufficient, but for larger the perturbations higher orders might be needed to accurately solve the problem.

7.7.1 Numerical Errors in LEO to GEO Transfer

Consider again a spacecraft going from a low-Earth orbit to a geostationary orbit on a Hohmann transfer. The oblateness parameter of the Earth is increased by a factor of 100 to show how the theory performs in a more perturbed environment, therefore the oblateness parameter $C_{20} = -0.1$. The orbit transfer occurs between a low-Earth orbit with a semimajor axis $a_0 = 0.01743$, to a geocentric orbit with a semimajor axis $a_1 = 0.11$. The transfer time is $T_H = 0.0508$. The goal is to show the accuracy of the theory for the first two orders compared to the true solution with different time steps for the numerical simulations.

The nominal solution is calculated by integrating the equations of motion using the *ode45* function in MATLAB, the integration time step is $\Delta t = 1E-6$, the relative tolerance of the numerical integration is $1E-13$. The correction to the initial velocity is calculated by quadratures of the integrals given by the perturbation theory. The quadratures are obtained using basic mid-point rule:

$$\int_{t_0}^{t_1} F(t)dt = (t_1 - t_0)F\left(\frac{t_0 + t_1}{2}\right). \quad (7.46)$$

For the 0^{th} order (nominal) solution, the final position error is $2.4516E4$ km, and the relative error of the initial velocity that solves the two-point boundary value problem is 0.0720.

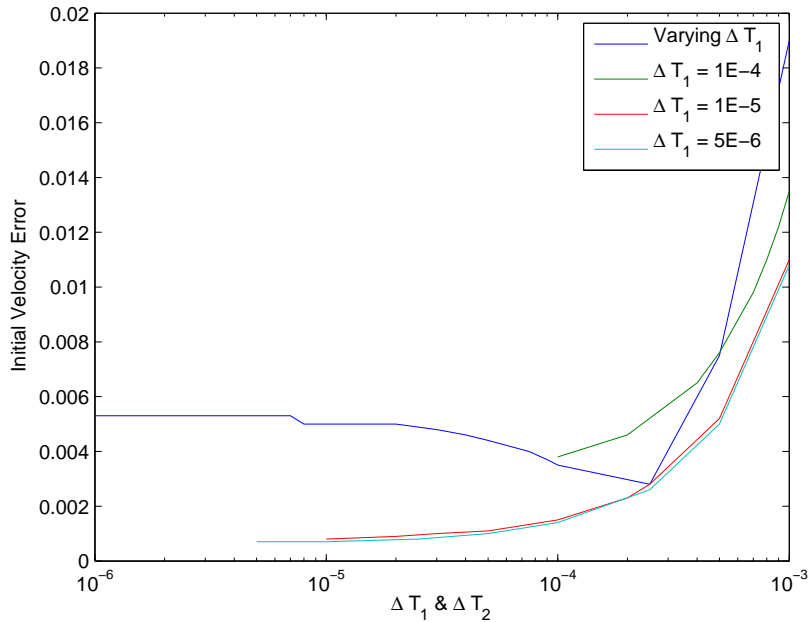


Figure 7.9: Semi-Log Plot of Initial Velocity Error

Fig. 7.9 shows the relative error in the calculation of the initial velocity for varying degrees of time-step calculations first-order and second order corrections using the perturbation theory. It's interesting to note that in order to obtain accurate results, small time steps must be taken in the quadrature scheme. This makes sense as the order of calculation is increased the correction term will be smaller for each order, resulting in more sensitive numerical accuracies.

Fig. 7.10 shows the error in the final spacecraft target as the integration step sizes vary for first and second order calculations for this highly perturbed example. The errors for both first and second orders level-off for $\Delta t_2 = 1E - 4$, however, Δt_1 is at least an order of magnitude smaller for these cases.

These results imply that the first order results need to be calculated very precisely in order to have accurate second order results. Second order calculations depend directly on the first order, so the larger the error in the first order, the worse the result.

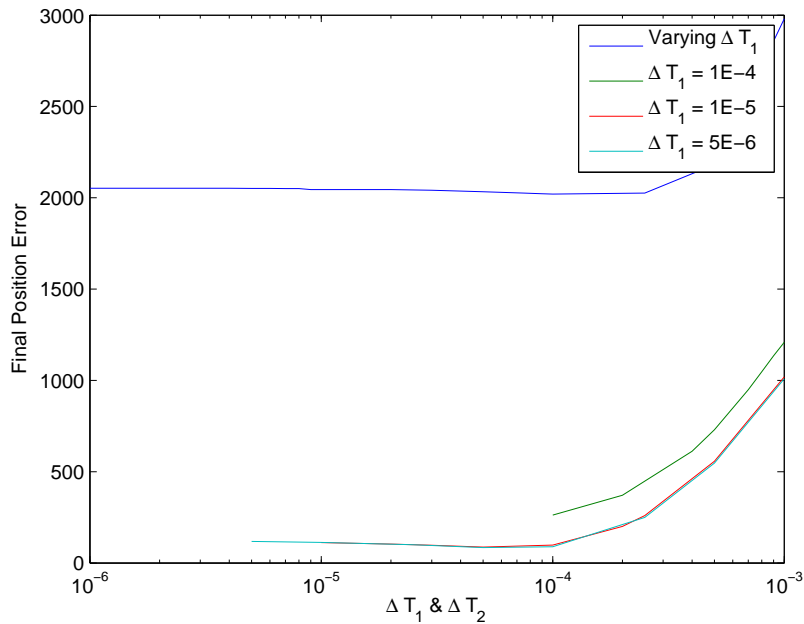


Figure 7.10: Final Position Error Semi-Log Plot

This can be seen for the case of $\Delta t_1 = 1E - 4$, where the initial velocity calculations actually diverges with the second order.

7.7.2 Example: Perturbed LEO to GEO Hohmann Transfer

Figs. 7.7 and 7.8 illustrate a contour plot of the nominal two-point boundary value problem for the two-body problem. The contour lines represent the total ΔV cost of the low-Earth orbit to geocentric orbit transfer for different coordinates transfer angle-time ($\Delta\Theta, \Delta T_H$). These coordinates represent the deviation in the transfer angle and transfer time from the Hohmann transfer. Note that despite this deviation, the transfer is still a low-Earth orbit to a geocentric orbit transfer, while finding solutions to the Lambert's problem instead of solving the Hohmann transfer. This is illustrated in Fig. 7.6.

For the Keplerian transfer, where no perturbing forces are present, it is expected

that the least costly two-impulse transfer will be the Hohmann transfer. Fig. 7.7 shows that this is the case on a larger scale where nonlinearities are present.

Fig 7.8 shows a more detailed illustration around the Hohmann transfer. The region around the nominal solution shows linear variations around the origin due to the very close proximity to the origin. Dynamics in this case are in the linear region, as the range of values for $\Delta\Theta$ and ΔT_H are very small. Recall that $T_H = 0.0508$ for the low-Earth orbit to geocentric orbit Hohmann transfer. The contour lines indicate the deviation from the total Hohmann transfer cost, ΔV . As expected, the minimum occurs at the Hohmann transfer point $(\Delta\Theta, \Delta T_H) = (0,0)$, and it increases as we deviate from the nominal solution.

The Moon's phase angle with respect to the inertial frame centered at the Earth influences the results obtained from the perturbation theory. The tidal effect of the Moon's gravity on the spacecraft differs as a function of the Moon's position with respect to the spacecraft. The initial phase angle of the Moon is chosen to be $\Phi = \frac{\pi}{2}$, since this phase angle has the most significant influence of all quarter-phase angles $(0, \frac{\pi}{2}, \pi, \frac{3\pi}{2})$. Earth's higher order gravitational field also causes the actual results to differ from the nominal solution. Earth's oblateness parameter is $C_{20} = -0.001$.

The correction terms for the initial impulses are calculated to the first order, as it is sufficient for a small perturbation such as Earth's oblateness. The quadratures are calculated using the midpoint rule given in Eq. 7.46. The time step used for the calculation is $\Delta t = 1E - 5$.

