ELECTROMAGNETIC SCATTERING FROM RADIALLY INHOMOGENEOUS MEDIA

by

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TABLE OF CONTENTS

Table of Contents (cont.)

ABSTRACT

This is a study of the scattering of electromagnetic waves by structures which are cylindrically or spherically symmetric, but inhomogeneous in the radial direction. It has applications to dielectric lenses at optical and microwave frequencies; to the propagation of radio waves in the ionosphere, and to their reflection by meteor trails; to the radar scattering behavior of, and to the radiation from, cylinders and spheres coated by layers of materials for ablating and camouflage purposes, or by a layer of plasma.

The first two chapters are devoted to the consideration of the scattering of a plane electromagnetic wave from a cylinder and a sphere on whose surfaces an impedance boundary condition holds, and which are coated by one or two layers of materials whose indexes of refraction are not large compared to unity. The high-frequency backscattered field is asymptotically determined in terms of geometric optics and creeping waves contributions. The results have applications to the radar scattering by space vehicles on which a highly absorbing ferrite layer is in turn covered by a dielectric ablative layer.

When a plane wave is diffracted by a radially inhomogeneous cylinder, the radial and circumferential field components are obtained by differentiation from the axial components of the electric and magnetic fields, which can be expressed as sums of solutions of certain second order differential equations. Explicit solutions of the boundary value problem for various cylindrical structures are given in Chapter Three, and similar studies are performed for spherical structures in Chapter Four. Also, resonance and dip conditions for the field backscattered from inhomogeneous cylinders and spheres are determined, in the Rayleigh approximation.

A fundamental theorem for oblique scattering by a certain class of cylindrical bodies is proven: a sufficient condition for the nth TE and the nth TM modes to be uncoupled for all n and all angles of incidence is that the index of refraction have no step discontinuities. Whenever this theorem is applicable, the field produced by a plane wave at oblique incidence with respect to the axis of the cylinder can be trivially derived from the field produced in the case of normal incidence.

If the electric permittivity and/or the magnetic permeability become infinite on the axis of the cylinder or at the center of the sphere, the appropriate radial eigenfunctions are chosen by means of a boundary condition of Meixner's type. An application is made to the spherical inverse-square-power dielectric lens, whose high-frequency backscattered field is evaluated in Chapter Five, in terms of geometric optics, creeping wave and evanescent wave contributions.

INTRODUCTION

I.1 General Considerations

The scattering of electromagnetic waves by structures which are cylindrically or spherically symmetric but inhomogeneous in the radial direction is a phenomenon which occurs in a variety of cases of practical interest. It has applications to dielectric lenses, both at optical and microwave frequencies; to the propagation of radio waves in the ionosphere, and to their reflection by meteor trails; to the radar scattering behavior of, and to the radiation from, cylinders and spheres coated by layers of materials for ablating and/or camouflage purposes, or by a layer of plasma, as for a space vehicle during the re-entry phase of its flight.

The propagation of waves in inhomogeneous media has been the subject of a few books, such as Brekhovskikh's (1960) and Wait's (1962), and of numerous journal articles. However, many interesting problems remain unsolved, especially at high frequencies; it is to the solution of some of these problems for cylindrically and spherically symmetric scatterers that the present work is devoted. After a few remarks of general interest, this introduction contains an outline of the research performed and of the principal results achieved, and a list of suggested topics for future research.

In the following chapters we consider the scattering of a plane electromagnetic wave by a cylinder or a sphere which are inhomogeneous in the radial direction. The choice of a plane wave as the primary field is not a strong limitation, because an arbitrary incident electromagnetic field can be decomposed into the sum of plane monochromatic waves by Fourier analysis. We adopt the rationalized MKSA system of units, and omit the time-dependence factor $e^{-i\omega t}$. The following notation will be used:

ω = angular frequency,

$k = \frac{2\pi}{\lambda} = \omega\sqrt{\epsilon_o \mu_o}$ = wave number in vacuo,

ϵ_o = electric permittivity (dielectric constant) in vacuo,

μ_o = magnetic permeability in vacuo,

$Z = Y^{-1} = \sqrt{\mu_o/\epsilon_o}$ = intrinsic impedance of free space (= 120π ohm),

ϵ and μ = relative permittivity and permeability inside the inhomogeneous medium (functions of the distance from the axis of the cylinder or from the center of the sphere),

$i = \sqrt{-1}$ = imaginary unit,

$\underline{E}$ and $\underline{H}$ = electric and magnetic field vectors,

x, y, z = rectangular Cartesian coordinates,

ρ, ϕ, z = circular cylindrical coordinates,

r, θ, ϕ = spherical polar coordinates,

vectors will be underlined (e.g. $\underline{E}$), and unit vectors will be denoted by carets (e.g. $\hat{r}$).

In our notation, Maxwell's equations are

$$\nabla x \underline{H} = -ikY\epsilon \underline{E} \quad ,$$
$$\nabla x \underline{E} = ikZ\mu \underline{H} \quad , \tag{I.1}$$

where ϵ and μ are, in general, complex quantities, and $\epsilon = \mu = 1$ in vacuo. At a surface of discontinuity in ϵ and/or μ one must apply the appropriate boundary conditions, and a boundary condition of Meixner type is needed wherever ϵ and/or μ become infinite (see Chapters Three and Four). The scattered (total minus incident) fields in the infinite free-space region surrounding the scattering body must satisfy Sommerfeld's radiation condition. Specifically, for the two-dimensional case of normal incidence on a cylinder of axis z, the scattered field components E_z^s and H_z^s must obey the condition

$$\lim_{\rho\to\infty} \rho^{1/2}\left(\frac{\partial}{\partial\rho} - ik\right) \begin{Bmatrix} E_z^s \\ H_z^s \end{Bmatrix} = 0 \qquad \text{uniformly in } \phi , \tag{I.2}$$

whereas in the spherical case the scattered fields $\underline{E}^s$ and $\underline{H}^s$ are required to satisfy the Silver-Müller condition

$$\lim_{r\to\infty} \left[\underline{r}\,\mathrm{x}(\nabla\mathrm{x}) + ikr\right] \begin{Bmatrix} \underline{E}^s \\ \underline{H}^s \end{Bmatrix} = 0 \qquad \text{uniformly in } \hat{r} . \tag{I.3}$$

From the radiation conditions and from Maxwell's equations it follows that the fields are of the form

$$\underline{f}(\phi)\,\frac{e^{ik\rho}}{\sqrt{k\rho}} , \qquad \text{as} \quad \rho \to \infty \tag{I.4}$$

in the cylindrical case, and

$$\underline{g}(\hat{r})\,\frac{e^{ikr}}{kr} , \qquad \text{as} \quad r \to \infty \tag{I.5}$$

in the spherical case, where $\underline{f}$ is independent of ρ and $\underline{g}$ is independent of r. This means that the radiation condition is satisfied if we choose the cylindrical or spherical Hankel functions of the <u>first</u> kind as the radial eigenfunctions in the free space surrounding the scatterer. For a given primary field and scattering body, the scattered field is uniquely determined by the appropriate boundary and radiation conditions.

In the spherical case, the differential scattering cross section or bistatic radar cross section $\sigma(\theta, \phi)$ is defined by

$$\sigma(\theta,\phi) = \lim_{r\to\infty} 4\pi r^2 \frac{|\underline{E}^s|^2}{|\underline{E}^i|^2} , \tag{I.6}$$

where $\underline{E}^i$ is the incident electric field and $\underline{E}^s$ is the scattered field at the observation point (r, θ, ϕ). The total scattering cross section σ_{total} is defined by the ratio of the time averaged total scattered power to the time averaged incident Poynting vector, and is related to the bistatic cross section by the equation

$$\sigma_{total} = \frac{1}{4\pi} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \sigma(\theta, \phi) \sin\theta \, d\theta \, d\phi \ . \qquad \text{(I. 7)}$$

A relation between σ_{total} and the far field coefficient $\underline{g}(\hat{r})$ of equation (I. 5) is provided by the forward scattering theorem:

$$\sigma_{total} = \frac{4\pi}{k^2} \operatorname{Im} g(\hat{r}_o) \ , \qquad \text{(I. 8)}$$

where $\underline{g}(\hat{r}_o) = g(\hat{r}_o)\hat{\tau}$ with $\hat{\tau}$ a unit vector, $\hat{r}_o$ is oriented in the direction of propagation of the incident wave, and g is normalized to the amplitude of the corresponding incident field.

In the case of normal incidence on an infinitely long cylinder, the bistatic cross section $\sigma(\phi)$ per unit length is defined by

$$\sigma(\phi) = \lim_{\rho \to \infty} 2\pi\rho \left| \frac{\psi^s}{\psi^i} \right|^2 \ , \qquad \text{(I. 9)}$$

where $\psi^s = E_z^s(\rho, \phi)$ and $\psi^i = E_z^i$ if the electric field is parallel to the cylinder axis, while $\psi^s = H_z^s(\rho, \phi)$ and $\psi^i = H_z^i$ if the magnetic field is parallel to the axis. The total scattering cross section per unit length is defined by the ratio of the time averaged total scattered power per unit length of cylinder to the time averaged incident Poynting vector.

In the case of either cylindrical or spherical symmetry, the vector scattering problem can always be reduced to the determination of two scalar functions and is therefore more cumbersome, but essentially not more difficult than the corresponding scalar problem. Many of the mathematical difficulties encountered in the analysis of the electromagnetic or acoustical scattering by radially inhomogeneous structures also arise in certain quantum mechanical problems, such as the solution of the radial Schrödinger equation for central potentials. Thus, for example, the scattering of a plane scalar wave by a penetrable sphere of refractive index N is equivalent to the quantum mechanical non-relativistic scattering of a particle by a square-well potential (Rubinow, 1961); if m is the mass of the particle, V_o the depth of the attractive well and k the wave number, then N stands for $\sqrt{1+2mV_o/(\hbar k)^2}$.

The solutions for the scattering problem under consideration, which have been adopted by various authors for different applications, are essentially of four types; geometrical optics method, exact solutions, stratification technique, and asymptotic evaluation of formal solutions.

The geometrical optics method is a ray-tracing technique which leads to results whose precision increases with the frequency of the radiation. It has been widely applied to optical systems, microwave dielectric lenses, coated metal cylinders, ionospheric radio propagation, etc. It is used not only to investigate the geometrical properties of the optical ray paths, but it can also account for amplitude, phase and polarization of the electromagnetic field. In all cases in which it is inapplicable or not sufficiently accurate, one of the other methods must be used.

The geometrical optics approximation does not account for the presence of nonzero scattered fields in the region of geometrical shadow, and often represents an insufficiently accurate approximation in the illuminated region. A better approximation is represented by the so-called geometrical theory of diffraction of Keller, which is an extension of geometrical optics. For a description of this theory, the reader is referred to a paper by Keller (1956), in which the extension of the laws of optics is presented in two equivalent forms. In the first form, the different situations in which diffracted rays are produced and the different kinds of diffracted rays which occur in each case are explicitly described. The second formulation is based on an extension of Fermat's principle. The equivalence of the two formulations follows from considerations of the calculus of variations. Keller's theory assigns a field value, which includes a phase, an amplitude and, in the electromagnetic case, a polarization to each point on a ray. The total field at a point is postulated to be the sum of the fields of all rays which pass through the point.

Keller's theory has been developed for both scalar and vector fields and for objects of various shape and type (e.g., acoustically hard and soft bodies, perfect conductors, dielectrics, inhomogeneous media). From its similarity to geometrical optics, Keller's method can be expected to yield good results when the wavelength is small compared to the obstacle dimensions. However, it has been found that in most cases the results are useful even for wavelengths as large as the relevant dimensions of the scatterer. An important advantage of the method is that it does not depend on separation of variables or any similar procedure, and it is therefore especially useful for shapes more complicated than a circular cylinder or a sphere.

Exact solutions of the electromagnetic boundary value problem for spherically and cylindrically symmetric structures can always be obtained by separation of variables. In general, however, the solutions are purely formal, since the ordinary differential equations for the radial eigenfunctions have been solved exactly only for certain radial variations of the inhomogeneities of the medium: see, for example,the list of references by Tai (1963) for spherical structures, and by Burman (1965, 1966) for cylindrical structures. Even in those cases in which the radial differential equations can be solved exactly, the solution is often of little or no practical usefulness owing to the insufficient theoretical and numerical data available for the radial eigenfunctions.

The stratification technique consists of replacing the radially inhomogeneous medium with a certain number of coaxial, or concentric, homogeneous layers, and in solving the boundary value problem for this modified structure. Although the infinite eigenfunction series for the electromagnetic field components are well-known in this case [see, for example, Kerker and Matijevic (1961) for cylinders, and Wait (1963) for spheres], they are so complicated that no information on their behavior can be derived by direct inspection. Thus, the stratification technique is simply a tool for obtaining numerical results by means of a computer; the complexity and cost of the computations increase rapidly with the number of layers and with the ratio between the diameter of the structure and the wavelength of the incident radiation.

Finally, the asymptotic evaluation of formal solutions at high frequencies consists of replacing the formal series solutions by contour integrals in the complex plane and in applying Cauchy's residue theorem, as in the Watson transformation (Watson, 1918; Laporte, 1923; Regge, 1959); the resulting line integral and residue series can be evaluated if the appropriate asymptotic expansions for the radial eigenfunctions are known. These asymptotic expansions can often be obtained without knowing the exact eigenfunctions, by operating directly on the radial differential equations, for example by the WBK method. However, the domain of validity of a classical WBK solution is limited by the Stokes phenomenon and special care must therefore be taken in using the appropriate connection formulas [see the general discussion in Fröman and Fröman (1965)]. This difficulty can be avoided, and the Stokes phenomenon circumvented, by means of the theory of transition points (i.e. turning points and singular points) which Langer has developed in a series of classical papers over the last thirty-five years [a bibliography on this subject is found in Cesari (1963)].

In the high-frequency analyses of the following chapters, the radial eigenfunctions for the inhomogeneous medium will be either Bessel functions (Chapters One and Two) or algebraic functions (Chapter Five), so that no direct application of the WBK method or of Langer's theory will be necessary. We shall need asymptotic expansions for Bessel functions of large order and argument; if the order and the argument are different, Debye's expansions are in order (see

Watson, 1958; chapter 8), whereas if they are nearly equal, Langer's uniform expansions in terms of Bessel functions of order one-third are to be used. These latter functions are easily related to the Airy integrals, which are well tabulated for all the practical needs of diffraction theory (Miller, 1946; Logan, 1959; Logan and Yee, 1962).

I.2 Outline of Research

The first two chapters are devoted to the determination of the high-frequency backscattered field from a cylinder and a sphere which are imperfectly conducting and are coated by layers of homogeneous materials. These are particular cases of radial inhomogeneities, in which the electromagnetic properties of the medium vary by steps, as functions of the radius (radially stratified medium).

In Chapters Three and Four a systematic presentation is given of the exact formal solutions and of low-frequency approximations for cylinders and spheres made of layers of different materials, within each of which the permittivity and/or permeability vary continuously with the radial distance. In Chapter Five, the high-frequency backscattering from the inverse-square-power dielectric lens is determined.

The main new results obtained are summarized in the following:

Exact Results

1) If the plane wave is obliquely incident on a radially inhomogeneous cylinder, the radial and circumferential field components are obtained by differentiation from the axial components of the electric and magnetic fields, which can be expressed as sums of solutions of certain second order differential equations (section 3.2). Explicit solutions of the boundary value problem for various cylindrical structures are given in sections 3.3 to 3.5, and similar studies are performed for spherical structures in sections 4.3 to 4.5.

2) A fundamental theorem on the uncoupling of TE and TM modes for the field scattered by a certain class of cylindrical bodies is proven in section 3.6.

This result has important applications to cylindrical dielectric lenses, because the field produced by a plane wave at oblique incidence with respect to the axis of the cylinder can be trivially derived from the field produced in the case of normal incidence.

Low-Frequency Results:

3) Resonance and dip conditions for the field backscattered from inhomogeneous cylinders and spheres are determined, in the Rayleigh approximation. In particular, the resonance condition (4.58) for an inverse-square-power lens is of special interest because it is valid for all modes; observe that (4.58) is also the resonance condition for all modes in the case of a homogeneous plasma cylinder.

4) The detailed results of the example of section 3.7 may have applications to graded absorbers.

High-Frequency Results:

5) The asymptotic determination of the backscattered field in terms of geometric optics and creeping wave contributions for impedance cylinders and spheres which are coated by layers of materials whose indexes of refraction are not large compared to unity (Chapters One and Two) has important applications to space vehicles on which a highly absorbing (ferrite) layer is in turn covered by a dielectric ablative layer.

6) The inverse-square-power lens of Chapter Five is one of the very few dielectric lenses whose behavior at high frequencies has been investigated by rigorous asymptotic theory.

The high-frequency research of Chapters One, Two and Five has been restricted to the backscattered field for various reasons; firstly, the backscattering cross section is of primary importance in practical applications; secondly, the most difficult part of the analysis is the determination of asymptotic expansions for quantities which contain the radial eigenfunctions of the inhomogeneous media, and since these quantities also appear when the scattered field is evaluated in an arbitrary direction, many asymptotic results obtained in the following can be used in studying, for example, bistatic cross sections.

I.3 Further Areas of Research

The general formal solutions for various boundary value problems which were given in sections 3.3 to 3.5 and 4.3 to 4.5 constitute the starting point for all high-frequency determinations of the fields scattered from cylindrical and spherical structures. Once the functional dependence of ϵ and μ on the radial distance is specified, the problem is essentially reduced to finding the appropriate asymptotic solutions of the radial differential equations, for various ranges of the parameters involved.

A rich field for applications is, for example, that of dielectric lenses. In addition to the lens considered in Chapter Five, many other lenses, which have been discussed in the literature from a geometrical optics viewpoint, are worth considering from the point of view of rigorous asymptotic theory. Among them, we mention the Luneburg lens $\left(N = \sqrt{2-(r/a)^2}\right.$, where N is the index of refraction and a the radius of the lens$\left.\right)$ and Maxwell's fish-eye $\left(N = 2/\left[1+(r/a)^2\right]\right)$; the radial differential equations for these two lenses can be solved explicitly in terms of confluent and generalized confluent hypergometric functions.

Chapter One

HIGH-FREQUENCY BACKSCATTERING FROM A COATED CYLINDER

1.1 Introduction

The determination of the high-frequency radar cross section of a smooth convex conducting body covered with one or more thin absorbing layers of materials with large complex indexes of refraction (e.g. ferrites) can be greatly simplified by observing that under certain general assumptions the total electric and magnetic fields satisfy an impedance boundary condition on the outer surface of the outer coating layer (Weston, 1963).

In certain practical applications, such a scatterer is in turn covered by another layer of material whose index of refraction is no longer large compared with unity. It is then of great practical importance to investigate the influence that this outer layer has on the magnitude of the far backscattered field, and therefore on the value of the monostatic radar cross section.

The analysis is complicated by the fact that the exact boundary conditions (i.e. the continuity of the tangential components of the total electric and magnetic fields) must be imposed at the outer surface of the outer layer, while an impedance boundary condition may still be assumed on its inner surface, as we shall see in the following.

In this chapter, the investigation is carried out for the case of an infinitely long circular coated cylinder. It is supposed that the material of the outer coating layer has a complex refractive index whose absolute value has a lower bound that is only moderately large compared with unity (e.g. 1.5 or 2), and whose argument is bounded away from both zero and $\pi/2$. An asymptotic evaluation of the far backscattered field is obtained in terms of the reflected field and of the creeping wave contributions, for small wavelengths and normal incidence.

The problem of scattering of plane electromagnetic waves by concentric infinite cylinders has been considered by many authors. The first calculated results for the case of a metal cylinder surrounded by a dielectric sleeve have been published by Adey (1956) who also gave a survey of the previous work on this subject. This case has been reconsidered by Kodis (1959, 1961, 1963) and by Helstrom (1963), among others. The boundary value problem for an arbitrary number of concentric cylinders has been solved by Kerker and Matijevic (1961).

The case in which an impedance boundary condition holds on the surface of the cylindrical core was first considered by the author (Uslenghi, 1964); more recently, Bowman and Weston (1966) have examined the reflected portion of the field scattered by a doubly-coated perfectly conducting cylinder.

1.2 The Infinite Series Solution

Consider an infinitely long cylinder of radius b, coated with a layer of constant thickness d and surrounded by free space; the radius a of the outer surface is then equal to (b+d). The geometry of the scatterer is illustrated in Fig. 1-1, which also shows the two systems of Cartesian (x, y, z) and cylindrical (ρ, ϕ, z) coordinates.

Let ϵ_o, μ_o, and $Z = \sqrt{(\mu_o/\epsilon_o)}$ be respectively the permittivity, the permeability, and the intrinsic impedance of free space; let ϵ and μ be the relative permittivity and permeability of the material of the layer, and suppose that on the surface $\rho = b$ of the cylinder the following impedance boundary condition holds:

$$\underline{E}_1 - (\underline{E}_1 \cdot \hat{\rho})\hat{\rho} = \eta Z \hat{\rho} \times \underline{H}_1 , \qquad (1.1)$$

where $\hat{\rho}$ is a unit vector directed radially from the axis z of the cylinder, $\underline{E}_1$ and $\underline{H}_1$ are the total electric and magnetic fields, and η is the relative surface impedance. The parameters ϵ, μ, and η are supposed constant in space and time.

Consider the plane incident electromagnetic wave:

$$E_z^i = -ZH_y^i = e^{ikx} , \qquad (1.2)$$

where $k = \omega\sqrt{\epsilon_o \mu_o} = 2\pi/\lambda$ is the free-space wave number.

The wave number k_1 of the coating is related to the index of refraction $N = \sqrt{\epsilon\mu}$ by the expression

$$k_1 = Nk .$$

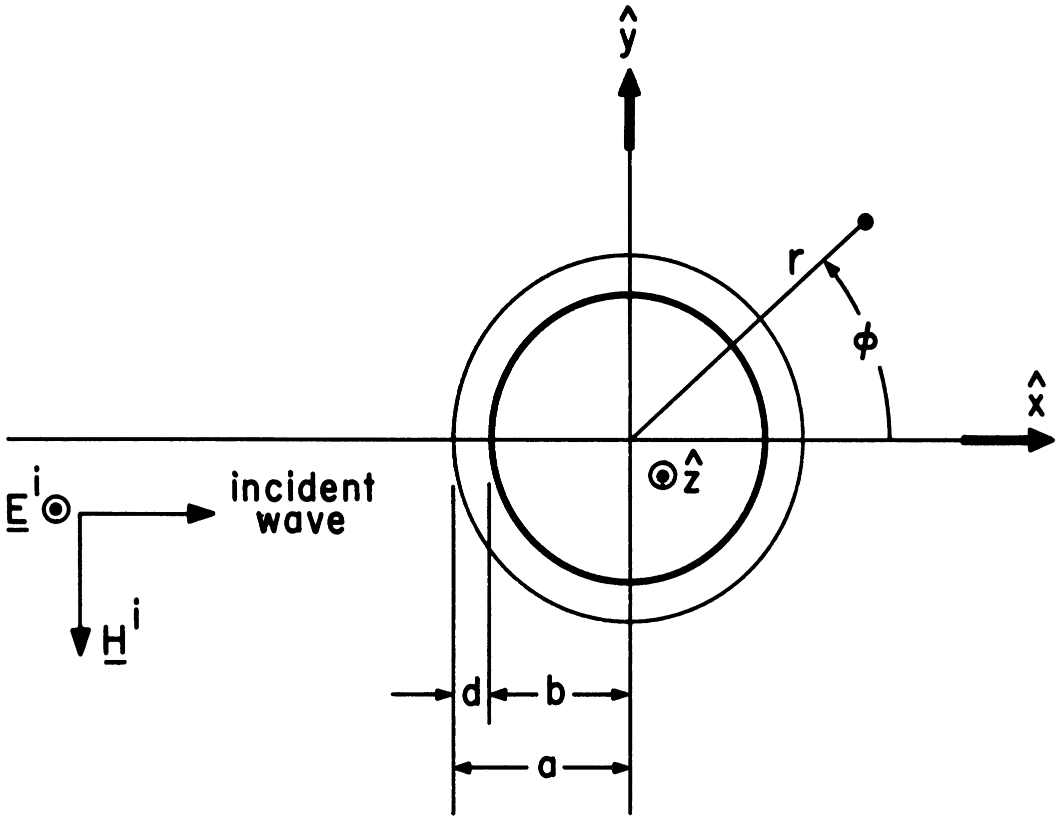

FIG. 1-1: GEOMETRY FOR THE SCATTERING PROBLEM

The incident wave is propagating in the positive x direction, perpendicularly to the axis z of the cylinder, and is polarized in the (x, z) plane. The results for the other polarization ($\underline{E}^i$ parallel to the y axis) may be easily obtained by replacing ϵ and ϵ_0 by μ and μ_0 and vice versa, E by H, H by -E, and η by η^{-1}, throughout the chapter (Senior, 1962).

For all $\rho \geqslant a$, the scattered electric field is given by

$$E_z^s = \sum_{n=0}^{\infty} h_n i^n a_n H_n^{(1)}(k\rho) \cos n\phi \ ,$$

$$E_x^s = E_y^s = 0 \ , \tag{1.3}$$

where $h_0 = 1$ and $h_n = 2$ for $n = 1, 2, \ldots$.

The constants a_n are determined by imposing the boundary conditions, i.e. the continuity of the tangential components of the total electric and magnetic fields across the outer surface $\rho = a$, and the impedance boundary condition (1.1) at the inner surface $\rho = b$. One finds:

$$a_n = - \frac{J_n'(ka) - A_n J_n(ka)}{H_n^{(1)'}(ka) - A_n H_n^{(1)}(ka)} \ , \tag{1.4}$$

where

$$A_n = \frac{N}{\mu} \frac{\partial(\ell n\, C_n)}{\partial(k_1 a)} \; \frac{1 - i\eta \dfrac{N}{\mu} \dfrac{\partial}{\partial(k_1 b)} \left[\ell n \dfrac{\partial C_n}{\partial(k_1 a)} \right]}{1 - i\eta \dfrac{N}{\mu} \dfrac{\partial(\ell n\, C_n)}{\partial(k_1 b)}} \ , \tag{1.5}$$

with

$$C_n = J_n(k_1 a) H_n^{(1)}(k_1 b) - J_n(k_1 b) H_n^{(1)}(k_1 a) \ . \tag{1.6}$$

By making use of the results of Leontovich as discussed by Weston (1963), one finds that the impedance boundary condition (1.1) at $\rho = b$ is a very good approximation provided that the index of refraction of the absorber is very large and has a large imaginary part, and that the radius b is large compared to the wavelength inside the absorber.

For an absorbing layer of thickness Δ, relative permittivity ϵ', and relative permeability μ' backed by a metal core, the relative impedance η on its outer surface $r = b$ is given by the expression:

$$\eta \sim -i \sqrt{(\mu'/\epsilon')} \tan\left[k\Delta \sqrt{(\epsilon'\mu')}\right] . \tag{1.7}$$

A rigorous derivation of (1.7) and of similar expressions for the case of several absorbing layers may be obtained by considering the exact solution of the corresponding boundary-value problem, and by applying to this solution a procedure similar to the one used by Weston and Hemenger (1962) for the coated sphere (see Bowman and Weston, 1966).

No particular expression for η will be assumed in this chapter; then η could not only represent the effect of absorbing layers, but could account for the finite conductivity of the core, or for the roughness of its surface (Senior, 1960).

In the following, it will be assumed that $|k_1 d|$ is not large compared with unity.