Fig. 7.11 shows the total ΔV contour of the corrected transfer in the presence of perturbing forces. The theory is used to find the necessary change in ΔV s to solve the perturbed two-point boundary value problem. As it can be seen, the contour has

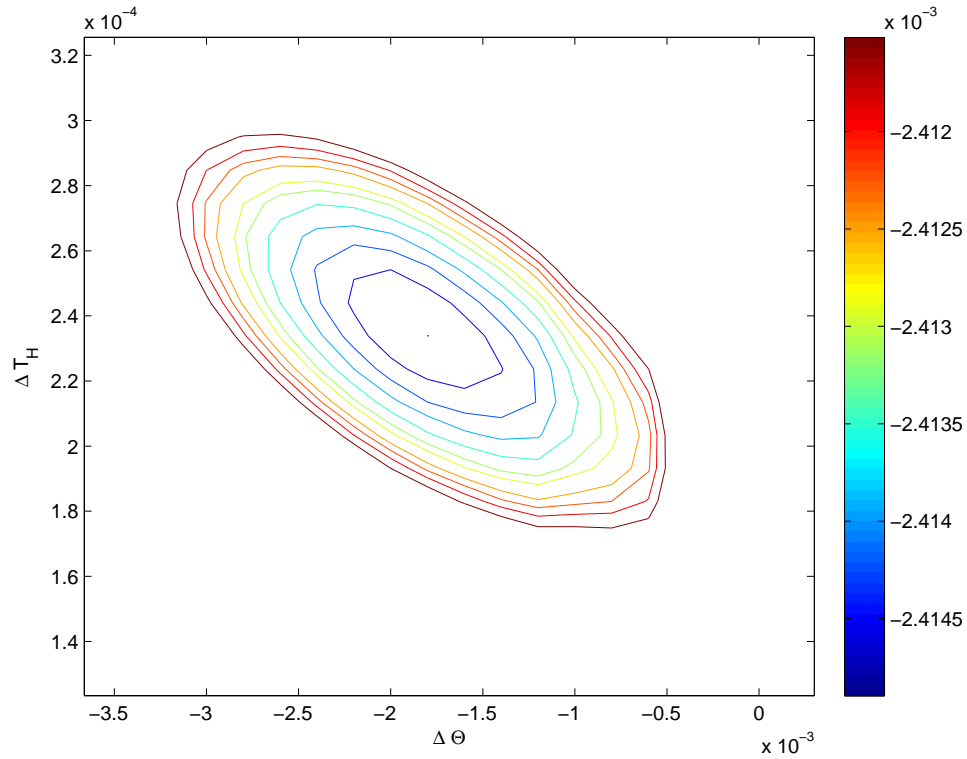


Figure 7.11: Detailed ΔV Contour map of the perturbed 2BVP solution

now shifted to another region of $(\Delta\Theta, \Delta T_H)$. The total ΔV of the transfer is also less than the Hohmann transfer ΔV . This implies that depending on the perturbing forces and their effects, one can find more cost efficient transfer using the theory and the subsequent approach to solving two-point boundary value problems.

7.7.3 Physical Explanation of Perturbed Contour Map Results

Fig. 7.8 shows smooth and consistent yet tilted ellipsoids around the Hohmann transfer solution $(\Delta\Theta, \Delta T_H = 0, 0)$. Although at first this "tilt" might not appear intuitive, it makes perfect sense from a dynamical standpoint.

The figure tells us that if the transfer angle is less than 180 degrees ($\Delta\Theta < 0$), the

optimal transfer time must be shorter than the Hohmann transfer time because it will require less change in energy to get there. Since the target final position is closer to the initial point, the time to get there should be less than the Hohmann transfer time.

Conversely, if the desired transfer angle is greater than 180 degrees ($\Delta\Theta > 0$) a longer transfer will require less change in energy, making it more fuel-efficient for two-impulse transfers. The further the target point is from the Hohmann transfer the greater the transfer time to get there.

Fig. 7.11 shows the total ΔV contour of the perturbed two-body two-point boundary value problem. The new fuel optimal target point is now located at $(\Delta\Theta, \Delta T_H) = (-0.00175, -0.00023)$, which implies that the solution has shifted to a target point further away from the Hohmann transfer final position and the transfer time is greater than T_H .

Note that this results are only an example as the perturbation effects depend on the phase angle of the Moon. Different phase angles will offer different results. However, for this particular case ($\Phi = \frac{\pi}{2}$) the shift in transfer angle and time also offer a more fuel optimal transfer than the Hohmann transfer. The contour lines indicate the relative total cost of the transfer. The transfer cost for this example is $\Delta V = -0.0024115$, which is about 0.06% lower cost than the nominal Hohmann transfer cost. The total cost of the perturbed Hohmann transfer is $\Delta V = 3.7905$, therefore the perturbation method offers a saving.

CHAPTER VIII

Example: Solving the Perturbed Two-Body Initial Value Problem

The initial value problem involves solving the variables of the equations of motion given a set of initial conditions for these variables. For a Hamiltonian dynamical system, solving this problem involves obtaining the generalized coordinates and momenta over time $[q_i(t), p_i(t)]$ given the initial set of conditions $[q_{0i}(t_0), p_{0i}(t_0)]$.

Solutions to the initial value usually involve numerical integration methods, however these methods do not give a qualitative interpretation of the system. Obtaining qualitative solutions allows to further analyze the system, and this can be applied to mission design analysis problems.

8.1 Perturbation Theory for the Two-Body Problem Hamilton's Principal and Characteristic Functions

As described in Chapter V, the difference between the nominal and perturbed generalized momenta at t_1 can be obtained from the perturbation theory developed for the principal function by differentiating with respect to the generalized coordinates

at t_1 :

$$p_{1i}^{(1)} = \int_{t_0}^{t_1} \frac{\partial H^{(1)}}{\partial q_j} \frac{\partial_b q_j}{\partial_b q_{1i}} dt, \tag{8.1}$$

$$p_{1j}^{(\alpha)} = \frac{1}{2} \int_{t_0}^{t_1} \left[\sum_{\beta=1}^{\alpha-1} \left(\frac{\partial p_i^{(\beta)}}{\partial q_{0j}} p_i^{(\alpha-\beta)} + \frac{\partial p_i^{(\alpha-\beta)}}{\partial q_{1j}} p_i^{(\beta)} \right) \right] dt.$$

Similarly, the difference in the generalized coordinates at t_1 can be obtained from the perturbation theory developed for the characteristic function, by differentiating the characteristic function with respect to p_{1i} :

$$q_{1i}^{(1)} = \int_{t_0}^{t_1} \frac{\partial H^{(1)}}{\partial p_j} \frac{\partial_b p_j}{\partial_b p_{1i}} dt, \tag{8.2}$$

$$q_{1j}^{(\alpha)} = \frac{1}{2} \int_{t_0}^{t_1} \left[\sum_{\beta=1}^{\alpha-1} \left(\frac{\partial q_i^{(\beta)}}{\partial p_{0j}} q_i^{(\alpha-\beta)} + \frac{\partial q_i^{(\alpha-\beta)}}{\partial p_{1j}} q_i^{(\beta)} \right) \right] dt.$$

Therefore, combining Eqs. 8.1 and 8.2 leads to the perturbed state of the system at t_1 .

8.1.1 First-Order Perturbation Theory for the Two-Body Initial Value Problem

Analyzing the perturbed initial value problem for the two-body problem to the full order can be burdensome and impractical. Therefore, the perturbed system can be conveniently analyzed with a first-order perturbation theory. In the two-body problem, the perturbation analysis becomes an easier problem, as it reduces the

perturbation integrals to a simple compact form. The first order perturbed state term for the two-body problem at state t_1 is:

$$x_{1i}^{(1)} = \int_{t_0}^{t_1} [\Phi(t_1, \tau) J H_x^{(1)}]^T d\tau, \quad (8.3)$$

where J is the identity block-matrix, and Φ is the state transition matrix. The first order solution to the perturbed state is therefore:

$$x_{1i} = \Phi(t_1, t_0) \vec{x}_0^{(0)} + \epsilon \left[\int_{t_0}^{t_1} N(t_1, \tau) d\tau \right], \quad (8.4)$$

where $N(t_1, \tau) = [\Phi(t_1, \tau) J H_x^{(1)}]^T$. Delaunay elements offer a practical set of canonical variables to solve the two-body initial value problem, as they are a function of the classical orbit elements. Recall the set of Delaunay elements that describe Keplerian orbits:

$$l_d = M \quad L_d = \sqrt{\mu a} \quad (8.5)$$

$$g_d = \omega \quad G_d = L_d \sqrt{1 - e^2} \quad (8.6)$$

$$h_d = \Omega \quad H_d = G_d \cos i, \quad (8.7)$$

The first order perturbation terms of the Delaunay elements with respect to the nominal system can be obtained by the following integral:

$$\begin{bmatrix} \Delta l_d \\ \Delta g_d \\ \Delta h_d \\ \Delta L_d \\ \Delta G_d \\ \Delta H_d \end{bmatrix} = \int_{t_0}^{t_1} \begin{bmatrix} \frac{\partial H^{(1)}}{\partial L_d} + \frac{3}{2}(t_1 - \tau) \frac{\partial H^{(1)}}{\partial l} \\ \frac{\partial H^{(1)}}{\partial G_d} \\ \frac{\partial H^{(1)}}{\partial H_d} \\ -\frac{\partial H^{(1)}}{\partial l} \\ -\frac{\partial H^{(1)}}{\partial g_d} \\ -\frac{\partial H^{(1)}}{\partial h_d} \end{bmatrix}_{x^{(0)}} d\tau. \quad (8.8)$$

Eq. 8.8 allows to predict the first-order perturbed Delaunay elements for the two-body problem.

8.2 Implementation of the First-Order Perturbation Theory to Numerical Simulations

The goal of applying the perturbation theory in the two-body problem is to accurately predict the Delaunay elements of the perturbed system for an arbitrary time. First of all the nominal system is defined, where the gravitational parameter of the central body is μ_1 . The initial orbit of the particle must be defined, which will yield the initial orbit elements. These orbit elements can be used to obtain the initial position and velocity vectors, as well as the Delaunay elements. With the given information, the nominal solution can be obtained from well-known two-body equations.

The nominal solution is then used for the perturbation theory, allowing to obtain the perturbed Delaunay elements for a specified time. For perturbations that are relatively small, the first order theory is sufficient to solve the problem. For larger perturbations, higher orders of the theory will solve the problem with better accuracy.

However, higher orders also mean more complex numerical algorithms involving state transition tensors, increasing the burden on the numerical calculations.

Despite the first order perturbation theory being best suited for small perturbations, it can also be used for larger perturbations or for simulations where the time period is large. Instead of carrying out the integral given by the perturbation theory from the initial time t_0 to the final time t_1 , the simulation time can be broken into smaller time frames, such that the perturbation is applied several times over that time span. From calculus, an integral can be divided into several integrals:

$$\int_0^N F(t)dt = \int_0^1 F(t)dt + \int_1^2 F(t)dt + \dots + \int_{N-1}^N F(t)dt. \quad (8.9)$$

For the perturbation theory, if the integration limits are divided into smaller intervals allows to create a feedback system that improves the first-order solution, especially when perturbations are relatively large. The feedback system can be summarized as follows:

1. Obtained the nominal solution from t_a to t_b .
2. Obtain the perturbed solution from t_a to t_b using the nominal solution.
3. Calculate the Delaunay elements of the perturbed system.
4. Calculate the perturbed orbit from the result given by the perturbation theory.
5. The perturbed orbit at this time step is the nominal orbit for the next time step.
6. Obtain the nominal solution from t_b to t_c using the perturbed orbit at t_b from the previous time step as the initial conditions.

7. Obtain the perturbed solution from t_b to t_c using the "new" nominal solution.
8. Calculate the Delaunay elements of the perturbed system.
9. Calculate the perturbed orbit from the result given by the perturbation theory.
10. The perturbed orbit at this time step is the nominal orbit for the next time step.
11. Repeat the process starting with t_c until the final time is reached.

The pattern shown above allows for the perturbation theory to obtain feedback on the "new" nominal solution for each time step and solve the perturbed problem for the next step. This allows to make the time steps arbitrarily small and therefore minimize the amount of error in the presence of a relatively large perturbation. The accuracy of results will depend on the amount of time steps taken. This method yields the perturbed Delaunay elements at each time step, and can be compared to a solution calculated by numerical integration of the equations of motion to check for accuracy.

8.2.1 Application of the Perturbation Theory to the Restricted Three-Body Problem

The restricted three-body problem is of particular interest for the initial value problem theory. As seen in Chapter VI, the restricted three-body problem can be viewed as a rotating two-body problem perturbed by the presence of a third body. The perturbation theory can be used to analytically solve the restricted three-body problem. In this case The nominal orbit is a Keplerian two-body orbit while the third body is introduced as a perturbation. All the orbit elements are calculated from the central body in the Keplerian problem, although that is not necessary.

The equations of motion for restricted three-body problem can be expressed from one of the bodies as the origin, as was shown in Chapter VI. This allows for a simple implementation of the theory using the Keplerian two-body problem as the nominal solution, while introducing a third body as the perturbation.

The perturbation theory is used to obtain the Delaunay elements from the central body in the presence of a perturbation. The coordinates and velocities of the restricted three-body problem can be obtained by viewing the problem as a perturbed two-body problem perturbed by the presence of the third body.

Analytically obtaining the Delaunay elements can be used to design low-energy spacecraft trajectories by developing a Keplerian map (17). The Keplerian map consists of a map of the argument of periapse versus the semimajor axis of the orbit at each periapse passage of the spacecraft. Fig. 8.1 shows a Keplerian map for an orbit with semimajor axis $a = 1.35$ and eccentricity $e = 0.23$. Both the semimajor axis and the argument of periapse can be obtained from the Delaunay elements, while the periapse passage occurs when the mean anomaly, which is the Delaunay element l , is equal to zero. The "holes" in the map correspond to stable resonant islands.

The Keplerian map allows to identify regions where transfer trajectories are possible. This application is most useful for long duration spacecraft missions where multiple orbit transfer are necessary. Examples include spacecraft missions to multi-moon planets and multiple fly-by missions. The goal of such maps is to minimize fuel use and use the gravitational pull of the bodies to perform orbit transfers. Ideally, these missions would require no fuel to accomplish objectives, however in real life some fuel would probably be used.

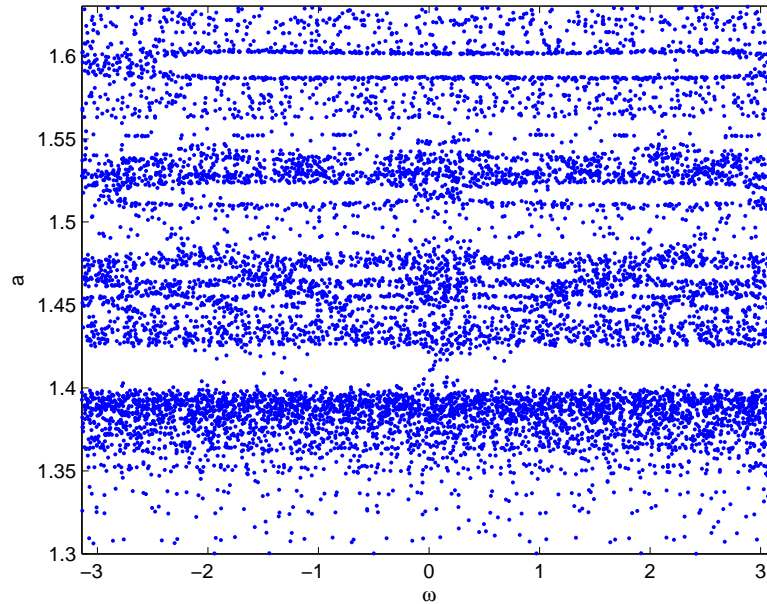


Figure 8.1: Keplerian Map for the Restricted Three-Body Problem: $a = 1.35$, $e = 0.23$

8.2.2 Low-Energy Spacecraft Mission Design

The results obtained by the perturbation theory for the initial value problem can be used to design low-energy spacecraft mission design. Ross and Scheeres (2007) developed a framework where they derive a Keplerian periapse kick map to find escape and capture trajectories for multi-moon missions (17).

Moreover, Grover and Ross (2008) further applied the results obtained by Ross and Scheeres to design realistic Jupiter-Europa-Ganymede mission (18). They acknowledge that no-fuel trajectories are not feasible due unrealistic time-of-flights, and develop a methodology to obtain low-thrust control inputs for desired targets.

However, the methods developed by Ross and Scheeres and Grover and Ross are best suited for qualitative analysis, which is important for mission design problems.

The proposed theory in this thesis leads to a method to accurately solve initial value problems that they are approximating for low-energy mission design problems. By obtaining solutions that are quantitatively accurate the mission design problem becomes more practical and the solution is no longer needed to use as an initial guess.

Additionally, their methods are restricted to the planar problem which limits mission design options by eliminating orbits that are inclined. The perturbation theory and method developed in this thesis are valid for the full dynamics of the system, including out of plane motion. This fact allows for a wider range of possible trajectories in low-energy mission design problems.

8.3 Numerical Example: Planar Perturbed Two-Body Problem

Consider a spacecraft orbiting the Jupiter-Europa planet-moon system, where the planet is of mass $m_1 = 0.99994333$, and the moon's mass is $m_2 = 0.00005667$. The nominal system is the Keplerian two-body problem where the particle orbits the planet with a semimajor axis $a = 1.35$ and eccentricity $e = 0.2$. The initial position occurs at the periapsis of the nominal orbit around Jupiter.

The nominal orbit is obtained by integrating the equation of motion using the 4th order Runge-Kutta integration function, *ode45* in MATLAB. The time step size for the numerical integration is $\Delta t = 1e - 5$, and the nominal orbit is updated 1, 10, and 100 times for different simulations, using the feedback algorithm described in the previous section. The relative tolerance of the numerical integration is $1E - 13$.

The perturbation theory is applied using the nominal solution obtained by Runge-Kutta numerical integrator described above. The quadratures in the perturbation theory are solved by using the extended Simpson's rule, and the time step size is $\Delta t = 1E - 5$. The perturbation theory is carried out using 1, 10, and 100 updates per orbit for different simulations.

The actual orbits of the spacecraft are calculated using the *ode45* using a time step size $\Delta t = 1E - 5$ and a relative tolerance of $1E - 13$. Each orbit is broken into 1000 segments of equal size in time, and the perturbed equations of motion are integrated from segment to segment.

Figs. 8.2, 8.3, 8.4 show the argument of periapse after 10 nominal orbits of the particle around the planet, with 1, 10, and 100 integration steps per orbit respectively. The figures contain the quantities nominal and perturbed quantities, along with the calculation by the theory. The figures show how the results improve by increasing the integral steps.

Similarly, Figs. 8.5, 8.6, 8.7 show the the Keplerian energy term, L , after 10 nominal orbits of the particle around the planet, with 1, 10, and 100 integration steps per orbit respectively. The figures contain the quantities nominal and perturbed quantities, along with the calculation by the theory. The figures show how the results improve by increasing the integral steps.

In order to build a periapsis Keplerian map the argument of periapsis and the semimajor axis of the orbit are calculated at each true (perturbed system) periapsis, or when the true mean anomaly is zero, $l = 0$. Fig. 8.8 shows the Keplerian map obtained from the theory, while Fig. 8.9 shows the actual Keplerian map. The maps are obtained through $1E4$ nominal Keplerian orbits.