1.3 Asymptotic Expansions of the Coefficients a_n.

Let us indicate with ν the order of the Bessel and Hankel functions, and let us suppose that ka and kb are large, and that the argument of N satisfies the inequalities:

$$\Phi < \arg N < \frac{\pi}{2} - \Phi' , \tag{1.8}$$

with

$$\Phi \gg |k_1 b|^{-2/3} , \qquad \Phi' \gg |k_1 b|^{-1} . \tag{1.9}$$

Moreover, let us assume:

$$|\nu - k_1 a| > |\nu^{1/3}| , \qquad |\nu - k_1 b| > |\nu^{1/3}| , \tag{1.10}$$

so that the Debye asymptotic expansions for the Bessel and Hankel functions may be used, yielding the following asymptotic expansion for the coefficients C_n of formula (1.6):

$$C_n \sim \frac{2}{\pi\nu} \frac{\sinh(\psi_2-\psi_1)}{\sqrt{(-i\tanh\gamma_1)(-i\tanh\gamma_2)}} , \tag{1.11}$$

where

$$\begin{aligned} \nu &= k_1 a\cosh\gamma_1, & \gamma_1 &= \alpha_1 + i\beta_1 , \\ \nu &= k_1 b\cosh\gamma_2, & \gamma_2 &= \alpha_2 + i\beta_2 , \\ \psi_1 &= \nu(\tanh\gamma_1 - \gamma_1) & \psi_2 &= \nu(\tanh\gamma_2 - \gamma_2) , \end{aligned} \tag{1.12}$$

and α_1, α_2, β_1 and β_2 are real quantities; β_1 and β_2 are supposed positive and less than π.

If $\nu/(k_1a)$ or $\nu/(k_1b)$ were close to $-1.5i$, then a modification of the expansion (1.11) would be required. This modification is avoided by imposing the upper bound (1.8) on the argument of N.

More details leading to the derivation of (1.11) can be found in Watson (1958) and in Weston and Hemenger (1962).

From (1.5) and (1.11) it follows that the asymptotic expansion of A_n under the hypotheses (1.8) and (1.9) for large ka and kb, is given by

$$A_n \sim -\frac{N}{\mu} \coth\sigma \sinh\gamma_1 , \tag{1.13}$$

where

$$\sigma = \psi_2 - \psi_1 - i\arctan(\eta \frac{N}{\mu} \sinh\gamma_2) . \tag{1.14}$$

In the particular case of a perfectly conducting cylinder ($\eta = 0$) coated with a material of very large index of refraction, it can be proven that A_n is independent of n, and that the coefficients a_n are given by

$$a_n \sim - \frac{J_n(ka) - i\eta' J_n'(ka)}{H_n^{(1)}(ka) - i\eta' H_n^{(1)'}(ka)} , \tag{1.15}$$

where

$$\eta' = -i \frac{\mu}{N} \tan k_1 d \ . \tag{1.16}$$

Approximation (1.15) can be obtained through a procedure very similar to that developed by Weston and Hemenger (1962) for the coated perfectly conducting sphere, and therefore will not be given. It is here sufficient to recall that in order that (1.15) and (1.16) be valid, the following additional assumptions must be made:

$$\begin{aligned} &\eta' \text{ is bounded away from the imaginary axis,} \\ &\left|\arg(N/\mu)\right| \leqslant \pi/4 , \\ &kd \ll 2M^2 \left|N\right| , \end{aligned} \tag{1.17}$$

where M is an undetermined real number satisfying the condition

$$1 + (ka)^{-2/3} \ll M \ll \left|N\right| \ . \tag{1.18}$$

The approximation (1.15) means that the field scattered by the coated cylinder is the same field that would be scattered by a cylinder with radius a and relative surface impedance η'.

Instead of making the approximation (1.15), in the following we shall consider the more general case in which the impedance η is not zero and $|N|$ is no longer restricted to large values.

The Debye expansions may be used for $J_\nu(ka)$ and $H_\nu^{(1)}(ka)$, provided that

$$\left|\nu - ka\right| > \left|\nu^{1/3}\right| \ . \tag{1.19}$$

In sections 1.4 and 1.5 we shall use the relations

$$\nu = ka \cosh\gamma_3, \qquad \gamma_3 = \alpha_3 + i\beta_3 \quad ,$$
$$\psi_3 = \nu(\tanh\gamma_3 - \gamma_3) \ , \tag{1.20}$$

with α_3 real, and β_3 real, positive and less than π.

1.4 High-Frequency Backscattered Field: Reflected Field Contribution

The intensity of the scattered electric field is given by relation (1.3), which in the case of the backscattered far field becomes:

$$E_z^{b.s.} \sim \sqrt{\frac{2}{\pi k\rho}}\, e^{ik\rho - i\frac{\pi}{4}} \left[a_o + 2\sum_{n=1}^{\infty} (-1)^n a_n \right] . \tag{1.21}$$

Treating the summation over n as a residue series, the summation is replaced by the contour integral C of Fig. 1-2, taken in the clockwise direction around the poles at $\nu = 1, 2, \ldots,$ giving:

$$E_z^{b.s.} \sim \sqrt{\frac{2}{\pi k\rho}}\, e^{ik\rho - i\frac{\pi}{4}} \left[a_o + \int_C \frac{ia_\nu}{\sin \pi\nu}\, d\nu \right] . \tag{1.22}$$

The poles of a_ν lie in the first and in the third quadrants of the complex ν plane. Following the Watson transform technique (Watson, 1918; Laporte, 1923), the contour integral C is replaced by the sum of the two following quantities:

1) A line integral whose contour Γ consists of the path Γ_1 extending from the fourth quadrant through the point $\nu = 1/2$ to the second quadrant, plus the arc Γ_2 of a circle of large radius R with center at the origin, extending from the second through the first to the fourth quadrant (Fig. 1-2). The sum of the asymptotic expressions for this line integral and for the term involving a_o in relation (1.22) gives the reflected field approximation for the far back-scattered field.

2) A residue series due to the poles of the integrand function which lie in the first quadrant. This series represents the creeping wave contribution to the far back scattered field, and will be considered in section 1.5.

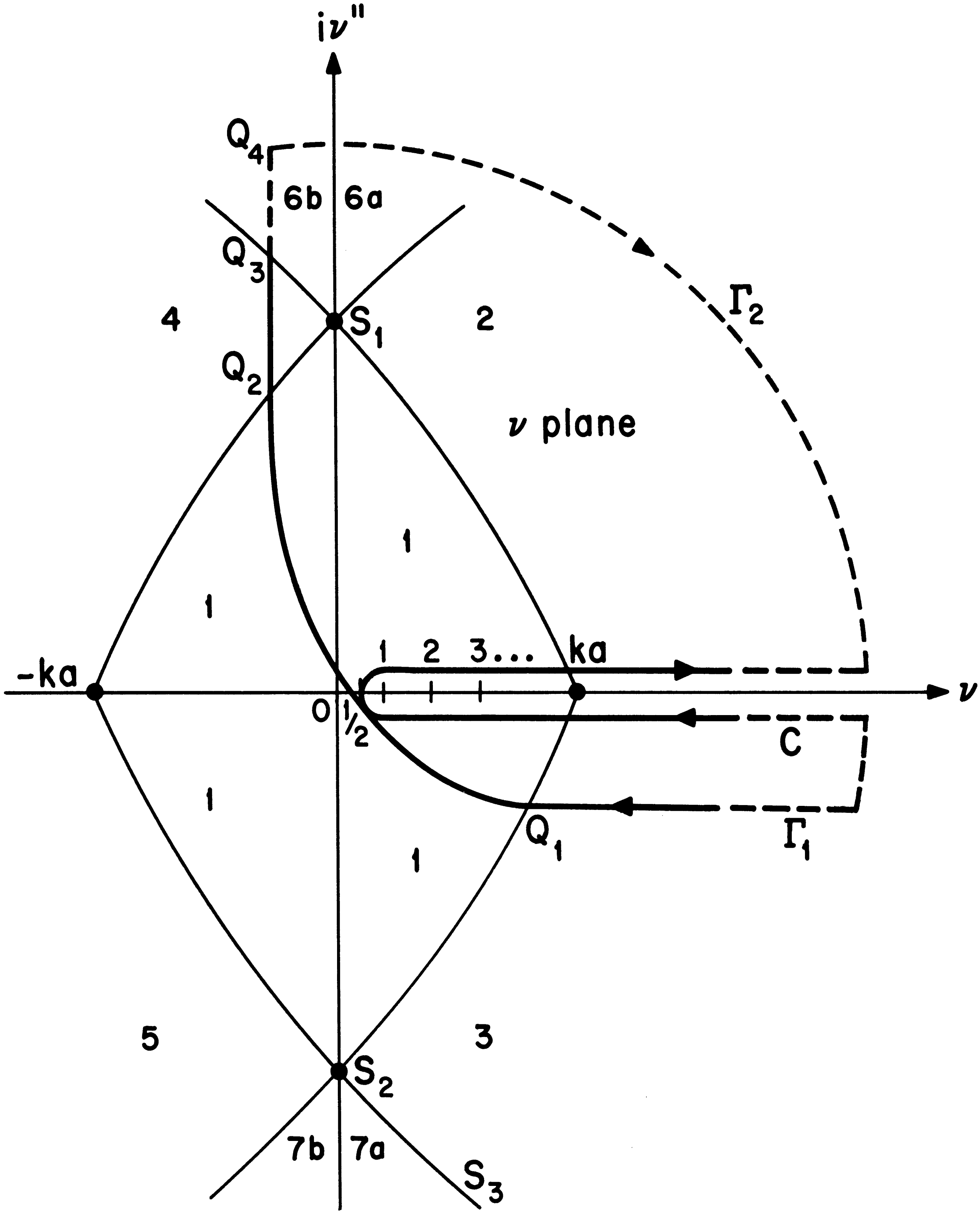

FIG. 1-2: CONTOURS OF INTEGRATION IN THE COMPLEX ν-PLANE

In this section we shall derive the reflected field approximation. Following Watson, we divide the ν plane into the various regions of Fig. 1-2, numbered from 1 to 7b and separated from one another by the coordinate axes and by the curves $S_1Q_2S_2S_3$ and $Q_3S_1Q_1S_2$, whose equations are respectively:

$$\alpha_3 \tanh\alpha_3 - (\pi - \beta_3) \cot\beta_3 - 1 = 0 \ ,$$

and

$$\alpha_3 \tanh\alpha_3 + \beta_3 \cot\beta_3 - 1 = 0 \ .$$

We shall now show that the integral along the arc Γ_2 vanishes in the limit where R tends to infinity. Let

$$\nu = \nu' + i\nu'' = R\,e^{\pm i\theta} \ ,$$

where $0 \leqslant \theta \leqslant \pi$, and the first (second) sign is valid when $\nu'' \geqslant 0$ ($\nu'' \leqslant 0$). Then the integrand function $ia_\nu/\sin\pi\nu$ behaves like

$$e^{-\pi R \sin\theta} \left(\frac{2R}{ka}\right)^{-2R\cos\theta}$$

in the regions 2, 3 and 6a, and like $e^{-\pi R \sin\theta}$ in the second quadrant.

Since, as we shall see, the contour Γ_1 is such that the angle θ is bounded away from π, then $ia_\nu/\sin\pi\nu$ tends exponentially to zero, and therefore the line integral along Γ_2 vanishes, provided that R goes to infinity through such values that no poles of the integrand ever lie on the contour Γ_2 (a general proof that this can be done was given by Goodrich and Kazarinoff, 1963).

The line integral along Γ_1 may be asymptotically evaluated by saddle point technique. Using the Debye asymptotic expansions derived in section 1.3, it is found that the integrand behaves as follows:

$$\frac{ia_\nu}{\sin\pi\nu} \sim f(\nu)\,\frac{e^{-2\psi_3}}{e^{i\pi\nu}-e^{-i\pi\nu}}\,, \qquad \text{in region 3,}$$

$$\sim f(\nu)\,\frac{e^{-2\psi_3}}{e^{i\pi\nu}-e^{-i\pi\nu}} - \frac{i}{2\sin\pi\nu}\,, \quad \text{in regions 1, 4, and 6b,} \tag{1.23}$$

where the slowly varying function $f(\nu)$ is given by

$$f(\nu) = i\,\frac{\sinh\gamma_1 - \frac{\mu}{N}\sinh\gamma_3\tanh\sigma}{\sinh\gamma_1 + \frac{\mu}{N}\sinh\gamma_3\tanh\sigma}\;. \tag{1.24}$$

Therefore we have that

$$\int_{\Gamma_1}\frac{ia_\nu}{\sin\pi\nu}\,d\nu = \int_{\Gamma_1}\frac{e^{-2\psi_3}}{e^{i\pi\nu}-e^{-i\pi\nu}}\,f(\nu)\,d\nu + \int_{Q_1Q_2Q_3Q_4}\frac{(-i)}{2\sin\pi\nu}\,d\nu\;. \tag{1.25}$$

In order to evaluate the first integral of the second member of relation (1.25), observe that with the substitution

$$\nu = u+\frac{1}{2}\;, \tag{1.26}$$

this is reduced to a form already investigated by Scott (1949). The main contribution arises from a pass near the point $u=0$; the contour Γ_1 runs at a suitably chosen distance below the real axis in region 3, and close to the imaginary axis in region 6b.

The asymptotic evaluation of the integrand for $|\nu| \ll ka$ yields:

$$\int_{\Gamma_1}\frac{e^{-2\psi_3}}{e^{i\pi\nu}-e^{-i\pi\nu}}\,f(\nu)d\nu \sim \frac{1}{2\pi}\,\frac{\tilde{\eta}-1}{\tilde{\eta}+1}\,e^{-i2ka}\int_{-2\pi\xi e^{i\delta}}^{+2\pi\xi e^{i\delta}}\frac{e^{-\epsilon w^2}}{1+e^{-w}}\,g(w)dw\;, \tag{1.27}$$

where:

$$g(w) = 1 - \frac{i}{ka}\frac{\tilde{\eta}^2}{\tilde{\eta}^2 - 1} - \frac{w}{2\pi ka} + \frac{1}{4\pi^2}\left[\frac{1}{2} + \frac{\tilde{\eta}}{\tilde{\eta}^2 - 1}\left(1 - i\frac{2p\tilde{\eta}}{N^2}\right)\right]\left(\frac{w}{ka}\right)^2 -$$

$$- \frac{i}{192\pi^4}\frac{w^4}{(ka)^3} + O\left[(ka)^{-2}\right] + O\left[\frac{w^5}{(ka)^4}\right] , \quad (1.28)$$

$$\epsilon = -\frac{i}{4\pi^2 ka} , \qquad \xi = (ka)^{\frac{1}{2}+\epsilon} , \quad (1.29)$$

$$\tilde{\eta} = -i\frac{\mu}{N}\tan\left[k_1 d + \arctan(i\eta\frac{N}{\mu})\right] , \quad (1.30)$$

$$p = \frac{\frac{N}{\mu}\cot k_1 d}{1 + i\eta\frac{N}{\mu}\cot k_1 d}\left[\frac{1}{2}\left\{1 + (\eta\frac{N}{\mu})^2\right\} - \frac{k_1 d}{\sin 2k_1 d}\left\{1 - (\eta\frac{N}{\mu})^2\right\} - i\eta\frac{N}{\mu}\tan k_1 d\right] , \quad (1.31)$$

and δ is an angle between zero and $\pi/2$. The starting point to obtain the asymptotic expansion of the integrand function is given by the relations (Watson, 1958):

$$H_\nu^{(1)}(z) = \sqrt{\frac{2i}{\pi\nu}\coth\gamma}\; e^{\psi - i\frac{\pi}{4}}\left[1 - \frac{i}{8z} + \frac{9}{128z^2} + O(z^{-3}) + O(\nu^2/z^3)\right] ,$$

$$H_\nu^{(2)}(z) = \sqrt{\frac{2i}{\pi\nu}\coth\gamma}\; e^{-\psi + i\frac{\pi}{4}}\left[1 + \frac{i}{8z} + \frac{9}{128z^2} + O(z^{-3}) + O(\nu^2/z^3)\right] ,$$

where $\nu = z\cosh\gamma$. The new variable of integration w is related to u by

$$u = -\frac{i}{2\pi} w .$$

Since ξ is large, the integral at the second member of relation (1.27) may be decomposed into a sum of integrals of the two following forms:

$$E_{o,n} = \int_{-\infty e^{i\delta}}^{\infty e^{i\delta}} \frac{e^{-\epsilon w^2}}{1+e^{-w}} w^{2n+1} dw , \qquad (n = 0, 1, 2, \dots),$$

$$E_{e,n} = \int_{-\infty e^{i\delta}}^{\infty e^{i\delta}} \frac{e^{-\epsilon w^2}}{1+e^{-w}} w^{2n} dw , \qquad (n = 0, 1, 2, \dots) . \tag{1.32}$$

The integrals $E_{o,n}$ were computed by Scott (1949):

$$E_{o,o} \sim \frac{1}{2\epsilon} - \frac{\pi^2}{6} + O(\epsilon^1) , \qquad E_{o,1} \sim \frac{1}{2\epsilon^2} + O(\epsilon^o) ,$$

$$E_{o,2} \sim \frac{1}{\epsilon^3} + O(\epsilon^o) . \tag{1.33}$$

The integrals $E_{e,n}$ can be easily reduced to Fresnel integrals:

$$E_{e,n} = \int_o^{\infty e^{i\delta}} e^{-\epsilon w^2} w^{2n} dw$$

$$= (-1)^n \frac{\partial^n}{\partial \epsilon^n} \left\{ \int_o^{\infty e^{i\delta}} e^{-\epsilon w^2} dw \right\} \tag{1.34}$$

$$= \frac{\sqrt{\pi}}{2} e^{i\frac{\pi}{4}} (-i)^n \frac{\partial^n}{\partial |\epsilon|^n} \left\{ |\epsilon|^{-1/2} \right\} , \quad (\text{for } \delta = \pi/4) .$$

Substitution of the above formulas in relation (1.27) yields:

$$\int_{\Gamma_1} \frac{e^{-2\psi_3}}{e^{i\pi\nu} - e^{-i\pi\nu}} f(\nu) d\nu \sim \frac{\tilde{\eta}-1}{\tilde{\eta}+1} \frac{\sqrt{\pi ka}}{2} e^{-i2ka + i\frac{\pi}{4}} \left[1 - \frac{e^{i\frac{\pi}{4}}}{\sqrt{\pi ka}} + \right.$$

$$\left. + \frac{i}{2ka} \frac{\tilde{\eta}}{\tilde{\eta}^2 - 1} \left\{ 1 - 2\tilde{\eta} \left(1 + \frac{ip}{N^2} \right) \right\} + O\left\{ (ka)^{-3/2} \right\} \right] \tag{1.35}$$

Let us now evaluate the second integral of the second member of relation (1.25). Since the integrand's only poles lie on the real ν axis, and since the integrand goes exponentially to zero when the absolute value of ν'' increases toward infinity, it is sufficient to remark that $|\nu''|$ at Q_1 is very large in order to be able to conclude that

$$\int_{Q_1Q_2Q_3Q_4} \frac{(-i)}{2\sin\pi\nu}\, d\nu \sim \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{(-i)}{2\sin\pi\nu}\, d\nu = \frac{1}{2}, \tag{1.36}$$

as it follows from the relation

$$2\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{(-i)}{2\sin\pi\nu}\, d\nu = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{(-i)}{2\sin\pi\nu}\, d\nu + \int_{\frac{3}{2}+i\infty}^{\frac{3}{2}-i\infty} \frac{(-i)}{2\sin\pi\nu}\, d\nu$$

$$= 2\pi i\left(\text{residue of } \frac{(-i)}{2\sin\pi\nu} \text{ at } \nu=1\right) = 1\ ;$$

in the preceding integrals the paths of integration are the straight lines $\nu' = 1/2$ and $\nu' = 3/2$.

The coefficient a_o has the asymptotic value

$$a_o + \frac{1}{2} = -\frac{1}{2}\, \frac{H_o^{(2)'}(ka) - A_o H_o^{(2)}(ka)}{H_o^{(1)'}(ka) - A_o H_o^{(1)}(ka)}$$

$$\sim \frac{i}{2}\, \frac{\tilde{\eta}-1}{\tilde{\eta}+1}\, e^{-i2ka} \left[1 - \frac{i}{4ka}\, \frac{3\tilde{\eta}^2+1}{\tilde{\eta}^2-1} + O\left\{(ka)^{-2}\right\}\right]. \tag{1.37}$$

Finally, the reflected field approximation to the far back-scattered field, as derived from relations (1.22), (1.25), (1.35), (1.36) and (1.37) is given by:

$$\left[E_z^{b.s.}\right]_{refl.} \sim \frac{\tilde{\eta}-1}{\tilde{\eta}+1}\sqrt{\frac{a}{2\rho}}\, e^{ik\rho - i2ka} \times$$

$$\times\left[1+\frac{i}{2ka}\left\{\frac{5}{8}+\frac{\tilde{\eta}}{\tilde{\eta}^2-1}\left[1-2\tilde{\eta}\left(1+\frac{i\rho}{N^2}\right)\right]\right\}+\ldots\right]. \qquad (1.38)$$

The first term of the Luneberg-Kline expansion (1.38) is the geometric optics contribution to the back scattered field.

If we let a go to infinity, we find the field that would be reflected by a plane of surface impedance η coated with a layer of thickness d and refractive index N, for normal incidence.

In the case of a perfectly conducting cylinder $(d=0,\ \eta=0)$, relation (1.38) becomes:

$$\left[E_z^{b.s.}\right]_{\substack{refl.\\ d=0,\ \eta=0}} \sim -\sqrt{\frac{a}{2\rho}}\, e^{ik\rho - i2ka}\left(1+\frac{5i}{16ka}+\ldots\right). \qquad (1.39)$$

This formula checks with the result obtained by Imai (1954).

If the core is perfectly conducting and the material of the coating is an absorbing dielectric $(\eta=0,\ \mu=1)$,

$$\left[E_z^{b.s.}\right]_{refl.} \sim -\sqrt{\frac{a}{2\rho}}\, e^{ik\rho - i2ka}\left[\frac{N+i\tan k_1 d}{N-i\tan k_1 d}+O(1/ka)\right]. \qquad (1.40)$$

In order to compare (1.40) with the reflected field of Kodis (1963),

$$\left[E_z^{b.s.}\right]_{\substack{refl.\\ Kodis}} \sim \sqrt{\frac{a}{2\rho}}\, e^{ik\rho - i2ka}\left[\frac{1-N}{1+N}+\frac{4N}{N^2-1}\sum_{s=1}^{\infty}\left(1+\frac{sd}{Nb}\right)^{-1/2} \times\right.$$

$$\left.\times\left\{\frac{1-N}{1+N}\, e^{i2k_1 d}\right\}^s\right], \qquad (1.41)$$

observe that since $\mathrm{Im}\, k_1$ is positive, only the lowest values of s are of importance, and therefore

$$\left(1+\frac{sd}{Nb}\right)^{-1/2} \sim 1+O(1/kb) \quad , \tag{1.42}$$

provides that s_{max} is not large compared to (2N)/(kd). Relation (1.40) is then easily obtained from (1.41) and (1.42).

1.5 High-Frequency Backscattered Field: Creeping Wave Contribution

The creeping wave contribution to the back scattering far field is given by the residue series:

$$\left[E_z^{b.s.}\right]_{cr.w.} \sim -\frac{4i}{ka}\sqrt{\frac{2}{\pi kr}}\, e^{ikr-i\frac{\pi}{4}} \times$$

$$\times \sum_s \left[\sin(\pi\nu)\, H_\nu^{(1)}(ka)\, \frac{\partial}{\partial\nu}\left\{H_\nu^{(1)'}(ka) - A_\nu H_\nu^{(1)}(ka)\right\}\right]^{-1}_{\nu=\nu_s} \tag{1.43}$$

where ν_s are the roots of

$$H_\nu^{(1)'}(ka) - A_\nu H_\nu^{(1)}(ka) = 0 \tag{1.44}$$

which have positive imaginary parts.

Since the main contribution arises from the roots of (1.44) which are close to ka, we introduce the Fock asymptotic approximation (Fock, 1945):

$$\begin{aligned} H_\nu^{(1)}(ka) &\sim -\frac{i}{\sqrt{\pi}}\, m^{-1} w_1(t) \quad , \\ H_\nu^{(1)'}(ka) &\sim \frac{i}{\sqrt{\pi}}\, m^{-2} w_1'(t) \ , \end{aligned} \tag{1.45}$$

where:

$$\nu = ka + mt, \qquad m = (ka/2)^{1/3} \quad , \tag{1.46}$$

and $w_1(t)$ is the Airy integral in the Fock notation, which is related to the functions Ai(t) and Bi(t) of Miller (1946) by the expression:

$$w_1(t) = \sqrt{\pi}\left[\mathrm{Bi}(t) + i\,\mathrm{Ai}(t)\right] ,$$

where: $$\mathrm{Ai}(t) = \frac{1}{\pi}\int_0^{\infty} \cos\left(\frac{u^3}{3} + tu\right) du ,$$

$$\mathrm{Bi}(t) = \frac{1}{\pi}\int_0^{\infty} \left[\exp\left(-\frac{u^3}{3} + tu\right) + \sin\left(\frac{u^3}{3} + tu\right)\right] du .$$

We can still use the Debye approximation for A_ν, provided that the absolute value of N is sufficiently large compared to unity. Specifically, we shall assume that $|N|$ is so large that inequalities (1.10) are satisfied by the first few roots of equation (1.44). Then the Debye approximation for A_ν gives:

$$A_\nu \sim -\frac{i}{\eta_1} + p_1 \frac{t}{m^2} , \tag{1.47}$$

where $$\eta_1 = -i\frac{\mu}{\sqrt{N^2-1}} \tan\left[kd\sqrt{N^2-1} + \arctan\left(i\eta\frac{\sqrt{N^2-1}}{\mu}\right)\right] , \tag{1.48}$$

$$p_1 = \frac{1}{2\mu\sqrt{N^2-1}}\left(1 + i\eta\frac{\beta\cot\beta}{\mu kd}\right)^{-2}\left[\frac{\beta}{\sin^2\beta} - \cot\beta + i\frac{2\eta\beta}{\mu kd}\left(1 + i\eta\frac{\beta\cot\beta}{\mu kd}\right)\right] \tag{1.49}$$

and $$\beta = kd\sqrt{N^2-1} . \tag{1.50}$$

Approximation (1.47) is valid under the assumptions

$$|t| \ll m^2 ,$$
$$\left|\frac{kd}{2\sqrt{N^2-1}}\frac{t}{m^2}\right| \ll 1 . \tag{1.51}$$

Observe that as $|N|$ increases, η_1 approaches the value $\tilde{\eta}$ given by relation (1.30).

With the approximations (1.45) and (1.47), the creeping wave contribution becomes:

$$\left[E_z^{b.s.}\right]_{cr.w.} \sim \frac{2\sqrt{2\pi}}{m} e^{-i\frac{3\pi}{4}} \frac{e^{ik\rho}}{\sqrt{k\rho}} \sum_s \left[\sin(\pi\nu_s) w_1^2(t_s) \left\{ \eta_1^{-2} + \frac{p_1}{m^3} + \right.\right.$$

$$\left.\left. + \frac{t_s}{m^2}\left(1 + i\frac{2p_1}{\eta_1}\right)\right\}\right]^{-1} , \qquad (1.52)$$

where t_s are the roots of

$$\frac{w_1'(t)}{w_1(t)} = \frac{im}{\eta_1} - p_1 \frac{t}{m} . \qquad (1.53)$$

An approximate evaluation of t_s gives

$$t_s \sim t_{os}\left(1 + \frac{p_1}{mt_{os} + m^3\eta_1^{-2}}\right)^{-1} , \qquad (1.54)$$

where t_{os} are the roots of the equation

$$\frac{w_1'(t)}{w_1(t)} = \frac{im}{\eta_1} , \qquad (1.55)$$

and may be obtained from the values of $w_1'(t)/w_1(t)$ which were computed by Logan and Yee (1962) when t lies in the first quadrant.

The total back scattered field is obtained by adding together the contributions (1.38) and (1.52).

If the parameter ka is not extremely large with respect to unity (for example $ka = 10$), then the approximations (1.45) are no longer sufficiently accurate, and must be replaced by the following relations:

$$H_\nu^{(1)}(ka) \sim -\frac{i}{\sqrt{\pi}} m^{-1} \left[w_1(t) - \frac{1}{60m^2}\left\{4tw_1(t) + t^2 w_1'(t)\right\}\right] ,$$

$$H_\nu^{(1)'}(ka) \sim \frac{i}{\sqrt{\pi}} m^{-2} \left[w_1'(t) + \frac{1}{60m^2}\left\{4tw_1'(t) + (6 - t^3) w_1(t)\right\}\right] . \qquad (1.56)$$

The right-hand sides of (1.52) and (1.53) are then replaced by more complicated expressions; the calculations were not explicitly performed for this case.