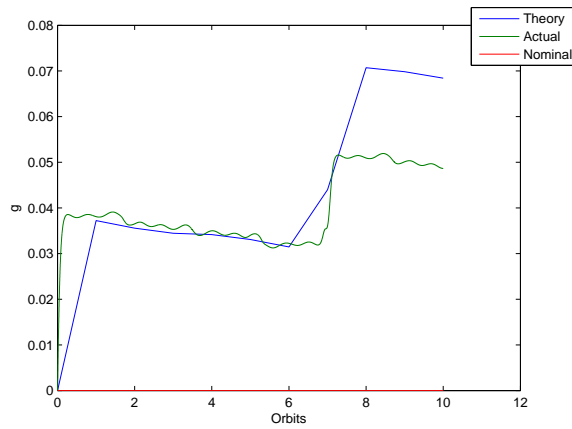


Figure 8.2: Argument of Periapse for Planar R3BP: 1 Step per Orbit

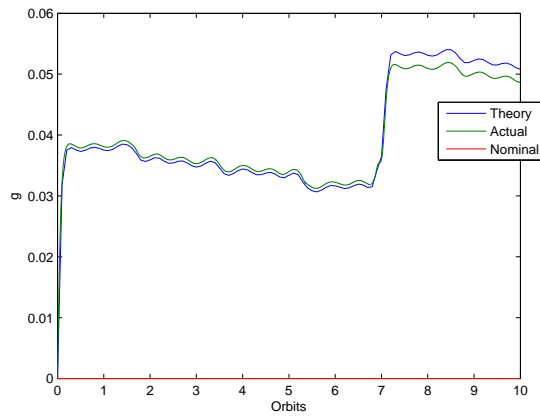


Figure 8.3: Argument of Periapse for Planar R3BP: 10 Steps per Orbit

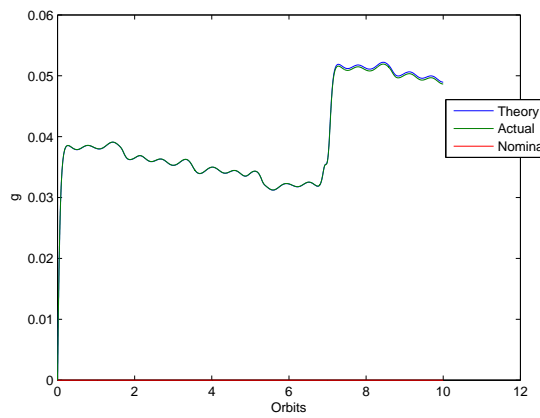


Figure 8.4: Argument of Periapse for Planar R3BP: 100 Steps per Orbit

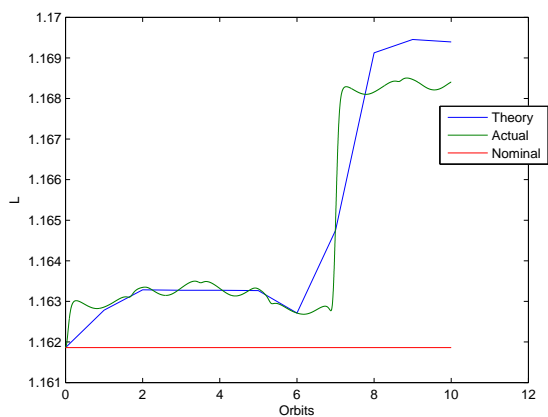


Figure 8.5: Keplerian Energy Term for Planar R3BP: 1 Step per Orbit

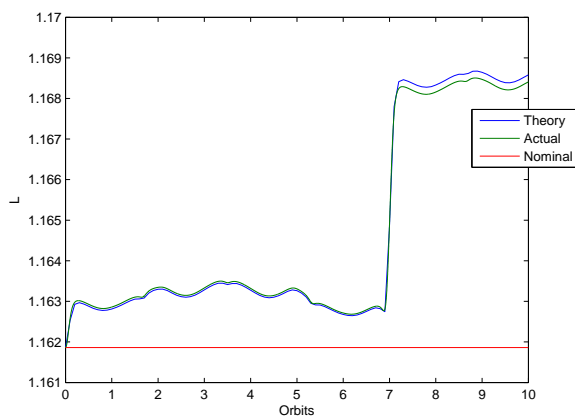


Figure 8.6: Keplerian Energy Term for Planar R3BP: 10 Steps per Orbit

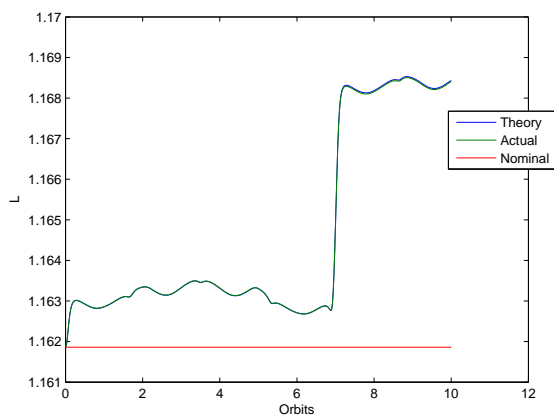


Figure 8.7: Keplerian Energy Term for Planar R3BP: 100 Steps per Orbit

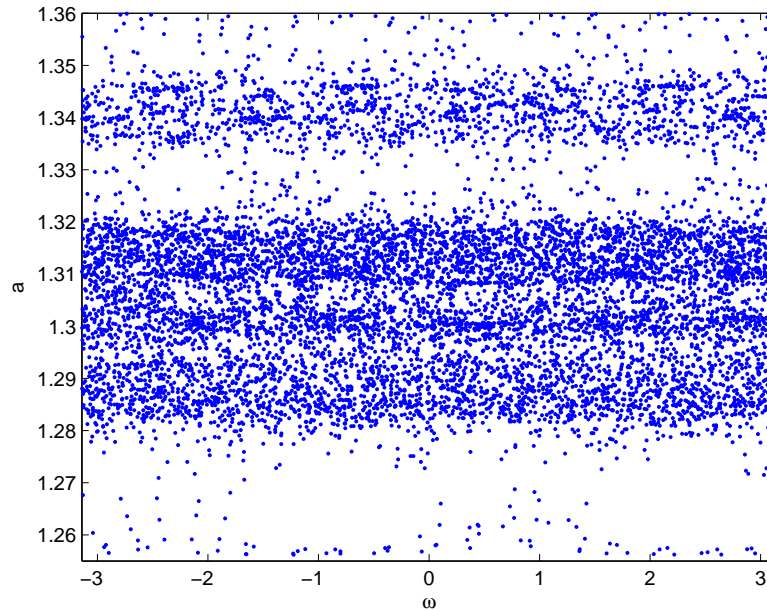


Figure 8.8: Theoretical Keplerian Map for the Restricted Three-Body Problem: $a = 1.35$, $e = 0.20$

However, these two figures are not as interesting as they lack the clear "swiss cheese" structure where resonant islands are clearly identified. Fig. 8.10 shown earlier offers a better illustration of the "swiss cheese" structure where the "holes" represent stable resonant islands. This latter case is for the same semimajor axis $a = 1.35$, but the eccentricity is $e = 0.23$, build through $1.5E4$ nominal orbits. Theorbits of the spacecraft are calculated by the extended Simpson's quadrature method for the theory and using a time step size $\Delta t = 1E - 4$ and a relative tolerance of $1E - 13$.

8.3.1 Numerical Example: Non-Planar Perturbed Two-Body Problem

Consider a spacecraft orbiting a central body of mass $m_1 = 0.99$. The semimajor axis of the orbit is $a = 0.5$, the eccentricity $e = 0.2$ and the inclination is $i = \frac{\pi}{4}$. The system is perturbed by the presence of a third body of mass $m_2 = 0.01$. The nominal system is a Keplerian orbit and the initial position is set at the periapsis of the orbit.

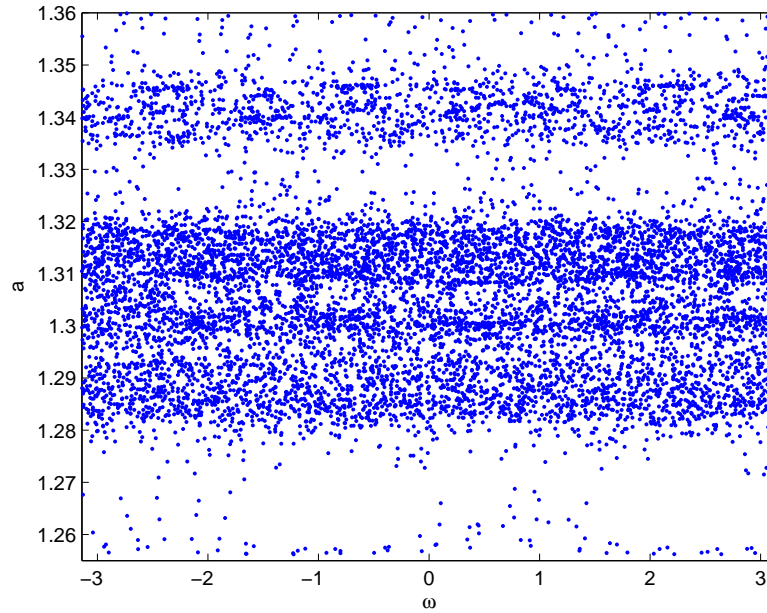


Figure 8.9: Actual Keplerian Map for the Restricted Three-Body Problem: $a = 1.35$, $e = 0.20$