1.6 Some Considerations for the Quasi-Optical Limit

The following considerations apply to a cylinder with a very large diameter. Since the creeping wave contribution to the backscattered field is proportional to $(ka)^{-1/3}$, the dominant part of the far backscattered field arises from the leading term in the asymptotic expansion (1.38). Therefore, the "reflection coefficient"

$$\tilde{R} = \frac{\tilde{\eta}-1}{\tilde{\eta}+1} \quad , \tag{1.57}$$

which depends upon the three parameters η, k_1d, and μ/N (see formula (1.30)), is the critical quantity which determines the magnitude of the backscattered field and of the monostatic radar cross section.

If $\epsilon = \mu$, we have

$$\tilde{R}_{\epsilon=\mu} = R e^{i2k_1d} \quad , \tag{1.58}$$

where

$$R = \frac{\eta-1}{\eta+1} \tag{1.59}$$

is the "reflection coefficient" for the uncoated cylinder, and therefore the strength of the backscattered field varies as

$$|R|\, e^{-2kd\,\mathrm{Im}\,N} \quad .$$

In particular, if

$$\tilde{R} = 0 \tag{1.60}$$

the radar cross section is $O(ka)^{-2/3}$, and therefore very small. Relation (1.60) is satisfied for all value of k_1d, provided that $\epsilon = \mu$ and $\eta = 1$.

If

$$\mathrm{th}(kd\,\mathrm{Im}\,N) \sim 1, \tag{1.61}$$

then R is given by the simple expression

$$\tilde{R} \sim \frac{\mu - N}{\mu + N} \quad , \tag{1.62}$$

which is independent of both η and k_1d, and equals zero when $\epsilon = \mu$.

In conclusion, we may say that the presence of the coating layer produces a modification in the far backscattered field, whose magnitude greatly depends upon the values of the parameters η, μ/N, and k_1d. In particular, the monostatic radar cross section can be reduced to very small values.

Chapter Two

HIGH-FREQUENCY BACKSCATTERING FROM A DOUBLY-COATED SPHERE

2.1 Introduction

The scattering of a plane electromagnetic wave by spheres composed of concentric layers of various materials has been previously investigated by Aden and Kerker (1951) and by Sharfman (1954), who derived the exact Mie series for the cases they considered. Weston and Hemenger (1962) performed an asymptotic evaluation of the backscattered field from a perfectly conducting sphere covered with a thin layer of material with a large complex index of refraction; they simplified their analysis by showing that an impedance boundary condition may be assumed on the outer surface of the coating layer. Recently, Bowman and Weston (1966) have considered the reflected portion of the field back scattered by a perfectly conducting sphere coated by one or two layers of absorbers.

In this chapter, the case of a sphere coated with two concentric layers of different materials is considered. It is supposed that the material of the inner layer has a refractive index whose absolute value has a lower bound which is only moderately large compared to unity, and whose argument is bounded away from both zero and $\pi/2$. The refractive index of the material of the outer layer is assumed very close to unity. An asymptotic evaluation of the far backscattered field is obtained in terms of the geometric optics and of the creeping wave contributions, for small wavelengths.

The analysis is performed by imposing the exact boundary conditions (i.e. the continuity of the tangential components of the total electric and magnetic fields) at the outer surfaces of both coating layers, while an impedance boundary condition is imposed on the surface of the spherical core. The procedure is similar to that one followed in Chapter One, and therefore the details of the derivations will not be given.

2.2 The Mie Series Solution

Consider a sphere of radius ρ coated with two concentric layers of different materials and surrounded by free space. The geometry of the scatterer is illustrated in Fig. 2-1; the radii of the outer surfaces of the two layers are given by $b = \rho + d_2$ and $a = b + d_1$, where d_1 and d_2 are the thicknesses of the outer and inner coatings, respectively.

Let ϵ_o, μ_o, $Z = \sqrt{\mu_o/\epsilon_o}$ and $k = \omega\sqrt{\epsilon_o\mu_o}$ be respectively the electric permittivity, the magnetic permeability, the intrinsic impedance and the wave number of free space, and let ϵ_j, μ_j, $Z_j = \sqrt{\mu_j/\epsilon_j}$ and $k_j = \omega\sqrt{\epsilon_j\mu_j}$, ($j = 1$ or 2) be the corresponding quantities for the two coating layers, where $j = 1$ for the outer coating.

The incident electromagnetic field

$$E_x^i = Z H_y^i = e^{ikz} \tag{2.1}$$

produces the far backscattered field

$$E_x^{b.s.} = -Z H_y^{b.s.} \sim -i \frac{e^{ikr}}{kr} \sum_{n=1}^{\infty} (-1)^n \left(n + \frac{1}{2}\right) (a_n - b_n) \,, \tag{2.2}$$

where $r = |z|$ is the distance of the field point from the center of the sphere.

The coefficients a_n and b_n are found by imposing the boundary conditions, i.e. the continuity of the tangential components of the total electric and magnetic fields across the surfaces $r = a$ and $r = b$, and the impedance boundary condition

$$\underline{E} - (\underline{E} \cdot \hat{r})\hat{r} = \eta \hat{r} \times \underline{H} \tag{2.3}$$

on the surface $r = \rho$ of the spherical core. In relation (2.3), η represents the surface impedance, $\hat{r}$ the radial unit vector, and $\underline{E}$ and $\underline{H}$ the total electric and magnetic fields at $r = \rho$. It is found that:

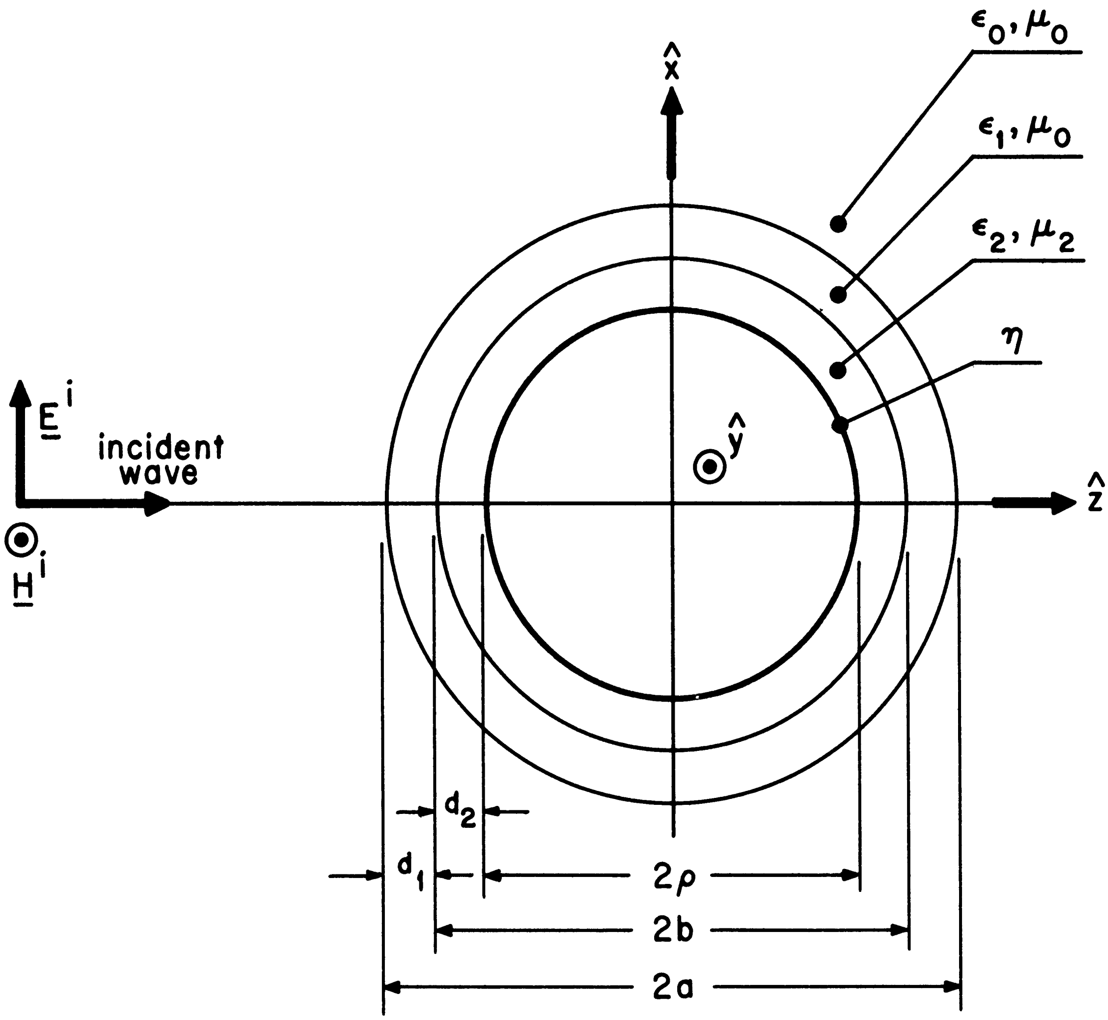

FIG. 2-1: GEOMETRY FOR THE SCATTERING PROBLEM.

$$a_n = -\frac{\psi_n'(ka) - A_n\psi_n(ka)}{\zeta_n^{(1)'}(ka) - A_n\zeta_n^{(1)}(ka)} \quad ,$$

$$b_n = -\frac{\psi_n'(ka) - B_n\psi_n(ka)}{\zeta_n^{(1)'}(ka) - B_n\zeta_n^{(1)}(ka)} \quad , \tag{2.4}$$

where:

$$A_n = \frac{Z}{Z_1}\frac{\partial}{\partial(k_1 a)} \ln\left[\frac{\partial C_n}{\partial(k_1 b)} - \frac{Z_1}{Z_2} C_n \frac{\partial}{\partial(k_2 b)} \ln\left\{D_n - i\frac{\eta}{Z_2}\frac{\partial D_n}{\partial(k_2\rho)}\right\}\right] ,$$

$$B_n = \frac{Z_1}{Z}\frac{\partial}{\partial(k_1 a)} \ln\left[\frac{\partial C_n}{\partial(k_1 b)} - \frac{Z_2}{Z_1} C_n \frac{\partial}{\partial(k_2 b)} \ln\left\{D_n - i\frac{Z_2}{\eta}\frac{\partial D_n}{\partial(k_2\rho)}\right\}\right] , \tag{2.5}$$

with

$$C_n = \zeta_n^{(2)}(k_1 a)\,\zeta_n^{(1)}(k_1 b) - \zeta_n^{(2)}(k_1 b)\,\zeta_n^{(1)}(k_1 a) \quad ,$$

$$D_n = \zeta_n^{(2)}(k_2 b)\,\zeta_n^{(1)}(k_2\rho) - \zeta_n^{(2)}(k_2\rho)\,\zeta_n^{(1)}(k_2 b) \quad . \tag{2.6}$$

The primes which appear in relation (2.4) denote derivatives with respect to the argument ka. The functions ψ_n and $\zeta_n^{(1),(2)}$ are given by

$$\psi_n(x) = \sqrt{\frac{\pi x}{2}}\, J_{n+1/2}(x) \quad ,$$

$$\zeta_n^{(1),(2)}(x) = \sqrt{\frac{\pi x}{2}}\, H_{n+1/2}^{(1),(2)}(x) \quad . \tag{2.7}$$

In particular, it follows from formula (2.2) that a sufficient condition to have a zero backscattered field is that $a_n = b_n$ for all values of n, i.e. that

$$Z_1 = Z_2 = Z \; , \qquad \eta = \pm Z \; . \tag{2.8}$$

This conclusion is easily extended to the case of an arbitrary number of concentric coating layers; a sufficient condition to have a zero backscattered field is that the relative permittivity be equal to the relative permeability for each layer, and that the relative surface impedance of the spherical core equals ± 1.

2.3 High-Frequency Backscattered Field: Geometric Optics Contribution

Setting $\nu = n + \frac{1}{2}$, and treating the summation of formula (2.2) as a residue series, the summation is replaced by a contour integral along a path C in the complex ν plane, taken around the poles at $\nu = 1/2, 3/2, \ldots,$ giving:

$$E_x^{b.s.} \sim \frac{e^{ikr}}{kr}\left\{\frac{i}{2}(a_o - b_o) - \frac{1}{2}\int_C \frac{\nu}{\cos(\pi\nu)}(a_{\nu-\frac{1}{2}} - b_{\nu-\frac{1}{2}})\,d\nu\right\} . \quad (2.9)$$

Following a Watson-type transformation, the contour C is deformed to include the poles of the integrand which lie in the first quadrant. The resulting line integral may be asymptotically evaluated by saddle point technique, and added to the asymptotic contribution arising from the term of formula (2.9) which contains $(a_o - b_o)$. The resulting sum represents the Luneberg-Kline asymptotic expansion of the reflected portion of the backscattered field; only the leading term of this expansion, corresponding to the geometric optics backscattered field, has been explicitly computed:

$$\left[E_x^{b.s.}\right]_{g.o.} \sim \frac{ka}{2}\left(\frac{\tilde{\eta} - Z}{\tilde{\eta} + Z}\right) e^{-i2ka}\,\frac{e^{ikr}}{kr}\left\{1 + O(1/ka)\right\} , \quad (2.10)$$

where the impedance $\tilde{\eta}$ is given by

$$\tilde{\eta} = -iZ_1 \tan\left\{k_1 d_1 + \arctan\left[\frac{Z_2}{Z_1}\tan\left\{k_2 d_2 + \arctan\left(i\frac{\eta}{Z_2}\right)\right\}\right]\right\} \quad (2.11)$$

We point out that formulas (1.30) and (2.11) have the same structure, and can be easily generalized to the case of an arbitrary number of coating layers.

2.4 High-Frequency Backscattered Field: Creeping Wave Contribution

The creeping wave contribution to the far backscattered field is given by the residue series:

$$\left[E_x^{b.s.}\right]_{cr.w.} \sim \pi \frac{e^{ikr}}{kr} \times$$

$$\times \left[\sum_{\ell} \left\{ \left[\frac{1}{\nu} \cos(\pi\nu)\, \zeta^{(1)}_{\nu-1/2}(ka) \frac{\partial}{\partial\nu} \left[\zeta^{(1)'}_{\nu-1/2}(ka) - A_{\nu-1/2}\, \zeta^{(1)}_{\nu-1/2}(ka) \right] \right] \right\}^{-1}_{\nu=\nu_\ell} - \right.$$

$$\left. - \sum_{s} \left\{ \left[\frac{1}{\nu} \cos(\pi\nu)\, \zeta^{(1)}_{\nu-1/2}(ka) \frac{\partial}{\partial\nu} \left[\zeta^{(1)'}_{\nu-1/2}(ka) - B_{\nu-1/2}\, \zeta^{(1)}_{\nu-1/2}(ka) \right] \right] \right\}^{-1}_{\nu=\nu_s} \right] , \tag{2.12}$$

where ν_ℓ and ν_s are respectively the roots of

$$\zeta^{(1)'}_{\nu-1/2}(ka) - A_{\nu-1/2}\, \zeta^{(1)}_{\nu-1/2}(ka) = 0 \tag{2.13}$$

and of

$$\zeta^{(1)'}_{\nu-1/2}(ka) - B_{\nu-1/2}\, \zeta^{(1)}_{\nu-1/2}(ka) = 0 \tag{2.14}$$

which have positive imaginary parts.

In order to determine these roots, we introduce the asymptotic approximations (Fock, 1945):

$$\begin{aligned} \zeta^{(1)}_{\nu-1/2}(ka) &\sim -i\, m^{1/2}\, w_1(t) \;, \\ \zeta^{(1)'}_{\nu-1/2}(ka) &\sim -i\, m^{-1/2}\, w_1'(t) \;, \end{aligned} \tag{2.15}$$

where m and t are defined by (1.46), and $w_1(t)$ is the Airy integral in Fock's notation.

The coefficients $A_{\nu-1/2}$ and $B_{\nu-1/2}$ have the asymptotic expansions:

$$A_{\nu-\frac{1}{2}} \sim \sigma_1 + \tau_1 \frac{t}{m^2} ,$$

$$B_{\nu-\frac{1}{2}} \sim \sigma_2 + \tau_2 \frac{t}{m^2} , \qquad (2.16)$$

where

$$\sigma_1 = \left(\eta_1 \frac{Z_2}{Z} + k_1 d_1 \frac{Z_1}{Z}\right)^{-1} ,$$

$$\sigma_2 = \left(\eta_2 \frac{Z}{Z_2} + k_1 d_1 \frac{Z}{Z_1}\right)^{-1} , \qquad (2.17)$$

$$\tau_1 = \sigma_1^{-1} \left\{ \xi_1 - \Delta + \eta_1 k_1 d_1 \frac{Z_2}{Z_1} + \sigma_1^{-1} \left(\eta_1 \Delta \frac{Z_2}{Z} - \xi_1 k_1 d_1 \frac{Z_1}{Z} \right) \right\} ,$$

$$\tau_2 = \sigma_2^{-1} \left\{ \xi_2 - \Delta + \eta_2 k_1 d_1 \frac{Z_1}{Z_2} + \sigma_2^{-1} \left(\eta_2 \Delta \frac{Z}{Z_2} - \xi_2 k_1 d_1 \frac{Z}{Z_1} \right) \right\} ,$$

with

$$\Delta = (ka - k_1 a)(ka - k_1 b) ,$$

$$\eta_{1,2} = \frac{k_2 d_2}{\beta} \tan(\beta + \arctan \delta_{1,2}) ,$$

$$\xi_{1,2} = \frac{1}{2} \left\{ (k_2/k_1)^2 - 1 \right\}^{-1} \left\{ \frac{\beta \tan\beta - 1 + \delta_{1,2}(\beta + 2\tan\beta)}{1 - \delta_{1,2}\tan\beta} + \frac{\beta + \delta_{1,2}(1 - \beta\tan\beta)}{\delta_{1,2} + \tan\beta} \right\} , \qquad (2.18)$$

and

$$\beta = k_1 d_2 \left\{ (k_2/k_1)^2 - 1 \right\}^{1/2} ,$$

$$\delta_1 = \frac{i\beta\eta}{k_2 d_2 Z_2} , \qquad \delta_2 = \frac{i\beta Z_2}{k_2 d_2 \eta} . \qquad (2.19)$$

The asymptotic expansions (2.16) are valid under the following assumptions:

(I) $k\rho \ggg 1$;

(II) $\phi < \arg(k_2/k) < \pi/2 - \phi'$,

with

$$\phi \gg |k_2\rho|^{-2/3}, \quad \phi' \gg |k_2\rho|^{-1} ;$$

(III) $|k_2/k|$ is so large that the inequalities:

$$|\nu - k_2 b| > |\nu|^{1/3}, \quad |\nu - k_2\rho| > |\nu|^{1/3}$$

are satisfied by the first few ν_ℓ's and ν_s's (i.e. by those values of ν_ℓ and ν_s that have a small imaginary part);

(IV) for the first few ν_ℓ's and ν_s's it is:

$$|t/m^2| << 1,$$

and

(V) $$\left| \frac{k_1 d_2}{2} \left\{ (k_2/k_1)^2 - 1 \right\}^{-1/2} t/m^2 \right| \ll 1 ;$$

(VI) the exception case in which $\beta = \pi(n + 1/2)$, with n integer, is excluded;

(VII) $$\left| \frac{ka - k_1 a}{m} \right| << 1 , \quad \left| \frac{ka - k_1 b}{m} \right| \ll 1 ;$$

(VIII) $$\left| 2\xi_{1,2} \left(1 - \frac{k_1 b}{ka} \right) \right| << 1 .$$

Restriction VI may be released provided that $\eta \neq 0$; in such a case expansions (2.16) are valid with the following values of $\eta_{1,2}$ and $\xi_{1,2}$:

$$(\eta_{1,2})_{\substack{\eta\neq 0\\ \beta=\pi(n+1/2)}} = -\frac{k_2 d_2}{\pi(n+1/2)\,\delta_{1,2}}\ , \tag{2.20}$$

$$(\xi_{1,2})_{\substack{\eta\neq 0\\ \beta=\pi(n+1/2)}} = \left\{(k_2/k_1)^2 - 1\right\}^{-1}\left\{-1+\frac{\pi}{2}\left(n+\frac{1}{2}\right)(\delta_{1,2}-\delta_{1,2}^{-1})\right\}, \tag{2.21}$$

where n is an integer.

Assumptions I and IV are satisfied for all large spheres. The first inequality of restriction VII limits the choice of k_1/k to values very close to unity; if the outer radius a is equal to ten wavelengths, then $\left|(k_1/k) - 1\right| \ll 0.05$, and the restriction becomes more severe as the frequency increases. The second inequality of VII establishes an upper bound on the permissible values of kd_1. Assumptions II, III and V place restrictions on the allowed values of k_2/k and kd_2. Finally, inequality of restriction VIII should be satisfied for the majority of cases in which the other seven conditions hold.

If the parameter $k\rho$ is only moderately large with respect to unity, then the expansions (2.15) must be modified by the inclusion of higher order terms; the calculations were not performed for this case.

With the approximations (2.15) and (2.16) the creeping wave contribution becomes:

$$\left[E_x^{b.s.}\right]_{cr.w.} \sim \frac{\pi}{m}\frac{e^{ikr}}{kr}\left[\sum_{\nu=\nu_\ell}\frac{\nu_\ell}{\cos(\pi\nu_\ell)}\left\{w_1(t_\ell)\right\}^{-2}\left\{\frac{\tau_1}{m^3}-\sigma_1^2 + \frac{t_\ell}{m^2}(1-2\sigma_1\tau_1)\right\}^{-1} - \sum_{\nu=\nu_s}\frac{\nu_s}{\cos(\pi\nu_s)}\left\{w_1(t_s)\right\}^{-2}\left\{\frac{\tau_2}{m^3}-\sigma_2^2 + \frac{t_s}{m^2}(1-2\sigma_2\tau_2)\right\}^{-1}\right]\ , \tag{2.22}$$

where t_ℓ and t_s are respectively the roots of

$$\frac{w_1'(t_\ell)}{w_1(t_\ell)} = -m\sigma_1 - \frac{\tau_1}{m} t_\ell \quad , \tag{2.23}$$

and

$$\frac{w_1'(t_s)}{w_1(t_s)} = -m\sigma_2 - \frac{\tau_2}{m} t_s \quad . \tag{2.24}$$

For large values of m, the t_ℓ's and t_s's are approximately given by

$$t_\ell \sim t_{o\ell} \left(1 + \frac{\tau_1/m}{t_{o\ell} - m^2 \sigma_1^2}\right)^{-1} \quad ,$$

$$t_s \sim t_{os} \left(1 + \frac{t_2/m}{t_{os} - m^2 \sigma_2^2}\right)^{-1} \quad , \tag{2.25}$$

where $t_{o\ell}$ and t_{os} are the roots of:

$$\frac{w_1'(t_{o\ell})}{w_1(t_{o\ell})} = -m\sigma_1, \qquad \frac{w_1'(t_{os})}{w_1(t_{os})} = -m\sigma_2 \quad . \tag{2.26}$$

The first few roots of equations (2.26) may be derived from the values of $w_1'(t)/w_1(t)$ which were computed by Logan and Yee (1962) when t lies in the first quadrant.

The total backscattered field is obtained by adding together the contributions (2.10) and (2.22).

Chapter Three

SCATTERING OF OBLIQUELY INCIDENT WAVES FROM RADIALLY INHOMOGENEOUS CYLINDERS. EXACT SOLUTIONS AND LOW-FREQUENCY APPROXIMATIONS

3.1 Introduction

The scattering of electromagnetic waves by structures which are cylindrically symmetric but inhomogeneous in the radial direction is considered in this chapter. The scatterer is an infinitely long circular cylindrical region with radially varying permittivity $\epsilon(\rho)\epsilon_0$ and permeability $\mu(\rho)\mu_0$, surrounded by free space. Since an arbitrary incident electromagnetic field can be decomposed into the sum of plane monochromatic waves by Fourier analysis, it is sufficient to consider the case of a time-harmonic plane wave at oblique incidence with respect to the axis z of the cylinder. The corresponding boundary value problem for a homogeneous cylinder has been solved by Wait (1955), and some considerations for the case $\mu(\rho) = 1$ have been developed by Farone and Querfeld (1966). The differential equations satisfied by the radial eigenfunctions in the more general case considered here were given by Uslenghi (1966). It appears that other authors have confined their attention to normal incidence [see lists of references in Farone and Querfeld (1966) and in Burman (1966)].

The components of the incident and scattered fields and of the fields inside the cylinder are given in section 3.2 as infinite series of eigenfunctions, and some considerations are developed for the low-frequency approximations. A few cases of practical interest are examined in detail: the scatterer is made of a radially inhomogeneous layer with an impenetrable cylindrical core (section 3.3), or with a free-space core (section 3.4), or is without a core (section 3.5).

For oblique incidence there is a coupling between the TM and TE modes of the electromagnetic field, that is even if the incident electric or magnetic field has a zero component along the cylinder axis, the axial components of the total electric and magnetic fields are, in general, both different from zero. However, if the index of refraction considered as a function of the distance ρ from the cylinder axis has no step discontinuities on the whole interval $0 < \rho < \infty$, then the TM and TE modes are uncoupled. This result is proven in section 3.6.

Finally, the particular case in which the permeability and the permittivity are, respectively, directly and inversely proportional to the distance from the cylinder axis is examined in detail in section 3.7. The corresponding boundary value problem is solved for an inhomogeneous shell with an imperfectly conducting core and normal incidence; low-frequency approximations are determined, and some considerations are developed on the high-frequency backscattered field, for normal incidence.

3.2 The General Case

Consider a radially inhomogeneous cylindrical region of outer radius $\rho = a$ and inner radius $\rho = b$, made of a material with relative permittivity $\epsilon(\rho)$ and relative permeability $\mu(\rho)$, and surrounded by free space; the cylindrical core $0 \leq \rho \leq b$ can be made of an imperfectly conducting material (section 3.3), or of free space (section 3.4), or be missing altogether if $b = 0$ (section 3.5).

Let us introduce two systems of rectangular Cartesian (x, y, z) and cylindrical polar (ρ, ϕ, z) coordinates, where z is the axis of symmetry of the scatterer. Let ϵ_o, μ_o, $Z = Y^{-1} = \sqrt{\mu_o/\epsilon_o}$ and $k = \omega\sqrt{\epsilon_o \mu_o}$ be respectively the permittivity, permeability, intrinsic impedance and wave number of free space, and consider an incident plane wave whose direction of propagation forms the angle α, $0 < \alpha < \pi$, with the positive z axis (Fig. 3-1).