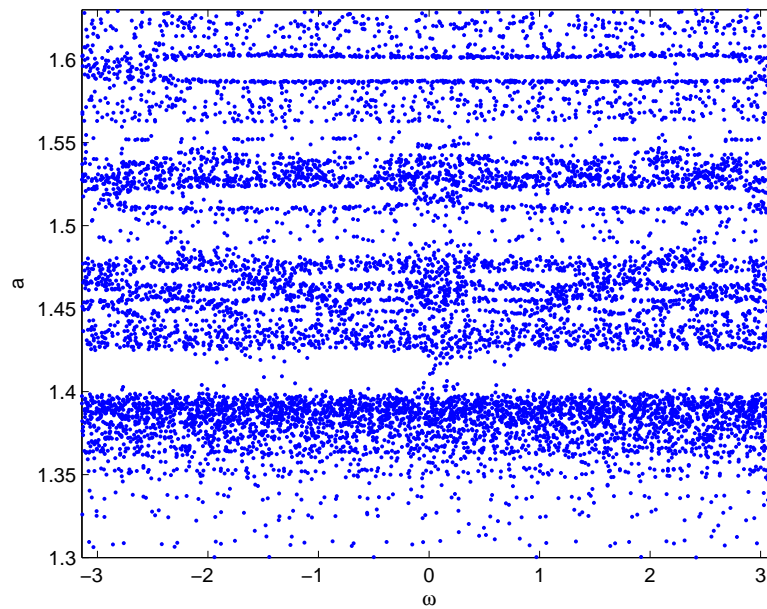


Figure 8.10: Keplerian Map for the Restricted Three-Body Problem: $a = 1.35$, $e = 0.23$

The nominal orbit is obtained by integrating the equation of motion using the 4th order Runge-Kutta integration function, *ode45* in MATLAB. The time step size for the numerical integration is $\Delta t = 1e - 5$, and the nominal orbit is updated 1, 10, and 100 times for different simulations, using the feedback algorithm described in the previous section. The relative tolerance of the numerical integration is $1E - 13$.

The perturbation theory is applied using the nominal solution obtained by Runge-Kutta numerical integrator described above. The quadratures in the perturbation theory are solved by using the extended Simpson's rule, and the time step size is $\Delta t = 1E - 5$. The perturbation theory is carried out using 1, 10, and 100 updates per orbit for different simulations.

The actual orbits of the spacecraft are calculated using the *ode45* using a time step size $\Delta t = 1E - 5$ and a relative tolerance of $1E - 13$. Each orbit is broken into 1000 segments of equal size in time, and the perturbed equations of motion are integrated from segment to segment.

Figs. 8.11, 8.12, 8.13 show the change in longitude of the ascending node of nominal system, the actual perturbed solution, and the solution calculated by the perturbation theory. Similarly, Figs. 8.14, 8.15, 8.16 show the change in the angular momentum projection element, H , which is directly proportional to the cosine of the inclination of the orbit.

The figures show that the accuracy of the theory increases as more steps per orbit are calculated.

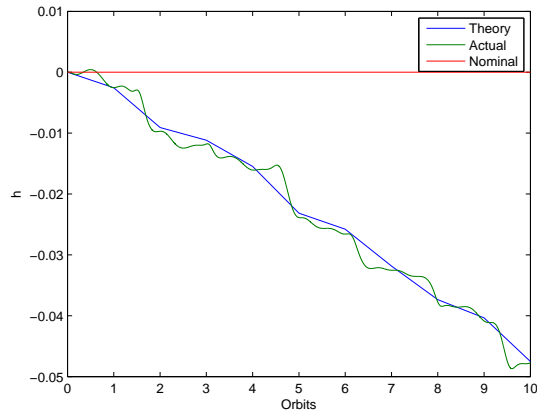


Figure 8.11: Longitude of Ascending Node for the 3-Dimensional R3BP: 1 Step per Orbit

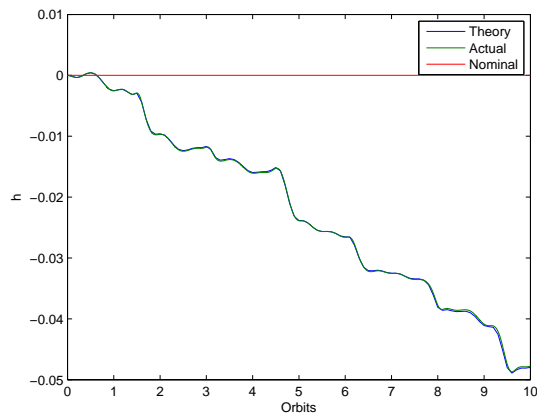


Figure 8.12: Longitude of Ascending Node for the 3-Dimensional R3BP: 10 Steps per Orbit

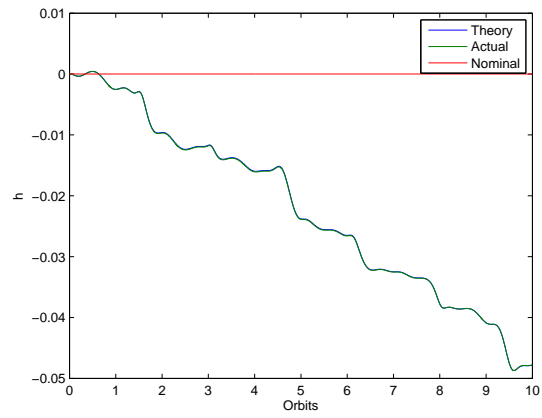


Figure 8.13: Longitude of Ascending Node for the 3-Dimensional R3BP: 100 Steps per Orbit

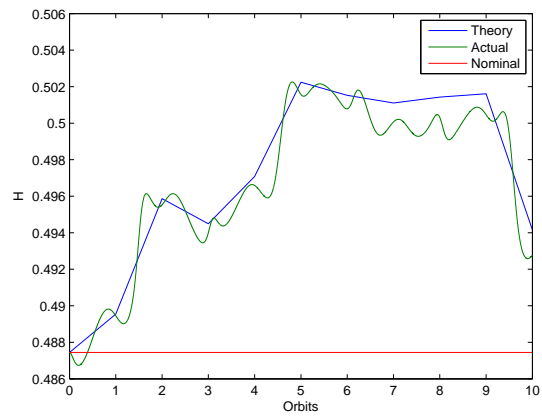


Figure 8.14: Angular Momentum Projection Term for the 3-Dimensional R3BP: 1 Step per Orbit

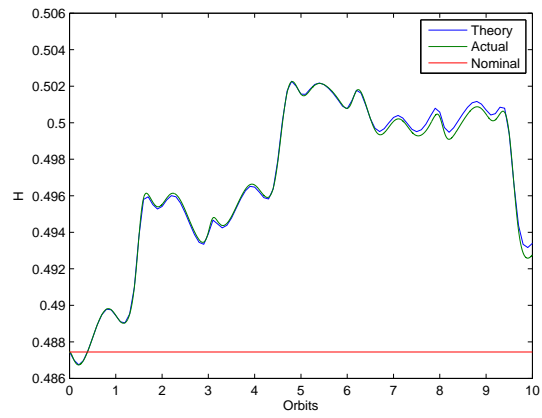


Figure 8.15: Angular Momentum Projection Term for the 3-Dimensional R3BP: 10 Steps per Orbit

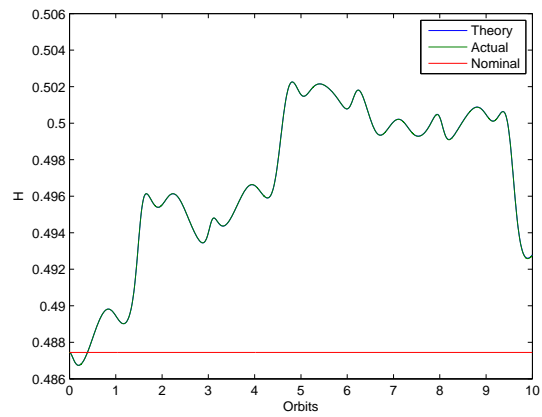


Figure 8.16: Angular Momentum Projection Term for the 3-Dimensional R3BP: 100 Steps per Orbit

8.4 Application of the First-Order Perturbation Theory to the Perturbed Restricted Three-Body Problem

The perturbed restricted three-body problem where one of the bodies has a higher order gravity field can be modeled as a perturbed rotating two-body problem, where the gravitational tidal force of one of the bodies and the higher order gravity field act as perturbing forces.

In order to illustrate the perturbed restricted three-body problem, let the central body be an oblate body with an oblateness parameter, C_{20} . The equations of motion of the particle with respect to the first primary are expressed as:

$$\ddot{\vec{r}} = -\mu_1 \frac{\vec{r}_i}{r_i^3} + \mu_2 \frac{\vec{r}_{2i} - \vec{r}_i - \vec{r}_{1i}}{|\vec{r}_{2i} - \vec{r}_i - \vec{r}_{1i}|^3} - \mu_2 \frac{\vec{r}_{2i} - \vec{r}_{1i}}{|\vec{r}_{2i} - \vec{r}_{1i}|^3} - \mu_1 \frac{3}{2} \frac{C_{20} R_B^2}{r_i^5}, \quad (8.10)$$

where \vec{r}_i is the position vector of the particle from the central body, \vec{r}_{1i} and \vec{r}_{2i} are the position vector of the first and second primaries from their barycenter, and R_B is the radius of the first primary. The nominal system is a Keplerian orbit of the particle around the first primary.

Consider a spacecraft orbiting a central body of mass $m_1 = 0.5$. The semimajor axis of the orbit is $a = 0.1$, the eccentricity $e = 0.1$, while the inclination is zero, making it an equatorial orbit. The system is perturbed by the presence of a third body of mass $m_2 = 0.5$, and by the oblateness parameter of the first primary $C_{20} = -0.1$, which is 100 times greater than the Earth's. The mean radius of the primary is, $R_B = 0.01665$ The nominal system is a Keplerian orbit and the initial position is set at the periapsis of the orbit.

The nominal orbit is obtained by integrating the equation of motion using the 4th order Runge-Kutta integration function, *ode45* in MATLAB. The time step size for the numerical integration is $\Delta t = 1e - 5$, and the nominal orbit is updated 1, 10, and 100 times for different simulations, using the feedback algorithm described in the previous section. The relative tolerance of the numerical integration is $1E - 13$.

The perturbation theory is applied using the nominal solution obtained by Runge-Kutta numerical integrator described above. The quadratures in the perturbation theory are solved by using the extended Simpson's rule, and the time step size is $\Delta t = 1E - 5$. The perturbation theory is carried out using 1, 10, and 100 updates per orbit for different simulations.

The actual orbits of the spacecraft are calculated using the *ode45* using a time step size $\Delta t = 1E - 5$ and a relative tolerance of $1E - 13$. Each orbit is broken into 1000 segments of equal size in time, and the perturbed equations of motion are integrated from segment to segment.

Figs. 8.17, 8.18, 8.19 show the change in argument of periapse of the nominal system, the actual perturbed solution, and the solution calculated by the perturbation theory. Similarly, Figs. 8.20, 8.21, 8.22 show the change in the angular momentum element, G , which is directly proportional to the cosine of the inclination of the orbit.

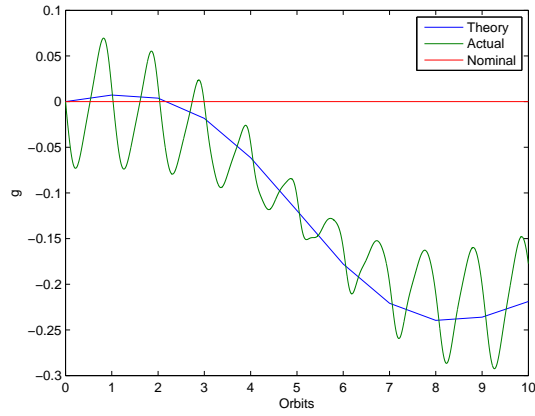


Figure 8.17: Argument of periapse for the Perturbed R3BP: 1 Step per Orbit

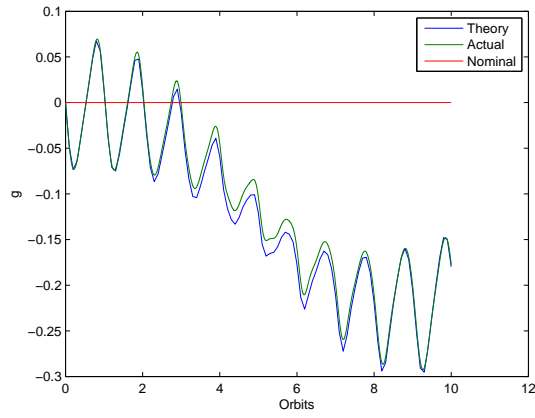


Figure 8.18: Argument of periapse for the Perturbed R3BP: 10 Steps per Orbit

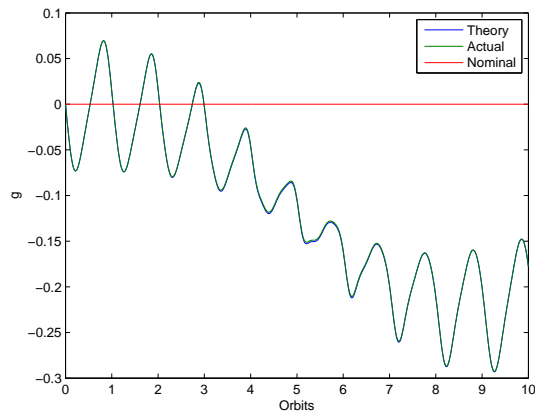


Figure 8.19: Argument of periapse for the Perturbed R3BP: 100 Steps per Orbit

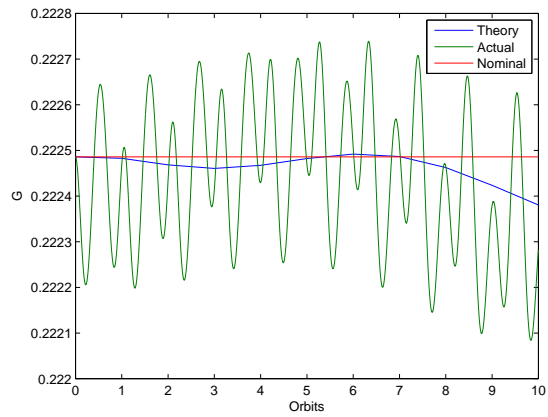


Figure 8.20: Angular Momentum Term for the Perturbed R3BP: 1 Step per Orbit

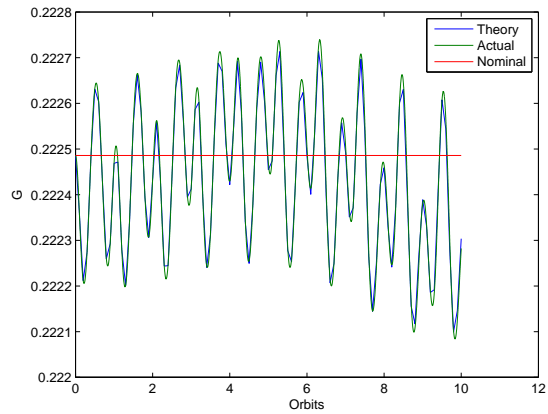


Figure 8.21: Angular Momentum Term for the Perturbed R3BP: 10 Steps per Orbit

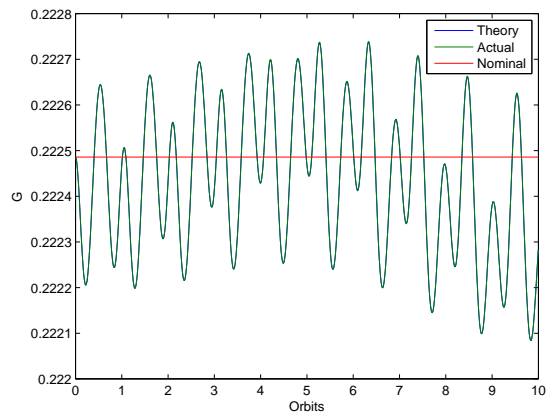


Figure 8.22: Angular Momentum Term for the Perturbed R3BP: 100 Steps per Orbit

8.5 Interpretation of Numerical Results from Perturbation Theory

The perturbation theory for the initial value problem was applied to the restricted circular three-body problem, which is viewed as a perturbed rotating two-body problem. The system is simulated as a two-body problem where the introduction of the third body acts as a perturbing potential.

The nominal and the perturbed system are obtained using the *ode45* 4th order Runge-Kutta integrator in MATLAB, with an integration time step size of $\Delta t = 1E - 5$. The quadratures in the perturbation theory are calculated using the extended Simpson's rule.

The values of the Delaunay elements using the perturbation theory were obtained by using 1, 10, and 100 steps per orbit, obtaining values of the Delaunay elements for each step. Taking several steps per orbit allows to create a feedback system for the perturbation theory by keeping the integration intervals small, therefore smaller orders of the perturbation theory can be used without incurring in significant errors.

It is shown in the numerical examples that the results from the first-order perturbation theory improve as more steps per orbit are taken. The examples presented show the validity of the first-order perturbation theory, even when the perturbing effects are not small and the simulation times are several orbits in length.

CHAPTER IX

Conclusions

The dissertation "A Perturbation Theory for Hamilton's Principal Function: Applications to Boundary Value Problems" presents a theory to obtain perturbed the Hamilton's principal and characteristic functions for a Hamiltonian dynamical system. The perturbation theory is further developed to solve two-point boundary value problems. The two-point boundary value problem can be reversed and converted into an initial value problem. Consequently the perturbation theory can be used to solve initial value problems.

The perturbation theory arises from Hamilton's principle and his original work on dynamics. Sir Rowan Hamilton discovered a fundamental function that contains the dynamics of a Hamiltonian system. The equations of motion can be obtained through simple differentiation and eliminations of this function, which is called Hamilton's principal function. A similar function called Hamilton's characteristic function can be used to obtain the dynamics of the system in a similar manner.

Chapter III describes the derivation procedure of the perturbation theory for Hamilton's principal and characteristic function. The principal function is expanded around the nominal solution by a small parameter ϵ . The Hamiltonian of the nominal

is also perturbed, and using the required boundary conditions and partial differential equations that the principal function must satisfy, it is shown how to obtain the perturbed principal function.

Similarly, Hamilton's characteristic function is expanded around the nominal solution by the same small parameter ϵ . The characteristic function can also be obtained by solving a different set of boundary conditions and partial differential equations.