The radial and circumferential components of the electric and magnetic fields $\underline{E}$ and $\underline{H}$ can be derived from the axial components through the relations:

$$E_\rho = \frac{i}{\tau}\left[\cos\alpha \frac{\partial E_z}{\partial(k\rho)} + Z\frac{\mu}{k\rho}\frac{\partial H_z}{\partial\phi}\right] , \tag{3.1}$$

$$E_\phi = \frac{i}{\tau}\left[\frac{\cos\alpha}{k\rho}\frac{\partial E_z}{\partial\phi} - Z\mu\frac{\partial H_z}{\partial(k\rho)}\right] , \tag{3.2}$$

$$H_\rho = \frac{i}{\tau}\left[\cos\alpha \frac{\partial H_z}{\partial(k\rho)} - Y\frac{\epsilon}{k\rho}\frac{\partial E_z}{\partial\phi}\right] , \tag{3.3}$$

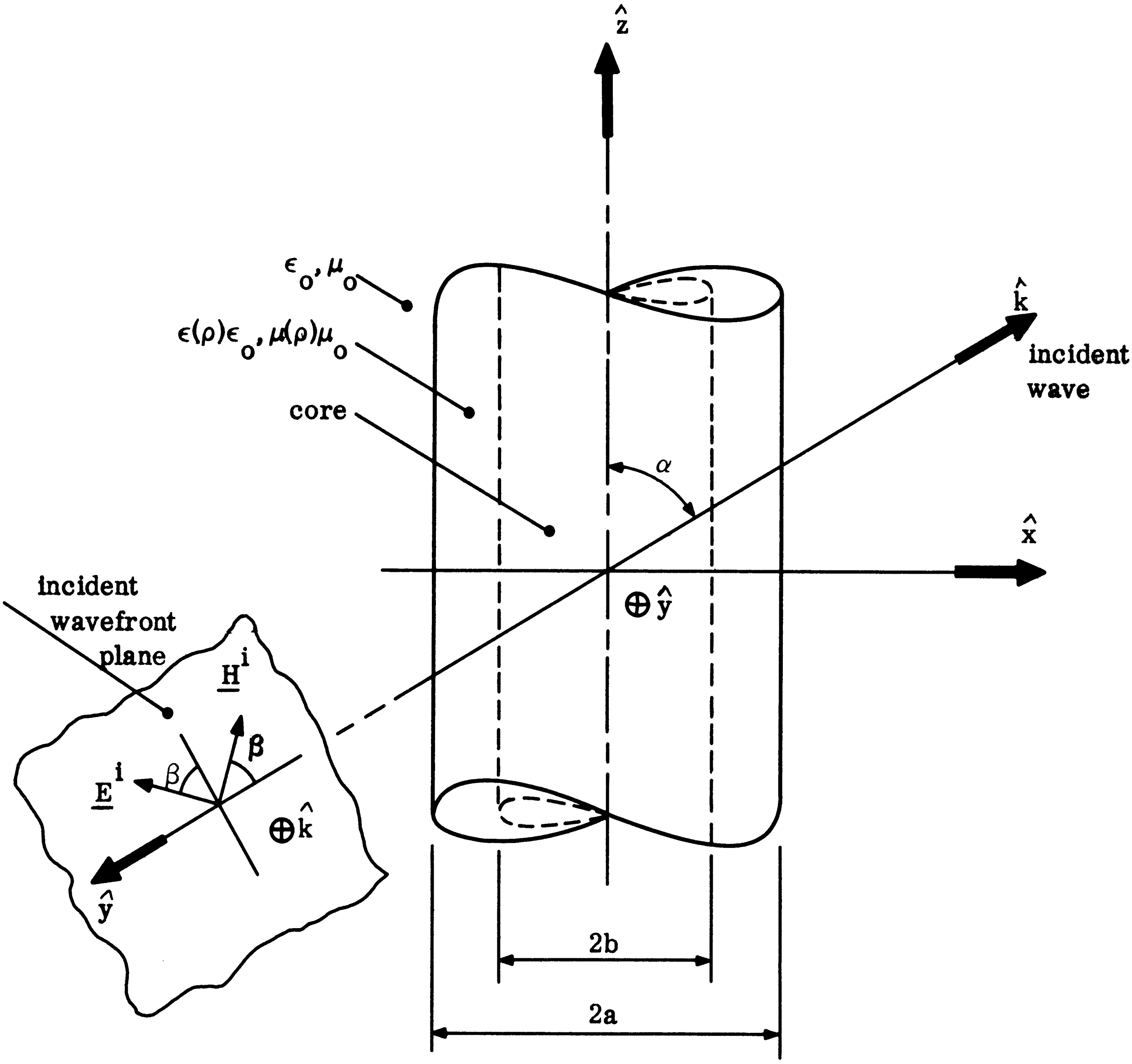

FIG. 3-1: GEOMETRY FOR PLANE WAVE AT OBLIQUE INCIDENCE.

$$H_\phi = \frac{i}{\tau}\left[\frac{\cos\alpha}{k\rho}\,\frac{\partial H_z}{\partial\phi} + Y\epsilon\frac{\partial E_z}{\partial(k\rho)}\right] , \tag{3.4}$$

where

$$\tau = \tau(\rho) = \epsilon(\rho)\mu(\rho) - \cos^2\alpha , \tag{3.5}$$

$\epsilon = \epsilon(\rho)$ and $\mu = \mu(\rho)$ inside the inhomogeneous region, and $\epsilon = \mu = 1$ in free space.

The axial components can be written in the forms

$$E_z = e^{ikz\cos\alpha} \sum_{n=-\infty}^{\infty} i^n U_n(\xi_\rho) e^{in\phi} , \tag{3.6}$$

$$H_z = Y e^{ikz\cos\alpha} \sum_{n=-\infty}^{\infty} i^n V_n(\xi_\rho) e^{in\phi} , \tag{3.7}$$

where

$$\xi_\rho = k\rho\sin\alpha . \tag{3.8}$$

The functional dependence on z is dictated by the incident field, and the dependence on ϕ by the physical consideration that the field must be periodic with period 2π. The radial eigenfunctions U_n and V_n are solutions of (Uslenghi, 1966):

$$\frac{d^2U_n}{d\xi_\rho^2} + \left[\frac{1}{\xi_\rho} + \frac{d}{d\xi_\rho}\ln\frac{\epsilon}{\tau}\right]\frac{dU_n}{d\xi_\rho} + \left[\frac{\tau}{\sin^2\alpha} - \frac{n^2}{\xi_\rho^2}\right]U_n = 0 , \tag{3.9}$$

$$\frac{d^2V_n}{d\xi_\rho^2} + \left[\frac{1}{\xi_\rho} + \frac{d}{d\xi_\rho}\ln\frac{\mu}{\tau}\right]\frac{dV_n}{d\xi_\rho} + \left[\frac{\tau}{\sin^2\alpha} - \frac{n^2}{\xi_\rho^2}\right]V_n = 0 . \tag{3.10}$$

These differential equations simplify for normal incidence ($\alpha = \pi/2$) with ϵ and μ arbitrary, and for arbitrary incidence with $\epsilon\mu$ = constant; in both cases:

$$\frac{d}{d\xi_\rho} \ln \frac{\epsilon}{\tau} \Bigg|_{\substack{\text{either } \alpha = \pi/2 \\ \text{or } \epsilon\mu = \text{constant}}} = - \frac{d \ln \mu}{d\xi_\rho} , \tag{3.11}$$

$$\frac{d}{d\xi_\rho} \ln \frac{\mu}{\tau} \Bigg|_{\substack{\text{either } \alpha = \pi/2 \\ \text{or } \epsilon\mu = \text{constant}}} = - \frac{d \ln \epsilon}{d\xi_\rho} . \tag{3.12}$$

The particular case of normal incidence and $\mu = 1$ for all ρ was previously considered, among others, by Yeh and Kaprielian (1963) whose radial functions U_n and V_n are equal to the product of $k\rho$ times our functions U_n and V_n, respectively.

The solution of a boundary value problem involving any number of coaxial layers, within each of which $\epsilon(\rho)$ and $\mu(\rho)$ are differentiable, is thus essentially reduced to the solution of (3.9) and (3.10) for each layer and to the imposition of the appropriate boundary conditions. Consider the incident plane electromagnetic wave

$$\underline{E}^i = (-\cos\alpha \cos\beta \hat{x} + \sin\beta \hat{y} + \sin\alpha \cos\beta \hat{z}) e^{ik(x \sin\alpha + z \cos\alpha)} , \tag{3.13}$$

$$\underline{H}^i = Y(-\cos\alpha \sin\beta \hat{x} - \cos\beta \hat{y} + \sin\alpha \sin\beta \hat{z}) e^{ik(x \sin\alpha + z \cos\alpha)} , \tag{3.14}$$

where $\underline{E}^i$ forms the angle β with the (x, z) plane and the angle $(\frac{\pi}{2} - \beta)$ with the positive y axis (Fig. 3-1).

The total electric and magnetic fields in the region $\rho > a$ are given by the sums of the incident fields $\underline{E}^i$ and $\underline{H}^i$ and of the scattered fields $\underline{E}^s$ and $\underline{H}^s$, whose components may be written in the form:

$$E^i_\rho = \sum_{n=-\infty}^{\infty} f_n \left[i \cos\alpha \cos\beta J'_n - \frac{n \sin\beta}{\xi_\rho} J_n \right] , \tag{3.15}$$

$$E^i_\phi = - \sum_{n=-\infty}^{\infty} f_n \left[\frac{n \cos\alpha \cos\beta}{\xi_\rho} J_n + i \sin\beta J'_n \right] , \tag{3.16}$$

$$E_z^i = \sin\alpha\cos\beta \sum_{n=-\infty}^{\infty} f_n J_n \,, \tag{3.17}$$

$$H_\rho^i = Y \sum_{n=-\infty}^{\infty} f_n\left[\frac{n\cos\beta}{\xi_\rho} J_n + i\cos\alpha\sin\beta J'_n\right] , \tag{3.18}$$

$$H_\phi^i = Y \sum_{n=-\infty}^{\infty} f_n\left[i\cos\beta J'_n - \frac{n\cos\alpha\sin\beta}{\xi_\rho} J_n\right] , \tag{3.19}$$

$$H_z^i = Y\sin\alpha\sin\beta \sum_{n=-\infty}^{\infty} f_n J_n \,, \tag{3.20}$$

$$E_\rho^s = \sum_{n=-\infty}^{\infty} f_n\left[i\cos\alpha\, a_n^s H_n^{(1)'} - \frac{n b_n^s}{\xi_\rho} H_n^{(1)}\right] , \tag{3.21}$$

$$E_\phi^s = - \sum_{n=-\infty}^{\infty} f_n\left[\frac{n\cos\alpha}{\xi_\rho} a_n^s H_n^{(1)} + i b_n^s H_n^{(1)'}\right] , \tag{3.22}$$

$$E_z^s = \sin\alpha \sum_{n=-\infty}^{\infty} f_n a_n^s H_n^{(1)} \,, \tag{3.23}$$

$$H_\rho^s = Y \sum_{n=-\infty}^{\infty} f_n\left[\frac{n a_n^s}{\xi_\rho} H_n^{(1)} + i\cos\alpha\, b_n^s H_n^{(1)'}\right] , \tag{3.24}$$

$$H_\phi^s = Y \sum_{n=-\infty}^{\infty} f_n\left[i a_n^s H_n^{(1)'} - \frac{n\cos\alpha}{\xi_\rho} b_n^s H_n^{(1)}\right] , \tag{3.25}$$

$$H_z^s = Y\sin\alpha \sum_{n=-\infty}^{\infty} f_n b_n^s H_n^{(1)} \,, \tag{3.26}$$

where

$$f_n = f_n(\phi, z) = i^n \exp(in\phi + ikz\cos\alpha) \quad , \tag{3.27}$$

the argument of the Bessel functions is ξ_ρ, and the prime indicates the derivative with respect to the argument.

The total electric and magnetic fields $\underline{E}_1$ and $\underline{H}_1$ inside the inhomogeneous region $b < \rho < a$ are given by:

$$E_{1\rho} = \frac{\sin\alpha}{\tau(\rho)} \sum_{n=-\infty}^{\infty} f_n \left[i\sin\alpha\cos\alpha\, U_n' - \frac{n\mu(\rho)}{k\rho} V_n \right] \quad , \tag{3.28}$$

$$E_{1\phi} = -\frac{\sin\alpha}{\tau(\rho)} \sum_{n=-\infty}^{\infty} f_n \left[\frac{n\cos\alpha}{k\rho} U_n + i\sin\alpha\,\mu(\rho) V_n' \right] \quad , \tag{3.29}$$

$$E_{1z} = \sin\alpha \sum_{n=-\infty}^{\infty} f_n U_n \quad , \tag{3.30}$$

$$H_{1\rho} = Y\frac{\sin\alpha}{\tau(\rho)} \sum_{n=-\infty}^{\infty} f_n \left[\frac{n\epsilon(\rho)}{k\rho} U_n + i\sin\alpha\cos\alpha\, V_n' \right] \quad , \tag{3.31}$$

$$H_{1\phi} = Y\frac{\sin\alpha}{\tau(\rho)} \sum_{n=-\infty}^{\infty} f_n \left[i\sin\alpha\,\epsilon(\rho) U_n' - \frac{n\cos\alpha}{k\rho} V_n \right] \quad , \tag{3.32}$$

$$H_{1z} = Y\sin\alpha \sum_{n=-\infty}^{\infty} f_n V_n \quad , \tag{3.33}$$

where the argument of U_n and V_n is ξ_ρ, the prime indicates the derivative with respect to ξ_ρ,

$$U_n(\xi_\rho) = a_{1n} U_n^{(1)}(\xi_\rho) + c_{1n} U_n^{(2)}(\xi_\rho) \quad , \tag{3.34}$$

$$V_n(\xi_\rho) = b_{1n} V_n^{(1)}(\xi_\rho) + d_{1n} V_n^{(2)}(\xi_\rho) \quad , \tag{3.35}$$

and $U_n^{(j)}$ and $V_n^{(j)}$ (j = 1 or 2) are two linearly independent solutions of (3.9) and (3.10), respectively. The Wronskians W_U of $U_n^{(j)}$ and W_V of $V_n^{(j)}$ are given by:

$$W_U(\rho) = U_n^{(1)}(\xi_\rho)U_n^{(2)'}(\xi_\rho) - U_n^{(1)'}(\xi_\rho)U_n^{(2)}(\xi_\rho) = \gamma\frac{\tau(\rho)}{\epsilon(\rho)\xi_\rho} \quad , \tag{3.36}$$

$$W_V(\rho) = V_n^{(1)}(\xi_\rho)V_n^{(2)'}(\xi_\rho) - V_n^{(1)'}(\xi_\rho)V_n^{(2)}(\xi_\rho) = \delta\frac{\tau(\rho)}{\mu(\rho)\xi_\rho} \quad , \tag{3.37}$$

where γ and δ are two constants whose values depend on the normalizations of the eigenfunctions.

In the case of the cylindrical shell of section 3.4, one must also consider the total electric and magnetic fields $\underline{E}_2$ and $\underline{H}_2$ inside the free-space region $0 \leqslant \rho < b$:

$$E_{2\rho} = \sum_{n=-\infty}^{\infty} f_n\left[i\cos\alpha\, a_{2n}J_n' - \frac{n b_{2n}}{\xi_\rho} J_n\right] , \tag{3.38}$$

$$E_{2\phi} = -\sum_{n=-\infty}^{\infty} f_n\left[\frac{n\cos\alpha}{\xi_\rho} a_{2n}J_n + i b_{2n}J_n'\right] , \tag{3.39}$$

$$E_{2z} = \sin\alpha \sum_{n=-\infty}^{\infty} f_n a_{2n} J_n \quad , \tag{3.40}$$

$$H_{2\rho} = Y\sum_{n=-\infty}^{\infty} f_n\left[\frac{-n\, a_{2n}}{\xi_\rho} J_n + i\cos\alpha\, b_{2n}J_n'\right] , \tag{3.41}$$

$$H_{2\phi} = Y\sum_{n=-\infty}^{\infty} f_n\left[i a_{2n}J_n' - \frac{n\cos\alpha}{\xi_\rho} b_{2n}J_n\right] , \tag{3.42}$$

$$H_{2z} = Y\sin\alpha \sum_{n=-\infty}^{\infty} f_n b_{2n} J_n \, , \tag{3.43}$$

where the argument of J_n is ξ_ρ, and the prime indicates the derivative with respect to ξ_ρ.

The constants a_n^s, b_n^s, a_{1n}, c_{1n}, b_{1n}, d_{1n}, and eventually a_{2n} and b_{2n}, are determined by imposing the appropriate boundary conditions at $\rho = a$ and $\rho = b$ (the radiation condition is already satisfied by the choices (3.23) and (3.26) for the scattered fields). In particular, a_n^s and b_n^s can always be written in the form:

$$a_n^s = -\frac{\left[J_n'(\xi_a) - M_n J_n(\xi_a)\right]\cos\beta + A_n \sin\beta}{H_n^{(1)'}(\xi_a) - M_n H_n^{(1)}(\xi_a)} \,, \tag{3.44}$$

$$b_n^s = -\frac{\left[J_n'(\xi_a) - \widetilde{M}_n J_n(\xi_a)\right]\sin\beta - \widetilde{A}_n \cos\beta}{H_n^{(1)'}(\xi_a) - \widetilde{M}_n H_n^{(1)}(\xi_a)} \,, \tag{3.45}$$

where $\xi_a = ka\sin\alpha$, and the quantities M_n, A_n, $\widetilde{M}_n$, $\widetilde{A}_n$ depend upon the structure of the scatterer and upon the angle α of incidence, but not upon the polarization angle β.

A resonance on the nth mode of the low-frequency scattered field occurs when the dominant term in the denominator of either a_n^s or b_n^s becomes zero, that is when either

$$\left[\frac{H_n^{(1)'}(\xi_a)}{H_n^{(1)}(\xi_a)}\right]_{LF} = (M_n)_{LF} \,, \tag{3.46}$$

or

$$\left[\frac{H_n^{(1)'}(\xi_a)}{H_n^{(1)}(\xi_a)}\right]_{LF} = (\widetilde{M}_n)_{LF} \,. \tag{3.47}$$

Here the subscript LF is used to indicate the Rayleigh approximation, i.e. the leading term in the low-frequency expansion. In particular,

$$\frac{H_o^{(1)'}(\xi_a)}{H_o^{(1)}(\xi_a)} \sim -\frac{2}{M}\xi_a^{-1}\left[1+O(\xi_a^2 \ln \xi_a)\right] , \tag{3.48}$$

$$\frac{H_1^{(1)'}(\xi_a)}{H_1^{(1)}(\xi_a)} \sim -\xi_a^{-1}\left[1+O(\xi_a^2 \ln \xi_a)\right] , \tag{3.49}$$

$$\left[\frac{H_n^{(1)'}(\xi_a)}{H_n^{(1)}(\xi_a)}\right]_{n>1} \sim -n\xi_a^{-1}\left[1+O(\xi_a^2)\right] , \tag{3.50}$$

where

$$M = i\pi - 2\gamma_o - 2\ln(\xi_a/2) \tag{3.51}$$

and $\gamma_o = 0.5772157..$ is Euler's constant.

3.3 The Coated Cylinder

In order to determine the coefficients a_n^s, b_n^s, a_{1n}, c_{1n}, b_{1n} and d_{1n}, one most impose the continuity of the tangential components of the total electric and magnetic fields across the surface $\rho = a$ and the impedance boundary condition

$$\underline{E}_1 - (\underline{E}_1 \cdot \hat{\rho})\hat{\rho} = \eta Z \hat{\rho} \times \underline{H}_1 , \quad \text{at } \rho = b , \tag{3.52}$$

where η is the relative surface impedance. One finds the system of six linear algebraic equations reproduced in Table I, where the abbreviations

$$J_\rho = J_n(\xi_\rho) , \qquad J'_\rho = \frac{\partial}{\partial \xi_\rho} J_n(\xi_\rho)$$

and similar ones for the other functions have been used; thus, for example, the equation corresponding to the first row is:

$$-H_n^{(1)}(ka\sin\alpha)a_n^s + U_n^{(1)}(ka\sin\alpha)a_{1n} + U_n^{(2)}(ka\sin\alpha)c_{1n} = J_n(ka\sin\alpha)\cos\beta .$$

TABLE I: COEFFICIENTS FOR COATED CYLINDER

a_n^s	b_n^s	a_{1n}	c_{1n}	b_{1n}	d_{1n}	right-hand side
$-H_a^{(1)}$	0	$U_a^{(1)}$	$U_a^{(2)}$	0	0	$\cos\beta J_a$
0	$-H_a^{(1)}$	0	0	$V_a^{(1)}$	$V_a^{(2)}$	$\sin\beta J_a$
$-\frac{n\cot\alpha}{ka} H_a^{(1)}$	$-iH_a^{(1)'}$	$\frac{n\sin 2\alpha}{2ka\,\tau(a)} U_a^{(1)}$	$\frac{n\sin 2\alpha}{2ka\,\tau(a)} U_a^{(2)}$	$\frac{i\sin^2\alpha}{\tau(a)} \mu(a) V_a^{(1)'}$	$\frac{i\sin^2\alpha}{\tau(a)} \mu(a) V_a^{(2)'}$	$\frac{n\cot\alpha\cos\beta}{ka} J_a + i\sin\beta J'_a$
$iH_a^{(1)'}$	$-\frac{n\cot\alpha}{ka} H_a^{(1)}$	$-\frac{i\sin^2\alpha}{\tau(a)} \epsilon(a) U_a^{(1)'}$	$-\frac{i\sin^2\alpha}{\tau(a)} \epsilon(a) U_a^{(2)'}$	$\frac{n\sin 2\alpha}{2ka\,\tau(a)} V_a^{(1)}$	$\frac{n\sin 2\alpha}{2ka\,\tau(a)} V_a^{(2)}$	$\frac{n\cot\alpha\sin\beta}{ka} J_a - i\cos\beta J'_a$
0	0	$U_b^{(1)} - \frac{i\eta\sin\alpha}{\tau(b)} \epsilon(b) U_b^{(1)'}$	$U_b^{(2)} - \frac{i\eta\sin\alpha}{\tau(b)} \epsilon(b) U_b^{(2)'}$	$\eta \frac{n\cos\alpha}{kb\,\tau(b)} V_b^{(1)}$	$\eta \frac{n\cos\alpha}{kb\,\tau(b)} V_b^{(2)}$	0
0	0	$-\eta^{-1} \frac{n\cos\alpha}{kb\,\tau(b)} U_b^{(1)}$	$-\eta^{-1} \frac{n\cos\alpha}{kb\,\tau(b)} U_b^{(2)}$	$V_b^{(1)} - \frac{i\eta^{-1}\sin\alpha}{\tau(b)} \mu(b) V_b^{(1)'}$	$V_b^{(2)} - \frac{i\eta^{-1}\sin\alpha}{\tau(b)} \mu(b) V_b^{(2)'}$	0

While the matrix form of Table I is useful for numerical purposes, the coefficients must be explicitly determined if analytical considerations are in order. Only the coefficients of the scattered field will be given here; it is found that a_n^s and b_n^s are of the forms (3.44) and (3.45) where

$$M_n = \frac{\sin^2\alpha}{\tau(a)K_n}\left\{\epsilon(a)\left(\partial_{aU}\ln R_n\right)\left[H_n^{(1)'}(\xi_a)-\frac{\mu(a)}{\tau(a)}\sin^2\alpha\, H_n^{(1)}(\xi_a)\left\{\partial_{aV}\ln\left(\partial_{aU}\ln R_n\right)\right\}\right]+\right.$$

$$\left.+\frac{1}{\tau(a)}H_n^{(1)}(\xi_a)\left[1-\frac{\tau(a)}{\sin^2\alpha}\right]\left[\frac{n\cos\alpha}{\tau(a)}\right]^2\left[1-\frac{\tau(a)}{\sin^2\alpha}+\frac{2\gamma\delta\tau(a)}{\tau(b)(kb)^2R_n}\right]\right\}, \quad (3.53)$$

$$A_n = \frac{2n\cos\alpha}{\pi\tau(a)(ka)^2K_n}\left[1-\frac{\tau(a)}{\sin^2\alpha}+\frac{\gamma\delta\tau(a)}{\tau(b)(kb)^2R_n}\right], \quad (3.54)$$

with

$$K_n = H_n^{(1)'}(\xi_a)-\frac{\mu(a)}{\tau(a)}\sin^2\alpha\, H_n^{(1)}(\xi_a)(\partial_{aV}\ln R_n), \quad (3.55)$$

$$R_n = E_nF_n+\left[\frac{n\cos\alpha}{kb\,\tau(b)}\right]^2 C_nD_n, \quad (3.56)$$

$$E_n = C_n - i\eta\sin\alpha\,\frac{\epsilon(b)}{\tau(b)}\,\partial_{bU}C_n, \quad (3.57)$$

$$F_n = D_n - i\eta^{-1}\sin\alpha\,\frac{\mu(b)}{\tau(b)}\,\partial_{bV}D_n, \quad (3.58)$$

$$C_n = U_n^{(1)}(\xi_a)U_n^{(2)}(\xi_b)-U_n^{(2)}(\xi_a)U_n^{(1)}(\xi_b), \quad (3.59)$$

$$D_n = V_n^{(1)}(\xi_a)V_n^{(2)}(\xi_b)-V_n^{(2)}(\xi_a)V_n^{(1)}(\xi_b), \quad (3.60)$$

$$\partial_{aU\,(V)} = \left(\frac{\partial}{\partial\xi_a}\right)_{\xi_b=\text{constant}}, \text{ operating on the } U_{(V)} \text{ functions only,} \quad (3.61)$$

$$\left[\text{e.g., } \partial_{aV}\ln R_n = \frac{1}{R_n}\left\{E_n\frac{\partial F_n}{\partial\xi_a}+\left[\frac{n\cos\alpha}{kb\,\tau(b)}\right]^2 C_n\frac{\partial D_n}{\partial\xi_a}\right\}\right],$$

$$\partial_{\substack{bU\\(V)}} = \left(\frac{\partial}{\partial \xi_b}\right)_{\xi_a = \text{constant}} \text{, operating on the } \underset{(V)}{U} \text{ functions only,} \tag{3.62}$$

$$\tau(a) = \lim_{\rho \to a-} \tau(\rho) = \epsilon(a)\mu(a) - \cos^2\alpha \;, \tag{3.63}$$

$$\tau(b) = \lim_{\rho \to b+} \tau(\rho) = \epsilon(b)\mu(b) - \cos^2\alpha \;, \tag{3.64}$$

ξ_a and ξ_b are given by (3.8), the prime indicates the derivative with respect to ξ_a, and γ and δ are the normalization constants of (3.36) and (3.37). The quantities $\widetilde{M}_n$ and $\widetilde{A}_n$ are obtained from M_n and A_n respectively, by replacing ϵ, μ, η, U and V respectively with μ, ϵ, η^{-1}, V and U in the above equations (3.53) to (3.64).

Observe that $A_n = \widetilde{A}_n = 0$ for all α if $n = 0$, and for all n if $\alpha = \pi/2$ (normal incidence); thus, in general, the coupling between TM and TE modes disappears only for normal incidence.

In the particular case in which $\epsilon(a) = \mu(a) = 1$, relations (3.53) and (3.54) become much simpler:

$$(M_n)_{\epsilon(a)=\mu(a)=1} = \partial_{aU}\left\{\ell n\left[H_n^{(1)'}(\xi_a)R_n - H_n^{(1)}(\xi_a)\partial_{aV}R_n\right]\right\} \;, \tag{3.65}$$

$$(A_n)_{\epsilon(a)=\mu(a)=1} = \frac{2n\gamma\delta\cos\alpha}{\pi\tau(b)(kakb)^2}\left[H_n^{(1)'}(\xi_a)R_n - H_n^{(1)}(\xi_a)\partial_{aV}R_n\right]^{-1} . \tag{3.66}$$

3.4 <u>The Cylindrical Shell</u>

By imposing the continuity of the tangential components of the total electric and magnetic fields across the surfaces $\rho = a$ and $\rho = b$, one finds that the various coefficients are given by the eight linear algebraic equations of Table II, whose symbols have the same interpretation of those of Table I in the previous section. In particular, the coefficients a_n^s and b_n^s of the scattered field are given by (3.44) and (3.45), where:

$$M_n = G_n^{-1}\left\{\frac{\epsilon(a)}{\tau(a)}\sin^2\alpha\, H_n^{(1)'}(\xi_a)\,\partial_{aU} L_n + H_n^{(1)}(\xi_a)\left[\Delta_a L_n - \right.\right.$$

$$\left.\left. - \frac{\gamma\delta}{2\,\tau(a)\tau(b)}\left(1-\frac{\tau(a)}{\sin^2\alpha}\right)\left(1-\frac{\tau(b)}{\sin^2\alpha}\right)\left\{\frac{n\sin\alpha\sin 2\alpha}{kakb}\,J_n(\xi_b)\right\}^2\right]\right\}, \quad (3.67)$$

$$A_n = \frac{2n\cos\alpha}{\pi\tau(a)(ka)^2 G_n}\left[\left(1-\frac{\tau(a)}{\sin^2\alpha}\right)L_n - \gamma\delta\frac{\tau(a)}{\tau(b)}\left(1-\frac{\tau(b)}{\sin^2\alpha}\right)\left(\frac{\sin\alpha}{kb}J_n(\xi_b)\right)^2\right], \quad (3.68)$$

with

$$G_n = H_n^{(1)'}(\xi_a)L_n - \frac{\mu(a)}{\tau(a)}\sin^2\alpha\, H_n^{(1)}(\xi_a)\,\partial_{aV} L_n, \quad (3.69)$$

$$L_n = \left\{\left[J_n(\xi_b)\right]^2\Delta_b + \frac{\sin^2\alpha}{\tau(b)}J_n(\xi_b)J_n'(\xi_b)\left[\epsilon(b)\partial_{bU} + \mu(b)\partial_{bV}\right] - \right.$$

$$\left. - \left[J_n'(\xi_b)\right]^2\right\}(C_n D_n), \quad (3.70)$$

$$\Delta_\rho = \left[\frac{n\sin 2\alpha}{2k\rho\tau(\rho)}\right]^2\left[1-\frac{\tau(\rho)}{\sin^2\alpha}\right]^2 - \frac{\epsilon(\rho)\mu(\rho)}{\tau(\rho)^2}\sin^4\alpha\,\partial_{\rho U}\partial_{\rho V}, \quad (\rho = a \text{ or } b); \quad (3.71)$$

the other quantities are given by (3.8), (3.62), (3.63) and (3.64), the prime indicates the derivative with respect to the argument of the Bessel function, and γ and δ are the normalization constants of (3.36) and (3.37). The quantities $\widetilde{M}_n$ and $\widetilde{A}_n$ are obtained from M_n and A_n respectively, by interchanging ϵ with μ and U with V (observe that these replacements leave Δ_a, Δ_b and L_n invariant).

Both A_n and $\widetilde{A}_n$ are zero for all α if $n = 0$, and for all n if $\alpha = \pi/2$. In the particular case $\epsilon(a) = \mu(a) = 1$, M_n of (3.67) reduces to the simpler form

$$(M_n)_{\epsilon(a)=\mu(a)=1} = \partial_{aU}\left\{\ln\left[H_n^{(1)'}(\xi_a)L_n - H_n^{(1)}(\xi_a)\,\partial_{aV} L_n\right]\right\}. \quad (3.72)$$

TABLE II: COEFFICIENTS FOR CYLINDRICAL SHELL.

a_n^s	b_n^s	a_{1n}	c_{1n}	b_{1n}	d_{1n}	a_{2n}	b_{2n}	right-hand side
$-H_a$	0	$U_a^{(1)}$	$U_a^{(2)}$	0	0	0	0	$J_a \cos\beta$
0	$-H_a$	0	0	$V_a^{(1)}$	$V_a^{(2)}$	0	0	$J_a \sin\beta$
$-\frac{n \cot\alpha}{ka} H_a$	$-iH'_a$	$\frac{n \sin 2\alpha}{2ka\,\tau(a)} U_a^{(1)}$	$\frac{n \sin 2\alpha}{2ka\,\tau(a)} U_a^{(2)}$	$\frac{i \sin^2\alpha\, \mu(a)}{\tau(a)} V_a^{(1)'}$	$\frac{i \sin^2\alpha\, \mu(a)}{\tau(a)} V_a^{(2)'}$	0	0	$i J'_a \sin\beta + \frac{n \cot\alpha}{ka} J_a \cos\beta$
$-iH'_a$	$\frac{n \cot\alpha}{ka} H_a$	$\frac{i \sin^2\alpha\, \epsilon(a)}{\tau(a)} U_a^{(1)'}$	$\frac{i \sin^2\alpha\, \epsilon(a)}{\tau(a)} U_a^{(2)'}$	$-\frac{n \sin 2\alpha}{2ka\,\tau(a)} V_a^{(1)}$	$-\frac{n \sin 2\alpha}{2ka\,\tau(a)} V_a^{(2)}$	0	0	$i J'_a \cos\beta - \frac{n \cot\alpha}{ka} J_a \sin\beta$
0	0	$U_b^{(1)}$	$U_b^{(2)}$	0	0	$-J_b$	0	0
0	0	0	0	$V_b^{(1)}$	$V_b^{(2)}$	0	$-J_b$	0
0	0	$\frac{n \sin 2\alpha}{2kb\,\tau(b)} U_b^{(1)}$	$\frac{n \sin 2\alpha}{2kb\,\tau(b)} U_b^{(2)}$	$\frac{i \sin^2\alpha\, \mu(b)}{\tau(b)} V_b^{(1)'}$	$\frac{i \sin^2\alpha\, \mu(b)}{\tau(b)} V_b^{(2)'}$	$-\frac{n \cot\alpha}{kb} J_b$	$-iJ'_b$	0
0	0	$\frac{i \sin^2\alpha\, \epsilon(b)}{\tau(b)} U_b^{(1)'}$	$\frac{i \sin^2\alpha\, \epsilon(b)}{\tau(b)} U_b^{(2)'}$	$-\frac{n \sin 2\alpha}{2kb\,\tau(b)} V_b^{(1)}$	$-\frac{n \sin 2\alpha}{2kb\,\tau(b)} V_b^{(2)}$	$-iJ'_b$	$\frac{n \cot\alpha}{kb} J_b$	0

If the conditions

$$\epsilon(a)\mu(a) = \epsilon(b)\mu(b) = 1 \tag{3.73}$$

are satisfied, then $A_n = \tilde{A}_n = 0$ for all n and all α. It is thus seen that the coupling between TM and TE modes disappears not only for normal incidence, but also for oblique incidence under conditions (3.73).

3.5 The Coreless Cylinder

The boundary conditions to be imposed in this case are the continuity of the tangential components of the electric and magnetic fields across the surface $\rho = a$, and a condition at $\rho = b = 0$ which leads to the proper choice between the solutions of (3.9) and (3.10). If $\epsilon(\rho)$ and $\mu(\rho)$ are finite at $\rho = 0$, then we shall require that $U_n(0)$ and $V_n(0)$ be finite, whereas if $\epsilon(\rho)$ and/or $\mu(\rho)$ are infinite at the origin, we ought to impose a boundary condition of the Meixner type to select the appropriate U_n and V_n (Meixner, 1949): the total energy of the electric and magnetic fields inside any cylinder of axis z, unit length and finite radius must be finite.

Let us indicate with $U_n^{(1)}$ and $V_n^{(1)}$ the radial eigenfunctions which have the appropriate behavior at $\rho = 0$; then

$$c_{1n} = d_{1n} = 0 \, , \tag{3.74}$$

and the coefficients a_n^s, b_n^s, a_{1n} and b_{1n} are given by the four equations of Table III, where the symbols used have the same interpretation as in Tables I and II. In particular, a_n^s and b_n^s are given by (3.44) and (3.45) where

$$M_n = g_n^{-1} \left\{ \frac{\epsilon(a)}{\tau(a)} \sin^2\alpha \, H_n^{(1)'}(\xi_a) U_n^{(1)'}(\xi_a) V_n^{(1)}(\xi_a) + \right.$$
$$+ H_n^{(1)}(\xi_a) \left[\left(\frac{n \sin 2\alpha}{2ka\,\tau(a)} \right)^2 \left(1 - \frac{\tau(a)}{\sin^2\alpha} \right)^2 U_n^{(1)}(\xi_a) V_n^{(1)}(\xi_a) - \right.$$
$$\left.\left. - \frac{\epsilon(a)\mu(a)}{\tau(a)^2} \sin^4\alpha \, U_n^{(1)'}(\xi_a) V_n^{(1)'}(\xi_a) \right] \right\} , \tag{3.75}$$

TABLE III: COEFFICIENTS FOR CORELESS CYLINDER

a_n^s	b_n^s	a_{1n}	b_{1n}	right-hand side
$-H_a$	0	$U_a^{(1)}$	0	$J_a \cos\beta$
0	$-H_a$	0	$V_a^{(1)}$	$J_a \sin\beta$
$-\frac{n\cot\alpha}{ka} H_a$	$-iH'_a$	$\frac{n\sin 2\alpha}{2ka\,\tau(a)} U_a^{(1)}$	$\frac{i\sin^2\alpha\,\mu(a)}{\tau(a)} V_a^{(1)'}$	$iJ'_a \sin\beta + \frac{n\cot\alpha}{ka} J_a \cos\beta$
$-iH'_a$	$\frac{n\cot\alpha}{ka} H_a$	$\frac{i\sin^2\alpha\,\epsilon(a)}{\tau(a)} U_a^{(1)'}$	$-\frac{n\sin 2\alpha}{2ka\,\tau(a)} V_a^{(1)}$	$iJ'_a \cos\beta - \frac{n\cot\alpha}{ka} J_a \sin\beta$

$$A_n = \frac{2n\cos\alpha}{\pi\tau(a)(ka)^2 g_n}\left[1 - \frac{\tau(a)}{\sin^2\alpha}\right] U_n^{(1)}(\xi_a)V_n^{(1)}(\xi_a) , \tag{3.76}$$

with

$$g_n = U_n^{(1)}(\xi_a)\left[H_n^{(1)'}(\xi_a)V_n^{(1)}(\xi_a) - \frac{\mu(a)}{\tau(a)} \sin^2\alpha H_n^{(1)}(\xi_a)V_n^{(1)'}(\xi_a)\right] ; \tag{3.77}$$

$\tau(a)$ is given by (3.63), $\xi_a = ka\sin\alpha$ and the prime indicates the derivative with respect to ξ_a. The quantities $\widetilde{M}_n$ and $\widetilde{A}_n$ are obtained from M_n and A_n respectively, by interchanging ϵ with μ and U with V.

As in the previous two sections, A_n and $\widetilde{A}_n$ are both zero for all α if $n=0$, and for all n if $\alpha = \pi/2$. Also, $A_n = \widetilde{A}_n = 0$ for all α and all n if $\epsilon(a)\mu(a) = 1$. If both $\epsilon(a)$ and $\mu(a)$ are equal to unity, then

$$(M_n)_{\epsilon(a)=\mu(a)=1} = \frac{\partial}{\partial\xi_a} \ln U_n^{(1)}(\xi_a) . \tag{3.78}$$

3.6 A General Result on Mode Coupling

As seen in the previous sections, the nth TM mode and the nth TE mode of the scattered field are generally coupled together; this coupling also presents itself for the fields inside the inhomogeneous region. If one could know a priori that the coupling does not occur for the particular scatterer under consideration (for example, if one could know a priori that A_n and $\widetilde{A}_n$ in (3.44) and (3.45) are zero for all n and all α), then the laborious calculations arising from the imposition of the boundary conditions would be greatly simplified. In the following, we derive sufficient conditions for the uncoupling of the TM and TE modes; the procedure followed in the proof is also useful in the practical determination of the constant coefficients.

Let us consider a radially inhomogeneous cylindrical region of outer radius $\rho = a$ on which a plane wave is obliquely incident, as shown in Fig. 3-1. Let us divide this region into coaxial cylindrical shells I, II, III, ... of radii $\rho_1, \rho_2, \ldots \rho = a$ (Fig. 3-2), within each of which $\epsilon(\rho)$ and $\mu(\rho)$ are continuous and

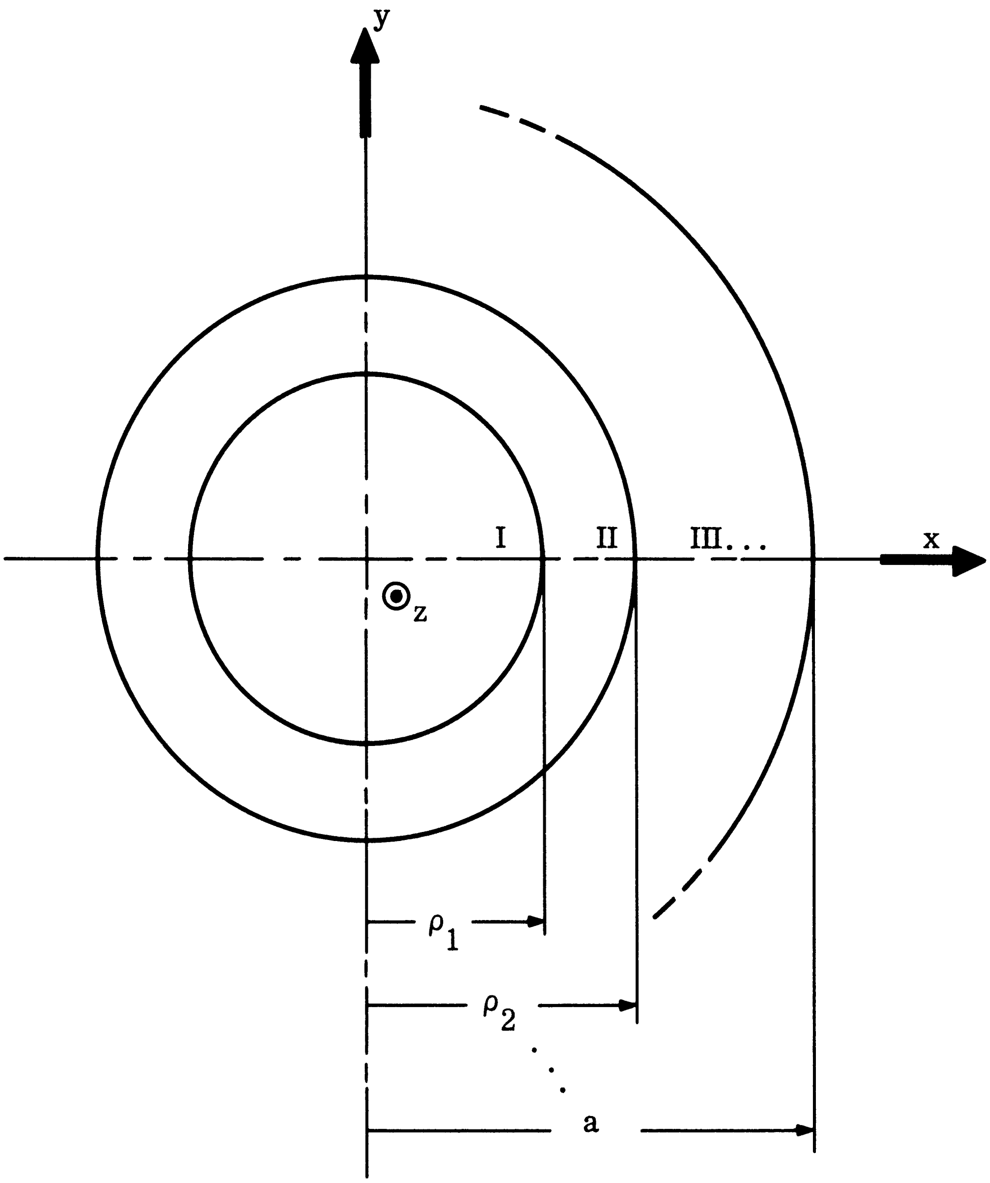

FIG. 3-2: MULTI-LAYERED CYLINDRICAL STRUCTURE.

differentiable functions of ρ; also, let us suppose that the solutions U_n and V_n of (3.9) and (3.10) are known for each shell.

Thus, in region I:

$$U_{n,I}(\xi_\rho) = a_{1n} U^{(1)}_{n,I}(\xi_\rho), \quad V_{n,I}(\xi_\rho) = b_{1n} V^{(1)}_{n,I}(\xi_\rho), \quad (0 \leqslant \rho < \rho_1) , \tag{3.79}$$

where $U^{(1)}_{n,I}$ and $V^{(1)}_{n,I}$ are those solutions of (3.9) and (3.10) for region I which satisfy the boundary conditions on the axis $\rho = 0$, and a_{1n} and b_{1n} are constants; in region II:

$$\left.\begin{aligned} U_{n,II}(\xi_\rho) &= a_{1n}\left[a_{n,II} U^{(1)}_{n,II}(\xi_\rho) + c_{n,II} U^{(2)}_{n,II}(\xi_\rho)\right] , \\ V_{n,II}(\xi_\rho) &= b_{1n}\left[b_{n,II} V^{(1)}_{n,II}(\xi_\rho) + d_{n,II} V^{(2)}_{n,II}(\xi_\rho)\right] , \end{aligned}\right\} (\rho_1 < \rho < \rho_2) , \tag{3.80}$$

where $U^{(j)}_{n,II}$ and $V^{(j)}_{n,II}$, (j = 1 or 2), are two linearly independent solutions of (3.9) and (3.10) for region II, and $a_{n,II}$, $c_{n,II}$, $b_{n,II}$ and $d_{n,II}$ are constants; expressions similar to (3.80) can be written for U_n and V_n in regions III, IV, etc. The boundary conditions at $\rho = \rho_1$ are

$$E_{z,I} = E_{z,II} , \quad H_{z,I} = H_{z,II} , \quad E_{\phi,I} = E_{\phi,II} , \quad H_{\phi,I} = H_{\phi,II} , \quad \text{for } \rho = \rho_1 . \tag{3.81}$$

The field components are given by expressions of the types (3.28) to (3.33), and therefore conditions (3.81) yield:

$$a_{n,II} U^{(1)}_{n,II}(\xi_{\rho_1}) + c_{n,II} U^{(2)}_{n,II}(\xi_{\rho_1}) = U^{(1)}_{n,I}(\xi_{\rho_1}) , \tag{3.82}$$

$$b_{n,II} V^{(1)}_{n,II}(\xi_{\rho_1}) + d_{n,II} V^{(2)}_{n,II}(\xi_{\rho_1}) = V^{(1)}_{n,I}(\xi_{\rho_1}) , \tag{3.83}$$

$$\frac{\mu(\rho_1-)}{\tau(\rho_1-)} V'_{n,I}(\xi_{\rho_1}) - \frac{\mu(\rho_1+)}{\tau(\rho_1+)} V'_{n,II}(\xi_{\rho_1}) = \frac{in\cot\alpha}{k\rho_1}\left[\frac{1}{\tau(\rho_1-)} - \frac{1}{\tau(\rho_1+)}\right] U_{n,I}(\xi_{\rho_1}), \tag{3.84}$$

$$\frac{\epsilon(\rho_1-)}{\tau(\rho_1-)} U'_{n,I}(\xi_{\rho_1}) - \frac{\epsilon(\rho_1+)}{\tau(\rho_1+)} U'_{n,II}(\xi_{\rho_1}) = -\frac{in\cot\alpha}{k\rho_1}\left[\frac{1}{\tau(\rho_1-)} - \frac{1}{\tau(\rho_1+)}\right] V_{n,I}(\xi_{\rho_1}), \tag{3.85}$$

where the meaning of the various symbols is obvious.

The nth TM and TE modes are uncoupled if the right-hand sides of both (3.84) and (3.85) are zero; this occurs in three cases: (i) for all α if $n=0$, (ii) for all n if $\alpha=\pi/2$, (iii) for all α and all n if $\tau(\rho_1-) = \tau(\rho_1+)$ or:

$$\epsilon(\rho_1-)\mu(\rho_1-) = \epsilon(\rho_1+)\mu(\rho_1+) . \tag{3.86}$$

In all three cases, (3.84) and (3.85) become:

$$b_{n,II} V^{(1)'}_{n,II}(\xi_{\rho_1}) + d_{n,II} V^{(2)'}_{n,II}(\xi_{\rho_1}) = \frac{\mu(\rho_1-)\tau(\rho_1+)}{\mu(\rho_1+)\tau(\rho_1-)} V^{(1)'}_{n,I}(\xi_{\rho_1}), \tag{3.87}$$

$$a_{n,II} U^{(1)'}_{n,II}(\xi_{\rho_1}) + c_{n,II} U^{(2)'}_{n,II}(\xi_{\rho_1}) = \frac{\epsilon(\rho_1-)\tau(\rho_1+)}{\epsilon(\rho_1+)\tau(\rho_1-)} U^{(1)'}_{n,I}(\xi_{\rho_1}). \tag{3.88}$$

The constants $a_{n,II}$ and $c_{n,II}$ are determined by solving (3.82) and (3.88); in fact, the determinant of the coefficients is the Wronskian of $U^{(1)}_{n,II}$ and $U^{(2)}_{n,II}$ which is non-zero by hypothesis, and the system is certainly non-homogeneous (if both $U^{(1)}_{n,I}$ and $U^{(1)'}_{n,I}$ were zero at ρ_1, then $U^{(1)}_{n,I}$ would be identically zero). Similarly, $b_{n,II}$ and $d_{n,II}$ are found by solving the system (3.83), (3.87).

It is clear that the above reasoning can be repeated in imposing the boundary conditions at $\rho=\rho_2$; the expressions inside the square brackets of (3.80) play the role of $U^{(1)}_{n,I}$ and $V^{(1)}_{n,I}$ in the previous discussion, ρ_1 is everywhere replaced by ρ_2, and so on. By repeating this process, we finally find, at the surface $\rho = a$, boundary conditions of the type discussed in section 3.5 for the coreless cylinder. Therefore, the nth TM and TE modes of the fields inside the inhomogeneous regions and of the scattered field are certainly uncoupled: (i) for all α if $n = 0$, (ii) for all n if $\alpha = \pi/2$, and (iii) for all α and all n if the product $\epsilon(\rho)\mu(\rho)$ has no step discontinuities. Case (iii) is obviously the most interesting one, and can be restated as follows:

Theorem: A sufficient condition for the TM and TE modes to be uncoupled for all n and all α is that the square $\epsilon(\rho)\mu(\rho)$ of the index of refraction have no step discontinuities in the interval $0 < \rho < \infty$.

3.7 An Example

The differential equations (3.9) and (3.10) for U_n and V_n can be solved exactly for special choices of $\epsilon(\rho)$ and $\mu(\rho)$. In this section, the case in which ϵ and μ are, respectively, inversely and directly proportional to ρ is considered. Let us set

$$\epsilon(\rho) = \epsilon(a)\frac{a}{\rho} , \qquad \mu(\rho) = \mu(a)\frac{\rho}{a} , \tag{3.89}$$

where a is the outer radius of the inhomogeneous region. It then follows from (3.9) to (3.12) that U_n and V_n are solutions of

$$\frac{d^2U_n}{d(h\rho)^2} + \left[1 - \frac{n^2}{(h\rho)^2}\right] U_n = 0 , \tag{3.90}$$

$$\frac{d^2V_n}{d(h\rho)^2} + \frac{2}{h\rho}\frac{dV_n}{d(h\rho)} + \left[1 - \frac{n^2}{(h\rho)^2}\right] V_n = 0 , \tag{3.91}$$

with

$$h = k\sqrt{\tau} = k\sqrt{\epsilon(a)\mu(a) - \cos^2\alpha} , \tag{3.92}$$

and are therefore given by

$$U_n = a_{1n}\psi_\sigma(h\rho) + c_{1n}\zeta_\sigma^{(1)}(h\rho) \ , \tag{3.93}$$

$$V_n = \frac{1}{h\rho}\left[b_{1n}\psi_\sigma(h\rho) + d_{1n}\zeta_\sigma^{(1)}(h\rho)\right] \ , \tag{3.94}$$

where

$$\sigma = -\frac{1}{2} + \sqrt{n^2 + \frac{1}{4}} \ , \tag{3.95}$$

$$\psi_\sigma(x) = \sqrt{\frac{\pi x}{2}}\, J_{\sigma+1/2}(x) \ , \qquad \zeta_\sigma^{(1)}(x) = \sqrt{\frac{\pi x}{2}}\, H^{(1)}_{\sigma+1/2}(x) \ . \tag{3.96}$$

In the following, a detailed analysis is performed for a scatterer made of an inhomogeneous layer of outer radius a, thickness d and parameters given by (3.89), which covers an imperfectly conducting cylindrical core of radius $b = a - d$. The calculations are carried out for normal incidence ($\alpha = \pi/2$), in which case the parameter h is given by

$$(h)_{\alpha=\pi/2} = k_1 = Nk \ , \tag{3.97}$$

where $N = \sqrt{\epsilon\mu}$ is the (constant) refractive index of the coating layer. The analysis of this section can be applied, with obvious modifications, to the case in which ϵ is directly and μ is inversely proportional to ρ.

Consider the incident plane electromagnetic wave

$$\begin{aligned} \underline{E}^i &= (\sin\beta\,\hat{y} + \cos\beta\,\hat{z})\, e^{ikx} \ , \\ \underline{H}^i &= Y(-\cos\beta\,\hat{y} + \sin\beta\,\hat{z})\, e^{ikx} \ , \end{aligned} \tag{3.98}$$

which propagates in the direction of the positive x axis of Fig. 3-1. The axial components of the scattered fields and of the total fields inside the coating layer are:

$$E_z^s = \sum_{n=-\infty}^{\infty} i^n a_n^s H_n^{(1)}(k\rho)\, e^{in\phi} \ , \tag{3.99}$$

$$H_z^s = Y \sum_{n=-\infty}^{\infty} i^n b_n^s H_n^{(1)}(k\rho)\, e^{in\phi} \,, \tag{3.100}$$

$$E_{1z} = \sum_{n=-\infty}^{\infty} i^n \left[a_{1n} \psi_\sigma(k_1\rho) + c_{1n} \zeta_\sigma^{(1)}(k_1\rho) \right] e^{in\phi} \,, \tag{3.101}$$

$$H_{1z} = \frac{Y}{k_1\rho} \sum_{n=-\infty}^{\infty} i^n \left[b_{1n} \psi_\sigma(k_1\rho) + d_{1n} \zeta_\sigma^{(1)}(k_1\rho) \right] e^{in\phi} \,. \tag{3.102}$$

The coefficients are given by:

$$a_n^s = -\cos\beta \frac{J_n'(ka) - M_n J_n(ka)}{H_n^{(1)'}(ka) - M_n H_n^{(1)}(ka)} \,, \tag{3.103}$$

$$a_{1n} = \frac{2i}{\pi ka} \cos\beta \frac{\zeta_\sigma^{(1)}(k_1 b) - i\eta \frac{N}{\mu(b)} \zeta_\sigma^{(1)'}(k_1 b)}{\left[H_n^{(1)'}(ka) - M_n H_n^{(1)}(ka) \right] \left[\Theta_n - i\eta \frac{N}{\mu(b)} \frac{\partial \Theta_n}{\partial(k_1 b)} \right]} \,, \tag{3.104}$$

$$c_{1n} = -\frac{2i}{\pi ka} \cos\beta \frac{\psi_\sigma(k_1 b) - i\eta \frac{N}{\mu(b)} \psi_\sigma'(k_1 b)}{\left[H_n^{(1)'}(ka) - M_n H_n^{(1)}(ka) \right] \left[\Theta_n - i\eta \frac{N}{\mu(b)} \frac{\partial \Theta_n}{\partial(k_1 b)} \right]} \,, \tag{3.105}$$

$$b_n^s = -\sin\beta \frac{J_n'(ka) - \widetilde{M}_n J_n(ka)}{H_n^{(1)'}(ka) - \widetilde{M}_n H_n^{(1)}(ka)} \,, \tag{3.106}$$

$$b_{1n} = \frac{2iN}{\pi} \sin\beta \frac{\zeta_\sigma^{(1)'}(k_1 b) - \frac{1 - i\eta kb\epsilon(b)}{k_1 b} \zeta_\sigma^{(1)}(k_1 b)}{\left[H_n^{(1)'}(ka) - \widetilde{M}_n H_n^{(1)}(ka) \right] \left[\frac{\partial \Theta_n}{\partial(k_1 b)} - \frac{1 - i\eta kb\epsilon(b)}{k_1 b} \Theta_n \right]} \,, \tag{3.107}$$

$$d_{1n} = -\frac{2iN}{\pi}\sin\beta \frac{\psi_\sigma'(k_1b) - \dfrac{1-i\eta kb\epsilon(b)}{k_1b}\psi_\sigma(k_1b)}{\left[H_n^{(1)'}(ka) - \widetilde{M}_n H_n^{(1)}(ka)\right]\left[\dfrac{\partial\Theta_n}{\partial(k_1b)} - \dfrac{1-i\eta kb\epsilon(b)}{k_1b}\Theta_n\right]} , \tag{3.108}$$

where

$$M_n = \frac{N}{\mu(a)}\frac{\partial}{\partial(k_1a)}\left\{\ln\left[\Theta_n - i\eta\frac{N}{\mu(b)}\frac{\partial\Theta_n}{\partial(k_1b)}\right]\right\} , \tag{3.109}$$

$$\widetilde{M}_n = \frac{N}{\epsilon(a)}\frac{\partial}{\partial(k_1a)}\left\{\ln\left[\frac{\partial\Theta_n}{\partial(k_1b)} - \frac{1-i\eta kb\epsilon(b)}{k_1b}\Theta_n\right]\right\} - \frac{1}{ka\,\epsilon(a)} , \tag{3.110}$$

with

$$\Theta_n = \psi_\sigma(k_1a)\zeta_\sigma^{(1)}(k_1b) - \psi_\sigma(k_1b)\zeta_\sigma^{(1)}(k_1a) , \tag{3.111}$$

and η is the relative surface impedance at $\rho = b$.