Chapter IV develops the perturbation theory in chapter III to solve perturbed boundary value problems. It is shown in chapter *II* that Hamilton's principal and characteristic functions can solve boundary value problems. Using this fact, it is shown that the perturbed Hamilton's principal and characteristic functions can each solve a specific type of principal function. In order to successfully solve perturbed two-point boundary value problems the endpoint constraint has to be taken into account, and it is shown how to solve the problem.

Chapter IV develops the perturbation theory for the two-point boundary value problems to solve the initial value problem. The endpoint constraints are removed and the results for the principal and characteristic functions are combined to solve the initial value problem.

In chapters VI, VII, and VIII the perturbation theories are applied to the two-body and three-body problems. It was shown that a Hohmann transfer might save fuel when perturbations are present by analytically finding a family of solutions around the nominal transfer.

Additionally, the restricted three-body problem is solved by obtaining analytical

solutions for the perturbed rotating two-body problem. These results can be used to derive a Keplerian periapse map for low-energy spacecraft mission design.

Hamilton's principal and characteristic functions have traditionally been used to solve the initial value problem, and generating functions have been favored to solve these type of problems. However, the principal and characteristic functions have different underlying structures and solve different problems.

Solving the perturbed two-point boundary value problem analytically offers a qualitative and quantitative tool that offers a unique insight to perturbed problems. Solving the initial value problem analytically also proves to be useful in understanding the behavior of perturbed systems, which can be applied to designing systems of interest.

Sir Hamilton developed a first order perturbation theory using Hamilton's principal function. This perturbation theory was used to solve the initial value problem. However, it makes assumptions about the system that are not valid for large perturbations, and the method is cumbersome and valid only to the first order. The proposed theory in this thesis completes Hamilton's work by expanding it to higher orders and developing it in a straight-forward manner.

9.1 Future Work

The work developed and presented in this dissertation can be further applied and developed to solve other problems, as well as applied in different areas.

Higher order solutions: The numerical simulations for the two-point boundary

value problem were developed to the second order, while the initial value problem was developed to first order. Introducing higher order solution could lead to more accurate solutions, especially for highly perturbed environments. However, this will lead to computational complexities as higher order state transition tensors have to be implemented. Park and Scheeres developed a method to do these calculations (39) .

Software package: The perturbation theory has been applied to solve specific problems, and computer algorithms were developed. A complete software package could be developed to solve any perturbed Hamiltonian dynamical system using higher order perturbation theory.

Formation flight problems: Guibout and Scheeres developed a method to solve formation flight problems using the generating function for the canonical transformation (38). The generating function was used to solve the two-point boundary value problem and design formation flight trajectories. Due to the similarities between the generating function and the principal function, the perturbation theory for hamilton's principal and characteristic functions can be applied to solve perturbed formation flight problems.

Optimal control problems: Optimal control problems can be modeled as Hamiltonian dynamical systems. Park and Scheeres used generating functions to solve optimal control problems (44). Due to similarities between the generating function and Hamilton's principal function, Hamilton's principal function can be used to solve optimal control problems.

APPENDICES

APPENDIX A

Partial Differential Equations of Parameters in Lambert's Theorem

The following equations show the relationship between the angles in Lambert's theorem and the position vectors through the chord c and the parameter s :

$$\sin \frac{\alpha}{2} = \sqrt{\frac{s}{2a}}, \quad (\text{A.1})$$

$$\sin \frac{\beta}{2} = \sqrt{\frac{s-c}{2a}}, \quad (\text{A.2})$$

$$\cos \frac{\alpha}{2} = \sqrt{1 - \frac{s}{2a}}, \quad (\text{A.3})$$

$$\cos \frac{\beta}{2} = \sqrt{1 - \frac{s-c}{2a}}, \quad (\text{A.4})$$

$$\sinh \frac{\gamma}{2} = \sqrt{\frac{s}{-2a}}, \quad (\text{A.5})$$

$$\sinh \frac{\delta}{2} = \sqrt{\frac{s-c}{-2a}}, \quad (\text{A.6})$$

$$\cosh \frac{\gamma}{2} = \sqrt{1 + \frac{s}{-2a}}, \quad (\text{A.7})$$

$$\cosh \frac{\delta}{2} = \sqrt{1 + \frac{s-c}{-2a}}. \quad (\text{A.8})$$

The following equations are the partial differential equations of the angles in Lambert's problem with respect to the initial and final position vectors, the semimajor axis,:

$$\frac{\partial \alpha}{\partial a} = \frac{2}{\sqrt{1 - \frac{s}{2a}}} \frac{1}{2} \left(\frac{s}{2a} \right)^{-1/2} \left(\frac{-s}{2a^2} \right) = -\frac{1}{a} \frac{\sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}} = -\frac{1}{a} \tan \frac{\alpha}{2} \quad (\text{A.9})$$

$$\frac{\partial \beta}{\partial a} = -\frac{1}{a} \tan \frac{\beta}{2} \quad (\text{A.10})$$

$$\frac{\partial \gamma}{\partial a} = \frac{2}{\sqrt{1 + \frac{s}{-2a}}} \frac{1}{2} \left(\frac{s}{-2a} \right)^{-1/2} \left(\frac{s}{2a^2} \right) = -\frac{1}{a} \frac{\sinh \frac{\gamma}{2}}{\cosh \frac{\gamma}{2}} = -\frac{1}{a} \tanh \frac{\gamma}{2} \quad (\text{A.11})$$

$$\frac{\partial \delta}{\partial a} = -\frac{1}{a} \tanh \frac{\delta}{2} \quad (\text{A.12})$$

$$\frac{\partial s}{\partial \vec{r}_1} = \frac{1}{2} \left[\frac{\vec{r}_1}{|\vec{r}_1|} - \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|} \right] \quad (\text{A.13})$$

$$\frac{\partial s}{\partial \vec{r}_2} = \frac{1}{2} \left[\frac{\vec{r}_2}{|\vec{r}_2|} + \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|} \right] \quad (\text{A.14})$$

$$\frac{\partial(s-c)}{\partial \vec{r}_1} = \frac{1}{2} \left[\frac{\vec{r}_1}{|\vec{r}_1|} + \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|} \right] \quad (\text{A.15})$$

$$\frac{\partial s - c}{\partial \vec{r}_2} = \frac{1}{2} \left[\frac{\vec{r}_2}{|\vec{r}_2|} - \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|} \right] \quad (\text{A.16})$$

$$\frac{\partial \alpha}{\partial s} = \frac{2}{\sqrt{1 - \frac{s}{2}}} \frac{1}{2} \left(\frac{s}{2a} \right)^{-1/2} \frac{1}{2a} = \frac{1}{2a} \frac{1}{\cos \frac{\alpha}{2} \sin \frac{\alpha}{2}} \quad (\text{A.17})$$

$$\frac{\partial \beta}{\partial(s-c)} = \frac{1}{2a} \frac{1}{\cos \frac{\beta}{2} \sin \frac{\beta}{2}} \quad (\text{A.18})$$

$$\frac{\partial \gamma}{\partial s} = \frac{2}{\sqrt{1 + \frac{s}{2}}} \frac{1}{2} \left(\frac{s}{-2a} \right)^{-1/2} \frac{1}{-2a} = \frac{1}{-2a} \frac{1}{\cosh \frac{\gamma}{2} \sinh \frac{\gamma}{2}} \quad (\text{A.19})$$

$$\frac{\partial \delta}{\partial(s-c)} = \frac{1}{-2a} \frac{1}{\cos \frac{\delta}{2} \sin \frac{\delta}{2}} \quad (\text{A.20})$$

$$\frac{\partial \alpha}{\partial \vec{r}_1} = \frac{\partial \alpha}{\partial s} \frac{\partial s}{\partial \vec{r}_1} = \frac{1}{4a \cos \frac{\alpha}{2} \sin \frac{\alpha}{2}} \left[\frac{\vec{r}_1}{|\vec{r}_1|} - \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|} \right] \quad (\text{A.21})$$

$$\frac{\partial \beta}{\partial \vec{r}_1} = \frac{\partial \beta}{\partial(s-c)} \frac{\partial(s-c)}{\partial \vec{r}_1} = \frac{1}{4a \cos \frac{\beta}{2} \sin \frac{\beta}{2}} \left[\frac{\vec{r}_1}{|\vec{r}_1|} + \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|} \right] \quad (\text{A.22})$$

$$\frac{\partial \gamma}{\partial \vec{r}_1} = \frac{\partial \gamma}{\partial s} \frac{\partial s}{\partial \vec{r}_1} = \frac{1}{-4a \cosh \frac{\gamma}{2} \sinh \frac{\gamma}{2}} \left[\frac{\vec{r}_1}{|\vec{r}_1|} - \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|} \right] \quad (\text{A.23})$$

$$\frac{\partial \delta}{\partial \vec{r}_1} = \frac{\partial \delta}{\partial (s-c)} \frac{\partial (s-c)}{\partial \vec{r}_1} = \frac{1}{-4a \cosh \frac{\delta}{2} \sinh \frac{\delta}{2}} \left[\frac{\vec{r}_1}{|\vec{r}_1|} + \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|} \right] \quad (\text{A.24})$$

$$\frac{\partial \alpha}{\partial \vec{r}_2} = \frac{\partial \alpha}{\partial s} \frac{\partial s}{\partial \vec{r}_2} = \frac{1}{4a \cos \frac{\alpha}{2} \sin \frac{\alpha}{2}} \left[\frac{\vec{r}_2}{|\vec{r}_2|} + \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|} \right] \quad (\text{A.25})$$

$$\frac{\partial \beta}{\partial \vec{r}_2} = \frac{\partial \beta}{\partial (s-c)} \frac{\partial (s-c)}{\partial \vec{r}_2} = \frac{1}{4a \cos \frac{\beta}{2} \sin \frac{\beta}{2}} \left[\frac{\vec{r}_2}{|\vec{r}_2|} - \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|} \right] \quad (\text{A.26})$$

$$\frac{\partial \gamma}{\partial \vec{r}_2} = \frac{\partial \gamma}{\partial s} \frac{\partial s}{\partial \vec{r}_2} = \frac{1}{-4a \cosh \frac{\alpha}{2} \sinh \frac{\alpha}{2}} \left[\frac{\vec{r}_2}{|\vec{r}_2|} + \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|} \right] \quad (\text{A.27})$$

$$\frac{\partial \delta}{\partial \vec{r}_2} = \frac{\partial \delta}{\partial (s-c)} \frac{\partial (s-c)}{\partial \vec{r}_2} = \frac{1}{-4a \cosh \frac{\delta}{2} \sinh \frac{\delta}{2}} \left[\frac{\vec{r}_2}{|\vec{r}_2|} - \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|} \right] \quad (\text{A.28})$$