In the preceding formulas, the prime indicates the derivative with respect to the argument of the primed function. Observe that a_n^s and b_n^s are unchanged whereas a_{1n}, c_{1n}, b_{1n} and d_{1n} must be multiplied by $(-1)^n$ when n is replaced by $(-n)$.

These exact results can be rewritten in a form which is especially useful for low-frequency approximations; thus we have that:

$$\Theta_n = \frac{\pi}{\cos\pi\sigma}\sum_{\ell=0}^{\infty} F_\ell^1\left(\frac{ik_1a}{2}\right)^{2\ell+1} , \tag{3.112}$$

$$\frac{\partial\Theta_n}{\partial(k_1a)} = \frac{i\pi}{2\cos\pi\sigma}\sum_{\ell=0}^{\infty} F_\ell^2\left(\frac{ik_1a}{2}\right)^{2\ell} , \tag{3.113}$$

$$\frac{\partial\Theta_n}{\partial(k_1b)} = \frac{i\pi}{2\cos\pi\sigma}\sum_{\ell=0}^{\infty} F_\ell^3\left(\frac{ik_1a}{2}\right)^{2\ell} , \tag{3.114}$$

$$\frac{\partial^2 \Theta_n}{\partial(k_1 a)\partial(k_1 b)} = -\frac{\pi}{4\cos\pi\sigma} \sum_{\ell=0}^{\infty} F_\ell^4 \left(\frac{ik_1 a}{2}\right)^{2\ell-1} , \tag{3.115}$$

where

$$F_\ell^1 = \sum_{m=0}^{\ell} \left[\Gamma_1 (b/a)^{\sigma+1} - \Gamma_2 (b/a)^{-\sigma}\right] , \tag{3.116}$$

$$F_\ell^2 = \sum_{m=0}^{\ell} \left[(2\ell - 2m - \sigma)\,\Gamma_1 (b/a)^{\sigma+1} - (2\ell - 2m + \sigma + 1)\,\Gamma_2 (b/a)^{-\sigma}\right] , \tag{3.117}$$

$$F_\ell^3 = \sum_{m=0}^{\ell} \left[(2m + \sigma + 1)\,\Gamma_1 (b/a)^{\sigma} - (2m - \sigma)\,\Gamma_2 (b/a)^{-\sigma-1}\right] , \tag{3.118}$$

$$F_\ell^4 = \sum_{m=0}^{\ell} \left[(2m+\sigma+1)(2\ell - 2m - \sigma)\,\Gamma_1 (b/a)^{\sigma} - (2m - \sigma)(2\ell - 2m + \sigma + 1)\,\Gamma_2 (b/a)^{-\sigma-1}\right] , \tag{3.119}$$

with

$$\Gamma_1 = (b/a)^{2m} \left[m!\,(\ell - m)!\,\Gamma(\sigma + m + \tfrac{3}{2})\,\Gamma(-\sigma + \ell - m + \tfrac{1}{2})\right]^{-1} , \tag{3.120}$$

$$\Gamma_2 = (b/a)^{2m} \left[m!\,(\ell - m)!\,\Gamma(-\sigma + m + \tfrac{1}{2})\,\Gamma(\sigma + \ell - m + \tfrac{3}{2})\right]^{-1} , \tag{3.121}$$

and Γ is the Gamma-function. In particular, it follows that

$$M_n = \frac{1}{\mu(a)ka} \cdot \frac{\sum_{\ell=0}^{\infty} \left[F_\ell^2 - \frac{i\eta}{\mu(b)ka} F_\ell^4\right] \left(\frac{ik_1 a}{2}\right)^{2\ell}}{\sum_{\ell=0}^{\infty} \left[F_\ell^1 - \frac{i\eta}{\mu(b)ka} F_\ell^3\right] \left(\frac{ik_1 a}{2}\right)^{2\ell}} , \tag{3.122}$$

$$\tilde{M}_n = \frac{1}{\epsilon(a)ka}\left\{-1+\frac{\sum_{\ell=0}^{\infty}\left[F_\ell^4+\left(i\eta\epsilon(b)ka-\frac{a}{b}\right)F_\ell^2\right]\left(\frac{ik_1a}{2}\right)^{2\ell}}{\sum_{\ell=0}^{\infty}\left[F_\ell^3+\left(i\eta\epsilon(b)ka-\frac{a}{b}\right)F_\ell^1\right]\left(\frac{ik_1a}{2}\right)^{2\ell}}\right\}. \tag{3.123}$$

With the aid of the preceding formulas, the low-frequency field components can be evaluated to any desired order in powers of ka. Here we limit our considerations to the scattered fields in the Rayleigh region, i.e. to the dominant terms in the coefficients a_n^s and b_n^s of (3.103) and (3.106) when $ka \ll 1$; since $a_n^s = a_{-n}^s$ and $b_n^s = b_{-n}^s$, we may take $n \geq 0$. For the E-modes ($\beta = 0$) we have that

$$a_o^s \sim i\pi \frac{ka}{2} \frac{\mu(b)/\mu(a)}{i\eta+\mu(b)kd}\left[1+O(ka\,\ell n\,ka)\right], \tag{3.124}$$

$$a_n^s \sim i\pi \frac{n}{(n!)^2}(ka/2)^{2n}\frac{n\mu(a)A_n^s+B_n^s}{n\mu(a)A_n^s-B_n^s}\left\{1+O\left[f_n(ka)\right]\right\}, \quad (n\geq 1), \tag{3.125}$$

where $d = a-b$,

$$A_n^s = \left[\mu(b)ka - i\eta\frac{a}{b}(\sigma+1)\right](b/a)^{\sigma+1} - \left[\mu(b)ka + i\eta\frac{a}{b}\sigma\right](b/a)^{-\sigma}, \tag{3.126}$$

$$B_n^s = \sigma\left[\mu(b)ka - i\eta\frac{a}{b}(\sigma+1)\right](b/a)^{\sigma+1} + (\sigma+1)\left[\mu(b)ka + i\eta\frac{a}{b}\sigma\right](b/a)^{-\sigma}, \tag{3.127}$$

$$\begin{aligned} f_n(ka) &= (ka)^2\ell n\,ka, &\quad \text{for } n=1, \\ &= (ka)^2, &\quad \text{for } n>1. \end{aligned} \tag{3.128}$$

If η, $\mu(a)$, $\mu(b)$ and kd are such that the coefficient of ka in the dominant term of (3.124) is not large compared to unity, then a_o^s is O(ka), hence smaller than in the case of a perfectly conducting uncoated cylinder ($\eta = kd = 0$) for which a_o^s is $O\left[(\ell n\,ka)^{-1}\right]$.

For the H-modes ($\beta = \pi/2$) and $ka \ll 1$, we have that

$$b_o^s \sim i\pi \frac{ka}{2} \frac{i\eta \frac{b}{a}\frac{\epsilon(b)}{\epsilon(a)} - \frac{ka}{2}}{\frac{a}{b} - i\eta\epsilon(b)kd} \left[1 + O(ka \ln ka)\right] , \tag{3.129}$$

$$b_n^s \sim i\pi \frac{n}{(n!)^2} (ka/2)^{2n} \frac{n\epsilon(a)\tilde{A}_n^s + \tilde{B}_n^s}{n\epsilon(a)\tilde{A}_n^s - \tilde{B}_n^s} \left\{1 + O\left[f_n(ka)\right]\right\}, \quad (n \geqslant 1) , \tag{3.130}$$

where $f_n(ka)$ is given by (3.128), and

$$\tilde{A}_n^s = \left[\sigma + i\eta\epsilon(b)kb\right] (b/a)^{\sigma} + \left[\sigma + 1 - i\eta\epsilon(b)kb\right] (b/a)^{-\sigma-1} , \tag{3.131}$$

$$\tilde{B}_n^s = -(\sigma+1) \left[\sigma + i\eta\epsilon(b)kb\right] (b/a)^{\sigma} + \sigma\left[\sigma + 1 - i\eta\epsilon(b)kb\right] (b/a)^{-\sigma-1} . \tag{3.132}$$

If η, $\epsilon(a)$, a/b and kd are such that the coefficient of ka in the dominant term of (3.129) is of the order of unity, then b_o^s is $O(ka)$, hence larger than in the case of a perfectly conducting uncoated cylinder ($\eta = kd = 0$) for which b_o^s is $O\left[(ka)^2\right]$.

As seen from the formulas (3.99) to (3.111), the infinite series solutions are so complicated that no information on their behavior can be derived by direct inspection; they are, therefore, simply a tool for numerical computations, whose complexity and cost increase rapidly with ka. Approximate expressions for the field components can be easily derived for long wavelengths, as we have just done, but high-frequency asymptotic expansions are much more difficult to obtain.

We limit our considerations to the backscattered field, under the hypothesis that the inhomogeneous coating layer is of moderate thickness, and that its material has a complex refractive index whose absolute value has a lower bound which is only moderately large compared to unity and whose argument is bounded away from both zero and $\pi/2$, Since $\epsilon(b) = \epsilon(a)(1 + d/b)$ and $\mu(b) = \mu(a)(1 - d/b)$, one would expect the leading terms in both the geometric optics and the creeping wave asymptotic expansions to be the same which would occur in the case of a homogeneous layer of relative permittivity $\epsilon(a)$ and relative permeability $\mu(a)$; this deduction has been confirmed by a rigorous analysis.

Starting from the exact infinite series representing the backscattered field for E-polarization ($\beta = 0$), and using the procedure and some of the results of Chapters One and Two, it can be proven that the dominant terms in the geometric optics and creeping wave contributions to the high-frequency backscattered field are the same which occur in the case of a homogeneous coating layer. Namely, the high-frequency backscattered field is given by the sum of the field of (1.52) and of the first term of the field of (1.38), where $\mu = \mu(a) \sim \mu(b)$. The precise conditions under which this result is valid are given in Chapter One.

The corresponding result for the other polarization ($\beta = \pi/2$) is trivally obtained by replacing ϵ with μ, μ with ϵ and η with η^{-1}. However, no such simple correspondence exists between the higher-order terms of the expansions for the two polarizations; furthermore, these higher-order terms cannot be inferred from the results valid for a homogeneous layer and, if needed, must be separately derived.

Tyras (1967) has recently considered the high-frequency radiation from a thin longitudinal slot in a metal cylinder coated with a layer of material having $\mu(\rho) = 1$ and $\epsilon(\rho) = (\rho/a)^{\alpha}$, where α is any real number. For this case, the differential equations for the radial eigenfunctions can be solved exactly in terms of Bessel functions; for positive α, the coating layer is taken to represent an inhomogeneous cold plasma.

Chapter Four

SCATTERING FROM RADIALLY INHOMOGENEOUS SPHERES. EXACT SOLUTIONS AND LOW-FREQUENCY APPROXIMATIONS

4.1 Introduction

In this chapter, the scattering of a plane electromagnetic wave by a sphere made of an inhomogeneous material is considered; exact solutions and low-frequency approximations are derived.

The general case in which the electric permittivity and the magnetic permeability are functions not only of the radial distance from the center of the sphere but also of the angular coordinates has been considered by Gutman (1965), who employed the Hansen-Stratton vector wave-function method in a modified form due to Kisun'ko. Gutman's general result is, however, of a formal nature, since it depends upon the solution of an infinite set of first-order linear ordinary differential equations. Explicit results can be obtained when the permittivity and the permeability are functions only of the radial distance; this is also the most interesting case in practice, and to this case our considerations will be limited.

The first general treatment of electromagnetic scattering from a radially stratified sphere is perhaps due to Marcuvitz (1951). Tai (1958a) extended the method of Hansen and Stratton to dielectric lenses and performed detailed calculations for the spherical Luneburg lens. Arnush (1964) gave an alternative formulation in terms of phase-shift analysis and examined the case when the dielectric constant vanishes on a spherical surface, as it happens in the scattering from a dense bounded collisionless plasma; he also gave a comprehensive list of bibliographical references. The Rayleigh-Gans approximation for the scattering by a radially inhomogeneous sphere with a refractive index close to unity was considered by Farone (1965), who also gave references to earlier works on this subject.

In section 4.2, the results of Tai are extended to the case in which also the permeability varies radially, and general conditions are established for the presence of resonances and dips in the low-frequency backscattering cross section. Certain assumptions, such as differentiability, are implicitly made on the functions representing the radial variations of permittivity and permeability.

In section 4.3, the fields produced by a plane wave incident on an imperfectly conducting sphere coated by a radially inhomogeneous layer are determined exactly by imposing the boundary conditions, i.e. the continuity of the tangential components of the total electric and magnetic fields across the outer surface of the coating layer, and an impedance boundary condition on the surface of the core. Low-frequency approximations, resonance and dip conditions for the backscattering cross section are obtained. The particular case of a relative permittivity equal to the relative permeability and inversely proportional to the distance from the center of the scatterer is considered in detail, as an example.

The analysis developed in section 4.3 for a coated sphere is repeated in sections 4.4 and 4.5 for an inhomogeneous spherical shell and a coreless inhomogeneous sphere, respectively. In both cases, a particular application is made to a dielectric material with relative permeability equal to unity and permittivity inversely proportional to the square of the radial distance. Special attention is devoted to the boundary condition at the center of a coreless sphere whose permittivity and/or permeability become infinite at the center.

4.2 The General Case

Consider a radially inhomogeneous spherical region of outer radius $r = a$ and inner radius $r = b$, made of a material with relative permittivity $\epsilon = \epsilon(r)$ and relative permeability $\mu = \mu(r)$, and surrounded by free space; the spherical core $0 \leq r \leq b$ can be made of an imperfectly conducting material (section 4.3), or of free space (section 4.4), or be missing altogether if $b = 0$ (section 4.5).

Let ϵ_o, μ_o, $Z = Y^{-1} = \sqrt{\mu_o/\epsilon_o}$ and $k = \omega\sqrt{\epsilon_o\mu_o}$ be respectively the permittivity, permeability, intrinsic impedance and wave number of free space. Let us introduce two systems of rectangular Cartesian (x, y, z) and spherical polar (r, θ, ϕ) coordinates with origin at the center of the scatterer, and consider the incident plane electromagnetic wave

$$\underline{E}^i = \hat{x} e^{ikz} \quad , \qquad \underline{H}^i = \hat{y} Y e^{ikz} \tag{4.1}$$

which propagates in the direction $\theta = 0$ of the positive z axis; here $\hat{x}$ and $\hat{y}$ are unit vectors parallel to the positive x and y axes.

The total electric and magnetic fields in the region $r > a$ are given by the sums of the incident fields (4.1) and of the scattered fields $\underline{E}^s$ and $\underline{H}^s$, which may be written in the form:

$$\underline{E}^i = \frac{1}{kr}\sum_{n=1}^{\infty} f(n)\left[\psi_n(kr)\underline{m}_{on} - \frac{i}{kr}\psi_n(kr)\underline{\ell}_{en} - i\psi_n'(kr)\hat{r}\times\underline{m}_{en}\right], \qquad (4.2)$$

$$\underline{H}^i = \frac{-Y}{kr}\sum_{n=1}^{\infty} f(n)\left[\psi_n(kr)\underline{m}_{en} + \frac{i}{kr}\psi_n(kr)\underline{\ell}_{on} + i\psi_n'(kr)\hat{r}\times\underline{m}_{on}\right], \qquad (4.3)$$

$$\underline{E}^s = \frac{1}{kr}\sum_{n=1}^{\infty} f(n)\left[a_n^s\zeta_n(kr)\underline{m}_{on} - \frac{ib_n^s}{kr}\zeta_n(kr)\underline{\ell}_{en} - ib_n^s\zeta_n'(kr)\hat{r}\times\underline{m}_{en}\right], \qquad (4.4)$$

$$\underline{H}^s = \frac{-Y}{kr}\sum_{n=1}^{\infty} f(n)\left[b_n^s\zeta_n(kr)\underline{m}_{en} + \frac{ia_n^s}{kr}\zeta_n(kr)\underline{\ell}_{on} + ia_n^s\zeta_n'(kr)\hat{r}\times\underline{m}_{on}\right], \qquad (4.5)$$

where

$$f(n) = \frac{2n+1}{n(n+1)}i^n, \qquad \psi_n(x) = \sqrt{\frac{\pi x}{2}}\,J_{n+1/2}(x), \qquad \zeta_n(x) = \sqrt{\frac{\pi x}{2}}\,H^{(1)}_{n+1/2}(x), \qquad (4.6)$$

$$\underline{m}_{\substack{e\\o}n} = \mp\frac{P_n^1(\cos\theta)}{\sin\theta}\begin{matrix}\sin\\\cos\end{matrix}\phi\hat{\theta} - \frac{dP_n^1(\cos\theta)}{d\theta}\begin{matrix}\cos\\\sin\end{matrix}\phi\hat{\phi}, \qquad (4.7)$$

$$\underline{\ell}_{\substack{e\\o}n} = n(n+1)\,P_n^1(\cos\theta)\begin{matrix}\cos\\\sin\end{matrix}\phi\hat{r}, \qquad (4.8)$$

$\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$ are unit vectors oriented in the positive r, θ and ϕ directions, the prime indicates the derivative with respect to the argument kr, and the definition of Stratton (1941) is used for $P_n^1(\cos\theta)$.

The total electric and magnetic fields $\underline{E}_1$ and $\underline{H}_1$ inside the inhomogeneous region $b < r < a$ are given by:

$$\underline{E}_1 = \frac{1}{kr}\sum_{n=1}^{\infty} f(n)\left[S_n(kr)\underline{m}_{on} - \frac{iT_n(kr)}{kr\epsilon}\,\underline{\ell}_{en} - \frac{i}{\epsilon}\,T'_n(kr)\,\hat{r}\times\underline{m}_{en}\right], \tag{4.9}$$

$$\underline{H}_1 = \frac{-Y}{kr}\sum_{n=1}^{\infty} f(n)\left[T_n(kr)\underline{m}_{en} + \frac{iS_n(kr)}{kr\mu}\,\underline{\ell}_{on} + \frac{i}{\mu}\,S'_n(kr)\,\hat{r}\times\underline{m}_{on}\right], \tag{4.10}$$

where

$$S_n(x) = a_{1n}S_n^{(1)}(x) + c_{1n}S_n^{(2)}(x), \tag{4.11}$$

$$T_n(x) = b_{1n}T_n^{(1)}(x) + d_{1n}T_n^{(2)}(x), \tag{4.12}$$

and $S_n^{(j)}(x)$ and $T_n^{(j)}(x)$ ($j = 1$ or 2) are, respectively, two linear independent solutions of the equations (see, also, for example, Wait, 1963):

$$S''_n - \frac{d\ln\mu}{dx}S'_n + \left[\epsilon\mu - \frac{n(n+1)}{x^2}\right]S_n = 0, \tag{4.13}$$

$$T''_n - \frac{d\ln\epsilon}{dx}T'_n + \left[\epsilon\mu - \frac{n(n+1)}{x^2}\right]T_n = 0, \tag{4.14}$$

where the primes indicate derivatives with respect to the argument $x = kr$. The Wronskians W_S of $S_n^{(j)}$ and W_T of $T_n^{(j)}$ are given by:

$$\begin{aligned} W_S(r) &= S_n^{(1)}(kr)S_n^{(2)'}(kr) - S_n^{(1)'}(kr)S_n^{(2)}(kr) = \alpha\mu(r), \\ W_T(r) &= T_n^{(1)}(kr)T_n^{(2)'}(kr) - T_n^{(1)'}(kr)T_n^{(2)}(kr) = \beta\epsilon(r), \end{aligned} \tag{4.15}$$

where α and β are two constants whose values depend on the normalizations of the eigenfunctions.

In the case of the spherical shell of section 4.4, one must also consider the total electric and magnetic fields $\underline{E}_2$ and $\underline{H}_2$ inside the free-space region $0 \leq r < b$:

$$\underline{E}_2 = \frac{1}{kr} \sum_{n=1}^{\infty} f(n) \left[a_{2n} \psi_n(kr) \underline{m}_{on} - ib_{2n} \frac{\psi_n(kr)}{kr} \underline{\ell}_{en} - ib_{2n} \psi_n'(kr) \hat{r} \times \underline{m}_{en} \right], \tag{4.16}$$

$$\underline{H}_2 = \frac{-Y}{kr} \sum_{n=1}^{\infty} f(n) \left[b_{2n} \psi_n(kr) \underline{m}_{en} + ia_{2n} \frac{\psi_n(kr)}{kr} \underline{\ell}_{on} + ia_{2n} \psi_n'(kr) \hat{r} \times \underline{m}_{on} \right] . \tag{4.17}$$

The constants a_n^s, b_n^s, a_{1n}, b_{1n}, c_{1n}, d_{1n}, and eventually a_{2n} and b_{2n}, are determined by imposing the appropriate boundary conditions at $r = a$ and $r = b$ (the radiation condition is already satisfied by the choices (4.4) and (4.5) for the scattered fields). In particular, a_n^s and b_n^s can always be written in the form:

$$a_n^s = -\frac{\psi_n'(ka) - M_n \psi_n(ka)}{\zeta_n'(ka) - M_n \zeta_n(ka)} \quad , \qquad b_n^s = -\frac{\psi_n'(ka) - \tilde{M}_n \psi_n(ka)}{\zeta_n'(ka) - \tilde{M}_n \zeta_n(ka)} \quad , \tag{4.18}$$

where the quantities M_n and $\tilde{M}_n$ depend upon the structure of the scatterer.

In the far field ($r \rightarrow \infty$):

$$\underline{E}^s \sim -i \frac{e^{ikr}}{kr} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left\{ \left[a_n^s \frac{P_n^1(\cos\theta)}{\sin\theta} + b_n^s \frac{dP_n^1}{d\theta} \right] \cos\phi \hat{\theta} - \right.$$

$$\left. - \left[a_n^s \frac{dP_n^1}{d\theta} + b_n^s \frac{P_n^1(\cos\theta)}{\sin\theta} \right] \sin\phi \hat{\phi} \right\}, \tag{4.19}$$

and, in particular, the backscattered field ($\theta = \pi$) is:

$$\underline{E}^{b.s.} \sim \hat{x}(-i) \frac{e^{ikr}}{kr} \sum_{n=1}^{\infty} (-1)^n (n + \tfrac{1}{2})(a_n^s - b_n^s) . \tag{4.20}$$

Therefore, in the low-frequency limit the nth mode of the backscattering cross section will present a dip whenever

$$(M_n)_{LF} = (\tilde{M}_n)_{LF} \quad ; \tag{4.21}$$

here and in the following, the subscript LF is used to indicate the Rayleigh approximation. A resonance on the nth mode of the low-frequency backscattered field occurs when the dominant term in the denominator of either a_n^s or b_n^s becomes zero, that is when either

$$(M_n)_{LF} = -n/ka \quad , \tag{4.22}$$

or

$$(\tilde{M}_n)_{LF} = -n/ka \quad . \tag{4.23}$$

Since the total scattering cross section is

$$\sigma_{total} = -\frac{\pi}{k^2} \operatorname{Re}\left\{\left[\sum_{n=1}^{\infty} (2n+1)(a_n^s + b_n^s)\right]\right\} \quad , \tag{4.24}$$

it follows that (4.22) and (4.23) are also the resonance conditions for σ_{total}.

The differential equations (4.13) and (4.14) can be solved exactly for special choices of the functions $\epsilon(r)$ and $\mu(r)$; some solutions are listed in the following:

(I) Luneburg lens: $\mu(r) = 1$, $\epsilon(r) = 2-(r/a)^2$; the functions S_n and T_n are given by products of powers of kr, exponentials, and, respectively, confluent and generalized confluent hypergeometric functions [for details see Tai (1958a)].

(II) Some considerations for the case $\mu(r) = 1$, $\epsilon(r) = \epsilon(\infty)(r+r_1)/(r+r_2)$ with $\epsilon(\infty)$, r_1 and r_2 constants, have been developed by Tai (1963).

(III) Maxwell fish-eye: $\mu(r) = 1$, $\epsilon(r) = 4\left[1+(r/a)^2\right]^{-2}$; S_n and T_n are given by products of powers of kr and hypergeometric functions (Tai, 1958b).

(IV) $\mu(r) = 1$, $\epsilon(r) = (r/a)^m$ with $m \neq -2$; this case has been investigated by Nomura and Takaku (see e.g. Tai, 1958a); S_n and T_n are products of powers of kr and Bessel functions of rather complicated order and argument.

(V) $$\epsilon(r) = \epsilon(a)(r/a)^{2\alpha-1} \quad , \qquad \mu(r) = \mu(a)(r/a)^{-2\alpha-1} \quad , \tag{4.25}$$

where $\epsilon(a)$, $\mu(a)$ and α are constants; then

$$S_n = (kr)^{-\alpha \pm \beta_n} \quad , \qquad T_n = (kr)^{\alpha \pm \beta_n} \quad , \tag{4.26}$$

with

$$\beta_n = \sqrt{\alpha^2 + n(n+1) - \epsilon(a)\mu(a)(ka)^2} \ ; \tag{4.27}$$

if $\beta_n = 0$ for a particular n, then a second independent solution is given by the product of (4.26) times $\ln(kr)$.

(VI) $$\epsilon(r) = \epsilon(a)(r/a)^{\alpha} \quad , \qquad \epsilon\mu = N^2 \ , \tag{4.28}$$

when $\epsilon(a)$, α and N are constants; then

$$S_n = (Nkr)^{-\alpha/2} \left[a_{1n} \psi_{\gamma_n}(Nkr) + c_{1n} \zeta_{\gamma_n}(Nkr) \right] \ , \tag{4.29}$$

$$T_n = (Nkr)^{\alpha/2} \left[b_{1n} \psi_{\delta_n}(Nkr) + d_{1n} \zeta_{\delta_n}(Nkr) \right] \ , \tag{4.30}$$

with

$$\gamma_n = -\frac{1}{2} + \sqrt{(n+\frac{1}{2})^2 + \frac{\alpha}{2}(\frac{\alpha}{2} - 1)} \quad , \tag{4.31}$$

$$\delta_n = -\frac{1}{2} + \sqrt{(n+\frac{1}{2})^2 + \frac{\alpha}{2}(\frac{\alpha}{2} + 1)} \quad . \tag{4.32}$$

4.3 The Coated Sphere

In order to determine the coefficients a_n^s, b_n^s, a_{1n}, c_{1n}, b_{1n} and d_{1n}, one must impose the continuity of the tangential components of the total electric and magnetic fields across the surface $r = a$ and the impedance boundary condition

$$\underline{E}_1 - (\underline{E}_1 \cdot \hat{r})\hat{r} = \eta Z \hat{r} \times \underline{H}_1 \ , \qquad \text{at } r = b \ , \tag{4.33}$$

where η is the relative surface impedance.

It is found that a_n^s and b_n^s are given by (4.18) where

$$M_n = \frac{1}{\mu(a)} \frac{\partial}{\partial(ka)} \ln\left[C_n - \frac{i\eta}{\mu(b)} \frac{\partial C_n}{\partial(kb)}\right], \tag{4.34}$$

$$\tilde{M}_n = \frac{1}{\epsilon(a)} \frac{\partial}{\partial(ka)} \ln\left[\tilde{C}_n - \frac{i\eta^{-1}}{\epsilon(b)} \frac{\partial \tilde{C}_n}{\partial(kb)}\right], \tag{4.35}$$

with

$$C_n = S_n^{(1)}(ka)S_n^{(2)}(kb) - S_n^{(1)}(kb)S_n^{(2)}(ka), \tag{4.36}$$

$$\tilde{C}_n = T_n^{(1)}(ka)T_n^{(2)}(kb) - T_n^{(1)}(kb)T_n^{(2)}(ka). \tag{4.37}$$

The other coefficients are:

$$a_{1n} = i \frac{S_n^{(2)}(kb) - \frac{i\eta}{\mu(b)} S_n^{(2)'}(kb)}{\left[C_n - \frac{i\eta}{\mu(b)} \frac{\partial C_n}{\partial(kb)}\right]\left[\zeta_n'(ka) - M_n\zeta_n(ka)\right]}, \tag{4.38}$$

$$c_{1n} = -i \frac{S_n^{(1)}(kb) - \frac{i\eta}{\mu(b)} S_n^{(1)'}(kb)}{\left[C_n - \frac{i\eta}{\mu(b)} \frac{\partial C_n}{\partial(kb)}\right]\left[\zeta_n'(ka) - M_n\zeta_n(ka)\right]}, \tag{4.39}$$

and b_{1n} and d_{1n} are obtained from a_{1n} and c_{1n} respectively, by replacing η, μ, S_n, C_n and M_n in (4.38) and (4.39) respectively with η^{-1}, ϵ, T_n, $\tilde{C}_n$ and $\tilde{M}_n$.

In particular, it follows from the previous formulas that a sufficient condition to have a zero backscattered field is that

$$\epsilon(r) = \mu(r), \qquad \eta = \pm 1 . \tag{4.40}$$

This result could have been predicted on the basis of two theorems by Weston (1963) or of a theorem by Wagner and Lynch (1963), or by approximating the inhomogeneous coating with an arbitrarily large number of concentric homogeneous layers and applying a result by Uslenghi (1965).

As an application, let us consider the particular case

$$\epsilon(r) = \mu(r) = N\frac{a}{r} \quad , \tag{4.41}$$

which is obtained from (4.25) by setting $\epsilon(a) = \mu(a) = N$ and $\alpha = 0$. Then

$$M_n = \frac{i\tau}{Nka} \cot\left[i\tau \ln\rho - \arctan\left(\eta\frac{\tau}{N}\right)\right] \quad , \tag{4.42}$$

where

$$\tau = \sqrt{\nu^2 - (Nka)^2} \; , \qquad \nu^2 = n(n+1) \; , \qquad \rho = a/b \; , \tag{4.43}$$

and $\tilde{M}_n$ is given by (4.42) with η replaced by η^{-1}. In the low-frequency approximation

$$M_n \sim \frac{\nu}{Nka} \cdot \frac{(1+i\eta\nu N^{-1})\rho^{2\nu} + (1-i\eta\nu N^{-1})}{(1+i\eta\nu N^{-1})\rho^{2\nu} - (1-i\eta\nu N^{-1})} \left\{1 + O\left[(Nka)^2\right]\right\} \; , \tag{4.44}$$

and it then follows from (4.21) that dips in the backscattered field occur when $\eta = \pm 1$, as one should have expected, since in this case the backscattered field is exactly zero. It also follows from (4.22) and (4.23) that a resonance for a_n^s occurs if

$$\eta = \frac{iN}{\nu} \cdot \frac{(n+\frac{\nu}{N})\rho^{2\nu} - (n-\frac{\nu}{N})}{(n+\frac{\nu}{N})\rho^{2\nu} + (n-\frac{\nu}{N})} \quad , \tag{4.45}$$

whereas a resonance for b_n^s occurs when (4.45) is satisfied with η replaced by η^{-1}. If, in particular, the core is a perfect conductor ($\eta = 0$), the backscattering cross section will present a resonance when either

$$N = -\frac{\nu}{n}\,\frac{\rho^{2\nu}+1}{\rho^{2\nu}-1} \quad , \tag{4.46}$$

or

$$N = -\frac{\nu}{n}\,\frac{\rho^{2\nu}-1}{\rho^{2\nu}+1} \quad . \tag{4.47}$$

Resonances and dips in the low-frequency backscattered field have been previously investigated by Murphy (1965) for the case of a perfectly conducting spherical core coated by a homogeneous dielectric (plasma) layer.

4.4 The Spherical Shell

By imposing the continuity of the tangential components of the total electric and magnetic fields across the surfaces $r = a$ and $r = b$, one finds that the coefficients which appear in (4.4), (4.5), (4.9), (4.10), (4.16) and (4.17) are:

$$a_{1n} = \frac{i}{A_n}\left[\psi_n'(kb)S_n^{(2)}(kb) - \psi_n(kb)\frac{1}{\mu(b)}S_n^{(2)'}(kb)\right], \tag{4.48}$$

$$c_{1n} = -\frac{i}{A_n}\left[\psi_n'(kb)S_n^{(1)}(kb) - \psi_n(kb)\frac{1}{\mu(b)}S_n^{(1)'}(kb)\right], \tag{4.49}$$

$$a_{2n} = -\frac{iW_S(b)}{\mu(b)A_n}, \tag{4.50}$$

where the Wronskian $W_S(b)$ is given by the first of (4.15) in which $r = b$, and

$$A_n = \left[\zeta_n'(ka) - M_n\zeta_n(ka)\right]\left[\psi_n'(kb)C_n - \psi_n(kb)\frac{1}{\mu(b)}\frac{\partial C_n}{\partial(kb)}\right]. \tag{4.51}$$

The coefficients a_n^s and b_n^s are given by (4.18), and b_{1n}, d_{1n} and b_{2n} are obtained from a_{1n}, c_{1n} and a_{2n} respectively, by replacing μ, S_n, C_n, M_n and $W_S(b)$ in (4.48), (4.49), (4.50) and (4.51) respectively with ϵ, T_n, $\tilde{C}_n$, $\tilde{M}_n$ and $W_T(b)$. The quantities C_n and $\tilde{C}_n$ are still given by (4.36) and (4.37), whereas now

$$M_n = \frac{1}{\mu(a)}\frac{\partial}{\partial(ka)}\ln\left[\psi_n'(kb)C_n - \psi_n(kb)\frac{1}{\mu(b)}\frac{\partial C_n}{\partial(kb)}\right] \tag{4.52}$$

and $\tilde{M}_n$ is obtained from (4.52) by replacing μ with ϵ and C_n with $\tilde{C}_n$. From the above formulas it is seen that the backscattering cross section is exactly zero if $\epsilon(r) = \mu(r)$; this result also follows as a particular case from the first theorem in Weston (1963).

As an application, let us consider the case

$$\mu(r) = 1, \qquad \epsilon(r) = \epsilon(a)(a/r)^2 \tag{4.53}$$

which follows from (4.25) when we set $\mu(a) = 1$ and $\alpha = -1/2$. It is found that

$$(M_n)_{LF} = \frac{n+1}{ka} , \tag{4.54}$$

$$(\tilde{M}_n)_{LF} = \frac{1}{\epsilon(a)ka} \cdot \frac{(n+1)\epsilon(a)\left[n\rho^{n+3}+(n+1)\rho^{-n+2}\right]+n(n+1)(\rho^{n+1}-\rho^{-n})}{(n+1)\epsilon(a)(\rho^{n+3}-\rho^{-n+2})+\left[(n+1)\rho^{n+1}+n\rho^{-n}\right]} , \tag{4.55}$$

where $\rho = a/b$.

From (4.22) and (4.54) it is seen that no resonances occur for a_n^s; thus it follows from (4.23) and (4.55) that the resonances in the backscattering cross section occur when $\epsilon(a)$ is a root of the equation:

$$\left[\epsilon(a)\right]^2 n(n+1)(\rho^{n+3}-\rho^{-n+2})+\epsilon(a)\left[(n+1)^2\rho^{-n+2}+n(n+1)(\rho^{n+3}+\rho^{n+1})+n^2\rho^{-n}\right]+$$
$$+\, n(n+1)(\rho^{n+1}-\rho^{-n}) = 0 ; \tag{4.56}$$

if the roots of (4.56) are real, then they are both negative. In agreement with (4.21), (4.54) and (4.55), the dips in the backscattered field occur when $\epsilon(a)$ is a root of the equation:

$$\left[\epsilon(a)\right]^2(n+1)(\rho^{n+3}-\rho^{-n+2})+\epsilon(a)\left[(n+1)(\rho^{n+1}-\rho^{-n+2})+n(\rho^{-n}-\rho^{n+3})\right]+$$
$$+\, n(\rho^{-n}-\rho^{n+1}) = 0 ; \tag{4.57}$$

if the roots of (4.57) are real, then they have opposite signs.

Conditions (4.56) and (4.57) simplify in the case $b=0$ (dielectric lens); by retaining only the dominant terms proportional to ρ^{n+3}, it is found that resonances occur when

$$\epsilon(a) = -1 , \tag{4.58}$$

and dips occur when

$$\epsilon(a) = \frac{n}{n+1} \quad . \tag{4.59}$$

The resonance condition (4.58) is of special interest because it is independent of n; we observe that (4.58) is also the resonance condition for all modes $n \neq 0$ in the case of a homogeneous plasma cylinder (see, for example, Murphy, 1965).

4.5 The Coreless Sphere

The boundary conditions to be imposed in this case are the continuity of the tangential components of the electric and magnetic fields across the surface $r = a$, and a condition at $r = b = 0$ which leads to the proper choice between the solutions of (4.13) and (4.14), and which will be discussed later. The coefficients in (4.4), (4.5), (4.9) and (4.10) are given by (4.18) and by:

$$a_{1n} = \frac{i}{S_n^{(1)}(ka)\left[\zeta_n'(ka) - M_n \zeta_n(ka)\right]} \quad , \tag{4.60}$$

$$b_{1n} = \frac{i}{T_n^{(1)}(ka)\left[\zeta_n'(ka) - \tilde{M}_n \zeta_n(ka)\right]} \quad , \tag{4.61}$$

$$c_{1n} = d_{1n} = 0 \, , \tag{4.62}$$

where:

$$M_n = \frac{1}{\mu(a)} \frac{\partial}{\partial(ka)} \ln S_n^{(1)}(ka) \; , \qquad \tilde{M}_n = \frac{1}{\epsilon(a)} \frac{\partial}{\partial(ka)} \ln T_n^{(1)}(ka) \; . \tag{4.63}$$

If $\epsilon(r)$ and $\mu(r)$ are finite at $r = 0$, then we may require that the field components be finite at $r = 0$ in order to select the appropriate solutions $S_n^{(1)}$ of (4.13) and $T_n^{(1)}$ of (4.14). However, if $\epsilon(r)$ and/or $\mu(r)$ present a singularity at the origin, it appears that the only sensible restriction which can be imposed on the solutions of (4.13) and (4.14) is a boundary condition of the Meixner type (Meixner, 1949): the total energy of the electric and magnetic fields inside any finite volume surrounding the origin $r=0$ must be finite. If we assume for simplicity that ϵ and μ are real, then sufficient conditions for the energy to be finite are:

$$\int_0^{r_o} F(r)dr = \text{finite} \, , \tag{4.64}$$

where r_o is any r such that $r_o > \delta > 0$ with δ arbitrarily small, and $F(r)$ is any of the following eight quantities (the asterisk denotes the complex conjugate, and n and m are positive integers):

$$F(r) = \epsilon S_n S_m^* \, ; \; \mu T_n T_m^* \, ; \; S_n T_m'^* ; \; T_n S_m'^* ; \; \frac{|S_n|^2}{r^2 \mu} \; ; \; \frac{|T_n|^2}{r^2 \epsilon} \; ;$$

$$\frac{1}{\mu} S_n' S_m'^* ; \; \frac{1}{\epsilon} T_n' T_m'^* \; . \tag{4.65}$$

Let us consider the particular case in which ϵ and μ are given by (4.53). The solutions of (4.13) and (4.14) which satisfy (4.64) - (4.65) are:

$$S_n^{(1)} = (kr)^{\frac{1}{2}+\beta_n} \, , \qquad T_n^{(1)} = (kr)^{-\frac{1}{2}+\beta_n} \, , \tag{4.66}$$

where $\beta_n = \sqrt{(n+\frac{1}{2})^2 - \epsilon(a)(ka)^2}$ has a positive real part; the other solutions $S_n^{(2)}$ and $T_n^{(2)}$, corresponding to a negative real part of β_n, do not obey (4.64) - (4.65). Actually, in order to satisfy (4.64) - (4.65) for all n when $\epsilon(a) \geqslant 9/(4k^2a^2)$, it is necessary to suppose that $\epsilon(a)$ has a small imaginary part, as always happens in practice. If $\epsilon(a)$ were a real quantity, then for all n such that

$$n \leqslant ka\sqrt{\epsilon(a)} - \frac{1}{2} \, ,$$

the sufficient conditions (4.64) - (4.65) would not be satisfied. In connection with this last remark, we observe that the trajectory of an optical ray which enters the dielectric lens (4.53) $\left[\epsilon(a) \geqslant 1\right]$ is a logarithmic spiral around the center $r = 0$, as is easily seen by the generalized Snell's law (Chapter Five); thus, from a geometric optics viewpoint all the electromagnetic energy entering the lens is accumulated toward the center and if the material were non-dissipative, the energy contained in any volume surrounding the center would be infinite.

Finally, one can easily verify that substitution of (4.66) into (4.63) yields the same resonance and dip conditions that were found in section 4.4 as a limiting case of the spherical shell, namely, resonances in the backscattered field occur for all modes if $\epsilon(a) = -1$, whereas a dip occurs for the nth mode if $\epsilon(a) = n/(n+1)$.

Chapter Five

HIGH-FREQUENCY BACKSCATTERING FROM A CERTAIN DIELECTRIC LENS

5.1 Introduction

In the following we consider the spherical dielectric lens of radius $r = a$, relative magnetic permeability equal to unity, and relative electric permittivity given by

$$\epsilon(r) = \epsilon(a)(a/r)^2 , \qquad \left(\operatorname{Re}\epsilon(a) \geqslant 1, \quad \operatorname{Im}\epsilon(a) \gtrsim 0\right) . \tag{5.1}$$

In the geometrical optics approximation, the problem is reduced to the consideration of ray paths in a plane passing through the center of the lens. With reference to the symbols of Fig. 5-1, the optical ray path is given by the equation

$$\frac{dr}{d\beta} = -r\cot\psi , \tag{5.2}$$

and the angle ψ is related to the angle α of incidence by the generalized Snell law

$$N(r) r \sin\psi = N(a) a \sin\alpha_1 \tag{5.3}$$

with

$$N(r) = \sqrt{\epsilon(r)} , \qquad \sin\alpha_1 = \frac{\sin\alpha}{\sqrt{\epsilon(a)}} . \tag{5.4}$$

Integration of (5.2) yields

$$r = a\exp\left\{-\beta\sqrt{\frac{\epsilon(a)}{\sin^2\alpha} - 1}\right\} \tag{5.5}$$

and in particular, for $\epsilon(a) = 1$:

$$r = a e^{-\beta\cot\alpha} , \qquad (0 \leqslant \alpha \leqslant \pi/2) ; \tag{5.6}$$

the optical ray describes a logarithmic spiral around the center $r = 0$.

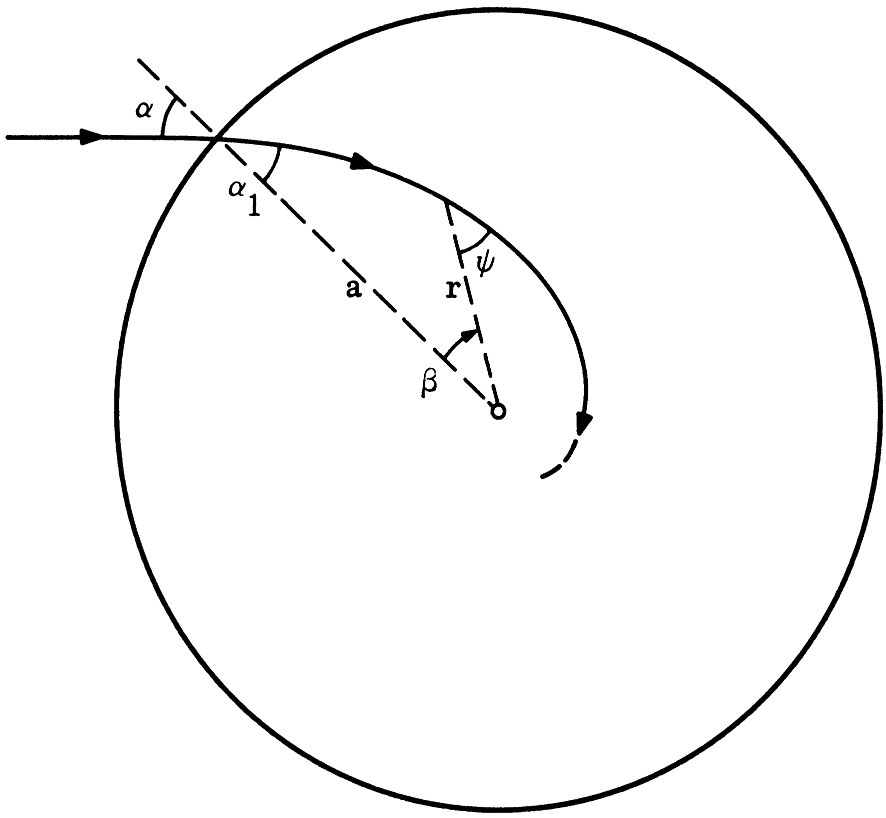

FIG. 5-1: GEOMETRY FOR OPTICAL RAY PATH.

Since none of the rays entering the lens ever leaves it, the geometrical optics scattered field for $r \geqslant a$ is obtained by applying the Fresnel reflection coefficient at the rim $r = a$; if, in particular, $\epsilon(a) = 1$, the geometrical optics scattered field is zero everywhere and a rigorous asymptotic analysis is needed to obtain even the leading term of the scattered field. Such analysis is performed in the following for the far backscattered field. The transformation of the infinite series solution into a contour integral and the choice of branch-cuts are given in section 5.2, whereas sections 5.3, 5.4 and 5.5 are respectively devoted to the determination of the reflected field, creeping waves and evanescent wave contributions to the backscattered field. A brief analysis of these results is included in section 5.6.

A lens which approximately obeys the functional relation (5.1) may be constructed by means of many concentric homogeneous layers of different materials; in particular, the lens may have a perfectly conducting core with a radius very small compared to a.

5.2 Infinite Series Solution and Contour Integral Representations

The incident plane electromagnetic wave

$$\underline{E}^i = \hat{x}\, e^{ikz} , \qquad \underline{H}^i = \hat{y}\, Y e^{ikz} , \qquad \left(Y = \sqrt{\epsilon_o/\mu_o}\right), \tag{5.7}$$

which propagates in the direction of the positive z-axis, produces the far backscattered field

$$\underline{E}^{b.s.} \sim \hat{x}(-i)\, \frac{e^{ikr}}{kr} \sum_{n=1}^{\infty} (-1)^n \left(n+\frac{1}{2}\right) (a_n^s - b_n^s) , \tag{5.8}$$

where a_n^s and b_n^s are given by (4.18) with

$$M_n = \frac{\beta_n + \frac{1}{2}}{ka} \, , \qquad \tilde{M}_n = \frac{\beta_n - \frac{1}{2}}{\epsilon(a)ka} \, , \tag{5.9}$$

$$\beta_n = \sqrt{\left(n+\frac{1}{2}\right)^2 - \epsilon(a)(ka)^2} \, , \qquad (\mathrm{Re}\,\beta_n > 0). \tag{5.10}$$

Formulas (5.9) and (5.10) were obtained by setting $\alpha = -1/2$ in (4.25) to (4.27) and by eliminating one of the radial eigenfunctions by means of a Meixner-type boundary condition about the center of the lens, as discussed in the last section of the previous chapter.

The infinite series (5.8) may be considered as a residue series in a complex plane ν, and may be replaced by a contour integral C taken in the clockwise direction around the poles $\nu = n+\frac{1}{2} = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, giving

$$E_x^{b.s.} \sim \frac{e^{ikr}}{kr} \left[\frac{i}{2}(a_o - b_o) - \frac{1}{2} \int_C \frac{\nu}{\cos \pi\nu} \left(a_{\nu-1/2} - b_{\nu-1/2} \right) d\nu \right] , \tag{5.11}$$

where, on the basis of relations (2.7), (4.18), (5.9) and (5.10),

$$a_{\nu-1/2} - b_{\nu-1/2} = \frac{2i}{\pi(ka)^2} \cdot \frac{\frac{1}{2}\left[1 + \frac{1}{\epsilon(a)}\right] + \left[1 - \frac{1}{\epsilon(a)}\right] ka\,C}{\left[H_\nu^{(1)\prime}(ka) - CH_\nu^{(1)}(ka)\right]\left[H_\nu^{(1)\prime}(ka) - \frac{1}{\epsilon(a)}\left(C - \frac{1+\epsilon(a)}{2ka}\right)H_\nu^{(1)}(ka)\right]} \tag{5.12}$$

with

$$C = \sqrt{(\nu/ka)^2 - \epsilon(a)} \; ; \tag{5.13}$$

here the prime indicates the derivative with respect to the argument ka.

In the complex ν-plane, the quantity $(a_{\nu-1/2} - b_{\nu-1/2})$ has two branch-points at $\nu = \pm ka\sqrt{\epsilon(a)}$. Let us set

$$\nu = \nu' + i\nu'' , \qquad \epsilon(a) = \epsilon' + i\epsilon'' \tag{5.14}$$

with ν', ν'', ϵ' and ϵ'' real, and let us choose the branch-cuts along the hyperbola

$$\nu'\nu'' = \frac{\epsilon''}{2}(ka)^2 \tag{5.15}$$

from $\nu = \pm ka\sqrt{\epsilon(a)}$ to $\nu = \pm i\infty$, as shown in Fig. 5-2. Along the cuts $\operatorname{Re} C = 0$, and elsewhere in the ν-plane $\operatorname{Re} C > 0$; on the remaining parts of the hyperbola (broken lines in Fig. 5-2) $\operatorname{Im} C = 0$; between the two branches of the hyperbola (hatched domain in Fig. 5-2) $\operatorname{Im} C < 0$, whereas in the remaining portions of the first and third quadrants $\operatorname{Im} C > 0$. The quantity C has a jump discontinuity in crossing a cut; with reference to the two points (+) just above and (-) just below the cut in the first quadrant of Fig. 5-2:

$$C_{\pm} = i(\operatorname{Im} C)_{\pm} = \pm i\sqrt{\epsilon' + \frac{\nu''^2 - \nu'^2}{(ka)^2}} , \tag{5.16}$$

where ν' and ν'' are linked by (5.15).

The choice of branch-cuts illustrated in Fig. 5-2 satisfies three requirements: (i) since $\operatorname{Re} C > 0$ on the real positive ν-axis, $\operatorname{Re} \beta_n > 0$ in agreement with (5.10); (ii) in the Watson-type transformation introduced in the following, the line integral along the circular arc Γ_2 tends to zero as the radius R of the circle tends to infinity (section 5.3); (iii) the branch-cut integral gives a contribution to the backscattered field which is finite and susceptible of a physical interpretation (section 5.5).

By deforming the path of integration, the contour integral C of (5.11) is replaced by the sum of an integral whose contour Γ_1 extends from the fourth

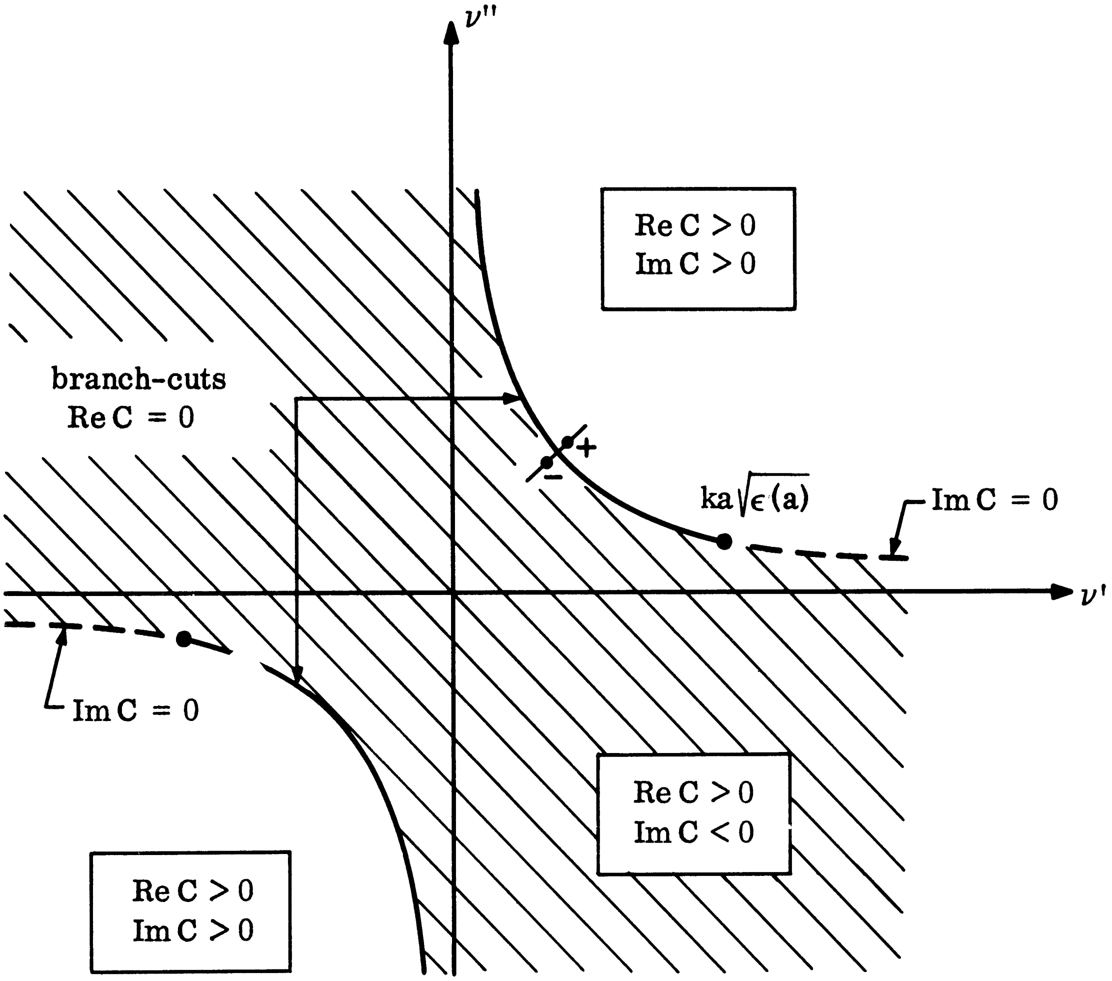

FIG. 5-2: THE CHOICE OF THE BRANCH-CUTS IN THE COMPLEX ν-PLANE.

quadrant through the origin $\nu = 0$ to the second quadrant, an integral along the arc Γ_2 of a circle of large radius R with center at the origin, an integral along the contour Γ_3 around the branch cut in the first quadrant, and a sum of residues due to the poles of the integrand which lie in the first quadrant:

$$\int_C = \int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3} + 2\pi i \sum (\text{residues in first quadrant}) \, . \tag{5.17}$$

The deformed path of integration is shown in Fig. 5-3. The following sections are devoted to the asymptotic evaluation as $ka \to \infty$ of the quantities appearing in (5.11) and (5.17), and to the physical interpretation of the results obtained.

5.3 Reflected Field Contribution

The reflected field approximation to the far backscattered field is obtained by adding together the asymptotic expressions for the line integral along the contour Γ_1 and for the term containing $(a_o - b_o)$ in relation (5.11). Firstly, however, we shall show that the line integral along the circular arc Γ_2 vanishes as the radius R of the circle tends to infinity.