APPENDIX B

Partial Differential Equations of Third Body Effects with Respect to Delaunay Elements

Consider the set of Delaunay variables for the two-body problem:

$$l = \sqrt{\frac{\mu}{a^3}}t \quad L = \sqrt{\mu a} \quad (\text{B.1})$$

$$g = \omega \quad G = L\sqrt{1 - e^2} \quad (\text{B.2})$$

$$h = \Omega \quad H = G \cos i. \quad (\text{B.3})$$

The third body tidal perturbation potential for the two-body problem can be written as

$$R = \mu_2 \left[\frac{1}{|\vec{r}_2 - \vec{r}|} - \frac{(\vec{r}_2 - \vec{r}_1) \cdot \vec{r}}{|\vec{r}_2 - \vec{r}_1|^3} \right], \quad (\text{B.4})$$

where \vec{r} is the distance from the main primary to the orbiting body, \vec{r}_1 is the position

vector of the primary with respect to the barycenter of the two massive bodies and \vec{r}_2 is the distance from the barycenter of the two massive bodies to the secondary body. The position vector of the orbiting particle with respect to the main primary can be expressed in terms of the Delaunay elements:

$$\vec{r} = r \begin{bmatrix} \cos(g+f) \cos h - \sin(g+f) \sin h \cos i \\ \cos(g+f) \sin h + \sin(g+f) \cos h \cos i \\ \sin(g+f) \sin i \end{bmatrix}, \quad (\text{B.5})$$

where the distance from the main primary to the orbiting particle is

$$r = \frac{a(1-e^2)}{1+e \cos f}. \quad (\text{B.6})$$

The true anomaly, f , can be obtained through the following relationships with the eccentric and mean anomalies:

$$\tan \frac{f}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2}, \quad (\text{B.7})$$

and

$$l = E - e \sin E. \quad (\text{B.8})$$

The partials of the perturbation with respect to the position coordinates and velocities are

$$\frac{\partial R}{\partial t} = \frac{\partial R}{\partial \vec{r}} \left[\frac{\partial \vec{r}}{\partial r} \frac{\partial r}{\partial f} + \frac{\partial \vec{r}}{\partial f} \right] \frac{\partial f}{\partial E} \frac{\partial E}{\partial t}, \quad (\text{B.9})$$

$$\frac{\partial R}{\partial g} = \frac{\partial R}{\partial \vec{r}} \frac{\partial \vec{r}}{\partial g}, \quad (\text{B.10})$$

$$\frac{\partial R}{\partial h} = \frac{\partial R}{\partial \vec{r}} \frac{\partial \vec{r}}{\partial h}, \quad (\text{B.11})$$

$$\frac{\partial R}{\partial L} = \frac{\partial R}{\partial \vec{r}} \left[\frac{\partial \vec{r}}{\partial r} \left[\frac{\partial r}{\partial a} \frac{\partial a}{\partial L} + \left(\frac{\partial r}{\partial e} + \frac{\partial r}{\partial f} \frac{\partial f}{\partial e} + \frac{\partial r}{\partial f} \frac{\partial f}{\partial E} \frac{\partial E}{\partial e} \right) \frac{\partial e}{\partial L} \right] + \frac{\partial \vec{r}}{\partial f} \left(\frac{\partial f}{\partial E} \frac{\partial E}{\partial e} + \frac{\partial f}{\partial e} \right) \frac{\partial e}{\partial L} \right], \quad (\text{B.12})$$

$$\frac{\partial R}{\partial G} = \frac{\partial R}{\partial \vec{r}} \left[\frac{\partial \vec{r}}{\partial r} \left(\frac{\partial r}{\partial e} + \frac{\partial r}{\partial f} \frac{\partial f}{\partial e} + \frac{\partial r}{\partial f} \frac{\partial f}{\partial E} \frac{\partial E}{\partial e} \right) \frac{\partial e}{\partial G} + \frac{\partial \vec{r}}{\partial f} \left(\frac{\partial f}{\partial E} \frac{\partial E}{\partial e} + \frac{\partial f}{\partial e} \right) \frac{\partial e}{\partial G} + \frac{\partial \vec{r}}{\partial i} \frac{\partial i}{\partial G} \right], \quad (\text{B.13})$$

$$\frac{\partial R}{\partial H} = \frac{\partial R}{\partial \vec{r}} \frac{\partial \vec{r}}{\partial i} \frac{\partial i}{\partial H}, \quad (\text{B.14})$$

where the following partials occur:

$$\frac{\partial R}{\partial \vec{r}} = \mu_2 \left[\frac{\vec{r}_2 - \vec{r}}{|\vec{r}_2 - \vec{r}|} - \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|} \right], \quad (\text{B.15})$$

$$\frac{\partial \vec{r}}{\partial r} = \begin{bmatrix} \cos(g+f) \cos h - \sin(g+f) \sin h \cos i \\ \cos(g+f) \sin h + \sin(g+f) \cos h \cos i \\ \sin(g+f) \sin i \end{bmatrix}, \quad (\text{B.16})$$

$$\frac{\partial \vec{r}}{\partial f} = r \begin{bmatrix} -\sin(g+f) \cos h - \cos(g+f) \sin h \cos i \\ -\sin(g+f) \sin h + \cos(g+f) \cos h \cos i \\ \cos(g+f) \sin i \end{bmatrix}, \quad (\text{B.17})$$

$$\frac{\partial \vec{r}}{\partial g} = r \begin{bmatrix} -\sin(g+f) \cos h - \cos(g+f) \sin h \cos i \\ -\sin(g+f) \sin h + \cos(g+f) \cos h \cos i \\ \cos(g+f) \sin i \end{bmatrix}, \quad (\text{B.18})$$

$$\frac{\partial \vec{r}}{\partial h} = r \begin{bmatrix} -\cos(g+f) \sin h - \sin(g+f) \cos h \cos i \\ \cos(g+f) \cos h - \sin(g+f) \sin h \cos i \\ 0 \end{bmatrix}, \quad (\text{B.19})$$

$$\frac{\partial \vec{r}}{\partial i} = r \begin{bmatrix} \sin(g+f) \sin h \sin i \\ -\sin(g+f) \cos h \sin i \\ \sin(g+f) \cos i \end{bmatrix}, \quad (\text{B.20})$$

$$\frac{\partial r}{\partial f} = \frac{ae(1 - e^2) \sin f}{(1 + e \cos f)^2}, \quad (\text{B.21})$$

$$\frac{\partial r}{\partial a} = \frac{1 - e^2}{1 + e \cos f}, \quad (\text{B.22})$$

$$\frac{\partial r}{\partial e} = \frac{-2ae(1 + e \cos f) - a(1 - e^2) \cos f}{(1 + e \cos f)^2}, \quad (\text{B.23})$$

$$\frac{\partial f}{\partial E} = \sqrt{\frac{1 + e \sec^2 \frac{E}{2}}{1 - e \sec^2 \frac{f}{2}}}, \quad (\text{B.24})$$

$$\frac{\partial f}{\partial e} = \frac{2}{(1 - e)^2} \sqrt{\frac{1 - e \tan \frac{E}{2}}{1 + e \sec^2 \frac{f}{2}}}, \quad (\text{B.25})$$

$$\frac{\partial E}{\partial e} = \frac{\sin E}{1 - e \cos E}, \quad (\text{B.26})$$

$$\frac{\partial E}{\partial l} = \frac{1}{1 - e \cos E}, \quad (\text{B.27})$$

$$\frac{\partial a}{\partial L} = \frac{2L}{\mu_1}, \quad (\text{B.28})$$

$$\frac{\partial e}{\partial L} = \frac{G^2}{eL^3}, \quad (\text{B.29})$$

$$\frac{\partial e}{\partial G} = \frac{-G}{eL^2}, \quad (\text{B.30})$$

$$\frac{\partial i}{\partial G} = \frac{H}{G^2} \csc i, \quad (\text{B.31})$$

$$\frac{\partial i}{\partial H} = -\frac{\csc i}{G}. \quad (\text{B.32})$$

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