The Debye expansions may be used for $H_\nu^{(1)}(ka)$ and its derivative provided that inequality (1.19) is satisfied; this is certainly true along the contour Γ_2, and also along Γ_1 if Γ_1 remains at a sufficient distance from $\nu = ka$ (see Fig. 5-3). The various regions of validity for the Debye expansions were described in sections 1.3 and 1.4 and illustrated in Fig. 1-2; a detailed derivation of the expansions is given by Watson (1958, chapter 8). We simply recall the relations:

$$\nu = ka \cosh\gamma \, , \qquad \psi = \nu(\tanh\gamma - \gamma) \tag{5.18}$$

where

$$-\frac{\pi}{2} < \arg(-i \sinh\gamma) < \frac{\pi}{2} \quad . \tag{5.19}$$

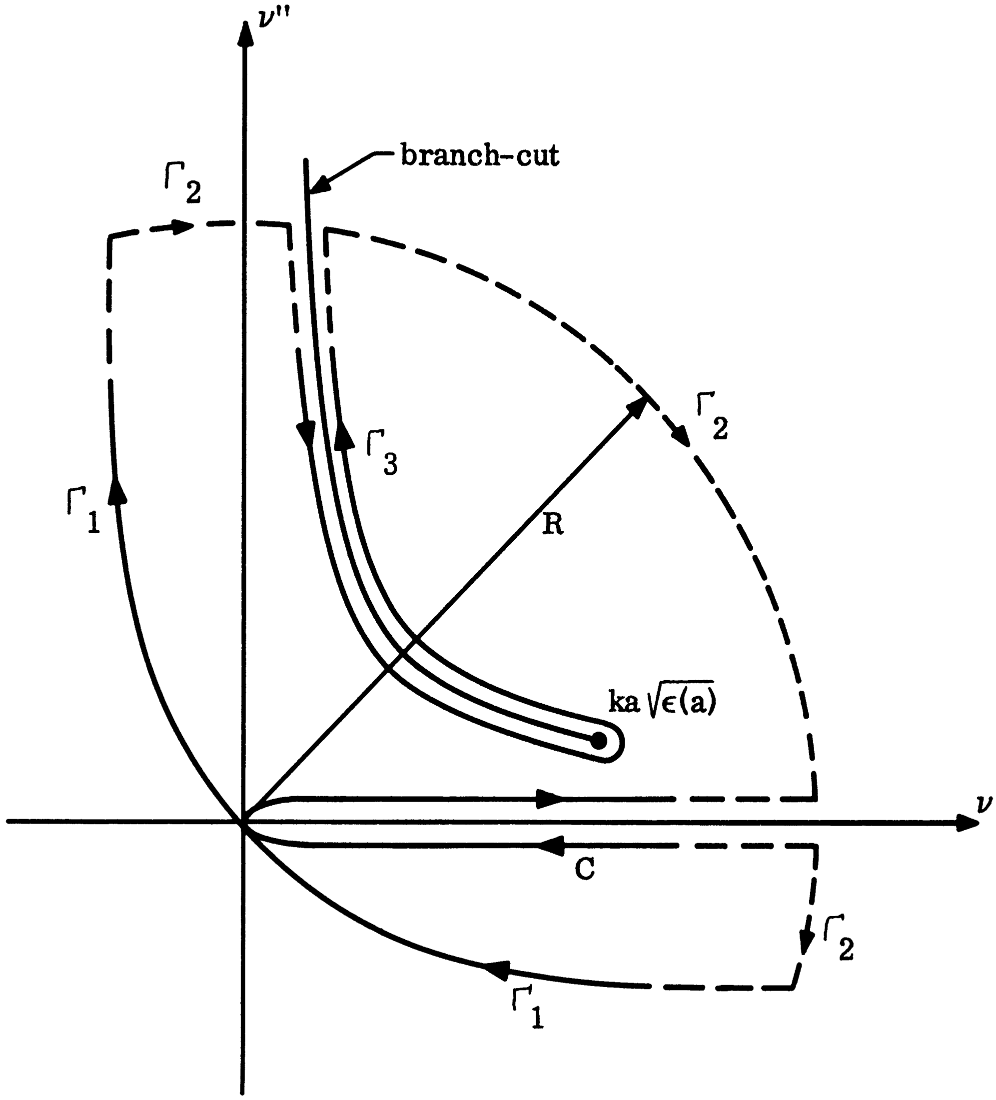

FIG. 5-3: CONTOURS OF INTEGRATION IN THE COMPLEX ν-PLANE.

With the choice of branch-cuts of the previous section

$$C_{|\nu|\to\infty} \sim \pm \frac{\nu}{ka} \quad , \qquad (\pm \text{ if } \nu' \gtrless 0) \, , \tag{5.20}$$

and therefore from (5.12):

$$\left(a_{\nu-1/2} - b_{\nu-1/2}\right)_{|\nu|\to\infty} \sim \frac{\pm \dfrac{i2\nu}{\pi(ka)^2}\left[1 - \dfrac{1}{\epsilon(a)}\right]}{\left[H_\nu^{(1)'}(ka) \mp \dfrac{\nu}{ka} H_\nu^{(1)}(ka)\right]\left[H_\nu^{(1)'}(ka) \mp \dfrac{\nu}{ka\,\epsilon(a)} H_\nu^{(1)}(ka)\right]} \, , \tag{5.21}$$

where the first (second) sign is valid if $\nu' > 0$ $(\nu' < 0)$. Let

$$\nu = R e^{\pm i\theta}$$

with $0 \leqslant \theta \leqslant \pi$ and the first (second) sign valid when $\nu'' > 0$ $(\nu'' < 0)$. Then along the arc Γ_2 the integrand function $\left[\nu(a_{\nu-1/2} - b_{\nu-1/2})/\cos \pi\nu\right]$ behaves like

$$R^2 e^{-\pi R \sin\theta} \left(\frac{2R}{ka}\right)^{-2R|\cos\theta|} \, ,$$

and therefore the integral along Γ_2 goes to zero as R tends to infinity.

The contour of integration Γ_1 extends through the Debye regions 3, 1, 4 and 6b of Fig. 1-2. If Γ_1 runs at a sufficient distance from $\nu = ka$, so that along the contour

$$\left|\left(\frac{\nu}{ka}\right)^2 - 1\right| \gg (2ka)^{-2/3} \, ,$$

and at a sufficient distance from $\nu = i1.51\,ka$ (point S_1 of Fig. 1-2), so that $|\nu| \gg ka$ along the portion of the contour lying in 6b, then the integrand becomes:

$$\frac{\nu}{\cos \pi\nu}\,(a_{\nu-1/2} - b_{\nu-1/2}) \sim f(\nu)\,\frac{e^{-2\psi}}{e^{i\pi\nu} + e^{-i\pi\nu}} \quad , \tag{5.22}$$

where

$$f(\nu) = i\pi\nu \sinh\gamma \frac{\frac{1}{2}\left[1 + \frac{1}{\epsilon(a)}\right] + \left[1 - \frac{1}{\epsilon(a)}\right] ka\,C}{\left[\sinh\gamma - C\right]\left[\sinh\gamma - \frac{1}{\epsilon(a)}\left(C - \frac{1+\epsilon(a)}{2ka}\right)\right]} \tag{5.23}$$

is a slowly varying function.

The main contribution to the integral along Γ_1 arises from a saddle-point at the origin $\nu = 0$. The integral function (5.22) can be asymptotically evaluated for $|\nu| \ll ka$ with the aid of the relations

$$e^{-2\psi} \sim \exp\left\{i\pi\nu - i\frac{\nu^2}{ka} - i2ka\right\}\left\{1 - \frac{i\nu^4}{12(ka)^3} + O\left[\frac{\nu^6}{(ka)^5}\right]\right\}, \tag{5.24}$$

$$\sinh\gamma \sim i\left\{1 - \frac{1}{2}\left(\frac{\nu}{ka}\right)^2 - \frac{1}{8}\left(\frac{\nu}{ka}\right)^4 + O\left[\left(\frac{\nu}{ka}\right)^6\right]\right\} , \tag{5.25}$$

$$C \sim -i\sqrt{\epsilon(a)} + \frac{i}{2\sqrt{\epsilon(a)}}\left(\frac{\nu}{ka}\right)^2 + \frac{i}{8}\left[\epsilon(a)\right]^{-3/2}\left(\frac{\nu}{ka}\right)^4 + O\left[\left(\frac{\nu}{ka}\right)^6\right] . \tag{5.26}$$

The integral along Γ_1 becomes:

$$\int_{\Gamma_1} \frac{\nu}{\cos \pi\nu}(a_{\nu-1/2} - b_{\nu-1/2})\,d\nu \sim \frac{i}{2\pi^2}\cdot\frac{1-\sqrt{\epsilon(a)}}{1+\sqrt{\epsilon(a)}}\, e^{-i2ka}\int_{-2\pi\xi e^{i\delta}}^{+2\pi\xi e^{i\delta}} \frac{e^{-\sigma w^2}}{1+e^{-w}}\, g(w)dw \ , \tag{5.27}$$

where

$$w = i2\pi\nu, \qquad \sigma = -\frac{i}{4\pi^2 ka}, \qquad \xi = (ka)^{\frac{1}{2}+\sigma}, \tag{5.28}$$

$$g(w) = 1 + \frac{i}{ka}\left[\frac{1}{4} + \frac{1}{1-\epsilon(a)}\right] - \frac{i}{192\pi^4}\,\frac{w^4}{(ka)^3} + O\left[(ka)^{-2}\right] + O\left[\frac{w^6}{(ka)^5}\right], \tag{5.29}$$

and δ is an angle between zero and $\pi/2$.

Since ξ is very large, the integral on the right-hand side of (5.27) may be decomposed into a sum of integrals $E_{o,n}$ defined by (1.32) and computed by Scott (1949). Specifically, in our notation:

$$\begin{aligned}
E_{o,o} &\sim i2\pi^2 ka\left\{1 + \frac{i}{12ka} + O\left[(ka)^{-2}\right]\right\}, \\
E_{o,1} &\sim -8\pi^4 (ka)^2\left\{1 + O\left[(ka)^{-2}\right]\right\}, \\
E_{o,2} &\sim -i64\pi^6 (ka)^3\left\{1 + O\left[(ka)^{-3}\right]\right\}, \\
E_{o,3} &\sim 768\pi^8 (ka)^4\left\{1 + O\left[(ka)^{-4}\right]\right\}.
\end{aligned} \tag{5.30}$$

Substitution of these formulas in relation (5.27) gives:

$$\int_{\Gamma_1} \frac{\nu}{\cos\pi\nu}\left(a_{\nu-1/2} - b_{\nu-1/2}\right) d\nu \sim ka\,\frac{\sqrt{\epsilon(a)}-1}{\sqrt{\epsilon(a)}+1}\,e^{-i2ka}\left\{1 + \frac{i}{ka}\left[\frac{1}{2} + \frac{1}{1-\epsilon(a)}\right] + O\left[(ka)^{-2}\right]\right\}. \tag{5.31}$$

In order to evaluate $(a_o - b_o)$ asymptotically as $ka \to \infty$, we use the relations

$$H^{(1)}_{1/2}(ka) = -i\sqrt{\frac{2}{\pi ka}}\; e^{ika} \,, \qquad H^{(1)'}_{1/2}(ka) = \sqrt{\frac{2}{\pi ka}}\; e^{ika}\left(1+\frac{i}{2ka}\right) \,, \tag{5.32}$$

$$\sqrt{\frac{1}{4ka} - \epsilon(a)} \;\sim\; -i\sqrt{\epsilon(a)}\left\{1 - \frac{1}{8\epsilon(a)(ka)^2} + O\left[(ka)^{-4}\right]\right\} \,, \tag{5.33}$$

in (5.12) and find

$$a_o - b_o \sim \frac{\sqrt{\epsilon(a)}-1}{\sqrt{\epsilon(a)}+1}\; e^{-i2ka}\left\{1+\frac{i}{\left[\epsilon(a)-1\right]ka} + O\left[(ka)^{-2}\right]\right\} \,. \tag{5.34}$$

The reflected field contribution to the far backscattered field is obtained by combining the asymptotic expansions (5.31) and (5.34) according to relation (5.11):

$$\left[E_x^{b.s.}\right]_{refl.} \sim \frac{1-\sqrt{\epsilon(a)}}{1+\sqrt{\epsilon(a)}}\left(\frac{a}{2r}\right) e^{ik(r-2a)}\left\{1+\frac{i}{2ka}\cdot\frac{1+\epsilon(a)}{1-\epsilon(a)} + O\left[(ka)^{-2}\right]\right\} \,. \tag{5.35}$$

In particular, in the limit $|\epsilon(a)| \rightarrow \infty$ one finds:

$$\left[E_x^{b.s.}\right]_{\substack{refl. \\ |\epsilon(a)|=\infty}} \sim -\frac{a}{2r}\, e^{ik(r-2a)}\left\{1-\frac{i}{2ka} + O\left[(ka)^{-2}\right]\right\} \,, \tag{5.36}$$

which is the well known expression for the field reflected by a perfectly conducting sphere, in the backscattering direction.

If $\epsilon(a) = 1$, expression (5.35) becomes

$$\left[E_x^{b.s.}\right]_{\substack{refl. \\ \epsilon(a)=1}} \sim \frac{i}{8kr}\, e^{ik(r-2a)}\left[1+O(1/ka)\right] \,. \tag{5.37}$$

5.4 Creeping Waves Contribution

The creeping waves contribution to the far backscattered field is given by the residue series:

$$\left[E_x^{b.s.}\right]_{cr.w.} \sim \pi \frac{e^{ikr}}{kr}\left\{\sum_{\nu=\nu_\ell} \frac{\nu}{\cos\pi\nu}\left[\zeta^{(1)}_{\nu-1/2}(ka)\frac{\partial}{\partial\nu}\left\{\zeta^{(1)'}_{\nu-1/2}(ka) - M_{\nu-1/2}\zeta^{(1)}_{\nu-1/2}(ka)\right\}\right]^{-1} - \sum_{\nu=\tilde{\nu}_\ell} \frac{\nu}{\cos\pi\nu}\left[\zeta^{(1)}_{\nu-1/2}(ka)\frac{\partial}{\partial\nu}\left\{\zeta^{(1)'}_{\nu-1/2}(ka) - \tilde{M}_{\nu-1/2}\zeta^{(1)}_{\nu-1/2}(ka)\right\}\right]^{-1}\right\}, \tag{5.38}$$

where ν_ℓ and $\tilde{\nu}_\ell$ are respectively the roots of

$$\frac{\zeta^{(1)'}_{\nu-1/2}(ka)}{\zeta^{(1)}_{\nu-1/2}(ka)} = C + \frac{1}{2ka} \tag{5.39}$$

and of

$$\frac{\zeta^{(1)'}_{\nu-1/2}(ka)}{\zeta^{(1)}_{\nu-1/2}(ka)} = \frac{1}{\epsilon(a)}\left[C - \frac{1}{2ka}\right] \tag{5.40}$$

which have positive imaginary parts, and C is given by (5.13).

If we define m and t by (1.46), and introduce the Fock asymptotic approximations (2.15), then the creeping waves contribution becomes:

$$\left[E_x^{b.s.}\right]_{cr.w.} \sim \frac{\pi}{m}\frac{e^{ikr}}{kr}\sum_{\ell}\left\{\frac{\nu_\ell}{\cos\pi\nu_\ell}\left[w_1(t_\ell)\right]^{-2}\left[\frac{t_\ell}{m^2}+\frac{1+\frac{t_\ell}{2m^2}}{ka\,C(t_\ell)}-\left(C(t_\ell)+\frac{1}{2ka}\right)^2\right]^{-1}-\frac{\tilde{\nu}_\ell}{\cos\pi\tilde{\nu}_\ell}\left[w_1(\tilde{t}_\ell)\right]^{-2}\left[\frac{\tilde{t}_\ell}{m^2}+\frac{1+\frac{\tilde{t}_\ell}{2m^2}}{ka\,\epsilon(a)C(\tilde{t}_\ell)}-\frac{1}{\left[\epsilon(a)\right]^2}\left(C(\tilde{t}_\ell)-\frac{1}{2ka}\right)^2\right]^{-1}\right\}, \tag{5.41}$$

where

$$C = C(t) = \sqrt{1-\epsilon(a)+\frac{t}{m^2}+\frac{t^2}{4m^4}}\ , \tag{5.42}$$

t_ℓ and $\tilde{t}_\ell$ are respectively the roots of

$$\frac{w_1'(t_\ell)}{w_1(t_\ell)} = -\frac{1}{4m^2} - m\,C(t_\ell) \tag{5.43}$$

and of

$$\frac{w_1'(\tilde{t}_\ell)}{w_1(\tilde{t}_\ell)} = \frac{1}{\epsilon(a)}\left[\frac{1}{4m^2} - mC(\tilde{t}_\ell)\right], \tag{5.44}$$

and $w_1(t)$ is the Airy function in Fock's notation.

The first few roots of equations (5.43) and (5.44) may be derived from the values of $w_1'(t)/w_1(t)$ which were computed by Logan and Yee (1962) when t lies in the first quadrant.

In particular, if m is large compared to unity and $\epsilon(a)$ is bounded away from unity, or more specifically

$$\frac{|t|}{4m^2} \ll 1 \ , \qquad \frac{|t|}{m^2} \ll |\epsilon(a) - 1| \ , \tag{5.45}$$

then t_ℓ and $\tilde{t}_\ell$ are approximately given by the roots of

$$\frac{w_1'(t_\ell)}{w_1(t_\ell)} \sim -im\sqrt{\epsilon(a)-1} + \frac{i}{2m\sqrt{\epsilon(a)-1}}\, t_\ell \ , \tag{5.46}$$

and

$$\frac{w_1'(\tilde{t}_\ell)}{w_1(\tilde{t}_\ell)} \sim -\frac{im}{\epsilon(a)}\sqrt{\epsilon(a)-1} + \frac{i}{2m\,\epsilon(a)\sqrt{\epsilon(a)-1}}\, \tilde{t}_\ell \ . \tag{5.47}$$

5.5 Evanescent Wave Contribution

We suppose that no poles of the quantity (5.12) lie on the branch-cut; then the integral along the contour Γ_3 of Fig. 5-3 becomes:

$$\int_{\Gamma_3} \frac{\nu}{\cos \pi\nu} (a_{\nu-\frac{1}{2}} - b_{\nu-\frac{1}{2}}) d\nu = \int\!\!\!\!\!\!\diagup_{ka\sqrt{\epsilon(a)}}^{+i\infty} \frac{\nu}{\cos \pi\nu} \Big[(a_{\nu-\frac{1}{2}} - b_{\nu-\frac{1}{2}})_+ - (a_{\nu-\frac{1}{2}} - b_{\nu-\frac{1}{2}})_- \Big] d\nu \ , \tag{5.48}$$

where $\int\!\!\!\!\diagup$ is taken along the cut in the first quadrant of Fig. 5-2, and $(a_{\nu-\frac{1}{2}} - b_{\nu-\frac{1}{2}})_\pm$ are given by (5.12) in which C has the values $C_\pm$ given by (5.16).

The calculation of the integral along the cut is complicated by the fact that different asymptotic expansions for the cylinder functions which appear in the integrand must be used in the Debye region 1 and in the regions 2 and 6a. Also, if ϵ'' is small enough, the cut crosses the region $|\nu - ka| < |\nu^{1/3}|$ in which Langer's uniform expansions must be employed. This latter inconvenience is avoided if

$$\epsilon'' > 2(ka)^{-2/3} \ . \tag{5.49}$$

In the following, an asymptotic estimate of the integral (5.48) is derived in the practically interesting case in which

$$\epsilon' \simeq 1 , \qquad 2(ka)^{-2/3} < \epsilon'' \ll 1 . \tag{5.50}$$

The contour of integration is replaced by the horizontal straight line $\nu'' = \frac{\epsilon''}{2} ka$ from the branch-point to the imaginary ν-axis, plus the portion of the imaginary ν-axis from $\frac{\epsilon''}{2}$ ka to infinity (broken lines in Fig. 5-4). The integral thus obtained is slightly in excess of the true value (5.48):

$$\left|\int_{ka\sqrt{\epsilon(a)}}^{+i\infty}\right| < \int_A^B + \int_B^C + \int_C^{S_1} + \int_{S_1}^{+i\infty} \;; \tag{5.51}$$

the points $A \equiv ka\sqrt{\epsilon(a)}$, B, C and S_1 are shown in Fig. 5-4.

Upper bounds for the integrals in (5.51) are obtained with the aid of the following relations:

$$\frac{1}{|\cos \pi\nu|} \lesssim 2 e^{-\frac{\pi}{2}\epsilon'' ka} , \quad \text{on ABC} ,$$
$$= \frac{1}{\cosh(\pi\nu')} , \quad \text{on the imaginary } \nu\text{-axis}; \tag{5.52}$$

$$\left|(a_{\nu-1/2} - b_{\nu-1/2})_+ - (a_{\nu-1/2} - b_{\nu-1/2})_-\right| \sim \frac{1}{2ka} , \quad \text{on AB} ,$$
$$\lesssim 2\left[1 - \left(\frac{\nu'}{ka}\right)^2\right]^{3/2} , \quad \text{on BC} ,$$
$$\sim 2\left[1 + \left(\frac{\nu''}{ka}\right)^2\right]^{3/2} e^{-\pi\nu''} , \quad \text{on the imaginary } \nu\text{-axis} , \tag{5.53}$$

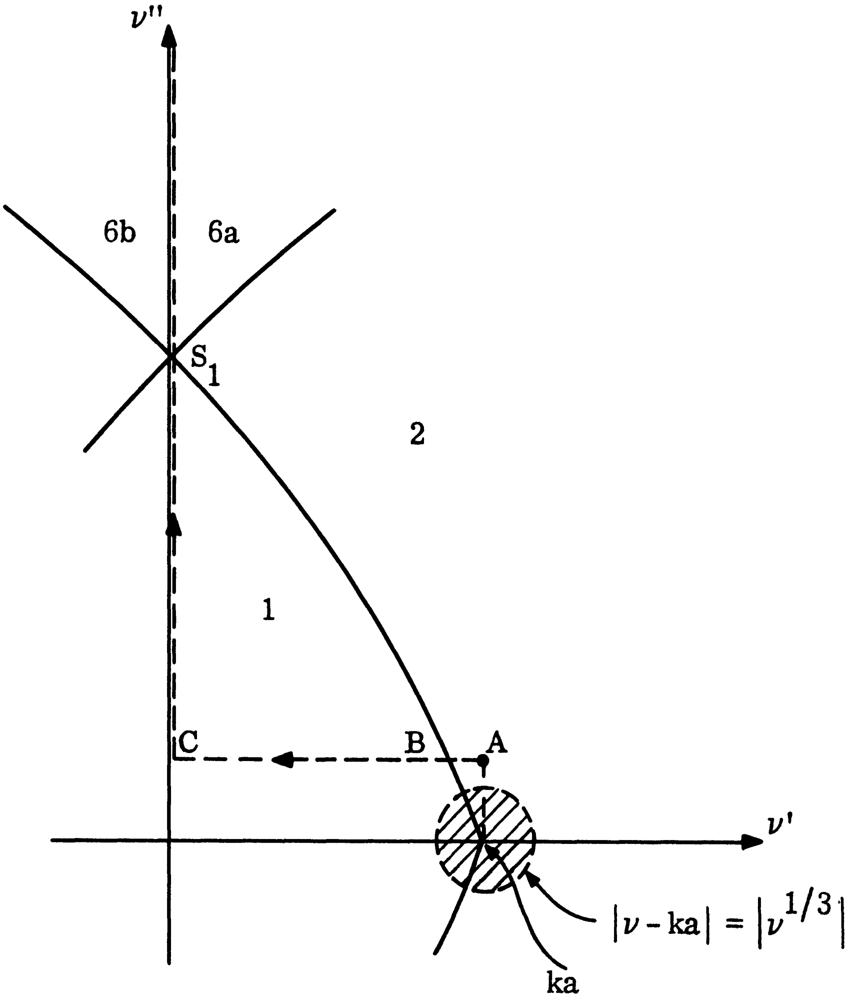

FIG. 5-4: CONTOUR OF INTEGRATION TO ESTIMATE THE BRANCH-CUT CONTRIBUTION.

where use has been made of the appropriate Debye expansions in regions 1, 2 and 6a. One then finds:

$$\left|\int_A^B\right| \sim e^{-\frac{\pi}{2}\epsilon'' ka} O(ka) , \tag{5.54}$$

$$\left|\int_B^C\right| \sim e^{-\frac{\pi}{2}\epsilon'' ka} O\left[(ka)^2\right] , \tag{5.55}$$

$$\left|\int_C^{S_1} + \int_{S_1}^{+i\infty}\right| \lesssim 4(ka)^2 \int_{\epsilon''/2}^{\infty} \frac{e^{-2\pi kax}}{1+e^{-2\pi kax}} (1+x^2)^{3/2} x\,dx <$$

$$< 4(ka)^2 e^{-\pi\epsilon'' ka} \int_0^{\infty} e^{-2\pi kay} \left[1+\left(y+\frac{\epsilon''}{2}\right)^2\right]^{3/2} \left(y+\frac{\epsilon''}{2}\right) dy <$$

$$< 128(ka)^2 e^{-\pi\epsilon'' ka} \int_0^{\infty} e^{-4\pi kaz} (1+z)^4 dx \sim$$

$$\sim e^{-\pi\epsilon'' ka} O(ka) ; \tag{5.56}$$

the last integral has been evaluated by observing that since

$$(1+z)^4 = \sum_{n=1}^{5} \binom{4}{n-1} z^{n-1} ,$$

one has by Watson's lemma (Erdélyi, 1956, section 2.2):

$$\int_0^{\infty} e^{-4\pi kaz} (1+z)^4 dz \sim \sum_{n=1}^{5} \binom{4}{n-1} \Gamma(n)(4\pi ka)^{-n} = O\left[(ka)^{-1}\right] .$$

On account of relations (5.11), (5.17), (5.51), (5.54), (5.55) and (5.56), the branch-cut contribution to the far backscattered field when $\epsilon(a)$ satisfies (5.50) becomes:

$$\left[E_x^{b.s.}\right]_{ev.w.} \sim \frac{e^{ikr}}{kr} e^{-\frac{\pi}{2}\epsilon''ka} O\left[(ka)^2\right] ; \tag{5.57}$$

since ϵ'' is kept fixed as $ka \to \infty$, this contribution decreases exponentially as ka increases and can therefore be physically interpreted as due to an evanescent wave (see, for example, Felsen, 1964).

5.6 Discussion of Results

The total high-frequency far backscattered field is obtained by adding together the reflected field contribution (5.35), the creeping waves contribution (5.41) and the evanescent wave contribution, which is given by (5.57) when $\epsilon(a)$ satisfies (5.50).

The far field coefficient, that is the coefficient of $e^{ikr}/(kr)$, has the following orders of magnitude:

1) reflected field: $O(ka)$, if $\epsilon(a) \neq 1$,

$O(1)$, if $\epsilon(a) = 1$;

2) creeping waves: $O\left[(ka)^{-1/3}\right]$;

3) evanescent wave: $O\left[(ka)^2 e^{-\frac{\pi}{2}\epsilon''ka}\right]$, if $\epsilon''ka \gg 1$;

therefore the dominant contribution to the backscattered field is due to reflection. The reflected field backscattering cross section $\sigma_{refl.}$ is given by:

$$\frac{\sigma_{refl.}}{\pi a^2} \sim \left|\frac{\sqrt{\epsilon(a)}-1}{\sqrt{\epsilon(a)}+1}\right|^2 \left[1+O(1/ka)\right] , \quad \text{if } \epsilon(a) \neq 1 ,$$

$$\sim (4ka)^{-2}\left[1+O(1/ka)\right] , \quad \text{if } \epsilon(a) = 1 , \tag{5.58}$$

where πa^2 is the cross section of a perfectly conducting sphere of radius a.

It should be pointed out that in the analysis of the previous two sections it was assumed that no pole lies at the branch-point $ka\sqrt{\epsilon(a)}$ or very near it. This hypothesis may not be satisfied for certain choices of $\epsilon(a)$, and the analysis should then be modified accordingly (see, for example, Van der Waerden, 1950).

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